QUANTUM DIFFERENCE EQUATION FOR THE AFFINE TYPE A QUIVER VARIETIES I: GENERAL CONSTRUCTION

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ABSTRACT. In this article we use the philosophy in [OS22] to construct the quantum difference equation of affine type A quiver varieties in terms of the quantum toroidal algebra $U_{q,t}(\hat{\hat{\mathfrak{sl}}}_r)$. In the construction, we define the set of wall for each quiver varieties by the action of the universal R-matrix, which is shown to be almost equivalent to that of the K-theoretic stable envelope on each interval in $H^2(X,\mathbb{Q})$. We also give the examples of the instanton moduli space $M(n,r)$ and the equivariant Hilbert scheme $\mathrm{Hilb}_n([\mathbb{C}^2/\mathbb{Z}_r])$ to show the explicit form of the quantum difference operator.

CONTENTS

1. INTRODUCTION

1.1. **Quantum differential equation for symplectic resolution X.** The enumerative geometry of rational curves in Nakajima quiver varieties is a captivating subject that elegantly connects counting problems in geometry to algebra. The countings involved in this field exhibit numerous surprising attributes, illustrating the profound connection that exists between geometry, algebra, and representation theory

In the equivariant cohomology setting, there are two enumerative counting of curves in Nakajima quiver varieties. One is the Gromov-Witten invariant, which is the counting of the holomorphic map $\Sigma \to X$. Another one is the Donaldson-Thomas invariants, which is the counting of the subscheme $Z \subset X$ of dimension 1. For the Calabi-Yau 3-fold, it is conjectured [MNO06A][MNO06B] that there is a correspondence between the Gromov-Witten invariants and the Donaldson-Thomas invariants for three-folds. The conjecture has been proved for toric 3-folds[MOO11]. Moreover, these enumerative invariants offer further perspectives that warrant exploration.

From the Gromov-Witten theory, one could also define the quantum cohomology $(H^*(X), \star)$ and the corresponding Dubrovin connection, or we shall call it as the quantum differential equation. We make some simple introduction for the quantum differential equation: Given X a Nakajima quiver varieties (actually, any symplectic varieties would be suitable for this setting), consider the cone of effective curves in $H_2(X,\mathbb{Z})$ and the completed (semi)group algebra $\mathbb{C}[[z^d]]$ spanned by z^d with $d \in H_2(X,\mathbb{Z})_{effective}$. And it is known that the genus 0 GW theory characterise the quantum cup product on $H^*(X,\mathbb{C})$ given as:

$$\alpha \star \beta = \alpha \cup \beta + O(z) \tag{1.1}$$

The quantum cup product is parametrised by $\mathbb{C}[[z^d]]$, where one counts the rational curves meeting three given cycles. In the context of topological strings, this quantum cup product corresponds to the three-point correlation function of the A-model over the variety X, further details can be found in [CK99]. The associated supercommutative algebra is known as the quantum cohomology of X. A similar setting is permissible for the equivariant case $(H^*_G(X)[[z^d]], \star)$.

Associated to the quantum cup product, one can construct a flat connection on the trivial $H^*_G(X,\mathbb{C})$-bundle over Spec $\mathbb{C}[[z^d]]_{d\in H_2(X,\mathbb{Z})_{eff}}$, which is known as the quantum connection, or the Dubrovin connection. It has the form

$$\frac{d}{d\lambda}\Psi(z) = \lambda \star \Psi(z), \qquad \Psi(z) \in H^*_G(X)[[z^d]], \qquad \frac{d}{d\lambda}z^d = (\lambda, d)z^d \tag{1.2}$$

with $\lambda \in H^2(X,\mathbb{C})$.

In our story, the flat sections of the quantum connection plays a crucial rule. In our case, the quantum connection exhibits regular singularity , and notably, the monodromy and the shifting properties impact both GW and PT theory. Surprisingly, it serves as a bridge connecting the enumerative geometry to the quantum algebra representations.

To provide a concise overview, let us consider the Gromov-Witten theory and the Donaldson-Thomas theory for $\mathbb{P}^1 \times \mathbb{C}^2$. These have been described in [BP08][OP10]. It is found that the genus 0 GW and DT theory can be used to describe the quantum cohomology of the Hilbert scheme $\text{Hilb}_d(\mathbb{C}^2)$. The quantum cohomology of the Hilbert scheme $\text{Hilb}_d(\mathbb{C}^2)$, with the torus action $(\mathbb{C}^\times)^2$ and the equivariant parametres (t_1, t_2), has been studied in [OP10B].

Another intriguing aspect in this story is the 1-legged vertex in the DT theory of $\mathbb{C}^2 \times \mathbb{P}^1$, which was initially introduced in [OP10]. This element plays a significant role in the computation of both GW and DT theory for $\mathbb{C}^2 \times \mathbb{P}^1$, which turned out to be the shift operators for the quantum connection of the Hilbert scheme $\text{Hilb}_d(\mathbb{C}^2)$. Here, shift operator is an operator intertwining the monodromy of the quantum connection for different equivariant parametres. Notably, it can be built up in terms of the solutions for the two quantum connections.

On the other hand, shift operators also possess representation-theoretic interpretations. It is known[SVC13][MO12] that the equivariant cohomology $\bigoplus_{d\geq 0} H_T^*(\text{Hilb}_d(\mathbb{C}^2))$ forms a Fock space of the affine Yangian $Y_{\hbar_1,\hbar_2}(\hat{\hat{\mathfrak{gl}}}_1)$. This action can be encoded into the unique map known as stable envelope, introduced by Maulik and Okounkov[MO12]. The stable envelope map is a Steinberg class in $H_T^*(X \times X^A)$ relating the quantum cohomology or quantum connection for the symplectic variety X to its fixed subvariety $X^A \subset X$. Additionally, it can be used to construct the geometric R-matrix for the geometric modules $\bigoplus_{d\geq 0} H_T^*(\text{Hilb}_d(\mathbb{C}^2)$. Furthermore, it has been shown in [MO12] that the shift operator can be computed via the geometric R-matrix for the $\bigoplus_{d\geq 0} H_T^*(\text{Hilb}_d(\mathbb{C}^2))$.

In fact, not only for the Hilbert scheme $\text{Hilb}_d(\mathbb{C}^2)$, but also for the general Nakajima quiver varieties, there exist enumerative invariants as the quantum cohomology and the quantum connection.

The similar result can also be applied to them that we can use the representation-theoretic viewpoint to construct the quantum connection, which is constructed in [MO12] via a certain Yangian $Y(\mathfrak{g})$. And in the finite ADE cases, $\mathfrak{g}$ coincides with the ADE Lie algebra, while in the other cases, it encompasses a slightly larger scope.

Here we list the corresponding quantum connections for different type of symplectic varieties.

Symplectic varieties	Quantum connections
Quiver Varieties $M_{\mathbf{v},\mathbf{w}}$	Casimir Trigonometric Connection[L11][MO12]
Affine Grassmannian Slices $\widehat{W}_{\lambda,\mu}$	Trigonometric Knizhnik-Zamolodchikov Connection[D22]
Springer Resolution $T^*(G/B)$	Affine Knizhnik-Zamolodchikov Connection[BMO11]

Remark.In the table there is something interesting that we can mention here. For the affine Grassmannian slices of type A, it is isomorphic to the Slodowy slices of type A which can be described in terms of the finite type A quiver varieties[MV07]. Thus there is an identification between the trigonometric KZ connection of type A and the Casimir trigonometric connection of type A. And actually they are related via the cohomological stable envelope [D22].

Another interesting story is about the monodromy of these quantum connections, and it turns out that they have the natural quantum Weyl group action on them, for details see [ATL15][BM15][EV02][GL11].

1.2. **Quantum difference equation and K-theoretic countings.** In the K-theory, one could also carry out the similar story by replacing the quantum differential equation with the quantum difference equation. The enumerative invariants here are replaced by the capped operators, capping operatos and vertex functions, which are the trigonometric countings of $\mathbb{P}^1$ into the symplectic varieties. And these invariants, especially for the capped operators and vertex functions, they are encoded into the solution of the specific type of difference equations.

The quantum difference equation is a flat q-difference connection

$$\Psi(p^{\mathscr{L}}z)=\mathbf{M}_{\mathscr{L}}(z)\Psi(z) \tag{1.3}$$

on functions of the Picard torus $\mathrm{Pic}(X)\otimes\mathbb{C}^{\times}$ with values in $K_G(X)$. In the case of X being the Nakajima quiver varities, with $\mathscr{L}\in\mathrm{Pic}(X)$ a line bundle on X, the regular part of the solution of the quantum difference equation near the zero point $z=0$ (Actually this "zero" point depends on the choice of the chamber) is the capping operator [O15].

The capping operators has really beautiful symmetry which is really unique in the K-theoretical enumerative geometry. Notably, for many capping operators, the connection matrix relating capping operators of two quiver varieties with different stability condition is characterised by the elliptic stable envelope, which is constructed in [AO21] as the elliptic cohomology analog of the stable envelope.

The foundational work on quantum difference equations was developed by Etingof and Varchenko [EV02], who constructed the dynamical Weyl group, subsequently employing it to construct difference operators for the intertwiners for the $U_q(\hat{\mathfrak{g}})$-modules of finite ADE type. These difference equations are the difference analog of the trigonometric Casimir connections, and will be referred to as quantum difference equations.

The quantum difference operator $\mathbf{M}_{\mathscr{L}}(z)$ has been constructed in [OS22] via the techniques of K-theoretic stable envelope. Once the formula for the K-theoretic stable envelope is known, in principle,it becomes possible to compute the explicit form of the monodromy operator. Since one could use K-theoretic stable envelope to construct the geometric R-matrix and use the FRT formalism to construct the geometric quantum group $U_q^{MO}(\hat{\mathfrak{g}}_Q)$ as elucidated in [OS22][MO12], it is anticipated that the monodromy operator could be expressed in terms of the generators of the geometric quantum group.

In the case of finite ADE types, the geometric quantum group is known to be isomorphic to the quantum affine algebra $U_q(\hat{\mathfrak{g}}_Q)$, as shown in [N23]. For the affine ADE types, the situation remains an open question; however, it is conjectured that the associated geometric quantum group should be isomorphic to the corresponding quantum toroidal algebra of ADE type. If this conjecture proves correct, It would be possible to represent the monodromy operator using the generators of the quantum toroidal algebra.

1.3. **Content of the paper.** In this paper we employ the approach in [OS22] to construct the quantum difference equation over the equivariant K-theory of the affine type A quiver varities using the quantum toroidal algebra $U_{q,t}(\hat{\hat{\mathfrak{sl}}}_r)$ action on them. In contrast to the employment of K-theoretical stable envelope, we adopt the methods of the factorization of the quantum toroidal algebra into the slope subalgebra, as developed by Negut[N15]. Using this factorisation, we systematically construct the monodromy operator $\mathbf{B}_{\mathbf{m}}(\lambda)$ for each slope subalgebra $\mathscr{B}_{\mathbf{m}}$ and the quantum difference opeartor $\mathbf{M}_{\mathscr{L}}(\lambda)$. Our construction is purely algebraic, relying on Nakajima's formulation of the quantum affine algebra action on the equivariant K-theory of quiver varieties. Additionally, we define the wall set for each quiver varieties in terms of those rational points $\mathbf{m} \in \mathbb{Q}^r$ whose associated reduced universal R-matrix of the corresponding root subalgebra $\mathscr{B}_{\mathbf{m}}$ is non-trivial when acting on some quiver varieties. We expect that the description of the wall set here will encompass the characteristics of the wall set defined in [OS22].

In our construction, we will see that the quantum difference opeartor $\mathbf{M}_{\mathscr{L}}$ with arbitrary $\mathscr{L} \in \mathrm{Pic}(M(\mathbf{v},\mathbf{w}))$ produces a holonomic system over $\mathrm{Pic}(M(\mathbf{v},\mathbf{w})) \otimes \mathbb{C}^{\times}$. The following two theorems are the main results of the paper:

Theorem 1.1. *(Theorem 3.15) Given an affine type A quiver variety $M(\mathbf{v},\mathbf{w})$ and line bundles $\mathscr{L} \in Pic(M(\mathbf{v},\mathbf{w}))$, there exists a holonomic system of operators $\mathbf{M}_{\mathscr{L}} \in End(K_T(M(\mathbf{v},\mathbf{w}))[[p^{\mathscr{L}}]]_{\mathscr{L}\in Pic(M(\mathbf{v},\mathbf{w}))}$ i.e.*

$$T_{p,\mathscr{L}}^{-1}\mathbf{M}_{\mathscr{L}}T_{p,\mathscr{L}'}^{-1}\mathbf{M}_{\mathscr{L}'} = T_{p,\mathscr{L}'}^{-1}\mathbf{M}_{\mathscr{L}'}T_{p,\mathscr{L}}^{-1}\mathbf{M}_{\mathscr{L}} \tag{1.4}$$

Moreover, $\mathbf{M}_{\mathscr{L}}$ can be constructed from $\widehat{U_{q,t}(\hat{\hat{\mathfrak{sl}}}_r)}$ of total degree 0.

In addition, we give a general formula for the monodromy operators $\mathbf{B}_{\mathbf{m}}(\lambda)$ of affine type A quiver varieties.

Theorem 1.2. *(Theorem 3.5) On the representation space $End(K_T(M(\mathbf{v},\mathbf{w})))$, the monodromy operator $\mathbf{B}_{\mathbf{m}}(\lambda)$ can be written as:*

$$\begin{aligned}
&\mathbf{B}_{\mathbf{m}}(\lambda)\\
=&\prod_{h=1}^{g} :(\textbf{\textit{Heisenberg algebra part}})\prod_{k=0}^{\overrightarrow{\infty}}\\
&\prod_{\substack{\gamma\in\Delta(A)\\ m\geq 0}}^{\leftarrow}(\exp_{q^2}(-(q-q^{-1})z^{k(-\mathbf{v}_{\gamma}+(m+1)\boldsymbol{\delta}_h)}p^{k\mathbf{m}\cdot(-\mathbf{v}_{\gamma}+(m+1)\boldsymbol{\delta}_h)}q^{\frac{k}{2}(\mathbf{v}^TC\mathbf{v}-\mathbf{w}^T\mathbf{w})^{`}\cdot(-\mathbf{v}_{\gamma}+(m+1)\boldsymbol{\delta}_h))}(\\
&q^{-2k(-\mathbf{v}_{\gamma}+(m+1)\boldsymbol{\delta})^TC(-\mathbf{v}_{\gamma}+(m+1)\boldsymbol{\delta})}\varphi^{-(k+1)(-\mathbf{v}_{\gamma}+(m+1)\boldsymbol{\delta})}f_{(\delta-\gamma)+m\delta}e'_{(\delta-\gamma)+m\delta}\varphi^{-(k+1)(-\mathbf{v}_{\gamma}+(m+1)\boldsymbol{\delta})})\\
&\times\exp(-(q-q^{-1})\sum_{m\in\mathbb{Z}_+}\sum_{i,j=1}^{l_h}z^{km\boldsymbol{\delta}_h}p^{km\mathbf{m}\cdot\boldsymbol{\delta}_h}u_{m,ij}q^{-2km^2\boldsymbol{\delta}^TC\boldsymbol{\delta}}\varphi^{-(k+1)m\boldsymbol{\delta}}f_{m\delta,\alpha_i}e'_{m\delta,\alpha_i}\varphi^{-(k+1)m\boldsymbol{\delta}})\\
&\times\prod_{\substack{\gamma\in\Delta(A)\\ m\geq 0}}^{\rightarrow}\exp_{q^2}(-(q-q^{-1})z^{k(\mathbf{v}_{\gamma}+m\boldsymbol{\delta}_h)}p^{k\mathbf{m}\cdot(\mathbf{v}_{\gamma}+m\boldsymbol{\delta}_h)}q^{\frac{k}{2}(\mathbf{v}^TC\mathbf{v}-\mathbf{w}^T\mathbf{w})^{`}\cdot(\mathbf{v}_{\gamma}+m\boldsymbol{\delta}_h))}\\
&q^{-2k(\mathbf{v}_{\gamma}+m\boldsymbol{\delta})^TC(\mathbf{v}_{\gamma}+m\boldsymbol{\delta})}(\varphi^{-(k+1)(\mathbf{v}_{\gamma}+m\boldsymbol{\delta})}f_{\gamma+m\delta}e'_{\gamma+m\delta}\varphi^{-(k+1)(\mathbf{v}_{\gamma}+m\boldsymbol{\delta})})):
\end{aligned}\tag{1.5}$$

And :: stands for the normal ordering.

As illustrative examples, we calculate the quantum difference operators of the equivariant Hilbert scheme $\mathrm{Hilb}_n([\mathbb{C}^2/\mathbb{Z}_r])$ of A_{r-1} type surfaces and $M(r,n)$ the instanton moduli space on $\mathbb{P}^2$. Upon examination, we observe that the quantum difference equation for the instanton moduli space $M(r,n)$ corresponds exactly to the one previously established by [OS22].

Let us make some comments on the proof of the Theorem 1.1, in [OS22], the analog of the theorem use the K-theoretic stable envelope to go through the key step, i.e. Proposition 2.2, Proposition 3.9, Theorem 3.13. In our paper, these propositions and theorems will be solved using the straight computation of the geometric action of the quantum toroidal algebra on the equivariant K-theory of the Nakajima quiver varieties.

For the Theorem 1.2, it should be noted that only a finite number of terms e_α are nonzero when applied to the space $K_T(M(\mathbf{v},\mathbf{w}))$. This means that the theorem yields a computable formula upon reduction within this equivariant K-theory framework.

In our next paper, we will use our construction to solve the quantum difference equation for $\text{Hilb}_n([\mathbb{C}_2/\mathbb{Z}_r])$ and study its degeneration to the quantum differential equation. We shall find that the limit properties of the connection matrix for the difference equation has a really simple form in terms of the monodromy operator $\mathbf{B}_{\mathbf{m}}(\lambda)$.

The connection between our construction of the quantum difference equation and the approach in [OS22] can be described as follows: In their method, they utilize the K-theoretic stable envelope to construct the MO quantum affine algebra $U_q^{MO}(\hat{\mathfrak{g}}_Q)$, and the stable envelope of different slope s to form the root subalgebra $U_q^{MO}(\mathfrak{g}_s)$. The relationship between the MO quantum affine algebra $U_q(\hat{\mathfrak{g}}_Q)$ and $U_q(\hat{\mathfrak{g}}_Q)$ and their root subalgebras are based on the following conjecture:

Conjecture 1.3. *There are isomorphisms of algebras:*

$$U_q^{MO}(\hat{\mathfrak{g}}_Q)\cong U_q(\hat{\mathfrak{g}}_Q), U_q(\mathfrak{g}_{\mathbf{m}})\cong\mathscr{B}_{\mathbf{m}} \tag{1.6}$$

The conjecture has been confirmed to be true for the case of finite ADE types. For the other types of quiver, it is known that $U_q(\hat{\mathfrak{g}}_Q)$ is a subalgebra of $U_q^{MO}(\hat{\mathfrak{g}}_Q)$ [N23]. As a result, the construction of the quantum difference equation in both situations have something in common. This might gives representation-theoretic interpretation of the K-theoretic countings, leading to a deeper understanding of the implications of the conjecture and its role in our construction.

For instance, in [AO21], it has been demonstrated that the connection matrix for the vertex functions and capping operators of quiver varieties can be expressed through the elliptic stable envelope, or the elliptic geometric R-matrix.

We anticipate using this construction to uncover intriguing and specific structures within both quantum difference and differential equations of the quiver varieties, such as monodromy representation. The degeneration from the quantum difference equation to the quantum differential equation has been demonstrated for Dynkin ADE type in [BM15], and we expect that the similar relation can be true for the quantum toroidal case

[BT19]. We give the detailed analysis of this degeneration in the work [Z24], and it will expand our understanding of these equations in diverse settings.

We also anticipate intriguing prospects in enumerative geometry, particularly under the auspices of Conjecture 1.3. We expect that, under this conjecture, the regular part of the solution for the quantum difference equation should correspond to the capping operator in the equivariant K-theory of the quiver varieties.

The structure of the paper goes in the following way. In section 2, we introduce some basics about the quantum toroidal algebra and the quantum affine algebra. We introduce the shuffle algebra realization of the quantum toroidal algebra and its slope subalgebra. Also we give the relation of the coproduct and the universal R-matrix between the quantum toroidal algebra and the slope subalgebra, which will be used to construct the quantum difference operator.

The heart of the paper lies in section 3, we introduce the construction of the quantum difference operator, and prove the theorem 3.15 and the theorem 1.2. In section 4, we give the examples of the instanton moduli space and the equivariant Hilbert scheme $\mathrm{Hilb}_n([\mathbb{C}^2/\mathbb{Z}_r])$. And we remark the claim about the relation between the wall set in our paper and the wall set of the K-theoretic stable envelope.

1.4. **Connection with qKZ.** In the construction of the quantum difference operators, we also prove that it commutes with the 2-point qKZ equation:

Theorem 1.4. *(See 3.12) For arbitrary line bundles $\mathscr{L}$, $\mathscr{L}' \in Pic(M(\mathbf{v},\mathbf{w}))$ and a slope s we have that the qKZ operators commute with the q-difference operators*

$$\Delta_s(\mathscr{B}^s_{\mathscr{L}})\mathscr{R}^s = \mathscr{R}^s\Delta_s(\mathscr{B}^s_{\mathscr{L}}) \tag{1.7}$$

Here qKZ equation is the difference equation over the equivariant parametre $u \in T^{\vee}$, while the quantum difference equation is the difference equation over the Kahler variable $z \in \mathrm{Pic}(X) \otimes \mathbb{C}^{\times}$. The commutativity of qKZ operator and quantum difference operator implies that the solution of the qKZ equation $\mathscr{R}^s\Psi(z,u) = \Psi(z,u)$ is also the solution of the coproduct of the quantum difference equations $\Delta_s(\mathscr{B}^s_{\mathscr{L}})\Psi(z,u) = \Psi(z,u)$.

For general, one may deduce that the quantum difference equation commute with arbitrary n-point qKZ equation of the quantum toroidal algebra $U_{q,t}(\hat{\hat{\mathfrak{sl}}}_n)$ in the sense of [FR92]. This has been confirmed to be true in the case of trigonometric Casimir connection and the difference Casimir connection in the case of ADE [EV02][L11]. The commutativity relation can give us different aspects about the connection matrix and the solution for the quantum difference equation from the side of qKZ equation. For example, the solution of the qKZ equation generally stands for the intertwiners of modules of quantum affine and toroidal algebra (See [AKMMMOZ17][AKMMS18][FR92]) , so this might give the interpretation of the solution for the quantum difference equation as intertwining operators for the integrable modules. For the intertwiners of quantum toroidal algebras of specific modules, one can see [AKMMS18][AFS12].

We shall not give detailed study of these relations in this paper and leave it for the further study.

On the side of the enumerative geometry, in the settings of [OS22], qKZ equation corresponds to the difference equation of equivariant variable of the capping operators as in 3.5. And the R-matrix $q_{(1)}^{(\lambda)}\mathscr{R}^{(s)}(u)$ corresponds to the shift operators $S(u,z)$ in 3.5. So we also expect to use these constructions to give some connection between the K-theoretic countings and the intertwiners for the modules of quantum algebras.

1.5. **Further directions.** This paper can be seen as the starting point of understanding the quantum difference equation of Nakajima quiver varieties, or other types of quiver varieties appearing in the K-theoretic enumerative geometry in the context of shuffle algebras. Many special things happen when we consider the K-theoretic quasi-map theory and the shuffle algebras in the context of K-theoretic Hall algebras.

In [Z24] we are going to analyze the quantum difference equation for the affine type A quiver varieties. The key ingredients for the quantum difference equation is the connection matrix for the solution in different asymptotic regions, and we can see that the connection matrix is strongly related to the ordered product of the monodromy operators $\mathbf{B}_{\mathbf{m}}(z)$. We can see that we can choose a generic path to represent the quantum difference operator $\mathbf{M}_{\mathscr{L}}(z)$ such that the monodromy operators $\mathbf{B}_{\mathbf{m}}(z)$ can be written in the simplest way, i.e. $U_q(\mathfrak{sl}_2)$-type or $U_q(\hat{\mathfrak{gl}}_1)$-type. This reveals the fact that the slope subalgebra $\mathscr{B}_{\mathbf{m}}$ has the "root decomposition" as in the case of the classical Lie algebra. Using these results, one can prove that the degenerate limit of the quantum difference equation is equal to the Dubrovin connection for the quantum cohomology of the affine type A quiver varieties, and the monodromy representation can be obtained via the monodromy operators $\mathbf{B}_{\mathbf{m}}(z)$.

The construction of the quantum difference equation can be really general for any other arbitrary types of quivers and weighted modules. In [Z24-02], we will generalize the construction to arbitrary types of quiver varieties and modules. Algebraically speaking, this can be thought of as a generalisation of the qKZ equation for other types of modules and quantum group, for which the solution to these equations will be really interested both in representation theory and enumerative geometry.

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2. Quantum toroidal algebra and geometric actions

2.1. Quantum affine groups $U_q(\hat{\mathfrak{sl}}_n)$. The quantum group $U_q(\hat{\mathfrak{sl}}_n)$ is a $\mathbb{Q}(q)$-Hopf algebra:

$$U_q(\hat{\mathfrak{sl}}_n) = \mathbb{Q}(q)\langle x_i^{\pm 1}, \psi_s^{\pm 1}, c^{\pm 1}\rangle_{s\in\{1,\cdots,n\}}^{i\in\mathbb{Z}/n\mathbb{Z}} \tag{2.1}$$

modulo the fact that c is central, as well as the following relations:

$$\psi_s\psi_{s'} = \psi_{s'}\psi_s \tag{2.2}$$

$$\psi_s x_i^{\pm} = q^{\pm(\delta_s^{i+1}-\delta_s^i)} x_i^{\pm}\psi_s \tag{2.3}$$

$$[x_i^{\pm}, x_j^{\pm}] = 0, \qquad \text{if } j \notin \{i-1, i+1\} \tag{2.4}$$

$$[x_i^{\pm}, [x_i^{\pm}, x_j^{\pm}]_q]_{q^{-1}} = 0 \qquad \text{if } j \in \{i-1, i+1\} \tag{2.5}$$

$$[x_i^+, x_j^-] = \frac{\delta_i^j}{q-q^{-1}}\left(\frac{\psi_{i+1}}{\psi_i} - \frac{\psi_i}{\psi_{i+1}}\right) \tag{2.6}$$

for all $i, j \in \mathbb{Z}/n\mathbb{Z}$ and $s, s' \in \{1, \cdots, n\}$. Here $[-,-]_q$ is the q-bracket:

$$[a, b]_q = ab - qba \tag{2.7}$$

We further require the following relation:

$$\psi_{s+n} = c\psi_s, \forall s \in \mathbb{Z} \tag{2.8}$$

The counit is given by $\epsilon(x_i^{\pm}) = 0, \epsilon(\psi_s) = 1$, and the coproduct given by:

$$\Delta(c) = c\otimes c, \Delta(\psi_s) = \psi_s \otimes \psi_s \tag{2.9}$$

$$\Delta(x_i^{\pm}) = \frac{\psi_{i+1}}{\psi_i} \otimes x_i^+ + x_i^+ \otimes 1 \tag{2.10}$$

$$\Delta(x_i^-) = 1 \otimes x_i^- + x_i^- \otimes \frac{\psi_i}{\psi_{i+1}} \tag{2.11}$$

One can also introduce its "half" subalgebras of $U_q(\hat{\mathfrak{sl}}_n)$:

$$U_q^{\geq}(\hat{\mathfrak{sl}}_n) = \mathbb{Q}(q)\langle x_i^+, \psi_s^{\pm 1}, c^{\pm 1}\rangle_{s\in\{1,\cdots,n\}}^{i\in\mathbb{Z}/n\mathbb{Z}} \subset U_q(\hat{\mathfrak{sl}}_n) \tag{2.12}$$

$$U_q^{\leq}(\hat{\mathfrak{sl}}_n) = \mathbb{Q}(q)\langle x_i^-, \psi_s^{\pm 1}, c^{\pm 1}\rangle_{s\in\{1,\cdots,n\}}^{i\in\mathbb{Z}/n\mathbb{Z}} \subset U_q(\hat{\mathfrak{sl}}_n) \tag{2.13}$$

Both of them are bialgebras, and there is a bialgebra pairing (Drinfeld pairing):

$$\langle -,-\rangle : U_q^{\geq}(\hat{\mathfrak{sl}}_n) \otimes U_q^{\leq}(\hat{\mathfrak{sl}}_n) \to \mathbb{Q}(q) \tag{2.14}$$

such that:

$$\langle x_i^+, x_j^-\rangle = \frac{\delta_j^i}{q^{-1}-q}, \qquad \langle \psi_s, \psi_{s'}\rangle = q^{-\delta_{s'}^s} \tag{2.15}$$

2.2. **The Heisenberg algebra.** We consider the q-deformed Heisenberg algebra:

$$U_q(\hat{\mathfrak{gl}}_1)=\mathbb{Q}(q)\langle p_{\pm k},c^{\pm1}\rangle_{k\in\mathbb{Z}} \tag{2.16}$$

where c is central and the generators $p_{\pm k}$ all commute, except for:

$$[p_k,p_{-k}]=k\cdot\frac{c^k-c^{-k}}{q^k-q^{-k}} \tag{2.17}$$

The counit ϵ is gvien by $\epsilon(p_k)=0,\epsilon(c)=1$. The coproduct given by $\nabla(c)=c\otimes c$ and:

$$\Delta(p_k)=c^k\otimes p_k+p_k\otimes 1 \tag{2.18}$$

$$\Delta(p_{-k})=1\otimes p_{-k}+p_{-k}\otimes c^{-k} \tag{2.19}$$

Similar to the previous subsection, the q-deformed Heisenberg algebra $U_q(\hat{\mathfrak{gl}}_1)$ is the Drinfeld double of its two halves:

$$U_q^{\geq}(\hat{\mathfrak{gl}}_1)=\mathbb{Q}(q)\langle p_k,c^{\pm1}\rangle_{k\in\mathbb{N}}\subset U_q(\hat{\mathfrak{gl}}_1) \tag{2.20}$$

$$U_q^{\leq}(\hat{\mathfrak{gl}}_1)=\mathbb{Q}(q)\langle p_{-k},c^{\pm1}\rangle_{k\in\mathbb{N}}\subset U_q(\hat{\mathfrak{gl}}_1) \tag{2.21}$$

with respect to the bialgebra pairing:

$$U_q^{\geq}(\hat{\mathfrak{gl}}_1)\otimes U_q^{\leq}(\hat{\mathfrak{gl}}_1)\to^{\langle-,-\rangle}\mathbb{Q}(q) \tag{2.22}$$

such that

$$\langle p_k,p_{-k}\rangle=\frac{k}{q^{-k}-q^k} \tag{2.23}$$

2.3. **Quantum affine group $U_q(\hat{\mathfrak{gl}}_n)$.** The quantum affine group $U_q(\hat{\mathfrak{gl}}_n)$ can be defined by the RTT formalism:

$$U_q(\hat{\mathfrak{gl}}_n):=\mathbb{Q}(q)\langle e_{\pm[i;j\rangle},\psi_s^{\pm1},c^{\pm1}\rangle^{s\in\{1,\cdots,n\}}_{(i,j)\in\mathbb{Z}^2/(n,n)\mathbb{Z}} \tag{2.24}$$

where the generators $e_{\pm[i;j\rangle}$ satisfy the quadratic $RTT=TTR$ relations and their standard coproduct is given by:

$$\Delta(e_{[i;j\rangle})=\sum_{s=i}^{j}e_{[s;j\rangle}\frac{\psi_s}{\psi_i}\otimes e_{[i;s\rangle} \tag{2.25}$$

$$\Delta(e_{-[i;j\rangle})=\sum_{s=i}^{j}e_{-[i;s\rangle}\otimes e_{-[s;j\rangle}\frac{\psi_i}{\psi_s} \tag{2.26}$$

The primitive part of $\Delta(e_{\pm[i;j\rangle})$ is given by:

$$\Delta_{prim}(e_{[i;j\rangle})=e_{[i;j\rangle}\otimes 1+\frac{\psi_j}{\psi_i}\otimes e_{[i;j\rangle} \tag{2.27}$$

$$\Delta_{prim}(e_{-[i;j\rangle})=e_{-[i;j\rangle}\otimes\frac{\psi_i}{\psi_j}+1\otimes e_{-[i;j\rangle} \tag{2.28}$$

There is an isomorphism of Hopf algebras:

$$U_q(\hat{\mathfrak{gl}}_n) \cong U_q(\hat{\mathfrak{sl}}_n) \otimes U_q(\hat{\mathfrak{gl}}_1)/(c \otimes 1 - 1 \otimes c) \tag{2.29}$$

which is given by:

$$e_{[i;i+1\rangle} = x_i^+(q - q^{-1}) \tag{2.30}$$

$$e_{-[i;i+1\rangle} = x_i^-(q^{-2} - 1) \tag{2.31}$$

The isomorphism identifies the quantum group $U_q(\hat{\mathfrak{gl}}_n)$ with $U_q(\hat{\mathfrak{sl}}_n) \otimes U_q(\hat{\mathfrak{gl}}_1)$. Here the Heisenberg generators $\{p_{\pm k}\}_{k\in\mathbb{Z}} \in U_q(\hat{\mathfrak{gl}}_1)$ in $U_q(\hat{\mathfrak{gl}}_n)$ satisfy the following relation:

$$[p_k, p_l] = k\delta^0_{k+l}\frac{(c^k - c^{-k})(q^{nk} - q^{-nk})}{(q^k - q^{-k})^2} \tag{2.32}$$

In this paper we use a smaller subalgebra of $U_q(\hat{\mathfrak{gl}}_n)$, which is almost the same as $U_q(\hat{\mathfrak{gl}}_n)$ besides that it has smaller set of Cartan elements:

$$U_q(\hat{\mathfrak{gl}}_n) := \mathbb{Q}(q)\langle e_{\pm[i;j\rangle}, \varphi_s^{\pm 1}, c^{\pm 1}\rangle^{s\in\{1,\cdots,n\}}_{(i,j)\in\mathbb{Z}^2/(n,n)\mathbb{Z}}, \qquad \varphi_s := \frac{\psi_{s+1}}{\psi_s} \tag{2.33}$$

We still denote it by $U_q(\hat{\mathfrak{gl}}_n)$. Note that we still have the isomorphism $U_q(\hat{\mathfrak{gl}}_n) \cong U_q(\hat{\mathfrak{sl}}_n) \otimes U_q(\hat{\mathfrak{gl}}_1)/(c \otimes 1 - 1 \otimes c)$, and here the Cartan elements ϕ_s of $U_q(\hat{\mathfrak{sl}}_n)$ are replaced by φ_s.

