Graded Semantics and Graded Logics for Eilenberg-Moore Coalgebras

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Abstract. Coalgebra, as the abstract study of state-based systems, comes naturally equipped with a notion of behavioural equivalence that identifies states exhibiting the same behaviour. In many cases, however, this equivalence is finer than the intended semantics. Particularly in automata theory, behavioural equivalence of nondeterministic automata is essentially bisimilarity, and thus does not coincide with language equivalence. Language equivalence can be captured as behavioural equivalence on the determinization, which is obtained via the standard powerset construction. This construction can be lifted to coalgebraic generality, assuming a so-called Eilenberg-Moore distributive law between the functor determining the type of accepted structure (e.g. word languages) and a monad capturing the branching type (e.g. nondeterministic, weighted, probabilistic). Eilenberg-Moore-style coalgebraic semantics in this sense has been shown to be essentially subsumed by the more general framework of graded semantics, which is centrally based on graded monads. Graded semantics comes with a range of generic results, in particular regarding invariance and, under suitable conditions, expressiveness of dedicated modal logics for a given semantics; notably, these logics are evaluated on the original state space. We show that the instantiation of such graded logics to the case of Eilenberg-Moore-style semantics works extremely smoothly, and yields expressive modal logics in essentially all cases of interest. We additionally parametrize the framework over a quantale of truth values, thus in particular covering both the two-valued notions of equivalence and quantitative ones, i.e. behavioural distances.

1 Introduction

When dealing with the logical aspects of state-based systems, one is particularly interested in the property of *expressiveness*, that is, the ability of a logic to differentiate between states that behave in different ways. The prototypical example of this property is captured by the *Hennessy-Milner theorem* [14], with modal logic distinguishing states in finitely branching transition systems precisely up to bisimilarity. There is, however, a wide array of equivalences of interest that are coarser than bisimilarity [12], each necessitating a different type of logic to stay invariant under the semantics while ensuring expressiveness. A similar story unfolds when state-based systems are studied abstractly as coalgebras for a given functor that encapsulates the transition type of systems [31]: The finest and mathematically most convenient type of equivalence is given by coalgebraic behavioural equivalence, with much of the literature on coalgebraic logic focusing on expressiveness with respect to this type of equivalence (e.g. [29,32,26,22,38,11]), though this might not necessarily be the equivalence the application demands. Consider for example nondeterministic automata, i.e. coalgebras for the **Set**-endofunctor $2 \times (\mathcal{P}-)^{\Sigma}$. The equivalence of interest in these systems is language equivalence, and as such is potentially much coarser than the coalgebraic notion of behavioural equivalence, which in this case instantiates to bisimilarity. A possible way to deal with this mismatch is to first transform the nondeterministic automaton into a deterministic one, that is, a coalgebra for the **Set**-endofunctor $F = 2 \times (-)^{\Sigma}$, via the powerset construction, obtaining language equivalence as behavioural equivalence in the determinized automaton. The powerset construction generalizes to coalgebras for functors of the form FT where F is a functor capturing the type of accepted structure (e.g. word languages for $F = 2\times(-)^{\Sigma}$ as above) and a monad T capturing the branching type of systems ($T = \mathcal{P}$ as above captures nondeterminism; other choices of T capture, e.g., probabilistic or weighted branching). To be applicable, this approach requires a so-called Eilenberg-Moore distributive law of T over F [33]; it then equips FT-coalgebras with a language-type semantics determined by F, to which we refer as *Eilenberg-Moore semantics*.

Our present aim is to obtain modal logics that are expressive and invariant for Eilenberg-Moore semantics, and at the same time can be seen as fragments of the standard expressive branching-time coalgebraic modal logics (in analogy to logics for the linear-time/branching-time spectrum of labelled transition system semantics [12], which are fragments of standard Hennessy-Milner logic). To this end, we exploit the machinery of *graded semantics* [27,8], in which notions of behavioural equivalence are modelled by mapping into a *graded monad* [34]; it has been shown that Eilenberg-Moore semantics can essentially be cast as a graded semantics [24,27]. Graded semantics comes with a general notion of invariant *graded logic* and a criterion for a graded logic to be expressive [27,8].

Contribution By instantiating the expressivity criterion of the graded semantics framework to Eilenberg-Moore semantics, we show that it is sufficient to provide a set of modal operators that separate the elements of FX, while the treatment of T is automatically provided by the framework. Separation of FX is typically easy to ensure, justifying the slogan that Eilenberg-Moore semantics essentially always admits an expressive invariant logic. We parametrize our results over the choice of a quantale that serves as a domain of both distances and truth values, allowing an instantiation to both the two-valued setting, where states are either equivalent or not and formulae take binary truth values, and to quantitative settings, where similarity of states is a continuum and formulae may take intermediate values, for instance in the real unit interval. We thus in particular cover notions of *behavioural distance* (e.g. [37,2,3]), providing logics that are expressive in the sense that the behavioural distance between states is always witnessed by differences in the evaluation of suitable formulae. We discuss a range of examples, in some cases obtaining new characteristic modal logics, e.g. for probabilistic trace equivalence of reactive probabilistic automata with black-hole termination.

Related work There has been a fair amount of work on the coalgebraic treatment of system semantics beyond branching time. Approaches using Kleisli-type distributive laws [13] and Eilenberg-Moore distributive laws (e.g. [33,19,6,17]) are subsumed by graded semantics [24,27]. The Kleisli approach has also been applied to infinite-trace semantics (e.g. [15,18,36,7]) and to trace semantics via steps [30]. Klin and Rot [20] present a notion of semantics based on selecting a modal logic, which is then expressive by definition of the semantics. For our present purposes, the most closely related piece of previous work uses corecursive algebras as a unifying concept subsuming the Kleisli-based, Eilenberg-Moore-based, and logic-based approaches to coalgebraic trace semantics [30]. In particular, the comparison between the Eilenberg-Moore-based and the logic-based semantics in this framework [30, Section 7.1] can be read as an expressiveness criterion for logics over Eilenberg-Moore semantics. In relation to this criterion, the distinguishing feature of our present main result lies in the concreteness of the construction of the logic in terms of modal and propositional operators, as well as the ease of checking our expressiveness criterion, which comes essentially for free in all cases of interest. We note also that our criterion works in quantalic generality, and thus applies also to notions of behavioural distance, which so far are not covered in the approach via corecursive algebras.

Through its applicability to behavioural distances, our work relates additionally to a spate of recent work on the coalgebraic treatment of characteristic logics for behavioural distances. For the most part, such results have been concerned mainly with branching-time distances (e.g. [22,38,21,11]).

Kupke and Rot [23] study logics for *coinductive predicates*, generalizing branching-time behavioural distances. Our overall setup differs from the one used in [23] by working with coalgebras for functors that live natively on metric spaces, including such functors that are not liftings of a set functor.

In recent work by König and (some of) the present authors [4,5], expressive logics for coalgebraic trace-type behavioural distances have been approached by setting up Galois connections between logics and distances. This concept is highly general (and in fact not even tied to models being coalgebras) but requires a comparatively high amount of effort for concrete instantiations. Moreover, its focus is on fixpoint characterizations of logical distance rather than on expressiveness w.r.t. a given notion of behavioural distance, and in fact the behaviour function inducing behavioural distance is defined directly via the logic.

2 Preliminaries

We assume basic familiarity with category theory (e.g. [1]). In the following, we recall requisite definitions and facts on universal coalgebra, quantales, and lifting functors to categories of monad algebras.

2.1 Universal Coalgebra

State-based systems of various types, such as non-deterministic, probabilistic, weighted, or game-based transition systems, are treated uniformly in the framework of *universal coalgebra* [31]. The branching type of a system is encapsulated as a functor $G\colon \mathbf{C} \to \mathbf{C}$ on a suitable base category $\mathbf{C}$, for instance on the category $\mathbf{Set}$ of sets and maps. A *G-coalgebra* (C, c) then consists of a $\mathbf{C}$-object C, thought of as an object of *states*, and a morphism $c\colon C \to GC$, thought of as a *transition map* that assigns to each state a structured collection of successor states, with structure determined by G. For instance, on $\mathbf{C} = \mathbf{Set}$, a $\mathcal{P}$-coalgebra for the covariant powerset functor is just a nondeterministic transition system, while a G-coalgebra for the functor G given by $GX = 2 \times X^{\Sigma}$, with Σ a fixed *alphabet*, is a deterministic automaton (without initial state), assigning to each state a finality status and a tuple of successors, one for every letter in Σ.

A *morphism* $h\colon (C, c) \to (D, d)$ of G-coalgebras is a $\mathbf{C}$-morphism $h\colon C \to D$ that is compatible with the transition maps in the sense that $d \cdot h = Gh \cdot c$. States $x, y \in C$ in a coalgebra (C, c) are *behaviourally equivalent* if there exist a G-coalgebra (D, d) and a morphism $h\colon (C, c) \to (D, d)$ such that $h(x) = h(y)$. For instance, two states in a labelled transition system (i.e. a coalgebra for $G = \mathcal{P}(\Sigma \times (-))$ where Σ is the set of labels) are behaviourally equivalent iff they are bisimilar in the usual sense.

