ON THE PLURICANONICAL MAP AND THE CANONICAL VOLUME OF PROJECTIVE 4-FOLDS OF GENERAL TYPE

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Abstract. For nonsingular projective 4-folds V of general type with plurigenus $P_{m_0}(V) \geq 2$ for some positive integer m_0, we show that φ_m is birational onto its image for all integers $m \geq 76m_0 + 77$ and the canonical volume $\mathrm{Vol}(V)$ has the lower bound $\frac{1}{1680m_0(m_0+1)^3}$. This improves earlier results of Meng Chen.

1. Introduction

Understanding the behavior of pluricanonical maps and the lowest bound of canonical volumes of projective varieties has been a major question in birational geometry. A crucial theorem given by Hacon-McKernan [13], Takayama [22] and Tsuji [23] shows that, for any integer $n > 0$, there are optimal constants $r_n \in \mathbb{Z}_{>0}$ and $v_n \in \mathbb{Q}_{>0}$ such that the pluricanonical map $\varphi_m := \varphi_{|mK_V|} : V \dashrightarrow \mathbb{P}(H^0(V, \mathcal{O}_V(mK_V)))$ is birational onto its image for all $m \geq r_n$ and the canonical volume $\mathrm{Vol}(V) := \lim\limits_{m\to\infty} \frac{n!h^0(V,mK_V)}{m^n} \geq v_n$ for all nonsingular projective n-folds V of general type. Therefore, it is interesting to know the values of the numbers r_n and v_n.

For $n = 1$, it is known that $r_1 = 3$. For $n = 2$, Bombieri showed in [2] that $r_2 = 5$. For 3-folds, Iano-Fletcher [14], Chen-Chen [3, 4, 5] and Chen [10] have shown that $27 \leq r_3 \leq 57$. As the classification of terminal singularities of 4-folds is still unclear, there is no effective Riemann-Roch formula for 4-folds to calculate $\chi(mK)$ explicitly. Consequently, it is still unknown when the plurigenus $P_m(V) := \dim H^0(V, \mathcal{O}_V(mK_V)) \geq 2$ holds for a 4-fold V of general type and very little is known about r_4. For a nonsingular projective 3-fold V of general type, suppose $P_{m_0}(V) \geq 2$, Kollár first proved in [19, Corollary 4.8] that the $(11m_0 + 5)$-canonical map is birational. Then Chen [6] improved the method of Kollár and showed that the m-canonical map of V is birational onto its image for all $m \geq 5m_0+6$. For a nonsingular projective 4-fold V of general type, granted that $P_{m_0}(V) \geq 2$ for some positive integer m_0, Meng Chen proved in [8] that φ_m is birational onto its image for all integers $m \geq 151m_0 + 77$.

As for the lower bound of the volume, given a projective variety V of general type, by the minimal model program(see [1, 17, 20, 21]), one can always find a minimal model X birational to V. For the canonical volume, one knows that $\mathrm{Vol}(V) = K_X^{\dim X}$. It is known that $v_1 = 2$ and $v_2 = 1$. As in dimension three or higher, a minimal model may have singularities, thus v_n is only a positive rational number. Chen-Chen ([3, 4, 5]) and Iano-Fletcher [14] showed that $\frac{1}{1680} \leq v_3 \leq \frac{1}{420}$. When $n = 4$, some partial results are known:

• In 1992, Kobayashi [18] proved that for a minimal projective 4-fold Y of general type with $p_g(Y) \geq 2$ and $\dim \varphi_{|K_Y|}(Y) = 4$, $\mathrm{Vol}(Y) \geq 2p_g(Y) - 8$.

• In 2007, Chen [7] proved that for a minimal projective 4-fold Y of general type with only canonical singularities and $p_g(Y) \geq 2$, if Y is not canonically fibered by either 3-folds of geometric genus 1 or any irrational pencil of 3-folds, then $\mathrm{Vol}(Y) \geq \frac{1}{81}$.

• In 2021, Yan [25] proved that for a nonsingular projective 4-fold V of general type with $p_g(V) \geq 2$, $\mathrm{Vol}(V) \geq \frac{1}{480}$.