2.4. **Quantum toroidal algebra $U_{q,t}(\hat{\hat{\mathfrak{sl}}}_n)$.** The quantum toroidal algebra $U_{q,t}(\hat{\hat{\mathfrak{sl}}}_n)$ is defined by:

$$U_{q,t}(\hat{\hat{\mathfrak{sl}}}_n) = \mathbb{C}\langle \{e^\pm_{i,d}\}^{d\in\mathbb{Z}}_{1\leq i\leq n}, \{\varphi^\pm_{i,d}\}^{d\in\mathbb{N}_0}_{1\leq i\leq n}\rangle/(2.36) \tag{2.34}$$

And the relation between the generators can be described as:

$$e_i^\pm(z) = \sum_{d\in\mathbb{Z}} e^\pm_{i,d} z^{-d} \quad \varphi_i^\pm(z) = \sum_{d=0}^{\infty} \varphi^\pm_{i,d} z^{\mp d} \tag{2.35}$$

with $\varphi^\pm_{i,d}$ commuting among themselves and:

$$\begin{gathered} e_i^\pm(z)\varphi_j^{\pm'}(w)\cdot\zeta\left(\frac{w^{\pm 1}}{z^{\pm 1}}\right) = \varphi_j^{\pm'}(w)e_i^\pm(z)\cdot\zeta\left(\frac{z^{\pm 1}}{w^{\pm 1}}\right) \\ e_i^\pm(z)e_j^\pm(w)\cdot\zeta\left(\frac{w^{\pm 1}}{z^{\pm 1}}\right) = e_j^\pm(w)e_i^\pm(z)\cdot\zeta\left(\frac{z^{\pm 1}}{w^{\pm 1}}\right) \\ \left[e_i^+(z), e_j^-(w)\right] = \delta_i^j\delta\left(\frac{z}{w}\right)\cdot\frac{\varphi_i^+(z) - \varphi_i^-(w)}{q - q^{-1}} \end{gathered} \tag{2.36}$$

Here $i,j\in\{1,\cdots,n\}$, and z,w are variables of color i and j. The rational function $\zeta(x_i/x_j)$ is defined by:

$$\zeta\left(\frac{x_i}{x_j}\right)=\frac{\left[\frac{x_j}{qtx_i}\right]^{\delta^i_{j-1}}\left[\frac{tx_j}{qx_i}\right]^{\delta^i_{j+1}}}{\left[\frac{x_j}{x_i}\right]^{\delta^i_j}\left[\frac{x_j}{q^2x_i}\right]^{\delta^i_j}} \tag{2.37}$$

Also we impose the Serre relation:

(2.38)

$$e_i^{\pm}(z_1)e_i^{\pm}(z_2)e^{\pm}_{i\pm'1}(w)+\left(q+q^{-1}\right)e_i^{\pm}(z_1)e^{\pm}_{i\pm'1}(w)e_i^{\pm}(z_2)+e^{\pm}_{i\pm'1}(w)e_i^{\pm}(z_1)e_i^{\pm}(z_2)+$$
$$+e_i^{\pm}(z_2)e_i^{\pm}(z_1)e^{\pm}_{i\pm'1}(w)+\left(q+q^{-1}\right)e_i^{\pm}(z_2)e^{\pm}_{i\pm'1}(w)e_i^{\pm}(z_1)+e^{\pm}_{i\pm'1}(w)e_i^{\pm}(z_2)e_i^{\pm}(z_1)=0$$

The standard coproduct structure is imposed as:

$$\Delta:U_{q,t}\left(\hat{\hat{\mathfrak{sl}}}_n\right)\longrightarrow U_{q,t}\left(\hat{\hat{\mathfrak{sl}}}_n\right)\widehat{\otimes}U_{q,t}\left(\hat{\hat{\mathfrak{sl}}}_n\right) \tag{2.39}$$

$$\begin{aligned}\Delta\left(e_i^+(z)\right)&=\varphi_i^+(z)\otimes e_i^+(z)+e_i^+(z)\otimes 1 & \Delta\left(\varphi_i^+(z)\right)&=\varphi_i^+(z)\otimes\varphi_i^+(z)\\ \Delta\left(e_i^-(z)\right)&=1\otimes e_i^-(z)+e_i^-(z)\otimes\varphi_i^-(z) & \Delta\left(\varphi_i^-(z)\right)&=\varphi_i^-(z)\otimes\varphi_i^-(z)\end{aligned} \tag{2.40}$$

Later, we will propose alternative coproduct structures on $U_{q,t}(\hat{\hat{\mathfrak{sl}}}_r)$ that are twisted by the universal R-matrix for the slope subalgebra.

2.5. **Shuffle algebra and slope factorization.** Here we review the reconstruction of the quantum toroidal algebra via the shuffle algebra and the induced slope factorization of the quantum toroidal algebra. For details see [N15].

Fix a fractional field $\mathbb{F}=\mathbb{Q}(q,t)$ and consider the space of symmetric rational functions:

$$\widehat{\mathrm{Sym}}(V):=\bigoplus_{\mathbf{k}=(k_1,\cdots,k_n)\in\mathbb{N}^n}\mathbb{F}(\cdots,z_{i1},\cdots,z_{ik_i},\cdots)^{\mathrm{Sym}}_{1\leq i\leq n} \tag{2.41}$$

Here "Sym" means to symmetrize the rational function for each $z_{i1},\cdots,z_{ik_i}$. We endow the vector space with the shuffle product:

$$F*G=\mathrm{Sym}[\frac{F(\cdots,z_{ia},\cdots)G(\cdots,z_{jb},\cdots)}{\mathbf{k}!\mathbf{l}!}\prod^{1\leq i\leq n}_{1\leq a\leq k_i}\prod^{1\leq j\leq n}_{k_j+1\leq b\leq k_j+l_j}\zeta(\frac{z_{ia}}{z_{jb}})] \tag{2.42}$$

We define the subspace $\mathcal{S}^{\pm}_{\mathbf{k}}\subset\widehat{\mathrm{Sym}}(V)$ by:

$$\mathcal{S}^+:=\{F(\cdots,z_{ia},\cdots)=\frac{r(\cdots,z_{ia},\cdots)}{\prod^{1\leq i\leq n}_{1\leq a\neq b\leq k_i}(qz_{ia}-q^{-1}z_{ib})}\} \tag{2.43}$$

where $r(\cdots, z_{i1}, \cdots, z_{ik_i}, \cdots)^{1\leq i\leq n}_{1\leq a\leq k_i}$ is any symmetric Laurent polynomial that satisfies the wheel conditions:

$$r(\cdots, q^{-1}, t^{\pm}, q, \cdots) = 0 \tag{2.44}$$

for any three variables of colors $i, \cdots, i\pm 1, i$.

It is easy to see that the shuffle algebra $\mathcal{S}^+$ is bigraded by the number of the variables $\mathbf{k}\in\mathbb{N}^n$ and the homogeneous degree of the rational functions $d\in\mathbb{Z}$, i.e.

$$\mathcal{S}^+ = \bigoplus_{(\mathbf{k},d)\in\mathbb{N}^n\times\mathbb{Z}} \mathcal{S}^+_{\mathbf{k},d} \tag{2.45}$$

Similarly we can define the negative shuffle algebra $\mathcal{S}^- := (\mathcal{S}^+)^{op}$ which is the same as $\mathcal{S}^+$ with the opposite shuffle product. The grading for $\mathcal{S}^-$ now is given by $\mathcal{S}^- = \bigoplus_{(\mathbf{k},d)\in\mathbb{N}^n\times\mathbb{Z}} \mathcal{S}^-_{-\mathbf{k},d}$. Now we slightly enlarge the positive and negative shuffle algebras by the generators $\{\varphi^{\pm}_{i,d}\}^{d\geq 0}_{1\leq i\leq n}$:

$$\mathcal{S}^{\geq} = \langle \mathcal{S}^+, \{(\varphi^+_{i,d})^{d\geq 0}_{1\leq i\leq n}\}\rangle, \qquad \mathcal{S}^{\leq} = \langle \mathcal{S}^-, \{(\varphi^-_{i,d})^{d\geq 0}_{1\leq i\leq n}\}\rangle \tag{2.46}$$

Here $\varphi^{\pm}_{i,d}$ commute with themselves and have the relation with $\mathcal{S}^{\pm}$ as follows:

$$\varphi^+_i(w)F = F\varphi^+_i(w)\prod^{1\leq j\leq n}_{1\leq a\leq k_j}\frac{\zeta(w/z_{ja})}{\zeta(z_{ja}/w)} \tag{2.47}$$

$$\varphi^-_i(w)G = G\varphi^-_i(w)\prod^{1\leq j\leq n}_{1\leq a\leq k_j}\frac{\zeta(z_{ja}/w)}{\zeta(w/z_{ja})} \tag{2.48}$$

The Drinfeld pairing $\langle -,-\rangle : \mathcal{S}^{\leq}\otimes\mathcal{S}^{\geq}\to\mathbb{C}(q,t)$ between the positive and negative shuffle algebras $\mathcal{S}^{\geq,\neq}$ is given by:

$$\langle \varphi^-_i(z), \varphi^+_j(w)\rangle = \frac{\zeta(w/z)}{\zeta(z/w)} \tag{2.49}$$

$$\langle G, F\rangle = \frac{1}{\mathbf{k}!}\int^{|q|<1|p|}_{|z_{ia}|=1}\frac{G(\cdots, z_{ia}, \cdots)F(\cdots, z_{ia}, \cdots)}{\prod^{1\leq i,j\leq n}_{a\leq k_i, b\leq k_j}\zeta_p(z_{ia}/z_{jb})}\prod^{1\leq i\leq n}_{1\leq a\leq k_i}\frac{dz_{ia}}{2\pi i z_{ia}}|_{p\mapsto q} \tag{2.50}$$

for $F\in\mathcal{S}^+$, $G\in\mathcal{S}^-$. Here ζ_p is defined as follows:

$$\zeta_p(\frac{x_i}{x_j}) = \zeta(\frac{x_i}{x_j})\frac{[\frac{x_j}{p^2x_i}]^{\delta^i_j}}{[\frac{x_j}{q^2x_i}]^{\delta^i_j}} \tag{2.51}$$

This defines the shuffle algebra $\mathcal{S} := \mathcal{S}^{\leq}\hat{\otimes}\mathcal{S}^{\geq}$. Also the coproduct $\Delta : \mathcal{S} \to \mathcal{S}\hat{\otimes}\mathcal{S}$ is given by:

(2.52)
$$\Delta(\varphi_i^{\pm}(w)) = \varphi_i^{\pm}(w) \otimes \varphi_i^{\pm}(w)$$

(2.53)
$$\Delta(F) = \sum_{0\leq\mathbf{l}\leq\mathbf{k}} \frac{[\prod_{1\leq j\leq n}^{b>l_j} \varphi_j^{+}(z_{jb})]F(\cdots, z_{i1}, \cdots, z_{il_i} \otimes z_{i,l_i+1}, \cdots, z_{ik_i}, \cdots)}{\prod_{1\leq i\leq n}^{a\leq l_i}\prod_{1\leq j\leq n}^{b>l_j}\zeta(z_{jb}/z_{ia})}, \qquad F \in \mathcal{S}^{+}$$

(2.54)
$$\Delta(G) = \sum_{0\leq\mathbf{l}\leq\mathbf{k}} \frac{G(\cdots, z_{i1}, \cdots, z_{il_i} \otimes z_{i,l_i+1}, \cdots, z_{ik_i}, \cdots)[\prod_{1\leq j\leq n}^{b>l_j} \varphi_j^{-}(z_{jb})]}{\prod_{1\leq i\leq n}^{a\leq l_i}\prod_{1\leq j\leq n}^{b>l_j}\zeta(z_{ia}/z_{jb})}, \qquad G \in \mathcal{S}^{-}$$

In human language, the above coproduct formula means the following: For the right hand side of the formula, in the limit $|z_{ia}|<<|z_{jb}|$ for all $a \leq l_i$ and $b > l_i$, and then place all monomials in $\{z_{ia}\}_{a\leq l_i}$ to the left of the $\otimes$ symbol and all monomials in $\{z_{jb}\}_{b>l_j}$ to the right of the $\otimes$ symbol. Also we have the antipode map $S : \mathcal{S} \to \mathcal{S}$ which is an anti-homomorphism of both algebras and coalgebras:

$$S(\varphi_i^{+}(z)) = (\varphi_i^{+}(z))^{-1}, \qquad S(F) = [\prod_{1\leq a\leq k_i}^{1\leq i\leq n}(-\varphi_i^{+}(z_{ia}))^{-1}] * F \tag{2.55}$$

$$S(\varphi_i^{-}(z)) = (\varphi_i^{-}(z))^{-1}, \qquad S(G) = G * [\prod_{1\leq a\leq k_i}^{1\leq i\leq n}(-\varphi_i^{-}(z_{ia}))^{-1}] \tag{2.56}$$

The following theorem has been proved in [N15]:

Theorem 2.1. *There is a bigraded isomorphism of bialgebras*

$$\Upsilon : U_{q,t}(\hat{\hat{\mathfrak{sl}}}_n) \to \mathcal{S} \tag{2.57}$$

given by:

$$\Upsilon(\varphi_{i,d}^{\pm}) = \varphi_{i,d}^{\pm}, \qquad \Upsilon(e_{i,d}^{\pm}) = \frac{z_{i1}^{d}}{[q^{-2}]} \tag{2.58}$$

2.5.1. *Slope subalgebra of the shuffle algebra.* Here we first review the definition of the slope subalgebra of the quantum toroidal algebra $U_{q,t}(\hat{\hat{\mathfrak{sl}}}_n)$ in section 5 of [N15].

For the positive half of the shuffle algebra $\mathcal{S}^{+}$, we define the subspace of slope $\leq \mathbf{m}$ as:

$$\mathcal{S}_{\leq\mathbf{m}}^{+} := \{F \in \mathcal{S}^{+} | \lim_{\xi\to\infty} \frac{F(z_{i1}, \cdots, z_{ia_i}, \xi z_{i,a_i+1}, \cdots, \xi z_{i,a_i+b_i})}{\xi^{\mathbf{m}\cdot\mathbf{b}-\frac{(\mathbf{a},\mathbf{b})}{2}}} < \infty\ \forall \mathbf{a}, \mathbf{b}\} \tag{2.59}$$

Similarly, for the negative half:

$$\mathcal{S}^{-}_{\leq \mathbf{m}} = \{G \in \mathcal{S}^{-} \mid \lim_{\xi \to 0} \frac{G(\xi z_{i1}, \cdots, \xi z_{ia_i}, z_{i,a_i+1}, \cdots, z_{i,a_i+b_i})}{\xi^{-\mathbf{m}\cdot\mathbf{a} + \frac{(\mathbf{a},\mathbf{b})}{2}}} < \infty \ \forall \mathbf{a}, \mathbf{b}\} \tag{2.60}$$

It can be checked that for the elements $F \in \mathcal{S}^{+}_{\leq \mathbf{m}}$, $G \in \mathcal{S}^{-}_{\leq \mathbf{m}}$:

$$\Delta(F) = \Delta_{\mathbf{m}}(F) + (\text{anything}) \otimes (\text{slope} < \mathbf{m}), \qquad \Delta_{\mathbf{m}}(F) = \sum_{\mathbf{a}+\mathbf{b}=\mathbf{k}} \lim_{\xi \to \infty} \frac{\Delta(F)_{\mathbf{a}\otimes\mathbf{b}}}{\xi^{\mathbf{m}\cdot\mathbf{b}}} \tag{2.61}$$

$$\Delta(G) = \Delta_{\mathbf{m}}(G) + (\text{slope} < \mathbf{m}) \otimes (\text{anything}), \qquad \Delta_{\mathbf{m}}(G) = \sum_{\mathbf{a}+\mathbf{b}=\mathbf{k}} \lim_{\xi \to \infty} \frac{\Delta(G)_{\mathbf{a}\otimes\mathbf{b}}}{\xi^{-\mathbf{m}\cdot\mathbf{a}}} \tag{2.62}$$

Thus now the slope subalgebra $\mathcal{B}^{\pm}_{\mathbf{m}}$ is defined as:

$$\mathcal{B}^{\pm}_{\mathbf{m}} := \bigoplus_{(\mathbf{k},d) \in \mathbb{N}^n \times \mathbb{Z}}^{\mathbf{m}\cdot\mathbf{k}=d} \mathcal{S}^{\pm}_{\leq \mathbf{m}} \cap \mathcal{S}^{\pm}_{\pm\mathbf{k},d} \tag{2.63}$$

It is easy to see that the coproduct $\Delta_{\mathbf{m}}$ descends to a true coproduct on $\mathcal{B}^{\geq,\leq}_{\mathbf{m}} := \mathcal{B}^{\pm}_{\mathbf{m}} \otimes \mathbb{F}[\varphi_i^{\pm 1}]$. Using the Drinfeld double defined for the shuffle algebra. We eventually obtain the slope subalgebra $\mathcal{B}_{\mathbf{m}} \subset \mathcal{S}$.

It is known that the slope subalgebra $\mathcal{B}_{\mathbf{m}}$ for the quantum toroidal algebra $U_{q,t}(\widehat{\widehat{\mathfrak{sl}}}_n)$ is isomorphic to:

$$\mathcal{B}_{\mathbf{m}} \cong \bigotimes_{h=1}^{g} U_q(\widehat{\mathfrak{gl}}_{l_h}) \tag{2.64}$$

For the proof of the above isomorphism 2.64 see [N15]. In the subsection 4.3 we will give a simple rule of determining l_h and g . In this section we use the generators of the slope subalgebra $\mathcal{B}_{\mathbf{m}}$ as the definition of the slope subalgebra.

Let us define the generators of the slope algebra $\mathcal{B}_{\mathbf{m}}$. For $\mathbf{m} \cdot [i;j\rangle \in \mathbb{Z}$, we denote the following elements:

$$P^{\pm\mathbf{m}}_{\pm[i;j)} = \mathrm{Sym}\left[\frac{\prod_{a=i}^{j-1} z_a^{\lfloor m_i+\ldots+m_a \rfloor - \lfloor m_i+\ldots+m_{a-1} \rfloor}}{t^{\mathrm{ind}\, d^{m}_{[i,j\rangle} q^{i-j}} \prod_{a=i+1}^{j-1}\left(1 - \frac{q_2 z_a}{z_{a-1}}\right)} \prod_{i \leq a < b < j} \zeta\left(\frac{z_b}{z_a}\right)\right] \tag{2.65}$$

$$Q^{\pm\mathbf{m}}_{\mp[i;j)} = \mathrm{Sym}\left[\frac{\prod_{a=i}^{j-1} z_a^{\lfloor m_i+\ldots+m_{a-1} \rfloor - \lfloor m_i+\ldots+m_a \rfloor}}{t^{-\mathrm{ind}^{m}_{[i,j)}} \prod_{a=i+1}^{j-1}\left(1 - \frac{q_1 z_{a-1}}{z_a}\right)} \prod_{i \leq a < b < j} \zeta\left(\frac{z_a}{z_b}\right)\right] \tag{2.66}$$

Here $\mathrm{ind}^{\mathbf{m}}_{[i;j)}$ is defined as:

$$\mathrm{ind}^{\mathbf{m}}_{[i;j)}=\sum_{a=i}^{j-1}(m_i+\cdots+m_a-\lfloor m_i+\cdots+m_{a-1}\rfloor) \tag{2.67}$$

It is proved in [N15] that the positive and negative part of the slope subalgebra $\mathscr{B}^{\pm}_{\mathbf{m}}$ for the shuffle algebra $\mathcal{S}^{\pm}$ are generated by $\{P^{\mathbf{m}}_{\pm[i;j)}\}_{i\leq j}$ and $\{Q^{\mathbf{m}}_{\pm[i;j)}\}_{i\leq j}$. And the slope subalgebra $\mathscr{B}_{\mathbf{m}}$ is the similarly the Drinfeld double of $\mathscr{B}^{\pm}_{\mathbf{m}}$ with the neutral elements $\{\varphi_{\pm[i;j)}\}$.

We can check that the antipode map $S_{\mathbf{m}}:\mathscr{B}_{\mathbf{m}}\to\mathscr{B}_{\mathbf{m}}$ has the following relation:

$$S_{\mathbf{m}}(P^{\mathbf{m}}_{[i;j)})=Q^{\mathbf{m}}_{[i;j)},\qquad S_{\mathbf{m}}(Q^{\mathbf{m}}_{-[i;j)})=P^{\mathbf{m}}_{-[i;j)} \tag{2.68}$$

$$\Delta_{\mathbf{m}}\left(P^{\mathbf{m}}_{[i;j)}\right)=\sum_{a=i}^{j}P^{\mathbf{m}}_{[a;j)}\varphi_{[i;a)}\otimes P^{\mathbf{m}}_{[i;a)}\quad \Delta_{\mathbf{m}}\left(Q^{\mathrm{m}}_{[i;j)}\right)=\sum_{a=i}^{j}Q^{\mathbf{m}}_{[i;a)}\varphi_{[a;j)}\otimes Q^{\mathbf{m}}_{[a;j)} \tag{2.69}$$

$$\Delta_{\mathbf{m}}\left(P^{\mathbf{m}}_{-[i;j)}\right)=\sum_{a=i}^{j}P^{\mathbf{m}}_{-[a;j)}\otimes P^{\mathbf{m}}_{-[i;a)}\varphi_{-[a;j)}\quad \Delta_{\mathbf{m}}\left(Q^{\mathbf{m}}_{-[i;j)}\right)=\sum_{a=i}^{j}Q^{\mathbf{m}}_{-[i;a)}\otimes Q^{\mathbf{m}}_{-[a;j)}\varphi_{-[i;a)} \tag{2.70}$$

Here $\Delta_{\mathbf{m}}$ is defined as (5.27) and (5.28) in [N15]. The isomorphism 2.64 is given by:

$$e_{[i;j)}=P^{\mathbf{m}}_{[i;j)_h},\qquad e_{-[i;j)}=Q^{\mathbf{m}}_{-[i;j)_h},\qquad \varphi_k=\varphi_{[k;v_{\mathbf{m}}(k))} \tag{2.71}$$

Here we show the explicit formula for the simplest primitive elements of $\mathscr{B}_{\mathbf{m}}$ for $\mathbf{m}=\mu\boldsymbol{\theta}$. Let $k=(j-i)\mu$ and we assume that for $0\leq n\leq j-i-1$, $\mu n\notin\mathbb{Z}$, in this case we have

$$\Delta_{\mathbf{m}}(P^{\mathbf{m}}_{[i;j)})=P^{\mathbf{m}}_{[i;j)}\otimes 1+\varphi_{[i;j)}\otimes P^{\mathbf{m}}_{[i;j)},\qquad \Delta_{\mathbf{m}}(Q^{\mathbf{m}}_{[i;j)})=Q^{\mathbf{m}}_{[i;j)}\otimes 1+\varphi_{[i;j)}\otimes Q^{\mathbf{m}}_{[i;j)} \tag{2.72}$$

$$\Delta_{\mathbf{m}}(P^{\mathbf{m}}_{-[i;j)})=P^{\mathbf{m}}_{-[i;j)}\otimes\varphi_{[i;j)}+1\otimes P^{\mathbf{m}}_{[i;j)},\qquad \Delta_{\mathbf{m}}(Q^{\mathbf{m}}_{-[i;j)})=Q^{\mathbf{m}}_{-[i;j)}\otimes\varphi_{[i;j)}+1\otimes Q^{\mathbf{m}}_{[i;j)} \tag{2.73}$$

Now we take $j=i+l\boldsymbol{\theta}$. Since $\varphi_{[i;i+l\boldsymbol{\theta})}=\varphi_1^l\cdots\varphi_n^l=c^l$, $P^{\mathbf{m}}_{\pm[i;j)}$ are primitive elements. Given $\mu=p/q$ with $\gcd(p,q)=1$, the first condition required for primitivity is that $ln\leq q$. In this way we can find some primitive elements in the generators $P^{\mathbf{m}}_{\pm[i;j)}$.

It is worth noting that there is no obvious formula for the Heisenberg operators $U_q(\hat{\mathfrak{gl}}_1)$ in $U_q(\hat{\mathfrak{gl}}_n)$.

Remark. The original motivation for the definition of the slope subalgebra $\mathscr{B}_{\mathbf{m}}$ can be thought as the analog of the root subalgebra $U_q(\mathfrak{g}_{\mathbf{m}})$ defined by the K-theoretic stable envelope in [OS22]. And it is conjectured that $\mathscr{B}_{\mathbf{m}}$ and $U_q(\mathfrak{g}_{\mathbf{m}})$ are isomorphic as Hopf algebras. In subsection 2.10 we will show that $\mathscr{B}_{\mathbf{m}}$ is a Hopf subalgebra in $U_q(\mathfrak{g}_{\mathbf{m}})$.

2.6. **Other coproduct structure on $U_{q,t}(\hat{\hat{\mathfrak{sl}}}_n)$.** In[N15], Negut proved that there is a factorization of $U_{q,t}(\hat{\hat{\mathfrak{sl}}}_n)$ into the product of the slope subalgebras:

$$U_{q,t}(\hat{\hat{\mathfrak{sl}}}_n) = \bigotimes_{m\in\mathbb{Q}}^{\rightarrow} \mathscr{B}^{+}_{s+m\boldsymbol{\theta}} \otimes \mathscr{B}_{\infty\boldsymbol{\theta}} \otimes \bigotimes_{m\in\mathbb{Q}}^{\rightarrow} \mathscr{B}^{-}_{s+m\boldsymbol{\theta}} \tag{2.74}$$

which preserves the Hopf pairing for arbitrary $s \in \mathbb{Q}^n$ and $\boldsymbol{\theta} \in (\mathbb{Q}^{+})^n$. Each slope subalgebra $\mathscr{B}_{\mathbf{m}}$ has the following isomorphic of Hopf algebras

$$\mathscr{B}_{\mathbf{m}} \cong \bigotimes_{h=1}^{g^{(m)}} U_q(\hat{\mathfrak{gl}}_{l_h^{(n)}}) \tag{2.75}$$

And $\mathscr{B}_{\infty\cdot\boldsymbol{\theta}} = \left\langle \varphi^{\pm}_{i,d} \right\rangle^{d\in\mathbb{N}_0}_{1\leq i\leq n}$ is the subalgebra of Cartan elements. This factorization leads to the following KT-type factorization formula for the universal R-matrix of $U_{q,t}(\hat{\hat{\mathfrak{sl}}}_r)$:

$$R^{\infty} = \prod_{m\in\mathbb{Q}}^{\rightarrow} (\hat{R}_{s+m\boldsymbol{\theta}})R_{\infty} = (\prod_{m\in\mathbb{Q}}^{\rightarrow}\prod_{h=1}^{g^{(m)}} \hat{R}_{U_q(\hat{\mathfrak{gl}}_{l_h^{(m)}})}) \cdot R^{\otimes n}_{\text{Heisenberg}} \tag{2.76}$$

For each $R_{s+m\boldsymbol{\theta}}$ it can be factorized as $\hat{R}_{s+m\boldsymbol{\theta}}\hat{K}$ for $\hat{K}$ the diagonal part, and we call $\hat{R}_{s+m\boldsymbol{\theta}}$ the **reduced part of the universal R-matrix**.

