# Geometric quantum complexity of bosonic oscillator systems 

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Abstract: According to the pioneering work of Nielsen and collaborators, the length of the minimal geodesic in a geometric realization of a suitable operator space provides a measure of the quantum complexity of an operation. Compared with the original concept of complexity based on the minimal number of gates required to construct the desired operation as a product, this geometrical approach amounts to a more concrete and computable definition, but its evaluation is nontrivial in systems with a high-dimensional Hilbert space. The geometrical formulation can more easily be evaluated by considering the geometry associated with a suitable finite-dimensional group generated by a small number of relevant operators of the system. In this way, the method has been applied in particular to the harmonic oscillator, which is also of interest in the present paper. However, subtle and previously unrecognized issues of group theory can lead to unforeseen complications, motivating a new formulation that remains on the level of the underlying Lie algebras for most of the required steps. Novel insights about complexity can thereby be found in a low-dimensional setting, with the potential of systematic extensions to higher dimensions as well as interactions. Specific examples include the quantum complexity of various target unitary operators associated with a harmonic oscillator, inverted harmonic oscillator, and coupled harmonic oscillators. The generality of this approach is demonstrated by an application to an anharmonic oscillator with a cubic term.

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## Contents

1 Introduction ..... 1
2 Overview of quantum complexity ..... 4
2.1 Gate approach to quantum complexity - Gate Complexity ..... 4
2.2 Geometrical approach to quantum complexity ..... 5
2.3 Recipe to determine complexity using the geometrical approach ..... 8
3 The harmonic oscillator group. ..... 9
3.1 Mathematical properties ..... 9
3.2 Computation of complexity of various target unitary operators ..... 11
3.2.1 Euler-Arnold equations and solutions ..... 12
3.2.2 Displacement operator ..... 13
3.2.3 Complexity of the time evolution operator ..... 14
3.2.4 Harmonic oscillator with an additional linear potential ..... 16
3.2.5 Geodesic equation for general frequencies ..... 18
4 Harmonic oscillator complexity from the $\mathrm{sp}(2, \mathbb{R})$ Lie algebra ..... 19
4.1 Time evolution operator of the harmonic oscillator - revisited ..... 21
4.2 Time evolution operator of the inverted harmonic oscillator ..... 22
4.3 Harmonic oscillator with an additional $Q^{2}$ term in the Hamiltonian. ..... 23
5 Coupled harmonic oscillators ..... 25
6 Anharmonic oscillator with the cubic term ..... 29
7 Discussion and Comments ..... 36
A Line element ..... 38
B Using a generic element of the suitable group to compute complexity ..... 39
C Commutation relations of the generators associated with the coupled har-monic oscillator46

## 1 Introduction

Quantum complexity, a notion of measuring the difficulty of carrying out a complicated task as a succession of simple operations, is an important concept in a variety of different areas. For instance, it has surprisingly helped to study important physical problems such as information processing by black holes. In this context, the famous Ryu-Takayanagi proposal
[1-3] of relating entanglement entropy (between two regions in the boundary conformal field theory) to minimal surfaces in AdS spaces proved to be extremely successful in interpreting spacetimes as emergent from quantum entanglement. Black holes posed serious challenges because the volume behind the horizon of the AdS black hole continues to grow for a very long time well after the saturation of its entanglement entropy [4]. Hence, it became essential to look for quantities that capture this long-term growth. Susskind and collaborators' [5-7] suggested that the growth of the volume should be dual to the quantum complexity of the boundary field theory that led to the "Complexity=Volume" (CV) conjecture [5-7], followed by the "Complexity=Action" (CA) conjecture, which related complexity of the field theory to the gravitational action of the Wheeler-de-Witt patch of the bulk gravity theory $[8,9]$. The most recent viewpoint of complexity is known as the "Complexity=Anything" [10, 11].

In quantum gravity, the motivation to study quantum complexity is primarily derived from the enigmatic black hole phenomena. Nevertheless, its importance extends to numerous other branches of physics, particularly those involving complex systems ${ }^{1}$. In broad terms, a complex system is generally characterized by a high number of degrees of freedom, non-linear interactions, and long-distance correlations. There exists a plethora of complex systems across various disciplines of physics, ranging from strongly correlated quantum systems to biological systems and even extending to social networks. In this discussion, however, our focus is specifically limited to systems that demonstrate quantum properties.

The notion of quantum complexity is relatively new in physics, but in computer science and information theory various relevant notions of complexity have been used for some time. All of them have something to do with how a complex object is built from simple or small constituents, and how many of these simple parts are needed to make the complex whole. Two of the most popular notions of complexity in computer science are the Kolmogorov complexity [13] and the stochastic complexity [14]. The Kolmogorov complexity of an object measures the length of the shortest program required to produce it. The stochastic complexity is applied to messages and the smallest code required to compress them. The general notion of complexity was inherited from computer science and information theory by quantum computation research in efforts to determine the complexity of quantum circuits.

In a groundbreaking series of papers, Nielsen and his collaborators [15-17] established a correlation between quantum complexity and the length of the minimal geodesic in the space of unitary operators. This geometric definition opened up an entirely new avenue of research, leading to significant advancements in understanding the role of quantum complexity in holography. Nielsen's formulation primarily focused on determining the minimal circuit size necessary to implement an $n$-qubit unitary operation, thereby determining the complexity of specific quantum states. However, its significance in the broader context of quantum mechanics remains largely unexplored. Several notable efforts have recently been made to quantify the complexity of individual states [18-27]. Reference [18] was the first paper to attempt to define complexity in quantum field theory. Specifically, they were interested in the complexity of the ground state in free scalar field theory. For this purpose,

[^1]choosing a suitable reference state is required. The approach of state complexity is then dependent on the choice of reference state. In practice, many of these attempts are limited by the requirement that the reference and target states be Gaussian. Several methods were developed and extensively applied to deal with the complexity of Gaussian states, like the covariance matrix method [19], Fubini-Study method [23]. An attempt to analyze complexity for a non-Gaussian state was made in [24]. A comparative study of different approaches to state complexity was carried out in [28]. While these achievements represent remarkable progress, they do not provide a comprehensive understanding of the complexity associated with quantum processes in general.

In our studies, we will be primarily interested in operator complexity, specifically in the complexity of time evolution operators associated with certain Hamiltonians rather than the complexity of a certain state. This article has two main motivations:

- Providing a general recipe to determine the complexity of any unitary operator without needing a separate definition of reference or target states. In the literature, the notion of operator complexity is mainly introduced by characterizing their action on Gaussian reference and target states. These limitations make it challenging to provide general statements on the complexity of the operator. A general state-independent approach is expected to be more feasible in extensions of the analysis to interacting systems or in applications to Hamiltonians beyond the quadratic level.
- Understanding how the complexity changes as a quantum system undergoes time evolution. For this purpose, the target unitary operators, whose complexity we will be interested in, are time-evolution operators for various systems.

The determination of the complexity of a certain state has its significance in the context of quantum simulations, particularly in many body systems and their wave functions. An important example is the complexity of specific states, such as the thermofield double state, which is conjectured to be dual to an eternal black hole in anti-de Sitter space-time and may, therefore, play a crucial role in the understanding of stationary black holes. By contrast, the complexity of time evolution addresses the question of the least complex way of evolving a quantum system: Is it possible to find a shorter path between two points in the evolution other than the one generated by the Hamiltonian itself? For our exploration, we will consider some of the widely considered models, including those considered in [18, 19, 29]. However, we will not be interested in the complexity of a particular state but rather in the complexity of the time-evolution operator.

Nielsen's general method, which can be applied to non-qubit systems by replacing the unitary groups $\operatorname{SU}(N)$ with other suitable Lie groups, is well-suited to this purpose. If the quantum system describes the motion of a point particle, the method requires a truncation of the space of all unitary operators on an infinite-dimensional Hilbert space to a suitable finite-dimensional group. In the case of the harmonic oscillator, with its wide-ranging applications in various fields, the groups $\mathrm{SU}(1,1)$ and $\mathrm{Sp}(2, \mathbb{R})$ play a prominent role (see for instance [30] for such applications to complexity). However, unlike $\operatorname{SU}(N)$, these groups are not compact, leading to mathematical subtleties that, to the best of our knowledge, have
not been appreciated in the existing literature. We will discuss these issues in the present paper - such as the non-existence of finite-dimensional unitary matrix representations, non-surjectivity of the exponential map, and geodesic incompleteness of group metrics and analyze to what degree they can be avoided by working on an algebraic rather than group-theoretic level whenever possible. We also demonstrate extensions of this method that allow us to go beyond quadratic Hamiltonians, providing the first results of complexity for anharmonic systems.

The rest of the paper is organized as follows: in Section 2 we review the essentials of quantum complexity. We start by introducing the conventional definition of quantum complexity described in terms of the number of universal gates in the desired quantum circuit. We explain the significance of the geometrical definition of quantum complexity by pointing out the drawbacks and limitations of the gate definition of quantum complexity. We then provide a quick review of the geometrical approach to quantum complexity with a complete recipe for a general target unitary operator. In Section 3, we briefly discuss the essential quantities corresponding to the harmonic oscillator group as a minimal implementation of the geometric method, highlighting several subtleties, and illustrate the recipe of the geometrical framework of quantum complexity bounds in this system by an explicit computation for the displacement operator and several examples of harmonic time evolution operators, which was the primary motivation of the paper. In Section 4, we show that not only the harmonic oscillator group but also the symplectic group $\operatorname{Sp}(2, \mathbb{R})$ is efficient in determining the complexity of the harmonic oscillator, and in addition of the inverted harmonic oscillator which cannot be described using the harmonic oscillator group. In Section 5, we consider the case of two coupled harmonic oscillators and study the complexity of time evolution in such coupled systems. Our final example discusses an anharmonic oscillator with a cubic term, after which we conclude with a brief discussion of our findings and future directions. Various calculations details are relegated to appendices.

## 2 Overview of quantum complexity

In this section, we briefly review Nielsen's geometric interpretation of quantum complexity. As discussed earlier, the notion of quantum complexity measures the minimum number of simple operations required to carry out a complex task. In terms of unitary operators (or quantum states), we might think of complexity as the minimum number of elementary gates required to construct the quantum circuit that will produce the desired unitary operator (or takes a reference state to the target state). This approach of counting gates is better known as gate complexity.

### 2.1 Gate approach to quantum complexity - Gate Complexity

Consider a target state $\left|\Psi_{T}\right\rangle$ to be constructed from a reference state $\left|\Psi_{R}\right\rangle$ :

$$
\begin{equation*}
\left|\Psi_{T}\right\rangle=U_{\text {target }}\left|\Psi_{R}\right\rangle . \tag{2.1}
\end{equation*}
$$

Here, $U_{\text {target }}$ represents the unitary operator that realizes the transformation $\left|\Psi_{R}\right\rangle \rightarrow\left|\Psi_{T}\right\rangle$. Thus, $U_{\text {target }}$ is our target unitary operator whose complexity we are interested in. An
actual construction of a quantum circuit would proceed by building it out of elementary operations $g_{i}$ that synthesize $U_{\text {target }}$ :

$$
\begin{equation*}
\left|\Psi_{T}\right\rangle=U_{\text {target }}\left|\Psi_{R}\right\rangle=g_{n} g_{n-1} \cdots g_{2} g_{1}\left|\Psi_{R}\right\rangle \tag{2.2}
\end{equation*}
$$

Out of all possible circuits built from the chosen gate set $\left\{g_{i}\right\}$, realizing $U_{\text {target }}$ (up to a certain level of accuracy $\epsilon$ ), the circuit requiring the minimal number of gates in $\left\{g_{i}\right\}$ is the optimal one, and this number of gates in the optimal circuit gives a measure of the complexity of $U_{\text {target }}$. Therefore,

## Gate Complexity $\equiv$ number of quantum gates used in the optimal circuit that implements the desired $U_{\text {target }}$ within $\epsilon$.

The gate approach of quantifying complexity is adapted to the question of how to build an actual quantum circuit out of component gates, but this definition is not suitable for quantum systems with continuous variables. One of the major drawbacks is the choice of a set of gates. For example, if two gate sets $\left\{g_{i}\right\}$ and $\left\{m_{i}\right\}$ can realize the same $U_{\text {target }}$, the numbers of gates in the optimal circuit required from $\left\{g_{i}\right\}\left(n_{1}\right)$ and from $\left\{m_{i}\right\}\left(n_{2}\right)$ are in general different. Therefore, a well-defined complexity of $U_{\text {target }}\left(n_{1}\right.$ or $\left.n_{2}\right)$ requires an a priori choice of the gate set, $\left\{g_{i}\right\}$ or $\left\{m_{i}\right\}$. The gate complexity is thus not uniquely defined for a target unitary.

Another significant drawback of gate complexity is its sensitivity to the level of accuracy $\left(\epsilon=\left|U_{\text {target }}-g_{n} g_{n-1} \ldots g_{2} g_{1}\right|\right.$ in some operator norm) required for the quantum circuit. Using discrete gates to build a circuit may result in producing a unitary $U_{\text {target }}^{A}$, which is close to but not exactly equal to the desired $U_{\text {target }}$. The need to use an operator norm renders this concept of closeness non-unique. In the above example, instead of $U_{\text {target }}$, we would apply $U_{\text {target }}^{A}$ to an initial state of a quantum system. The dynamics produced by $U_{\text {target }}^{A}$ is not identical to the desired dynamics, generated by $U_{\text {target }}$, and will have some correction terms which might play a significant role for instance in quantum computation. The correction terms can be reduced by making $U_{\text {target }}^{A}$ as close as possible to $U_{\text {target }}$, using smaller and smaller $\epsilon$, but the outcomes of the actual target operator may be different in each case. Therefore, a dependence of complexity on the sensitivity $\epsilon$ is not desirable. This level of non-uniqueness in the definition of quantum complexity calls for a better and more concrete alternative. A geometrical viewpoint of quantum complexity was given by Nielsen and his collaborators, which we discuss in the next subsection.

### 2.2 Geometrical approach to quantum complexity

In a series of papers, Nielsen et. al [15-17] proposed a transition from the discrete picture of gate complexity to a continuous description, making new connections between quantum complexity and differential geometry. They observed that determining the quantum complexity of a unitary operation is closely related to the problem of finding minimal-length geodesics in a certain curved geometry. The original motivation for introducing a geometrical notion of quantum complexity was to use it as a tool to bound the value of gate complexity. From this initial definition, it has evolved into a candidate for a fundamental
and unique definition of quantum complexity, a viewpoint that is supported by its smaller degree of ambiguity. Therefore, from now on, we will cease thinking of complexity geometry as a continuous approximation to gate complexity but rather view gate complexity as a discrete approximation of geometric complexity.

Let us briefly review the overall idea of the geometrical method to quantum complexity. In this approach, the complexity of a unitary operator $U$ is the length of the minimal geodesic on the unitary group manifold joining the identity to $U$. In the original manifestation of the geometric approach, unitaries acting on $n$-qubit systems were investigated, and the framework relied on the special unitary groups $\mathrm{SU}\left(2^{n}\right)$. An extension of the basic idea of the entire framework to a general unitary is initially straightforward, but it does lead to several mathematical subtleties, some of which will be described in more detail in the main part of our paper. For instance, a general discussion of complexity in quantum mechanics would require differential geometry on infinite-dimensional manifolds. Properties of geodesics are then hard to analyze, not only because solving an infinite set of coupled differential equations is usually difficult, but also because a geodesic between two given points is then not guaranteed to exist (since the Hopf-Rinow theorem no longer applies). Instead, geodesic distance is defined as the infimum (not necessarily a minimum) of the distance on the space of all possible curves connecting the two points, and there may be no curve that has the resulting distance. For a tractable application of the method, one should therefore, first reduce the infinite-dimensional problem to a finite-dimensional one, depending on the target unitary of interest. Even then, properties of Lie groups may lead to further subtleties.

Given the target unitary operator whose complexity is of interest, one may identify a set of fundamental operators related in some way to it. In particular, the Lie algebra generated by a suitable choice of fundamental operators of the quantum system (of which there are finitely many ones in quantum mechanics, as opposed to quantum field theory) can be exponentiated to a group of which the target unitary is an element. If the target unitary is defined as the exponential of a Hamiltonian (times $i$ ), the task in this step is to find a suitable set of other operators, including additional observables of interest such as position and momentum, that, together with the Hamiltonian, have a closed set of commutators. After identifying these generators of a Lie algebra, one classifies them as "easy" or "hard" for an application in the given quantum system. For instance, the closure condition on the brackets may require one to use additional generators that are physically less motivated than the original observables, which accordingly would be considered "hard" to construct. Their contributions are then suppressed in the geodesic distance by assigning large metric components to their directions. The Lie algebra generated by all the operators, easy and hard ones, is exponentiated to a Lie group. If the generators are self-adjoint, the Lie group is unitary.

In order to define a geometry, one then considers a right-invariant metric that accurately captures the hardness property by penalizing the directions along the hard operators such that moving in their direction is discouraged for geodesics on the Lie group. In the literature on quantum complexity, the set of hardness coefficients is known as the penalty factor matrix, denoted by $G_{I J}$. The choice of the matrix $G_{I J}$ is usually motivated by
phenomenological considerations [31, 32], inspired by difficulties of performing certain operations during an experiment [33]. Sometimes theoretical bias is also used in this choice. Assuming different penalties or operational costs in different directions $\left(G_{I J} \neq \delta_{I J}\right)$ introduces anisotropy in the resulting operator space geometry. The choice of the metric leads to a notion of distance on the unitary space, which, as recalled in Appendix A, is given by:

$$
\begin{equation*}
d s^{2}=\frac{1}{\operatorname{Tr}\left(\mathcal{O}_{I} \mathcal{O}_{I}^{\dagger}\right) \operatorname{Tr}\left(\mathcal{O}_{J} \mathcal{O}_{J}^{\dagger}\right)}\left[G_{I J} \operatorname{Tr}\left[i U^{-1} \mathcal{O}_{I}^{\dagger} d U\right] \operatorname{Tr}\left[i U^{-1} \mathcal{O}_{J}^{\dagger} d U\right]\right] \tag{2.3}
\end{equation*}
$$

where the $\mathcal{O}_{I}$ represent the generators of the unitary group and $U$ plays the role of a point on the manifold. The trace Tr is taken in a matrix representation of the generators. For geodesics, only the right-invariance of the line element matters, but not the specific form on the entire group.

An efficient way of determining geodesics on Lie groups equipped with a right-invariant metric was given by Arnold and is known as the Euler-Arnold equation [34]:

$$
\begin{equation*}
G_{I J} \frac{d V^{J}(s)}{d s}=f_{I J}^{K} V^{J}(s) G_{K L} V^{L}(s) \tag{2.4}
\end{equation*}
$$

where $f_{I J}^{K}$ are the structure constants of the Lie algebra, defined by

$$
\begin{equation*}
\left[\mathcal{O}_{I}, \mathcal{O}_{J}\right]=i f_{I J}^{K} \mathcal{O}_{K} \tag{2.5}
\end{equation*}
$$

The components $V^{I}(s)$ represent the tangent vector (or the velocity) at each point along the geodesic, defined by:

$$
\begin{equation*}
\frac{d U(s)}{d s}=-i V^{I}(s) \mathcal{O}_{I} U(s) \tag{2.6}
\end{equation*}
$$

The coupled differential equations (2.4) do not depend on the position $U(s)$ along the geodesic, and can therefore be solved independently of (2.6). Given a solution $V^{I}(s)$, a further integration of (2.6) results in the path (or trajectory) in the group, guided by the velocity vector $V^{I}(s)$. Generically, this solution can be written as the path-ordered exponential:

$$
\begin{equation*}
U(s)=\mathcal{P} \exp \left(-i \int_{0}^{s} d s^{\prime} V^{I}\left(s^{\prime}\right) \mathcal{O}_{I}\right) \tag{2.7}
\end{equation*}
$$

on which we impose the boundary conditions

$$
\begin{equation*}
U(s=0)=\mathbb{I} \quad \text { and } \quad U(s=1)=U_{\mathrm{target}} \tag{2.8}
\end{equation*}
$$

where $U_{\text {target }}$ is some target unitary whose complexity we wish to study.
In general, equation (2.4) defines a family of geodesics $\left\{V^{I}(s)\right\}$ on the unitary space. The boundary condition $U(s=1)=U_{\text {target }}$ filters out those geodesics that can realize the target unitary operator by fixing the magnitude of the tangent vector $V^{I}$ at $s=0$ (at the identity operator). This procedure is analogous to the shooting method in which the trajectory followed by a particle with a certain initial velocity is required to hit the target. The initial velocity for which the target is reached is determined by the boundary condition at $s=1$.