The (initial ω-segment of) the *final chain* of G is the sequence $(G^n 1)_{n<\omega}$ of $\mathbf{C}$-objects. Given a G-coalgebra (C, c), we have the *canonical cone* of maps $c_n\colon C \to G^n 1$, defined by c_0 being the unique map $C \to 1$ and by $c_{n+1} = C \xrightarrow{c} GC \xrightarrow{Gc_n} G^{n+1}1$. When $\mathbf{C}$ is a concrete category over $\mathbf{Set}$, states $x, y \in C$ are termed *finite-depth behaviourally equivalent* if $c_n(x) = c_n(y)$ for all $n \in \mathbb{N}$. For finitary set functors, finite-depth behavioural equivalence and behavioural equivalence coincide [39].

2.2 Quantales

We use (symmetrized) *quantale-enriched categories* as a joint generalization of equivalence relations and pseudometric spaces; this enables us to cover both two-valued and quantitative semantics and logics uniformly in one framework. In a nutshell, a quantale is a monoid in the category of complete join semilattices. Explicitly, this notion expands as follows:

Definition 1. A (commutative unital) *quantale* $\mathcal{V} = (V, \otimes, k, \leq)$ consists of a set V that carries both the structure of a complete lattice $(V, \leq)$ and the structure of a commutative monoid $(V, \otimes, k)$ such that for all $v \in V$, the operation $- \otimes v$ is join-continuous; that is,

$$\left(\bigvee\nolimits_{i \in I} u_i\right) \otimes v = \bigvee\nolimits_{i \in I} (u_i \otimes v)$$

where we use $\bigvee$ to denote joins.

By the standard equivalence between join preservation and adjointness for functions on complete lattices, it follows that for every $b \in V$, the map $- \otimes b$ has a right adjoint $[b, -]$, with defining property

$$a \otimes b \leq c \Leftrightarrow a \leq [b, c]$$

As first observed by Lawvere [25], metric spaces can be seen as enriched categories, which leads to the notion of categories enriched in a quantale $\mathcal{V}$, or briefly $\mathcal{V}$-categories, as a generalized notion of (pseudo-)metric space:

Definition 2. A $\mathcal{V}$*-category* is a pair (X, d_X) consisting of a set X and a function $d_X : X \times X \to V$ such that for all $x, y, z \in X$ we have $d_X(x, y) \otimes d_X(y, z) \leq d_X(x, z)$, as well as $k \leq d_X(x, x)$. A $\mathcal{V}$-category (X, d_X) is *symmetric* if $d_X(x, y) = d_X(y, x)$ for all $x, y \in X$, and *separated* if $k \leq d_X(x, y)$ implies $x = y$. A function $f : X \to Y$ is a $\mathcal{V}$*-functor* between $\mathcal{V}$-categories (X, d_X) and (Y, d_Y) if $d_X(a, b) \leq d_Y(f(a), f(b))$ for all $a, b \in X$.

We fix a quantale $\mathcal{V}$ for the rest of the technical development. We write $\mathbf{DPMet}_{\mathcal{V}}$ for the category of $\mathcal{V}$-categories and $\mathcal{V}$-functors, which we view as generalized directed pseudometric spaces, with distance values in $\mathcal{V}$. Further, we write $\mathbf{PMet}_{\mathcal{V}}$ for the full subcategory of symmetric $\mathcal{V}$-categories, viewed as generalized pseudometric spaces, and $\mathbf{Met}_{\mathcal{V}}$ for the full subcategory of symmetric and separated $\mathcal{V}$-categories, viewed as generalized metric spaces. The quantale $\mathcal{V}$ itself has the structure of an object in $\mathbf{DPMet}_{\mathcal{V}}$, where $d(x, y) = [x, y]$ for all $x, y \in \mathcal{V}$. It may also be viewed as an object in $\mathbf{Met}_{\mathcal{V}}$ through symmetrization: $d_{\mathrm{sym}}(x, y) = [x, y] \wedge [y, x]$. In this way, we will often use $\mathcal{V}$ as the codomain of evaluation morphisms of our logics. We will focus on the following two examples:

Example 3. 1. The lattice $2 = \{\bot, \top\}$ carries a quantale $\mathbf{2} = (2, \wedge, \top, \leq)$. In this case, $[b, c]$ is just the Boolean implication $b \to c$. The category $\mathbf{PMet_2}$ is isomorphic to the category of setoids, i.e. of equivalence relations and equivalence-preserving maps, while the category $\mathbf{Met_2}$ is isomorphic to the category of sets and functions. We use this quantale to cover two-valued equivalences, used in situations where one is only interested in determining whether states behave in precisely the same way or not.

2. We use the quantale $[0, 1]_\oplus = ([0, 1], \oplus, 0, \geq)$, where $\oplus$ is truncated addition ($a \oplus b = \min(a + b, 1)$), to cover cases where one wishes to measure differences in the behaviour of states in a continuous manner. In this case, $[-, -]$ is truncated subtraction ($[b, c] = \max(c - b, 0)$). Indeed, taking $[-, 1]$ as negation makes $[0, 1]_\oplus$ into an MV-algebra, providing a domain of truth values for multi-valued Łukasiewicz logic. The category $\mathbf{Met}_{[0,1]_\oplus}$ is isomorphic to the usual category of 1-bounded metric spaces and non-expansive maps [25], while $\mathbf{PMet}_{[0,1]_\oplus}$ is isomorphic to the category of pseudometric spaces (that is, distinct elements may take distance 0). Note that the ordering on the set $[0, 1]$ is reversed compared to its natural ordering. This is necessary, since otherwise $\oplus$ does not distribute over the empty join.

We will use the concept of initiality (in the concrete case of $\mathcal{V}$-categories) to describe the fact that a set of morphisms is large enough to witness the distances in its domain. Later, expressivity demands that the set of evaluation morphisms of formulae form an initial source.

Definition 4. A source $\mathfrak{A}$ of $\mathcal{V}$-functors $f_i\colon (X, d_X) \to (Y_i, d_{Y_i})$ is *initial* if $d_X(x, y) = \bigwedge_{i\in I} d_{Y_i}(f_i(x), f_i(y))$ for all $x, y \in X$.

2.3 Lifting Functors to Eilenberg-Moore Categories

Recall that a *monad* (T, μ, η), denoted just T by abuse of notation, on a base category $\mathbf{C}$ consists of a functor $T\colon \mathbf{C} \to \mathbf{C}$ and natural transformations $\mu\colon TT \Rightarrow T$, as well as $\eta\colon Id \Rightarrow T$ (the *multiplication* and *unit* of T) satisfying natural laws. Monads on **Set** may be thought of as encapsulating algebraic theories, with TX being terms over X modulo provable equality, μ collapsing layered terms into terms, thus abstracting substitution, and η converting variables into terms. We call a monad T *affine* [16] when T preserves the terminal object, that is $T1 \cong 1$. For instance, the *distribution monad* $\mathcal{D}$, given by $\mathcal{D}X$ being the set

$$\{f\colon X \to [0,1] \mid f(x) = 0 \text{ for almost all } x \in X,\ \textstyle\sum_{x\in X} f(x) = 1\}$$

of finitely supported probability distributions on X, is affine. Monads induce a natural notion of algebra: A *monad algebra* or *Eilenberg-Moore algebra* (A, a) for T consists of a $\mathbf{C}$-object A and a morphism $a\colon TA \to A$ making the left and middle diagrams below commute.

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, "id_A"'] & TA \arrow[d, "a"] \\ & A \end{tikzcd} \qquad \begin{tikzcd} TTA \arrow[r, "Ta"] \arrow[d, "\mu_A"'] & TA \arrow[d, "a"] \\ TA \arrow[r, "a"'] & A \end{tikzcd} \qquad \begin{tikzcd} TA \arrow[r, "Tf"] \arrow[d, "a"'] & TB \arrow[d, "b"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

A $\mathbf{C}$-morphism $f\colon A \to B$ is a morphism between algebras $f\colon (A, a) \to (B, b)$ if the right diagram commutes. We write $\mathbf{EM}(T)$ for the category of Eilenberg-Moore algebras for T and their morphisms. We denote the functor that takes a $\mathbf{C}$-object A to the free T-algebra (TA, μ) over A by $L\colon \mathbf{C} \to \mathbf{EM}(T)$. This functor is left adjoint to the forgetful functor $R\colon \mathbf{EM}(T) \to \mathbf{C}$ that takes algebras (A, a) to their carrier A. The category $\mathbf{EM}(T)$ has all limits that $\mathbf{C}$ has [1, Proposition 20.12]. We occasionally need the n-fold power $(A, a)^n$ of an algebra (A, a), whose carrier is the $\mathbf{C}$-object A^n. We denote its algebra structure by $a^{(n)}\colon T(A^n) \to A^n$.