Now we choose arbitrary $\mathbf{m} \in \mathbb{Q}^n$, we can twist the coproduct by:

$$\Delta_{(\mathbf{m})}(a) = [\prod_{r\in\mathbb{Q}_{>0}\cup\{\infty\}}^{\rightarrow} \hat{R}_{\mathbf{m}+r\boldsymbol{\theta}}] \cdot \Delta(a) \cdot [\prod_{r\in\mathbb{Q}_{>0}\cup\{\infty\}}^{\rightarrow} \hat{R}_{\mathbf{m}+r\boldsymbol{\theta}}]^{-1} \tag{2.77}$$

Now we prove the following important proposition:

Proposition 2.2. $\Delta_{(\mathbf{m})}(a) = \Delta_{\mathbf{m}}(a)$ *when* $a \in \mathscr{B}_{\mathbf{m}}$.

Proof. This proposition actually comes from a much more general statement for the Drinfeld double of the quasi-triangular Hopf algebra:

Theorem 2.3. *Let $A = A^{\geq} \otimes A^{\leq}$ be the Drinfeld double with respect to the nondegenerate bilinear pairing $\langle -,-\rangle : A^{\geq} \otimes A^{\leq} \to \mathbb{C}$. Now suppose that $A^{\geq,\leq} \cong \bigotimes_{i\in I} \mathscr{B}_i^{\geq,\leq}$ and $A^{\pm} \cong \bigotimes^{\rightarrow}_{i\in I} \mathscr{B}_i^{\pm}$ the Hopf factorization as a vector space with respect to the bilinear pairing of $\langle -,-\rangle_i : \mathscr{B}_i^{\geq} \otimes \mathscr{B}_i^{\leq} \to \mathbb{C}$ and such that the elements in $A^{\geq,\leq}$ can be written as $\prod_{i\in I} a_i$, and here $\mathscr{B}_i^{\geq,\leq} = \mathscr{B}_i^{\pm} \otimes Z_0^{\pm}$ with an ordering on I with a maximal element u. $Z_0^{\pm} \subset Z_{\pm}$ are the elements such that $\epsilon(a) = 1$ for the generators of $Z_0^{\pm}$. Then $\Delta(a) = \Delta_u(a)$ for $a \in \mathscr{B}_u$.*

Proof. For each $\mathscr{B}_i^{\pm}$ one can write down the orthogonal basis and dual basis $\{F_{(i)}^m\},\{G_{(i)}^m\}$, here $m\in S_i$ and S_i is the set of basis for $\mathscr{B}_i^{\pm}$. And using these basis, one could write down the universal R-matrix for $\mathscr{B}_i$ as:

$$R_i=\sum_{m\in S_i}F_{(i)}^m\otimes G_{(i)}^m \tag{2.78}$$

And here we denote $m=0$ for the basis $1\otimes 1$.

Now via the factorization $A^{\pm}\cong\bigotimes_{i\in I}^{\rightarrow}\mathscr{B}_i^{\pm}$, we have the factorization for the universal R-matrix for A:

$$R=\prod_{i\in I}^{\rightarrow}R_i \tag{2.79}$$

And also via the factorization, we have that $\{\prod_{i\in I}F_{(i)}^{m_i}\}$ and $\{\prod_{i\in I}G_{(i)}^{m_i}\}$ is a basis for $A^{\pm}$. Since u is the maximal element in I, we will denote the basis by $\prod_{i\in I}F_{(i)}^{m_i}F_{(u)}^{m_u}$ and $\prod_{i\in I}G_{(i)}^{m_i}G_{(u)}^{m_u}$.

To prove the statement, we need to use the following formula:

$$(\Delta\otimes 1)R=R_{13}R_{23},\qquad (1\otimes\Delta)R=R_{12}R_{13} \tag{2.80}$$

The first formula is used to construct the coproduct for A^+, and the second is used to construct the coproduct for A^-. For simplicity, we will only show the computation for A^+. The computation for A^- is completely similar.

Now we write down the product $R_{13}R_{23}$ in the explicit form:

$$\begin{aligned}R_{13}R_{23}&=\sum_{\substack{m_i,n_i\in S_i\\ i\in I}}(\prod_{i\in I}F_{(i)}^{m_i}F_{(u)}^{m_u}\otimes 1\otimes\prod_{i\in I}G_{(i)}^{m_i}G_{(u)}^{m_u})(1\otimes\prod_{i\in I}F_{(i)}^{n_i}F_{(u)}^{n_u}\otimes\prod_{i\in I}G_{(i)}^{n_i}G_{(u)}^{n_u})\\&=\sum_{\substack{m_i,n_i\in S_i\\ i\in I}}\prod_{i\in I}F_{(i)}^{m_i}F_{(u)}^{m_u}\otimes\prod_{i\in I}F_{(i)}^{n_i}F_{(u)}^{n_u}\otimes\prod_{i\in I}G_{(i)}^{m_i}G_{(u)}^{m_u}\prod_{i\in I}G_{(i)}^{n_i}G_{(u)}^{n_u})\\&=\sum_{\substack{m_i,n_i,l_i\in S_i\\ i\in I}}\prod_{i\in I}F_{(i)}^{m_i}F_{(u)}^{m_u}\otimes\prod_{i\in I}F_{(i)}^{n_i}F_{(u)}^{n_u}\otimes\prod_{i\in I}G_{(i)}^{m_i}G_{(u)}^{m_u}\prod_{i\in I}G_{(i)}^{n_i}G_{(u)}^{n_u})\\&=\sum_{\substack{m_i,n_i,l_i\in S_i\\ i\in I}}a_{\mathbf{m},\mathbf{n}}^{\mathbf{l}}\prod_{i\in I}F_{(i)}^{m_i}F_{(u)}^{m_u}\otimes\prod_{i\in I}F_{(i)}^{n_i}F_{(u)}^{n_u}\otimes\prod_{i\in I}G_{(i)}^{l_i}G_{(u)}^{l_u}\end{aligned} \tag{2.81}$$

Here

$$a_{\mathbf{m},\mathbf{n}}^{\mathbf{l}}=\langle\prod_{i\in I}F_{(i)}^{l_i}F_{(u)}^{l_u},\prod_{i\in I}G_{(i)}^{m_i}G_{(u)}^{m_u}\prod_{i\in I}G_{(i)}^{n_i}G_{(u)}^{n_u})\rangle \tag{2.82}$$

And the coproduct $(\Delta\otimes 1)R$ can be written as:

$$(\Delta\otimes 1)R=\sum_{\substack{l_i\in S_i\\ i\in I}}\Delta(\prod_{i\in I}F_{(i)}^{l_i}F_{(u)}^{l_u})\otimes\prod_{i\in I}G_{(i)}^{l_i}G_{(u)}^{l_u} \tag{2.83}$$

This gives the general coproduct formula:

$$\Delta(\prod_{i\in I}F_{(i)}^{l_i}F_{(u)}^{l_u})=\sum_{\substack{m_i,n_i\in S_i\\ i\in I}}a_{\mathbf{m},\mathbf{n}}^{\mathbf{l}}\prod_{i\in I}F_{(i)}^{m_i}F_{(u)}^{m_u}\otimes\prod_{i\in I}F_{(i)}^{n_i}F_{(u)}^{n_u} \tag{2.84}$$

Now we compare the coproduct for $G_{(u)}^{m_u}$, note that:

$$\begin{aligned}&\langle F_{(u)}^{l_u},\prod_{i\in I}G_{(i)}^{m_i}G_{(u)}^{m_u}\prod_{i\in I}G_{(i)}^{n_i}G_{(u)}^{n_u})\rangle\\ =&\sum_{r_i\in S_i,i\in I}a_{\mathbf{m},\mathbf{n}}^{\mathbf{r}}\langle F_{(u)}^{l_u},\prod_{i\in I}G_{(i)}^{r_i}G_{(u)}^{r_u}\rangle\\ =&\sum_{r_i\in S_i,i\in I}a_{\mathbf{m},\mathbf{n}}^{\mathbf{r}}\prod_{i\in I}\langle 1,G_{(i)}^{r_i}\rangle\langle F_{(u)}^{l_u},G_{(u)}^{r_u}\rangle\\ =&\sum_{r_i\in S_i,i\in I}a_{\mathbf{m},\mathbf{n}}^{\mathbf{r}}\delta_{l_u,r_u}\prod_{i\in I}\epsilon(G_{(i)}^{r_i})\end{aligned} \tag{2.85}$$

And the product is nonzero if and only if $G_{(i)}^{m_i}$ are all 1 or in Z_0^-, and the elements in Z_0^- is still in $\mathscr{B}_i$, hence this requires that $G_{(i)}^{m_i}=G_{(i)}^{n_i}=1$ and the coproduct formula only contains the element in $\mathscr{B}_i$, and it can be written as:

$$\Delta(F_{(u)}^{l_u})=\sum_{m_u,n_u\in S_u}a_{m_u,n_u}^{l_u}F_{(u)}^{m_u}\otimes F_{(u)}^{n_u} \tag{2.86}$$

which coincides with the coproduct Δ_u of $\mathscr{B}_u$. □

Now back to our case, in this case the corresponding universal R-matrix is

$$R^{\mathbf{m}}=\overleftarrow{\prod_{r\in\mathbb{Q}_{>0}\cup\{\infty\}}}\hat{R}_{\mathbf{m}+r\boldsymbol{\theta}}^{-}\overrightarrow{\prod_{r\in\mathbb{Q}_{\leq 0}}}\hat{R}_{\mathbf{m}+r\boldsymbol{\theta}} \tag{2.87}$$

Here $\hat{R}_{\mathbf{m}}^{-}:=\hat{R}_{\mathbf{m}}^{21}$.

Thus we can see that in this order, the maximal element is $\mathbf{m}$, thus the proof is finished. □

A useful consequence for the universal R-matrix is the following:

Proposition 2.4. *Given $End(K_G(M(\mathbf{v}_1,\mathbf{w}_1))\otimes K_G(M(\mathbf{v}_2,\mathbf{w}_2))\subset End(K_G(M(\mathbf{w}_1))\otimes K_G(M(\mathbf{w}_2)))$, Fix a $s\in\mathbb{Q}^n$ and a compact subset in $s+\mathbb{Q}\boldsymbol{\theta}$ with $\boldsymbol{\theta}\in(\mathbb{Q}^+)^n$, there are only finitely many $R_{s+m\boldsymbol{\theta}}$ such that their reduced part acting on $End(K_G(M(\mathbf{v}_1,\mathbf{w}_1)\otimes K_G(M(\mathbf{v}_2,\mathbf{w}_2))$ which is nonzero.*

Proof. This follows from the definition of the shuffle algebra representation for each $\mathscr{B}_{\mathbf{m}}$, and the fact that $\mathrm{End}(K_G(M(\mathbf{v}_1,\mathbf{w}_1)\otimes K_G(M(\mathbf{v}_2,\mathbf{w}_2))$ is a finitely generated module over $K_G(pt)$. Also the reduced part of the universal R-matrix $R_{\mathbf{m}}$ of $\mathscr{B}_{\mathbf{m}}$ is expressed as:

$$R_{\mathbf{m}}=1\otimes 1+\sum_{\alpha}e_{[i;i+1)_{\mathbf{m}}}\otimes e_{-[i;i+1)_{\mathbf{m}}}+\cdots \tag{2.88}$$

And $e_{[i;i+1)_{\mathbf{m}}}$ sends $K_G(M(\mathbf{v}_1,\mathbf{w}_1))$ to $K_G(M(\mathbf{v}_1-[i;i+1)_{\mathbf{m}},\mathbf{w}_1))$. So as long as $\mathbf{m}$ is "complicated" enough, (i.e. $\mathbf{m}=(m_1,\cdots,m_n)$ is more complicated if it has the bigger greatest common multiple denominator of the rational numbers $(m_1,\cdots,m_n)$). $[i;i+1)_{\mathbf{m}}$ can have larger positive integers than $\mathbf{v}_1$ on each component. i.e. $R_{\mathbf{m}}$ acts on $K_G(M(\mathbf{v}_1,\mathbf{w}_1)\otimes K_G(M(\mathbf{v}_2,\mathbf{w}_2)$ trivially as $\mathbf{m}$ complicated enough.

On each compact subset of $s+\mathbb{Q}\boldsymbol{\theta}$, we fix a maximal number n of the common denominator of the rational point $\mathbf{m}=(m_1,\cdots,m_n)\in s+\mathbb{Q}\boldsymbol{\theta}$ and denote the corresponding set by N_n. Since the compact set is a bounded subset in the 1-dimensional vector space $s+\mathbb{Q}\boldsymbol{\theta}$, N_n is a finite set. Thus the lemma is proved. $\square$

This result can be seen as the definition of **wall set** for each quiver variety $M(\mathbf{v},\mathbf{w})$, which can be written as follows:

$$\mathrm{Walls}(M(\mathbf{v},\mathbf{w}))=\{\mathbf{m}\in\mathbb{Q}^n|\text{Reduced part of } R_{\mathbf{m}} \text{ acts on } K_T(M(\mathbf{v},\mathbf{w}))\otimes K_T(M(\mathbf{v},\mathbf{w})) \text{ trivially}\} \tag{2.89}$$

And from the above proposition, the intersection $\mathrm{Walls}(M(\mathbf{v},\mathbf{w}))\cap(s+\mathbb{Q}\boldsymbol{\theta})$ is a finite set for $\boldsymbol{\theta}\in(\mathbb{Q}^+)^r$.

For our situation, we choose the following coproduct:

$$\Delta_{(\mathbf{s})}(a)=\prod^{\rightarrow}_{r\in\mathbb{Q}_{\geq 0}}R^{-1}_{\mathbf{s}+r\boldsymbol{\theta}}\Delta_{\infty}(a)\prod^{\leftarrow}_{r\in\mathbb{Q}_{\geq 0}}R_{\mathbf{s}+r\boldsymbol{\theta}} \tag{2.90}$$

And here $\Delta_{\infty}:=R_{\infty}\Delta^{op}R_{\infty}^{-1}$. And the corresponding universal R-matrix for $\Delta_{(\mathbf{s})}$ is given by:

$$R^{\mathbf{s}}=\prod^{\leftarrow}_{r\in\mathbb{Q}_{<0}}R^{-}_{\mathbf{s}+r\boldsymbol{\theta}}\cdot R_{\infty}\cdot\prod^{\leftarrow}_{r\in\mathbb{Q}_{\geq 0}}R_{\mathbf{s}+r\boldsymbol{\theta}} \tag{2.91}$$

Here $R^{-}_{\mathbf{m}}:=(R_{\mathbf{m}})_{21}$. Now given two slopes s and s', choose a vector $\mathbb{Q}\boldsymbol{\theta}$ such that $s,s'\in\boldsymbol{\theta}$, define Γ to be the path from s to s' along $\mathbb{Q}\boldsymbol{\theta}$. Now for s and s', we denote:

$$T^{+}_{s,s'}=\prod^{\leftarrow}_{\mathbf{m}\in\Gamma}R^{+}_{\mathbf{m}},\qquad T^{-}_{s,s'}=\prod^{\rightarrow}_{\mathbf{m}\in\Gamma}R^{-1}_{\mathbf{m}} \tag{2.92}$$

And comparing the two formula, we have that:

Lemma 2.5. *The coproduct $\Delta_{(\mathbf{s})}$ and $\Delta_{(\mathbf{s}')}$ are related by the following formula*

$$\Delta_{(\mathbf{s})}(a)=T^{-}_{s,s'}\Delta_{(\mathbf{s}')}(a)T^{+}_{s,s'} \tag{2.93}$$

Moreover, if we restrict the element on the action of $End(K_T(\mathbf{v},\mathbf{w})\otimes K_T(\mathbf{v},\mathbf{w}))$, the number of the universal R-matrix in $T^{-}_{s,\mathbf{m}}$ is finite.

A useful result of this lemma would be the following:

Corollary 2.6. *Let $\mathscr{L}\in Pic(M(\mathbf{v},\mathbf{w}))$, then we have:*

$$\Delta_{\infty}(\mathscr{L})=R^{+}_{\mathbf{m}_{m-1}}\cdots R^{+}_{\mathbf{m}_0}\Delta_s(\mathscr{L}) \tag{2.94}$$

in $Hom(K(\mathbf{v},\mathbf{w})\otimes K(\mathbf{v},\mathbf{w}),K(\mathbf{w})\otimes K(\mathbf{w}))$. Here Δ_{∞} is the standard coproduct on $U_{q,t}(\hat{\hat{\mathfrak{sl}}}_n)$. $\mathbf{m}_0,\cdots,\mathbf{m}_{m-1}$ are the slope points between s and $s+\mathscr{L}$.

Proof. Note that $\mathscr{L}$ acts on $K_T(M(\mathbf{v},\mathbf{w}))$ as the Cartan element of $U_{q,t}(\hat{\hat{\mathfrak{sl}}}_n)$, and thus we have

$$\Delta_{\infty}(\mathscr{L})=\mathscr{L}\otimes\mathscr{L} \tag{2.95}$$

And use the formula, we have that:

$$\Delta_s(\mathscr{L})=(R^{+}_{\mathbf{m}_0})^{-1}\cdots(R^{+}_{\infty})^{-1}\Delta_{\infty}(\mathscr{L})(R^{+}_{\infty}\cdots R^{+}_{\mathbf{m}_0}) \tag{2.96}$$

From the proposition 3.9 we have that

$$\Delta_{\infty}(\mathscr{L})R^{+}_{\mathbf{m}_k}\Delta_{\infty}(\mathscr{L})^{-1}=R^{+}_{\mathbf{m}_k+\mathscr{L}}=R_{\mathbf{m}_{k+m}} \tag{2.97}$$

And thus obtain the result. □

Remark. In the context of our study, the wall set is defined as the point $\mathbf{m}\in\mathrm{Pic}(X)\otimes\mathbb{Q}$ whose corresponding reduced part of the universal R-matrix $R_{\mathbf{m}}\subset\mathscr{B}_{\mathbf{m}}\hat{\otimes}\mathscr{B}_{\mathbf{m}}$ act trivially on $K_T(M(\mathbf{v},\mathbf{w}))\otimes K_T(M(\mathbf{v},\mathbf{w}))$. This is different from the one in [OS22] where the definition of the walls are the real hyperplanes in $\mathrm{Pic}(X)\otimes\mathbb{C}$. Nonetheless, we expect that this definition will coincide with the set of walls depicted in [OS22] when these are intersected with any bounded intervals. We will discuss more about these structures with those in the stable envelope settings in the section 3 for the case of equivariant Hilbert scheme of A_r singularity $\mathrm{Hilb}_n([\mathbb{C}^2/\mathbb{Z}_{r+1}])$. We will do further study of the corresponding between these two definitions in our next series of papers.

2.7. Geometric action on $K_T(M(\mathbf{w}))$.

We first fix the notation for the Nakajima quiver varieties.

Given a finite quiver $Q=(I,E)$, consider the following quiver representation space

$$\mathrm{Rep}(\mathbf{v},\mathbf{w}):=\bigoplus_{h\in E}\mathrm{Hom}(V_{i(h)},V_{o(h)})\oplus\bigoplus_{i\in I}\mathrm{Hom}(V_i,W_i) \tag{2.98}$$

here $\mathbf{v}=(\dim(V_1),\cdots,\dim(V_I))$ is the dimension vector for the vector spaces at the vertex, $\mathbf{w}=(\dim(W_1),\cdots,\dim(W_I))$ is the dimension vector for the framing vector spaces.

Denote $G_{\mathbf{v}} := \prod_{i\in I} GL(V_i)$ and $G_{\mathbf{w}} := \prod_{i\in I} GL(W_i)$. There is a natural action of $G_{\mathbf{v}}$ and $G_{\mathbf{w}}$ on $\text{Rep}(\mathbf{v},\mathbf{w})$, which induces a natural Hamiltonian action on $T^*\text{Rep}(\mathbf{v},\mathbf{w})$ with respect to the standard symplectic form ω, now we have the moment map

$$\mu : T^*\text{Rep}(\mathbf{v},\mathbf{w}) \to \mathfrak{g}_{\mathbf{v}}^* \tag{2.99}$$

And thus with suitable stability condition θ, i.e. $\theta \in (\mathbb{Q}^+)^I$ one can define **Nakajima variety**:

$$M_{\theta,0}(\mathbf{v},\mathbf{w}) := \mu^{-1}(0)//_{\theta}G_{\mathbf{v}} \tag{2.100}$$

The Nakajima varieties also has a natural action $\mathbb{C}_q^{\times}$ which scales the cotangent fibre with a character q^{-1}.

We will abbreviate $M(\mathbf{v},\mathbf{w})$ as the Nakajima quiver variety defined in 2.100.

In this paper, we will mainly focus on the finite quiver $Q = (I, E)$ of affine type A, i.e. the corresponding space of quiver representation is defined as:

$$\text{Rep}(\mathbf{v},\mathbf{w}) = \bigoplus_{i\in\mathbb{Z}/n\mathbb{Z}} \text{Hom}(V_i, V_{i+1}) \oplus \bigoplus_{i\in I} \text{Hom}(V_i, W_i) \tag{2.101}$$

The stability condition is chosen as $\theta = (1, \cdots, 1)$.

The action of $\mathbb{C}_q^* \times \mathbb{C}_t^* \times G_{\mathbf{w}}$ on $M(\mathbf{v},\mathbf{w})$ is given by:

$$(q, t, U_i) \cdot (X_e, Y_e, A_i, B_i)_{e\in E, i\in I} = (qtX_e, qt^{-1}Y_e, qA_iU_i, qU_i^{-1}B_i) \tag{2.102}$$

The fixed points set of the torus action $\mathbb{C}_q^* \times \mathbb{C}_t^*$ on $M_{\theta,0}(\mathbf{v},\mathbf{w})$ is indexed by the $|\mathbf{w}|$-partitions $\boldsymbol{\lambda} = (\lambda_1, \cdots, \lambda_{\mathbf{w}})$. If we choose the cocharacter $\sigma : \mathbb{C}^* \to G_{\mathbf{w}}$ such that $\mathbf{w} = u_1\mathbf{w}_1 + \cdots + u_k\mathbf{w}_k$, we have that:

$$M(\mathbf{v},\mathbf{w})^{\sigma} = \bigsqcup_{\mathbf{v}_1+\cdots+\mathbf{v}_k=\mathbf{v}} M(\mathbf{v}_1,\mathbf{w}_1) \times \cdots \times M(\mathbf{v}_k,\mathbf{w}_k) \tag{2.103}$$

We denote:

$$K_T(M(\mathbf{w})) := \bigoplus_{\mathbf{w}} K_{\mathbb{C}_q^*\times\mathbb{C}_t^*\times G_{\mathbf{w}}}(M(\mathbf{v},\mathbf{w}))_{loc} \tag{2.104}$$

Thus we can choose the fixed point basis $|\boldsymbol{\lambda}\rangle$ of the corresponding partition $\boldsymbol{\lambda}$ to span the vector space $K_T(M(\mathbf{w}))$.

Here we briefly review the geometric action of $U_{q,t}(\hat{\hat{\mathfrak{sl}}}_r)$ on $K_T(M(\mathbf{w}))$. The construction is based on Nakajima's simple correspondence.

Consider a pair of vectors $(\mathbf{v}_+, \mathbf{v}_-)$ such that $\mathbf{v}_+ = \mathbf{v}_- + \mathbf{e}_i$. There is a simple correspondence

$$Z(\mathbf{v}_+, \mathbf{v}_-, \mathbf{w}) \hookrightarrow M(\mathbf{v}_+, \mathbf{w}) \times M(\mathbf{v}_-, \mathbf{w}) \tag{2.105}$$

parametrises pairs of quadruples $(X^{\pm}, Y^{\pm}, A^{\pm}, B^{\pm})$ that respect a fixed collection of quotients $(V^{+} \to V^{-})$ of codimension δ^i_j with only the semistable and zeros part for the moment map μ for each $M(\mathbf{v}_+, \mathbf{w})$. The variety $Z(\mathbf{v}_+, \mathbf{v}_-, \mathbf{w})$ is smooth with a tautological line bundle:

$$\mathscr{L}|_{V^+ \to V^-} = \mathrm{Ker}(V_i^+ \to V_i^-) \tag{2.106}$$

There are natural projection maps:

$$\begin{tikzcd} & Z(\mathbf{v}_+, \mathbf{v}_-, \mathbf{w}) \arrow[dl, "\pi_+"'] \arrow[dr, "\pi_-"] & \\ M(\mathbf{v}_+, \mathbf{w}) & & M(\mathbf{v}_-, \mathbf{w}) \end{tikzcd} \tag{2.107}$$

Using these (π_+, π_-) we could consturct the operator:

(2.108)
$$e^{\pm}_{i,d} : K_T(M(\mathbf{v}^{\mp}, \mathbf{w})) \to K_T(M(\mathbf{v}^{\pm}, \mathbf{w})), \qquad e^{\pm}_{i,d}(\alpha) = \pi_{\pm *}(\mathrm{Cone}(d\pi_{\pm})\mathscr{L}^d \pi^*_{\mp}(\alpha))$$

Here $\mathrm{Cone}(d\pi_{\pm})$ is defined in [N15]. Take all $\mathbf{v}$ we have the operator $e^{\pm}_{i,d} : K_T(M(\mathbf{w})) \to K_T(M(\mathbf{w}))$. Also we have the action of $\varphi^{\pm}_{i,d}$ given by the multiplication of the tautological class, which means that:

$$\varphi^{\pm}_i(z) = \overline{\frac{\zeta(\frac{z}{X})}{\zeta(\frac{X}{z})}} \prod_{1 \le j \le \mathbf{w}}^{u_j \equiv i} \frac{[\frac{u_j}{\hbar z}]}{[\frac{z}{\hbar u_j}]} \tag{2.109}$$

In particular, the element $\varphi_{i,0}$ acts on $K_T(M(\mathbf{v}, \mathbf{w}))$ as $q^{\alpha_i^T(\mathbf{w} - C\mathbf{v})}$.

The above construction gives the following well-known result:

Theorem 2.7. *For all $\mathbf{w} \in \mathbb{N}^r$, the operator $e^{\pm}_{i,d}$ and $\varphi^{\pm}_{i,d}$ give rise to an action of $U_{q,t}(\hat{\hat{\mathfrak{sl}}}_r)$ on $K_T(M(\mathbf{w}))$.*

In terms of the shuffle algebra, we can give the explicit formula of the action of the quantum toroial algebra $U_{q,t}(\hat{\hat{\mathfrak{sl}}}_n)$ on $K_T(M(\mathbf{w}))$:

Given $F \in \mathcal{S}^+_{\mathbf{k}}$, we have that

$$\langle \boldsymbol{\lambda} | F | \boldsymbol{\mu} \rangle = F(\chi_{\boldsymbol{\lambda} \backslash \boldsymbol{\mu}}) \prod_{\blacksquare \in \boldsymbol{\lambda} \backslash \boldsymbol{\mu}} [\prod_{\square \in \boldsymbol{\mu}} \zeta(\frac{\chi_\blacksquare}{\chi_\square}) \prod_{i=1}^{\mathbf{w}} [\frac{u_i}{q\chi_\blacksquare}]] \tag{2.110}$$

Similarly, for $G \in \mathcal{S}^-_{-\mathbf{k}}$, we have

$$\langle \boldsymbol{\mu} | G | \boldsymbol{\lambda} \rangle = G(\chi_{\boldsymbol{\lambda} \backslash \boldsymbol{\mu}}) \prod_{\blacksquare \in \boldsymbol{\lambda} \backslash \boldsymbol{\mu}} [\prod_{\square \in \boldsymbol{\lambda}} \zeta(\frac{\chi_\square}{\chi_\blacksquare}) \prod_{i=1}^{\mathbf{w}} [\frac{\chi_\blacksquare}{q u_i}]]^{-1} \tag{2.111}$$

The following theorem might be useful for expressing the quantum algebra operators in terms of the geometric objects:

Theorem 2.8. *There is an algebra injection*

$$U_{q,t}(\hat{\hat{\mathfrak{sl}}}_n) \hookrightarrow \prod_{\mathbf{w}} End(K_T(M(\mathbf{w}))) \tag{2.112}$$

Proof. We will give a sketch of the proof, for the detail see [N23].

Choose the following factorisation:

$$U_{q,t}(\hat{\hat{\mathfrak{sl}}}_n) \cong \overset{\rightarrow}{\bigotimes_{\mu\in\mathbb{Q}\sqcup\infty}} \mathscr{B}_{\mu\boldsymbol{\theta}}, \qquad \boldsymbol{\theta} = (1,1,\cdots,1) \tag{2.113}$$

Thus elements in $U_{q,t}(\hat{\hat{\mathfrak{sl}}}_r)$ can be written as:

$$\prod_{\mu\in\mathbb{Q}} a_\mu * b * \prod_{\mu\in\mathbb{Q}} b_\mu, \qquad b\in\mathscr{B}_\infty, a_\mu\in\mathscr{B}^+_{\mu\boldsymbol{\theta}}, b_\mu\in\mathscr{B}^-_{\mu\boldsymbol{\theta}} \tag{2.114}$$

such that:

$$\mathscr{B}_{b/a\boldsymbol{\theta}} \cong U_q(\hat{\mathfrak{gl}}_{n/g})^{\otimes g}, \qquad g := \gcd(a,b) \tag{2.115}$$

For each $\mathscr{B}_{b/a}$, it has the generator $P^{\mu\boldsymbol{\theta}}_{[i;j)}$, $Q^{\mu\boldsymbol{\theta}}_{-[i;j)}$ and $\varphi_{[i;j)}$, and the PBW basis for $\mathscr{B}_{b/a}$ is based on the following factorization:

$$\mathscr{B}_{b/a\boldsymbol{\theta}} \cong \mathscr{B}^+_{b/a\boldsymbol{\theta}}\hat{\otimes}\mathscr{B}^0_{b/a\boldsymbol{\theta}}\hat{\otimes}\mathscr{B}^-_{b/a\boldsymbol{\theta}} \tag{2.116}$$

And thus can be written as:

(2.117)

$$P^{\mu\boldsymbol{\theta}}_{[i_1;i_1+l_1)} * P^{\mu\boldsymbol{\theta}}_{[i_2;i_2+l_2)} * \cdots * P^{\mu\boldsymbol{\theta}}_{[i_d;i_d+l_d)} * \varphi * Q^{\mu\boldsymbol{\theta}}_{-[j_1;j_1+r_1)} * Q^{\mu\boldsymbol{\theta}}_{-[j_2;j_2+r_2)} * \cdots * Q^{\mu\boldsymbol{\theta}}_{-[j_n;j_n+r_n)}$$

with φ being the product of φ_i.

Note that the elements of the form $Q^{\mu\boldsymbol{\theta}}_{-[j_1;j_1+r_1)}$ and φ would not annihlate the vector $|\lambda\rangle$. Thus it remains to prove that for the element of the form:

$$\prod_{\mu\in\mathbb{Q}}\prod_{k=1}^{n_\mu} P^{\mu\boldsymbol{\theta}}_{[i_1;i_1+l_1)} \tag{2.118}$$

which has finitely many terms in the product. Fix the element written as above, there exists a vector $|\lambda\rangle \in K_T(M(\mathbf{v},\mathbf{w}))$ such that $\mathbf{v},\mathbf{w}$ is large enough that would not vanish after the action of the element. This is obvious since the vector $|\lambda\rangle$ is labelled by the partition, and all we have to do is just to make sure that the partition is large enough. This finishes the proof. $\square$

2.8. **Khoroshkin-Tolstoy construction of R-matrix.** The Khoroshkin-Tolstoy construction[KT92] of universal R-matrix is, roughly speaking, a direct generalization of the method in [KT91] via consturcting the Cartan-Weyl basis with respect to the root system of the corresponding quantum group. The Cartan-Weyl basis can be constructed via the q-Weyl group method in [KR90].