In general, there might be more than one value of the initial velocity for which the point of the target unitary is reached. The complexity of the target unitary operator is then given by the length of the shortest geodesic realizing the target unitary operator:

$$
\begin{equation*}
C\left[U_{\mathrm{target}}\right]=\min _{\left\{V^{I}(s)\right\}} \int_{0}^{1} d s \sqrt{G_{I J} V^{I}(s) V^{J}(s)}, \tag{2.9}
\end{equation*}
$$

where the minimization is over all geodesics $\left\{V^{I}(s)\right\}$ from the identity to $U_{\text {target }}$. This equation makes use of the right-invariance of the line element.

The Euler-Arnold equations has been extensively used recently to compute the geodesics on unitary manifolds, see Refs. [31, 34-38].

### 2.3 Recipe to determine complexity using the geometrical approach

In this subsection, we provide a summary in the form of a simple recipe that can be applied to compute the complexity of an operator:

- From the operator whose complexity is to be determined, identify a basis of the generators $\left(\mathcal{O}_{I}\right)$ that form a closed commutator algebra and hence specify a Lie group. A commutator algebra is said to be closed if taking the commutation of any two elements produces an element that is also part of the algebra. It might be possible that the generators of some target operators do not form a closed algebra. The simplest example is that of a unitary operator, which is generated from non-quadratic Hamiltonians like an anharmonic oscillator. In that case, by penalizing the generators of higher orders, we can neglect their contribution in order to regain a closed set of commutators. Geometrically, large penalties restrict the movement in the direction of higher-order generators. Alternatively, the complexity resulting from a calculation that ignores higher-order generators can be interpreted as an upper bound on the exact complexity because it ignores possible shortcuts that might be taken in the direction of higher-order generators. We will also encounter additional subtleties related to group-theoretical properties of non-compact Lie groups that indicate that the computed distances should be considered upper bounds on the complexity rather than strict values.
- The commutators of the generators determine the structure constants of their Lie algebra. Using the structure constants, solve the Euler-Arnold equation to get the set of geodesics $\left\{V^{I}(s)\right\}$ in the corresponding Lie group.
- Having obtained the $V^{I}(s)$, use them to compute the path-ordered exponential (2.7).
- Implement the boundary conditions $U(s=0)=\mathbb{I}$ and $U(s=1)=U_{\text {target }}$ by fixing the initial value of the components of the velocity vector $V^{I}(s)$ in terms of the parameters of the target unitary. This step ensures that the geodesic characterized by those initial values reaches the target.
- Compute the length of the geodesics for all values of the initial tangent vector determined in the previous step with respect to the chosen right-invariant metric.
- The length of the minimal geodesic determines the complexity of the target unitary operator.

It is worth repeating our new viewpoint that the geometrical method applied to an originally infinite-dimensional quantum system is, in general, expected to provide an upper bound on the complexity. Computational tractability often requires one to ignore directions in the full group of unitaries that may seem unrelated to the problem at hand but might still be relevant for geodesics as a shortcut between the initial and target operator. We will, therefore, refer to our results as "complexity bounds."

We will follow this recipe for various harmonic oscillator systems, using two different non-compact groups that can be interpreted as containing the harmonic oscillator Hamiltonian as a generator. We first introduce and apply the harmonic oscillator groups, in which the Hamiltonian is accompanied by position, momentum, and the identity, and then turn to the symplectic group.

## 3 The harmonic oscillator group.

We begin with a brief review of the harmonic oscillator group. This group is based on four generators $Q, P, H$ and $E$, which satisfy the commutation relations

$$
\begin{equation*}
[Q, P]=i E, \quad[H, Q]=-i P, \quad[H, P]=i Q . \tag{3.1}
\end{equation*}
$$

Upon exponentiation, the generators $Q, P, H$, and $E$ specify a Lie group, which is popularly known as the harmonic oscillator group, studied extensively in [39]. If we represent the generators $Q, P, H$ and $E$ as

$$
\begin{equation*}
Q=x, \quad P=-i \frac{\partial}{\partial x}, \quad H=-\frac{\partial^{2}}{\partial x^{2}}+x^{2} \quad \text { and } E=\mathbb{I}, \tag{3.2}
\end{equation*}
$$

on the Hilbert space of square-integrable functions of $x$, they can be recognized as the position, momentum, Hamiltonian and the identity operator of a harmonic oscillator. The generators $P, Q$ and $E$ form a subalgebra isomorphic to the Heisenberg algebra.

### 3.1 Mathematical properties

Exponentiation of all four generators results in the harmonic oscillator group. This group is not exponential, as shown in [39], which means that there are some group elements that cannot be written as $\exp \left(-i\left(\alpha_{1} E+\alpha_{2} P+\alpha_{3} Q+\alpha_{4} H\right)\right)$ with real numbers $\alpha_{j}$. By definition, the Lie group of the corresponding Lie algebra is the group generated by $\exp \left(-i \alpha_{1} E\right)$, $\exp \left(-i \alpha_{2} P\right), \exp \left(-i \alpha_{3} Q\right)$ and $\exp \left(-i \alpha_{4} H\right)$. Trying to rewrite a generic product of these exponentials as an exponentiated sum of the generators, using the Baker-Campbell-Hausdorff formula, may result in an infinite series of the generators that is not contained in the Lie algebra. A generic element of the Lie group is, therefore, not guaranteed to be the exponential of some element of the Lie algebra. Mathematically, this property is related to the fact that the harmonic oscillator group is solvable and, unlike the Heisenberg group, not nilpotent. If complexity calculations are performed by exponentiating all elements of
the Lie algebra and not considering products, one does not cover the full Lie group if it is not exponential. Geodesics in the covered part of the group may then be shortened if one makes it possible to include also the missing directions. As already mentioned, we for this and other reasons consider our results to be upper bounds on the complexity rather than strict values.

Another subtlety is that the harmonic oscillator algebra does not permit a finitedimensional representation by Hermitian matrices, which is well known from the Heisenberg subalgebra. We will therefore use a representation- independent derivation of some relevant properties of geodesics, based on the right-invariance of suitable line elements and derivations in the Lie algebra. However, certain topological properties of the desired Lie group, such as periodic directions, cannot be captured in this way. Yet another subtlety then appears, related to the possible existence of different covering groups that have the same Lie algebra. We will make an attempt to highlight such features by carefully separating derivations that can be performed at the Lie algebra level from those that require additional properties of the group manifold.

From the commutation relation satisfied by the generators of the harmonic oscillator group, it is easy to see that the non-zero structure constants $f_{I J}^{K}$ are:

$$
\begin{equation*}
f_{Q P}^{E}=1, \quad f_{P Q}^{E}=-1, \quad f_{H Q}^{P}=-1, \quad f_{Q H}^{P}=1, \quad f_{H P}^{Q}=1, \quad f_{P H}^{Q}=-1 \tag{3.3}
\end{equation*}
$$

The structure constants allow us to look for directions of geodesics in the harmonic oscillator group, given by components of the Euler-Arnold equation (2.4). Choosing a diagonal penalty matrix $G_{I J}$, these equations read

$$
\begin{align*}
G_{H H} \frac{d V^{H}}{d s} & =\left(G_{Q Q}-G_{P P}\right) V^{P} V^{Q}  \tag{3.4}\\
G_{P P} \frac{d V^{P}}{d s} & =-G_{Q Q} V^{H} V^{Q}+G_{E E} V^{Q} V^{E},  \tag{3.5}\\
G_{Q Q} \frac{d V^{Q}}{d s} & =G_{P P} V^{H} V^{P}-G_{E E} V^{P} V^{E}  \tag{3.6}\\
G_{E E} \frac{d V^{E}}{d s} & =0 . \tag{3.7}
\end{align*}
$$

As a specific example of a diagonal penalty factor matrix, we implement equal penalties for all the generators, choosing $G_{I J}=\delta_{I J}$, such that the Euler-Arnold equations decouple:

$$
\begin{align*}
\frac{d V^{H}}{d s} & =0  \tag{3.8}\\
\frac{d V^{P}}{d s} & =-V^{H} V^{Q}+V^{Q} V^{E}  \tag{3.9}\\
\frac{d V^{Q}}{d s} & =V^{H} V^{P}-V^{P} V^{E},  \tag{3.10}\\
\frac{d V^{E}}{d s} & =0, \tag{3.11}
\end{align*}
$$

to which we have complete solutions given by:

$$
\begin{align*}
V^{H}(s) & =v_{H}, \quad V^{E}(s)=v_{E},  \tag{3.12}\\
V^{P}(s) & =v_{P} \cos \left(s\left(v_{E}-v_{H}\right)\right)+v_{Q} \sin \left(s\left(v_{E}-v_{H}\right)\right),  \tag{3.13}\\
V^{Q}(s) & =v_{Q} \cos \left(s\left(v_{E}-v_{H}\right)\right)-v_{P} \sin \left(s\left(v_{E}-v_{H}\right)\right) . \tag{3.14}
\end{align*}
$$

More generally, to decouple the equations, it is sufficient to have $G_{Q Q}=G_{P P}$ while $G_{H H}$ and $G_{E E}$ may be chosen independently (the equations do not depend on $G_{H H}$ in this case, but they do depend on $G_{E E}$ ).

Thus we have the tangent vectors $\left\{V^{I}(s)\right\}$ along all geodesics in the harmonic oscillator group manifold. The constants $v_{i}$ determine the magnitude of the velocity vectors at $s=0$, which will be fixed by the target unitary. The length of the resulting curve, starting from the identity, equals:

$$
\begin{equation*}
\int_{0}^{1} \sqrt{G_{I J} V^{I} V^{J}} d s=\sqrt{v_{H}^{2}+v_{P}^{2}+v_{Q}^{2}+v_{E}^{2}} . \tag{3.15}
\end{equation*}
$$

The integrand is independent of $s$ in this case, and therefore the information about the path length is contained entirely in the magnitude of the tangent vector $V^{I}$ at the identity given by $v_{i}$. This initial magnitude is expressed in terms of the parameters of the target unitary operator. Implementing the boundary condition $U(1)=U_{\text {target }}$ in order to derive the $v_{i}$ for a specified target unitary operator requires dealing with the path-ordered exponential (2.7), which is a notoriously tricky task. The usual way of deriving it is an iterative approach, expressed as a Dyson series:

$$
\begin{equation*}
U(s)=\mathbb{I}-i \int_{0}^{s} V^{I}\left(s^{\prime}\right) O_{I} d s^{\prime}+(-i)^{2} \int_{0}^{s} V^{I}\left(s^{\prime}\right) O_{I} d s^{\prime} \int_{0}^{s^{\prime}} V^{I}\left(s^{\prime \prime}\right) O_{I} d s^{\prime \prime}+\cdots \tag{3.16}
\end{equation*}
$$

In a first approximation, we will keep only the leading order term in this series. In the following section, we will explain the implications of neglecting the higher-order terms in the interpretation of our results.

As a final step, we should then look at the group manifold and determine if topological properties imply that the same target unitary can be reached in multiple ways, in which case we would identify the shortest possible connection with the complexity bound. Such considerations cannot be performed at the Lie algebra level and depend on the specific choice of a target unitary.

Furthermore, let us note that the complexity bound is not invariant under a canonical transformation of the classical pair $(Q, P)$ even if it simply rescales the variables by a constant, such as $Q \mapsto \lambda Q$ and $P \mapsto P / \lambda$. At the quantum level, such a transformation redefines the penalty coefficients of $Q$ and $P$ and therefore changes the complexity bound.

### 3.2 Computation of complexity of various target unitary operators

We now discuss the methodology of solving equation (2.6) subject to the boundary conditions (2.8). To begin with, we should find a sufficiently generic expression for elements of the harmonic oscillator group. At this initial step, the non-exponential nature of the
group is already relevant. If the Lie algebra has a matrix or operator representation of $N$ Hermitian generators $\hat{\mathcal{O}}_{I}$ satisfying

$$
\begin{equation*}
\left[\hat{\mathcal{O}}_{I}, \hat{\mathcal{O}}_{J}\right]=i f_{I J}^{K} \hat{\mathcal{O}}_{K}, \tag{3.17}
\end{equation*}
$$

a large number of elements of the corresponding Lie group are obtained as

$$
\begin{equation*}
U=\exp \left(-i \sum_{I}^{N} \alpha_{I} \hat{\mathcal{O}}_{I}\right) \tag{3.18}
\end{equation*}
$$

with $N$ real numbers $\alpha_{I}$. The Lie group is generated by all elements of the form $U$, taking all possible products, but it may contain other elements not included in the image of the exponential map. If the image is the whole group, the Lie group is called exponential, which is the case in several well-known examples, including connected compact Lie groups (such as $\operatorname{SU}(N)$ ) and nilpotent Lie groups (such as the Heisenberg group). If the Lie algebra is nilpotent, for instance, the Baker-Campbell-Hausdorff formula can always be used to bring a product of elements $U$ to the form of a single exponentiated sum of generators because iterating the commutator then, by definition, always produces zero after a finite number of iterations. The harmonic oscillator group, however, is not nilpotent but solvable, and it is not exponential, as shown in [39].

Nevertheless, we will use the form written in (3.18), which is sufficient for a large set of target unitaries, although not all possible ones. Throughout the central part of this paper, we will assume group elements of the form $U$ because it greatly simplifies computations. For comparison, we provide a detailed example using a product of exponentiated generators for the generic element in Appendix B. In general, our results then provide upper bounds on the complexity but not necessarily its actual value because the geodesic distance through the image of the exponential map could be further shortened by moving through its complement in the full Lie group.

### 3.2.1 Euler-Arnold equations and solutions

Having obtained the $V^{I}(s)$ for the Harmonic oscillator group by solving the Euler-Arnold equation, the next step is to use the obtained $V^{I}(s)$ in the Dyson series, whose leading order term can be written as:

$$
\begin{aligned}
-i \int_{0}^{s} V^{I}\left(s^{\prime}\right) O_{I} d s^{\prime}= & -i \int_{0}^{s}\left(v_{H} H+\left(v_{P} \cos \left(s\left(v_{E}-v_{H}\right)\right)+v_{Q} \sin \left(s\left(v_{E}-v_{H}\right)\right)\right) P\right. \\
& \left.+\left(v_{Q} \cos \left(s\left(v_{E}-v_{H}\right)\right)-v_{P} \sin \left(s\left(v_{E}-v_{H}\right)\right)\right) Q+v_{E} E\right) \\
= & -i\left(\left\{\frac{1}{v_{E}-v_{H}}\left(v_{Q}-v_{Q} \cos \left(s\left(v_{E}-v_{H}\right)\right)+v_{P} \sin \left(s\left(v_{E}-v_{H}\right)\right)\right\} P\right.\right. \\
& +\left\{\frac{1}{v_{E}-v_{H}}\left(-v_{P}+v_{P} \cos \left(s\left(v_{E}-v_{H}\right)\right)+v_{Q} \sin \left(s\left(v_{E}-v_{H}\right)\right)\right)\right\} Q \\
& \left.+s v_{H} H+s v_{E} E\right) .
\end{aligned}
$$

Keeping only up to the leading order term in the Dyson series, the path-ordered exponential can be approximately written as:

$$
\begin{align*}
U(s) & \approx \exp \left(-i\left(\left\{\frac{1}{v_{E}-v_{H}}\left(v_{Q}-v_{Q} \cos \left(s\left(v_{E}-v_{H}\right)\right)+v_{P} \sin \left(s\left(v_{E}-v_{H}\right)\right)\right\} P\right.\right.\right. \\
& +\left\{\frac{1}{v_{E}-v_{H}}\left(-v_{P}+v_{P} \cos \left(s\left(v_{E}-v_{H}\right)\right)+v_{Q} \sin \left(s\left(v_{E}-v_{H}\right)\right)\right)\right\} Q \\
& \left.\left.+s v_{H} H+s v_{E} E\right)\right) \tag{3.19}
\end{align*}
$$

Let us rewrite the above equation as:

$$
\begin{equation*}
U(s) \approx \exp \left(-i\left(\alpha_{1}(s) E+\alpha_{2}(s) P+\alpha_{3}(s) Q+\alpha_{4}(s) H\right)\right) \tag{3.20}
\end{equation*}
$$

where the $\alpha_{I}$ are:

$$
\begin{align*}
\alpha_{1}(s) & =s v_{E}  \tag{3.21}\\
\alpha_{2}(s) & =\frac{1}{v_{E}-v_{H}}\left(v_{Q}-v_{Q} \cos \left(s\left(v_{E}-v_{H}\right)\right)+v_{P} \sin \left(s\left(v_{E}-v_{H}\right)\right)\right)  \tag{3.22}\\
\alpha_{3}(s) & =\frac{1}{v_{E}-v_{H}}\left(-v_{P}+v_{P} \cos \left(s\left(v_{E}-v_{H}\right)\right)+v_{Q} \sin \left(s\left(v_{E}-v_{H}\right)\right)\right)  \tag{3.23}\\
\alpha_{4}(s) & =s v_{H} \tag{3.24}
\end{align*}
$$

In principle, the complete $s$-dependent unitary with a given tangent direction should be the path-ordered exponential (which involves taking into account all the terms in the Dyson series), and is not necessarily equal to the simple exponential as written in equation (3.20). In writing $U(s)$ as the simple exponential (by considering only the leading order term in the Dyson series), we are implicitly assuming that the deviations between the simple exponential and the path-ordered one remain small for sufficiently short geodesic distances. As we will see, the expressions are equal for our leading-order results in the case of harmonic oscillators, but not necessarily when we introduce additional terms by perturbation theory.

Imposing the boundary condition at $s=1$, setting $U(s=1)=U_{\text {target }}$, allows us to determine the geodesic constants $v_{I}$ in terms of the target operator quantities. In the following sections, we consider various target unitary operators for the purpose of illustration and computing their complexities or upper bound on complexities.

### 3.2.2 Displacement operator

We illustrate the methodology by explicitly computing the complexity of an operator constructed out of the harmonic oscillator group generators, the displacement operator

$$
\begin{equation*}
U_{\text {target }}=\exp \left(\alpha a^{\dagger}-\alpha^{*} a\right) \tag{3.25}
\end{equation*}
$$

for a complex number $\alpha$. The $a^{\dagger}$ and $a$ are creation and annihilation operators, satisfying the standard commutation relation $\left[a, a^{\dagger}\right]=\mathbb{I}$.