• In 2020, Chen, Jiang and Li [11] proved that for a minimal projective 4-fold Y of general type with $p_g(Y) \geq 2$, when $\dim \overline{\varphi_{|K_Y|}(Y)} = 3$ (or $\dim \overline{\varphi_{|K_Y|}(Y)} = 2$ respectively), $\mathrm{Vol}(Y) \geq \frac{2}{3}$ (or $\mathrm{Vol}(Y) \geq \frac{1}{6}$ respectively).

We can see that the partial results known for the volume all have the prerequisite that the geometric genus of the 4-fold is greater than or equal to 2, which does not always hold. So we want to consider the more general case.

This paper aims to improve the result of Meng Chen in [8] on the birationality of the pluricanonical map φ_m and to give a lower bound of the volume of nonsingular projective 4-folds of general type, in terms of m_0.

Theorem 1.1. *Let V be a nonsingular projective 4-fold of general type with $P_{m_0}(V) \geq 2$ for some positive integer m_0. Then*

(1) *$P_m(V) \geq 2$ for all $m \geq 38m_0 + 39$;*
(2) *φ_m is birational onto its image for all integers $m \geq 76m_0 + 77$;*
(3) $\mathrm{Vol}(V) \geq \frac{1}{1680m_0(m_0+1)^3}$.

2. **Preliminaries**

Throughout we work over an algebraically closed field k of characteristic 0. We use the following notations:

$\sim$	linear equivalence
$\sim_{\mathbb{Q}}$	$\mathbb{Q}$-linear equivalence
$\equiv$	numerical equivalence
$\lvert A\rvert \succcurlyeq \lvert B\rvert$ or equivalently $\lvert B\rvert \preccurlyeq \lvert A\rvert$	$\lvert A\rvert \supseteq \lvert B\rvert +$ fixed effective divisors.

2.1. **Convention.**

For an arbitrary linear system $|D|$ of positive dimension on a normal projective variety Z, we may define *a generic irreducible element of* $|D|$ in the following way. We have $|D| = \mathrm{Mov}|D| + \mathrm{Fix}|D|$, where $\mathrm{Mov}|D|$ and $\mathrm{Fix}|D|$ denote the moving part and the fixed part of $|D|$ respectively. Consider the rational map $\varphi_{|D|} = \varphi_{\mathrm{Mov}|D|}$. We say that $|D|$ *is composed of a pencil* if $\dim \overline{\varphi_{|D|}(Z)} = 1$; otherwise, $|D|$ *is not composed of a pencil. A generic irreducible element of* $|D|$ is defined to be an irreducible component of a general member in $\mathrm{Mov}|D|$ if $|D|$ is composed of a pencil or, otherwise, a general member of $\mathrm{Mov}|D|$.

Keep the above settings. We say that $|D|$ can *distinguish different generic irreducible elements* X_1 *and* X_2 *of a linear system* $|M|$ if neither X_1 nor X_2 is contained in $\mathrm{Bs}|D|$, and if $\overline{\varphi_{|D|}(X_1)} \nsubseteq \overline{\varphi_{|D|}(X_2)}$, $\overline{\varphi_{|D|}(X_2)} \nsubseteq \overline{\varphi_{|D|}(X_1)}$.

2.2. **Set up for the pluricanonical map $\varphi_{m_0,Y}$.**

Let V be a nonsingular projective 4-fold of general type. By the minimal model program (see, for instance [1, 17, 20, 21]), one can always pick a minimal model Y of V with at worst $\mathbb{Q}$-factorial terminal singularities. As the plurigenus, the canonical volume and the behavior of the pluricanonical map are all birationally invariant in the category of normal varieties with canonical singularities, we may just study on Y instead.