Coalgebraic determinization [33] is concerned with coalgebras for functors of the form $G = FT$ where T is a monad, thought of as capturing the branching type of systems, and F is a functor determining the system semantics. As indicated in the introduction, the basic example is given by nondeterministic automata over an alphabet Σ, which are coalgebras for the set functor $G = FT$ with $FX = 2 \times X^{\Sigma}$ and $T = \mathcal{P}$, while F-coalgebras are deterministic automata. The coalgebraic generalization of the powerset construction that determinizes nondeterministic automata relies on having a suitable type of distributive law between F and T:

Definition 5. An *Eilenberg-Moore distributive law*, or just *EM law*, of a monad (T, μ, η) over a functor F is a natural transformation $\zeta\colon TF \Rightarrow FT$ such that the following diagrams commute:

$$\begin{tikzcd} F \arrow[r, "\eta F"] \arrow[dr, "F\eta"'] & TF \arrow[d, "\zeta"] \\ & FT \end{tikzcd} \qquad \begin{tikzcd} TTF \arrow[r, "T\zeta"] \arrow[d, "\mu F"] & TFT \arrow[r, "\zeta T"] & FTT \arrow[d, "F\mu"] \\ TF \arrow[rr, "\zeta"] & & FT \end{tikzcd}$$

It is well-known (cf. [28]) that EM laws $\zeta\colon TF \Rightarrow FT$ are in 1-1 correspondence with liftings $\tilde{F}$ of the functor F to the Eilenberg-Moore category $\mathbf{EM}(T)$. Given ζ, the functor $\tilde{F}$ maps the T-algebra (A, a) to $(FA, Fa \cdot \zeta_A)$. As a result, every FT-coalgebra $c\colon X \to FTX$ can be determinized in the presence of an EM law [33], yielding an $\tilde{F}$-coalgebra $c^{\#}\colon TX \to FTX$ in $\mathbf{EM}(T)$ as follows:

$$TX \xrightarrow{Tc} TFTX \xrightarrow{\zeta_{TX}} FTTX \xrightarrow{F\mu_X} FTX$$

Taking a more abstract perspective, $c^{\#}$ is the adjoint transpose of c under $L \dashv R$. We say that states $c, d \in C$ are *EM-equivalent* if $\eta_C(c), \eta_C(d) \in TC$ are behaviourally equivalent in the $\tilde{F}$-coalgebra $(TC, c^{\#})$. We refer to this equivalence as *EM semantics*; when this equivalence can be captured as the kernel of a suitable map (in this case, the map assigning to each state its accepted language), we also refer to this map as the EM semantics. We will later encounter situations where the codomain of the semantics carries a generalized metric structure, in which case we will also subsume the induced generalized pseudometric on C under the moniker 'EM semantics'.

The standard powerset construction for determinizing nondeterministic automata is recovered by the following EM law:

Example 6. In **Set** (i.e. $\mathbf{Met_2}$), let $T = \mathcal{P}$ be the powerset monad and $F = 2 \times -^{\Sigma}$. The determinization $(\mathcal{P}C, c^{\#})$ of an FT-coalgebra (C, c) w.r.t. the EM law $\zeta\colon TF \Rightarrow FT$ defined by

$$\zeta(t) = \Big(\bigvee\nolimits_{(v,f)\in t} v, \quad \lambda a.\{f(a) \mid (v, f) \in t\}\Big)$$

for $t \in \mathcal{P}(2 \times X^{\Sigma})$ is precisely the powerset construction. Thus, the language semantics of nondeterministic automata is an instance of EM semantics.

Example 7. More generally, let $F = A \times -^{\Sigma}$ where Σ is discrete, let T be a monad on **Set**, and suppose that A carries a T-algebra structure $a\colon TA \to A$. Define a natural transformation $\delta\colon T(-^{\Sigma}) \Rightarrow (T-)^{\Sigma}$ by $\delta(t)(\sigma) = T(\lambda f. f(\sigma))(t)$. We then have an EM law $\zeta\colon T(A \times -^{\Sigma}) \Rightarrow A \times (T-)^{\Sigma}$ given (componentwise) by

$$\pi_1 \cdot \zeta_X = (T(A \times X^{\Sigma}) \xrightarrow{T\pi_1} TA \xrightarrow{a} A)$$
$$\pi_2 \cdot \zeta_X = (T(A \times X^{\Sigma}) \xrightarrow{T\pi_2} T(X^{\Sigma}) \xrightarrow{\delta} (TX)^{\Sigma}).$$

The arising EM semantics assigns to each state x a map $\Sigma^* \to A$, which may be thought of as assigning to each word $w \in \Sigma^*$ the degree (a value in A) to which x accepts w.

3 Graded Semantics and Graded Logics

Graded semantics [27,8] uniformly captures a wide range of semantics on various system types and of varying degrees of granularity as found, for instance, on the linear-time/branching-time spectrum of labelled transition system semantics [12]. Here, we are interested primarily in applying general results provided by the framework of graded semantics to the setting of EM semantics, which is, in essence, subsumed by graded semantics [24,27]. We recall the basic definition of graded semantics as such, and then give a new perspective on a general notion of characteristic modal logics for graded semantics, so-called graded logics.

3.1 Graded Semantics

The concepts central to graded semantics are those of graded monads and graded algebras. These are very similar to those of monads and monad algebras as recalled in Section 2.3 but, in the mentioned analogy with universal algebra, equip operations and terms with a *depth* that, in the application to system semantics, records the depth of look-ahead; that is, the depth corresponds to the (exact) number of transition steps considered. We briefly review the formal definitions.

Definition 8 (Graded Monad). A *graded monad* $\mathbb{M}$ on a category $\mathbf{C}$ consists of a family of functors $M_n\colon \mathbf{C} \to \mathbf{C}$ for $n \in \mathbb{N}$, a natural transformation $\eta\colon Id \Rightarrow M_0$ (the *unit*), and a family of natural transformations $\mu^{n,k}\colon M_nM_k \Rightarrow M_{n+k}$ for all $n, k \in \mathbb{N}$ (the *multiplication*) such that for all $n, k, m \in \mathbb{N}$ the following diagrams commute:

$$\begin{tikzcd}
& M_n \arrow[dl, "\eta M_n"'] \arrow[d, "id_{M_n}"] \arrow[dr, "M_n\eta"] & \\
M_0M_n \arrow[r, "\mu^{0,n}"] & M_n & M_nM_0 \arrow[l, "\mu^{n,0}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
M_nM_kM_m \arrow[r, "M_n\mu^{k,m}"] \arrow[d, "\mu^{n,k}M_m"] & M_nM_{k+m} \arrow[d, "\mu^{n,k+m}"] \\
M_{n+k}M_m \arrow[r, "\mu^{n+k,m}"] & M_{n+k+m}
\end{tikzcd}$$

Definition 9 (Graded semantics). A *graded semantics* $(\alpha, \mathbb{M})$ for an endofunctor $G\colon \mathbf{C} \to \mathbf{C}$ consists of a graded monad $\mathbb{M}$ on $\mathbf{C}$ and a natural transformation $\alpha\colon G \Rightarrow M_1$. If (C, c) is a G-coalgebra, then we define the n-step behaviour $c^{(n)}\colon C \to M_n1$, for $n \in \mathbb{N}$, by

$$c^{(0)} = (X \xrightarrow{M_0!\cdot\eta} M_01) \qquad c^{(n+1)}(X \xrightarrow{\alpha\cdot c} M_1X \xrightarrow{M_1c^{(n)}} M_1M_n1 \xrightarrow{\mu^{1n}} M_{n+1}1).$$

We think of $c^{(n)}$ as assigning to a state in C its behaviour after n steps. We illustrate this more concretely in Example 12. We are mainly interested in the case where the base category $\mathbf{C}$ is a category of generalized (directed) pseudo-metric spaces (Section 2.2). In this case, a graded semantics induces a notion of behavioural distance:

Definition 10 (Behavioural distance). When $\mathbf{C}$ is $\mathbf{DPMet}_\mathcal{V}$ (or $\mathbf{PMet}_\mathcal{V}$, $\mathbf{Met}_\mathcal{V}$), then we define the *behavioural distance* of two states $x, y \in C$ of a G-coalgebra (C, c) under a graded semantics $(\alpha, \mathbb{M})$ to be

$$d^b(x, y) = \bigwedge_{n\in\mathbb{N}} d_{M_n 1}(c^{(n)}(x), c^{(n)}(y)).$$

Remark 11. In case $\mathcal{V} = \mathbf{2}$ (Example 3.1), behavioural distance is two-valued, and thus in fact constitutes either a preorder (if $\mathbf{C}$ is $\mathbf{DPMet}_\mathcal{V}$) or an equivalence (if $\mathbf{C}$ is $\mathbf{PMet}_\mathcal{V}$).

Example 12. We recall two basic examples of graded monads [27] and associated graded semantics, capturing branching-time semantics and EM semantics, respectively. In both cases, it happens that α is identity; this need not always be the case, however [8].

1. Any functor F induces a graded monad $\mathbb{M}_F$ where the functor parts $M_n = F^n$ are given via repeated application of F and both multiplication and unit are identity. The arising graded semantics of F-coalgebras is branching-time semantics, specifically finite-depth behavioural equivalence (which coincides with behavioural equivalence if F is finitary).