In terms of the generators of the harmonic oscillator group, this operator takes the form:

$$
\begin{align*}
U_{\text {target }} & =\exp \left(\frac{\alpha}{\sqrt{2}}(Q-i P)-\frac{\alpha^{*}}{\sqrt{2}}(Q+i P)\right)=\exp \left(\frac{1}{\sqrt{2}}\left(\alpha-\alpha^{*}\right) Q-\frac{i}{\sqrt{2}}\left(\alpha+\alpha^{*}\right) P\right) \\
& =\exp (i \sqrt{2}(\operatorname{Im}(\alpha) Q-\operatorname{Re}(\alpha) P)) \tag{3.26}
\end{align*}
$$

Matching the boundary conditions, we obtain:

$$
\begin{equation*}
\exp \left(-i\left\{\alpha_{1}(1) E+\alpha_{2}(1) P+\alpha_{3}(1) Q+\alpha_{4}(1) H\right\}\right)=\exp (i \sqrt{2}(\operatorname{Im}(\alpha) Q-\operatorname{Re}(\alpha) P)) \tag{3.27}
\end{equation*}
$$

Comparing the coefficients of the generators on both sides, we find:

$$
\begin{align*}
& \alpha_{1}(1)=0 \text { and } \alpha_{4}(1)=0  \tag{3.28}\\
& \alpha_{2}(1)=\sqrt{2} \operatorname{Re}(\alpha)=-\frac{v_{P} \sin \left(v_{E}-v_{H}\right)+v_{Q} \cos \left(v_{E}-v_{H}\right)-v_{Q}}{v_{E}-v_{H}},  \tag{3.29}\\
& \alpha_{3}(1)=-\sqrt{2} \operatorname{Im}(\alpha)=-\frac{v_{P}\left(-\cos \left(v_{E}-v_{H}\right)\right)+v_{Q} \sin \left(v_{E}-v_{H}\right)+v_{P}}{v_{E}-v_{H}} . \tag{3.30}
\end{align*}
$$

The first equation requires $v_{E}=0$ and $v_{H}=0$, such that we have to take the appropriate limit of $v_{E}-v_{H} \rightarrow 0$ when we evaluate the other two equations. Using

$$
\begin{equation*}
\lim _{v_{E}-v_{H} \rightarrow 0} \frac{1-\cos \left(v_{E}-v_{H}\right)}{v_{E}-v_{H}}=0 \tag{3.31}
\end{equation*}
$$

and

$$
\begin{equation*}
\lim _{v_{E}-v_{H} \rightarrow 0} \frac{\sin \left(v_{E}-v_{H}\right)}{v_{E}-v_{H}}=1 \tag{3.32}
\end{equation*}
$$

we obtain:

$$
\begin{equation*}
v_{P}=-\sqrt{2} \operatorname{Re}(\alpha), \quad \text { and } \quad v_{Q}=\sqrt{2} \operatorname{Im}(\alpha) \tag{3.33}
\end{equation*}
$$

Therefore, the complexity bound of the displacement operator is given by:

$$
\begin{equation*}
C[D]=\sqrt{v_{E}^{2}+v_{P}^{2}+v_{Q}^{2}+v_{H}^{2}}=\sqrt{2}|\alpha| \tag{3.34}
\end{equation*}
$$

### 3.2.3 Complexity of the time evolution operator

In our main examples, we consider the unitary operator produced by exponentiation the harmonic oscillator Hamiltonian:

$$
\begin{equation*}
H_{\omega}=\frac{P^{2}}{2 m}+\frac{m \omega^{2}}{2} Q^{2} \tag{3.35}
\end{equation*}
$$

We first assume that $m=\omega^{-1}$, which happens to simplify the resulting complexity bound. Classically, this relationship can always be achieved by a canonical transformation: $Q \mapsto$
$\sqrt{m \omega} Q, P \mapsto P / \sqrt{m \omega}$. However, as already mentioned, such a transformation upon quantization changes the penalty factors and, therefore the complexity bound.

In terms of the generators of the harmonic oscillator group, the Hamiltonian $H_{\omega}$ with $m=\omega^{-1}$ can be written as:

$$
\begin{equation*}
H_{\omega}=\omega H . \tag{3.36}
\end{equation*}
$$

The associated unitary operator is:

$$
\begin{equation*}
U_{\omega}(t)=\exp (-i \omega t H) \tag{3.37}
\end{equation*}
$$

Imposing the boundary condition $U(s=1)=U_{\omega}(t)$, we have:

$$
\begin{equation*}
\exp \left(-i\left\{\alpha_{1}(1) E+\alpha_{2}(1) P+\alpha_{3}(1) Q+\alpha_{4}(1) H\right\}\right)=\exp (-i \omega t H) \tag{3.38}
\end{equation*}
$$

and therefore:

$$
\begin{align*}
& v_{E}=0, \quad \frac{1}{v_{E}-v_{H}}\left(v_{Q}-v_{Q} \cos \left(v_{E}-v_{H}\right)+v_{P} \sin \left(v_{E}-v_{H}\right)\right)=0,  \tag{3.39}\\
& \frac{1}{v_{E}-v_{H}}\left(-v_{P}+v_{P} \cos \left(v_{E}-v_{H}\right)+v_{Q} \sin \left(v_{E}-v_{H}\right)\right)=0, \quad v_{H}=\omega t . \tag{3.40}
\end{align*}
$$

We obtain the solutions:

$$
\begin{equation*}
v_{E}=0, \quad v_{P}=0, \quad v_{Q}=0, \quad v_{H}=\omega t . \tag{3.4.4}
\end{equation*}
$$

The corresponding length of a curve from the identity to the harmonic oscillator evolution operator with $m=\omega^{-1}$ is given by:

$$
\begin{equation*}
L_{\left[U_{\text {target }}\right]}(\omega, t)=\sqrt{v_{E}^{2}+v_{P}^{2}+v_{Q}^{2}+v_{H}^{2}}=\omega t . \tag{3.42}
\end{equation*}
$$

This value may be improved further as a complexity bound because of possible periodic directions in the group manifold. If this length would always equal the complexity, the latter would grow linearly in time, which should not be the case according to previous derivations, such as [30]. Periodicity properties cannot be determined solely at the Lie algebra level because they depend on topological properties of the Lie group and the specific covering space suitable for physical properties of the system. In the present case, we know that the harmonic oscillator Hamiltonian corresponding to our $H$ has spectrum $n+1 / 2$ with integer $n$. Therefore, if the finite-dimensional Lie group used here is embedded in the infinite-dimensional Hilbert space of quantum mechanics, $\exp (-i \omega t H)$ as a function of $\omega t$ has a period of $4 \pi$. The length $L_{\left[U_{\text {target }}\right]}(\omega, t)$, therefore, equals $\omega t$ only as long as it is less than half the period, in which case there is no shortcut to the same target unitary. For $2 \pi<\omega t<4 \pi$, however, the target unitary can be reached in a shorter distance by moving in the opposite direction, and for $\omega t=4 \pi$, the target unitary equals the identity and has zero complexity. This process can be repeated for larger values of $\omega t$, resulting in the complexity bound

$$
\begin{equation*}
C_{\left[U_{\text {target }}\right]}(\omega, t)=\sqrt{v_{E}^{2}+v_{P}^{2}+v_{Q}^{2}+v_{H}^{2}}=|\omega t-4 \pi\lfloor(\omega t+2 \pi) /(4 \pi)\rfloor| . \tag{3.43}
\end{equation*}
$$



Figure 1. Complexity of the time evolution operator of a harmonic oscillator.

The plot of this function is shown in Fig. 1.
It is worth emphasizing that up to the value of the period, our result agrees with the complexity of the harmonic oscillator derived in Ref. [30]. In that article, the period $T=2 \pi / \omega$ of the classical harmonic oscillator was used. However, as indicated here, the actual quantum periodicity for the infinite-dimensional Hilbert space of the full quantum harmonic oscillator is doubling that period due to the presence of the $1 / 2$ term in its spectrum, corresponding to the non-zero ground state energy.

Furthermore, the specific solutions of the Euler-Arnold equations considered here are also periodic. However, this property is independent of the Lie group or the embedding in the full space of unitaries, and, therefore, cannot be considered an indicator of the periodicity of the complexity.

### 3.2.4 Harmonic oscillator with an additional linear potential

As a combination of two different generators, we now consider the time evolution operator of the harmonic oscillator with an added linear potential as the target unitary operator, given by:

$$
\begin{equation*}
H_{\lambda}=\omega\left(\frac{P^{2}}{2}+\frac{Q^{2}}{2}\right)+\lambda Q, \tag{3.44}
\end{equation*}
$$

if we still assume $m=\frac{1}{\omega}$. Classically, the added term merely shifts the origin of the harmonic oscillator because the potential equals $\frac{1}{2} \omega Q^{2}+\lambda Q=\frac{1}{2} \omega(Q+\lambda / \omega)^{2}-\frac{1}{2} \lambda^{2} / \omega^{2}$. As an expression in the harmonic oscillator algebra, however, the constant terms require the generator $E$, and since the product $E Q$ is not defined in the Lie algebra, the quadratic completion cannot be done at this level. Within the truncated setting, the complexity
bound of a harmonic oscillator with an additional linear term therefore need not equal the complexity bound of an individual oscillator just computed.

The time evolution operator in this case is:

$$
\begin{equation*}
U_{\text {target }}(t)=\exp \left(-i H_{\lambda} t\right)=\exp \left(-i\left\{\omega\left(\frac{P^{2}}{2}+\frac{Q^{2}}{2}\right)+\lambda Q\right\} t\right), \tag{3.45}
\end{equation*}
$$

or

$$
\begin{equation*}
U_{\text {target }}=\exp (-i(\omega H+\lambda Q) t), \tag{3.46}
\end{equation*}
$$

in terms of the generators of the Harmonic oscillator group. Substituting the boundary condition that $U(s=1)=U_{\text {target }}$, we have:

$$
\begin{equation*}
\exp \left(-i\left\{\alpha_{1}(1) E+\alpha_{2}(1) P+\alpha_{3}(1) Q+\alpha_{4}(1) H\right\}\right)=\exp (-i(\omega H+\lambda Q) t) \tag{3.47}
\end{equation*}
$$

which implies the conditions:

$$
\begin{align*}
& v_{E}=0, \quad \frac{1}{v_{E}-v_{H}}\left(v_{Q}-v_{Q} \cos \left(v_{E}-v_{H}\right)+v_{P} \sin \left(v_{E}-v_{H}\right)\right)=0,  \tag{3.48}\\
& \frac{1}{v_{E}-v_{H}}\left(-v_{P}+v_{P} \cos \left(v_{E}-v_{H}\right)+v_{Q} \sin \left(v_{E}-v_{H}\right)\right)=\lambda t, \quad v_{H}=\omega t . \tag{3.49}
\end{align*}
$$

The generator $H$ still belongs to the periodic direction in the group, with the period of $4 \pi$ for $v_{H}$ if the group is to be embedded in the full space of unitaries. Within a period, there are now values, specifically given by $v_{H}=2 \pi n$, with $n \in \mathbb{Z}$, for which the conditions have no solutions because the first equation in (3.49) then reads $0=\lambda t$. Here, we encounter another subtlety: A metric on a non-compact group manifold is not guaranteed to be geodesically complete. There are then pairs of endpoints for which the geodesic equation has no solutions, as seen here in a specific example. Formally, the geodesic distance between these points is then infinite, and while it may be finite if the endpoints are moved slightly, the divergence shows, again, that the result can only be considered an upper bound on the complexity. This notion of interpreting geodesic distance as the upper bound on complexity was previously done in [40, 41]. The solutions

$$
\begin{equation*}
v_{Q}=\frac{1}{2} v_{H} \lambda t \cot \left(v_{H} / 2\right), \tag{3.50}
\end{equation*}
$$

and

$$
\begin{equation*}
v_{P}=\frac{1}{2} v_{H} \lambda t \tag{3.51}
\end{equation*}
$$

where $v_{H}$ is periodic as in (3.43) explicitly show the divergence of $v_{Q}$ at $v_{H}=2 \pi n$, with $n \in \mathbb{Z}$, and so does the complexity bound:

$$
\begin{equation*}
C_{\left[U_{\text {target }}\right]}(\omega, \lambda, t) \leq v_{H} \sqrt{1+\frac{\lambda^{2} t^{2}}{4 \sin ^{2}\left(v_{H} / 2\right)}}, \tag{3.52}
\end{equation*}
$$

with $v_{H}$ as in (3.43). So that in the $\lambda \rightarrow 0$ limit the bound reduces to $\omega t$, as expected. Examples of the bound are shown in Figure 2.


Figure 2. Complexity bound of the time evolution operator of a harmonic oscillator with an additional linear term in Q.

As a result, introducing an extra direction associated with the generator $Q$ can potentially lead to increased complexity, possibly even approaching infinity. From a mathematical perspective, such a scenario cannot be excluded when dealing with non-compact groups. Nevertheless, it is important to note that infinite complexity values are not expected to arise in physical situations. Such extreme values would violate Lloyd's computational bound [8, 42], making them highly unlikely to be encountered. In our context, non-compact groups can lead to geodesic incompleteness such that there may be no finite-length geodesic between a given pair of two points. Interpreting our results as upper bounds on the complexity rather than precise values, which is also indicated by the truncations we use of the original infinite-dimensional group of unitary operators as well as the non-exponential nature of the groups used here, is consistent with both the mathematical and physical viewpoints. This question will be discussed in more detail in our conclusions.

### 3.2.5 Geodesic equation for general frequencies

So far, we have restricted our attention to harmonic oscillators with the condition $m=\omega^{-1}$. At the quantum level, as already noted, it is not possible to use a canonical transformation in order to bring any harmonic oscillator to this form because this step would modify the penalty factors. Alternatively, the harmonic oscillator algebra can be generalized to include an arbitrary Hamiltonian with a generic frequency $\omega$ as well as an independent mass $m$ by using:

$$
\begin{equation*}
H=\frac{1}{2} \frac{P^{2}}{m}+\frac{1}{2} m \omega^{2} Q^{2}, \tag{3.53}
\end{equation*}
$$

as a generator, together, with the previous $Q, P$, and $E$. The structure constants are now modified because the relevant commutation relations are: $[H, Q]=-i P / m$ and $[H, P]=$ $i m \omega^{2} Q$.

As a consequence, the Euler-Arnold equations take the modified form:

$$
\begin{align*}
G_{H H} \frac{d V^{H}}{d s} & =\left(m \omega^{2} G_{Q Q}-m^{-1} G_{P P}\right) V^{P} V^{Q},  \tag{3.54}\\
G_{P P} \frac{d V^{P}}{d s} & =-m \omega^{2} G_{Q Q} V^{H} V^{Q}+G_{E E} V^{Q} V^{E},  \tag{3.55}\\
G_{Q Q} \frac{d V^{Q}}{d s} & =m^{-1} G_{P P} V^{H} V^{P}-G_{E E} V^{P} V^{E},  \tag{3.56}\\
G_{E E} \frac{d V^{E}}{d s} & =0 . \tag{3.57}
\end{align*}
$$

If we continue to use the penalty factor matrix $G_{I J}=\delta_{I J}$, these equations do not decouple, and they are non-linear and much harder to solve.

Another possibility to change the frequency is to add a term quadratic in $Q$ to the harmonic oscillator Hamiltonian, instead of a linear term as in the preceding example. This term is not contained in the harmonic oscillator algebra, but it can be included in a $\operatorname{sp}(2, \mathbb{R})$ algebra, which we will describe in detail in the next section.

## 4 Harmonic oscillator complexity from the $\mathrm{sp}(2, \mathbb{R})$ Lie algebra

The quadratic nature of the harmonic oscillator Hamiltonian makes it possible to view it as a generator of various Lie algebras. Let us again consider the Hamiltonian:

$$
\begin{equation*}
\hat{H}=\frac{P^{2}}{2 m}+\frac{1}{2} m \omega^{2} Q^{2}, \tag{4.1}
\end{equation*}
$$

with independent mass and frequency.
A natural set of operators that form a closed Lie algebra includes $\frac{1}{2} \hat{P}^{2}, \frac{1}{2} \hat{Q}^{2}$ and their commutator. This observation suggests using the generators:

$$
\begin{align*}
& K_{1}=\frac{\hat{Q}^{2}}{2},  \tag{4.2}\\
& K_{2}=\frac{\hat{P}^{2}}{2},  \tag{4.3}\\
& K_{3}=\frac{1}{2}(\hat{Q} \hat{P}+\hat{P} \hat{Q}), \tag{4.4}
\end{align*}
$$

which satisfy the following commutation relations:

$$
\begin{equation*}
\left[K_{1}, K_{2}\right]=i K_{3} \quad\left[K_{3}, K_{1}\right]=-2 i K_{1}, \quad\left[K_{3}, K_{2}\right]=2 i K_{2} . \tag{4.5}
\end{equation*}
$$

Since these three operators form a closed Lie algebra, given by $\operatorname{sp}(2, \mathbb{R})$, they specify a Lie group, $\mathrm{Sp}(2, \mathbb{R})$ or one of its covering groups, part of which can be parameterized by the group elements

$$
\begin{equation*}
U(s)=\exp \left(-i\left(\gamma_{1}(s) K_{1}+\gamma_{2}(s) K_{2}+\gamma_{3}(s) K_{3}\right)\right) . \tag{4.6}
\end{equation*}
$$

(The Lie group $\operatorname{SU}(1,1)$, which has been used in the past for this system, is a different real form.) According to [43], the Lie group $\operatorname{Sp}(2, \mathbb{R})$ is not exponential, but the projective
version, isomorphic to $\mathrm{SO}(2,1)$ of which $\mathrm{Sp}(2, \mathbb{R})$ is a 2 -fold covering, is. A large part of $\operatorname{Sp}(2, \mathbb{R})$, but not all of it, can therefore be parameterized by exponentiated sums of the generators.