Denote by K_Y the canonical divisor of Y. Let m_0 be a positive integer such that $P_{m_0}(Y) = h^0(Y, \mathcal{O}_Y(m_0K_Y)) \geq 2$. Fix an effective divisor $K_{m_0} \sim m_0K_Y$. Guaranteed by Hironaka's theorem, we may take a series of blow-ups $\pi : Y' \longrightarrow Y$ such that:

(i) Y' is nonsingular and projective;

(ii) the moving part of $|m_0K_{Y'}|$ is base point free so that

$$g_{m_0} = \varphi_{m_0,Y} \circ \pi : Y' \longrightarrow \overline{\varphi_{m_0,Y}(Y)} \subseteq \mathbb{P}^{P_{m_0}(Y)-1}$$

is a non-trivial morphism;

(iii) the support of the union of $\pi^*(K_{m_0})$ and all those exceptional divisors of π is of simple normal crossings.

Taking the Stein factorization of g_{m_0}, we get $Y' \xrightarrow{f_{m_0}} \Gamma \xrightarrow{s} \overline{\varphi_{m_0,Y}(Y)}$ and the following commutative diagram:

$$\begin{tikzcd}
Y' \arrow[r, "f_{m_0}"] \arrow[d, "\pi"'] \arrow[dr, "g_{m_0}"] & \Gamma \arrow[d, "s"] \\
Y \arrow[r, dashed, "\varphi_{m_0,Y}"'] & \overline{\varphi_{m_0,Y}(Y)}.
\end{tikzcd}$$

We may write

$$K_{Y'} = \pi^*(K_Y) + E_\pi, \tag{2.1}$$

where E_π is a sum of distinct exceptional divisors with positive rational coefficients. Denote by $|M_m|$ the moving part of $|mK_{Y'}|$ for any positive integer m. We may write

$$m_0\pi^*(K_Y) \sim_{\mathbb{Q}} M_{m_0} + E_{m_0},$$

where E_{m_0} is an effective $\mathbb{Q}$-divisor.

If $\dim(\Gamma) = 1$, we have $M_{m_0} \sim \sum_{i=1}^{b} F_i \equiv bF$, where F_i and F are general fibers of f_{m_0} and $b = \deg f_{m_0*}\mathcal{O}_{Y'}(M_{m_0}) \geq P_{m_0}(Y) - 1$. More specifically, when $g(\Gamma) = 0$, we say that $|M_{m_0}|$ *is composed of a rational pencil* and when $g(\Gamma) > 0$, we say that $|M_{m_0}|$ *is composed of an irrational pencil*.

If $\dim(\Gamma) > 1$, by Bertini's theorem, we know that a general member of $|M_{m_0}|$ is nonsingular and irreducible.

Denote by T' a generic irreducible element of $|M_{m_0}|$. Set

$$\theta_{m_0} = \theta_{m_0,|M_{m_0}|} = \begin{cases} b, & \text{if } \dim(\Gamma) = 1; \\ 1, & \text{if } \dim(\Gamma) \geq 2. \end{cases}$$

So we naturally get

$$m_0\pi^*(K_Y) \equiv \theta_{m_0} T' + E_{m_0}. \tag{2.2}$$

2.3. **Fixed notation.**

Pick a generic irreducible element T' of $|M_{m_0}|$. Let t_1 be a positive integer such that $P_{t_1}(T') \geq 2$. Let $\pi_T : T' \dashrightarrow T$ be the contraction map onto its minimal model T. Fix an effective divisor $K_{t_1} \sim t_1 K_T$. Let $\nu : T'' \longrightarrow T'$ be the birational modification of T' such that when $\vartheta := \pi_T \circ \nu$, $\vartheta^*(K_{t_1}) \cup \{\text{Exceptional divisors of } \vartheta\}$ has simple normal crossing supports, $\mathrm{Mov}|t_1 K_{T''}|$ is base point free and T is also a minimal model of T''. We may take blow-ups $\eta : Y'' \longrightarrow Y$ of Y such that $Y'' \longrightarrow \Gamma$ is a morphism and T'' is a generic irreducible element of $\mathrm{Mov}|m_0 K_{Y''}|$. So we may work on Y'' instead. Without loss of generality, we may and do assume from the very beginning that Y' (and T' respectively) satisfies all the properties of Y'' (and T'' respectively) and $\pi_T = \vartheta$.