2. Any EM law $\zeta\colon TF \Rightarrow FT$ induces a graded monad $\mathbb{M}_\zeta$ where $M_n = F^nT$. The unit of $\mathbb{M}_\zeta$ is the unit of T. We define an iterated distributive law $\zeta^{(n)}\colon TF^n \Rightarrow F^nT$ by putting

$$\zeta^{(0)} = id \quad \text{and} \quad \zeta^{(n+1)} = TF^{n+1} \xrightarrow{\zeta F^n} FTF^n \xrightarrow{F\zeta^{(n)}} F^{(n+1)}T.$$

The multiplications of the graded monad $\mathbb{M}_\zeta$ are then given by $\mu^{m,n} = F^{n+m}\mu \cdot F^m\zeta^{(n)}T$. The arising graded semantics is essentially EM semantics, in the sense that the latter is obtained by erasing further information by postcomposing the maps $c^{(n)}\colon C \to F^nT1$ (in the notation of Definition 9) with $F^n!$ where $!$ is the unique map $T1 \to 1$. In particular, the EM semantics and the graded semantics introduced by an EM law for T agree exactly if T is affine (Section 2.3). Otherwise, the information erased by $F^n!$ essentially concerns the possibility of executing certain words, without regard to their acceptance [24, Section 5]. For a concrete example where T is affine, consider $T = \mathcal{D}$ (the distribution monad, cf. Section 2.3) and $FX = [0, 1] \times X^\Sigma$, with an EM law ζ as per Example 7. Then FT-coalgebras are reactive probabilistic automata, and for a state x in an FT-coalgebra, $c^{(n)}(x) \in F^n\mathcal{D}1 \cong F^n1 \cong [0, 1]^{\Sigma^{<n}}$ assigns to each word of length $< n$ over Σ its probability of being accepted.

Note that 1. is the special case of 2. where $T = Id$.

We will in fact be interested exclusively in graded monads that are, in the universal-algebraic view [27,8], presented by operations and equations of depth at most 1, which intuitively means that identifications among behaviours do not depend on looking more than one step ahead. Categorically, this notion is captured as follows [27, Proposition 7.3]:

Definition 13. We say that a graded monad is *depth*-1 if for all $n \in \mathbb{N}$, $\mu^{1,n}$ is a coequalizer in the following diagram:

$$M_1 M_0 M_n X \underset{\mu^{1,0} M_n}{\overset{M_1 \mu^{0,n}}{\rightrightarrows}} M_1 M_n X \xrightarrow{\mu^{1,n}} M_{1+n} X.$$

Example 14. All graded monads described in Example 12 are depth-1.

The semantics of modalities in graded logics will rely on a graded variant of the notion of monad algebra:

Definition 15 (Graded algebra). Let $\mathbb{M}$ be a graded monad in $\mathbf{C}$, and $n \in \mathbb{N} \cup \{\omega\}$. A *graded M_n-algebra* $A = ((A_k)_{k \leq n}, (a^{m,k})_{m+k \leq n})$ consists of a family of $\mathbf{C}$-objects A_k and morphisms $a^{m,k} \colon M_m A_k \to A_{m+k}$ satisfying the following conditions: For $m \leq n$, we have $a^{0,m} \cdot \eta_{A_m} = id_{A_m}$ and additionally, if $m + r + k \leq n$, then the left diagram below commutes:

$$\begin{array}{ccc} M_m M_r A_k & \xrightarrow{M_m a^{r,k}} & M_m A_{r+k} \\ \downarrow{\scriptstyle \mu^{m,r}_{A_k}} & & \downarrow{\scriptstyle a^{m,r+k}} \\ M_{m+r} A_k & \xrightarrow{a^{m+r,k}} & A_{m+r+k} \end{array} \qquad \begin{array}{ccc} M_m A_k & \xrightarrow{M_m f_k} & M_m B_k \\ \downarrow{\scriptstyle a^{m,k}} & & \downarrow{\scriptstyle b^{m,k}} \\ A_{m+k} & \xrightarrow{f_{m+k}} & B_{m+k} \end{array}$$

A *homomorphism* of M_n-algebras A and B is a family of maps $f_k \colon A_k \to B_k$ such that the above right diagram commutes for all $m + k \leq n$. For all $n \in \mathbb{N} \cup \{\omega\}$, the collection of M_n-algebras and their morphisms forms a category $\mathrm{Alg}_n(\mathbb{M})$.

The category $\mathrm{Alg}_0(\mathbb{M})$ is the Eilenberg-Moore category $\mathbf{EM}(M_0)$ for the (non-graded) monad $(M_0, \eta, \mu^{0,0})$. The semantics of modalities in graded logics will involve a special type of M_1-algebras [8]:

Definition 16 (Canonical algebras). For $i \in \{0, 1\}$, let $(-)_i : \mathrm{Alg}_1(\mathbb{M}) \to \mathrm{Alg}_0(\mathbb{M})$ be the functor taking an M_1-algebra $A = ((A_k)_{k \leq 1}, (a^{m,k})_{m+k \leq 1})$ to the M_0-algebra $(A_i, a^{0,i})$. We say that an M_1-algebra A is *canonical* if it is free over $(-)_0$, i.e. if for all M_1-algebras B and M_0-homomorphisms $f : (A)_0 \to (B)_0$ there is a unique M_1-homomorphism $g : A \to B$ such that $(g)_0 = f$.

Lemma 17. *([8, Lemma 5.3]) An M_1-algebra A is canonical iff the following diagram is a coequalizer diagram in the category of M_0-algebras:*

$$M_1 M_0 A_0 \underset{\mu^{1,0}}{\overset{M_1 a^{0,0}}{\rightrightarrows}} M_1 A_0 \xrightarrow{a^{1,0}} A_1$$

Combining Definition 13 with Lemma 17 immediately gives us the following fact [8], which is a crucial ingredient for invariance of graded logics:

Proposition 18. *If $\mathbb{M}$ is a depth-1 graded monad, then for every $n \in \mathbb{N}$ and every object X, the M_1-algebra with carriers $M_n X$, $M_{n+1} X$ and multiplications as algebra structure is canonical.*

3.2 Graded Logics as a Fragment of Branching-Time Logic

We proceed to recall the general framework of (branching time) coalgebraic modal logic [29,32] and show that graded logics [27,8] are naturally viewed as a fragment of coalgebraic modal logic.

Syntactically, a logic is a triple $\mathcal{L} = (\Theta, \mathcal{O}, \Lambda)$ where Θ is a set of truth constants, $\mathcal{O}$ is a set of propositional operators, each with associated finite arity, and Λ is a set of modal operators, also each with an associated finite arity. The set of formulae of $\mathcal{L}$ is given by the grammar

$$\phi ::= \theta \mid p(\phi_1, \dots, \phi_n) \mid \lambda(\phi_1, \dots, \phi_m)$$

where $p \in \mathcal{O}$ is n-ary, $\lambda \in \Lambda$ is m-ary and $\theta \in \Theta$.

Semantically, formulae are interpreted in coalgebras of some functor $G\colon \mathbf{C} \to \mathbf{C}$, taking values in a truth-value object Ω of $\mathbf{C}$. We assume that $\mathbf{C}$ has finite products and a terminal object. The semantics of a formula ϕ in a G-coalgebra (C, c) is a morphism $[\![\phi]\!]_c\colon C \to \Omega$. The semantics is parametric in the following components:

- For each $\theta \in \Theta$ a $\mathbf{C}$-morphism $\hat{\theta}\colon 1 \to \Omega$.
- For each $p \in \mathcal{O}$ with arity n a $\mathbf{C}$-morphism $[\![p]\!]\colon \Omega^n \to \Omega$
- For each $\lambda \in \Lambda$ a $\mathbf{C}$-morphism $[\![\lambda]\!]\colon G(\Omega^n) \to \Omega$

The semantics of formulae is then defined inductively:

- For $\theta \in \Theta$ we define $[\![\theta]\!]_c = C \xrightarrow{!} 1 \xrightarrow{\hat{\theta}} \Omega$
- For $p \in \mathcal{O}$ we define $[\![p(\phi_1, \dots, \phi_n)]\!]_c = [\![p]\!] \cdot \langle [\![\phi_1]\!]_c, \dots, [\![\phi_n]\!]_c \rangle$
- For $\lambda \in \Lambda$ we define $[\![\lambda(\phi_1, \dots, \phi_m)]\!]_c = [\![\lambda]\!] \cdot G\langle [\![\phi_1]\!]_c, \dots, [\![\phi_m]\!]_c \rangle \cdot c$

The following definition of logical distance quantifies over all formulae ϕ of *uniform depth*, meaning that all occurrences of truth constants in ϕ are under the same number of nested modal operators. This is a mild restriction; in fact, for the above version of coalgebraic logic, truth constants can always be modelled as 0-ary propositional operators, for which there is no uniformity restriction. Uniform depth does come to play a role once we talk about graded logics, where propositional operators are additionally required to be gM_0-algebra homomorphisms, while truth constants are not. If M_0 is affine, then all $\mathbf{C}$-morphisms $1 \to A$ into M_0-algebras A are M_0-algebra homomorphisms.

For the rest of the paper, assume that $\mathbf{C}$ is one of $\mathbf{Met}_{\mathcal{V}}$, $\mathbf{PMet}_{\mathcal{V}}$ or $\mathbf{DPMet}_{\mathcal{V}}$; in particular, the truth value object Ω carries the structure of a $\mathcal{V}$-category.