The structure constants associated with the Lie algebra (4.5) are:

$$
\begin{equation*}
f_{12}^{3}=1, \quad f_{31}^{1}=-2, \quad f_{32}^{2}=2 \tag{4.7}
\end{equation*}
$$

In consequence, the Euler-Arnold equations (choosing $G_{I J}=\delta_{I J}$ as before) can be written as:

$$
\begin{align*}
\frac{d V^{1}}{d s} & =V^{2} V^{3}+2 V^{1} V^{3}  \tag{4.8}\\
\frac{d V^{2}}{d s} & =-V^{1} V^{3}-2 V^{2} V^{3}  \tag{4.9}\\
\frac{d V^{3}}{d s} & =-2\left(V^{1}\right)^{2}+2\left(V^{2}\right)^{2} \tag{4.10}
\end{align*}
$$

The system at hand consists of three coupled non-linear differential equations. We can significantly simplify these equations by choosing an alternative basis of the Lie algebra, which is given by:

$$
\begin{align*}
& J_{1}=K_{3}=\frac{1}{2}(Q P+P Q)  \tag{4.11}\\
& J_{2}=K_{1}-K_{2}=\frac{1}{2}\left(Q^{2}-P^{2}\right),  \tag{4.12}\\
& J_{3}=K_{1}+K_{2}=\frac{1}{2}\left(Q^{2}+P^{2}\right), \tag{4.13}
\end{align*}
$$

with commutation relations:

$$
\begin{equation*}
\left[J_{1}, J_{2}\right]=-2 i J_{3}, \quad\left[J_{2}, J_{3}\right]=2 i J_{1}, \quad\left[J_{3}, J_{1}\right]=2 i J_{2} \tag{4.14}
\end{equation*}
$$

In this form, the Euler-Arnold equations (still with $G_{I J}=\delta_{I J}$ ) reduce to:

$$
\begin{align*}
\frac{d V^{1}}{d s} & =-4 V^{2} V^{3}  \tag{4.15}\\
\frac{d V^{2}}{d s} & =4 V^{1} V^{3}  \tag{4.16}\\
\frac{d V^{3}}{d s} & =0 \tag{4.17}
\end{align*}
$$

for which the solutions are given by:

$$
\begin{align*}
& V^{1}(s)=v_{1} \cos \left(4 s v_{3}\right)-v_{2} \sin \left(4 s v_{3}\right)  \tag{4.18}\\
& V^{2}(s)=v_{2} \cos \left(4 s v_{3}\right)+v_{1} \sin \left(4 s v_{3}\right)  \tag{4.19}\\
& V^{3}(s)=v_{3} \tag{4.20}
\end{align*}
$$

Keeping up to leading-order terms in the Dyson series, the path-ordered exponential can be approximately written as:

$$
\begin{equation*}
U(s) \approx \exp \left(-i\left(\gamma_{1}(s) J_{1}+\gamma_{2}(s) J_{2}+\gamma_{3}(s) J_{3}\right)\right) \tag{4.21}
\end{equation*}
$$

where:

$$
\begin{align*}
\gamma_{1}(s) & =\frac{1}{4 v_{3}}\left(-v_{2}+v_{2} \cos \left(4 s v_{3}\right)+v_{1} \sin \left(4 s v_{3}\right)\right)  \tag{4.22}\\
\gamma_{2}(s) & =\frac{1}{4 v_{3}}\left(v_{1}-v_{1} \cos \left(4 s v_{3}\right)+v_{2} \sin \left(4 s v_{3}\right)\right)  \tag{4.23}\\
\gamma_{3}(s) & =s v_{3} \tag{4.24}
\end{align*}
$$

### 4.1 Time evolution operator of the harmonic oscillator - revisited

Employing the $\operatorname{sp}(2, \mathbb{R})$ algebra, let us now revise the geometric complexity of the time evolution operator generated by the harmonic oscillator Hamiltonian (4.1). Again, we first choose $m=\omega^{-1}$, so that:

$$
\begin{equation*}
\mathbf{H}=\omega\left(\frac{Q^{2}}{2}+\frac{P^{2}}{2}\right)=\omega J_{3} . \tag{4.25}
\end{equation*}
$$

In consequence, the target unitary operator is:

$$
\begin{equation*}
U(s=1)=U_{\text {target }}=\exp \left(-i \omega t J_{3}\right) \tag{4.26}
\end{equation*}
$$

where $U(s=1)$ can be found by substituting the solutions (4.24) in (4.21):

$$
\begin{equation*}
\exp \left(-i\left(\gamma_{1}(1) J_{1}+\gamma_{2}(1) J_{2}+\gamma_{3}(1) J_{3}\right)\right)=\exp \left(-i \omega t J_{3}\right) \tag{4.27}
\end{equation*}
$$

which implies the conditions:

$$
\begin{equation*}
\gamma_{1}(1)=0, \quad \gamma_{2}(1)=0, \quad \gamma_{3}(1)=\omega t . \tag{4.28}
\end{equation*}
$$

Employing Eqs. $(4.22,4.23,4.24)$ this leads to:

$$
\begin{equation*}
v_{3}=\omega t, \quad v_{1}=0, \quad v_{2}=0 . \tag{4.29}
\end{equation*}
$$

The final step requires consideration of periodicity properties in possible exponentiations of the Lie algebra $\operatorname{sp}(2, \mathbb{R})$. The immediate choice, $\mathrm{Sp}(2, \mathbb{R})$, would represent the generator $J_{3}$ by the matrix:

$$
J_{3}=\left(\begin{array}{cc}
1 & 0  \tag{4.30}\\
0 & -1
\end{array}\right)
$$

such that

$$
\exp \left(-i \omega t J_{3}\right)=\left(\begin{array}{cc}
e^{-i \omega t} & 0  \tag{4.31}\\
0 & e^{i \omega t}
\end{array}\right)
$$

as also used in [30]. However, the resulting period of $2 \pi$ in $\omega t$ obtained for $\exp (-i \omega t H)$ is not compatible with the spectrum $n+1 / 2$ of $\hat{H}$ in the full infinite-dimensional space of unitaries. As an abstract group, $\operatorname{Sp}(2, \mathbb{R})$ has infinitely many covering groups, including a 2 -fold covering which results in a compatible period of $4 \pi$ in $\omega t$. This covering group
is the metaplectic group $\mathrm{Mp}(2, \mathbb{R})$, but it is not a matrix group and does not have finitedimensional matrix representations. Therefore, it is not possible to obtain the period of $4 \pi$ from a matrix calculation, following the methods of [30], but it is consistent within our treatment based mainly on the Lie algebra plus a final periodicity condition that requires only abstract group properties but no matrix representation. The resulting complexity bound is the same as in (3.43),

$$
\begin{equation*}
C_{\left[U_{\mathrm{H} . \mathrm{O} .]}\right]}(\omega, t)=\sqrt{v_{E}^{2}+v_{P}^{2}+v_{Q}^{2}+v_{H}^{2}}=|\omega t-4 \pi\lfloor(\omega t+2 \pi) /(4 \pi)\rfloor|, \tag{4.32}
\end{equation*}
$$

with a period of $4 \pi$ in $\omega t$.

### 4.2 Time evolution operator of the inverted harmonic oscillator

The same method allows us to consider the time evolution operator of the inverted harmonic oscillator as the target unitary,

$$
\begin{equation*}
\mathbf{H}_{\mathbf{I H O}}=\frac{P^{2}}{2 m}-\frac{1}{2} m \Omega^{2} Q^{2} . \tag{4.33}
\end{equation*}
$$

This model was also considered in [44, 45] from the perspective of complexity. Again for the sake of simplicity, we assume $m=\Omega^{-1}$. In that case the Hamiltonian in terms of the generators of the group $\operatorname{Sp}(2, \mathbb{R})$ can be written as

$$
\begin{equation*}
\mathbf{H}_{\mathbf{I H O}}=\frac{\Omega}{2}\left(P^{2}-Q^{2}\right)=-\Omega J_{2}, \tag{4.34}
\end{equation*}
$$

such that

$$
\begin{equation*}
U_{\text {target }}=U(s=1)=\mathbf{U}_{\mathbf{I H O}}=\exp \left(i \Omega J_{2}\right) \tag{4.35}
\end{equation*}
$$

The equation

$$
\begin{equation*}
\exp \left(-i\left(\gamma_{1}(1) J_{1}+\gamma_{2}(1) J_{2}+\gamma_{3}(1) J_{3}\right)\right)=\exp \left(i \Omega t J_{2}\right) \tag{4.36}
\end{equation*}
$$

implies

$$
\begin{equation*}
\gamma_{1}(1)=0, \quad \gamma_{2}(1)=-\Omega t, \quad \gamma_{3}(1)=0 . \tag{4.37}
\end{equation*}
$$

Using equation (4.24), we obtain

$$
\begin{equation*}
v_{3}=0, \quad v_{2}=-\Omega t, \quad v_{1}=0, \tag{4.38}
\end{equation*}
$$

with

$$
\begin{equation*}
\lim _{v_{3} \rightarrow 0} \frac{1-\cos \left(4 v_{3}\right)}{4 v_{3}}=0, \text { and } \lim _{v_{3} \rightarrow 0} \frac{\sin \left(4 v_{3}\right)}{4 v_{3}}=1 . \tag{4.39}
\end{equation*}
$$

The resulting complexity bound equals:

$$
\begin{equation*}
C\left[U_{\text {I.H.O }}\right] \leq \sqrt{v_{1}^{2}+v_{2}^{2}+v_{3}^{2}}=|\Omega t| . \tag{4.40}
\end{equation*}
$$

Since $J_{2}$ does not correspond to a periodic direction in $\mathrm{Sp}(2, \mathbb{R})$ or its covering groups, there is no periodicity condition on the complexity, which instead grows linearly in time.

### 4.3 Harmonic oscillator with an additional $Q^{2}$ term in the Hamiltonian.

Within $\operatorname{Sp}(2, \mathbb{R})$, we can change the frequency of the harmonic oscillator Hamiltonian by adding an additional quadratic potential, which in contrast to the harmonic oscillator group is now one of the generators. If we still assume $m=\omega^{-1}$ for the original Hamiltonian, adding a quadratic term with a free coefficient, as in

$$
\begin{equation*}
\mathbf{H}_{\mathbf{H . O} . \lambda}=\omega\left(\frac{P^{2}}{2}+\frac{1}{2} Q^{2}\right)+\lambda Q^{2} \tag{4.41}
\end{equation*}
$$

allows us to interpret this system as having independent mass $m=\omega^{-1}$ and frequency:

$$
\begin{equation*}
\bar{\omega}=\sqrt{\frac{\omega+2 \lambda}{m}}=\omega \sqrt{1+2 \frac{\lambda}{\omega}} . \tag{4.42}
\end{equation*}
$$

In terms of the generators $J_{i}$, the new Hamiltonian can be written as

$$
\begin{equation*}
\mathbf{H}_{\mathbf{H . O .} . \lambda}=\omega J_{3}+\lambda\left(J_{2}+J_{3}\right)=(\omega+\lambda) J_{3}+\lambda J_{2} . \tag{4.43}
\end{equation*}
$$

Therefore, the target unitary operator is:

$$
\begin{equation*}
U_{\text {target }}=\exp \left(-i\left\{(\omega+\lambda) J_{3}+\lambda J_{2}\right\} t\right) \tag{4.44}
\end{equation*}
$$

Identifying the two expressions:

$$
\begin{equation*}
\exp \left(-i\left(\gamma_{1}(1) J_{1}+\gamma_{2}(1) J_{2}+\gamma_{3}(1) J_{3}\right)\right)=\exp \left(-i\left\{(\omega+\lambda) J_{3}+\lambda J_{2}\right\} t\right) \tag{4.45}
\end{equation*}
$$

implies the boundary conditions:

$$
\begin{equation*}
\gamma_{1}(1)=0, \quad \gamma_{2}(1)=\lambda t, \quad \gamma_{3}(1)=(\omega+\lambda) t \tag{4.46}
\end{equation*}
$$

which lead to:

$$
\begin{align*}
& \gamma_{1}(1)=\frac{1}{4 v_{3}}\left(-v_{2}+v_{2} \cos \left(4 v_{3}\right)+v_{1} \sin \left(4 v_{3}\right)\right)=0  \tag{4.47}\\
& \gamma_{2}(1)=\frac{1}{4 v_{3}}\left(v_{1}-v_{1} \cos \left(4 v_{3}\right)+v_{2} \sin \left(4 v_{3}\right)\right)=\lambda t  \tag{4.48}\\
& \gamma_{3}(1)=v_{3}=(\omega+\lambda) t \tag{4.49}
\end{align*}
$$

The periodicity argument is now more complicated for general $\lambda$ because a full representation of the Hamiltonian operator on the infinite-dimensional Hilbert space of quantum mechanics shows that $U_{\text {target }}$ should have period $4 \pi$ in $\bar{\omega} t$. However, the linear combination (4.43) in terms of $\operatorname{sp}(2, \mathbb{R})$ generators does not exponentiate to a periodic expression because it contains $J_{2}$, and the coefficient of $J_{3}$ which does belong to a periodic direction in the group has coefficient $\omega+\lambda \neq \bar{\omega}$. A reliable periodicity argument can be given only approximately for small $\lambda$, in which case the $J_{2}$-contribution to the Hamiltonian has a small coefficient, and $\omega+\lambda \approx \bar{\omega}$. Periodicity in the $J_{3}$-direction then approximates the period of $4 \pi$ in the full representation, and we can solve the equations by:

$$
\begin{equation*}
v_{1}=2 v_{3} \lambda t, \quad \text { and } \quad v_{2}=2 v_{3} \lambda t \cot \left(2 v_{3}\right) \tag{4.50}
\end{equation*}
$$



Figure 3. Complexity bounds on time evolution of a harmonic oscillator with quadratic term in Q.
where:

$$
\begin{equation*}
v_{3}=|(\omega+\lambda) t-4 \pi\lfloor((\omega+\lambda) t+2 \pi) /(4 \pi)\rfloor|, \tag{4.51}
\end{equation*}
$$

for minimized geodesics. The complexity bound is the given by:

$$
\begin{equation*}
C_{\left[U_{\text {target }}\right]}(\omega, \lambda, t) \leq \sqrt{v_{1}^{2}+v_{2}^{2}+v_{3}^{2}}=v_{3} \sqrt{1+\frac{4 \lambda^{2} t^{2}}{\sin ^{2}\left(2 v_{3}\right)}} . \tag{4.52}
\end{equation*}
$$

Since $\sqrt{1+2 \lambda / \omega}<1+\lambda / \omega$ for $\lambda \neq 0$, the complexity bound in the first branch of $|\bar{\omega} t|<2 \pi$ is greater than $\bar{\omega} t$. The fact that we are using two generators, $H$ and $Q^{2}$, in order to construct the Hamiltonian means that both penalty factors are relevant. The complexity bound is, therefore, greater than expected for an individual oscillator of frequency $m=\omega^{-1}$, for which the additional $Q$-term is not needed.

Going beyond the range of small $\lambda$, we can consider two instructive special cases. First, applying our result to $\lambda=-\omega / 2$, we have the Hamiltonian for a free particle of mass $m=\omega^{-1}$. The complexity bound in this case is:

$$
\begin{equation*}
C_{\left[U_{\text {target }]}\right.}\left(m^{-1},-(2 m)^{-1}, t\right) \leq \sqrt{v_{1}^{2}+v_{2}^{2}+v_{3}^{2}}=v_{3} \sqrt{1+\frac{t^{2}}{m^{2} \sin ^{2}\left(2 v_{3}\right)}}, \tag{4.53}
\end{equation*}
$$

where

$$
\begin{equation*}
v_{3}=\frac{1}{2}|\omega t-8 \pi\lfloor(\omega t+4 \pi) /(8 \pi)\rfloor| . \tag{4.54}
\end{equation*}
$$

In the limit of $\lambda \rightarrow-\omega$, the Hamiltonian describes the inverted Harmonic oscillator. Equation (4.51) then implies that $v_{3} \rightarrow 0$, which in (4.52) results in the complexity bound:

$$
\begin{equation*}
C_{\left[U_{\text {target }}\right]}(\omega,-\omega, t) \leq|\omega t|, \tag{4.55}
\end{equation*}
$$

consistent with our previous result for this system, equation (4.40). In this limit, the periodicity argument simplifies because $J_{3}$ disappears from the Hamiltonian (4.43), and only the non-compact $J_{2}$ remains. The complexity bound is, therefore, reliable even though $\lambda=-\omega$ is not small.

## 5 Coupled harmonic oscillators

A model of two coupled oscillators was considered in [18], but with a coupling term different from what we consider here. More importantly, the previous paper was interested in the complexity of the ground state of this Hamiltonian, as briefly reviewed here. The authors of [18] were interested in the complexity of:

$$
\begin{equation*}
\Psi_{T}=\frac{\left(\omega_{1} \omega_{2}-\beta^{2}\right)^{1 / 4}}{\sqrt{\pi}} \exp \left(-\frac{\omega_{1}}{2} x_{1}^{2}-\frac{\omega_{2}}{2} x_{2}^{2}-\beta x_{1} x_{2}\right), \tag{5.1}
\end{equation*}
$$

referred to as the target state, relative to the reference state:

$$
\begin{equation*}
\Psi_{R}=\sqrt{\frac{\omega_{0}}{\pi}} \exp \left(-\frac{\omega_{0}}{2}\left(x_{1}^{2}+x_{2}^{2}\right)\right) \tag{5.2}
\end{equation*}
$$

Once the target and the reference states are fixed, the next step involves identifying simple gates used to construct the unitary (or quantum circuit) that can implement the transformation $\Psi_{T}=U \Psi_{R}$. They chose the following simple gates:

$$
\begin{equation*}
H=e^{i \epsilon x_{0} p_{0}}, \quad J_{a}=e^{i \epsilon x_{0} p_{a}}, \quad K_{a}=e^{i \epsilon x_{a} p_{0}}, \quad Q_{a b}=e^{i \epsilon x_{a} p_{b}}, \quad Q_{a a}=e^{\frac{\epsilon}{2}} e^{i \epsilon x_{a} p_{a}} . \tag{5.3}
\end{equation*}
$$

Out of the chosen gates, it was realized that $Q_{a b}$ and $Q_{a a}$ would suffice for their purpose, considering the target state they were interested in, acting on a generic wave function as follows:

$$
\begin{equation*}
Q_{11} \Psi\left(x_{1}, x_{2}\right)=e^{\epsilon / 2} \Psi\left(e^{\epsilon} x_{1}, x_{2}\right), \quad Q_{21} \Psi\left(x_{1}, x_{2}\right)=\Psi\left(x_{1}+\epsilon x_{2}, x_{2}\right) . \tag{5.4}
\end{equation*}
$$

These two gates played a crucial role in the circuits they constructed. However, finding the optimal circuit is necessary to determine the complexity. Nielsen's geometric approach was then utilized to complete this step. To apply Nielsen's geometric approach, it is necessary to understand that both the target and the reference states are Gaussian, such that Gaussian wave functions represent the endpoints of any curve connecting the two states. Furthermore, the scaling and entangling gate actions preserve the Gaussian structure of the wave functions. Thus, it was concluded that the circuit constructed out of the $Q$ gates forms a representation of $\mathrm{GL}(2, \mathbb{R})$. Hence, finding the optimal circuit required finding the shortest geodesic in the space of $\operatorname{GL}(2, \mathbb{R})$ transformations. Of course, there is some arbitrariness in the obtained complexity, as it is dependent on the choice of the reference state and the choice of gates.