Set $|N| = \mathrm{Mov}|t_1 K_{T'}|$ and let $\varphi_{t_1,T}$ be the t_1-canonical map: $T \dashrightarrow \mathbb{P}^{P_{t_1}(T)-1}$. Similar to the 4-fold case, take the Stein factorization of the composition:

$$\varphi_{t_1,T} \circ \pi_T : T' \xrightarrow{j} \Gamma' \longrightarrow \overline{\varphi_{t_1,T}(T)}.$$

Denote by j the induced projective morphism with connected fibers from $\varphi_{t_1,T} \circ \pi_T$ by Stein factorization.

Set

$$a_{t_1,T} = \begin{cases} c, & \text{if } \dim(\Gamma') = 1 \ ; \\ 1, & \text{if } \dim(\Gamma') \geq 2 \ , \end{cases}$$

where $c = \deg j_*\mathcal{O}_{T'}(N) \geq P_{t_1}(T) - 1$. Let S be a generic irreducible element of $|N|$. Then S is nonsingular and we have

$$t_1\pi_T^*(K_T) \equiv a_{t_1,T}S + E_N, \tag{2.3}$$

where E_N is an effective $\mathbb{Q}$-divisor. Denote by $\sigma : S \longrightarrow S_0$ the contraction morphism of S onto its minimal model S_0.

For a projective variety X of general type with at worst $\mathbb{Q}$-factorial terminal singularities, define the *pluricanonical section index of X* to be

$$\delta(X) := min\{m|m \in \mathbb{Z}_{>0}, P_m(X) \geq 2\},$$

which is a birational invariant.

2.4. **Technical preparation.**

We will use the following theorem which is a special form of Kawamata's extension theorem (see [16, Theorem A]).

Theorem 2.1. *(cf.* [12, Theorem 2.2]*) Let Z be a nonsingular projective variety on which D is a smooth divisor. Assume that $K_Z + D \sim_{\mathbb{Q}} A + B$ where A is an ample $\mathbb{Q}$-divisor and B is an effective $\mathbb{Q}$-divisor such that $D \not\subseteq Supp(B)$. Then the natural homomorphism*

$$H^0(Z, m(K_Z + D)) \longrightarrow H^0(D, mK_D)$$

is surjective for any integer $m > 1$.

In particular, when Z is of general type and D, as a generic irreducible element, moves in a base point free linear system, the conditions of Theorem 2.1 are automatically satisfied. Keep the settings as in 2.2 and 2.3. Take $Z = Y'$ and $D = T'$.

If $|M_{m_0}|$ (resp. $\mathrm{Mov}|t_1K_{T'}|$) is composed of an irrational pencil, by [8, Lemma 2.5], we have

$$\pi^*(K_Y)|_{T'} = \pi_T^*(K_T) \tag{2.4}$$

(resp.

$$\pi_T^*(K_T)|_S = \sigma^*(K_{S_0}). \tag{2.5}$$

).

If $|M_{m_0}|$ is not composed of an irrational pencil, we have $M_{m_0} \geq \theta_{m_0}T'$. For a sufficiently large and divisible integer $n > 0$, we have

$$|n(\frac{m_0}{\theta_{m_0}} + 1)K_{Y'}| \succcurlyeq |n(K_{Y'} + T')|$$

and the natural restriction map

$$H^0(Y', n(K_{Y'} + T')) \longrightarrow H^0(T', nK_{T'})$$

is surjective. By [20, Theorem 3.3], $|nK_T|$ is base point free, which implies that $\mathrm{Mov}|nK_{T'}| = |n\pi_T^*(K_T)|$. We deduce that

$$n(\frac{m_0}{\theta_{m_0}} + 1)\pi^*(K_Y)|_{T'} \geq M_{n(\frac{m_0}{\theta_{m_0}}+1)}|_{T'} \geq n\pi_T^*(K_T).$$

Together with (2.4), we deduce the *canonical restriction inequality*:

$$\pi^*(K_Y)|_{T'} \geq \frac{\theta_{m_0}}{m_0+\theta_{m_0}}\pi_T^*(K_T). \tag{2.6}$$

Similarly, one has

$$\pi_T^*(K_T)|_S \geq \frac{a_{t_1,T}}{t_1+a_{t_1,T}}\sigma^*(K_{S_0}). \tag{2.7}$$

We will tacitly use the birationality principle of the following type.