We recommend the reference [BGKNV10] as a good reference for an in-depth and comprehensive treatment of the universal R-matrix for the quantum affine algebra $U_q(\hat{\mathfrak{g}})$.

Here we use the notation in [BGKNV10] in the following subsection.

The formula for the universal R-matrix of $U_q(\hat{\mathfrak{g}})$ is given by:

$$R = R_{<\delta} R_{\cong\delta} R_{>\delta} K \tag{2.119}$$

Here:

$$R_{<\delta} = \prod_{m\geq 0}^{\rightarrow} \prod_{\gamma\in\Delta(A)}^{\leftarrow} \exp_{q^{-(\gamma,\gamma)}}((q-q^{-1}) s_{m,\gamma}^{-1} e_{\gamma+m\delta} \otimes f_{\gamma+m\delta}) \tag{2.120}$$

$$R_{\cong\delta} = \exp((q-q^{-1}) \sum_{m\in\mathbb{Z}_+} \sum_{i,j=1}^{r} u_{m,ij} e_{m\delta,\alpha_i} \otimes f_{m\delta,\alpha_j}) \tag{2.121}$$

$$R_{>\delta} = \prod_{m\geq 0}^{\rightarrow} \prod_{\gamma\in\Delta(A)}^{\rightarrow} \exp_{q^{-(\gamma,\gamma)}}((q-q^{-1}) s_{m,\delta-\gamma}^{-1} e_{(\delta-\gamma)+m\delta} \otimes f_{(\delta-\gamma)+m\delta}) \tag{2.122}$$

And here $\Delta(A)$ are the set of positive roots for $\mathfrak{g}$, and $\delta = \alpha_0 + \cdots + \alpha_n$ is the imaginary roots. e_α and f_α are the Cartan-Weyl basis constructed in [KT92].

2.9. **R-matrix presentation in the geometric moudle.** In this subsection we prove the Laurent polynomiality of the R-matrix in each module $K_{T\times\mathbb{C}_q^\times}(M(\mathbf{w}))$.

Proposition 2.9. *Given $K_T(pt) := \mathbb{C}[u_i^{\pm 1}]_{i\in I}$, and we set the equivariant parametre of $M(\mathbf{w}_1)$ and $M(\mathbf{w}_2)$ be u_1 and u_2. Then the matrix coefficients of the representation of $R_{\mathbf{m}} \in \mathscr{B}_{\mathbf{m}} \hat{\otimes} \mathscr{B}_{\mathbf{m}}$ in $End(K_{T\times\mathbb{C}_q^\times}(M(\mathbf{w}_1))_{loc} \otimes K_{T\times\mathbb{C}_q^\times}(M(\mathbf{w}_2)))_{loc}$ is a Laurent monomial in u_1/u_2, with coefficients in $\mathbb{Q}(q,t)$. Moreover, the natural map:*

$$\mathscr{B}_{\mathbf{m}} \hookrightarrow \prod_{\mathbf{w}} End(K_{T\times\mathbb{C}_q^\times}(M(\mathbf{w}))_{loc})[u_i^{\pm}]_{i\in I}$$

is injective

Proof. For the first statement, fix two quiver varieties $M(\mathbf{w}_1)$ and $M(\mathbf{w}_2)$ and the equivariant parametre torus on $M(\mathbf{w}_1)$ and $M(\mathbf{w}_2)$ to be u_1 and u_2, and fix the representation:

$$(\pi_1 \otimes \pi_2) : \mathscr{B}_{\mathbf{m}} \hat{\otimes} \mathscr{B}_{\mathbf{m}} \to \mathrm{End}(K_{T\times\mathbb{C}_q^\times}(M(\mathbf{w}_1))_{loc} \otimes K_{T\times\mathbb{C}_q^\times}(M(\mathbf{w}_2))_{loc}) \tag{2.123}$$

Now using the KT factorization 2.119 and the isomorphism of $\mathscr{B}_{\mathbf{m}}$ in 2.64. We can know that only finitely many terms in the reduced part of 2.119 are non-vanishing for arbitrary $v_1 \otimes v_2 \in K_{T\times\mathbb{C}_q^\times}(M(\mathbf{w}_1))_{loc} \otimes K_{T\times\mathbb{C}_q^\times}(M(\mathbf{w}_2))_{loc}$. Consequently, to show that $(\pi_1 \otimes \pi_2)(R_{\mathbf{m}})$ is a Laurent polynomial in u_1/u_2, it suffices to show that $(\pi_1 \otimes \pi_2)(P^{\mathbf{m}}_{[i;j)} \otimes Q^{\mathbf{m}}_{[i;j)})$ is a Laurent polynomial in u_1/u_2. This follows from the fact that $P^{\mathbf{m}}_{[i;j)}|\boldsymbol{\lambda}\rangle$ is a polynomial in u_1 and $Q^{\mathbf{m}}_{[i;j)}|\boldsymbol{\lambda}\rangle$ is a polynomial in u_2^{-1} such that $\deg_{u_1}(P^{\mathbf{m}}_{[i;j)}|\boldsymbol{\lambda}\rangle) + \deg_{u_2}(Q^{\mathbf{m}}_{[i;j)}|\boldsymbol{\lambda}\rangle) = 0$.

Now the second statement comes from the first statement and the injective theorem 2.8 and the RTT formalism. □

An easy consequence of the above proposition can be stated in the following:

Corollary 2.10. *Fix the representation*

$$(\pi_1 \otimes \pi_2) : \mathscr{B}_{\mathbf{m}}\hat{\otimes}\mathscr{B}_{\mathbf{m}} \to End(K_{T\times\mathbb{C}_q^\times}(M(\mathbf{w}_1))_{loc} \otimes K_{T\times\mathbb{C}_q^\times}(M(\mathbf{w}_2))_{loc}) \tag{2.124}$$

And the condition in the proposition in 2.9. Then the R-matrix $(\pi_1 \otimes \pi_2)(R_{\mathbf{m}})$ is a polynomial in u_1/u_2. Similarly, the R-matrix $(\pi_1 \otimes \pi_2)(R^-_{\mathbf{m}})$ is a polynomial in u_2/u_1 and satisfy the trigonometric Yang-Baxter equation.

Using this result, we can prove the rationality of the R-matrix $\mathscr{R}^s(u)$ valued in $\mathrm{End}(K(\mathbf{w}_1) \otimes K(\mathbf{w}_2))$ such that $u = u_1/u_2$:

Proposition 2.11. *The matrix coefficients of the R-matrix $\mathscr{R}^{\mathbf{m}}(u) := (\pi_1 \otimes \pi_2)(R^{\mathbf{m}})$ valued in $End(K(\mathbf{w}_1) \otimes K(\mathbf{w}_2))$ are rational functions of u. Moreover, $R(0)$ and $R(\infty)$ are well-defined in $\mathbb{Q}(u)$.*

Proof. We will prove this proposition in the Appendix 5. □

2.10. **Relation with the quantum algebra of** [OS22]**.** The connection between the quantum toroidal algebra $U_{q,t}(\hat{\hat{\mathfrak{sl}}}_r)$ and the RTT stable envelope quantum algebra $U_q(\hat{\mathfrak{g}}_Q)$ is controlled by the following theorem in [N23]:

Theorem 2.12. *There exists a natural algebra injection map:*

$$U_{q,t}(\hat{\hat{\mathfrak{sl}}}_n) \hookrightarrow U_q^{MO}(\hat{\mathfrak{g}}_Q) \hookrightarrow \prod_{\mathbf{w}} End(K_{T_{\mathbf{w}}}(M(\mathbf{w}))) \tag{2.125}$$

Recall that the wall subalgebra $U_q(\mathfrak{g}_{\mathbf{m},\boldsymbol{\theta}})$ is defined via the FRT formalism of the wall R-matrix $R^{\pm}_{\mathbf{m},\mathbf{m}+\epsilon\boldsymbol{\theta}} := \lim_{\epsilon\to 0}\mathrm{Stab}^{-1}_{\mathbf{m}+\epsilon\boldsymbol{\theta},\pm\mathscr{C}} \circ \mathrm{Stab}_{\mathbf{m},\pm\mathscr{C}}$ with $\boldsymbol{\theta} \in \mathbb{Z}^n$. We can see that $U_q(\mathfrak{g}_{\mathbf{m},\boldsymbol{\theta}})$ depends on the choice of $\boldsymbol{\theta}$, and using these wall subalgebra with fixed $\mathbf{m}$, we define the **root subalgebra** $U_q(\mathfrak{g}_{\mathbf{m}})$ as the algebra generated by FRT formalism of all the wall R-matrix $R_{\mathbf{m},\mathbf{m}+\epsilon\boldsymbol{\theta}}$ with fixed $\mathbf{m} \in \mathbb{Q}^n$ and arbitrary $\boldsymbol{\theta} \in \mathbb{Z}^n$, i.e. In the language of [OS22], $U_q^{MO}(\mathfrak{g}_{\mathbf{m}})$ is generated by all the wall subalgebra $U_q^{MO}(\mathfrak{g}_w)$ such that the wall w contains the point $\mathbf{m}$.

Proposition 2.13. *There is a Hopf algebra embedding*

$$\mathscr{B}_{\mathbf{m}} \hookrightarrow U_q^{MO}(\mathfrak{g}_{\mathbf{m}}) \tag{2.126}$$

which is independent of the choice of $\boldsymbol{\theta} \in \mathbb{Q}^n$.

Proof. First we prove that the coproduct $\Delta_{\mathbf{m}}$ on $\mathscr{B}_{\mathbf{m}}$ intertwine the $\mathrm{Stab}_{\mathbf{m}}$, i.e.

$$\begin{tikzcd} \mathscr{B}_{\mathbf{m}} \arrow[r, hook] \arrow[d, "\Delta_{\mathbf{m}}"] & \prod_{\mathbf{w}} \mathrm{End}(K_{T_{\mathbf{w}}}(M(\mathbf{w}))) \arrow[d, "\mathrm{Stab}_{\mathbf{m}}^{\dagger}\circ(-)\circ\mathrm{Stab}_{\mathbf{m}}"] \\ \mathscr{B}_{\mathbf{m}}\hat{\otimes}\mathscr{B}_{\mathbf{m}} \arrow[r, hook] & \prod_{\mathbf{w}_1,\mathbf{w}_2} \mathrm{End}(K_{T_{\mathbf{w}_1}}(M(\mathbf{w}_1)))\hat{\otimes}(K_{T_{\mathbf{w}_2}}(M(\mathbf{w}_2))) \end{tikzcd} \tag{2.127}$$

For simplicity, we only show the proof for $P^{\mathbf{m}}_{[i;j)}$, and this means that we need to prove the following identity:

$$P^{\mathbf{m}}_{[i;j)} \circ \mathrm{Stab}_{\mathbf{m}}(s^{\mathbf{m}}_{\boldsymbol{\lambda}_1} \otimes s^{\mathbf{m}}_{\boldsymbol{\lambda}_2}) = \mathrm{Stab}_{\mathbf{m}}(\Delta_{\mathbf{m}}(P^{\mathbf{m}}_{[i;j)})(s^{\mathbf{m}}_{\boldsymbol{\lambda}_1} \otimes s^{\mathbf{m}}_{\boldsymbol{\lambda}_2})) \tag{2.128}$$

For simplicity, using the formula for the coproduct, we can reduce the proof to the case $P^{\mathbf{m}}_{[i;i+1)_h}$.

Recall the formula of $P^{\mathbf{m}}_{[i;i+1)_h}$ on the stable basis $s^{\mathbf{m}}_{\boldsymbol{\lambda}}$ in [N15]:

$$P^{\mathbf{m}}_{[i;i+1)_h} s^{\mathbf{m}}_{\boldsymbol{\mu}} = \sum_{\boldsymbol{\lambda}} s^{\mathbf{m}}_{\boldsymbol{\lambda}} \mathrm{l.d.}(P^{\mathbf{m}}_{[i;i+1)_h}(\boldsymbol{\lambda}\backslash\boldsymbol{\mu})\boldsymbol{\Theta}_{\boldsymbol{\lambda}\backslash\boldsymbol{\mu}}) \frac{\mathrm{l.d.}(\mathfrak{K}_{\boldsymbol{\mu}})}{\mathrm{l.d.}(\mathfrak{K}_{\boldsymbol{\lambda}})} \tag{2.129}$$

And here:

$$\mathfrak{K}_{\boldsymbol{\lambda}} = \prod_{\square\in\boldsymbol{\lambda}} [\prod_{\blacksquare\in\boldsymbol{\lambda}} \zeta(\frac{\chi_{\blacksquare}}{\chi_{\square}}) \prod_{k=1}^{\mathbf{w}} [\frac{u_k}{q\chi_{\square}}][\frac{\chi_{\square}}{qu_k}]]^{(-)} \tag{2.130}$$

$$\boldsymbol{\Theta}_{\boldsymbol{\lambda}\backslash\boldsymbol{\mu}} = \prod_{\blacksquare\in\boldsymbol{\lambda}\backslash\boldsymbol{\mu}} [\prod_{\square\in\boldsymbol{\mu}} \zeta(\frac{\chi_{\blacksquare}}{\chi_{\square}}) \prod_{k=1}^{\mathbf{w}} [\frac{u_k}{q\chi_{\blacksquare}}]] \tag{2.131}$$

Observe that the lowest degree part of $P^{\mathbf{m}}_{[i;i+1)_h}(\boldsymbol{\lambda}\backslash\boldsymbol{\mu})$ does not affect the term, so we only consider the following part:

$$\begin{aligned} &\mathrm{l.d.}(\boldsymbol{\Theta}_{\boldsymbol{\lambda}\backslash\boldsymbol{\mu}}) \frac{\mathrm{l.d.}(\mathfrak{K}_{\boldsymbol{\mu}})}{\mathrm{l.d.}(\mathfrak{K}_{\boldsymbol{\lambda}})} \\ =&\frac{\mathrm{l.d.}(\prod_{\square\in\boldsymbol{\mu}}^{\blacksquare\in\boldsymbol{\lambda}\backslash\boldsymbol{\mu}} \zeta(\frac{\chi_{\blacksquare}}{\chi_{\square}}) \prod_{i=1}^{\mathbf{w}}[\frac{u_i}{q\chi_{\blacksquare}}])^{(+)}}{\mathrm{l.d.}(\prod_{\square\in\boldsymbol{\mu}}^{\blacksquare\in\boldsymbol{\lambda}\backslash\boldsymbol{\mu}} \zeta(\frac{\chi_{\square}}{\chi_{\blacksquare}}) \prod_{i=1}^{\mathbf{w}}[\frac{\chi_{\blacksquare}}{qu_i}])^{(-)}} \cdot \frac{\prod_{\blacksquare\in\boldsymbol{\lambda}\backslash\boldsymbol{\mu}}(\prod_{\square\in\boldsymbol{\mu}} \zeta(\frac{\chi_{\blacksquare}}{\chi_{\square}}) \prod_{i=1}^{\mathbf{w}}[\frac{u_i}{q\chi_{\blacksquare}}])^{(0)}}{\prod_{\blacksquare,\blacksquare'\in\boldsymbol{\lambda}\backslash\boldsymbol{\mu}} \zeta(\frac{\chi_{\blacksquare}}{\chi_{\blacksquare'}})^{(-)}} \end{aligned} \tag{2.132}$$

The superscripts $(+)$, (0) and $(-)$ refer to that we only retain those factors $[x]$ such that $\deg(x) > 0$, $\deg(x) = 0$ and $\deg(x) < 0$. We can see that the degree 0 part does not contribute to the lowest degree.

We want to analyze the behavior of the formula in terms of the power of q. For the first fraction, by calculation we can see that:

$$\frac{\text{l.d.}\prod_{\square\in\boldsymbol{\mu}}^{\blacksquare\in\boldsymbol{\lambda}\backslash\boldsymbol{\mu}}(\prod_{i=1}^{\mathbf{w}}[\frac{u_i}{q\chi_{\blacksquare}}])^{(+)}}{\text{l.d.}(\prod_{\square\in\boldsymbol{\mu}}^{\blacksquare\in\boldsymbol{\lambda}\backslash\boldsymbol{\mu}}\prod_{i=1}^{\mathbf{w}}[\frac{\chi_{\blacksquare}}{qu_i}])^{(-)}}=\prod_{\square\in\boldsymbol{\mu}}^{\blacksquare\in\boldsymbol{\lambda}\backslash\boldsymbol{\mu}}q^{-\sum_{i=w_{\blacksquare}+1}^{\mathbf{w}}} \tag{2.133}$$

by the following formula:

$$\text{l.d.}\prod_{\square\in\boldsymbol{\mu}}^{\blacksquare\in\boldsymbol{\lambda}\backslash\boldsymbol{\mu}}(\prod_{i=1}^{\mathbf{w}}[\frac{u_i}{q\chi_{\blacksquare}}])^{(+)}=\text{l.d.}\prod_{\square\in\boldsymbol{\mu}}^{\blacksquare\in\boldsymbol{\lambda}\backslash\boldsymbol{\mu}}(\prod_{i=w_{\blacksquare}+1}^{\mathbf{w}}[\frac{u_i}{q\chi_{\blacksquare}}])=\prod_{\square\in\boldsymbol{\mu}}^{\blacksquare\in\boldsymbol{\lambda}\backslash\boldsymbol{\mu}}q^{-\frac{1}{2}\sum_{i=w_{\blacksquare}+1}^{\mathbf{w}}} \tag{2.134}$$

$$\text{l.d.}(\prod_{\square\in\boldsymbol{\mu}}^{\blacksquare\in\boldsymbol{\lambda}\backslash\boldsymbol{\mu}}\prod_{i=1}^{\mathbf{w}}[\frac{\chi_{\blacksquare}}{qu_i}])^{(-)}=\text{l.d.}(\prod_{\square\in\boldsymbol{\mu}}^{\blacksquare\in\boldsymbol{\lambda}\backslash\boldsymbol{\mu}}\prod_{i=w_{\blacksquare}+1}^{\mathbf{w}}[\frac{\chi_{\blacksquare}}{qu_i}])=\prod_{\square\in\boldsymbol{\mu}}^{\blacksquare\in\boldsymbol{\lambda}\backslash\boldsymbol{\mu}}\prod_{i=w_{\blacksquare}+1}^{\mathbf{w}}q \tag{2.135}$$

And for

$$\frac{\text{l.d.}(\prod_{\square\in\boldsymbol{\mu}}^{\blacksquare\in\boldsymbol{\lambda}\backslash\boldsymbol{\mu}}\zeta(\frac{\chi_{\blacksquare}}{\chi_{\square}}))^{(+)}}{\text{l.d.}(\prod_{\square\in\boldsymbol{\mu}}^{\blacksquare\in\boldsymbol{\lambda}\backslash\boldsymbol{\mu}}\zeta(\frac{\chi_{\square}}{\chi_{\blacksquare}}))^{(-)}}=\prod_{\blacksquare\in\boldsymbol{\lambda}\backslash\boldsymbol{\mu}}q^{(\sum_{\square\in\boldsymbol{\mu}}^{\sigma_{\square}=\sigma_{\blacksquare}+1,,c_{\square}>c_{\blacksquare}}+\sum_{\square\in\boldsymbol{\mu}}^{\sigma_{\square}=\sigma_{\blacksquare}-1,c_{\square}>c_{\blacksquare}}-2\sum_{\square\in\boldsymbol{\mu}}^{\sigma_{\square}=\sigma_{\blacksquare},,c_{\square}>c_{\blacksquare}})} \tag{2.136}$$

Note that

$$\begin{aligned}\text{l.d.}(\prod_{\square\in\boldsymbol{\mu}}^{\blacksquare\in\boldsymbol{\lambda}\backslash\boldsymbol{\mu}}\zeta(\frac{\chi_{\blacksquare}}{\chi_{\square}}))^{(+)}&=\text{l.d.}(\prod_{\blacksquare\in\boldsymbol{\lambda}\backslash\boldsymbol{\mu}}\frac{\prod_{\square\in\boldsymbol{\mu}}^{\sigma_{\square}=\sigma_{\blacksquare}+1}[\frac{t\chi_{\square}}{q\chi_{\blacksquare}}]\prod_{\square\in\boldsymbol{\mu}}^{\sigma_{\square}=\sigma_{\blacksquare}-1}[\frac{\chi_{\square}}{qt\chi_{\blacksquare}}]}{\prod_{\square\in\boldsymbol{\mu}}^{\sigma_{\square}=\sigma_{\blacksquare}}[\frac{\chi_{\square}}{\chi_{\blacksquare}}][\frac{\chi_{\square}}{q^2\chi_{\blacksquare}}]})^{(+)}\\&=\prod_{\blacksquare\in\boldsymbol{\lambda}\backslash\boldsymbol{\mu}}q^{1/2(\sum_{\square\in\boldsymbol{\mu}}^{\sigma_{\square}=\sigma_{\blacksquare}+1,,c_{\square}>c_{\blacksquare}}+\sum_{\square\in\boldsymbol{\mu}}^{\sigma_{\square}=\sigma_{\blacksquare}-1,,c_{\square}>c_{\blacksquare}}-2\sum_{\square\in\boldsymbol{\mu}}^{\sigma_{\square}=\sigma_{\blacksquare},,c_{\square}>c_{\blacksquare}})}\end{aligned} \tag{2.137}$$

Now use the formula for the stable envelope:

$$P_{[i;i+1)_h}^{\mathbf{m}}\text{Stab}_{\mathbf{m}}(s_{\boldsymbol{\mu}_1}^{\mathbf{m}}\otimes s_{\boldsymbol{\mu}_2}^{\mathbf{m}})=P_{[i;i+1)_h}^{\mathbf{m}}s_{(\boldsymbol{\mu}_1,\boldsymbol{\mu}_2)}^{\mathbf{m}} \tag{2.138}$$

$$\text{Stab}_{\mathbf{m}}(\Delta_{\mathbf{m}}(P_{[i;i+1)_h}^{\mathbf{m}})s_{\boldsymbol{\mu}_1}^{\mathbf{m}}\otimes s_{\boldsymbol{\mu}_2}^{\mathbf{m}})=\text{Stab}_{\mathbf{m}}(P_{[i;i+1)_h}^{\mathbf{m}}s_{\boldsymbol{\mu}_1}^{\mathbf{m}}\otimes\varphi^{[i;i+1)_h}s_{\boldsymbol{\mu}_2}^{\mathbf{m}}+s_{\boldsymbol{\mu}_1}^{\mathbf{m}}\otimes P_{[i;i+1)_h}^{\mathbf{m}}s_{\boldsymbol{\mu}_2}^{\mathbf{m}}) \tag{2.139}$$

Now given our chosen ordering the order $(\boldsymbol{\mu}_1,\boldsymbol{\mu}_2)$, $P_{[i;i+1)_h}^{\mathbf{m}}s_{\boldsymbol{\mu}_1}^{\mathbf{m}}$ must have some remaining power of q left, which coincides with the one given by $\varphi^{[i;i+1)_h}$. For the $\frac{1}{\prod_{\blacksquare,\blacksquare'\in\boldsymbol{\lambda}\backslash\boldsymbol{\mu}}\zeta(\frac{\chi_{\blacksquare}}{\chi_{\blacksquare'}})^{(-)}}$ part, this does not affect since $\frac{1}{\prod_{\blacksquare,\blacksquare'\in\boldsymbol{\lambda}\backslash\boldsymbol{\mu}}\zeta(\frac{\chi_{\blacksquare}}{\chi_{\blacksquare'}})^{(-)}}=\frac{1}{\prod_{\blacksquare,\blacksquare'\in\boldsymbol{\lambda}_1\backslash\boldsymbol{\mu}_1}\zeta(\frac{\chi_{\blacksquare}}{\chi_{\blacksquare'}})^{(-)}\prod_{\blacksquare,\blacksquare'\in\boldsymbol{\lambda}_2\backslash\boldsymbol{\mu}_2}\zeta(\frac{\chi_{\blacksquare}}{\chi_{\blacksquare'}})^{(-)}}$, thus completing the proof of the above commutative diagram.

For the general $P^{\mathbf{m}}_{[i,j)}$, using the coproduct formula:

$$\Delta_{\mathbf{m}}(P^{\mathbf{m}}_{[i;j)})=\sum_{a=i}^{j}P^{\mathbf{m}}_{[a,j)}\varphi_{[i,a)}\otimes P^{\mathbf{m}}_{[i,a)} \tag{2.140}$$

The argument is the same as those of $P^{\mathbf{m}}_{[i;i+1)_h}$.

For the algebra map $\mathscr{B}_{\mathbf{m}}\to U_q(\mathfrak{g}_{\mathbf{m}})$, we claim the following thing:

Lemma 2.14. *If $a\in\mathscr{B}_{\mathbf{n}}$ such that $\Delta_{\mathbf{m}}(a)\subset\mathscr{B}_{\mathbf{n}}\hat{\otimes}\mathscr{B}_{\mathbf{n}}$, $\mathbf{m}=\mathbf{n}$. Then similarly for $b\in U_q(\mathfrak{g}_{\mathbf{n}})$ such that $\Delta_{\mathbf{m}}(b)\in U_q(\mathfrak{g}_{\mathbf{n}})\hat{\otimes}U_q(\mathfrak{g}_{\mathbf{n}})$, $\mathbf{n}=\mathbf{m}$.*

Proof. Now suppose that $a\in\mathscr{B}_{\mathbf{n}}$ such that $\Delta_{\mathbf{m}}(a)\subset\mathscr{B}_{\mathbf{n}}\hat{\otimes}\mathscr{B}_{\mathbf{n}}$, now choose a path from $\mathbf{n}$ to $\mathbf{m}$, and now use the formula:

$$\Delta_{\mathbf{m}}(a)=T_{\mathbf{m},\mathbf{n}}\Delta_{\mathbf{n}}(a)T_{\mathbf{m},\mathbf{n}}^{-1} \tag{2.141}$$

And since $\Delta_{\mathbf{n}}(a)\subset\mathscr{B}_{\mathbf{n}}\hat{\otimes}\mathscr{B}_{\mathbf{n}}$, now we write it as:

$$\Delta_{\mathbf{n}}(a)=\sum a^{(1)}\otimes a^{(2)} \tag{2.142}$$

And without loss of generality, we could reduce the case to $P^{\mathbf{n}}_{[i;j)}$ and thus by the coproduct formula one could further reduce the proof to $P^{\mathbf{n}}_{[i;i+1)_h}$, and in this case:

$$\Delta_{\mathbf{n}}(P^{\mathbf{n}}_{[i;i+1)_h})=P^{\mathbf{n}}_{[i;i+1)_h}\otimes\varphi_{[i;i+1)_h}+1\otimes P^{\mathbf{n}}_{[i;i+1)_h} \tag{2.143}$$

Then the first part of the lemma is true following from the fact that $[a,P^{\mathbf{n}}_{[i;i+1)_h}]\notin\mathscr{B}_{\mathbf{n}}$ if $a\in\mathscr{B}_{\mathbf{m}}$ with $\mathbf{m}\neq\mathbf{n}$ without φ_i kind of elements.

For the second part of the lemma, recall that $U_q(\mathfrak{g}_{\mathbf{m}})$ is defined as the matrix coefficients of the geometric R-matrix $R_{\mathbf{m},\mathbf{m}+\epsilon\boldsymbol{\theta}}:=\lim_{\epsilon\to 0}\mathrm{Stab}^{-1}_{\mathbf{m}+\epsilon\boldsymbol{\theta}}\circ\mathrm{Stab}_{\mathbf{m}}$ with $\boldsymbol{\theta}\in\mathbb{Z}^r$. By simple calculation, one can show that the matrix elements of $R_{\mathbf{m},\mathbf{m}+\epsilon\boldsymbol{\theta}}$ acting on the corresponding stable basis $s^{\mathbf{m}}_{\boldsymbol{\lambda}_1}\otimes s^{\mathbf{m}}_{\boldsymbol{\lambda}_2}$ depends on the value over the fixed point geometric R-matrix, i.e.

$$R_{\mathbf{m},\mathbf{m}+\epsilon\boldsymbol{\theta}}(s^{\mathbf{m}}_{\boldsymbol{\lambda}_1}\otimes s^{\mathbf{m}}_{\boldsymbol{\lambda}_2})=\lim_{\epsilon\to 0}\sum_{\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}(R^{\sigma}_{\mathbf{m},\mathbf{m}+\epsilon\boldsymbol{\theta}})_{(\lambda_1,\lambda_2),(\mu_1,\mu_2)}s^{\mathbf{m}+\epsilon\boldsymbol{\theta}}_{\boldsymbol{\mu}_1}\otimes s^{\mathbf{m}+\epsilon\boldsymbol{\theta}}_{\boldsymbol{\mu}_2} \tag{2.144}$$

Thus one could use the result of [D21] to write down the explicit formula of the fixed point R-matrix.

Also the R-matrix can be factorized as follows:

$$R_{\mathbf{m},\mathbf{m}+\epsilon\boldsymbol{\theta}}=1+\sum_{\langle\alpha,\theta\rangle>0}(R_{\mathbf{m},\mathbf{m}+\epsilon\boldsymbol{\theta}})_{\alpha},\qquad (R_{\mathbf{m},\mathbf{m}+\epsilon\boldsymbol{\theta}})^{-1}=1+\sum_{\langle\alpha,\theta\rangle>0}(R^{-1}_{\mathbf{m},\mathbf{m}+\epsilon\boldsymbol{\theta}})_{\alpha} \tag{2.145}$$

And we have the formula:

$$\Delta_{\mathbf{m}}\langle\boldsymbol{\lambda}|((R_{\mathbf{m},\mathbf{m}+\epsilon\boldsymbol{\theta}})_{\alpha})|\boldsymbol{\mu}\rangle\in\bigoplus_{\beta+\gamma=\alpha}(R_{\mathbf{m},\mathbf{m}+\epsilon\boldsymbol{\theta}})_{\beta}\otimes(R_{\mathbf{m},\mathbf{m}+\epsilon\boldsymbol{\theta}})_{\gamma} \tag{2.146}$$

Thus:

(2.147)
$$\Delta_{\mathbf{n}}\langle\boldsymbol{\lambda}|(R_{\mathbf{m},\mathbf{m}+\epsilon\boldsymbol{\theta}})_{\alpha}|\boldsymbol{\mu}\rangle=\bigoplus_{\beta_1+\beta_2+\beta_3+\beta_4=\alpha}(T_{\mathbf{n},\mathbf{m}})_{\beta_1}(R_{\mathbf{m},\mathbf{m}+\epsilon\boldsymbol{\theta}})_{\beta_2}(R_{\mathbf{m},\mathbf{m}+\epsilon\boldsymbol{\theta}})_{\beta_3}(T^{-1}_{\mathbf{n},\mathbf{m}})_{\beta_4}$$

Here $T_{\mathbf{n},\mathbf{m}}:=\mathrm{Stab}_{\mathbf{n}}^{-1}\circ\mathrm{Stab}_{\mathbf{m}}$. Then one can show that given an order on the set of roots $\{\alpha\}$, $U_q(\mathfrak{g}_{\mathbf{m}})$ is generated by elements of R_α with α of the lowest order, i.e. those elements corresponding to the simple roots. So we have reduced the proof to the case for the lowest order elements in $(R_{\mathbf{m}})_\alpha$. And in this case, the above formula implies that:

(2.148)
$$\Delta_{\mathbf{n}}\langle\boldsymbol{\lambda}|(R_{\mathbf{m},\mathbf{m}+\epsilon\boldsymbol{\theta}})_{\alpha}|\boldsymbol{\mu}\rangle=\bigoplus_{\beta_1+\beta_2=\alpha}(T_{\mathbf{n},\mathbf{m}})_{\beta_1}(T^{-1}_{\mathbf{n},\mathbf{m}})_{\beta_2}+(R_{\mathbf{m},\mathbf{m}+\epsilon\boldsymbol{\theta}})_{\alpha}\otimes 1+1\otimes(R_{\mathbf{m},\mathbf{m}+\epsilon\boldsymbol{\theta}})_{\alpha}$$

Now by computations the first term does not vanish due to the formula 2.144. And thus this proves the second claim.