As a first step towards understanding the quantum complexity of the time evolution operator of a simple interacting system in a state-independent way, we consider the case of two coupled harmonic oscillators with time-independent coupling. Although the model
considered here is quite similar to what was considered in [18], our interest lies in the complexity of the time evolution operator. Our results then tell us how the complexity changes with time as the system evolves. As the Hamiltonian of such a system, we choose:

$$
\begin{equation*}
\mathbf{H}_{\mathbf{C} . \mathbf{O}}=\frac{P_{1}^{2}}{2 m_{1}}+\frac{P_{2}^{2}}{2 m_{2}}+\frac{1}{2 m_{1}} \omega_{1}^{2} Q_{1}^{2}+\frac{1}{2 m_{2}} \omega_{2}^{2} Q_{1}^{2}+\mu^{2}\left(Q_{1} Q_{2}+P_{1} P_{2}\right) . \tag{5.5}
\end{equation*}
$$

For the sake of simplicity, we again assume $\omega_{i}=m_{i}^{-1}$, for $i=1,2$, which simplifies the Hamiltonian to:

$$
\begin{equation*}
\mathbf{H}_{\mathbf{C} . \mathbf{O}}=\omega_{1}\left(\frac{P_{1}^{2}}{2}+\frac{Q_{1}^{2}}{2}\right)+\omega_{2}\left(\frac{P_{2}^{2}}{2}+\frac{Q_{2}^{2}}{2}\right)+\mu^{2}\left(Q_{1} Q_{2}+P_{1} P_{2}\right), \tag{5.6}
\end{equation*}
$$

and defines the target unitary operator:

$$
\begin{equation*}
U_{\text {target }}=\exp \left(-i \mathbf{H}_{\mathbf{C} . \mathbf{O}} t\right) \tag{5.7}
\end{equation*}
$$

A natural choice of generators that form a closed Lie algebra for the target unitary operator (5.7) is $T_{1}=H_{1}=\frac{1}{2}\left(Q_{1}^{2}+P_{1}^{2}\right), T_{2}=H_{2}=\frac{1}{2}\left(Q_{2}^{2}+P_{2}^{2}\right), T_{3}=\frac{1}{2}\left(Q_{1}^{2}-P_{1}^{2}\right)$, $T_{4}=\frac{1}{2}\left(Q_{2}^{2}-P_{2}^{2}\right), T_{5}=\left(Q_{1} P_{1}+P_{1} Q_{1}\right), T_{6}=\left(Q_{2} P_{2}+P_{2} Q_{2}\right), T_{7}=\left(Q_{1} Q_{2}+P_{1} P_{2}\right)$, $T_{8}=\left(Q_{1} P_{2}+P_{1} Q_{2}\right) T_{9}=\left(Q_{1} Q_{2}-P_{1} P_{2}\right), T_{10}=\left(Q_{1} P_{2}-P_{1} Q_{2}\right)$. The generators form a ten-dimensional Lie algebra isomorphic to $\operatorname{sp}(4, \mathbb{R})$, which can be derived in the same way as used for second-order central moments in [46]. For our target unitary operator, it is sufficient to consider the subalgebra formed by the four generators $T_{1}, T_{2}, T_{5}$ and $T_{10}$, which we relabel as $M_{i}$ :

$$
\begin{equation*}
M_{1}=H_{1}, \quad M_{2}=H_{2}, \quad M_{3}=Q_{1} Q_{2}+P_{1} P_{2}, \quad M_{4}=P_{1} Q_{2}-Q_{1} P_{2} . \tag{5.8}
\end{equation*}
$$

Their commutation relations are:

$$
\begin{align*}
& {\left[M_{1}, M_{2}\right]=0, \quad\left[M_{1}, M_{3}\right]=-i M_{4}, \quad\left[M_{1}, M_{4}\right]=i M_{3},}  \tag{5.9}\\
& {\left[M_{2}, M_{3}\right]=i M_{4}, \quad\left[M_{2}, M_{4}\right]=-i M_{3}, \quad\left[M_{3}, M_{4}\right]=-2 i M_{1}+2 i M_{2} .} \tag{5.10}
\end{align*}
$$

It follows that $M_{1}+M_{2}$ commutes with all other generators in this subalgebra, while $M_{1}-M_{2}$ together with $M_{3}$ and $M_{4}$ obey the commutation relations of $\mathrm{su}(2)$ :

$$
\begin{equation*}
\left[M_{1}-M_{2}, M_{3}\right]=-2 i M_{4}, \quad\left[M_{1}-M_{2}, M_{4}\right]=2 i M_{3}, \quad\left[M_{3}, M_{4}\right]=-2 i\left(M_{1}-M_{2}\right) \tag{5.11}
\end{equation*}
$$

The Lie algebra is, therefore, a direct product of the Abelian $u(1)$ with the $s u(2)$ subalgebra of $\operatorname{sp}(4, \mathbb{R})$. Compactness and periodicity properties of possible exponentiations are simpler for this algebra than in the case of $\operatorname{sp}(2, \mathbb{R})$ or the full $\operatorname{sp}(4, \mathbb{R})$. (The universal covering group of $\operatorname{sp}(4, \mathbb{R})$ has been studied, for instance in [47].) In particular, the Abelian generator $M_{1}+M_{2}=H_{1}+H_{2}$, which has an integer spectrum in quantum mechanics, should exponentiate to an operator with period $2 \pi$, and $M_{1}-M_{2}$ has the same period.

In terms of the new generators, the target unitary operator is given by:

$$
\begin{equation*}
U_{\text {target }}=\exp \left(-i\left(\omega_{1} M_{1}+\omega_{2} M_{2}+\mu^{2} M_{3}\right) t\right) \tag{5.12}
\end{equation*}
$$

The Euler-Arnold equations are:

$$
\begin{align*}
\frac{d V^{1}}{d s} & =f_{13}^{4} V^{3} \frac{G_{44}}{G_{11}} V^{4}+f_{14}^{3} V^{4} \frac{G_{33}}{G_{11}} V^{3}=-\frac{G_{44}}{G_{11}} V^{3} V^{4}+\frac{G_{33}}{G_{11}} V^{3} V^{4},  \tag{5.13}\\
\frac{d V^{2}}{d s} & =f_{23}^{4} V^{3} \frac{G_{44}}{G_{22}} V^{4}+f_{24}^{3} V^{4} \frac{G_{33}}{G_{22}} V^{3}=\frac{G_{44}}{G_{22}} V^{3} V^{4}-\frac{G_{33}}{G_{22}} V^{3} V^{4},  \tag{5.14}\\
\frac{d V^{3}}{d s} & =f_{31}^{4} V^{1} \frac{G_{44}}{G_{33}} V^{4}+f_{32}^{4} V^{2} \frac{G_{44}}{G_{33}} V^{4}+f_{34}^{1} V^{4} \frac{G_{11}}{G_{33}} V^{1}+f_{34}^{2} V^{4} \frac{G_{22}}{G_{33}} V^{2} \\
& =\frac{G_{44}}{G_{33}} V^{1} V^{4}-\frac{G_{44}}{G_{33}} V^{2} V^{4}-2 \frac{G_{11}}{G_{33}} V^{4} V^{1}+2 \frac{G_{22}}{G_{33}} V^{4} V^{2},  \tag{5.15}\\
\frac{d V^{4}}{d s} & =f_{41}^{3} V^{1} \frac{G_{33}}{G_{44}} V^{3}+f_{42}^{3} V^{2} \frac{G_{33}}{G_{44}} V^{3}+f_{43}^{1} V^{3} \frac{G_{11}}{G_{44}} V^{1}+f_{43}^{2} V^{3} \frac{G_{22}}{G_{44}} V^{2} \\
& =-\frac{G_{33}}{G_{44}} V^{1} V^{3}+\frac{G_{33}}{G_{44}} V^{2} V^{3}+2 \frac{G_{11}}{G_{44}} V^{3} V^{1}-2 \frac{G_{22}}{G_{44}} V^{3} V^{2} . \tag{5.16}
\end{align*}
$$

As before, we will consider the case of all generators with equal penalties, $G_{I J}=\delta_{I J}$. Then the Euler-Arnold equations reduce to:

$$
\begin{align*}
\frac{d V^{1}}{d s} & =0  \tag{5.17}\\
\frac{d V^{2}}{d s} & =0  \tag{5.18}\\
\frac{d V^{3}}{d s} & =-V^{1} V^{4}+V^{2} V^{4}  \tag{5.19}\\
\frac{d V^{4}}{d s} & =V^{1} V^{3}-V^{2} V^{3} \tag{5.20}
\end{align*}
$$

The equations can easily be solved by:

$$
\begin{align*}
& V^{1}(s)=v_{1}, \quad V^{2}(s)=v_{2}, \quad V^{4}(s)=v_{4} \cos \left(s\left(v_{1}-v_{2}\right)\right)+v_{3} \sin \left(s\left(v_{1}-v_{2}\right)\right), \\
& V^{3}(s)=v_{3} \cos \left(s\left(v_{1}-v_{2}\right)\right)-v_{4} \sin \left(s\left(v_{1}-v_{2}\right)\right), \tag{5.21}
\end{align*}
$$

and the length of the geodesics characterized by the $v_{i}$ equals:

$$
\begin{equation*}
\int_{0}^{1} \sqrt{G_{I J} V^{I} V^{J}} d s=\sqrt{v_{1}^{2}+v_{2}^{2}+v_{3}^{2}+v_{4}^{2}} . \tag{5.22}
\end{equation*}
$$

Another interesting and physically justifiable case of the penalty factor matrix is motivated by the specific setting of two coupled harmonic oscillators. Out of the four generators, $M_{1}$ and $M_{2}$ consist of terms associated with only one individual oscillator, while the generators $M_{3}$ and $M_{4}$ involve terms that act on both oscillators. It is natural to assign higher penalties to generators that involve two oscillator terms compared to the terms that act on only one of them. We may, therefore, choose:

$$
\begin{equation*}
G_{11}=G_{22}:=q, \quad \text { and } \quad G_{33}=G_{44}:=p \quad(p>q) . \tag{5.23}
\end{equation*}
$$

With this choice, the Euler-Arnold equations are given by:

$$
\begin{align*}
\frac{d V^{1}}{d s} & =0  \tag{5.24}\\
\frac{d V^{2}}{d s} & =0  \tag{5.25}\\
\frac{d V^{3}}{d s} & =\left(1-\frac{2 q}{p}\right) V^{1} V^{4}-\left(1-\frac{2 q}{p}\right) V^{2} V^{4}  \tag{5.26}\\
\frac{d V^{4}}{d s} & =-\left(1-\frac{2 q}{p}\right) V^{1} V^{3}+\left(1-\frac{2 q}{p}\right) V^{2} V^{3} \tag{5.27}
\end{align*}
$$

and can be solved as:

$$
\begin{align*}
& V^{1}(s)=v_{1}  \tag{5.28}\\
& V^{2}(s)=v_{2}  \tag{5.29}\\
& V^{3}(s)=v_{3} \cos \left(\frac{s(p-2 q)\left(v_{1}-v_{2}\right)}{p}\right)+v_{4} \sin \left(\frac{s(p-2 q)\left(v_{1}-v_{2}\right)}{p}\right)  \tag{5.30}\\
& V^{4}(s)=v_{4} \cos \left(\frac{s(p-2 q)\left(v_{1}-v_{2}\right)}{p}\right)-v_{3} \sin \left(\frac{s(p-2 q)\left(v_{1}-v_{2}\right)}{p}\right) . \tag{5.31}
\end{align*}
$$

Equal penalty factors clearly correspond to the limit $p=q=1$. In the more general setting of $q \neq p$, the length of the geodesics characterized by the $v_{i}$ is given by:

$$
\begin{equation*}
\int_{0}^{1} \sqrt{G_{I J} V^{I} V^{J}} d s=\sqrt{q\left(v_{1}^{2}+v_{2}^{2}\right)+p\left(v_{3}^{2}+v_{4}^{2}\right)} \tag{5.32}
\end{equation*}
$$

Substituting the obtained $V^{I}(s)$ and keeping only the leading order term in the Dyson series, $U(s)$ can be written as:

$$
\begin{equation*}
U(s) \approx \exp \left(-i\left(\beta_{1}(s) M_{1}+\beta_{2}(s) M_{2}+\beta_{3}(s) M_{3}+\beta_{4}(s) M_{4}\right)\right) \tag{5.33}
\end{equation*}
$$

where the $\beta_{I}$ are as follows

$$
\begin{align*}
\beta_{1} & =s v_{1}  \tag{5.34}\\
\beta_{2} & =s v_{2}  \tag{5.35}\\
\beta_{3} & =\frac{1}{(p-2 q)\left(v_{1}-v_{2}\right)}\left(-p v_{4}+p v_{4} \cos \left(\frac{s(p-2 q)\left(v_{1}-v_{2}\right)}{p}\right)\right. \\
& \left.+p v_{3} \sin \left(\frac{s(p-2 q)\left(v_{1}-v_{2}\right)}{p}\right)\right)  \tag{5.36}\\
\beta_{4} & =\frac{1}{(p-2 q)\left(v_{1}-v_{2}\right)}\left(p v_{3}-p v_{3} \cos \left(\frac{s(p-2 q)\left(v_{1}-v_{2}\right)}{p}\right)\right. \\
& \left.+p v_{4} \sin \left(\frac{s(p-2 q)\left(v_{1}-v_{2}\right)}{p}\right)\right) . \tag{5.37}
\end{align*}
$$

The final condition $U(s=1)=U_{\text {target }}$ implies:
$\exp \left(-i\left(\beta_{1}(1) M_{1}+\beta_{2}(1) M_{2}+\beta_{3}(1) M_{3}+\beta_{4}(1) M_{4}\right)\right)=\exp \left(-i\left(\omega_{1} M_{1}+\omega_{2} M_{2}+\mu^{2} M_{3}\right) t\right)$,
such that:

$$
\begin{equation*}
\beta_{1}(1)=\omega_{1} t, \quad \beta_{2}(1)=\omega_{2} t, \quad \beta_{3}(1)=\mu^{2} t, \quad \beta_{4}(1)=0 . \tag{5.39}
\end{equation*}
$$

These equations are solved by:

$$
\begin{align*}
& v_{1}=\omega_{1} t,  \tag{5.40}\\
& v_{2}=\omega_{2} t  \tag{5.41}\\
& v_{3}=\mu^{2} t \frac{p-2 q}{2 p}\left(v_{1}-v_{2}\right) \cot \left(\frac{p-2 q}{2 p}\left(v_{1}-v_{2}\right)\right),  \tag{5.42}\\
& v_{4}=\mu^{2} t \frac{p-2 q}{2 p}\left(v_{1}-v_{2}\right) . \tag{5.43}
\end{align*}
$$

Using

$$
\begin{equation*}
v_{1} M_{1}+v_{2} M_{2}=\frac{1}{2}\left(\left(v_{1}+v_{2}\right)\left(M_{1}+M_{2}\right)+\left(v_{1}-v_{2}\right)\left(M_{1}-M_{2}\right)\right), \tag{5.44}
\end{equation*}
$$

and the $2 \pi$-periodicity in the exponentiated $M_{1}+M_{2}$ and $M_{1}-M_{2}$, we need a period of $4 \pi$ for $v_{1}-v_{2}$ and $v_{1}+v_{2}$. Therefore:

$$
\begin{equation*}
v_{1} \pm v_{2}=\left|\left(\omega_{1} \pm \omega_{2}\right) t-4 \pi\left\lfloor\left(\left(\omega_{1} \pm \omega_{2}\right) t+2 \pi\right) /(4 \pi)\right\rfloor\right| . \tag{5.45}
\end{equation*}
$$

Finally, the complexity bound is given by:

$$
\begin{align*}
& C_{\left[U_{\text {target }}( \right.}\left(\omega_{1}, \omega_{2}, \mu, t\right) \leq \sqrt{v_{1}^{2}+v_{2}^{2}+v_{3}^{2}+v_{4}^{2}}  \tag{5.46}\\
& =\sqrt{\frac{1}{2}\left(\left(v_{1}+v_{2}\right)^{2}+\left(v_{1}-v_{2}\right)^{2}\right)+\mu^{4} t^{2} \frac{(p-2 q)^{2}}{4 p^{2}} \frac{\left(v_{1}-v_{2}\right)^{2}}{\sin ^{2}\left((p-2 q)\left(v_{1}-v_{2}\right) /(2 p)\right)}} .
\end{align*}
$$

Examples are shown in Figs. 4 and 5. For $\omega_{1}=\omega_{2}$, the result simplifies to

$$
\begin{equation*}
C_{\mathrm{U}_{\mathrm{target}}}(\omega, \omega, \mu, t) \leq \sqrt{\frac{1}{2}\left(v_{1}+v_{2}\right)^{2}+\mu^{4} t^{2}}, \tag{5.47}
\end{equation*}
$$

and is independent of $q$ and $p$. Here,

$$
\begin{equation*}
v_{1}+v_{2}=2|\omega t-2 \pi\lfloor(\omega t+\pi) / 2 \pi\rfloor| . \tag{5.48}
\end{equation*}
$$

## 6 Anharmonic oscillator with the cubic term

With an additional ingredient in the motivation of penalty factors, our methods can be applied even to anharmonic systems. As an example, we choose the Hamiltonian:

$$
\begin{equation*}
\mathbf{H}_{A \cdot H O}=\omega\left(\frac{P^{2}}{2}+\frac{Q^{2}}{2}\right)+\lambda Q^{3} . \tag{6.1}
\end{equation*}
$$

In this case, the generators can be considered to be the original:

$$
\begin{equation*}
M_{1}=\frac{1}{2} Q^{2}+\frac{1}{2} P^{2}, \quad M_{2}=\frac{1}{2} Q^{2}-\frac{1}{2} P^{2}, \quad M_{3}=\frac{1}{2}(Q P+P Q), \tag{6.2}
\end{equation*}
$$



Figure 4. Behavior of complexity bounds for the time evolution operator of two coupled oscillators as a function of time for different values of the penalty factor $p$. The frequencies of the two oscillators are fixed at $\omega_{1}=2$ and $\omega_{2}=1$, and the coupling constant is fixed at $\mu=3$. The penalty factor $q$ is fixed at 1 .


Figure 5. Behavior of complexity bounds for the time evolution operator of two coupled oscillators as a function of time for different values of the coupling constant $\mu$. The frequencies of the two oscillators are fixed at $\omega_{1}=2$ and $\omega_{2}=1$. The plots have been made by choosing $p=10$ and $q=1$.
already used for the harmonic oscillator, together with:

$$
\begin{equation*}
M_{4}=Q^{3}, \quad M_{5}=P^{3}, \quad M_{6}=Q^{2} P+Q P Q+P Q^{2}, \quad M_{7}=Q P^{2}+P Q P+P^{2} Q \tag{6.3}
\end{equation*}
$$

and so on with higher powers in $Q$ and $P$.

In terms of these generators, the Hamiltonian (6.1) can be written as:

$$
\begin{equation*}
\mathbf{H}_{A \cdot H O}=\omega M_{1}+\lambda M_{4}, \tag{6.4}
\end{equation*}
$$

defining the target unitary:

$$
\begin{equation*}
U_{\text {target }}=\exp \left(-i\left(\omega M_{1}+\lambda M_{4}\right) t\right) . \tag{6.5}
\end{equation*}
$$

The generators $M_{2}$ and $M_{3}$ do not appear in commutators of the other ones, $M_{1}, M_{4}$, $M_{5}, M_{6}, M_{7}$. They are, therefore, not required for Euler-Arnold equations with the desired target unitary. Commutators of the latter generators are given by:

$$
\begin{align*}
& {\left[M_{1}, M_{4}\right]=-i\left(P Q^{2}+Q P Q+Q^{2} P\right)=-i M_{6},}  \tag{6.6}\\
& {\left[M_{1}, M_{6}\right]=-2 i\left(P^{2} Q+P Q P+Q P^{2}\right)+3 i Q^{3}=-2 i M_{7}+3 i M_{4},}  \tag{6.7}\\
& {\left[M_{1}, M_{7}\right]=2 i\left(Q^{2} P+Q P Q+P Q^{2}\right)-3 i P^{3}=2 i M_{6}-3 i M_{5},}  \tag{6.8}\\
& {\left[M_{1}, M_{5}\right]=i\left(Q P^{2}+P Q P+P Q^{2}\right)=i M_{7},}  \tag{6.9}\\
& {\left[M_{4}, M_{5}\right]=3 i\left(Q^{2} P^{2}+P Q^{2} P+P^{2} Q^{2}\right),}  \tag{6.10}\\
& {\left[M_{4}, M_{6}\right]=9 i Q^{4},}  \tag{6.11}\\
& {\left[M_{4}, M_{7}\right]=6 i Q^{3} P+6 i P Q^{3}+3 i Q P Q^{2}+3 i Q^{2} P Q,}  \tag{6.12}\\
& {\left[M_{5}, M_{6}\right]=\mathcal{O}\left(M_{I}^{4}\right),}  \tag{6.13}\\
& {\left[M_{5}, M_{7}\right]=\mathcal{O}\left(M_{I}^{4}\right),}  \tag{6.14}\\
& {\left[M_{6}, M_{7}\right]=\mathcal{O}\left(M_{I}^{4}\right),} \tag{6.15}
\end{align*}
$$

where the higher-order terms in the last three equations require the introduction of additional independent generators. Using those new generators in commutators with the original ones requires even higher orders. Iterating this procedure does not result in a finite-dimensional closed algebra suitable for this system.

Using suitable penalty factors, we can nevertheless propose a method to deal with target unitary operators whose generators are not part of a finite-dimensional closed commutator algebra. Illustrating the general method for the example just introduced, our proposal consists of two steps:

- Step 1: The generators of order higher than that appearing in the target unitary operators are assigned prohibitively large penalties, such that geodesics will not move in their direction on the operator space. We can neglect their contribution to the Euler-Arnold equation and the resulting complexity.