Theorem 2.2. *(cf.* [4, 2.7]*) Let Z be a nonsingular projective variety, A and B be two divisors on Z with $|A|$ being a base point free linear system. Take the Stein factorization of $\varphi_{|A|}: Z \xrightarrow{h} W \longrightarrow \mathbb{P}^{h^0(Z,A)-1}$, where h is a fibration onto a normal variety W. Then the rational map $\varphi_{|B+A|}$ is birational onto its image if one of the following conditions is satisfied:*

(i) $\dim \varphi_{|A|}(Z) \geq 2$, $|B| \neq \emptyset$ *and* $\varphi_{|B+A|}|_D$ *is birational for a general member D of $|A|$.*
(ii) $\dim \varphi_{|A|}(Z) = 1$, $\varphi_{|B+A|}$ *can distinguish different general fibers of h and $\varphi_{|B+A|}|_F$ is birational for a general fiber F of h.*

2.5. **Some useful lemmas.**
The following results on surfaces and 3-folds will be used in our proof.

Lemma 2.3. *Let S be a smooth surface of general type. Denote by $\sigma: S \to S_0$ the contraction morphism onto its minimal model. Let Q be a nef and big $\mathbb{Q}$-divisor on S. Then $\varphi_{|K_S+3\sigma^*(K_{S_0})+\lceil Q\rceil|}$ is birational.*

Proof. If $p_g(S) > 0$, $\varphi_{|K_S+3\sigma^*(K_{S_0})+\lceil Q\rceil|}$ is birational by [8, Theorem 3.2(2)].

If $p_g(S) = 0$, by [5, Lemma 2.5], we have $(\sigma^*(K_{S_0})\cdot C) \geq 2$ for any irreducible curve C passing through a very general point of S. So $((3\sigma^*(K_{S_0})+Q)\cdot C) > 6$. Note that $(3\sigma^*(K_{S_0})+Q)^2 > 9K_{S_0}^2 > 8$. By [9, Lemma 2.5], $\varphi_{|K_S+3\sigma^*(K_{S_0})+\lceil Q\rceil|}$ is birational. □

Lemma 2.4. *(*[4, Lemma 2.14]*) Let S be a nonsingular projective surface of general type. Denote by $\sigma: S \longrightarrow S_0$ the blow-down onto its minimal model S_0. Let Q be a $\mathbb{Q}$-divisor on S. Then $h^0(S, K_S+\lceil Q\rceil) \geq 2$ under one of the following conditions:*

(1) $p_g(S) > 0, Q \equiv \sigma^*(K_{S_0}) + Q_1$ *for some nef and big $\mathbb{Q}$-divisor Q_1 on S;*
(2) $p_g(S) = 0, Q \equiv 2\sigma^*(K_{S_0}) + Q_2$ *for some nef and big $\mathbb{Q}$-divisor Q_2 on S.*

The following lemma is just a slight modification of [8, Proposition 3.3].

Lemma 2.5. *Let T be a minimal 3-fold of general type with $P_{t_1}(T) \geq 2$. Take a birational modification $\pi_T : T' \to T$ as in 2.3. Keep the same notation and setting as in 2.3. Let S be a generic irreducible element of $|N| = \mathrm{Mov}|t_1 K_{T'}|$, which is base point free by our assumption. Suppose that $Q_\lambda \equiv \lambda \pi_T^*(K_T)$ is a nef $\mathbb{Q}$-divisor on T'. If $\lambda > 3t_1 + 2$, we have $h^0(T', K_{T'} + \lceil Q_\lambda \rceil) \geq 2$. Moreover, if $p_g(S) > 0$, we have $h^0(T', K_{T'} + \lceil Q_\lambda \rceil) \geq 2$ for all $\lambda > 2t_1 + 1$.*