Definition 19. The *logical distance* of states $x, y \in C$ in a G-coalgebra (C, c) under the logic $\mathcal{L}$ is

$$d^{\mathcal{L}}(x,y) = \bigwedge\nolimits_{n\in\mathbb{N},\phi\in\mathcal{L}_n} d_\Omega(\llbracket\phi\rrbracket_c(x), \llbracket\phi\rrbracket_c(y))$$

where $\mathcal{L}_n$ is the set of all uniform depth-n $\mathcal{L}$ formulae. We say that $\mathcal{L}$ is *invariant* for $(\alpha, \mathbb{M})$ if $d^b \leq d^{\mathcal{L}}$ and *expressive* if $d^b \geq d^{\mathcal{L}}$.

It is straightforward to show that the logic defined above is invariant under behavioural equivalence, i.e. the graded equivalence induced by $\mathbb{M}_G$ (Example 12.1). We want to identify logics that are invariant not only under behavioural equivalence, but under an arbitrary graded semantics. To this end, we define graded logics:

Definition 20. Let $(\alpha, \mathbb{M})$ be a graded semantics for G and $o\colon M_0\Omega \to \Omega$ an M_0-algebra structure on Ω. A logic $\mathcal{L}$ is a *graded logic* (for $(\alpha, \mathbb{M})$) if the following hold:

1. For every n-ary $p \in \mathcal{O}$, the morphism $\llbracket p\rrbracket$ is an M_0-algebra homomorphism $(\Omega, o)^n \to (\Omega, o)$.
2. The semantics of $\lambda \in \Lambda$ factors as $\llbracket\lambda\rrbracket = f \cdot \alpha_{\Omega^n}$ such that the tuple $(\Omega^n, \Omega, o^{(n)}, o, f)$ constitutes an M_1-algebra. More concretely, this means that it satisfies $f \cdot \mu^{1,0} = f \cdot M_1 o^{(n)}$ (we refer to this property as *coequalization*), as well as $f \cdot \mu^{0,1} = o \cdot M_0 f$ (*homomorphy*), or written diagrammatically:

$$M_1 M_0 \Omega^n \underset{M_1 o^{(n)}}{\overset{\mu^{1,0}}{\rightrightarrows}} M_1\Omega^n \xrightarrow{f} \Omega \qquad \begin{array}{ccc} M_0 M_1 \Omega^n & \xrightarrow{M_0 f} & M_0\Omega \\ {\scriptstyle \mu^{0,1}}\downarrow & & \downarrow{\scriptstyle o} \\ M_1\Omega^n & \xrightarrow[f]{} & \Omega \end{array}$$

In many examples (including those discussed in this work), the factorization in Condition 2 is simplified by the fact that $\alpha = id$, and just requires that $(\Omega^n, \Omega, o^{(n)}, o, \llbracket\lambda\rrbracket)$ is an M_1-algebra. For readability, we restrict the technical development to unary modalities from now on; treating modalities of arbitrary arity is simply a matter of adding indices. In examples, modalities will have arity either 1 or 0.

Proposition 21. *Let $\mathcal{L}$ be a graded logic for the semantics $(\alpha, \mathbb{M})$ on $G\colon \mathbf{C} \to \mathbf{C}$ and (C, c) a G-coalgebra. For two states $x, y \in C$ we have that $d^b(x, y) \leq d^{\mathcal{L}}(x, y)$.*

Proof (Sketch). The proof is based on showing, by induction on ϕ, the stronger property that the evaluation functions $\llbracket\phi\rrbracket_c$ of depth-n formulae ϕ factor through M_0-homomorphisms

$$\llbracket\phi\rrbracket_{\mathbb{M}}\colon M_n 1 \to \Omega, \tag{1}$$

as used in earlier formulations of the semantics [8,9], with canonicity of $M_n 1$ (Lemma 17) being the key property in the step for modalities. □

The proof uses uniformity to enable the factorization of formula evaluation via a single $M_n 1$, which in general is possible only for uniform-depth formulae. In general, non-uniform depth formulae of graded logics fail to be invariant. We provide an example for this fact in the appendix. Recall that when M_0 is affine, then uniform depth is not an actual restriction. Having established invariance, we next generalize the expressivity criterion for graded logics [8] to our present quantitative setting:

Definition 22. A graded logic $\mathcal{L}$ consisting of Θ, $\mathcal{O}$, Λ is *depth-0 separating* if the family of maps $\{o \cdot M_0\hat{\theta}\colon M_0 1 \to \Omega \mid c \in \Theta\}$ is initial. Moreover, $\mathcal{L}$ is *depth-1 separating* if for all canonical M_1-algebras A and initial sources $\mathfrak{A}$ of M_0-homomorphisms $(A_0, a^{0,0}) \to (\Omega, o)$, closed under the propositional operators in $\mathcal{O}$, the set

$$\Lambda(\mathfrak{A}) := \{[\![\lambda]\!](f) : A_1 \to \Omega \mid \lambda \in \Lambda, f \in \mathfrak{A}\}$$

is initial, where $[\![\lambda]\!](f)$ is the by canonicity unique morphism such that $[\![\lambda]\!](f) \cdot a^{1,0} = [\![\lambda]\!] \cdot M_1 f$.

Essentially, the above conditions encapsulate what is needed to push initiality through an induction on the depth of formulae. We thus obtain

Theorem 23. *Suppose that a graded logic $\mathcal{L}$ is both depth-0 separating and depth-1 separating. Then $\mathcal{L}$ is expressive.*

4 Graded Semantics via Coalgebraic Determinization

From now on, fix a $\mathbf{C}$-endofunctor F, a monad T on $\mathbf{C}$, and an EM law $\zeta\colon TF \Rightarrow FT$. The objective of this section is to show that behavioural equivalences, respectively metrics, on a determinized coalgebra agree with the equivalences/metrics induced by the graded semantics (Lemma 27), and that graded logics for FT may be reduced to coalgebraic logics for F. We recall the notion of predeterminization under a graded semantics [10] and show that this is the same concept as determinization under an EM law, under the condition that the monad T is affine.

Let $\mathbb{M}$ be a graded monad. We have a functor $E\colon \mathrm{Alg}_0(\mathbb{M}) \to \mathrm{Alg}_1(\mathbb{M})$ that takes an M_0-algebra A to the free M_1-algebra over A with respect to $(-)_0$ (which is then canonical, cf. Definition 16). This gives rise to a functor

$$\overline{M}_1 = (\mathrm{Alg}_0(\mathbb{M}) \xrightarrow{E} \mathrm{Alg}_1(\mathbb{M}) \xrightarrow{(-)_1} \mathrm{Alg}_0(\mathbb{M})),$$

which intuitively takes an M_0-algebra of behaviours to the M_0-algebra of behaviours having absorbed one more step. Since $(M_0X, M_1X, \mu_X^{0,0}, \mu_X^{0,1}, \mu_X^{1,0})$ is canonical (Proposition 18), we have $\overline{M}_1(M_0X, \mu^{0,0}) = (M_1X, \mu^{0,1})$, or stated slightly differently, if we denote the free-forgetful adjunction on $\mathrm{Alg}_0(\mathbb{M})$ by $L \dashv R$, then $M_1 = R\overline{M}_1L$. For a graded semantics $(\alpha\colon G \to M_1, \mathbb{M})$ and a coalgebra $c\colon C \to GC$, we have $C \xrightarrow{\alpha \cdot c} M_1C = R\overline{M}_1LC$. The adjunction then yields a unique morphism $c^\dagger\colon LC \to \overline{M}_1LC$, defining a form of determinization under the graded semantics, similar to the generalized powerset construction. Specifically, if $M_0 1 = 1$, then for $x, y \in C$, $\eta(x)$ and $\eta(y)$ are behaviourally equivalent in $c^\dagger$ iff x and y are identified under the graded semantics $(\alpha, \mathbb{M})$. We show next that

Lemma 24. *If* $\mathbb{M} = \mathbb{M}_\zeta$ *then* $\overline{M}_1 = \tilde{F}$.

Proof. Let (A, a) be a T-algebra. Then $\tilde{F}(A, a) = (Fa, Fa \cdot \zeta_A)$. On the other hand, by Lemma 17, the 1-part of the canonical algebra of $\overline{M}_1(A, a)$ is given by the following (split) coequalizer:

$$\begin{tikzcd} FTTA \arrow[r, shift left, "F\mu_A"] \arrow[r, shift right, "FTa"'] & FTA \arrow[r, "Fa"] \arrow[l, bend left=60, "F\eta_{TA}"] & FA \arrow[l, bend left=60, "F\eta_A"] \end{tikzcd}$$

Commutativity of all relevant paths is obvious from the algebra and monad axioms, implying that the diagram is a coequalizer diagram by virtue of being a split coequalizer. Then $(A, FA, a, Fa \cdot \zeta_A, Fa)$ defines a canonical M_1-algebra where coequalization, as well as canonicity (due to Lemma 17), are by the above coequalizer, and homomorphy instantiates to the outer paths of the following diagram:

$$\begin{tikzcd} TFTA \arrow[r, "\zeta_{TA}"] \arrow[d, "TFa"] & FTTA \arrow[r, "F\mu"] \arrow[d, "FTa"] & FTA \arrow[d, "Fa"] \\ TFA \arrow[r, "\zeta_A"] & FTA \arrow[r, "Fa"] & FA \end{tikzcd}$$

Commutativity of the outer rectangle follows from the fact that the left square commutes by naturality of ζ and the right square commutes by virtue of (A, a) being a T-algebra. Taking the 1-part of this canonical algebra then leaves us with $(Fa, Fa \cdot \zeta_A)$. On morphisms $h \colon (A, a) \to (B, b)$, the lifting $\tilde{F}$ acts by sending h to Fh. Commutativity of the relevant diagram making Fh a T-algebra morphism between FA and FB is easily checked, as is the fact that (h, Fh) constitutes a morphism between the canonical M_1-algebras. □

Lemma 25. *Let* (C, c) *be an* FT*-coalgebra and* $c^\dagger$ *the predeterminization under the graded semantics* M_ζ. *Then* $c^\# = c^\dagger$.