□

Now since the coproduct $\Delta_{\mathbf{m}}$ coincides on both $\mathscr{B}_{\mathbf{m}}$ and $U_q(\mathfrak{g}_{\mathbf{m}})$, we can see that $\mathscr{B}_{\mathbf{m}}\subset U_q(\mathfrak{g}_{\mathbf{m}})$.

□

Remark. We expect that the coproduct $\Delta_{(\mathbf{m})}$ on $U_{q,t}(\hat{\hat{\mathfrak{sl}}}_r)$ defined in 2.2 should coincide with the coproduct $\Delta_{(\mathbf{m})}$ on $U_q(\hat{\mathfrak{g}}_Q)$ defined by the stable envelope $\mathrm{Stab}_{\mathbf{m}}$ of the slope $\mathbf{m}$. But here one should note that the R-matrix $R_{\mathbf{m}}\in\mathscr{B}_{\mathbf{m}}\hat{\otimes}\mathscr{B}_{\mathbf{m}}$ might not be an integral K-theory class by the definition of the generators $P^{\mathbf{m}}_{[i;j)}$, $Q^{\mathbf{m}}_{-[i;j)}$. This is different from the wall subalgebra $U_q^{MO}(\mathfrak{g}_w)$ since the generators of the wall subalgebra are in the integral K-theory.

3. Quantum difference equation

3.1. **Background: Stable quasimaps to Nakajima varieties.** A quasimap $f:C\to M(\mathbf{v},\mathbf{w})$ with C the irreducible component isomorphic to $\mathbb{P}^1$ consists of the following data:

- A collection of vector bundles $\mathscr{V}_i, i\in I$ on C with ranks v_i. (This is the one induced by those V_i on $M(\mathbf{v},\mathbf{w})$).
- A collection of trivial vector bundles $\mathscr{W}_i$ of rank w_i.
- A section

(3.1)
$$f\in H^0(C,\mathscr{M}\oplus\mathscr{M}^*\otimes q^{-1})$$

satisfying the moment map condition $\mu=0$, here

(3.2)
$$\mathscr{M}=(\bigoplus_{h\in E}\mathrm{Hom}(\mathscr{V}_{i(h)},\mathscr{V}_{o(h)})\oplus\bigoplus_{i\in I}\mathrm{Hom}(\mathscr{V}_i,\mathscr{W}_i))$$

And the degree of a quasimap is given as $d = (\deg(\mathscr{V}_i))_{i=1}^I \in \mathrm{Pic}(M(\mathbf{v},\mathbf{w})) \cong \mathbb{Z}^I$.

Thus we can define the moduli space of stable quasimaps of degree d as $QM^d(X)$. And let $QM^d(X)_{\text{nonsing }p} \subset QM^d(X)$ be the open subset of the moduli space corresponding to the stable quasimaps nonsingular at a point p. It has a natural evaluation morphism $ev_p : QM^d(X)_{\text{nonsing }p} \to X$ and also we could define the moduli space of **relative quasimaps** $QM^d(X)_{\text{rel }p}$ is a compactification of the map ev_p, i.e.

$$\begin{tikzcd} & QM^d_{\text{rel }p}(X) \arrow[dr, "\hat{ev}_p"] & \\ QM^d_{\text{nonsing }p}(X) \arrow[ur] \arrow[rr] & & X \end{tikzcd} \tag{3.3}$$

And this moduli space can be constructed the natural virtual structure sheaves $\hat{\mathscr{O}}_{vir}$.

Consider the moduli space $QM^d_{\text{rel }p_1,\text{nonsing }p_2}(X)$ of quasimaps with relative conditions at $p_1 \in C$ and nonsingular at $p_2 \in C$. Thus we can define the evaluation map:

$$ev = \hat{ev}_{p_1} \times ev_{p_2} : QM^d_{\text{rel }p_1,\text{nonsing }p_2}(X) \to X \times X \tag{3.4}$$

This moduli space is equipped with an action of $G \times \mathbb{C}_p^\times$ with $G = \prod_i GL_{v_i}$ comes from X and $\mathbb{C}_q^\times$ comes from C.

Note that this map ev is not proper, but it becomes proper on the subset of fixed points $QM^d_{\text{rel }p_1,\text{nonsing }p_2}(X)^{G\times\mathbb{C}_p^\times}$

And thus we can define the **Capping operator**:

$$\mathbf{J}(u,z) = \sum_{d\in\mathbb{Z}^n} z^d \mathrm{ev}_*(QM^d_{\text{rel }p_1,\text{nonsing }p_2}(X), \hat{\mathscr{O}}_{vir}) \in K_{G\times\mathbb{C}_p^\times}(X)^{\otimes 2}_{loc} \otimes \mathbb{Q}[[z]] \tag{3.5}$$

With $z \in \mathbb{C}[\mathrm{Pic}(X) \otimes \mathbb{C}^\times]$ called the **Kahler parametre**, and $u \in \mathbb{C}[T^\vee]$ is called the **equivariant parametre**.

It is proved in[AO21] that capping operator is the matrix of fundamental solution of a system of q-difference equations:

$$\mathbf{J}(u, zp^{\mathscr{L}})\mathscr{L} = \mathbf{M}_{\mathscr{L}}(u,z)\mathbf{J}(u,z), \qquad \mathbf{J}(up,z)E(u,z) = S(u,z)\mathbf{J}(u,z) \tag{3.6}$$

In this paper we focus on the operator $\mathbf{M}_{\mathscr{L}}(u,z)$, which is called the **quantum difference operator**.

3.2. **Quantum difference equation in the algebraic setting.** Here in our settings, we will construct the quantum difference equation without using the K-theoretical stable envelope, which is purely algebraic.

The construction of the quantum differece operator can be described as follows, which is similar to that in [OS22].

Denote the translation operator with respect to $\mathscr{L}$ by $T_{\mathscr{L}}$:

$$T_{\mathscr{L}}f(u,z)=f(u,zp^{\mathscr{L}}) \tag{3.7}$$

And define the difference connection:

$$A_{\mathscr{L}}=T_{\mathscr{L}}^{-1}\mathbf{M}_{\mathscr{L}}(u,z) \tag{3.8}$$

This operators have the commutativity condition

$$[A_{\mathscr{L}},A_{\mathscr{L}'}]=0 \tag{3.9}$$

The stable envelope version of the result above has been proved in [OS22]. In this section, we will prove the algebraic version of the result.

Now introduce a vector $\lambda=(\mathbf{t}_1,\cdots,\mathbf{t}_n)\in H^2(X,\mathbb{C})$ with $n=\#I$ such that $q^{\mathbf{t_i}}=z_i$ is the Kahler parametre for the corresponding quiver variety $M(\mathbf{v},\mathbf{w})$. And for simplicity, we shall use both $\{z_i\}\in\mathrm{Pic}(X)\otimes\mathbb{C}^\times\cong(\mathbb{C}^\times)^{\#I}$ and λ as the same meaning, i.e. $f(z_i)=f(\lambda)$.

Now let $X=M(\mathbf{v},\mathbf{w})$ and $A\subset T_{\mathbf{w}}$ a subtorus of the framing torus corresponding to a decomposition:

$$X^A=\sqcup_{\mathbf{v}_1+\cdots+\mathbf{v}_n=\mathbf{v}}M(\mathbf{v}_1,\mathbf{w}_1)\times\cdots\times M(\mathbf{v}_n,\mathbf{w}_n) \tag{3.10}$$

First we define the operator $q^{(\lambda)}_{(k)}\in\mathrm{End}(K_G(F))$ with $F=M(\mathbf{v}_1,\mathbf{w}_1)\times\cdots\times M(\mathbf{v}_n,\mathbf{w}_n)\subset X^A$ to be diagonal in the basis supported on the set fixed points:

$$q^{\lambda}_{(k)}(\gamma)=q^{(\lambda,\mathbf{v}_k)}\gamma=z_1^{v_{k,1}}\cdots z_n^{v_{k,n}}\gamma \tag{3.11}$$

Also note that $q^{\lambda}_{(k)}$ can be thought of as the Cartan element in $U_{q,t}(\hat{\hat{\mathfrak{sl}}}_r)^{\otimes n}$ with the Kahler variable multiplications.

Also we define $q^{\Omega}\in\mathrm{End}(K_G(F))$ is defined as:

$$q^{\Omega}(\gamma)=q^{\mathrm{codim}(F)}(\gamma),\mathrm{codim}(F)=\dim(M(\sum_i\mathbf{v}_i,\sum_i\mathbf{w}_i))-\sum_i\dim(M(\mathbf{v}_i,\mathbf{w}_i)) \tag{3.12}$$

Similarly the antipode map can be written as:

$$(S_w\otimes\cdots\otimes S_w)(q^{\Omega})=q^{\Omega} \tag{3.13}$$

Remark. Note that the coproduct Δ_w for $q^{\lambda}_{(k)}$ and q^{Ω} is also group-like. See [N15].

The framework for constructing quantum difference equation can be encoded into the construction of the solution for the ABRR equation within the root subalgebra[OS22]. In this context, we extend the conceptual parallels to furnish a comparable algebraic construction. Specifically, we give an analogous construction for the quantum difference equation in the algebraic settings for the affine type A quiver varieties.

For each slope subalgebra $\mathscr{B}_{\mathbf{m}}$, one could associate the ABRR element $J^{\pm}_{\mathbf{m}}(\lambda)$ in the following:

Proposition 3.1. *There exist unique strictly upper triangular $J^{+}_{\mathbf{m}}(\lambda)$ and stirctly lower triangular $J^{-}_{\mathbf{m}}(\lambda)$ solutions of the following ABRR equations:*

$$J^{+}_{\mathbf{m}}(\lambda)q^{-\lambda}_{(1)}q^{\Omega}R^{+}_{\mathbf{m}} = q^{-\lambda}_{(1)}q^{\Omega}J^{+}_{\mathbf{m}}(\lambda), \qquad q^{\Omega}R^{-}_{\mathbf{m}}q^{-\lambda}_{(1)}J^{-}_{\mathbf{m}}(\lambda) = J^{-}_{\mathbf{m}}(\lambda)q^{\Omega}q^{-\lambda}_{(1)} \tag{3.14}$$

where $R^{-}_{\mathbf{m}} := (R_{\mathbf{m}})_{21}$ is the transposition of $R_{\mathbf{m}}$. Moreover, $J^{\pm}_{\mathbf{m}}(\lambda)$ are elements in a completion $\mathscr{B}_{\mathbf{m}}\hat{\otimes}\mathscr{B}_{\mathbf{m}}$ satisfying:

$$S_{\mathbf{m}} \otimes S_{\mathbf{m}}((J^{+}_{\mathbf{m}}(\lambda))_{21}) = J^{-}_{\mathbf{m}}(\lambda) \tag{3.15}$$

Proof. For a detailed derivation of the proof, readers are refered to Proposition 7 in [OS22]. Within the referenced work, it is important to note that their root subalgebra $U_q(\mathfrak{g}_w)$ and the quantum algebra $U_{q,t}(\hat{\mathfrak{g}}_Q)$ is defined via the wall and geometric R-matrix, but the core argument presented in their proof maintains can be applied to our setting. □

The reason why this is called the ABRR equation is that it is of the form of the ABRR equation in [ABRR97].

Remark. Here note that this proposition is also true for arbitrary quantum affine algebra or toroidal algebra for arbitrary quiver type by the construction in [N22]

An easy corollary of this is the following:

Lemma 3.2.

$$\lim_{z\to\infty} J^{+}_{\mathbf{m}}(z^{\theta}) = 1, \qquad \lim_{z\to 0} J^{+}_{\mathbf{m}}(z^{\theta}) = R^{+}_{\mathbf{m}} \tag{3.16}$$

here θ is the parametre above in the root factorization of the quantum toroidal algebra. Here we require that $\theta \in (\mathbb{Q}^{+})^{n}$.

Proof. This is mostly the same as the proof in [OS22]. Here we provide a proof given by the direct computation.

We do the factorization of $J^{+}_{\mathbf{m}}$ and $R^{+}_{\mathbf{m}}$ with respect to the shuffle degree, i.e. the shuffle degree of the slope subalgebra:

$$J_{\mathbf{m}} = 1 + \sum_{\mathbf{n}>0} J_{\mathbf{m}|\mathbf{n}}, \qquad (R^{+}_{\mathbf{m}})^{-1} = 1 + \sum_{\mathbf{n}>0} R^{+}_{\mathbf{m}|\mathbf{n}} \tag{3.17}$$

Using the ABRR equation we have the following recursion equations:

$$J_{\mathbf{m}|\mathbf{n}}(z) = \frac{1}{z^{\mathbf{n}}q^{k} - 1} \sum_{\substack{\mathbf{n}_1+\mathbf{n}_2=\mathbf{n} \\ \mathbf{n}_1<\mathbf{n}}} J_{\mathbf{m}|\mathbf{n}_1}(z)R^{+}_{\mathbf{m}|\mathbf{n}_2} \tag{3.18}$$

Now we take $(z_1,\cdots,z_n)=(z^{\theta_1},\cdots,z^{\theta_n})$. As $z\to\infty$, it is obvious that $J_{\mathbf{m}|\mathbf{n}}=0$.

If $z\to 0$, we have the following expression for $J_{\mathbf{m}|\mathbf{n}}(0^{\theta})$:

$$J_{\mathbf{m}|\mathbf{n}}(0^{\theta})=\sum_{k}\sum_{\mathbf{n}_1+\cdots+\mathbf{n}_k=\mathbf{n}}(-1)^kR^+_{\mathbf{m}|\mathbf{n}_1}R^+_{\mathbf{m}|\mathbf{n}_2}\cdots R^+_{\mathbf{m}|\mathbf{n}_k} \tag{3.19}$$

Thus we have that:

$$\begin{aligned}J_{\mathbf{m}}(0^{\theta})=&1+\sum_{k}\sum_{\mathbf{n}_1,\cdots,\mathbf{n}_k}(-1)^kR^+_{\mathbf{m}|\mathbf{n}_1}R^+_{\mathbf{m}|\mathbf{n}_2}\cdots R^+_{\mathbf{m}|\mathbf{n}_k}\\ =&(1+\sum_{\mathbf{n}>0}R^+_{\mathbf{m}})^{-1}\\ =&R^+_{\mathbf{m}}\end{aligned} \tag{3.20}$$

□

Remark. It is worth noting that if we take $\theta=(\theta_1,\cdots,\theta_n)$ to be the vector of different signs for each components. In this case the asymptotic behavior of $J_{\mathbf{m}}(0_{\theta})$ and $J_{\mathbf{m}}(\infty_{\theta})$ could be different. The following lemma gives a description of the asymptotic behavior in terms of the Hopf subalgebra of the slope subalgebra:

Lemma 3.3. *Let $s=(s_1,\cdots,s_n)$ be a real vector in $\mathbb{R}^n$, and define:*

$$J_{\mathbf{m}}(s):=\lim_{p\to 0}J_{\mathbf{m}}(z_1p^{s_1},\cdots,z_np^{s_n}) \tag{3.21}$$

Then $J_{\mathbf{m}}(s)$ is an element in $\mathcal{B}_{\mathbf{m}}\hat{\otimes}\mathcal{B}_{\mathbf{m}}$ independent of the variables $z_1,\cdots,z_n$ and locally constant on $s=(s_1,\cdots,s_n)$.

We would not give a refined version of the proof of the lemma here. It will be heavily used in the next paper when we start to analyze the quantum difference equation.

Let $\tau_{\mathbf{m}}=\mathbf{s}\mathscr{L}_{\mathbf{m}}$ be such that $q^s=p$, thus we can check that

$$q^{\tau_{\mathbf{m}}}_{(1)}R^+_{\mathbf{m}}(u)q^{-\tau_{\mathbf{m}}}_{(1)}=R^+_{\mathbf{m}}(uq) \tag{3.22}$$

By the previous proposition we have

$$q^{\tau_{\mathbf{m}}}_{(1)}J^+_{\mathbf{m}}(u)q^{-\tau_{\mathbf{m}}}_{(1)}=J^+_{\mathbf{m}}(uq) \tag{3.23}$$

Thus we can rewrite the above equation as:

$$J^+_{\mathbf{m}}(u,\lambda-\tau_{\mathbf{m}})q^{-\lambda}_{(1)}q^{\Omega}R^+_{\mathbf{m}}(uq)=q^{-\lambda}_{(1)}q^{\Omega}J^+_{\mathbf{m}}(uq,\lambda-\tau_{\mathbf{m}}) \tag{3.24}$$

And define

$$\mathbf{J}^{\pm}_{\mathbf{m}}(\lambda)=J^{\pm}_{\mathbf{m}}(\lambda-\tau_{\mathbf{m}}),\qquad \tau_{\mathbf{m}}=s\mathscr{L}_{\mathbf{m}} \tag{3.25}$$

And the above relation can be concluded as follows:

Proposition 3.4. *There exist unique strictly upper triangular* $\mathbf{J}^+_{\mathbf{m}}(\lambda) \in \mathscr{B}_{\mathbf{m}}^{\otimes 2}$ *and strictly lower triangular* $\mathbf{J}^-_{\mathbf{m}}(\lambda) \in \mathscr{B}_{\mathbf{m}}^{\otimes 2}$ *solutions of the following shifted ABRR equation:*

$$\mathbf{J}^+_{\mathbf{m}}(\lambda) q_{(1)}^{-\lambda} T_u q^{\Omega} R^+_{\mathbf{m}} = q_{(1)}^{-\lambda} T_u q^{\Omega} \mathbf{J}^+_{\mathbf{m}}(\lambda) \tag{3.26}$$

$$R^-_{\mathbf{m}} q_{(1)}^{-\lambda} T_u q^{\Omega} \mathbf{J}^-_{\mathbf{m}}(\lambda) = \mathbf{J}^-_{\mathbf{m}}(\lambda) q_{(1)}^{-\lambda} T_u q^{\Omega} \tag{3.27}$$

In [OS22], they called these two equations as **wall Knizhnik-Zamolodchikov equations**

Now we define the dynamical operator as:

$$\mathbf{B}_{\mathbf{m}}(\lambda) = m(1 \otimes S_{\mathbf{m}}(\mathbf{J}^-_{\mathbf{m}}(\lambda)^{-1}))|_{\lambda \to \lambda + \kappa} \tag{3.28}$$

Here $\kappa = \frac{C\mathbf{v}-\mathbf{w}}{2}$. We shall call this dynamical operator the **monodromy operator**[1]. Here m is the multiplication map.

Note that if $\mathbf{m}$ is not the wall for the corresponding quiver variety $M(\mathbf{v}, \mathbf{w})$, $R^{\pm}_{\mathbf{m}}$ is trivial. From the ABRR equation in the proposition 3.4, $J^{\pm}_{\mathbf{m}}$ is also trivial, thus we can restrict our definition of $\mathbf{B}_{\mathbf{m}}(\lambda)$ to the case when $\mathbf{m}$ is on the wall set for the quiver variety $M(\mathbf{v}, \mathbf{w})$ such that $R^{\pm}_{\mathbf{m}}$ is nontrivial.

Let $\mathscr{L} \in Pic(X)$ be a line bundle. Now we fix a slope $s \in H^2(X, \mathbb{R})$ and choose a path in $H^2(X, \mathbb{R})$ from s to $s - \mathscr{L}$. This path crosses finitely many slope points in some order $\{\mathbf{m}_1, \mathbf{m}_2, \cdots, \mathbf{m}_m\}$ by the proposition 2.4. And for this choice of a slope, line bundle and a path we associate the following operator:

$$\mathbf{B}^s_{\mathscr{L}}(\lambda) = \mathscr{L} \mathbf{B}_{\mathbf{m}_m}(\lambda) \cdots \mathbf{B}_{\mathbf{m}_1}(\lambda) \tag{3.29}$$

We define the q-difference operators:

$$\mathscr{A}^s_{\mathscr{L}} = T_{\mathscr{L}}^{-1} \mathbf{B}^s_{\mathscr{L}}(\lambda) \tag{3.30}$$

Remark. It is notable that the term q^{Ω} in the ABRR equation represents the scaling of the solution. In the context of enumerative geometry background, this corresponds to an additional scaling grading for the capping operators, so one can also consider the ABRR equation without the scaling operator q^{Ω}.

3.3. **Monodromy operator for the root subalgebra $\mathscr{B}_{\mathbf{m}}$.** In this subsection we compute the monodromy operator $\mathbf{B}_{\mathbf{m}} \in \mathscr{B}_{\mathbf{m}}$ for the slope subalgebra $\mathbf{B}_{\mathbf{m}}$ with $\mathbf{m} \in \mathbb{Q}^r$

[1] The reason for this name is that, by [OS22][S21] that these dynamical operators represents the q-monodromy for the quantum difference equation. We will further explain this terminology in our next paper

By [N15] the slope subalgebra $\mathscr{B}_{\mathbf{m}}$ for the quantum toroidal algebra $U_{q,t}(\hat{\hat{\mathfrak{sl}}}_r)$ is isomorphic to:

$$\mathscr{B}_{\mathbf{m}}\cong\bigotimes_{h=1}^{g}U_q(\widehat{\mathfrak{gl}}_{l_h}) \tag{3.31}$$

The isomorphism is constructed in the following way. For $\mathbf{m}\in\mathbb{Q}^r$, it is proved in [N15] that as the shuffle algebra, it can be written as the Drinfeld double of $\mathscr{B}^{\pm}_{\mathbf{m}}$:

$$\mathscr{B}_{\mathbf{m}}:=\mathscr{B}^{+}_{\mathbf{m}}\otimes\mathbb{C}[\varphi_{[i;j)}]\otimes\mathscr{B}^{-}_{\mathbf{m}}/(\text{relations}) \tag{3.32}$$

The generators for the the slope subalgebra can be written as the following: Recall the elements in 2.65 and 2.66, for $\mathbf{m}\cdot[i;j\rangle\in\mathbb{Z}$:

$$P^{\pm\mathbf{m}}_{\pm[i;j)}=\text{Sym}\left[\frac{\prod_{a=i}^{j-1}z_a^{\lfloor m_i+\dots+m_a\rfloor-\lfloor m_i+\dots+m_{a-1}\rfloor}}{t^{\text{ind }d^{m}_{[i,j\rangle}}q^{i-j}\prod_{a=i+1}^{j-1}\left(1-\frac{q_2z_a}{z_{a-1}}\right)}\prod_{i\leq a<b<j}\zeta\left(\frac{z_b}{z_a}\right)\right] \tag{3.33}$$

$$Q^{\pm\mathbf{m}}_{\mp[i;j)}=\text{Sym}\left[\frac{\prod_{a=i}^{j-1}z_a^{\lfloor m_i+\dots+m_{a-1}\rfloor-\lfloor m_i+\dots+m_a\rfloor}}{t^{-\text{ind}^{m}_{[i,j)}}\prod_{a=i+1}^{j-1}\left(1-\frac{q_1z_{a-1}}{z_a}\right)}\prod_{i\leq a<b<j}\zeta\left(\frac{z_a}{z_b}\right)\right] \tag{3.34}$$

And

$$\text{ind}^{\mathbf{m}}_{[i;j)}=\sum_{a=i}^{j-1}(m_i+\dots+m_a-\lfloor m_i+\dots+m_{a-1}\lfloor) \tag{3.35}$$

We can check that the antipode map $S_{\mathbf{m}}:\mathscr{B}_{\mathbf{m}}\to\mathscr{B}_{\mathbf{m}}$ has the following relation:

$$S_{\mathbf{m}}(P^{\mathbf{m}}_{[i;j)})=Q^{\mathbf{m}}_{[i;j)},\qquad S_{\mathbf{m}}(Q^{\mathbf{m}}_{-[i;j)})=P^{\mathbf{m}}_{-[i;j)} \tag{3.36}$$

Now the slope subalgebra $\mathscr{B}_{\mathbf{m}}$ is generated by $Q^{\pm\mathbf{m}}_{\mp[i;j)}$ and $P^{\pm\mathbf{m}}_{\pm[i;j)}$ and φ_k. And the isomorphism 2.64 is given by:

$$e_{[i;j)}=P^{\mathbf{m}}_{[i;j)_h},\qquad e_{-[i;j)}=Q^{\mathbf{m}}_{-[i;j)_h},\qquad \varphi_k=\varphi_{[k;v_{\mathbf{m}}(k))} \tag{3.37}$$

Such that as the morphism:

$$e_{[i;j)}:K_T(M(\mathbf{v},\mathbf{w}))\to K_T(M(\mathbf{v}-[i;j)_h,\mathbf{w})) \tag{3.38}$$

$$e_{-[i;j)}:K_T(M(\mathbf{v},\mathbf{w}))\to K_T(M(\mathbf{v}+[i;j)_h,\mathbf{w})) \tag{3.39}$$

In particular:

$$e_{[i;i+1)}:K_T(M(\mathbf{v},\mathbf{w}))\to K_T(M(\mathbf{v}-[i;i+1)_h,\mathbf{w})) \tag{3.40}$$

$$e_{-[i;i+1)}:K_T(M(\mathbf{v},\mathbf{w}))\to K_T(M(\mathbf{v}+[i;i+1)_h,\mathbf{w})) \tag{3.41}$$

To solve the monodromy operator $\mathbf{B_m}(\lambda)$, we need to solve the ABRR equation 3.14. Now we write down the formal solution for the monodromy operator $\mathscr{B}_{\mathbf{m}}$:

$$J^-_{\mathbf{m}}(\lambda) = \prod_{k=0}^{\overrightarrow{\infty}} \mathrm{Ad}_{(q^{\Omega}q^{-\lambda}_{(1)})^k} \mathrm{Ad}_{q^{\Omega}} R^-_{\mathbf{m}}, \qquad R^-_{\mathbf{m}} = (R_{\mathbf{m}})_{21} \tag{3.42}$$

$$\mathbf{B_m}(\lambda) = m(1 \otimes S_{\mathbf{m}}(J^-_{\mathbf{m}}(\lambda - s\mathscr{L}_{\mathbf{m}})^{-1}))|_{\lambda \to \lambda+\kappa} \tag{3.43}$$

By the isomorphism 2.29 the universal R-matrix is written as:

$$R_{\mathbf{m}} = \prod_{h=1}^{g} R_{\mathfrak{sl}_{l_h}} R_{\mathfrak{gl}_1} \tag{3.44}$$

Using the KT factorisation of $R_{\mathbf{m}}$, now the computation is reduced to the following pieces:

$$\exp_{q^{-2}}((q-q^{-1}) f_{\gamma+m\delta} \otimes e_{\gamma+m\delta}) \tag{3.45}$$

$$\exp((q-q^{-1}) \sum_{m\in\mathbb{Z}_+} \sum_{i,j=1}^{l_h} u_{m,ij} f_{m\delta,\alpha_i} \otimes e_{m\delta,\alpha_i}) \tag{3.46}$$

$$\exp_{q^{-2}}((q-q^{-1}) f_{(\delta-\gamma)+m\delta} \otimes e_{(\delta-\gamma)+m\delta}) \tag{3.47}$$

$$R_{\mathfrak{gl}_1} = \exp(\sum_{k=1}^{\infty} n_k p_k \otimes p_{-k}) \tag{3.48}$$

Now we choose specific $U_q(\hat{\mathfrak{gl}}_{l_h})$, which corresponds to the cycle $(i_1, i_2, \cdots, i_{l_h})$. And now $\delta = \sum_{k=1}^{l_h} [i_k; i_{k+1})$, the computation shows that:

$$\begin{aligned} &\mathrm{Ad}_{(q^{\Omega}q^{-\lambda}_{(1)})^k} \exp_{q^{-2}}((q-q^{-1}) f_{\gamma+m\delta} \otimes e_{\gamma+m\delta}) \\ =&\exp_{q^{-2}}((q-q^{-1}) z^{k(\mathbf{v}_\gamma+m\boldsymbol{\delta})} q^{-2k(\mathbf{v}_\gamma+m\boldsymbol{\delta})^T C(\mathbf{v}_\gamma+m\boldsymbol{\delta})} (\varphi^{-k(\mathbf{v}_\gamma+m\boldsymbol{\delta})} \otimes \varphi^{k(\mathbf{v}_\gamma+m\boldsymbol{\delta})}) f_{\gamma+m\delta} \otimes e_{\gamma+m\delta}) \end{aligned} \tag{3.49}$$

$$\begin{aligned} &\mathrm{Ad}_{(q^{\Omega}q^{-\lambda}_{(1)})^k} \exp((q-q^{-1}) \sum_{m\in\mathbb{Z}_+} \sum_{i,j=1}^{l_h} u_{m,ij} f_{m\delta,\alpha_i} \otimes e_{m\delta,\alpha_i}) \\ =&\exp((q-q^{-1}) \sum_{m\in\mathbb{Z}_+} \sum_{i,j=1}^{l_h} z^{km\boldsymbol{\delta}} q^{-2km^2\boldsymbol{\delta}^T C\boldsymbol{\delta}} (\varphi^{-km\boldsymbol{\delta}} \otimes \varphi^{km\boldsymbol{\delta}}) u_{m,ij} f_{m\delta,\alpha_i} \otimes e_{m\delta,\alpha_i}) \end{aligned} \tag{3.50}$$

$$\begin{aligned}&\mathrm{Ad}_{(q^{\Omega}q_{(1)}^{-\lambda})^k}\exp_{q^{-2}}((q-q^{-1})f_{(\delta-\gamma)+m\delta}\otimes e_{(\delta-\gamma)+m\delta})\\=&\exp_{q^{-2}}((q-q^{-1})z^{k(-\mathbf{v}_{\gamma}+(m+1)\boldsymbol{\delta})}q^{-2k(-\mathbf{v}_{\gamma}+(m+1)\boldsymbol{\delta})^TC(-\mathbf{v}_{\gamma}+(m+1)\boldsymbol{\delta})}\\&(\varphi^{-k(-\mathbf{v}_{\gamma}+(m+1)\boldsymbol{\delta})}\otimes\varphi^{k(-\mathbf{v}_{\gamma}+(m+1)\boldsymbol{\delta})})f_{(\delta-\gamma)+m\delta}\otimes e_{(\delta-\gamma)+m\delta})\end{aligned}\tag{3.51}$$

$$\begin{aligned}&\mathrm{Ad}_{(q^{\Omega}q_{(1)}^{-\lambda})^k}\exp(\sum_{k=1}^{\infty}n_kp_k\otimes p_{-k})\\=&\exp(\sum_{k=1}^{\infty}z^{km\boldsymbol{\delta}}(\varphi^{-km\boldsymbol{\delta}}\otimes\varphi^{km\boldsymbol{\delta}})n_kp_k\otimes p_{-k})\end{aligned}\tag{3.52}$$

Here the operator $\varphi^{\boldsymbol{\alpha}}\in\mathrm{End}(K_T(M(\mathbf{v},\mathbf{w}))$ acts on $K_T(M(\mathbf{v},\mathbf{w})$ as:

$$\varphi^{\boldsymbol{\alpha}}v=q^{2\alpha^T(\mathbf{w}-C\mathbf{v})}v\tag{3.53}$$

And via the shift $\lambda\to\lambda-s\mathscr{L}_{\mathbf{m}}$, this corresponds to $z\to zp^{-\mathbf{m}}$. And the shift $\lambda\to\lambda+\kappa$ corresponds to $z\to zq^{\frac{1}{2}(\mathbf{v}^TC\mathbf{v}-\mathbf{w}^T\mathbf{w})}$. And the antipode map send the $(e_{\pm[i;j)})$ basis to the $(f_{\mp[i;j)})$ basis.