Proof. Without loss of generality(by the same argument as in the proof of [8, Lemma 2.8]), we may assume that $\mathrm{Supp}(Q_\lambda) \cup \mathrm{Supp}(E_{\pi_T} + E_N)$ has only simple normal crossings, where $\mathrm{Supp}(E_{\pi_T})$ is the support of all π_T-exceptional divisors. Note that we always have $\lambda > 2t_1 + 1$ by our assumption. By (2.3),

$$Q_\lambda - \frac{1}{a_{t_1,T}} E_N - S \equiv (\lambda - \frac{t_1}{a_{t_1,T}}) \pi_T^*(K_T)$$

is nef and big and it has simple normal crossing support by assumption. By Kawamata-Viehweg vanishing theorem([15, 24]), we have

$$\begin{aligned} |K_{T'} + \lceil Q_\lambda \rceil||_S \succcurlyeq & |K_{T'} + \lceil Q_\lambda - \frac{1}{a_{t_1,T}} E_N \rceil||_S \\ \succcurlyeq & |K_S + \lceil (Q_\lambda - \frac{1}{a_{t_1,T}} E_N - S)|_S \rceil|. \end{aligned} \tag{2.8}$$

By (2.7), we may write $\pi_T^*(K_T)|_S \sim_{\mathbb{Q}} \frac{a_{t_1,T}}{t_1 + a_{t_1,T}} \sigma^*(K_{S_0}) + H$ for an effective $\mathbb{Q}$-divisor H. Thus we have

$$(Q_\lambda - \frac{1}{a_{t_1,T}} E_N - S)|_S - (\lambda - \frac{t_1}{a_{t_1,T}})H \equiv (\lambda - \frac{t_1}{a_{t_1,T}}) \cdot \frac{a_{t_1,T}}{t_1 + a_{t_1,T}} \sigma^*(K_{S_0}).$$

Note that $(\lambda - \frac{t_1}{a_{t_1,T}}) \cdot \frac{a_{t_1,T}}{t_1 + a_{t_1,T}} \geq \frac{\lambda - t_1}{t_1 + 1}$. If $\lambda > 3t_1 + 2$ (resp. $\lambda > 2t_1 + 1$ and $p_g(S) > 0$), we have $(\lambda - \frac{t_1}{a_{t_1,T}}) \cdot \frac{a_{t_1,T}}{t_1 + a_{t_1,T}} > 2$ (resp. $(\lambda - \frac{t_1}{a_{t_1,T}}) \cdot \frac{a_{t_1,T}}{t_1 + a_{t_1,T}} > 1$ and $p_g(S) > 0$). By Lemma 2.4, if $\lambda > 3t_1 + 2$ (resp. $\lambda > 2t_1 + 1$ and $p_g(S) > 0$), we have $h^0(S, K_S + \lceil (Q_\lambda - \frac{1}{a_{t_1,T}} E_N - S)|_S - (\lambda - \frac{t_1}{a_{t_1,T}})H \rceil) \geq 2$, which implies that $h^0(S, K_S + \lceil (Q_\lambda - \frac{1}{a_{t_1,T}} E_N - S)|_S \rceil) \geq 2$ (resp. the same inequality holds when $p_g(S) > 0$ and $\lambda > 2t_1 + 1$). We deduce that $h^0(T', K_{T'} + \lceil Q_\lambda \rceil) \geq 2$ by (2.8). □

Remark 2.6. For any minimal 3-fold T of general type, by [8, 3.6], one of the following holds:

(a) $\chi(\mathcal{O}_T) > 1$ and $q(T) = 0$. In this case, we have $\delta(T) \leq 18$ and $p_g(S) > 0$;
(b) $\delta(T) \leq 10$.

By Lemma 2.5, we always have $h^0(T', K_{T'} + \lceil Q_\lambda \rceil) \geq 2$ if $\lambda > 37$.

By adopting the idea in the proof of [8, Theorem 3.5], we have the following lemma.

