WEAK UNIQUENESS BY NOISE FOR SINGULAR STOCHASTIC PDES

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Abstract. We prove weak uniqueness of mild solutions for a general class of SPDEs on a Hilbert space. The main novelty is that the drift is only defined on a Sobolev-type subspace and no Hölder-continuity assumptions are required. This allows us to cover examples such as equations with divergence-form and non-divergence-form drift, and Cahn–Hilliard-type equations with possibly singular perturbations. The main idea is to consider a suitable coloured Wiener noise so that both the solvability of the SPDE and the regularising effect of the Kolmogorov operator are preserved via stochastic maximal regularity results. As a by-product, this also allows us to generalise the available results of uniqueness by noise for perturbations of the heat equation to higher dimensions.

1. INTRODUCTION

We are interested in weak uniqueness by noise for stochastic evolution equations of the form

$$\mathrm{d}X + AX\,\mathrm{d}t = B(X)\,\mathrm{d}t + G\,\mathrm{d}W\,, \qquad X(0) = x\,, \qquad (1.1)$$

where A is a linear self-adjoint maximal monotone operator on a Hilbert space H with an effective domain $D(A)$, W is an H-cylindrical Wiener process, and $x \in H$. As far as the drift B is concerned, we assume that it is defined on a subspace of H of Sobolev-type and may take values in a dual space larger than H. More precisely, we suppose that

$$B : D(A^{\alpha}) \to D(A^{-\beta})$$

for some constants $\alpha \in [0, 1)$ and $\beta \in [0, \frac{1}{2})$. Regarding the covariance operator G, we require that it is linear and continuous from H to $D(A^{\delta})$ for some $\delta \in [0, \frac{1}{2})$ and nondegenerate, with the main prototype being $G = A^{-\delta}$. Precise assumptions on the setting are presented in Section 2.

1.1. **Existing literature.** Starting from the pioneering contribution by Veretennikov [Ver80], the literature on uniqueness by noise has witnessed important developments in the infinite-dimensional case in recent years. The first results in this direction were obtained in [GP93] in the context of quasilinear equations, and in [Zam00, ABGP06], where weak uniqueness was studied for equations with nondegenerate multiplicative noise and Hölder-continuous bounded drift. This was then refined in [DP03], where the drift term has the specific form $B = A^{\frac{1}{2}}F$ for some $F : H \to H$ that is Hölder continuous and bounded with a sufficiently small Hölder norm. The most recent result on weak uniqueness that we are aware of is [Pri21, Pri22], in which the particular case $B = A^{\frac{1}{2}}F$ is considered where F is merely locally Hölder-continuous and the assumptions of smallness and boundedness on F are removed. Besides weak uniqueness,

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pathwise uniqueness by noise has also been investigated in the infinite-dimensional setting. In this direction, it is worth mentioning [DPF10], which deals with equations with Hölder continuous drift defined everywhere on H, and [DPFPR13, DPFPR15] for equations with Borel-measurable drift, bounded and unbounded, respectively. For completeness, we also refer to the works [AB23, FF11, FGP10, KR05]. Strong uniqueness by noise has also been studied in [MP17, AMP23] using techniques from backward stochastic equations.

The addition of noise has also been shown to be effective in providing refined regularisation phenomena. In this spirit, we mention the contributions [GG19] on stochastic Hamilton-Jacobi equations, [GS17, GM18] for stochastic scalar conservation laws, [Mau20, BMX23] on non-blow-up effects for stochastic equations, [BW22] on p-Laplace equations, and [GL23, FL21] in the context of fluid dynamics.