In all, we have the following theorem:

Theorem 3.5. *On the representation space $End(K_T(M(\mathbf{v},\mathbf{w})))$, the monodromy operator $\mathbf{B}_{\mathbf{m}}(\lambda)$ defined in 3.28 can be written as:*

$$\begin{aligned}&\mathbf{B}_{\mathbf{m}}(\lambda)\\=&\mathbf{m}(\prod_{h=1}^{g}(\textbf{\textit{Heisenberg algebra part}})\prod_{k=0}^{\overrightarrow{\infty}}\\&\prod_{\substack{\gamma\in\Delta(A)\\m\geq0}}^{\leftarrow}(\exp_{q^2}(-(q-q^{-1})z^{k(-\mathbf{v}_{\gamma}+(m+1)\boldsymbol{\delta}_h)}p^{k\mathbf{m}\cdot(-\mathbf{v}_{\gamma}+(m+1)\boldsymbol{\delta}_h)}q^{\frac{k}{2}(\mathbf{v}^TC\mathbf{v}-\mathbf{w}^T\mathbf{w})^{\grave{}}\cdot(-\mathbf{v}_{\gamma}+(m+1)\boldsymbol{\delta}_h))}(\\&q^{-2k(-\mathbf{v}_{\gamma}+(m+1)\boldsymbol{\delta})^TC(-\mathbf{v}_{\gamma}+(m+1)\boldsymbol{\delta})}\varphi^{-(k+1)(-\mathbf{v}_{\gamma}+(m+1)\boldsymbol{\delta})}f_{(\delta-\gamma)+m\delta}\otimes e'_{(\delta-\gamma)+m\delta}\varphi^{-(k+1)(-\mathbf{v}_{\gamma}+(m+1)\boldsymbol{\delta})})\\&\times\exp(-(q-q^{-1})\sum_{m\in\mathbb{Z}_+}\sum_{i,j=1}^{l_h}z^{km\boldsymbol{\delta}_h}p^{km\mathbf{m}\cdot\boldsymbol{\delta}_h}u_{m,ij}q^{-2km^2\boldsymbol{\delta}^TC\boldsymbol{\delta}}\varphi^{-(k+1)m\boldsymbol{\delta}}f_{m\delta,\alpha_i}\otimes e'_{m\delta,\alpha_i}\varphi^{-(k+1)m\boldsymbol{\delta}})\\&\times\prod_{\substack{\gamma\in\Delta(A)\\m\geq0}}^{\rightarrow}\exp_{q^2}(-(q-q^{-1})z^{k(\mathbf{v}_{\gamma}+m\boldsymbol{\delta}_h)}p^{k\mathbf{m}\cdot(\mathbf{v}_{\gamma}+m\boldsymbol{\delta}_h)}q^{\frac{k}{2}(\mathbf{v}^TC\mathbf{v}-\mathbf{w}^T\mathbf{w})^{\grave{}}\cdot(\mathbf{v}_{\gamma}+m\boldsymbol{\delta}_h))}\\&q^{-2k(\mathbf{v}_{\gamma}+m\boldsymbol{\delta})^TC(\mathbf{v}_{\gamma}+m\boldsymbol{\delta})}(\varphi^{-(k+1)(\mathbf{v}_{\gamma}+m\boldsymbol{\delta})}f_{\gamma+m\delta}\otimes e'_{\gamma+m\delta}\varphi^{-(k+1)(\mathbf{v}_{\gamma}+m\boldsymbol{\delta})})))\end{aligned}\tag{3.54}$$

Here $e'_v = S(e_v)$, $f'_v = S(f_v)$ are the image of the antipode map. And e_α, f_α here stands for the generators in $\mathscr{B}_{\mathbf{m}}$ without using $\mathbf{m}$*. The* ***Heisenberg algebra part*** *denotes the formula of the form 4.25 And it can be constructed iteratively by the following formula:*

$$S(P^{\pm\mathbf{m}}_{\pm[i;j\rangle}) = Q^{\pm\mathbf{m}}_{\pm[i;j\rangle}, \qquad S(Q^{\pm\mathbf{m}}_{\pm[i;j\rangle}) = P^{\pm\mathbf{m}}_{\pm[i;j\rangle} \tag{3.55}$$

with $P^{\pm\mathbf{m}}_{\pm[i;j\rangle}$ and $Q^{\pm\mathbf{m}}_{\pm[i;j\rangle}$ defined in 3.33 and 3.34.

Remark. Note that in the formula 4.43 only for finitely many roots $\gamma \in \Delta(A)$, the corresponding generator e_γ have nontrivial action on $K_G(M(\mathbf{v}, \mathbf{w}))$.

A useful consequence about the convergence of the monodromy operator $\mathbf{B}_{\mathbf{m}}(\lambda)$ can be stated in the following, it has the potential application of analyzing the convergence of the solution for the difference equation. We won't use the result in this paper.

Proposition 3.6. *The monodromy operator $\mathbf{B}_{\mathbf{m}}$ is convergent in each $K_T(M(\mathbf{v}, \mathbf{w}))$ of $\mathscr{B}_{\mathbf{m}}$ for $\{z_i\}$ sufficiently small.*

Proof. This is basically the imitation of the proof of the proposition 5 in [ABRR97] with using the isomorphism 2.64 □

3.4. **Some properties of $\mathbf{B}_{\mathbf{m}}(\lambda)$.** We revisit essential properties of $\mathbf{B}_{\mathbf{m}}(\lambda)$. Most of these properties have been established in the K-theoretic stable envelope language, as detailed in [OS22]. It is noteworthy that we have successfully adapted and replicated similar formulations within the language of the quantum toroidal algebra by requiring some modification in the construction.

We present the following theorem without proof in our current framework, for the proof, see Theorem 5 and theorem 6 in [OS22], where the same principles apply within the quantum toroidal algebra setting.

Theorem 3.7. *The operators $J^{\pm}_{\mathbf{m}}(\lambda)$ satisfy the dynamical cocycle conditions on $End(K_G(M(\mathbf{v}_1, \mathbf{w}_1) \times M(\mathbf{v}_2, \mathbf{w}_2) \times M(\mathbf{v}_3, \mathbf{w}_3)))$:*

$$\begin{aligned} J^{-}_{\mathbf{m}}(\lambda)^{12,3} J^{-}_{\mathbf{m}}(\lambda + \kappa_{(3)})^{12} &= J^{-}_{\mathbf{m}}(\lambda)^{1,23} J^{-}_{\mathbf{m}}(\lambda - \kappa_{(1)})^{23} \\ J^{+}_{\mathbf{m}}(\lambda + \kappa_{(3)})^{12} J^{+}_{\mathbf{m}}(\lambda)^{12,3} &= J^{+}_{\mathbf{m}}(\lambda - \kappa_{(1)})^{23} J^{+}_{\mathbf{m}}(\lambda)^{1,23} \end{aligned} \tag{3.56}$$

in $\mathscr{B}_{\mathbf{m}} \hat{\otimes} \mathscr{B}_{\mathbf{m}} \hat{\otimes} \mathscr{B}_{\mathbf{m}}$. Here $\kappa(\mathbf{v}, \mathbf{w}) = (C\mathbf{v} - \mathbf{w})/2$ is the dynamical shift operator, which is the product of $\varphi_{i,0}$ in $U_{q,t}(\hat{\hat{\mathfrak{sl}}}_r)$. Moreover, we set

$$\tilde{B}_{\mathbf{m}}(\lambda) = \mathbf{m}(1 \otimes S_{\mathbf{m}}(J^{-}_{\mathbf{m}}(\lambda)^{-1}))|_{\lambda \mapsto \lambda + \kappa}, \qquad B_{\mathbf{m}}(\lambda) = \mathbf{m}_{21}(S^{-1}_{\mathbf{m}} \otimes 1(J^{-}_{\mathbf{m}}(\lambda)^{-1}))|_{\lambda \mapsto \lambda - \kappa} \tag{3.57}$$

We have the following coproduct formula:

$$\Delta_{\mathbf{m}} \tilde{B}_{\mathbf{m}}(\lambda) = J^{-}_{\mathbf{m}}(\lambda)(\tilde{B}_{\mathbf{m}}(\lambda + \kappa_{(2)}) \otimes \tilde{B}_{\mathbf{m}}(\lambda - \kappa_{(1)})) J^{+}_{\mathbf{m}}(\lambda) \tag{3.58}$$

$$\Delta_{\mathbf{m}} B_{\mathbf{m}}(\lambda) = J^{-}_{\mathbf{m}}(\lambda)(B_{\mathbf{m}}(\lambda + \kappa_{(2)}) \otimes B_{\mathbf{m}}(\lambda - \kappa_{(1)})) J^{+}_{\mathbf{m}}(\lambda) \tag{3.59}$$

Now with the shift $\lambda \to \lambda - \tau_{\mathbf{m}}$, we have the coproduct formula for $\mathbf{B}_{\mathbf{m}}(\lambda)$:

$$\Delta_{\mathbf{m}}(\mathbf{B}_{\mathbf{m}}(\lambda)) = \mathbf{J}^{-}_{\mathbf{m}}(\lambda)(\mathbf{B}_{\mathbf{m}}(\lambda + \kappa_{(2)}) \otimes \mathbf{B}_{\mathbf{m}}(\lambda - \kappa_{(1)}))\mathbf{J}^{+}_{\mathbf{m}}(\lambda) \tag{3.60}$$

Now given a root subalgebra $\mathscr{B}_{\mathbf{m}} \subset U_{q,t}(\hat{\hat{\mathfrak{sl}}}_r)$ and its corresponding universal R-matrix $R^+_{\mathbf{m}}$ and $R^-_{\mathbf{m}}$, we set the qKZ equation for each slope subalgebra $\mathscr{B}_{\mathbf{m}}$.:

$$\mathscr{R}^+_{\mathbf{m}} = T_u q^{-\lambda}_{(1)} q^{\Omega} R^+_{\mathbf{m}}, \qquad \mathscr{R}^-_{\mathbf{m}} = R^-_{\mathbf{m}} q^{\Omega} T_u q^{-\lambda}_{(1)} \tag{3.61}$$

It can be checked that it commutes with the monodromy operator:

Proposition 3.8.

$$\mathscr{R}^-_{\mathbf{m}} \Delta_{\mathbf{m}}(\mathbf{B}_{\mathbf{m}}(\lambda)) = \Delta_{\mathbf{m}}(\mathbf{B}_{\mathbf{m}}(\lambda))\mathscr{R}^+_{\mathbf{m}} \tag{3.62}$$

Proof. From the ABRR equation and the above coproduct formula, we have the formula, for details see proposition 11 in [OS22]. □

Proposition 3.9. *For $\mathscr{L} \in Pic(X)$, where $X = M(\mathbf{v}, \mathbf{w})$ is an affine type A quiver variety, we have the following translation formula:*

$$\mathscr{L}\mathbf{B}_{\mathbf{m}}(\lambda - s\mathscr{L}) = \mathbf{B}_{\mathbf{m}+\mathscr{L}}(\lambda)\mathscr{L} \tag{3.63}$$

Proof. The stable envelope version of this formula was proved in the proposition 12 of [OS22] , for which the result holds for all type quiver varieties. Our proof here is to compare the matrix coefficients on both sides of the equality.

For two wall elements $\mathbf{m}$ and $\mathbf{m}'$, it is known that their corresponding slope subalgebra $\mathscr{B}_{\mathbf{m}}$ and $\mathscr{B}_{\mathbf{m}'}$ are isomorphic if $\mathbf{m}' - \mathbf{m} \in \mathbb{Z}^{|I|}$.

Let $X = M(\mathbf{v}, \mathbf{w})$ and $A \subset T$ such that $X^A = \sqcup_{\mathbf{v}_1+\mathbf{v}_2=\mathbf{v}} M(\mathbf{v}_1, \mathbf{w}_1) \times M(\mathbf{v}_2, \mathbf{w}_2) = \sqcup_i F_i$.

Now for $\mathscr{L} \in \text{Pic}(X)$, we compare the action of $\mathscr{L}\mathbf{B}_{\mathbf{m}}(\lambda - s\mathscr{L})\mathscr{L}^{-1}$ on $K_T(X)$ and that of $\mathbf{B}_{\mathbf{m}+\mathscr{L}}(\lambda)$, via computation we can see that $\mathscr{L}\mathbf{B}_{\mathbf{m}}(\lambda - s\mathscr{L})\mathscr{L}^{-1} \in U_q(\mathfrak{g}_{\mathbf{m}+\mathscr{L}})$. Thus it is reduced to prove that $\mathbf{B}_{\mathbf{m}+\mathscr{L}}(\lambda)$ and $\mathscr{L}\mathbf{B}_{\mathbf{m}}(\lambda - s\mathscr{L})\mathscr{L}^{-1}$' corresponding element $U(\mathscr{L})J^-_{\mathbf{m}}(\lambda)U(\mathscr{L})^{-1}$ and $J^-_{\mathbf{m}+\mathscr{L}}(\lambda)$ solve the same ABRR equation for $R^-_{\mathbf{m}+\mathscr{L}}$. And here $U(\mathscr{L}) \in \text{End}(K_T(X^A))$ is the block diagonal operator such that:

$$U(\mathscr{L})|_{F_i \times F_i} = \mathscr{L}|_{F_i} \tag{3.64}$$

We need to show that for $R_{\mathbf{m}}$ acting on $K_T(X^A)$, $R_{\mathbf{m}}$ has the following translation formula:

$$U(\mathscr{L})R^{\pm}_{\mathbf{m}}U(\mathscr{L})^{-1} = R^{\pm}_{\mathbf{m}+\mathscr{L}} \tag{3.65}$$

Using the expression of the KT factorisation of $R_{\mathbf{m}}$, it is equivalent to prove the following identity:

$$\mathscr{L}P^{\mathbf{m}}_{\pm[i;j)}\mathscr{L}^{-1} = P^{\mathbf{m}+\mathscr{L}}_{\pm[i;j)}, \qquad \mathscr{L}Q^{\mathbf{m}}_{\pm[i;j)}\mathscr{L}^{-1} = Q^{\mathbf{m}+\mathscr{L}}_{\pm[i;j)} \tag{3.66}$$

Here we only show the proof for the case of $P^{\mathbf{m}}_{\pm[i;j)}$. The proof for the case $Q^{\mathbf{m}}_{\pm[i;j)}$ is the same. We examine this via the matrix element of $\mathscr{L}P^{\mathbf{m}}_{[i;j)}\mathscr{L}^{-1}$ in $K(\mathbf{w})$. Recall that the matrix element of $P^{\mathbf{m}}_{[i;j)}$ is given by [N15]:

$$\langle\boldsymbol{\lambda}|P^{\mathbf{m}}_{[i;j)}|\boldsymbol{\mu}\rangle = P^{\mathbf{m}}_{[i;j)}(\boldsymbol{\lambda}\backslash\boldsymbol{\mu})\prod_{\blacksquare\in\boldsymbol{\lambda}\backslash\boldsymbol{\mu}}[\prod_{\square\in\boldsymbol{\mu}}\zeta(\frac{\chi_{\blacksquare}}{\chi_{\square}})\prod_{k=1}^{\mathbf{w}}[\frac{u_k}{q\chi_{\blacksquare}}]] \tag{3.67}$$

Here $P^{\mathbf{m}}_{[i;j)}(\boldsymbol{\lambda}\backslash\boldsymbol{\mu})$ is given by:

$$P^{\mathbf{m}}_{[i;j)}(\boldsymbol{\lambda}\backslash\boldsymbol{\mu}) = \sum_{\text{ASYT}^+}\prod_{a=i}^{j-1}\chi_a^{m_a}\cdot\frac{\prod_{a=i}^{j-1}(\chi_a t^{-a})^{s_a}}{q^{-1}\prod_{a=i+1}^{j-1}\left(\frac{1}{q}-\frac{\chi_a}{t\chi_{a-1}}\right)}\prod_{i\leq a<b<j}\zeta\left(\frac{\chi_b}{\chi_a}\right) \tag{3.68}$$

And here:

$$s_a = \lfloor m_i+\ldots+m_a\rfloor - \lfloor m_i+\ldots+m_{a-1}\rfloor - m_a \tag{3.69}$$

And ASYT$^+$ is the set of bijection $\psi : \{\boldsymbol{\lambda}\backslash\boldsymbol{\mu}\} \to \{i,\cdots,j-1\}$ such that $\psi(\square) =$ color of $\square$

Now since $\mathscr{L}$ is the product of the tautological bundle, for simplicity, we assume that $\mathscr{L} = \mathscr{L}_l$ is the generator corresponding to the vector space V_l in the quiver. In this case we have that:

$$\langle\boldsymbol{\lambda}|\mathscr{L}P^{\mathbf{m}}_{\pm[i;j)}\mathscr{L}^{-1}|\boldsymbol{\mu}\rangle = \prod_{\square\in\boldsymbol{\lambda}\backslash\boldsymbol{\mu}}^{c_\square=l}\chi_\square P^{\mathbf{m}}_{[i;j)}(\boldsymbol{\lambda}\backslash\boldsymbol{\mu})\prod_{\blacksquare\in\boldsymbol{\lambda}\backslash\boldsymbol{\mu}}[\prod_{\square\in\boldsymbol{\mu}}\zeta(\frac{\chi_{\blacksquare}}{\chi_{\square}})\prod_{k=1}^{\mathbf{w}}[\frac{u_k}{q\chi_{\blacksquare}}]] \tag{3.70}$$

Since the expreesion after $P^{\mathbf{m}}_{[i;j)}(\boldsymbol{\lambda}\backslash\boldsymbol{\mu})$ only depends on $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$, we only need to compute $\prod_{\square\in\boldsymbol{\lambda}\backslash\boldsymbol{\mu}}\chi_\square P^{\mathbf{m}}_{[i;j)}(\boldsymbol{\lambda}\backslash\boldsymbol{\mu})$, note that:

$$\begin{aligned}&\prod_{\square\in\boldsymbol{\lambda}\backslash\boldsymbol{\mu}}\chi_\square P^{\mathbf{m}}_{[i;j)}(\boldsymbol{\lambda}\backslash\boldsymbol{\mu})\\ =&\prod_{\square\in\boldsymbol{\lambda}\backslash\boldsymbol{\mu}}^{c_\square=l}\chi_\square\sum_{\text{ASYT}^+}\prod_{a=i}^{j-1}\chi_a^{m_a}\cdot\frac{\prod_{a=i}^{j-1}(\chi_a t^{-a})^{s_a}}{q^{-1}\prod_{a=i+1}^{j-1}\left(\frac{1}{q}-\frac{\chi_a}{t\chi_{a-1}}\right)}\prod_{i\leq a<b<j}\zeta\left(\frac{\chi_b}{\chi_a}\right)\end{aligned} \tag{3.71}$$

We can observe that in the expreesion, $\chi_\square$ contains all the box in $\boldsymbol{\lambda}\backslash\boldsymbol{\mu}$ such that they are in the color l, while the expression $\prod_{a=i}^{j-1}\chi_a^{m_a}$ contains the product $\prod_{i\leq a\leq j-1}^{c_{\psi^{-1}(a)}=l}\chi_{\psi^{-1}(a)}^{m_a} =$

$\prod_{\square\in\boldsymbol{\lambda}\backslash\boldsymbol{\mu}}^{c_\square=l}\chi_\square^{m_{\psi(\square)}}$, thus from this expression we have that:

$$\begin{aligned}&\prod_{\square\in\boldsymbol{\lambda}\backslash\boldsymbol{\mu}}^{c_\square=l}\chi_\square\sum_{\mathrm{ASYT}^+}\prod_{a=i}^{j-1}\chi_a^{m_a}\cdot\frac{\prod_{a=i}^{j-1}(\chi_at^{-a})^{s_a}}{q^{-1}\prod_{a=i+1}^{j-1}\left(\frac{1}{q}-\frac{\chi_a}{t\chi_{a-1}}\right)}\prod_{i\leq a<b<j}\zeta\left(\frac{\chi_b}{\chi_a}\right)\\ =&\sum_{\mathrm{ASYT}^+}\prod_{a=i}^{j-1}\chi_a^{m_a+\delta_{ai}}\cdot\frac{\prod_{a=i}^{j-1}(\chi_at^{-a})^{s_a}}{q^{-1}\prod_{a=i+1}^{j-1}\left(\frac{1}{q}-\frac{\chi_a}{t\chi_{a-1}}\right)}\prod_{i\leq a<b<j}\zeta\left(\frac{\chi_b}{\chi_a}\right)\end{aligned}\tag{3.72}$$

Also note that doing the integral translation $\mathbf{m}\mapsto\mathbf{m}+\mathscr{L}_i$ doesn't change s_a , thus we have proved that:

$$\mathscr{L}P_{\pm[i;j)}^{\mathbf{m}}\mathscr{L}^{-1}=P_{\pm[i;j)}^{\mathbf{m}+\mathscr{L}}\tag{3.73}$$

And similar proof also applied for $Q_{\pm[i;j)}^{\mathbf{m}}$.

In conclusion, we can see that $U(\mathscr{L})J_{\mathbf{m}}^-(\lambda)U(\mathscr{L})^{-1}$ and $J_{\mathbf{m}+\mathscr{L}}^-(\lambda)$ solve the same ABRR equation. By the uniqueness of the solution of the ABRR equation, we have that:

$$U(\mathscr{L})J_{\mathbf{m}}^-(\lambda)U(\mathscr{L})^{-1}=J_{\mathbf{m}+\mathscr{L}}^-(\lambda)\tag{3.74}$$

Thus finish the proof. □

Remark. We will give a more general construction of the proposition 3.9 in our next paper. The proof actually can be more intrinsic without using the specific basis.

3.5. **Main result for the quantum difference operator $\mathbf{B}_{\mathscr{L}}^s(\lambda)$.** In this subsection we prove that the quantum difference operator $\mathbf{B}_{\mathscr{L}}^s(\lambda)$ is independent of the choice of the slope generically.

Now we fix a universal R-matrix R^s for $U_{q,t}(\hat{\hat{\mathfrak{sl}}}_n)$ with respect to the factorization

$$U_{q,t}(\hat{\hat{\mathfrak{sl}}}_n)=\bigotimes_{m\in\mathbb{Q}}^{\rightarrow}\mathscr{B}_{s+m\boldsymbol{\theta}}^+\otimes\mathscr{B}_{\infty\boldsymbol{\theta}}\otimes\bigotimes_{m\in\mathbb{Q}}^{\rightarrow}\mathscr{B}_{s+m\boldsymbol{\theta}}^-\tag{3.75}$$

And $\mathscr{R}^s(u)$ be the image of R^s acting on $K_G(M(\mathbf{w}'))\otimes K_G(M(\mathbf{w}''))$. Define the qKZ operator with the slope s by

$$\mathscr{R}^s=q_{(1)}^{(\lambda)}T_u^{-1}\mathscr{R}^s(u)\tag{3.76}$$

and we set $\mathscr{B}_{\mathscr{L}}^s=T_{\mathscr{L}}^{-1}\mathbf{B}_{\mathscr{L}}^s(\lambda)$.

Proposition 3.10. *On $K_T(M(\mathbf{v}_1,\mathbf{w}_1))\otimes K_T(M(\mathbf{v}_2,\mathbf{w}_2))$ we have that*

$$\Delta_s(\mathscr{B}_{\mathscr{L}}^s)=W_{\mathbf{m}_0}(\lambda)W_{\mathbf{m}_1}(\lambda)\cdots W_{\mathbf{m}_{m-1}}(\lambda)\Delta_\infty(\mathscr{L})T_{\mathscr{L}}^{-1}\tag{3.77}$$

with $\mathbf{m}_0, \cdots, \mathbf{m}_{m-1}$ *the ordered set of slope points from* s *to* $s+\mathscr{L}$ *with slope having no nonzero component.* $W_{\mathbf{m}}(\lambda) = \Delta_{\mathbf{m}}(\mathbf{B}_{\mathbf{m}}(\lambda))(R^+_w)^{-1}$ *and* Δ_∞ *is the original coproduct of* $U_{q,t}(\hat{\hat{\mathfrak{sl}}}_r)$.

Proof. By definition we know that:

$$\mathscr{B}^s_{\mathscr{L}} = T^{-1}_{\mathscr{L}}\mathscr{L}\mathbf{B}_{\mathbf{m}_{-m}}(\lambda)\cdots\mathbf{B}_{\mathbf{m}_{-2}}(\lambda)\mathbf{B}_{\mathbf{m}_{-1}}(\lambda) \tag{3.78}$$

Here $\mathbf{m}_{-1}, \cdots, \mathbf{m}_{-m}$ is the ordered set of slope points between the s and $s-\mathscr{L}$. By the translation symmetry in the proposition 3.9 we know that:

$$\mathscr{B}^s_{\mathscr{L}} = \mathbf{B}_{\mathbf{m}_0}(\lambda)\mathbf{B}_{\mathbf{m}_1}(\lambda)\cdots\mathbf{B}_{\mathbf{m}_{m-1}}(\lambda)\mathscr{L}T^{-1}_{\mathscr{L}} \tag{3.79}$$

And here $\mathbf{m}_{k+m} = \mathbf{m}_k + \mathscr{L}$.

Now use the coproduct formula:

$$\Delta_s(\mathscr{B}^s_{\mathscr{L}}) = \Delta_s(\mathbf{B}_{\mathbf{m}_0}(\lambda)\mathbf{B}_{\mathbf{m}_1}(\lambda)\cdots\mathbf{B}_{\mathbf{m}_{m-1}}(\lambda)\mathscr{L})T^{-1}_{\mathscr{L}} \tag{3.80}$$

Note that the coproduct Δ_s and $\Delta_{\mathbf{m}_k}$ is differed as follows:

$$\Delta_s(\mathbf{B}_{\mathbf{m}_k}(\lambda)) = (T^-_{s,\mathbf{m}_k})\Delta_{\mathbf{m}_k}(\mathbf{B}_{\mathbf{m}_k}(\lambda))T^+_{s,\mathbf{m}_k} \tag{3.81}$$

And write it explicitly we have that:

$$\Delta_s(\mathbf{B}_{\mathbf{m}_k}(\lambda)) = (R^+_{\mathbf{m}_0})^{-1}\cdots(R^+_{\mathbf{m}_{k-1}})^{-1}\Delta_{\mathbf{m}_k}(\mathbf{B}_{\mathbf{m}_k}(\lambda))R^+_{\mathbf{m}_{k-1}}\cdots R^+_{\mathbf{m}_0} \tag{3.82}$$

Thus we have that:

$$\Delta_s(\mathscr{B}^s_{\mathscr{L}}) = \Delta_{\mathbf{m}_0}(\mathbf{B}_{\mathbf{m}_0}(\lambda))(R^+_{\mathbf{m}_0})^{-1}\cdots\Delta_{\mathbf{m}_{m-1}}(\mathbf{B}_{\mathbf{m}_{m-1}})(R^+_{\mathbf{m}_{m-1}})^{-1}R^+_{\mathbf{m}_{m-1}}\cdots R^+_{\mathbf{m}_0}\Delta_s(\mathscr{L})T^{-1}_{\mathscr{L}} \tag{3.83}$$

$$= \Delta_{\mathbf{m}_0}(\mathbf{B}_{\mathbf{m}_0}(\lambda))(R^+_{\mathbf{m}_0})^{-1}\cdots\Delta_{\mathbf{m}_{m-1}}(\mathbf{B}_{\mathbf{m}_{m-1}})(R^+_{\mathbf{m}_{m-1}})^{-1}\Delta_\infty(\mathscr{L})T^{-1}_{\mathscr{L}} \tag{3.84}$$

Thus finish the proof. ∎

Theorem 3.11. *On the space* $K_T(M(\mathbf{v},\mathbf{w}))\otimes K_T(M(\mathbf{v},\mathbf{w}))$, *for two slopes* s *and* s' *separated by a single slope points* $\mathbf{m}$, *we have:*

$$W_{\mathbf{m}}(\lambda)^{-1}\mathscr{R}^s W_{\mathbf{m}}(\lambda) = \mathscr{R}^{s'}, \qquad W_{\mathbf{m}}(\lambda)^{-1}\Delta_s(\mathscr{B}^s_{\mathscr{L}})W_{\mathbf{m}}(\lambda) = \Delta_{s'}(\mathscr{B}^{s'}_{\mathscr{L}}) \tag{3.85}$$

Proof. By the computation:

$$\begin{aligned}
\mathscr{R}^s W_{\mathbf{m}}(\lambda) =& q^\lambda_{(1)}T_u^{-1}\mathscr{R}^s(u)\Delta_{\mathbf{m}}(\mathbf{B}_{\mathbf{m}}(\lambda))(R^+_{\mathbf{m}})^{-1} \\
=& q^\lambda_{(1)}T_u^{-1}\Delta^{op}_{\mathbf{m}}(\mathbf{B}_{\mathbf{m}}(\lambda))\mathscr{R}^s(u)(R^+_{\mathbf{m}})^{-1} \\
=& q^\lambda_{(1)}T_u^{-1}(R^-_{\mathbf{m}})^{-1}(R^-_{\mathbf{m}})\Delta^{op}_{\mathbf{m}}(\mathbf{B}_{\mathbf{m}}(\lambda))\mathscr{R}^s(u)(R^+_{\mathbf{m}})^{-1} \\
=& q^\lambda_{(1)}T_u^{-1}(R^-_{\mathbf{m}})^{-1}\Delta_{\mathbf{m}}(\mathbf{B}_{\mathbf{m}}(\lambda))(R^-_{\mathbf{m}})\mathscr{R}^s(u)(R^+_{\mathbf{m}})^{-1} \\
=& \Delta_{\mathbf{m}}(\mathbf{B}_{\mathbf{m}}(\lambda))(R^+_{\mathbf{m}})^{-1}q^\lambda_{(1)}T_u^{-1}\mathscr{R}^{s'}(u) = W_{\mathbf{m}}(\lambda)\mathscr{R}^{s'}
\end{aligned} \tag{3.86}$$

Now for the second equality, we leave it as an exercise for the readers. ∎

Theorem 3.12. *For arbitrary line bundles $\mathscr{L},\mathscr{L}'\in Pic(M(\mathbf{v},\mathbf{w}))$ and a slope s we have that the qKZ operators commute with the q-difference operators*

$$\Delta_s(\mathscr{B}^s_{\mathscr{L}})\mathscr{R}^s=\mathscr{R}^s\Delta_s(\mathscr{B}^s_{\mathscr{L}}) \tag{3.87}$$

Proof.

$$\begin{aligned}\mathscr{R}^s\Delta_s(\mathscr{B}^s_{\mathscr{L}})=&\mathscr{R}^sW_{\mathbf{m}_0}(\lambda)W_{\mathbf{m}_1}(\lambda)\cdots W_{\mathbf{m}_{m-1}}(\lambda)\Delta_{\infty}(\mathscr{L})T_{\mathscr{L}}^{-1}\\ =&W_{\mathbf{m}_0}(\lambda)W_{\mathbf{m}_1}(\lambda)\cdots W_{\mathbf{m}_{m-1}}(\lambda)\mathscr{R}^{s+\mathscr{L}}\Delta_{\infty}(\mathscr{L})T_{\mathscr{L}}^{-1}\\ =&\Delta_s(\mathscr{B}^s_{\mathscr{L}})\mathscr{R}^s\end{aligned} \tag{3.88}$$

□

Now we come to the uniqueness of the monodromy operator:

Theorem 3.13. *The monodromy operator $\mathbf{B}^s_{\mathscr{L}}$ is independent of the choice of the path from s to $s-\mathscr{L}$ as long as the real slope of the path contains no zero.*

Proof. Given $\mathbf{B}^s_{\mathscr{L}}(u,\lambda)'$ and $\mathbf{B}^s_{\mathscr{L}}(u,\lambda)$ two quantum difference operators with different paths acting on $K_{T\times\mathbb{C}_q^\times}(M(\mathbf{v},\mathbf{w}))_{loc}$ with u is the equivariant parametre from the torus. Now consider

$$D(u)=\mathbf{B}^s_{\mathscr{L}}(u,\lambda)^{-1}\mathbf{B}^s_{\mathscr{L}}(u,\lambda)'\in\text{End}(K_{T\times\mathbb{C}_q^\times}(M(\mathbf{v},\mathbf{w}))_{loc}) \tag{3.89}$$

Since $\mathscr{B}^s_{\mathscr{L}}$ commute with $\mathscr{R}^s$, we have that $[\Delta_s(D(u)),\mathscr{R}^s]=0$, and this means that:

$$\Delta_s(D(up))=q^{(\lambda)}_{(1)}\mathscr{R}^s(u)\Delta_s(D(u))(q^{(\lambda)}_{(1)}\mathscr{R}^s(u))^{-1} \tag{3.90}$$

And $D(u)$ is a Laurent polynomials in u by the proposition 2.9 and the ABRR equation in the proposition 3.1.