Lemma 2.7. *Keep the same notation as in Lemma 2.5. If* $\lambda > 4t_1+3$, $\varphi_{|K_{T'}+\lceil Q_\lambda\rceil|}$ *is birational onto its image. In particular, for any* $\lambda > 75$, $\varphi_{|K_{T'}+\lceil Q_\lambda\rceil|}$ *is birational onto its image.*

Proof. By [8, Lemma 2.8], we can assume that $\mathrm{Supp}(Q_\lambda)\cup\mathrm{Supp}(E_{\pi_T}+E_N)$ has only simple normal crossings, where $\mathrm{Supp}(E_{\pi_T})$ is the support of all π_T-exceptional divisors.

By Lemma 2.5, if $\lambda > 4t_1+2$, we have

$$h^0(T', K_{T'}+\lceil Q_\lambda - E_N - a_{t_1,T}S\rceil) \geq 2.$$

By [8, Lemma 2.7](take $V=T'$, $R=Q_\lambda - E_N - a_{t_1,T}S$, $L=\mathrm{Mov}|t_1K_{T'}|$), $|K_{T'}+\lceil Q_\lambda - E_N\rceil|$ separates different generic irreducible elements in $\mathrm{Mov}|t_1K_{T'}|$. Thus $|K_{T'}+\lceil Q_\lambda\rceil|$ separates different generic irreducible elements in $\mathrm{Mov}|t_1K_{T'}|$.

By Theorem 2.2, we only need to show that $\varphi_{|K_{T'}+\lceil Q_\lambda\rceil|}|_S$ is birational. By the same argument as in the proof of Lemma 2.5, we have

$$\begin{aligned}|K_{T'}+\lceil Q_\lambda\rceil||_S \succcurlyeq & |K_S+\lceil (Q_\lambda - \frac{1}{a_{t_1,T}}E_N - S)|_S\rceil| \\ \succcurlyeq & |K_S+\lceil (Q_\lambda - \frac{1}{a_{t_1,T}}E_N - S)|_S - (\lambda - \frac{t_1}{a_{t_1,T}})H\rceil|,\end{aligned}$$

where H is the same effective $\mathbb{Q}$-divisor as in the proof of Lemma 2.5. Note that we have

$$(Q_\lambda - \frac{1}{a_{t_1,T}}E_N - S)|_S - (\lambda - \frac{t_1}{a_{t_1,T}})H \equiv (\lambda - \frac{t_1}{a_{t_1,T}})\cdot \frac{a_{t_1,T}}{t_1+a_{t_1,T}}\sigma^*(K_{S_0})$$

and

$$(\lambda - \frac{t_1}{a_{t_1,T}})\cdot \frac{a_{t_1,T}}{t_1+a_{t_1,T}} \geq \frac{\lambda - t_1}{t_1+1} > 3,$$

where the last inequality holds by our assumption $\lambda > 4t_1+3$. Thus $\varphi_{|K_S+\lceil (Q_\lambda - \frac{1}{a_{t_1,T}}E_N - S)|_S - (\lambda - \frac{t_1}{a_{t_1,T}})H\rceil|}$ is birational by Lemma 2.3. We deduce that $\varphi_{|K_{T'}+\lceil Q_\lambda\rceil|}$ is birational if $\lambda > 4t_1+3$. By Remark 2.6, we may take $t_1 \leq 18$. Thus $\varphi_{|K_{T'}+\lceil Q_\lambda\rceil|}$ is birational onto its image if $\lambda > 75$.

□

3. **Proof of the main theorem**

As an overall discussion, we keep the notation and settings in 2.2 and 2.3.