Proof. This follows from the fact that $c^\#$ can equivalently be defined as the adjoint transpose of c under the free-forgetful adjunction of $\mathbf{EM}(T)$ [33]. Then $c^\#$ and $c^\dagger$ agree by definition. □

Definition 26. Let $T \colon \mathbf{C} \to \mathbf{C}$ be a monad and $H \colon \mathbf{EM}(T) \to \mathbf{EM}(T)$ a functor on the corresponding Eilenberg-Moore category. Further, let $c \colon ((A, d_A), a) \to H((A, d_A), a)$ be an H-coalgebra. The *finite-depth behavioural distance* of two states $x, y \in A$ is given by $d^H(x, y) = \bigwedge_{i \in \mathbb{N}} d_{H^i 1}(f_i(x), f_i(y))$, where the $f_i \colon A \to H^i 1$ are the projections into the final H-chain.

Lemma 27. *Let* $(\alpha \colon G \to M_1, \mathbb{M})$ *be a graded semantics on* $\mathbf{C}$ *with* M_0 *affine, and let* (C, c) *be a* G*-coalgebra. Then* $d^{\overline{M}_1}(\eta(x), \eta(y)) = d^b(x, y)$ *for all* $x, y \in C$.

Proof. One shows by induction on n that $f_n \cdot \eta = c^{(n)}$ for all n, where affinity is needed for the base case $n = 0$. □

Remark 28. In the case where the graded monad is $\mathbb{M}_\zeta$, if T is affine, then the final chain of $\overline{M}_1$ lives over the final chain of F. In particular, if F is finitary and $\mathcal{V} = 2$, then finite-depth behavioural equivalence agrees with behavioural equivalence, for both F-coalgebras and $\overline{M}_1$-coalgebras.

Remark 29. As noted in Example 12.2, finite-depth behavioural distance in $\mathbf{EM}(T)$ may be coarser than the graded semantics but may then be canonically recovered from the graded semantics.

From now on, we notationally conflate modalities $\lambda \in \Lambda$ and their interpretations $[\![\lambda]\!]\colon FT\Omega \to \Omega$. The following result completely characterizes the modal operators of graded logics for the semantics $(id, \mathbb{M}_\zeta)$:

Theorem 30. *Let $\lambda\colon FT\Omega \to \Omega$ be a modal operator for a graded logic with truth value object (Ω, o). Then $\lambda = ev_\lambda \cdot Fo$ for some algebra homomorphism $ev_\lambda\colon \tilde{F}(\Omega, o) \to (\Omega, o)$. On the other hand, every algebra homomorphism $\tilde{F}(\Omega, o) \to (\Omega, o)$ yields a modal operator in this way.*

As our second main result, we next show that a logic is depth-1 separating for the semantics of $\mathbb{M}_\zeta$ if the F-algebra part of its modal operators is expressive for F. This criterion is typically very easy to establish and can be shown for general classes of functors, which is what we mean by our slogan that expressive graded logics for EM semantics come essentially for free.

Theorem 31. *Let $\mathcal{L} = (\Theta, \mathcal{O}, \Lambda)$ be a graded logic for $\mathbb{M}_\zeta$ and $\mathcal{L}' = (\Theta, \mathcal{O}, \Lambda')$ the (graded) logic for $\mathbb{M}_F$ with $\Lambda' = \{f\colon F\Omega \to \Omega \mid f \cdot Fo \in \Lambda\}$. Then $\mathcal{L}$ is depth-1 separating for $\mathbb{M}_\zeta$ if $\mathcal{L}'$ is depth-1 separating for $\mathbb{M}_F$.*

Proof. Let A be a canonical M_1-algebra. Since $\overline{M}_1 = \tilde{F}$, we know that A has the form $(A_0, FA_0, a^{0,0}, Fa^{0,0} \cdot \zeta, Fa^{0,0})$. For a homomorphism $h\colon (A_0, a^{0,0}) \to (\Omega, o)$ of T-algebras and $\lambda \in \Lambda$ where $\lambda = f \cdot Fo$, $\lambda(h)$ is, by definition, the unique morphism that makes the outer rectangle in the following diagram commute:

$$\begin{tikzcd}
FTA_0 \arrow[r, "FTh"] \arrow[dd, "Fa^{0,0}"'] & FT\Omega \arrow[d, "Fo"] \\
 & F\Omega \arrow[d, "f"] \\
FA_0 \arrow[ru, "Fh"] \arrow[r, dashed, "\lambda(h)"'] & \Omega
\end{tikzcd}$$

The top square commutes since it is just F applied to the homomorphism square of h. Since $a^{0,0}$ is a split epimorphism (by virtue of being an algebra for a monad), $Fa^{0,0}$ is also a split epimorphism. Therefore, $\lambda(h) = f \cdot Fh$. Let $\mathfrak{A} \subseteq \mathbf{EM}(T)((A_0, a^{0,0}), (\Omega, o))$ be a separating set of algebra homomorphisms; we have to show that $\Lambda(\mathfrak{A})$ is separating. But since $\lambda(h) = f \cdot Fh$ for all $h \in \mathfrak{A}$ and $\lambda = f \cdot Fo \in \Lambda$, we have $\Lambda'(\mathfrak{A}) = \{f \cdot Fh \mid f \in \Lambda', h \in \mathfrak{A}\} = \Lambda(\mathfrak{A})$, and $\Lambda'(\mathfrak{A})$ is spearating by depth-1 separation for $\mathcal{L}'$. □

5 Examples

In our central examples, F takes the form $\mathcal{V} \times (-)^{\Sigma}$ while T varies. In these cases, we always have a set of separating modalities: We have the set $\Lambda' = \{\mathrm{ev}_\sigma \mid \sigma \in \Sigma\} \cup \{\mathrm{ev}_\top\}$ of modalities for F, where $\mathrm{ev}_\sigma \colon \mathcal{V} \times \mathcal{V}^{\Sigma} \to \mathcal{V}$ is a unary operator defined by $(v, f) \mapsto f(\sigma)$, and $\mathrm{ev}_\top \colon \mathcal{V} \times 1^{\Sigma} \to \mathcal{V}$ is the 0-ary operator defined by $(v, f) \mapsto v$. For a monad T and an algebra structure $o \colon T\mathcal{V} \to \mathcal{V}$, the semantics of each $\mathrm{ev}_\lambda \in \Lambda'$ extends to a modal operator $\langle\lambda\rangle$ for FT, given by $\langle\lambda\rangle = \mathrm{ev}_\lambda \cdot Fo$. We thus have coalgebraic logics $\mathcal{L}' = (\emptyset, \emptyset, \Lambda')$ for F and $\mathcal{L} = (\emptyset, \emptyset, \Lambda)$ for FT.

Lemma 32. *Let $F = \mathcal{V} \times -^{\Sigma}$. Let T be a monad and $\tilde{F} \colon \mathbf{EM}(T) \to \mathbf{EM}(T)$ a lifting of F. Moreover, suppose that $\mathcal{V}$ carries a T-algebra structure $o \colon T\mathcal{V} \to \mathcal{V}$. Then, for every $ev_\lambda \in \Lambda'$, the semantics ev_λ is a homomorphism of algebras $\tilde{F}(\mathcal{V}, o) \to (\mathcal{V}, o)$*

Proof. Since the ev_λ are just product projections, this follows from the fact that the forgetful functor $U \colon \mathbf{EM}(T) \to \mathbf{C}$ creates limits [1, Proposition 20.12]. $\square$

Corollary 33. *Let ζ be defined as in Example 7. The logic $\mathcal{L}$ as defined above is a graded logic for the graded semantics $(id, \mathbb{M}_\zeta)$.*

Lemma 34. *The logic $\mathcal{L}'$ as defined above is depth-1 separating for the graded semantics $(id, \mathbb{M}_F)$.*

Proof. By Proposition 18, canonical M_1-algebras have the form $A = (A_0, FA_0, id, id, id)$. Let A be a canonical M_1-algebra and $\mathfrak{A}$ an initial source $A_0 \to \mathcal{V}$; we then need to show that the lower edges in the following diagram collectively form an initial source, where f ranges over $\mathfrak{A}$:

$$\begin{tikzcd} FA_0 \arrow[r, "Ff"] \arrow[d, "id"'] & F\mathcal{V} \arrow[d, "\mathrm{ev}_\lambda"] \\ FA_0 \arrow[r, dashed, "\mathrm{ev}_\lambda(f)"] & \mathcal{V} \end{tikzcd}$$

Since the modal operators ev_λ are precisely the projections of the product $F\mathcal{V}$, they constitute an initial source; moreover, again since F is a product, it preserves initial sources, so the source of all Ff is initial. Thus, $\Lambda(\mathfrak{A})$ is a composite of initial sources, hence itself initial. $\square$