For example, in the commutator algebra shown above, we will assign prohibitive penalties to the generators with quartic or higher powers, such as $Q^{4}, Q^{2} P^{2}$, and $P Q^{3}$. For purposes of geodesic distance, their contributions to the algebra are therefore
neglected. Under this assumption, the Euler-Arnold equations can be written as:

$$
\begin{align*}
& \frac{d V^{1}}{d s} \approx-\frac{G_{66}}{G_{11}} V^{4} V^{6}-2 \frac{G_{77}}{G_{11}} V^{6} V^{7}+3 \frac{G_{44}}{G_{11}} V^{4} V^{6} \\
&+2 \frac{G_{66}}{G_{11}} V^{7} V^{6}-3 \frac{G_{55}}{G_{11}} V^{7} V^{5}+\frac{G_{77}}{G_{11}} V^{5} V^{7},  \tag{6.16}\\
& \frac{d V^{4}}{d s} \approx \frac{G_{66}}{G_{44}} V^{1} V^{6},  \tag{6.17}\\
& \frac{d V^{5}}{d s} \approx-\frac{G_{77}}{G_{55}} V^{1} V^{7},  \tag{6.18}\\
& \frac{d V^{6}}{d s} \approx 2 \frac{G_{77}}{G_{66}} V^{1} V^{7}-3 \frac{G_{44}}{G_{66}} V^{1} V^{4},  \tag{6.19}\\
& \frac{d V^{7}}{d s} \approx-2 \frac{G_{66}}{G_{77}} V^{1} V^{6}+3 \frac{G_{55}}{G_{77}} V^{1} V^{5} . \tag{6.20}
\end{align*}
$$

- Step 2: The generators of higher order have comparatively much higher penalties, i.e., the generators $M_{4}, M_{5}, M_{6}$, and $M_{7}$ which are of cubic order have sufficiently large penalties compared to $M_{1}$, which is quadratic.

A large penalty factor, such as $G_{66}$, means that a geodesic will not move in the corresponding direction, and therefore the component $V^{6}$ remains small. We can then ignore products of the prohibited components even if they are multiplied by a large penalty factor, such as $\left(G_{66} / G_{11}\right) V^{6} V^{7}$, as well as terms with a single factor of a prohibited component as long as it is not multiplied by a large penalty factor, such as $\left(G_{55} / G_{11}\right) V^{5} V^{7}$. Following this procedure, the equations simplify slightly to:

$$
\begin{align*}
& \frac{d V^{1}}{d s} \approx-\frac{G_{66}}{G_{11}} V^{4} V^{6}+\frac{G_{77}}{G_{11}} V^{5} V^{7},  \tag{6.21}\\
& \frac{d V^{4}}{d s} \approx \frac{G_{66}}{G_{44}} V^{1} V^{6},  \tag{6.22}\\
& \frac{d V^{5}}{d s} \approx-\frac{G_{77}}{G_{55}} V^{1} V^{7},  \tag{6.23}\\
& \frac{d V^{6}}{d s} \approx 2 \frac{G_{77}}{G_{66}} V^{1} V^{7}-3 \frac{G_{44}}{G_{66}} V^{1} V^{4},  \tag{6.24}\\
& \frac{d V^{7}}{d s} \approx-2 \frac{G_{66}}{G_{77}} V^{1} V^{6}+3 \frac{G_{55}}{G_{77}} V^{1} V^{5}, \tag{6.25}
\end{align*}
$$

if $M_{6}$ and $M_{7}$ have prohibitive penalties. If $M_{5}$ has a prohibitive penalty as well, we obtain

$$
\begin{align*}
\frac{d V^{1}}{d s} & \approx-\frac{G_{66}}{G_{11}} V^{4} V^{6}  \tag{6.26}\\
\frac{d V^{4}}{d s} & \approx \frac{G_{66}}{G_{44}} V^{1} V^{6}  \tag{6.27}\\
\frac{d\left(G_{55} V^{5}\right)}{d s} & \approx-G_{77} V^{1} V^{7}  \tag{6.28}\\
\frac{d\left(G_{66} V^{6}\right)}{d s} & \approx 2 G_{77} V^{1} V^{7}-3 G_{44} V^{1} V^{4}  \tag{6.29}\\
\frac{d\left(G_{77} V^{7}\right)}{d s} & \approx-2 G_{66} V^{1} V^{6}+3 G_{55} V^{1} V^{5} \tag{6.30}
\end{align*}
$$

Similarly, if $M_{4}$ has prohibitive penalties, we can ignore the term $V^{4} V^{6}$ and simplify the Euler-Arnold equations further. We will consider the penalties $G_{44}=G_{55}=G_{66}=$ $G_{77}=p \gg G_{11}$. In that case, the Euler-Arnold equations simplify to:

$$
\begin{align*}
\frac{d V^{1}}{d s} & \approx 0  \tag{6.31}\\
\frac{d V^{4}}{d s} & \approx V^{1} V^{6}  \tag{6.32}\\
\frac{d V^{5}}{d s} & \approx-V^{1} V^{7}  \tag{6.33}\\
\frac{d V^{6}}{d s} & \approx 2 V^{1} V^{7}-3 V^{1} V^{4}  \tag{6.34}\\
\frac{d V^{7}}{d s} & \approx-2 V^{1} V^{6}+3 V^{1} V^{5} \tag{6.35}
\end{align*}
$$

The solutions of the equations can be written as:

$$
\begin{align*}
V^{1}(s) & =v_{1}  \tag{6.36}\\
V^{4}(s) & =\frac{1}{4} v_{4}\left(3 \cos \left(s v_{1}\right)+\cos \left(3 s v_{1}\right)\right)+\frac{1}{4} v_{5}\left(3 \sin \left(s v_{1}\right)-\sin \left(3 s v_{1}\right)\right) \\
& +\frac{1}{4} v_{6}\left(\sin \left(s v_{1}\right)+\sin \left(3 s v_{1}\right)\right)+\frac{1}{4} v_{7}\left(\cos \left(s v_{1}\right)-\cos \left(3 s v_{1}\right)\right)  \tag{6.37}\\
V^{5}(s) & =\frac{1}{4} v_{4}\left(\sin \left(3 s v_{1}\right)-3 \sin \left(s v_{1}\right)\right)+\frac{1}{4} v_{5}\left(3 \cos \left(s v_{1}\right)+\cos \left(3 s v_{1}\right)\right) \\
& +\frac{1}{4} v_{6}\left(\cos \left(s v_{1}\right)-\cos \left(3 s v_{1}\right)\right)+\frac{1}{4} v_{7}\left(-\sin \left(s v_{1}\right)-\sin \left(3 s v_{1}\right)\right)  \tag{6.38}\\
V^{6}(s) & =-\frac{3}{4} v_{4}\left(\sin \left(s v_{1}\right)+\sin \left(3 s v_{1}\right)\right)+\frac{3}{4} v_{5}\left(\cos \left(s v_{1}\right)-\cos \left(3 s v_{1}\right)\right) \\
& +\frac{1}{4} v_{6}\left(\cos \left(s v_{1}\right)+3 \cos \left(3 s v_{1}\right)\right)+\frac{1}{4} v_{7}\left(3 \sin \left(3 s v_{1}\right)-\sin \left(s v_{1}\right)\right)  \tag{6.39}\\
V^{7}(s) & =\frac{3}{4} v_{4}\left(\cos \left(s v_{1}\right)-\cos \left(3 s v_{1}\right)\right)+\frac{3}{4} v_{5}\left(\sin \left(s v_{1}\right)+\sin \left(3 s v_{1}\right)\right) \\
& +\frac{1}{4} v_{6}\left(\sin \left(s v_{1}\right)-3 \sin \left(3 s v_{1}\right)\right)+\frac{1}{4} v_{7}\left(\cos \left(s v_{1}\right)+3 \cos \left(3 s v_{1}\right)\right) \tag{6.40}
\end{align*}
$$

Proceeding as before, we would keep only up to the leading order term in the Dyson series, which helps us to write the path-ordered exponential as follows:

$$
\begin{equation*}
U(s) \approx \exp \left(-i\left(\gamma_{1}(s) M_{1}+\gamma_{4}(s) M_{4}+\gamma_{5}(s) M_{5}+\gamma_{6}(s) M_{6}+\gamma_{7}(s) M_{7}\right)\right) \tag{6.41}
\end{equation*}
$$

where the following formulas give the $\gamma_{I}$ functions:

$$
\begin{align*}
\gamma_{1}(s) & =s v_{1}  \tag{6.42}\\
\gamma_{4}(s) & =\frac{1}{12 v_{1}}\left(9 v_{4} \sin \left(s v_{1}\right)+v_{4} \sin \left(3 s v_{1}\right)-3\left(3 v_{5}+v_{6}\right) \cos \left(s v_{1}\right)+\left(v_{5}-v_{6}\right) \cos \left(3 s v_{1}\right)\right. \\
& \left.+3 v_{7} \sin \left(s v_{1}\right)-v_{7} \sin \left(3 s v_{1}\right)+8 v_{5}+4 v_{6}\right)  \tag{6.43}\\
\gamma_{5}(s) & =\frac{1}{12 v_{1}}\left(3\left(3 v_{4}+v_{7}\right) \cos \left(s v_{1}\right)+\left(v_{7}-v_{4}\right) \cos \left(3 s v_{1}\right)+9 v_{5} \sin \left(s v_{1}\right)\right. \\
& \left.+v_{5} \sin \left(3 s v_{1}\right)+3 v_{6} \sin \left(s v_{1}\right)-v_{6} \sin \left(3 s v_{1}\right)-8 v_{4}-4 v_{7}\right)  \tag{6.44}\\
\gamma_{6}(s) & =\frac{1}{4 v_{1}}\left(\left(3 v_{4}+v_{7}\right) \cos \left(s v_{1}\right)+\left(v_{4}-v_{7}\right) \cos \left(3 s v_{1}\right)+3 v_{5} \sin \left(s v_{1}\right)-v_{5} \sin \left(3 s v_{1}\right)\right. \\
& \left.+v_{6} \sin \left(s v_{1}\right)+v_{6} \sin \left(3 s v_{1}\right)-4 v_{4}\right)  \tag{6.45}\\
\gamma_{7}(s) & =\frac{1}{4 v_{1}}\left(3 v_{4} \sin \left(s v_{1}\right)-v_{4} \sin \left(3 s v_{1}\right)-\left(3 v_{5}+v_{6}\right) \cos \left(s v_{1}\right)+\left(v_{6}-v_{5}\right) \cos \left(3 s v_{1}\right)\right. \\
& \left.+v_{7} \sin \left(s v_{1}\right)+v_{7} \sin \left(3 s v_{1}\right)+4 v_{5}\right) \tag{6.46}
\end{align*}
$$

Upon implementing the boundary condition at $s=1$ by setting $U(s=1)$ equal to the target unitary operator written in (6.5), we obtain:

$$
\begin{equation*}
\gamma_{1}(1)=\omega t, \quad \gamma_{4}(1)=\lambda t, \quad \gamma_{5}(1)=0, \quad \gamma_{6}(1)=0, \quad \gamma_{7}(1)=0 \tag{6.47}
\end{equation*}
$$

Our treatment of higher-order contributions by suppressing them through large penalty factors also simplifies the periodicity argument. The commutators of $M_{1}$ through $M_{7}$ with neglected fourth-order terms take the form of a semidirect product of the Lie algebra $\mathrm{sp}(2, \mathbb{R})$ spanned by $M_{1}, M_{2}$ and $M_{3}$ with an Abelian 4-dimensional Lie algebra spanned by $M_{4}$ through $M_{7}$. The precise form of the semi-direct product can be determined by the same methods used in [46] for the algebra formed by Poisson brackets of third-order central moments: It is given by $\operatorname{sp}(2, \mathbb{R}) \ltimes \mathbb{R}^{4}$ where $\operatorname{sp}(2, \mathbb{R})$ acts on $\mathbb{R}^{4}$ according to the spin- $3 / 2$ representation of $\operatorname{sp}(2, \mathbb{R})$. Imposing periodicity can then be done by the same arguments used for a harmonic oscillator, based on properties of $\operatorname{sp}(2, \mathbb{R})$, requiring $4 \pi$-periodicity of $v_{1}$ in $\omega t$ :

$$
\begin{equation*}
v_{1}=|\omega t-4 \pi\lfloor(\omega t+2 \pi) /(4 \pi)\rfloor| . \tag{6.48}
\end{equation*}
$$

With this result, the other coefficients are:

$$
\begin{align*}
& v_{4}=\frac{3 \lambda v_{1} t \cos \left(v_{1}\right) \cot \left(v_{1} / 2\right)}{2\left(1+2 \cos \left(v_{1}\right)\right)},  \tag{6.49}\\
& v_{5}=0,  \tag{6.50}\\
& v_{6}=\frac{3 v_{1} \lambda t}{2},  \tag{6.51}\\
& v_{7}=\frac{3 v_{1} \lambda t \sin \left(v_{1}\right)}{2\left(1+2 \cos \left(v_{1}\right)\right)} . \tag{6.52}
\end{align*}
$$

The initial velocity components $v_{4}$ and $v_{7}$ diverge at $\omega t= \pm \frac{2 n \pi}{3}$, which means that the complexity bound also diverges at these points. Since we are using a compact group for this system, it is exponential, and we are not missing parts of the manifold. The rightinvariant metric used here does not seem to be complete. Around a divergence, neglecting higher-order terms in the full algebra may not be justified because, compared with an infinite distance, they could certainly contribute to geodesics even if they are subject to prohibitive penalties. If one removes suitable regions around the divergences, the finite geodesic distances are reliable as upper bounds on the complexity.

The complexity bound of the time evolution operator keeping up to cubic order terms can therefore be written as:

$$
\begin{align*}
C\left[U_{A . H O}\right] & \leq \int_{0}^{1} d s \sqrt{G_{I J} V^{I}(s) V^{J}(s)} \\
& =\int_{0}^{1} d s \sqrt{G_{11}\left(V^{1}\right)^{2}+p\left(\left(V^{4}\right)^{2}+\left(V^{5}\right)^{2}+\left(V^{6}\right)^{2}+\left(V^{7}\right)^{2}\right)} \\
& =\int_{0}^{1} d s\left(\frac { 1 } { 2 } \left[4 G_{11} v_{1}^{2}+p \cos \left(4 s v_{1}\right)\left(-3 v_{4}^{2}+2 v_{4} v_{7}-3 v_{5}^{2}+2 v_{5} v_{6}+v_{6}^{2}+v_{7}^{2}\right)\right.\right. \\
& \left.\left.+4 p \sin \left(4 s v_{1}\right)\left(v_{5} v_{7}-v_{4} v_{6}\right)+p\left(7 v_{4}^{2}-2 v_{4} v_{7}+7 v_{5}^{2}-2 v_{5} v_{6}+3\left(v_{6}^{2}+v_{7}^{2}\right)\right)\right]^{1 / 2}\right) . \tag{6.53}
\end{align*}
$$

The integrand is no longer constant, but it can be integrated upon using $A+B \cos (x)+$ $C \sin (x)=A+\sqrt{B^{2}+C^{2}} \sin (x+\phi)$ with $\sin \phi=B / \sqrt{B^{2}+C^{2}}$ and $\int \sqrt{a+b \sin y} d y=$ $-2 \sqrt{a+b} E\left(\left.\frac{1}{4}(\pi-2 x) \right\rvert\, 2 b /(a+b)\right)$ with an elliptic function of the second kind. Here, we have:

$$
\begin{align*}
A & =2 G_{11} v_{1}^{2}+\frac{p}{2}\left(7 v_{4}^{2}-2 v_{4} v_{7}+7 v_{5}^{2}-2 v_{5} v_{6}+3\left(v_{6}^{2}+v_{7}^{2}\right)\right),  \tag{6.54}\\
B & =\frac{p}{2}\left(-3 v_{4}^{2}+2 v_{4} v_{7}-3 v_{5}^{2}+2 v_{5} v_{6}+v_{6}^{2}+v_{7}^{2}\right),  \tag{6.55}\\
C & =2 p\left(v_{5} v_{7}-v_{4} v_{6}\right),  \tag{6.56}\\
x & =4 s v_{1}, \tag{6.57}
\end{align*}
$$

with the above identification, $a$ and $b$ can be written as:

$$
\begin{align*}
& a=A=2 G_{11} v_{1}^{2}+\frac{p}{2}\left(7 v_{4}^{2}-2 v_{4} v_{7}+7 v_{5}^{2}-2 v_{5} v_{6}+3\left(v_{6}^{2}+v_{7}^{2}\right)\right)  \tag{6.58}\\
& b=\sqrt{B^{2}+C^{2}}=\frac{p}{2} \sqrt{\left(-3 v_{4}^{2}+2 v_{4} v_{7}-3 v_{5}^{2}+2 v_{5} v_{6}+v_{6}^{2}+v_{7}^{2}\right)^{2}+16\left(v_{4} v_{6}-v_{5} v_{7}\right)^{2}} . \tag{6.59}
\end{align*}
$$

Substituting $y=x+\phi$, the complexity bound can be explicitly evaluated, as used in the plot for Fig. 6.

In the harmonic limit $\lambda \rightarrow 0$ (with $G_{11}=1$ as used earlier), we have:

$$
\begin{equation*}
C\left[\left.U_{A . H O}\right|_{\lambda \rightarrow 0}=|\omega t-4 \pi\lfloor(\omega t+2 \pi) /(4 \pi)\rfloor|\right. \tag{6.60}
\end{equation*}
$$

in agreement with our direct derivation for the harmonic oscillator. In this limit, the initial velocities $v_{4}, v_{6}$, and $v_{7}$ go to zero, and the only contribution comes from $v_{1}$.


Figure 6. Complexity of the time evolution operator of an anharmonic oscillator.

## 7 Discussion and Comments

The geometrical approach to quantum complexity is intriguing because it expresses a complicated optimization problem with a standard procedure of finding the geodesic distance between two given points in a suitable curved space, as proposed in [15-17]. In the application of geodesic distance to complexity, the two endpoints are represented by the identity and a target unitary operator, respectively, embedded in a suitable geometrical formulation of the group of unitary operators on a Hilbert space. On a given Lie group equipped with a right-invariant metric, the problem of finding geodesics, whose lengths then determine the complexity of operations, is reduced to solving the Euler-Arnold equations.

Initial subtleties that immediately arise in an infinite-dimensional setting can be evaded by using a truncation of the full group of unitary operators to a suitable finite-dimensional
subgroup that includes operators of interest, including the target unitary as well as additional basic operations that form a closed group together with the target unitary. In this spirit, we have revisited the question of the quantum complexity of the harmonic oscillator within various finite-dimensional groups, which had already been discussed in other papers, in order to understand its properties and with an aim toward possible generalizations to related evolution operators. In this process, we found and highlighted several additional mathematical subtleties, even in a finite-dimensional construction that require a proper consideration of interesting properties of the group theory. In particular, the question of properly embedding the finite-dimensional truncation in the full group of unitary operators, which had not been considered before, requires information about topological properties of Lie groups and their covering groups.

We studied the harmonic oscillator from the perspective of two different Lie groups, the solvable harmonic oscillator group and the semisimple $\operatorname{Sp}(2, \mathbb{R})$. Our results qualitatively agree with previous work, showing a piecewise linear oscillating behavior of the quantum complexity of the evolution operator. However, based on the condition that the finitedimensional group should be embeddable in the full infinite-dimensional group of unitary operators with the well-known spectrum for the evolution operator of the harmonic oscillator, we concluded that the period of oscillations should be doubled. This condition takes into account the half-integer nature of the ground-state energy. The correct group is, therefore not directly $\mathrm{Sp}(2, \mathbb{R})$ but rather its 2 -fold covering, the metaplectic group $\mathrm{Mp}(2, \mathbb{R})$. This group is not a matrix group, and therefore previous methods which explicitly used finite-dimensional matrix representations cannot be applied. Our methods, by contrast, work mainly at an algebraic level and bring in group-theoretical properties only in the final step in order to determine the periodicity properties. Our new methods, therefore, present crucial generalizations and clarifications even for the well-studied harmonic oscillator. While doubling the period does not change the short-term behavior of the complexity while the evolution operator is still close to the identity, it can have large effects after several periods of the system.