Let $m \geq 38m_0+39$ be a positive integer. Since

$$(m-1)\pi^*(K_Y) - \frac{1}{\theta_{m_0}}E_{m_0} - T' \equiv (m-1-\frac{m_0}{\theta_{m_0}})\pi^*(K_Y)$$

is nef and big and it has only simple normal crossings, by Kawamata-Viehweg vanishing theorem and (2.1), we have

$$\begin{aligned}|mK_{Y'}||_{T'} \succcurlyeq& |K_{Y'}+\lceil (m-1)\pi^*(K_Y)-\frac{1}{\theta_{m_0}}E_{m_0}\rceil||_{T'}\\ \succcurlyeq& |K_{T'}+\lceil ((m-1)\pi^*(K_Y)-\frac{1}{\theta_{m_0}}E_{m_0}-T')|_{T'}\rceil| \qquad (3.1)\end{aligned}$$

By (2.6), we may write

$$\pi^*(K_Y)|_{T'}\equiv \frac{\theta_{m_0}}{m_0+\theta_{m_0}}\pi_T^*(K_T)+G,$$

where G is an effective $\mathbb{Q}$-divisor. By (3.1), we have

$$|mK_{Y'}||_{T'}\succcurlyeq |K_{T'}+\lceil ((m-1)\pi^*(K_Y)-\tfrac{1}{\theta_{m_0}}E_{m_0}-T')|_{T'}-(m-1-\tfrac{m_0}{\theta_{m_0}})G\rceil|. \qquad (3.2)$$

Note that we have

$$\begin{aligned}&((m-1)\pi^*(K_Y)-\frac{1}{\theta_{m_0}}E_{m_0}-T')|_{T'}-(m-1-\frac{m_0}{\theta_{m_0}})G\\ \equiv&(m-1-\frac{m_0}{\theta_{m_0}})\pi^*(K_Y)|_{T'}-(m-1-\frac{m_0}{\theta_{m_0}})G\\ \equiv&(m-1-\frac{m_0}{\theta_{m_0}})\cdot\frac{\theta_{m_0}}{m_0+\theta_{m_0}}\pi_T^*(K_T).\end{aligned}$$

Since

$$(m-1-\frac{m_0}{\theta_{m_0}})\cdot\frac{\theta_{m_0}}{m_0+\theta_{m_0}}\geq\frac{m-1-m_0}{m_0+1},$$

we have $(m-1-\frac{m_0}{\theta_{m_0}})\cdot\frac{\theta_{m_0}}{m_0+\theta_{m_0}}>37$ if $m>38(m_0+1)$, and $(m-1-\frac{m_0}{\theta_{m_0}})\cdot\frac{\theta_{m_0}}{m_0+\theta_{m_0}}>75$ if $m>76(m_0+1)$.

By Remark 2.6 and (3.2), we have $P_m(Y)\geq 2$ for all $m\geq 38m_0+39$. This proves (1).

For (2), by Lemma 2.7 and (3.2), $\varphi_{|mK_{Y'}|}|_{T'}$ is birational for all $m\geq 76m_0+77$. By Theorem 2.2, we only need to show that $|mK_{Y'}|$ separates different generic irreducible elements of $|M_{m_0}|$. By the same argument as in the proof of (1), we have

$$h^0(Y',K_{Y'}+\lceil (m-1)\pi^*(K_Y)-E_{m_0}-\theta_{m_0}T'\rceil)\geq 2$$

for all $m\geq 38m_0+39$. By [8, Lemma 2.7], $|K_{Y'}+\lceil (m-1)\pi^*(K_Y)-E_{m_0}\rceil|$ separates different generic irreducible elements of $|M_{m_0}|$ for all $m\geq 38m_0+39$ (set $V=Y'$, $R=(m-1)\pi^*(K_Y)-E_{m_0}-\theta_{m_0}T'$ and $L=M_{m_0}$). We conclude that $\varphi_{|mK_Y|}$ is birational for all $m\geq 76m_0+77$. This proves (2).

For (3), we have $\pi^*(K_Y)|_{T'}\geq\frac{1}{1+m_0}\pi_T^*(K_T)$ by (2.6). We deduce that

$$K_Y^4\geq\frac{1}{m_0}(\pi^*(K_Y)|_{T'})^3\geq\frac{1}{m_0(m_0+1)^3}K_T^3\geq\frac{1}{1680m_0(m_0+1)^3},$$

where the first inequality follows by (2.2) and the last inequality follows by [5, Theorem 1.6]. The proof is completed.

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