We can write down $D(u)=\sum_{n=-m_1}^{m_2}D_nu^n$

Now use the KT-type factorization for the slope $\mathbf{m}$ R-matrix $R_{\mathbf{m}}\in\mathscr{B}_{\mathbf{m}}\hat{\otimes}\mathscr{B}_{\mathbf{m}}$ and the rationality of the R-matrix $\mathscr{R}^s(u)$ in the Proposition 2.11, we have that:

$$\mathscr{R}^s(\infty)=q^{\Omega}\overleftarrow{\prod}R_{\mathbf{m}}(\infty),\qquad\mathscr{R}^s(0)=\overrightarrow{\prod}(R^-_{\mathbf{m}})^{-1}q^{\Omega} \tag{3.91}$$

Note that $R_{\mathbf{m}}$'s are strictly upper and lower triangular wall R-matrices with diagonal being 1. Therefore the eigenvalues of the adjoint by $q^{\lambda}_{(1)}\mathscr{R}^s(u)$ are either 1 or z^mq^n, and here $m\neq0$. And also it is regular at $u=0$ and $u=\infty$. Thus now $D(u)=D$ is a constant matrix in u and it is block-diagonal with respect to the decomposition:

$$K_{T\times\mathbb{C}_q^\times}(M(\mathbf{w}))_{loc}=\bigoplus_{\mathbf{v}}K_{T\times\mathbb{C}_q^\times}(M(\mathbf{v},\mathbf{w}))_{loc} \tag{3.92}$$

Now we can rewrite the equation 3.90 as:

$$\Delta_s(D) = q^{(\lambda)}_{(1)}\mathscr{R}^s(u)\Delta_s(D)(q^{(\lambda)}_{(1)}\mathscr{R}^s(u))^{-1} \tag{3.93}$$

If we restrict the operator $\Delta_s(D)$ to $K_{T\times\mathbb{C}^\times_q}(M(\mathbf{v}_1,\mathbf{w}_1))_{loc}\otimes K_{T\times\mathbb{C}^\times_q}(M(\mathbf{v}_2,\mathbf{w}_2))_{loc}$, the equation can be restricted to the following:

$$\Delta_s(D) = q^{(\lambda)}_{(1)}\mathscr{R}^s(u)_{(\mathbf{v}_1,\mathbf{w}_1),(\mathbf{v}_2,\mathbf{w}_2)}\Delta_s(D)(q^{(\lambda)}_{(1)}\mathscr{R}^s(u)_{(\mathbf{v}_1,\mathbf{w}_1),(\mathbf{v}_2,\mathbf{w}_2)})^{-1} \tag{3.94}$$

and $\mathscr{R}^s(u)_{(\mathbf{v}_1,\mathbf{w}_1),(\mathbf{v}_2,\mathbf{w}_2)}$ is just the multiplication of the tautological bundles, which is diagonal for $K_{T\times\mathbb{C}^\times_q}(M(\mathbf{v}_1,\mathbf{w}_1))_{loc}\otimes K_{T\times\mathbb{C}^\times_q}(M(\mathbf{v}_2,\mathbf{w}_2))_{loc}$. And since $\mathscr{R}^s(u)_{(\mathbf{v}_1,\mathbf{w}_1),(\mathbf{v}_2,\mathbf{w}_2)}$ has its diagonal element generically different for $\forall\mathbf{v}_1,\mathbf{w}_1,\mathbf{v}_2,\mathbf{w}_2$, thus we can conclude that $\Delta_s(D)$ is a diagonal matrix.

Next by the formula in the proposition 4.1 we know that only the tautological bundle term $\mathscr{L}$ contribute to the digonal part, thus we have that $\Delta_s(D)$ is a constant diagonal matrix.

Finally we know that the monodromy operators $\mathbf{B}_w(\lambda)$ are normally ordered with respect to the path. Thus there exists a component of minimal weight γ such that $\mathbf{B}_w(\lambda)\gamma=\gamma$, which means $D\gamma=\gamma$, and thus D is the idenitity matrix.

Thus the proof is finished. □

There is digression for this lemma. Despite the fact that the lemma tells us that we can approach the vertical or the horizontal slope, but one by computation can find that there might be approximating infinitely many walls. So on the horizontal and the vertical walls, we need some modified definition of the wall, which will be discussed in the next paper.

Theorem 3.14. *For arbitrary line bundles $\mathscr{L}$, $\mathscr{L}'\in Pic(M(\mathbf{v},\mathbf{w}))$ and slopes $s\in\mathbb{Q}^r$ the corresponding q-difference operators commute:*

$$\mathscr{B}^s_{\mathscr{L}}\mathscr{B}^s_{\mathscr{L}'} = \mathscr{B}^s_{\mathscr{L}'}\mathscr{B}^s_{\mathscr{L}} \tag{3.95}$$

Proof. This follows from the fact that $\mathscr{B}^s_{\mathscr{L}}\mathscr{B}^s_{\mathscr{L}'} = \mathscr{B}^s_{\mathscr{L}+\mathscr{L}'} = \mathscr{B}^s_{\mathscr{L}'}\mathscr{B}^s_{\mathscr{L}}$ □

Now in conclusion, for a given quiver variety $M(\mathbf{v},\mathbf{w})$ of affine type A_{r-1}, let $\mathscr{L}_i$, $i=1,\cdots,r$ be the generators of $\mathrm{Pic}(M(\mathbf{v},\mathbf{w}))$. We have the following main result in this paper:

Theorem 3.15. *Given an affine type A quiver variety $M(\mathbf{v},\mathbf{w})$ and line bundles $\mathscr{L}\in Pic(M(\mathbf{v},\mathbf{w}))$, there exists a holonomic system of operators $\mathbf{M}_{\mathscr{L}}\in End(K_T(M(\mathbf{v},\mathbf{w}))[[p^{\mathscr{L}}]]_{\mathscr{L}\in Pic(M(\mathbf{v},\mathbf{w}))})$, which is defined as $Const\cdot\mathbf{B}_{\mathscr{L}}(\lambda)$ defined in 3.29, i.e.*

$$T^{-1}_{p,\mathscr{L}}\mathbf{M}_{\mathscr{L}}T^{-1}_{p,\mathscr{L}'}\mathbf{M}_{\mathscr{L}'} = T^{-1}_{p,\mathscr{L}'}\mathbf{M}_{\mathscr{L}'}T^{-1}_{p,\mathscr{L}}\mathbf{M}_{\mathscr{L}} \tag{3.96}$$

Moreover, $\mathbf{M}_{\mathscr{L}}$ can be constructed from $\widehat{U_{q,t}(\hat{\hat{\mathfrak{sl}}}_n)}$ of total degree 0.

Proof. This is a direct corollary of theorem 3.14 □

Remark.

Here we make some conclusion about the reason for the construction in this section. In the [OS22], the root subalgebra $U_q(\mathfrak{g}_w)$ is defined via the RTT formalism of the geometric R-matrix constructed by the K-theoretic stable envelope. Using this construction, one could have the following consequence for the quantum difference operator $\mathbf{M}_{\mathscr{L}}(u,z)$:

Theorem 3.16. *(See* [OS22]*) Under the settings and notation in* [OS22]. *Let $\nabla \subset H^2(X,\mathbb{R})$ be the alcove uniquely defined by the conditions:*

(1) $0 \in H^2(X,\mathbb{R})$ *is one of the vertices of* ∇.
(2) $\nabla \subset -C_{ample}$.

Then for $s \in \nabla$ we have:

$$Stab^{-1}_{+,T^{1/2},s}\mathscr{A}_{\mathscr{L}}Stab_{+,T^{1/2},s} = \mathscr{A}^s_{\mathscr{L}} \tag{3.97}$$

In other words,

$$\mathbf{M}_{\mathscr{L}}(u,z) = Const \cdot T_{\mathscr{L}}Stab_{+,T^{1/2},s}T^{-1}_{\mathscr{L}}\mathbf{B}^s_{\mathscr{L}}(u,z)Stab^{-1}_{+,T^{1/2},s} \tag{3.98}$$

A good consequence of this result is that the operator $\mathbf{B}^s_{\mathscr{L}}(\lambda)$ does not depend on the choice of arbitrary path, while the diffculty is that it is difficult to compute $\mathbf{B}_w(\lambda)$.

Another way to define the root subalgebra is given by Negut[N15] via the shuffle algebra techniques, as is used in this paper. Both the construction admits the following thing: Suppose that the quantum affine algebra $U_q(\hat{\hat{\mathfrak{g}}})$ defined via the K-theoretic stable envelope and the quantum toroidal algebra $U_q(\hat{\hat{\mathfrak{gl}}}_r)$ are isomorphic up to a centre. Then the universal R-matrix for both Hopf algebras admit the following KT factorization:

$$R = \prod_{w \subset \mathbf{h}_{\mathbb{Q}}} R_w \tag{3.99}$$

And this factorization is unique. Thus once we prove that the two Hopf algebra are isomorphic, it is immediate that the root subalgebra are both sides are actually isomorphic.

Thus in this assumption, all the ABBR equation that we are concerned here are the ABRR equation for the quantum toroidal algebra $U_{q,t}(\hat{\hat{\mathfrak{sl}}}_r)$ and its root subalgebra $\mathscr{B}_{\mathbf{m}}$.

So here the quantum difference operator we assume is:

$$\mathbf{M}_{\mathscr{L}}(u,z) = \mathrm{Const} \cdot \mathbf{B}^s_{\mathscr{L}}(u,z) \tag{3.100}$$

The constant term is determined by the normalization condition for the solution. Also note that under this notation, $\mathbf{M}_{\mathscr{L}}(u,z)$ depends on s actually, but here we still omit the slope s in the notation. Generally we assume that $\mathbf{M}_{\mathscr{L}}(u,z)$ starts at s which is sufficiently close to 0.

In addition, if we want to prove 3.16 algebraically, it is actually equivalent to prove that the R-matrix $R^s(u)$ of the quantum toroidal algebra $U_{q,t}(\hat{\hat{\mathfrak{sl}}}_n)$should coincide with the one computed by the K-theoretic stable envelope.

3.6. **Stable basis and quantum difference operators.** In this subsection we will introduce how to relate the stable basis with the quantum difference operators.

Now we fix an affine type A quiver variety $M(\mathbf{v},\mathbf{w})$ and a slope $\mathbf{m}\in\mathbb{Q}^r$. There is a well known formula for the operator $P^{\mathbf{m}}_{[i;j)}$ acting on the stable basis $\mathrm{Stab}_{\mathbf{m},\mathfrak{C}}(\boldsymbol{\lambda})$ given by Negut [N15]:

(3.101)
$$P^{\mathbf{m}}_{[i;j)}\mathrm{Stab}_{\mathbf{m},\mathfrak{C}}(\boldsymbol{\mu})=\sum_{\text{cavalcade of }\mathbf{m}-\text{ribbons}}^{\boldsymbol{\lambda}\backslash\boldsymbol{\mu}=C\text{ is a type }[i;j)}(1-q^2)^{\#_C}(-q)^{N_C^+}q^{\mathrm{ht}(C)+\mathrm{ind}_C^{\mathbf{m}}}\mathrm{Stab}_{\mathbf{m},\mathfrak{C}}(\boldsymbol{\lambda})$$

(3.102)
$$Q^{\mathbf{m}}_{-[i;j)}\mathrm{Stab}_{\mathbf{m},\mathfrak{C}}(\boldsymbol{\lambda})==\sum_{\text{stampede of }\mathbf{m}-\text{ribbons}}^{\boldsymbol{\lambda}\backslash\boldsymbol{\mu}=S\text{ is a type }[i;j)}(1-q^2)^{\#_S}(-q)^{N_S^-}q^{\mathrm{wd}(S)-\mathrm{ind}_S^{\mathbf{m}}-j+i}\mathrm{Stab}_{\mathbf{m},\mathfrak{C}}(\boldsymbol{\mu})$$

with $\mathrm{Stab}_{\mathbf{m},\mathfrak{C}}(\boldsymbol{\mu})$ being normalized.

Now given the quantum difference operator $\mathbf{M}_{\mathscr{L}}(z)$ being written as:

$$\mathbf{M}_{\mathscr{L}}(z)=\mathscr{L}\mathbf{B}_{w_m}(z)\cdots\mathbf{B}_{w_0}(z)\tag{3.103}$$

Choose the stable basis $\mathrm{Stab}_{w_0,\mathfrak{C}}(\boldsymbol{\mu})$ of slope w_0, we can have that:

(3.104)
$$\begin{aligned}&\mathscr{L}\mathbf{B}_{w_m}(z)\cdots\mathbf{B}_{w_0}(z)\mathrm{Stab}_{w_0,\mathfrak{C}}(\boldsymbol{\mu})\\&=\sum_{\boldsymbol{\lambda}_0,\boldsymbol{\lambda}_1}\mathscr{L}\mathbf{B}_{w_m}(z)\cdots\mathbf{B}_{w_1}(z)\mathbf{B}_{w_0}(z)^{stab,w_0}_{\boldsymbol{\lambda}_0,\boldsymbol{\mu}}\mathrm{Stab}_{w_1}(\boldsymbol{\lambda}_1)R_{w_1,w_0}(\boldsymbol{\lambda}_0)_{\boldsymbol{\lambda}_0,\boldsymbol{\lambda}_1}\\&=\sum_{\boldsymbol{\lambda}_0,\boldsymbol{\lambda}_1}\cdots\sum_{\boldsymbol{\lambda}_{2m-2},\boldsymbol{\lambda}_{2m-1}}\sum_{\boldsymbol{\lambda}_{2m}}\mathscr{L}\mathrm{Stab}_{w_m}(\boldsymbol{\lambda}_{2m})\prod_{i=0}^{m}\mathbf{B}_{w_i}(z)^{stab,w_i}_{\boldsymbol{\lambda}_{2i},\boldsymbol{\lambda}_{2i-1}}\prod_{i=0}^{m-1}R_{w_{i+1},w_i}(\boldsymbol{\lambda}_{\mathbf{2i}})_{\boldsymbol{\lambda}_{2i},\boldsymbol{\lambda}_{2i+1}}\end{aligned}$$

And here $R_{w_i,w_j}:=\mathrm{Stab}^{-1}_{w_i,\mathfrak{C}}\circ\mathrm{Stab}_{w_j,\mathfrak{C}}$ is the geometric R-matrix. And the K-theoretic stable envelope can be computed via the formula of the elliptic stable envelope as in [D21]. We expect that such computation would be helpful to write down the quantum difference operator explicitly with respect to the stable basis.

4. Examples

4.1. **Cotangent bundle of the Grassmannians.** Now we consider the case such that we take the finite type A quiver varieties into the affine type A quiver varieties via taking some vector space of the vertices to be zero. For example,

(4.1)

$$
\begin{tikzcd}
V_1 \arrow[r, shift left] \arrow[d, shift right] & V_2 \arrow[l, shift left] \arrow[r, shift left] \arrow[d, shift right] & V_3 \arrow[l, shift left] \arrow[r, shift left] \arrow[d, shift right] & V_4 \arrow[l, shift left] \arrow[d, shift right] \\
W_1 \arrow[u, shift right] & W_2 \arrow[u, shift right] & W_3 \arrow[u, shift right] & W_4 \arrow[u, shift right]
\end{tikzcd}
\hookrightarrow
\begin{tikzcd}
 & & 0 \arrow[d, shift left] & & \\
 & & 0 \arrow[u, shift left] \arrow[dl, shift left] \arrow[dr, shift left] & & \\
W_1 \arrow[r, shift left] & V_1 \arrow[l, shift left] \arrow[ur, shift left] \arrow[d, shift right] & & 0 \arrow[ul, shift left] \arrow[r, shift left] \arrow[d, shift right] & 0 \arrow[l, shift left] \\
W_2 \arrow[r, shift left] & V_2 \arrow[l, shift left] \arrow[u, shift right] \arrow[dr, shift left] & & V_4 \arrow[u, shift right] \arrow[r, shift left] \arrow[dl, shift left] & W_4 \arrow[l, shift left] \\
 & & V_3 \arrow[ul, shift left] \arrow[ur, shift left] \arrow[d, shift left] & & \\
 & & W_3 \arrow[u, shift left] & &
\end{tikzcd}
$$

In this case we have that the monodromy operator can be reduced to:

$$
\begin{aligned}
\mathbf{B}_{\mathbf{m}}(\lambda) = \prod_{k=0}^{\overrightarrow{\infty}} \overrightarrow{\prod_{\gamma\in\Delta(A)}} \exp_{q^2}(-(q-q^{-1}) z^{k(\mathbf{v}_\gamma+m\delta_h)} p^{k\mathbf{m}\cdot(\mathbf{v}_\gamma)} q^{\frac{k}{2}(\mathbf{v}^T C\mathbf{v}-\mathbf{w}^T\mathbf{w})^{\grave{}}\cdot(\mathbf{v}_\gamma))} \\
q^{-2k(\mathbf{v}_\gamma)^T C(\mathbf{v}_\gamma)}(\varphi^{-(k+1)(\mathbf{v}_\gamma)} f_\gamma \otimes e'_\gamma \varphi^{-(k+1)(\mathbf{v}_\gamma)})))
\end{aligned} \tag{4.2}
$$

For the case of the cotangent bundle of the Grassmannians.

$$
T^*Gr(k,n) = \begin{tikzcd} \mathbb{C}^k \arrow[r, shift left] & \mathbb{C}^n \arrow[l, shift left] \end{tikzcd} \tag{4.3}
$$

It can be easily checked that for $\mathbf{m} \notin \mathbb{Z}$ and generic, the monodromy operator $\mathbf{B}_{\mathbf{m}}(\lambda)$ acts as zero on $K_T(T^*Gr(k,n))$. Then the quantum difference operator is:

$$
\mathbf{M}_{\mathcal{O}(1)}(z) = \mathcal{O}(1)\mathbf{B}_{-e_1}(z) \tag{4.4}
$$

And $\mathbf{B}_{-e_1}(z)$ can be written as:

$$
\begin{aligned}
\mathbf{B}_{-e_1}(z) &= : \prod_{k=0}^{\infty} \exp_{q^2}(-(q-q^{-1}) z^{-k} p^{-k} q^{-2k} f \otimes e) : \\
&= \sum_{n=0}^{\infty} \frac{(q-q^{-1})^n}{[n]_{q^2}!} \frac{(-1)^n}{\prod_{k=1}^{n}(1-z^{-1}p^{-1}q^{2k})} f^n e^n
\end{aligned} \tag{4.5}
$$

And this coincides with the one given by [ABRR97][EV02][OS22].

4.2. **Instanton moduli space.** This example has already been shown in [OS22], and now we review it for the completeness.

The instanton moduli space can be constructed via the Jordan quiver $\mathrm{Rep}(n,r) = \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^n) \oplus \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^n)$, the quiver can be drawn as follows:

$$\begin{tikzcd} \mathbb{C}^n \arrow[loop left] \arrow[r, bend right] & \mathbb{C}^r \arrow[l, bend right] \end{tikzcd} \tag{4.6}$$

For the case $r = 1$, the instanton moduli space is isomorphic to the Hilbert scheme of points over $\mathbb{C}^2$ $\mathrm{Hilb}_m(\mathbb{C}^2)$. As a vector space, the equivariant K-theory of Hilbert schemes can be identified with the space of symmetric polynomials in an infinite number of variables.

$$\oplus_{m=0}^{\infty} K_T(\mathrm{Hilb}_m(\mathbb{C}^2))_{loc} = F(u_1) := \mathbb{Q}(q_1, q_2)[x_1, x_2, \cdots]^{Sym} \tag{4.7}$$

The fixed point basis corresponds to the Macdonald polynomial P_λ, which is the orthogonal basis of the vector space of the symmetric polynomial over $\mathbb{Q}(q,t)$.

Similarly, for $M(n,r)$ the instanton moduli space of rank r and instanton number n, choose the whole framed torus $T = (\mathbb{C}^\times)^r$ such that $\mathbf{w} = u_1 + \cdots + u_r$, and we have that:

$$\oplus_{n=0}^{\infty} K_G(M(n,r))_{loc} = F(u_1) \otimes \cdots \otimes F(u_r) \tag{4.8}$$

For convenience, we first introduce the elliptic hall algebra description of the quantum toroidal $U_{q,t}(\hat{\hat{\mathfrak{gl}}}_1)$.

4.2.1. *Quantum toroidal $U_{q,t}(\hat{\hat{\mathfrak{gl}}}_1)$.* Set

$$n_k = \frac{(q_1^{k/2} - q_1^{-k/2})(q_2^{k/2} - q_2^{-k/2})(q^{-k/2} - q^{k/2})}{k}, \qquad q = q_1 q_2 \tag{4.9}$$

For simplicity, we can use the elliptic hall algebra description for the quantum toroidal algbera $U_q(\hat{\hat{\mathfrak{gl}}}_1)$:

$$U_q(\hat{\hat{\mathfrak{gl}}}_1) = \mathbb{Q}(q_1, q_2)\langle K_{\mathbf{a}}, e_{\mathbf{a}} \rangle / \{\text{relation in 1.1 of [SV13] or 7.1 of [OS22]}\} \tag{4.10}$$

With $\mathbf{a} = (a_1, a_2) \in \mathbb{Z}^2 \backslash \{(0,0)\}$. For $w \in \mathbb{Q} \cup \{\infty\}$ we denote by $d(w)$ and $n(w)$ the denominator and numerator of w. We set $d(\infty) = 0$ and $n(\infty) = 1$. And it is easy to deduce that

$$\alpha_k^w = e_{(d(w)k, n(w)k)}, \qquad k \in \mathbb{Z} \backslash \{0\} \tag{4.11}$$

generate a Heisenberg subalgebra of $\mathscr{B}_w \subset U_q(\hat{\hat{\mathfrak{gl}}}_1)$ with the following relations:

$$[\alpha^w_{-k}, \alpha^w_k] = \frac{q^{-kd(w)} - q^{kd(w)}}{n_k} \tag{4.12}$$

And thus we have the slope factorization:

$$U_{q,t}(\hat{\hat{\mathfrak{gl}}}_1) \cong \bigotimes^{\rightarrow}_{w \in \mathbb{Q} \sqcup \{\infty\}} \mathscr{B}_w \tag{4.13}$$

With each $\mathscr{B}_w$ being isomorphic to $U_q(\hat{\mathfrak{gl}}_1)$ for $w \in \mathbb{Q}$. And we can construct the action of $U_{q_1,q_2}(\hat{\hat{\mathfrak{gl}}}_1)$ on $\oplus_m K_T(\mathrm{Hilb}_m(\mathbb{C}^2))_{loc}$[SV13]. The central elements act in this representation by:

$$K_{(1,0)} = q_1^{-1/2} q_2^{-1/2}, K_{(0,1)} = 1 \tag{4.14}$$

And the vertical generators is commutative with each other:

$$[e_{(0,m)}, e_{(0,n)}] = 0 \tag{4.15}$$

And $\mathscr{B}_0$ corresponds to the horizontal Heisenberg subalgebra

$$[e_{(m,0)}, e_{(n,0)}] = -\frac{m}{(q_1^{m/2} - q_1^{-m/2})(q_2^{m/2} - q_2^{-m/2})}\delta_{m+n,0} \tag{4.16}$$

The vertical subalgebra $\{e_{(m,0)}\}$ is commutative, and it acts diagonally in the fixed point basis, i.e. the basis of Macdonald polynomials:

$$e_{(0,l)}(P_\lambda) = u_1^{-l}\mathrm{sign}(l)(\frac{1}{1-q_1^{-l}}\sum_{i=1}^{\infty} q_1^{-l\lambda_i} q_2^{-l(i-1)})P_\lambda \tag{4.17}$$

And thus the reduced part of the universal R-matrix of $U_q(\hat{\hat{\mathfrak{gl}}}_1)$ can be written as

$$R = \prod^{\rightarrow}_{a/b \in \mathbb{Q} \cup \{\infty\}} \exp(-\sum_{k=1}^{\infty} n_k \alpha^{a/b}_{-k} \otimes \alpha^{a/b}_k) \tag{4.18}$$

Now we derive the quantum difference operator $\mathbf{M}_{\mathscr{L}}(z)$.

We first derive the solution of the ABRR equation. We choose a subtorus A such that it splits the framing by $r = r_1 u_1 + r_2 u_2$ so that

$$K_G(M(r)^A)_{loc} = F^{\otimes r_1}(u_1) \otimes F^{\otimes r_2}(u_2) \tag{4.19}$$

Let $F = M(m_1, r_1) \times M(m_2, r_2)$ be a component of $M(m,r)^A$, and as $\dim(M(m,r)) = 2mr$ we obtain that the corresponding eigenvalue of Ω equals:

$$\Omega = \frac{\mathrm{codim}(F)}{4} = \frac{m_1 r_1 + m_2 r_2}{2} \tag{4.20}$$

the ABRR equation for a wall $w \in \mathbb{Q}$ takes the form:

$$q^{\Omega}R_w^{-}q_{(1)}^{-\lambda}J_w^{-}(z) = J_w^{-}(z)q^{\Omega}q_{(1)}^{-\lambda} \tag{4.21}$$

And for the deduction one can see the section 7 of [OS22]. In the end one can obtain that:

$$J_w^{-}(z) = \exp(-\sum_{k=1}^{\infty}\frac{n_kK_{(1,0)}^{kd(w)}\otimes K_{(1,0)}^{-kd(w)}}{1-z^{-kd(w)}K_{(1,0)}^{kd(w)}\otimes K_{(1,0)}^{-kd(w)}}\alpha_{-k}^{w}\otimes\alpha_k^{w}) \tag{4.22}$$

And we obtain the monodromy operators:

$$\mathbf{B}_w(z) = \exp(-\sum_{k=1}^{\infty}\frac{n_kK_{(1,0)}^{kd(w)}}{1-z^{-kd(w)}p^{kn(w)}K_{(1,0)}^{kd(w)}}\alpha_{-k}^{w}\alpha_k^{w}) \tag{4.23}$$

Now let $\mathscr{L} = \mathscr{O}(1)$ be the generator of the Picard group. Then the interval $(s, s-\mathscr{O}(1))$ containing all the walls $w \in \mathbb{Q}$ is in $-1 \leq w < 0$. And by the definition of the wall set it is easy to deduce that the wall set is:

$$\text{Walls}(M(n,r)) := \{\frac{a}{b}|a \in \mathbb{Z}, |b| \leq n\} \cap [-1,0) \tag{4.24}$$

This coincides with the one given by the computation of the K-theoretic stable envelope in [S20].