We modified the original Hamiltonian by adding terms linear or quadratic in the position operator, which can be done without enlarging the original finite-dimensional groups. Classically, such a procedure merely shifts the origin of the oscillator or changes the frequency, but there are stronger effects in quantum mechanics. In particular, rewriting the added operator as a shift in one of the real-valued parameters might require operations that are not contained in the finite-dimensional group chosen for calculations, or the addition might implicitly change penalty factors assigned to operations by choosing a specific right-invariant metric on the group. As we found, the quantum complexity may well depend on such innocuous-looking modifications. In particular, such terms might change the piecewise linear nature of the complexity of a harmonic oscillator. The inverted harmonic oscillator can be treated with similar methods. Since its Hamiltonian corresponds to a noncompact direction in the finite-dimensional group used here, the complexity is not periodic and increases linearly, as seen in previous work.

We generalized the methods in two ways: by using larger but still finite-dimensional groups and estimating the complexity of two coupled oscillators and of an anharmonic os-
cillator. We were able to formulate these problems within subgroups with easily identifiable periodicity properties. In all cases, we encountered additional subtleties because the groups involved are not exponential, and because the right-invariant metrics motivated by physical arguments are not guaranteed to be geodesically complete. These properties, together with the overall truncation to a finite-dimensional group, imply that a given calculation does not take into account directions in the full infinite-dimensional group of unitary operators through which a geodesic might be able to take a shortcut. Such results can therefore be considered only upper bounds on the quantum complexity rather than strict values. This caveat applies in particular to divergences of length that could be implied by geodesic incompleteness.

We encountered non-exponential groups and possible geodesic incompleteness because useful groups for bosonic oscillator systems are not compact. This property is different for fermionic oscillators, which lead to orthogonal groups generated by fermion bilinears. Some of the derivations may, therefore simplify for fermions, which we are planning to analyze in future work.

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## A Line element

In order to derive the expression for a right-invariant line element on a Lie group with generators $\mathcal{O}_{I}$, we first compute the differential:

$$
\begin{equation*}
d U=-i \mathcal{O}_{I} U d x^{I} \tag{A.1}
\end{equation*}
$$

of a group element $U=\exp \left(-i x^{I} \mathcal{O}_{I}\right)$ that exponentiates a fixed generator $\mathcal{O}_{I}$ (no summation over $I$ ). The line element requires us to solve for $d x^{I}$, for which we first multiply (A.1) with $U^{-1}$ from the right:

$$
\begin{equation*}
i d U U^{-1}=\mathcal{O}_{I} d x^{I} \tag{A.2}
\end{equation*}
$$

The generator $\mathcal{O}_{I}$ does not have an inverse, but if we use a matric representation of the Lie algebra, we can first multiply by $\mathcal{O}_{I}^{\dagger}$, take a trace in the representation, and divide the equation by the resulting number:

$$
\begin{equation*}
d x^{I}=\frac{1}{\operatorname{Tr}\left[\mathcal{O}_{I} \mathcal{O}_{I}^{\dagger}\right]}\left[\operatorname{Tr}\left[i d U U^{-1} \mathcal{O}_{I}^{\dagger}\right]\right] \tag{A.3}
\end{equation*}
$$

(If the trace happens to be zero, it effectively puts a prohibitive penalty on the corresponding generator.)

The generator $\mathcal{O}_{I}$ commutes with $U$, and any ordering could have been chosen on the right of (A.1). The way in which we wrote (A.1) is suitable for the construction of a right-invariant line element by transporting the resulting $d x^{I}$ to generic group elements: The product $d U U^{-1}$ is right-invariant because for any constant $g$ in the Lie group, $d(U g)(g U)^{-1}=d U U^{-1}$. After such a right translation on the group, the ordering is relevant because $d(U g)$ is no longer guaranteed to commute with $(U g)^{-1}$, unlike $d U$ and $U^{-1}$ for our specific $U$.

After the right transportation, we can use (A.3) at any group element $U$. Using the penalty matrix $G_{I J}$, the line element on the entire group then takes the form:

$$
\begin{align*}
d s^{2} & =G_{I J} d x^{I} d x^{J}=G_{I J} \frac{1}{\operatorname{Tr}\left[\mathcal{O}_{I} \mathcal{O}_{I}^{\dagger}\right]}\left[\operatorname{Tr}\left[i d U U^{-1} \mathcal{O}_{I}^{\dagger}\right]\right] \frac{1}{\operatorname{Tr}\left[\mathcal{O}_{J} \mathcal{O}_{J}^{\dagger}\right]}\left[\operatorname{Tr}\left[i d U U^{-1} \mathcal{O}_{J}^{\dagger}\right]\right] \\
& =G_{I J} \frac{1}{\operatorname{Tr}\left[\mathcal{O}_{I} \mathcal{O}_{I}^{\dagger}\right]} \frac{1}{\operatorname{Tr}\left[\mathcal{O}_{J} \mathcal{O}_{J}^{\dagger}\right]}\left[\operatorname{Tr}\left[i U^{-1} \mathcal{O}_{I}^{\dagger} d U\right]\right]\left[\operatorname{Tr}\left[i U^{-1} \mathcal{O}_{J}^{\dagger} d U\right]\right], \tag{A.4}
\end{align*}
$$

where we brought $d U$ to the right using cyclic commutation in the trace.

## B Using a generic element of the suitable group to compute complexity

In this appendix, we show that instead of trying to write down the Dyson series for the path-ordered exponential, we can make use of the differential equation (2.6) it satisfies in order to derive the complexity. In fact, this method is usually adopted in the circuit complexity literature. Let us explain the steps usually followed, taking the example of the model of two coupled oscillators studied in [18]. It was realized that for the given purpose, it is necessary to look for geodesics in the $\mathrm{GL}(2, \mathbb{R})$ group. One may choose an explicit parametrization of a general element $U \in \mathrm{GL}(2, \mathbb{R})$, equation (3.18) of [18], and a suitable finite-dimensional matrix representation of the generators. However, in general, there may be obstacles to finding a finite-dimensional matrix representation of the generators, for instance, for certain universal covering groups. In this appendix, we, therefore, use an algebraic method based on a suitable parametrization of a general element of the desired group without using any matrix representation of the generators.

We will explicitly calculate the complexity of the displacement operator using the product form (disentangled form) of the generic element of the Harmonic oscillator group. To remind, our intention is to solve:

$$
\begin{equation*}
\frac{d U(s)}{d s}=-i V^{I}(s) \mathcal{O}_{I} U(s) \tag{B.1}
\end{equation*}
$$

subject to the boundary conditions:

$$
\begin{equation*}
U(s=0)=\mathbb{I}, \quad \text { and } \quad U(s=1)=U_{\text {target }} . \tag{B.2}
\end{equation*}
$$

To solve the above equation, we discussed that we need to introduce a generic element of the Lie group under consideration, and there are two ways to represent the generic element:

$$
\begin{equation*}
U(s)=\exp \left(-i \sum_{i=1}^{n} \beta_{i}(s) \hat{\mathcal{O}}_{i}\right)=\prod_{i=1}^{N} \exp \left(-i \alpha_{i}(s) \hat{\mathcal{O}}_{i}\right) . \tag{B.3}
\end{equation*}
$$

In the main text, we have worked with the form $\exp \left(-i \sum_{i=1}^{n} \beta_{i}(s) \hat{\mathcal{O}}_{i}\right)$. Here we do the computation for the product form, i.e. we take:

$$
\begin{equation*}
U(s)=\prod_{i=1}^{N} \exp \left(-i \alpha_{i}(s) \hat{\mathcal{O}}_{i}\right) . \tag{B.4}
\end{equation*}
$$

For the Harmonic oscillator group with generators $H, P, Q$ and $E$, the generic element can be written as:

$$
\begin{equation*}
U(s)=\exp \left(-i \alpha_{1}(s) E\right) \exp \left(-i \alpha_{2}(s) P\right) \exp \left(-i \alpha_{3}(s) Q\right) \exp \left(-i \alpha_{4} H\right) \tag{B.5}
\end{equation*}
$$

Substituting the above equation in B.1, the LHS can be written as:

$$
\begin{align*}
\frac{d U(s)}{d s} & =-i\left[\exp \left\{-i \alpha_{1}(s) E\right\} \exp \left\{-i \alpha_{2}(s) P\right\} \exp \left\{-i \alpha_{3}(s) Q\right\}\left(\exp \left\{-i \alpha_{4}(s) H\right\} \alpha_{4}^{\prime}(s) H\right)\right. \\
& +\exp \left\{-i \alpha_{1}(s) E\right\} \exp \left\{-i \alpha_{2}(s) P\right\}\left(\exp \left\{-i \alpha_{3}(s) Q\right\} \alpha_{3}^{\prime}(s) Q\right) \exp \left\{-i \alpha_{4}(s) H\right\} \\
& +\exp \left\{-i \alpha_{1}(s) E\right\}\left(\exp \left\{-i \alpha_{2}(s) P\right\} \alpha_{2}^{\prime}(s) P\right) \exp \left\{-i \alpha_{3}(s) Q\right\} \exp \left\{-i \alpha_{4}(s) H\right\} \\
& \left.+\left(\exp \left\{-i \alpha_{1}(s) E\right\} \alpha_{1}^{\prime}(s) E\right) \exp \left\{-i \alpha_{2}(s) P\right\} \exp \left\{-i \alpha_{3}(s) Q\right\} \exp \left\{-i \alpha_{4}(s) H\right\}\right] \tag{B.6}
\end{align*}
$$

whereas the RHS (for $G_{I J}=\delta_{I J}$ ) can be written as:

$$
\begin{align*}
-i V^{I}(s) \mathcal{O}_{I} U(s)= & -i\left(v_{H} H+\left\{v_{P} \cos \left(s\left(v_{E}-v_{H}\right)\right)+v_{Q} \sin \left(s\left(v_{E}-v_{H}\right)\right)\right\} P\right. \\
& \left.+\left\{v_{Q} \cos \left(s\left(v_{E}-v_{H}\right)\right)-v_{P} \sin \left(s\left(v_{E}-v_{H}\right)\right)\right\} Q+v_{E} E\right) U(s) . \tag{B.7}
\end{align*}
$$

Before proceeding, let us make a simplification by realizing the action of the generator $E$. The generator $E$ being a non-trivial center, commutes with all other generators, and its role is just to produce a phase factor when applied to states. Therefore to simplify, we can choose $v_{E}=0$ for which eqn B. 7 becomes:

$$
\begin{align*}
-i V^{I}(s) \mathcal{O}_{I} U(s)= & -i\left(v_{H} H+\left\{v_{P} \cos \left(s\left(v_{H}\right)\right)-v_{Q} \sin \left(s\left(v_{H}\right)\right)\right\} P\right.  \tag{B.8}\\
& \left.+\left\{v_{Q} \cos \left(s\left(v_{H}\right)\right)+v_{P} \sin \left(s\left(v_{H}\right)\right)\right\} Q\right) U(s) \tag{B.9}
\end{align*}
$$

To equate the coefficients of the generators in B. 6 and B. 7 and derive the differential equations for the $s$ dependent parameters $\alpha_{i}$ 's, we have to express B. 6 in way such that it resembles B. 7 in its form i.e it should be written as $(\quad) \times U(s)$. For that purpose, let us begin by writing equation B. 6 as:

$$
\begin{equation*}
\frac{d U(s)}{d s}=-i[\text { Term } 1+\text { Term } 2+\text { Term } 3+\text { Term } 4], \tag{B.10}
\end{equation*}
$$

where Term 1, Term 2, Term 3, and Term 4 are as follows:

$$
\begin{equation*}
\text { Term1 }=\exp \left\{-i \alpha_{1}(s) E\right\} \exp \left\{-i \alpha_{2}(s) P\right\} \exp \left\{-i \alpha_{3}(s) Q\right\}\left(\exp \left\{-i \alpha_{4}(s) H\right\} \alpha_{4}^{\prime}(s) H\right) \tag{B.11}
\end{equation*}
$$

Term $2=\exp \left\{-i \alpha_{1}(s) E\right\} \exp \left\{-i \alpha_{2}(s) P\right\}\left(\exp \left\{-i \alpha_{3}(s) Q\right\} \alpha_{3}^{\prime}(s) Q\right) \exp \left\{-i \alpha_{4}(s) H\right\}$,
Term $3=\exp \left\{-i \alpha_{1}(s) E\right\}\left(\exp \left\{-i \alpha_{2}(s) P\right\} \alpha_{2}^{\prime}(s) P\right) \exp \left\{-i \alpha_{3}(s) Q\right\} \exp \left\{-i \alpha_{4}(s) H\right\}$,

Term $4=\left(\exp \left\{-i \alpha_{1}(s) E\right\} \alpha_{1}^{\prime}(s) E\right) \exp \left\{-i \alpha_{2}(s) P\right\} \exp \left\{-i \alpha_{3}(s) Q\right\} \exp \left\{-i \alpha_{4}(s) H\right\}$.

Let us start the simplification with the easiest case, which is Term 4. We can easily shift the generator $E$ to the left of $\exp \left\{-i \alpha_{1}(s) E\right\}$ as $E$ commutes with it. Thus, Term 4 can be written as:

$$
\begin{align*}
\text { term } 4 & =\alpha_{1}^{\prime}(s) E \exp \left\{-i \alpha_{1}(s) E\right\} \exp \left\{-i \alpha_{2}(s) P\right\} \exp \left\{-i \alpha_{3}(s) Q\right\} \exp \left\{-i \alpha_{4}(s) H\right\} \\
& =\alpha_{1}^{\prime}(s) E U(s) \tag{B.15}
\end{align*}
$$

Now let us consider Term 3. Again, since $P$ commutes with $\exp \left\{-i \alpha_{2}(s) P\right\}$, we can shift $P$ to the left of the exponential i.e. we can rewrite Term 3 as:

$$
\begin{align*}
\text { Term } 3= & \alpha_{2}^{\prime}(s)\left(\exp \left\{-i \alpha_{1}(s) E\right\} P\right) \exp \left\{-i \alpha_{2}(s) P\right\} \exp \left\{-i \alpha_{3}(s) Q\right\} \exp \left\{-i \alpha_{4}(s) H\right\} \\
= & \alpha_{2}^{\prime}(s)\left(\left[\exp \left\{-i \alpha_{1}(s) E\right\}, P\right]+P \exp \left\{-i \alpha_{1}(s) E\right\}\right) \exp \left\{-i \alpha_{2}(s) P\right\} \\
& \quad \exp \left\{-i \alpha_{3}(s) Q\right\} \exp \left\{-i \alpha_{4}(s) H\right\} \tag{B.16}
\end{align*}
$$

So, we have to evaluate the commutator of $\exp \left\{-i \alpha_{1}(s) E\right\}$ with $P$. Since $E$ commutes with $P$, all the higher orders of $E$ will also commute with $P$, therefore:

$$
\begin{equation*}
\left[\exp \left\{-i \alpha_{1}(s) E\right\}, P\right]=0 \tag{B.17}
\end{equation*}
$$

and in consequence:

$$
\begin{align*}
\text { Term3 } & =\alpha_{2}^{\prime}(s) P \exp \left\{-i \alpha_{1}(s) E\right\} \exp \left\{-i \alpha_{2}(s) P\right\} \exp \left\{-i \alpha_{3}(s) Q\right\} \exp \left\{-i \alpha_{4}(s) H\right\} \\
& =\left(\alpha_{2}^{\prime}(s) P\right) U(s) \tag{B.18}
\end{align*}
$$

Now let us try to simply Term 2 :

$$
\begin{align*}
\text { Term2 } & =\exp \left\{-i \alpha_{1}(s) E\right\} \exp \left\{-i \alpha_{2}(s) P\right\}\left(\exp \left\{-i \alpha_{3}(s) Q\right\} \alpha_{3}^{\prime}(s) Q\right) \exp \left\{-i \alpha_{4}(s) H\right\} \\
& =\alpha_{3}^{\prime}(s) \exp \left\{-i \alpha_{1}(s) E\right\}\left(\exp \left\{-i \alpha_{2}(s) P\right\} Q\right) \exp \left\{-i \alpha_{3}(s) Q\right\} \exp \left\{-i \alpha_{4}(s) H\right\} \tag{B.19}
\end{align*}
$$

We manipulate the term $\left(\exp \left\{-i \alpha_{2}(s) P\right\} Q\right)$ to shift the generator $Q$ to the left of the exponential. For this purpose, let us write $\left(\exp \left\{-i \alpha_{2}(s) P\right\} Q\right)$ as:

$$
\begin{align*}
\left(\exp \left\{-i \alpha_{2}(s) P\right\} Q\right) & =\left(\exp \left\{-i \alpha_{2}(s) P\right\} Q\right) \underbrace{\exp \left\{i \alpha_{2}(s) P\right\} \exp \left\{-i \alpha_{2}(s) P\right\}}_{\text {Identity operator }} \\
& =(\underbrace{\exp \left\{-i \alpha_{2}(s) P\right\} Q \exp \left\{i \alpha_{2}(s) P\right\}}_{\text {Apply Baker-Campbell Hausdorff }}) \exp \left\{-i \alpha_{2}(s) P\right\} . \tag{B.20}
\end{align*}
$$

Applying Baker-Campbell-Hausdroff (BCH) lemma:

$$
\begin{equation*}
e^{\lambda B} A e^{-\lambda B}=A+\lambda[B, A]+\frac{\lambda^{2}}{2!}[B,[B, A]]+\ldots \tag{B.21}
\end{equation*}
$$

we can write the term $\left(\exp \left\{-i \alpha_{2}(s) P\right\} Q \exp \left\{i \alpha_{2}(s) P\right\}\right)$ as:

$$
\begin{equation*}
\left(\exp \left\{-i \alpha_{2}(s) P\right\} Q \exp \left\{i \alpha_{2}(s) P\right\}\right)=Q-\alpha_{2} E . \tag{B.22}
\end{equation*}
$$

Therefore, Term 2 can be written as:

$$
\begin{align*}
\operatorname{term} 2 & =\alpha_{3}^{\prime}(s) e^{-i \alpha_{1}(s) E}\left(Q-\alpha_{2}(s) E\right) e^{-i \alpha_{2}(s) P} e^{-i \alpha_{3}(s) Q} e^{-i \alpha_{4}(s) H} \\
& =\alpha_{3}^{\prime}(s) Q U(s)-\alpha_{3}^{\prime}(s) \alpha_{2}(s) E U(s) \tag{B.23}
\end{align*}
$$