Words in Σ^* can be viewed as formulae of $\mathcal{L}$ in the obvious way. The evaluation $[\![\phi]\!]_c$ then captures the notion of acceptance in the automaton given by the FT-coalgebra (C, c). Logics in general however allow to express far more interesting statements, since on the one hand formulae may specify words only up to a suffix, and on the other hand the logic may include propositional operators. We consider a few concrete examples:

Example 35 (Deadlock-free nondeterministic automata). We take $\mathcal{V} = \mathbf{2}$, and work in the category $\mathbf{Met_2}$ of sets and functions. Concretely, this means that all objects carry the discrete equivalence relation, and initiality of a source is joint injectivity. If T is the nonempty powerset monad $\mathcal{P}^+$, then coalgebras $c\colon C \to 2 \times (\mathcal{P}^+C)^\Sigma$ are deadlock-free nondeterministic automata. With the algebra structure $o\colon \mathcal{P}^+2 \to 2$ defined by $o(X) = \top$ if $\top \in X$ and $o(X) = \bot$ otherwise, we can construct a distributive law ζ as in Example 7. Since, by Lemma 34, $\mathcal{L}$ is depth-1 separating for $2 \times -^\Sigma$, we have that by Theorem 31 the logic $\mathcal{L}$ is expressive for $(id, \mathbb{M}_\zeta)$. We can add disjunction as a propositional operator, preserving invariance of the logic, since disjunction preserves joins (i.e. is a homomorphism of $\mathcal{P}^+$-algebras).

Example 36 (Reactive probabilistic automata). For $\mathcal{V} = [0,1]_\oplus$, we consider reactive probabilistic automata. Let T be the (finitely supported) probability distribution monad $\mathcal{D}$ on $\mathbf{PMet}_{[0,1]_\oplus}$, which equips the set of distributions with the Kantorovich metric (e.g. [2]). We put $\Omega = [0,1]$, equipped with the symmetrized metric $d(x,y) = |x-y|$. We have an algebra $o\colon \mathcal{D}[0,1] \to [0,1]$ taking expected values: $o(\mu) = \sum_{v\in[0,1]} v\mu(v)$. The construction in Example 7 then yields a semantics where, intuitively, the first component of F determines the probability of a state to accept. Upon reading a letter a, the automaton moves to a random successor state according to the probability distribution on states associated with a. The evaluation $[\![\phi]\!]_c(x)$ is then the expected probability of the state $x \in C$ of an automaton (C,c) accepting the word corresponding to ϕ. The distance of two states $x, y \in C$ is the supremum in difference of acceptance across all words in Σ^*. Again we have expressivity of $\mathcal{L}$ by combining Lemma 34 and Theorem 31. The logic remains invariant w.r.t. the semantics when extended with propositional operators that are homomorphisms $[0,1]^n \to [0,1]$ of $\mathcal{D}$-algebras, which in this case means they are affine maps, such as convex combinations or fuzzy negation $x \mapsto 1-x$.

Example 37 (Reactive probabilistic automata with black hole termination). Going beyond the leading example $F = \mathcal{V} \times (-)^\Sigma$, we add explicit failure in the vein of [35] to reactive probabilistic automata: We now take $\mathcal{V} = 2$, and again view $\mathbf{Met_2}$ as the category of sets and functions (Example 35). Let $\Omega = [0,1]$, equipped with the $\mathcal{D}$-algebra structure $o\colon \mathcal{D}[0,1] \to [0,1]$ that takes expected values. We obtain a distributive law $\mathcal{D}(2 \times -+1)^\Sigma \Rightarrow 2 \times ((\mathcal{D}-)+1)^\Sigma$ by composing the distributive law from Example 7 with the law $\lambda\colon \mathcal{D}(-+1) \Rightarrow (\mathcal{D}-)+1$ that maps $\mu \in \mathcal{D}(X+1)$ to $*$ iff $\mu(*) \neq 0$, and to μ otherwise, where $*$ denotes the unique element of 1. The semantics for this type of automaton is like that of probabilistic automata, with the exception that if a run leads to the "state" $*$ with non-zero probability, then the automaton immediately gets stuck and rejects the word. For the logic, we consider the same operators as in the previous examples, with the modification that $\mathrm{ev}_\sigma(v,f) = \bot$ if $f(\sigma) = *$. Additionally we introduce the modal operator $\mathrm{ev}_{\bar\sigma}$, which carries the semantics $\mathrm{ev}_{\bar\sigma}(v,f) = \top$ if $f(\sigma) = *$ and $\mathrm{ev}_{\bar\sigma}(v,f) = \bot$ otherwise. It is straightforward to check that these operations define $\mathcal{D}$-algebra homomorphisms, making them valid modalities according to Theorem 30.

To verify expressivity, it is sufficient by Theorem 31 to prove separation of elements of $2 \times (X+1)^{\Sigma}$, so let $\mathfrak{A}$ be an initial set of morphisms of type $X \to 2$. Given $s, t \in 2 \times (X+1)^{\Sigma}$ such that $d(s,t) = \bot$, we need to find ev_λ and $h \in \mathfrak{A}$ (or just ev_λ if ev_λ is 0-ary) such that $\mathrm{ev}_\lambda(h)(s) \neq \mathrm{ev}_\lambda(h)(t)$. If $s = (v, f)$ and $t = (w, g)$ differ in their first component $v \neq w$, we can choose $\mathrm{ev}_\top$. If the elements differ in one of the other components σ, we distinguish cases: If $x = f(\sigma) \neq * \neq g(\sigma) = y$, then there is $h \in \mathfrak{A}$ separating x from y, thus $\mathrm{ev}_\sigma(h)(x) \neq \mathrm{ev}_\sigma(h)(y)$. Otherwise, if $f(\sigma) = * \neq g(\sigma)$, we can choose $\mathrm{ev}_{\bar{\sigma}}$ to separate s and t, and similarly for the symmetric case. We thus obtain expressiveness in the two-valued sense, i.e. the logic distinguishes non-equivalent states. Like in the previous example, the logic remains invariant when extended with propositional operators that are affine maps $[0,1]^n \to [0,1]$.

6 Conclusion

We have discussed characteristic logics for system semantics arising via determinization in the coalgebraic powerset construction, so-called Eilenberg-Moore semantics, which relies on a distributive law of a functor representing the language type of a system over a monad representing the branching type [33]. Leading examples are languages semantics for various forms of automata. As our main technical tool, we have exploited that Eilenberg-Moore semantics may be cast as an instance of graded semantics, which provides generic mechanisms for designing invariant modal logics and establishing their expressiveness. Our first main result establishes an overview of all graded modalities available for Eilenberg-Moore semantics, showing that these are canoincally obtained from modalities for the language type and a single modality for the branching type. Our second main result shows that expressivity of such a logic follows from branching-time expressivity of the same collection of operators with respect to the language type. Our results are stated in quantalic generality, allowing for instantiation to both two-valued and quantitative types of semantics and logics.

An important next step in the programme of developing graded logics into a verification framework is the question of how graded semantics relates to fixpoint logics. While we have focused on Eilenberg-Moore semantics in the present work, graded semantics does also subsume Kleisli-style trace semantics [13], which poses additional challenges for the design of characteristic modal logics, in particular in the quantitative setting.

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A Proofs for Section 3 (Graded Semantics and Graded Logics)

Proposition 21. *Let $\mathcal{L}$ be a graded logic for the semantics $(\alpha, \mathbb{M})$ on $G\colon \mathbf{C} \to \mathbf{C}$ and (C, c) a G-coalgebra. For two states $x, y \in C$ we have that $d^b(x,y) \le d^{\mathcal{L}}(x,y)$.*

Proof. We define an evaluation of formulae on the semantic objects as morphisms $[\![\phi]\!]_{\mathbb{M}}\colon M_n 1 \to \Omega$, and show that $[\![\phi]\!]_c = [\![\phi]\!]_{\mathbb{M}} \cdot c^{(n)}$. Let $x, y \in C$ be states of a coalgebra (C, c). The claim then follows from the fact that $d^b(x,y) \le d_{M_n 1}(c^{(n)}(x), c^{(n)}(y))$ and the $[\![\phi]\!]_{\mathbb{M}}$ are $\mathcal{V}$-functors. We define the semantics $[\![\cdot]\!]_{\mathbb{M}}$:

- $[\![\theta]\!]_{\mathbb{M}} = M_0 1 \xrightarrow{M_0\hat{\theta}} M_0 \Omega \xrightarrow{o} \Omega$ for $\theta \in \Theta$
- $[\![p(\phi_1, \dots, \phi_n)]\!]_{\mathbb{M}} = [\![p]\!] \cdot \langle [\![\phi_1]\!]_{\mathbb{M}}, \dots, [\![\phi_n]\!]_{\mathbb{M}} \rangle$ for $p \in \mathcal{O}$ n-ary
- $[\![\lambda\phi]\!]_{\mathbb{M}} = f([\![\phi]\!]_{\mathbb{M}})$ for $\lambda \in \Lambda$ (we continue to restrict to unary modalities)

where f in the clause for modal operators comes from canonicity of $(M_n, M_{n+1}, \mu^{0,n}, \mu^{0,n+1}, \mu^{1,n})$ (Proposition 18), that is, $f(\langle [\![\phi_1]\!]_{\mathbb{M}}, \dots, [\![\phi_m]\!]_{\mathbb{M}} \rangle)$ is the, by freeness unique, morphism that makes the following square commute:

$$\begin{tikzcd} M_1 M_n 1 \arrow[r, "M_1 [\![\phi]\!]_{\mathbb{M}}"] \arrow[d, "\mu^{1n}"] & M_1(\Omega) \arrow[d, "f"] \\ M_{n+1} 1 \arrow[r, "f([\![\phi]\!]_{\mathbb{M}})"] & \Omega \end{tikzcd} \tag{2}$$

It is straightforward to show by induction on the depth of ϕ that the morphism $[\![\phi]\!]_{\mathbb{M}}$ defines a homomorphism of M_0-algebras from $(M_n 1, \mu^{0,n})$ to (Ω, o), which is needed for $[\![\lambda\phi]\!]_{\mathbb{M}}$ to be defined.