Now we have that the quantum difference operator can be written as:

$$\mathbf{M}_{\mathscr{O}(1)}(z) = \mathscr{O}(1)\prod_{\substack{w\in\mathbb{Q}\\-1\leq w<0}}^{\leftarrow} :\exp(-\sum_{k=1}^{\infty}\frac{n_kq^{-\frac{krd(w)}{2}}}{1-z^{-kd(w)}p^{kn(w)}q^{-krd(w)}}\alpha_{-k}^{w}\alpha_k^{w}): \tag{4.25}$$

Example. For $M(2,5)$, the quantum difference operator can be written as:

(4.26)
$$\mathbf{M}_{\mathscr{O}(1)}(z) = \mathscr{O}(1)\mathbf{B}_{-1}(z)\mathbf{B}_{-4/5}(z)\mathbf{B}_{-3/4}(z)\mathbf{B}_{-2/3}(z)\mathbf{B}_{-3/5}(z)\mathbf{B}_{-1/2}(z)\mathbf{B}_{-2/5}(z)\mathbf{B}_{-1/3}(z)\mathbf{B}_{-1/4}(z)\mathbf{B}_{-1/5}(z$$

And we write down the expression for the first few terms:

(4.27)
$$\mathbf{B}_{-1/5}(z) = 1 - \frac{n_1q^{-5}}{1-z^{-5}pq^{-5}}\alpha_{-1}^{1/5}\alpha_1^{1/5}$$

(4.28)
$$\mathbf{B}_{-1/4}(z) = 1 - \frac{n_1q^{-4}}{1-z^{-4}pq^{-4}}\alpha_{-1}^{1/4}\alpha_1^{1/4}$$

(4.29)
$$\mathbf{B}_{-1/3}(z)=1-\frac{n_1q^{-3}}{1-z^{-3}pq^{-3}}\alpha_{-1}^{1/3}\alpha_1^{1/3}$$
(4.30)
$$\mathbf{B}_{-2/5}(z)=1-\frac{n_1q^{-5}}{1-z^{-5}p^2q^{-5}}\alpha_{-1}^{1/5}\alpha_1^{1/5}$$
(4.31)
$$\mathbf{B}_{-1/2}(z)=1-\frac{n_1q^{-2}}{1-z^{-2}pq^{-2}}\alpha_{-1}^{1/2}\alpha_1^{1/2}-\frac{n_2q^{-4}}{1-z^4p^2q^{-4}}\alpha_{-2}^{1/2}\alpha_2^{1/2}+\frac{n_1^2q^{-4}}{2(1-z^{-2}pq^{-2})^2}\alpha_{-1}^{1/2}\alpha_{-1}^{1/2}\alpha_1^{1/2}\alpha_1^{1/2}$$

(4.32)
$$\mathbf{B}_{-1}(z)=:\exp(-\frac{n_1q^{-1}}{1-z^{-1}pq^{-1}}\alpha_{-1}^{-1}\alpha_1^{-1}-\frac{n_2q^{-2}}{1-z^{-2}p^2q^{-2}}\alpha_{-2}^{-1}\alpha_2^{-1}-\frac{n_3q^{-3}}{1-z^{-3}p^3q^{-3}}\alpha_{-3}^{-1}\alpha_3^{-1}$$
$$-\frac{n_4q^{-4}}{1-z^{-4}p^4q^{-4}}\alpha_{-4}^{-1}\alpha_4^{-1}-\frac{n_5q^{-5}}{1-z^{-5}p^5q^{-5}}\alpha_{-5}^{-1}\alpha_5^{-1}):$$

The formula for the the operator $\alpha_{-k}^w\alpha_k^w$ acting on the Fock space can be written in the following way:

(4.33)
$$\begin{aligned}X^-X^+|\lambda\rangle=&\sum_{\substack{\lambda\supset\mu\subset\nu\\|\lambda|=|\nu|}}R(\nu\backslash\mu)R(\lambda\backslash\mu)q^{\frac{1}{2}|\lambda\backslash\mu|}\prod_{\blacksquare\in\lambda\backslash\mu}\frac{\prod_{\square\text{ i.c. of }\mu}(1-\frac{\chi_\square}{q\chi_\blacksquare})}{\prod_{\square\text{ o.c. of }\mu}(1-\frac{\chi_\square}{q\chi_\blacksquare})}\\&\times\prod_{\blacksquare\in\nu\backslash\mu}\frac{\prod_{\square\text{ o.c. of }\nu}(1-\frac{\chi_\square}{\chi_\blacksquare})}{\prod_{\square\in\text{ i.c. of }\nu}(1-\frac{\chi_\square}{\chi_\blacksquare})}|\nu\rangle\end{aligned}$$

with $X^+\in\mathcal{S}^+$ and $X^-\in\mathcal{S}^-$. And in our case, the operator $\alpha_k^{a/b}$ can be written in terms of the shuffle algebra element as:

(4.34)
$$\alpha_k^{a/b}=\frac{(q_1-1)^{bk}(1-q_2)^{bk}}{(q_1^k-1)(1-q_2^k)}\mathrm{Sym}[\frac{\prod_{i=1}^{ak}z_i^{\lfloor\frac{ib}{a}\rfloor-\lfloor\frac{(i-1)b}{a}\rfloor}}{\prod_{i=1}^{ak-1}(1-\frac{qz_{i+1}}{z_i})}\sum_{s=0}^{k-1}q^s\frac{z_{a(k-1)+1}\cdots z_{a(k-s)+1}}{z_{a(k-1)}\cdots z_{a(k-s)}}\prod_{i<j}\zeta(\frac{z_i}{z_j})]$$

4.3. **Hilbert scheme $\mathbf{Hilb}_n([\mathbb{C}^2/\mathbb{Z}_{r+1}])$.** For the equivariant Hilbert scheme $\mathrm{Hilb}_n([\mathbb{C}^2/\mathbb{Z}_{r+1}])$, it is known that it is of the quiver type $\hat{A}_r$, and the corresponding quiver is drawn as the following:

$$\begin{tikzcd}
 & \cdots \arrow[dl, shift left] \arrow[dr, shift left] & \\
\mathbb{C}^n \arrow[ur, shift left] \arrow[d, shift left] & & \mathbb{C}^n \arrow[ul, shift left] \arrow[d, shift left] \\
\mathbb{C}^n \arrow[u, shift left] \arrow[dr, shift left] & & \mathbb{C}^n \arrow[u, shift left] \arrow[dl, shift left] \\
 & \mathbb{C}^n \arrow[ul, shift left] \arrow[ur, shift left] \arrow[d, shift left] & \\
 & \mathbb{C} \arrow[u, shift left] &
\end{tikzcd} \tag{4.35}$$

with $r+1$ vertices and 1 framed vertices. Recall that for the slope subalgebra $\mathscr{B}_{\mathbf{m}}$, we have an isomorphism:

$$\mathscr{B}_{\mathbf{m}} \cong \bigotimes_{h=1}^{g} U_q(\hat{\mathfrak{gl}}_{l_h}) \tag{4.36}$$

It has the slope decomposition as:

$$U_{q,t}(\hat{\hat{\mathfrak{sl}}}_{r+1}) \cong \bigotimes_{r\in\mathbb{Q}} \mathscr{B}_{\mathbf{m}+r\boldsymbol{\theta}} \tag{4.37}$$

Here $\mathbf{m} = (m_1, \cdots, m_{r+1}) \in \mathbb{Q}^{r+1}$ and $\boldsymbol{\theta} = (1, \cdots, 1)$, and for $\mathbf{m} \in \mathbb{Q}^{r+1}$, we have the following result.

We simply explain how to construct the natural number $g, l_1, \cdots, l_g$. Set

$$e_i = (\underbrace{0, \cdots, 1,}_{i \text{ numbers}} \cdots, 0) \tag{4.38}$$

The set of vectors $\{e_i\}_{1\le i\le r+1}$ span the $\mathbb{Q}$-vector space $\mathbb{Q}^{r+1}$.

We define:

$$[i; j\rangle := e_i + e_{i+1} + \cdots + e_{j-1} \tag{4.39}$$

with $e_i := e_{i \bmod (r+1)}$. We call this vector as **arc vector**. We say that an arc vector $[i; j\rangle$ is **m-integral** if

$$\mathbf{m} \cdot [i; j\rangle \in \mathbb{Z} \tag{4.40}$$

.

It is easily proved that there exists **m**-integral arc vector as above, starting at each vertex i. Therefore there exists a well-defined minimal **m**-integral arc vector starting at each vertex i, and we will denote it by

$$[i;v_{\mathbf{m}}(i)\rangle \tag{4.41}$$

Now since we have only n vertices in the quiver with modulo relations, the uniqueness of the minimal $\mathbf{m}$-integral arc vector implies that the resulting graph will be a union of oriented cycles:

$$C_1\sqcup\cdots\sqcup C_g \tag{4.42}$$

where $C_h=\{i_1,\cdots,i_h\}$ such that $i_{e+1}=v_{\mathbf{m}}(i_e)$ and $C_h=\{i_1,\cdots,i_{l_h}\}$.

For the solution of $\mathbf{B}_w(\lambda)$, and now for our case of $\text{Hilb}_n([\mathbb{C}^2/\mathbb{Z}_r])$, $\mathbf{v}_2=\mathbf{w}_2=0$, $\mathbf{v}_1=n\boldsymbol{\theta}$, $\mathbf{w}_1=\mathbf{e}_1$, and thus we have that $\mathbf{v}_2-\mathbf{v}_1=-m\boldsymbol{\theta}$, $\mathbf{w}_2-\mathbf{w}_1=-\mathbf{e}_1$. And use the theorem 3.5 we obtain the following result:

Proposition 4.1. *For the monodromy operator $\mathbf{B}_{\mathbf{m}}(\lambda)\in\mathscr{B}_{\mathbf{m}}$, where $\mathbf{B}_{\mathbf{m}}(\lambda)$ is defined in 3.29 and $\mathscr{B}_{\mathbf{m}}$ is defined as 2.64, its representation in $End(K_T(Hilb_n([\mathbb{C}^2/\mathbb{Z}_r]))$ is given by:*
(4.43)

$$\begin{aligned}
&\mathbf{B}_{\mathbf{m}}(\lambda)\\
=&\mathbf{m}\Big(\prod_{h=1}^{g}(\exp(-\sum_{k=1}^{\infty}\frac{n_kq^{-\frac{k|\boldsymbol{\delta_h}|}{2}}}{1-z^{-k|\boldsymbol{\delta_h}|}p^{k\mathbf{m}\cdot\boldsymbol{\delta}_h}q^{-\frac{k|\boldsymbol{\delta_h}|}{2}}}\alpha_{-k}^{\mathbf{m},h}\otimes\alpha_k^{\mathbf{m},h})\prod_{k=0}^{\overrightarrow{\infty}}\\
&\times\prod^{\leftarrow}_{\substack{\gamma\in\Delta(A)\\ m\geq0}}(\exp_{q^2}(-(q-q^{-1})z^{k(-\mathbf{v}_\gamma+(m+1)\boldsymbol{\delta}_h)}p^{k\mathbf{m}\cdot(-\mathbf{v}_\gamma+(m+1)\boldsymbol{\delta}_h)}q^{k(-\mathbf{v}_\gamma+(m+1)\boldsymbol{\delta}_h)^T((\frac{n^2r-1}{2})\boldsymbol{\theta}+\mathbf{e}_1)-2-2\delta_{1\gamma}}\\
&f_{(\delta-\gamma)+m\delta}\otimes e'_{(\delta-\gamma)+m\delta})\exp(-(q-q^{-1})\sum_{m\in\mathbb{Z}_+}\sum_{i,j=1}^{l_h}z^{km\boldsymbol{\delta}_h}p^{km\mathbf{m}\cdot\boldsymbol{\delta}_h}q^{-km\boldsymbol{\delta}_h^T((\frac{n^2r-1}{2})\boldsymbol{\theta}+\mathbf{e}_1)-2}u_{m,ij}f_{m\delta,\alpha_i}\otimes e'_{m\delta,\alpha_i})\\
&\times\prod^{\rightarrow}_{\substack{\gamma\in\Delta(A)\\ m\geq0}}\exp_{q^2}(-(q-q^{-1})z^{k(\mathbf{v}_\gamma+m\boldsymbol{\delta}_h)}p^{k\mathbf{m}\cdot(\mathbf{v}_\gamma+m\boldsymbol{\delta}_h)}q^{-k(\mathbf{v}_\gamma+m\boldsymbol{\delta}_h)^T((\frac{n^2r-1}{2})\boldsymbol{\theta}+\mathbf{e}_1)-2-2\delta_{1\gamma}}f_{\gamma+m\delta}\otimes e'_{\gamma+m\delta}))\Big)
\end{aligned}$$

and $e'_v=S(e_v)$, $f'_v=S(f_v)$ are the image of the antipode map. n_k is an element in $\mathbb{Q}(q,t)$ such that $[\alpha_{-k}^{\mathbf{m},h},\alpha_k^{\mathbf{m},h}]=\frac{1}{n_k}$.

To determine the quantum difference operator, we choose the slope from 0 to $-\mathscr{L}$, with $\mathscr{L}=\mathscr{L}_1\cdots\mathscr{L}_{r+1}$, we choose a straight line of the equation $L_t:=(ts,\cdots,ts)$ to connect from $-\boldsymbol{\theta}$ to 0. And we can see that the quantum difference equation over this line has the form:

$$\mathbf{M}_{\mathscr{O}(1)}(z)=\mathscr{O}(1)\prod^{\rightarrow}_{w\in\text{Walls}\cap L_t}\mathbf{B}_w(z) \tag{4.44}$$

And for the finite set $\text{Walls}\cap L_t$, we have the following:

Lemma 4.2. *The finite set Walls $\cap L_t$ has the following elements:*

$$Walls_0 := Walls \cap L_t = \{(\frac{a}{b}, \cdots, \frac{a}{b}) | |a| < |b|, b \in [1, n(r+1)]\} \tag{4.45}$$

which coincides with the one defined by the K-theoretic stable envelope.

Proof. Note that in the root subalgebra $\mathscr{B}_{\mathbf{m}}$ with the slope $\mathbf{m} = (\frac{a}{b}, \cdots, \frac{a}{b})$, the basic generator $e_{[i;i+1)}$ sends $K_T(M(n\boldsymbol{\theta}, e_j))$ to $K_T(M(n\boldsymbol{\theta} - [i; i+b) - e_i, e_i))$. And for each vertices v_i, under the operation $e_{[i;i+1)}$, it can be sent to $v_i - [b/(r+1)] - 1$, now we take $v_i = n$, and to make sure that $M(n\boldsymbol{\theta} - [i; i+b) - e_i, e_i)$ is non-empty, we require that:

$$n - [\frac{b-1}{r+1}] - 1 \geq 0 \tag{4.46}$$

And we obtain that $b \leq n(r+1)$.

□

We give the example of the quantum difference operator at $\mathscr{O}(1) = \mathscr{L}_1 \cdots \mathscr{L}_{r+1} \in \text{Pic}(\text{Hilb}_n([\mathbb{C}^2/\mathbb{Z}_{r+1}])$, which means that choose a path from 0 to $-\mathscr{O}(1)$.

Let us choose the straight line from 0 to $\mathscr{O}(1)$. For $-\frac{a}{b}\boldsymbol{\theta} \in \text{Walls} \cap L_t$, we know that by the result 2.64 the corresponding root subalgebra should be:

$$\mathscr{B}_{-\frac{a}{b}\boldsymbol{\theta}} = U_q(\hat{\mathfrak{gl}}_{\frac{r+1}{g}})^{\otimes g}, \qquad g = \gcd(b, r+1) \tag{4.47}$$

Thus finally we can write down the monodromy operator for the tautological bundle as:

$$\mathbf{M}_{\mathscr{O}(1)}(z) = \mathscr{O}(1) \prod^{\rightarrow}_{-\frac{a}{b}\boldsymbol{\theta} \in \text{Walls} \cap L_t, a \neq 0} \mathbf{B}_{-\frac{a}{b}\boldsymbol{\theta}}(z) \tag{4.48}$$

As an exercise, we choose the slope to be $\mathscr{L}_i\mathscr{O}(1)$, and now we fix $\mathscr{L}_i\mathscr{O}(1)$ and the interval $I_i := (\underbrace{w, \cdots, 2w}_{i \text{ numbers}}, \cdots w)$ and $-1 < w \leq 0$, it is easy to compute that:

(4.49)

$$\text{Walls}_{i,0}(\text{Hilb}_n([\mathbb{C}^2/\mathbb{Z}_{r+1}])) \cap I_i = \{(\underbrace{\frac{a}{b}, \cdots, \frac{2a}{b}}_{i \text{ numbers}}, \cdots \frac{a}{b}) | a \in \mathbb{Z}, 1 \leq b \leq n(r+1)+1, -1 < \frac{a}{b} \leq 0\}$$

Then the quantum difference operator can be written in the following way:

$$\mathbf{B}^s_{\mathscr{L}_i\mathscr{O}(1)}(\lambda) = \mathscr{L}_i\mathscr{O}(1) \prod^{\rightarrow}_{w \in \text{Walls}_{i,0}} \mathbf{B}_w(\lambda) \tag{4.50}$$

For each $w \in \text{Walls}_{i,0}$, the corresponding root subalgebra $\mathscr{B}_w$ can be computed using the formula above. It is worth noting that the corresponding wall structure of $\text{Hilb}_n([\mathbb{C}^2/\mathbb{Z}_{r+1}])$ above can be computed from the elliptic stable envelope formula using the formula in [D21].

Remark. For those who are familiar with the K-theoretic stable envelope of affine type A quiver varieties [D21], via computation you can see that the wall set here just coincide with the wall set defined by the jump of the K-theoretic stable envelope. And generally we claim that the wall set in this paper is the same as the wall set of the K-theoretic stable envelope for the affine typa A quiver varieties for the compact interval of the form $\mathbf{m}+s\boldsymbol{\theta}$ with $\boldsymbol{\theta} \in (\mathbb{Q}^+)^r$.

Examples: The A_{r-1} surfaces $\widehat{\mathbb{C}^2/\mathbb{Z}_r}$. The monodromy operator $\mathbf{B}_0(\lambda)$ can be written as:

(4.51)

$$\begin{aligned}\mathbf{B}_0(\lambda) =&1-\frac{n_1q^{-r/2}}{1-z^{-r}q^{-kr/2}}\alpha^0_{-1}\alpha^0_1-\sum_{\gamma\in\Delta(A)}\frac{(q-q^{-1})q^{-2-2\delta_{1\gamma}}}{1-z^{-\mathbf{v}_\gamma+\boldsymbol{\delta}_h}q^{(-\mathbf{v}_\gamma+\boldsymbol{\delta}_h)^T((\frac{r-1}{2})\boldsymbol{\theta}+\mathbf{e}_1)}}f_{(\delta-\gamma)}e'_{(\delta-\gamma)}\\&-(q-q^{-1})\sum_{i,j=1}^{r}\frac{1}{1-z^{\boldsymbol{\delta}_h}q^{-\boldsymbol{\delta}_h^T((\frac{r-1}{2})\boldsymbol{\theta}+\mathbf{e}_1)-2}}u_{1,ij}f_{\delta,\alpha_i}e'_{\delta,\alpha_i}\\&-\sum_{\gamma\in\Delta(A)}\frac{(q-q^{-1})q^{-2-2\delta_{1\gamma}}}{1-z^{\mathbf{v}_\gamma}q^{(\mathbf{v}_\gamma)^T((\frac{r-1}{2})\boldsymbol{\theta}+\mathbf{e}_1)}}f_\gamma e'_\gamma\end{aligned}$$

In our next paper, we will focus on solving the quantum difference equation for the equivariant Hilbert scheme $\text{Hilb}_n([\mathbb{C}^2/\mathbb{Z}_r])$. We would see that the connection matrix for the difference equation has lots of simplified description in terms of the monodromy operator, and thus have simple forms in terms of the universal R-matrix.

It is also an interesting problem to generalize these quantum difference equation construction to arbitrary type of quiver varieties and the weight spaces of other types of representations. These directions would be continued in our future work.

5. Appendix: proof of the Proposition 2.11

In this appendix we prove the rationality of the matrix coefficients of the R-matrix $\mathscr{R}^{\mathbf{m}}(u)$ for arbitrary slope $\mathbf{m} \in \mathbb{Q}^r$.

Recall first that the universal R-matrix $R^{\mathbf{m}}$ of slope $\mathbf{m} \in \mathbb{Q}^r$ can be written as:

$$R^{\mathbf{m}} = \prod^{\leftarrow}_{\mu\in\mathbb{Q}_{<0}} R^-_{\mathbf{m}+\mu\boldsymbol{\theta}} \cdot R_\infty \cdot \prod^{\leftarrow}_{\mu\in\mathbb{Q}_{\geq 0}} R_{\mathbf{m}+\mu\boldsymbol{\theta}} \tag{5.1}$$

Thus to prove the rationality of $\mathscr{R}^{\mathbf{m}}(u)$, we need to prove the rationality of $(\pi_1 \otimes \pi_2)(\prod^{\leftarrow}_{\mu\in\mathbb{Q}_{<0}} R^-_{\mathbf{m}+\mu\boldsymbol{\theta}})$, $(\pi_1\otimes\pi_2)(R_\infty)$ and $(\pi_1\otimes\pi_2)(\prod^{\leftarrow}_{\mu\in\mathbb{Q}_{\geq 0}} R_{\mathbf{m}+\mu\boldsymbol{\theta}})$.

First we fix $\mathbf{m} \in \mathbb{Q}^r$ in the region $|\mathbf{m}| < 1$. The rationality of $(\pi_1 \otimes \pi_2)(R_\infty)(u)$ and the existence of the limit $(\pi_1 \otimes \pi_2)(R_\infty)(0, \infty)$ comes from the definition of the action given by the Cartan currents $\varphi_i^{\pm}(z)$ in 2.109.

Now it remains to prove the rationality of $(\pi_1 \otimes \pi_2)(\prod^{\leftarrow}_{\mu \in \mathbb{Q}_{\geq 0}} R_{\mathbf{m}+\mu\boldsymbol{\theta}})$ and $(\pi_1 \otimes \pi_2)(\prod^{\leftarrow}_{\mu \in \mathbb{Q}_{<0}} R^-_{\mathbf{m}+\mu\boldsymbol{\theta}})$. The proof of both are similar, so here we only prove the rationality of $(\pi_1 \otimes \pi_2)(\prod^{\leftarrow}_{\mu \in \mathbb{Q}_{\geq 0}} R_{\mathbf{m}+\mu\boldsymbol{\theta}})$.

Recall that we have the following translation formula 3.65 for the R-matrix:

$$R_{\mathbf{m}+\mathscr{L}} = (\mathscr{L} \otimes \mathscr{L}) R_{\mathbf{m}} (\mathscr{L} \otimes \mathscr{L})^{-1} =: \mathrm{Ad}_{\mathscr{L}} R_{\mathbf{m}} \tag{5.2}$$

By the Proposition 2.9, it is known that $(\pi_1 \otimes \pi_2)(R_{\mathbf{m}})$ is the upper-triangular matrix written as $\mathrm{Id} + U_{\mathbf{m}}(u)$ with $U_{\mathbf{m}}(u)$ the strictly-upper triangular matrix with matrix coefficients as the Laurent monomials in u. By the translation formula of R-matrix 3.65, we have that:

$$U_{\mathbf{m}+\mathscr{L}}(u) = \mathrm{Ad}_{\mathscr{L}} U_{\mathbf{m}}(u) \tag{5.3}$$

Thus we choose $\mathscr{L}$ corresponding to $\boldsymbol{\theta}$, we can write down $(\pi_1 \otimes \pi_2)(\prod^{\leftarrow}_{\mu \in \mathbb{Q}_{\geq 0}} R_{\mathbf{m}+\mu\boldsymbol{\theta}})$ in the following way:

$$(\pi_1 \otimes \pi_2)(\prod^{\leftarrow}_{\mu \in \mathbb{Q}_{\geq 0}} R_{\mathbf{m}+\mu\boldsymbol{\theta}}) = \prod_{n=0}^{\infty} \prod^{\leftarrow}_{\mu \in \mathbb{Q}_{\geq 0} \cap [0,1)} (\mathrm{Id} + \mathrm{Ad}_{\mathscr{L}^n} U_{\mathbf{m}+\mu\boldsymbol{\theta}}(u)) \tag{5.4}$$

Now if we fix the matrix coefficients as $\langle \boldsymbol{\mu}_1 | \otimes \langle \boldsymbol{\mu}_2 | (\pi_1 \otimes \pi_2)(\prod^{\leftarrow}_{\mu \in \mathbb{Q}_{\geq 0}} R_{\mathbf{m}+\mu\boldsymbol{\theta}}) | \boldsymbol{\lambda}_1 \rangle \otimes | \boldsymbol{\lambda}_2 \rangle$, it is observed that $\prod^{\leftarrow}_{\mu \in \mathbb{Q}_{\geq 0} \cap [0,1)} (\mathrm{Id} + u^{n|\boldsymbol{\theta}|} U_{\mathbf{m}+\mu\boldsymbol{\theta}}(u))$ will only contain finitely many products. So the proof of the rationality relies on the following lemma:

Lemma 5.1. *Suppose that given finitely many strictly-upper triangular matrices $U(u)$ such that the matrix coefficients of these matrices are Laurent polynomials in u. Then the matrix coefficients of the infinite product:*

$$\prod_{n=0}^{\infty} (Id + u^n U) \tag{5.5}$$

are rational functions in u.

Proof. Choose two index $i < j$, we consider the matrix coeffients $(\prod_{n=0}^{\infty}(1 + u^n U))_{ij}$. It can be expanded as:

$$\begin{aligned}
(\prod_{n=0}^{\infty}(1 + u^n U))_{ij} &= \sum_{\substack{k_1, \cdots, k_n, \cdots \\ i \leq k_1 \leq k_2 \leq \cdots \leq j}} (1 + U)_{ik_1} (1 + uU)_{k_1 k_2} \cdots (1 + u^n U)_{k_n k_{n+1}} \cdots \\
&= \sum_{\substack{k_1, \cdots, k_n, \cdots \\ i \leq k_1 \leq k_2 \leq \cdots \leq j}} (\delta_{ik_1} + U_{ik_1})(\delta_{k_1 k_2} + uU_{k_1 k_2}) \cdots (\delta_{k_n k_{n+1}} + u^n U_{k_n k_{n+1}}) \cdots
\end{aligned} \tag{5.6}$$

Since U is strictly upper-triangular, there are only finitely many terms like U_{kl} or $U_{k_0k_1}\cdots U_{k_{l-1}k_l}$ for $i\leq k\leq l\leq j$, $i\leq k_0<k_1<\cdots<k_{l-1}<k_l\leq j$. So for now we just need to find out the terms of the form $U_{k_0k_1}\cdots U_{k_{l-1}k_l}u^m$ in $(\prod_{n=0}^{\infty}(1+u^nU))_{ij}$. From the expansion 5.6, the terms of the form $U_{k_0k_1}\cdots U_{k_{l-1}k_l}$ can be written as:

$$U_{k_0k_1}\cdots U_{k_{l-1}k_l}u^{\frac{l(l-1)}{2}}\sum_{n=0}^{\infty}a_{n,l}u^n \tag{5.7}$$

Here $a_{n,l}$ is the number of partitions $(\lambda_1,\cdots,\lambda_l)$ of n such that $(\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_l)$ and $\sum_{i=1}^{l}\lambda_i=n$. It is known that

$$\sum_{n=0}^{\infty}a_{n,l}u^n=\prod_{i=1}^{l}\frac{1}{1-u^i}=\frac{1}{(1-u)(1-u^2)\cdots(1-u^l)} \tag{5.8}$$

Since the finite product $U_{k_0k_1}\cdots U_{k_{l-1}k_l}$ is the Laurent polynomial. We have shown that $U_{k_0k_1}\cdots U_{k_{l-1}k_l}u^{\frac{l(l-1)}{2}}\sum_{n=0}^{\infty}a_{n,l}u^n$ is a rational function. Now since $(\prod_{n=0}^{\infty}(1+u^nU))_{ij}$ has finitely many these terms, $(\prod_{n=0}^{\infty}(1+u^nU))_{ij}$ is thus a rational function. □

Using this lemma, we prove the following lemma:

Lemma 5.2. *Given a strictly upper-triangular matrix U with matrix coefficients in the Laurent polynomial of u, with the decomposition $U=\sum_kU_k$ given by the smaller-diagonal terms away from the main-diagonal. Then the matrix coefficients of the infinite product:*

$$\prod_{n=0}^{\infty}(Id+\sum_k u^{nk}U_k) \tag{5.9}$$

are rational functions in u

Proof. Similarly one could do the expansion:

$$\begin{aligned}(\prod_{n=0}^{\infty}(1+u^nU))_{ij}&=\sum_{\substack{k_1,\cdots,k_n,\cdots\\ i\leq k_1\leq k_2\leq\cdots\leq j}}(1+U)_{ik_1}(1+uU)_{k_1k_2}\cdots(1+u^nU)_{k_nk_{n+1}}\cdots\\&=\sum_{\substack{k_1,\cdots,k_n,\cdots\\ i\leq k_1\leq k_2\leq\cdots\leq j}}(\delta_{ik_1}+U_{ik_1})(\delta_{k_1k_2}+u^{k_2-k_1}U_{k_1k_2})\cdots(\delta_{k_nk_{n+1}}+u^{n(k_{n+1}-k_n)}U_{k_nk_{n+1}})\cdots\end{aligned} \tag{5.10}$$

Since U is strictly upper-triangular, there are only finitely many terms like U_{kl} or $U_{k_0k_1}\cdots U_{k_{l-1}k_l}$ for $i\leq k\leq l\leq j$, $i\leq k_0<k_1<\cdots<k_{l-1}<k_l\leq j$. So for now we just need to find out the terms of the form $U_{k_0k_1}\cdots U_{k_{l-1}k_l}u^m$ in $(\prod_{n=0}^{\infty}(1+u^nU))_{ij}$. From the expansion 5.6, the terms of the form $U_{k_0k_1}\cdots U_{k_{l-1}k_l}$ can be written as:

$$U_{k_0k_1}\cdots U_{k_{l-1}k_l}u^{\sum_{i=0}^{l-1}i(k_{i+1}-k_i)}\sum_{n=0}^{\infty}b_{n,l}^{\mathbf{w}}u^n \tag{5.11}$$

Here $\mathbf{w} = (w_1, \cdots, w_l) = (k_1 - k_0, k_2 - k_1, \cdots, k_l - k_{l-1})$. $b^{\mathbf{w}}_{n,l}$ stands for the number of partition of n such that there are columns of length $w_1 + \cdots + w_i$. It is known that the corresponding generating function can be written as

$$\sum_{n=0}^{\infty} b^{\mathbf{w}}_{n,l} u^n = \prod_{i=1}^{l} \frac{1}{1 - u^{w_1 + \cdots + w_i}} \tag{5.12}$$

Thus we can see that $U_{k_0 k_1} \cdots U_{k_{l-1} k_l} u^{\sum_{i=0}^{l-1} i(k_{i+1} - k_i)} \sum_{n=0}^{\infty} b^{\mathbf{w}}_{n,l} u^n$ is a rational function. $\square$

Now back to our case, note that the formula 5.11 in our case can be written as:

$$U_{k_0 k_1} \cdots U_{k_{l-1} k_l} u^{|\boldsymbol{\theta}| \sum_{i=0}^{l-1} i(k_{i+1} - k_i)} \sum_{n=0}^{\infty} b^{\mathbf{w}}_{n,l} u^{n|\boldsymbol{\theta}|} = U_{k_0 k_1} \cdots U_{k_{l-1} k_l} u^{|\boldsymbol{\theta}| \sum_{i=0}^{l-1} i(k_{i+1} - k_i)} \prod_{i=1}^{l} \frac{1}{1 - u^{(k_i - k_0)|\boldsymbol{\theta}|}} \tag{5.13}$$

It is easy to see that the polynomial degree of $U_{k_0 k_1} \cdots U_{k_{l-1} k_l} u^{|\boldsymbol{\theta}| \sum_{i=0}^{l-1} i(k_{i+1} - k_i)}$ is smaller or equal to $\sum_{i=1}^{l} (k_i - k_0)|\boldsymbol{\theta}|$. Thus we can see that for $(\pi_1 \otimes \pi_2)(\prod^{\leftarrow}_{\mu \in \mathbb{Q}_{\geq 0}} R_{\mathbf{m}+\mu\boldsymbol{\theta}})$, the limit $u = 0, \infty$ exists and $(\pi_1 \otimes \pi_2)(\prod^{\leftarrow}_{\mu \in \mathbb{Q}_{\geq 0}} R_{\mathbf{m}+\mu\boldsymbol{\theta}})(0) = 1$.

Similar proof can be applied to $(\pi_1 \otimes \pi_2)(\prod^{\leftarrow}_{\mu \in \mathbb{Q}_{<0}} R^{-}_{\mathbf{m}+\mu\boldsymbol{\theta}})$ to see that it is a rational function of u, and the limit $u = 0, \infty$ exists with $(\pi_1 \otimes \pi_2)(\prod^{\leftarrow}_{\mu \in \mathbb{Q}_{<0}} R^{-}_{\mathbf{m}+\mu\boldsymbol{\theta}})(\infty) = 1$.

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