Finally, let us simplify Term 1 :

$$
\begin{align*}
& \text { Term1 }=\exp \left\{-i \alpha_{1}(s) E\right\} \exp \left\{-i \alpha_{2}(s) P\right\} \exp \left\{-i \alpha_{3}(s) Q\right\}\left(\exp \left\{-i \alpha_{4}(s) H\right\} \alpha_{4}^{\prime}(s) H\right) \\
& =\alpha_{4}^{\prime}(s) \exp \left\{-i \alpha_{1}(s) E\right\} \exp \left\{-i \alpha_{2}(s) P\right\}\left(\exp \left\{-i \alpha_{3}(s) Q\right\} H\right) \exp \left\{-i \alpha_{4}(s) H\right\} \\
& =\alpha_{4}^{\prime}(s) e^{-i \alpha_{1}(s) E} e^{-i \alpha_{2}(s) P}\left(\exp \left\{-i \alpha_{3}(s) Q\right\} H\right) \underbrace{e^{i \alpha_{3}(s) Q} e^{-i \alpha_{3}(s) Q}}_{\text {Identity operator inserted }} \\
& \exp \left\{-i \alpha_{4}(s) H\right\} \\
& =\alpha_{4}^{\prime}(s) \exp \left\{-i \alpha_{1}(s) E\right\} \exp \left\{-i \alpha_{2}(s) P\right\}(\underbrace{\exp \left\{-i \alpha_{3}(s) Q\right\} H \exp \left\{i \alpha_{3}(s) Q\right\}}_{\text {Apply Baker-Campbell Hausdorff formula }}) \\
& \exp \left\{-i \alpha_{3}(s) Q\right\} \exp \left\{-i \alpha_{4}(s) H\right\} \\
& =\alpha_{4}^{\prime}(s) \exp \left\{-i \alpha_{1}(s) E\right\} \exp \left\{-i \alpha_{2}(s) P\right\}\left(H+\alpha_{3}(s) P+\frac{\alpha_{3}(s)^{2}}{2} E\right) \\
& \exp \left\{-i \alpha_{3}(s) Q\right\} \exp \left\{-i \alpha_{4}(s) H\right\} \\
& =\alpha_{4}^{\prime}(s) \exp \left\{-i \alpha_{1}(s) E\right\} \exp \left\{-i \alpha_{2}(s) P\right\} H \exp \left\{-i \alpha_{3}(s) Q\right\} \exp \left\{-i \alpha_{4}(s) H\right\} \\
& +\alpha_{4}^{\prime}(s) \exp \left\{-i \alpha_{1}(s) E\right\} \exp \left\{-i \alpha_{2}(s) P\right\} \alpha_{3}(s) P \exp \left\{-i \alpha_{3}(s) Q\right\} \exp \left\{-i \alpha_{4}(s) H\right\} \\
& +\alpha_{4}^{\prime}(s) \exp \left\{-i \alpha_{1}(s) E\right\} \exp \left\{-i \alpha_{2}(s) P\right\} \frac{\alpha_{3}(s)^{2}}{2} E \exp \left\{-i \alpha_{3}(s) Q\right\} \exp \left\{-i \alpha_{4}(s) H\right\} \\
& =\alpha_{4}^{\prime}(s) \exp \left\{-i \alpha_{1}(s) E\right\}\left(H-\alpha_{2}(s) Q+\frac{\alpha_{2}(s)^{2}}{2} E\right) \\
& \exp \left\{-i \alpha_{2}(s) P\right\} \exp \left\{-i \alpha_{3}(s) Q\right\} \exp \left\{-i \alpha_{4}(s) H\right\} \\
& +\alpha_{4}^{\prime}(s) \alpha_{3}(s) P U(s)+\alpha_{4}^{\prime}(s) \frac{\alpha_{3}(s)^{2}}{2} E U(s) \\
& =\alpha_{4}^{\prime}(s) H U(s)-\alpha_{4}^{\prime}(s) \alpha_{2}(s) Q U(s)+\frac{1}{2} \alpha_{4}^{\prime}(s) \alpha_{2}(s)^{2} E U(s) \\
& +\alpha_{4}^{\prime}(s) \alpha_{3}(s) P U(s)+\frac{1}{2} \alpha_{4}^{\prime}(s) \alpha_{3}(s)^{2} E U(s) \\
& =\left(\alpha_{4}^{\prime}(s) H-\alpha_{4}^{\prime}(s) \alpha_{2}(s) Q+\frac{1}{2} \alpha_{4}^{\prime}(s) \alpha_{2}(s)^{2} E\right. \\
& \left.+\alpha_{4}^{\prime}(s) \alpha_{3}(s) P+\frac{1}{2} \alpha_{4}^{\prime}(s) \alpha_{3}(s)^{2} E\right) U(s) . \tag{B.24}
\end{align*}
$$

The above form of $d U(s) / d s$ is exactly what was required. Now we one simply match the coefficients of the corresponding generators on both sides and arrive at the following
differential equations of the $\alpha_{i}$ 's. The terms in B. 10 can be written as:

$$
\begin{align*}
& \text { Term1 }=\left(\alpha_{4}^{\prime}(s) H-\alpha_{4}^{\prime}(s) \alpha_{2}(s) Q+\frac{1}{2} \alpha_{4}^{\prime}(s) \alpha_{2}(s)^{2} E\right. \\
&\left.+\alpha_{4}^{\prime}(s) \alpha_{3}(s) P+\frac{1}{2} \alpha_{4}^{\prime}(s) \alpha_{3}(s)^{2} E\right) U(s),  \tag{B.25}\\
& \text { Term } 2=\left(\alpha_{3}^{\prime}(s) Q-\alpha_{3}^{\prime}(s) \alpha_{2}(s) E\right) U(s),  \tag{B.26}\\
& \text { Term3 }= \alpha_{2}^{\prime}(s) P U(s),  \tag{B.27}\\
& \text { Term } 4=\alpha_{1}^{\prime}(s) E U(s) . \tag{B.28}
\end{align*}
$$

In consequence, Eq. B. 10 can be written as:

$$
\begin{align*}
\frac{d U(s)}{d s}=- & i\left[\alpha_{4}^{\prime}(s) H-\alpha_{4}^{\prime}(s) \alpha_{2}(s) Q+\frac{1}{2} \alpha_{4}^{\prime}(s) \alpha_{2}(s)^{2} E+\alpha_{4}^{\prime}(s) \alpha_{3}(s) P\right. \\
& \left.+\frac{1}{2} \alpha_{4}^{\prime}(s) \alpha_{3}(s)^{2} E+\alpha_{3}^{\prime}(s) Q-\alpha_{3}^{\prime}(s) \alpha_{2}(s) E+\alpha_{2}^{\prime}(s) P+\alpha_{1}^{\prime}(s) E\right] U(s) \\
=- & i\left[\alpha_{4}^{\prime}(s) H+\left\{\alpha_{4}^{\prime}(s) \alpha_{3}(s)+\alpha_{2}^{\prime}(s)\right\} P+\left\{-\alpha_{4}^{\prime}(s) \alpha_{2}(s)+\alpha_{3}^{\prime}(s)\right\} Q\right. \\
& \left.+\left\{\alpha_{1}^{\prime}(s)-\alpha_{3}^{\prime}(s) \alpha_{2}(s)+\frac{1}{2} \alpha_{4}^{\prime}(s) \alpha_{3}(s)^{2}+\frac{1}{2} \alpha_{4}^{\prime}(s) \alpha_{2}(s)^{2}\right\} E\right] U(s) \tag{B.29}
\end{align*}
$$

Equating the coefficients of the generators from B. 29 and B.7, we get the following:

$$
\begin{align*}
& \alpha_{1}^{\prime}(s)=\alpha_{2}(s) v_{P} \sin \left(s v_{Q}\right)+\alpha_{2}(s) v_{Q} \cos \left(s v_{Q}\right)+\frac{\alpha_{2}(s)^{2} v_{Q}}{2}-\frac{\alpha_{3}(s)^{2} v_{Q}}{2},  \tag{B.30}\\
& \alpha_{2}^{\prime}(s)=v_{P} \cos \left(s v_{Q}\right)-v_{Q} \sin \left(s v_{Q}\right)+\alpha_{3}(s)\left(-v_{Q}\right),  \tag{B.31}\\
& \alpha_{3}^{\prime}(s)=v_{P} \sin \left(s v_{Q}\right)+v_{Q} \cos \left(s v_{Q}\right)+\alpha_{2}(s) v_{Q}  \tag{B.32}\\
& \alpha_{4}^{\prime}(s)=v_{H} . \tag{B.33}
\end{align*}
$$

The solutions of the above equations are as follows:

$$
\begin{align*}
\alpha_{1}(s)= & \frac{1}{2}\left(\frac{1}{2} \sin \left(2 s v_{Q}\right)\left(C_{1}^{2}+2 s\left(C_{1} v_{P}-C_{2} v_{Q}\right)-C_{2}^{2}+s^{2}\left(v_{P}^{2}-v_{Q}^{2}\right)\right)\right. \\
& \left.\quad+\left(C_{1}+s v_{P}\right)\left(C_{2}+s v_{Q}\right) \cos \left(2 s v_{Q}\right)+s\left(C_{1} v_{Q}-C_{2} v_{P}\right)\right)+C_{3},  \tag{B.34}\\
\alpha_{2}(s)= & C_{1} \cos \left(s v_{Q}\right)-C_{2} \sin \left(s v_{Q}\right)+s v_{P} \cos \left(s v_{Q}\right)-s v_{Q} \sin \left(s v_{Q}\right),  \tag{B.35}\\
\alpha_{3}(s)= & C_{1} \sin \left(s v_{Q}\right)+C_{2} \cos \left(s v_{Q}\right)+s v_{P} \sin \left(s v_{Q}\right)+s v_{Q} \cos \left(s v_{Q}\right),  \tag{B.36}\\
\alpha_{4}(s)= & s v_{H}+C_{4} . \tag{B.37}
\end{align*}
$$

Imposing the boundary condition $U(s=0)=\mathbb{I}$, the constants $C_{i}$ 's can be fixed to be:

$$
\begin{equation*}
C_{1}=0, \quad C_{2}=0, \quad C_{3}=0, \quad C_{4}=0 . \tag{B.38}
\end{equation*}
$$

which simplifies the solution as follows:

$$
\begin{align*}
& \alpha_{1}(s)=\frac{1}{4} s^{2}\left(\left(v_{P}^{2}-v_{Q}^{2}\right) \sin \left(2 s v_{Q}\right)+2 v_{P} v_{Q} \cos \left(2 s v_{Q}\right)\right),  \tag{B.39}\\
& \alpha_{2}(s)=s\left(v_{P} \cos \left(s v_{Q}\right)-v_{Q} \sin \left(s v_{Q}\right)\right),  \tag{B.40}\\
& \alpha_{3}(s)=s\left(v_{P} \sin \left(s v_{Q}\right)+v_{Q} \cos \left(s v_{Q}\right)\right),  \tag{B.41}\\
& \alpha_{4}(s)=s v_{H} . \tag{B.42}
\end{align*}
$$

Now let us compute the complexity of the displacement operator as an illustration. The displacement operator in terms of the generators of the Harmonic oscillator group has been written in Eq. 3.26:

$$
\begin{equation*}
U_{\text {target }}=\hat{D}=\exp (\sqrt{2} \operatorname{Re}(\alpha) Q+\sqrt{2} \operatorname{Im}(\alpha) P) . \tag{B.43}
\end{equation*}
$$

Looking at $U_{\text {target }}$, we immediately realize that it is not given in the form that we desire i.e., we have to disentangle the operator and express it as a product of the exponentials of the generators. Since the generators of Q and P commute with the commutator of Q and P (which is E ), it is not difficult to see that $\hat{D}$ can be written as:

$$
\begin{align*}
\hat{D} & =\exp (\sqrt{2} \operatorname{Im}(\alpha) P+\sqrt{2} \operatorname{Re}(\alpha) Q) \\
& =\exp (\sqrt{2} \operatorname{Im}(\alpha) P) \exp (\sqrt{2} \operatorname{Re}(\alpha) Q) \exp (-\operatorname{Re}(\alpha) \operatorname{Im}(\alpha)[P, Q]) \\
& =\exp (\sqrt{2} \operatorname{Im}(\alpha) P) \exp (\sqrt{2} \operatorname{Re}(\alpha) Q) \exp (-\operatorname{Re}(\alpha) \operatorname{Im}(\alpha)(-i E)) . \tag{B.44}
\end{align*}
$$

In the above derivation, we have used the fact that if two operators $A$ and $B$ commutes with the commutator of $A$ and $B$, then we can write (from the BCH formula):

$$
\begin{equation*}
\exp (A+B)=\exp (A) \exp (B) \exp \left(-\frac{1}{2}[A, B]\right) \tag{B.45}
\end{equation*}
$$

However, before comparing $\hat{D}$ with $U(s=1)$, we have to take into consideration the ordering of the generators, which should be identical to the ordering used while defining the generic element. In the generic element written in B.5, we see that the generator $E$ was at the beginning. Hence we have to reorder the exponentials in $\hat{D}$. Since $E$ commutes with both $Q$ and $P$, it can be shifted. Hence, $\hat{D}$ can be written as:

$$
\begin{equation*}
\hat{D}=\exp (i \operatorname{Re}(\alpha) \operatorname{Im}(\alpha) E) \exp (\sqrt{2} \operatorname{Im}(\alpha) P) \exp (\sqrt{2} \operatorname{Re}(\alpha) Q) . \tag{B.46}
\end{equation*}
$$

Therefore, we get the following:

$$
\begin{align*}
U(s=1) & =\hat{D} \\
& =\exp \left(-i \alpha_{1}(1) E\right) \exp \left(-i \alpha_{2}(1) P\right) \exp \left(-i \alpha_{3}(1) Q\right) \exp \left(-i \alpha_{4}(1) H\right)  \tag{B.47}\\
& =\exp (i \operatorname{Re}(\alpha) \operatorname{Im}(\alpha) E) \exp (\sqrt{2} \operatorname{Im}(\alpha) P) \exp (\sqrt{2} \operatorname{Re}(\alpha) Q), \tag{B.48}
\end{align*}
$$

from which we obtain the four conditions:

$$
\begin{equation*}
\alpha_{1}(1)=-\operatorname{Re}(\alpha) \operatorname{Im}(\alpha), \quad \alpha_{2}(1)=i \sqrt{2} \operatorname{Im}(\alpha), \quad \alpha_{3}(1)=i \sqrt{2} \operatorname{Re}(\alpha), \quad \alpha_{4}(1)=0 . \tag{B.49}
\end{equation*}
$$

From the above conditions, we find that:

$$
\begin{equation*}
v_{H}=0, \quad v_{P}=i \sqrt{2} \operatorname{Im}(\alpha), \quad v_{Q}=i \sqrt{2} \operatorname{Re}(\alpha) \tag{B.50}
\end{equation*}
$$

Therefore, the complexity of the displacement operator can be written a:

$$
\begin{equation*}
C[\hat{D}]=\sqrt{v_{P}^{2}+v_{Q}^{2}+v_{H}^{2}}=2|\alpha| \tag{B.51}
\end{equation*}
$$

which is exactly what we found in Sec. 3.2.2.

## C Commutation relations of the generators associated with the coupled harmonic oscillator

As discussed in the main text (see Sec. 5), a basis of generators associated with a coupled harmonic oscillator that forms a closed Lie algebra are:

$$
\begin{align*}
T_{1} & =H_{1}=\frac{1}{2}\left(Q_{1}^{2}+P_{1}^{2}\right),  \tag{C.1}\\
T_{2} & =H_{2}=\frac{1}{2}\left(Q_{2}^{2}+P_{2}^{2}\right),  \tag{C.2}\\
T_{3} & =\frac{1}{2}\left(Q_{1}^{2}-P_{1}^{2}\right),  \tag{C.3}\\
T_{4} & =\frac{1}{2}\left(Q_{2}^{2}-P_{2}^{2}\right),  \tag{C.4}\\
T_{5} & =\left(Q_{1} P_{1}+P_{1} Q_{1}\right),  \tag{C.5}\\
T_{6} & =\left(Q_{2} P_{2}+P_{2} Q_{2}\right),  \tag{C.6}\\
T_{7} & =\left(Q_{1} Q_{2}+P_{1} P_{2}\right),  \tag{C.7}\\
T_{8} & =\left(Q_{1} P_{2}+P_{1} Q_{2}\right),  \tag{C.8}\\
T_{9} & =\left(Q_{1} Q_{2}-P_{1} P_{2}\right),  \tag{C.9}\\
T_{10} & =\left(Q_{1} P_{2}-P_{1} Q_{2}\right) \tag{C.10}
\end{align*}
$$

The commutation relations satisfied by the generators are as follows:

$$
\begin{aligned}
& {\left[T_{1}, T_{2}\right]=0, \quad\left[T_{1}, T_{3}\right]=-i T_{5}, \quad\left[T_{1}, T_{4}\right]=0, \quad\left[T_{1}, T_{5}\right]=i T_{3}, \quad\left[T_{1}, T_{6}\right]=0,} \\
& {\left[T_{1}, T_{8}\right]=i T_{9}, \quad\left[T_{1}, T_{10}\right]=-i T_{7}, \quad\left[T_{1}, T_{7}\right]=i T_{10}, \quad\left[T_{1}, T_{9}\right]=-i T_{8},} \\
& {\left[T_{2}, T_{3}\right]=0, \quad\left[T_{2}, T_{4}\right]=-i T_{6}, \quad\left[T_{2}, T_{7}\right]=-i T_{10}, \quad\left[T_{2}, T_{9}\right]=-i T_{8},} \\
& {\left[T_{2}, T_{8}\right]=i T_{9}, \quad\left[T_{2}, T_{10}\right]=i T_{7}, \quad\left[T_{2}, T_{6}\right]=i T_{4}, \quad\left[T_{2}, T_{5}\right]=0,} \\
& {\left[T_{3}, T_{4}\right]=0, \quad\left[T_{3}, T_{5}\right]=i T_{1}, \quad\left[T_{4}, T_{5}\right]=0, \quad\left[T_{3}, T_{6}\right]=0, \quad\left[T_{4}, T_{6}\right]=i T_{2},} \\
& {\left[T_{3}, T_{7}\right]=i T_{8}, \quad\left[T_{4}, T_{7}\right]=i T_{8}, \quad\left[T_{3}, T_{8}\right]=i T_{7}, \quad\left[T_{4}, T_{8}\right]=i T_{7},} \\
& {\left[T_{3}, T_{9}\right]=-i T_{10}, \quad\left[T_{4}, T_{9}\right]=i T_{10}, \quad\left[T_{3}, T_{10}\right]=-i T_{9}, \quad\left[T_{4}, T_{10}\right]=i T_{9},} \\
& {\left[T_{5}, T_{1}\right]=-i T_{3}, \quad\left[T_{6}, T_{2}\right]=-i T_{4}, \quad\left[T_{5}, T_{3}\right]=-i T_{1}, \quad\left[T_{6}, T_{4}\right]=-i T_{2}, \quad\left[T_{5}, T_{6}\right]=0,} \\
& {\left[T_{5}, T_{7}\right]=-2 i T_{9}, \quad\left[T_{6}, T_{7}\right]=-2 i T_{9}, \quad\left[T_{5}, T_{8}\right]=-2 i T_{10}, \quad\left[T_{6}, T_{8}\right]=-2 i T_{10},} \\
& {\left[T_{5}, T_{9}\right]=-2 i T_{7}, \quad\left[T_{6}, T_{9}\right]=-2 i T_{7}, \quad\left[T_{5}, T_{10}\right]=-2 i T_{8}, \quad\left[T_{6}, T_{10}\right]=-2 i T_{8},}
\end{aligned}
$$

$$
\begin{aligned}
& {\left[T_{7}, T_{1}\right]=-i T_{10}, \quad\left[T_{7}, T_{2}\right]=i T_{10}, \quad\left[T_{7}, T_{3}\right]=-i T_{8}, \quad\left[T_{7}, T_{4}\right]=-i T_{8},} \\
& {\left[T_{7}, T_{5}\right]=2 i T_{9}, \quad\left[T_{7}, T_{6}\right]=2 i T_{9}, \quad\left[T_{7}, T_{8}\right]=2 i T_{3}+2 i T_{4}, \quad\left[T_{7}, T_{9}\right]=-i T_{6}-i T_{5},} \\
& {\left[T_{7}, T_{10}\right]=2 i T_{1}-2 i T_{2}, \quad\left[T_{8}, T_{9}\right]=-2 i T_{1}-2 i T_{2}, \quad\left[T_{8}, T_{10}\right]=-i T_{6}+i T_{5},} \\
& {\left[T_{9}, T_{10}\right]=2 i T_{3}-2 i T_{4} .}
\end{aligned}
$$

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[^1]:    ${ }^{1} \mathrm{~A}$ comprehensive review of recent advancements in quantum complexity across the field of physics is presented in [12].