Now fix a coalgebra (C, c) and a uniform-depth formula ϕ of $\mathcal{L}$. We prove the claim that $[\![\phi]\!]_c = [\![\phi]\!]_{\mathbb{M}} \cdot c^{(n)}$ by structural induction on ϕ.

For the case of $\phi = \theta \in \Theta$ we have, by unrolling definitions, that $[\![\theta]\!]_c = \hat{\theta} \cdot !_X$ and $[\![\phi]\!]_{\mathbb{M}} \cdot c^{(n)} = o \cdot M_0\hat{\theta} \cdot M_0 !_X \cdot \eta_X$, which are the outer paths in the following diagram:

$$\begin{tikzcd} X \arrow[r, "!"] \arrow[d, "\eta_X"] & 1 \arrow[r, "\hat{\theta}"] \arrow[d, "\eta_1"] & \Omega \arrow[d, "\eta_\Omega"] \arrow[dr, "id"] & \\ M_0 X \arrow[r, "M_0 !"] & M_0 1 \arrow[r, "M_0\hat{\theta}"] & M_0 \Omega \arrow[r, "o"] & \Omega \end{tikzcd}$$

The squares commute due to naturality of η, while commutativity of the triangle is implied by o being an M_0-algebra. The step for formulae of the form $\phi = p(\phi_1, \dots, \phi_n)$ is immediate from definitions. For $\phi = \lambda\phi'$ with ϕ' of uniform depth n, we have

$$\begin{aligned}
[\![\phi]\!]_c &= [\![\lambda]\!] \cdot G[\![\phi']\!]_c \cdot c \\
&= f \cdot \alpha_\Omega \cdot G[\![\phi']\!]_c \cdot c \\
&= f \cdot M_1[\![\phi']\!]_c \cdot \alpha_X \cdot c && \text{(naturality of } \alpha\text{)} \\
&= f \cdot M_1[\![\phi']\!]_{\mathbb{M}} \cdot M_1 c^{(n)} \cdot \alpha_X \cdot c && \text{(IH)} \\
&= f([\![\phi']\!]_{\mathbb{M}}) \cdot \mu^{1n} \cdot M_1 c^{(n)} \cdot \alpha_X \cdot c && (2) \\
&= [\![\phi]\!]_{\mathbb{M}} \cdot c^{(n+1)}
\end{aligned}$$

□

Details for failure of invariance of non-uniform depth fomulae As a counterexample, consider the Kleisli-style graded monad, i.e. $M_n = TF^n$ where the monad part $T = \mathcal{P}$, functor $FX = X \times X$, and the Kleisli distributive law $\zeta\colon FT \to TF$ given by $\zeta(A, B) = X \times Y$ for $A, B \subseteq X$. These data induce a minimal form of *tree-shaped-trace* semantics: For a state x in a TF-coalgebra (C, c), $c^{(n)}(x) \in TF^n 1 = \mathcal{P}1$ records whether the complete binary tree of depth n can be executed at x. We define a graded logic $\mathcal{L}$ over (Ω, o), where $\Omega = \{\bot, \top\}$ and $o\colon \mathcal{P}\Omega \to \Omega$ takes suprema. The logic contains a truth constant $\top$, where $\hat{\top}\colon 1 \to \Omega$ is the constant map to $\top$. We also have a binary modal operator $\Diamond$, with $[\![\Diamond]\!]\colon \mathcal{P}(\Omega^2 \times \Omega^2) \to \Omega$ defined as $[\![\Diamond]\!](S) = o(\mathcal{P}\pi_1(S)) \wedge o(\mathcal{P}\pi_4(S))$ where π_i is the i-th projection of $\Omega^2 \times \Omega^2 \cong \Omega^4$; that is, $\Diamond(\phi_1, \phi_2)$ evaluates to $\top$, if there is a successor pair whose first component satisfies ϕ_1 and whose second component satisfies ϕ_2. We define a TF-coalgebra $(\{x, y, z\}, c)$ where $c(x) = \{(z, z)\}$, $c(y) = \{(x, z)\}$, and $c(z) = \emptyset$. Then x and y disagree on the non-uniform formula $\Diamond(\Diamond(\top, \top), \top)$, even though x and y are equivalent under the graded semantics.

Theorem 23. *Suppose that a graded logic $\mathcal{L}$ is both depth-0 separating and depth-1 separating. Then $\mathcal{L}$ is expressive.*

Proof. We utilize the semantics $[\![-]\!]_{\mathbb{M}}$, defined in the proof of Proposition 21. It suffices to show that the family of maps

$$\{[\![\phi]\!]_{\mathbb{M}} : M_n 1 \to \Omega \mid \phi \text{ is a uniform depth-}n\ \mathcal{L} \text{ formula}\}$$

is initial for each n. We proceed by induction on n. The base case $n = 0$ is immediate by depth-0 separation. For the inductive step, let $\mathfrak{A}$ denote the set of evaluations $M_n 1 \to \Omega$ of depth-n formulas. By the induction hypothesis, $\mathfrak{A}$ is initial. By definition, $\mathfrak{A}$ is closed under propositional operators in $\mathcal{O}$. By depth-1 separation, it follows that set

$$\{[\![\lambda]\!]([\![\phi]\!])\colon M_{n+1} \to \Omega \mid \lambda \in \Lambda, \phi \text{ a uniform depth-}n \text{ formula}\}$$

is initial, proving the claim.

B Proofs for Section 4 (Graded Semantics via Coalgebraic Determinization)

Theorem 30. *Let $\lambda\colon FT\Omega \to \Omega$ be a modal operator for a graded logic with truth value object (Ω, o). Then $\lambda = ev_\lambda \cdot Fo$ for some algebra homomorphism $ev_\lambda\colon \tilde{F}(\Omega, o) \to (\Omega, o)$. On the other hand, every algebra homomorphism $\tilde{F}(\Omega, o) \to (\Omega, o)$ yields a modal operator in this way.*

Proof. Since λ is an M_1-algebra structure and thus satisfies the coequalization property, it factors through the coequalizer of $\mu^{1,0}_\Omega\colon M_1M_0\Omega \to M_1\Omega$ and $M_1o\colon M_1M_0\Omega \to M_1\Omega$, which, by definition, is given by $\overline{M}_1(\Omega, o) = (F\Omega, Fo \cdot \zeta_\Omega)$, as displayed in the following diagram:

$$\begin{tikzcd}
FTT\Omega \arrow[r, shift left, "F\mu_\Omega"] \arrow[r, shift right, "FTo"'] & FT\Omega \arrow[rr, "\lambda"] \arrow[dr, "Fo"'] & & \Omega \\
& & F\Omega \arrow[ur, "ev_\lambda"'] &
\end{tikzcd}$$

To show that ev_λ is a homomorphism of T-algebras consider the following diagram:

$$\begin{tikzcd}
FT\Omega \arrow[rr, "\lambda"] \arrow[dr, "Fo"] & & \Omega \\
& F\Omega \arrow[ur, "ev_\lambda"'] & \\
FTT\Omega \arrow[uu, "F\mu"] \arrow[r, "FTo"] & FT\Omega \arrow[u, "Fo"'] & \\
& TF\Omega \arrow[u, "\zeta_\Omega"'] \arrow[dr, "Tev_\lambda"] & \\
TFT\Omega \arrow[uu, "\zeta_{T\Omega}"] \arrow[ur, "TFo"] \arrow[rr, "T\lambda"] & & T\Omega \arrow[uuuu, "o"']
\end{tikzcd}$$

The outer square commutes by homomorphy of λ. The top triangle commutes by coequalization, as does the bottom triangle, since it is just T applied to the top triangle. The left top square commutes since o is a T-algebra structure and the left bottom square commutes by naturality of ζ. It follows that the right hand square precomposed with TFo commutes. TFo is a split coequalizer, and therefore an epimorphism. Therefore by canceling TFo we have that the right hand square commutes, which is precisely the condition for ev_λ to be a homomorphism of T-algebras.

Conversely, given a morphism $ev_\lambda\colon \tilde{F}(\Omega, o) \to (\Omega, o)$ it is straight forward to check that $\lambda = ev_\lambda \cdot Fo$ satisfies the laws necessary to make it an M_1-algebra main structure. $\square$