1.2. **Novelty and main goals.** The above-mentioned results on weak uniqueness by noise for infinite-dimensional equations in the form (1.1) only deal with the particular case $B : H \to H$, which corresponds to the specific choice $\alpha = \beta = 0$. The case $\beta > 0$ can be handled only if $\alpha = 0$, i.e., when the drift B takes the specific form $A^\beta F$, where $F : H \to H$ (see, e.g., [Pri21, Pri22]). Roughly speaking, if A is a classical second-order differential operator on some domain, the limitation $\alpha = \beta = 0$ means that the perturbation B acts as an operator of order zero, e.g., the superposition operator associated with a measurable real function. This excludes the possibility of considering singular perturbation terms in differential form arising in several applications, which are dominated by A itself. For example, this is the case for drifts B in the form $B(\cdot) = F(A^\alpha \cdot)$, which may depend on derivatives of the underlying process in a nonlinear way.

The second main limitation of the available literature consists in the form of the covariance operator of the noise, which is chosen to be the identity in the vast majority. One of the reasons for such a specific setting is due to the formal intuition that "*the rougher the noise, the more regularising the transition semigroup*", in the sense that the resolvent operator associated with the Kolmogorov equation of the SPDE is more smoothing when G is close to the identity. However, the choice $G = \mathrm{Id}$ (or equivalently $\delta = 0$) comes with an important drawback in terms of the actual solvability of the SPDE in the mild sense. For example, it imposes strong constraints on the spatial dimension of the domain, allowing only dimension one in the case of second-order operators. Moreover, if G is the identity, no additional space regularity can be obtained for the stochastic convolution in general, so one is forced to work on the whole space H by considering the choice $\alpha = 0$. More precisely, if the noise is not coloured (i.e. $\delta = 0$) and $\alpha > 0$, existence of solutions with values in the relevant space $D(A^\alpha)$ fails in general due to the lack of regularity on the stochastic convolution.

At an intuitive level, the main idea is that, on the one hand, if δ is "close" to zero, it is simpler to obtain uniqueness results via the Kolmogorov equation, but the existence of mild solutions for the SPDE requires several restrictions (e.g, $\alpha = 0$ and low dimensions). On the other hand, if δ is "larger", then one loses some regularisation effects on the Kolmogorov operator but gains in the range of solvable SPDEs that can be considered (e.g., $\alpha > 0$ and higher dimensions).

The main goal of this article is to prove weak uniqueness of mild solutions for stochastic equations in the form (1.1) with possibly singular drift B (i.e. $\alpha > 0$) and a possibly coloured noise G (i.e. $\delta > 0$). In particular, this contribution can be regarded as a first step toward developing a general approach for singular equations. Naturally, such a general framework can be further refined when dealing with specific examples by exploiting the typical features of the equation

under consideration, such as Burgers-type and fluid-dynamical models. This could be the subject of future, more specific investigations.

In the present paper, we show that by carefully calibrating the parameter δ, it is possible to achieve uniqueness of solutions to the SPDE in the more general case of $\alpha, \beta > 0$. This is carried out in a wide generality for local-in-time solutions, in order to cover as many examples as possible. More precisely, the main relation between the parameters involved turns out to be

$$0 \leq \delta + \beta < \frac{1}{2}\,,$$

which comes naturally from the regularising properties of the associated Ornstein-Uhlenbeck semigroup (see (3.8) with $\gamma = \beta$). The restrictions on the parameter α are more subtle as they are evident only in the case of unbounded drift in the form

$$0 \leq \alpha + \beta \leq \frac{1}{2}$$

and are related to maximal regularity arguments (see Section 4.2). The two limiting cases where either $(\alpha, \beta) = (0, \frac{1}{2})$ or $(\alpha, \beta) = (\frac{1}{2}, 0)$ will be referred to as *critical cases*. The former case is not covered here, but has been investigated with the specific choice $\delta = 0$ in [Pri21, Pri22] under a local Hölder-type condition on the drift. The latter case is covered instead by the present paper.

We emphasise that the focus of the paper is *not* to provide a unifying condition that guarantees existence of solutions, rather to establish uniqueness as soon as any local solution exists. Here, we only provide a sufficient condition for global-in-time existence of solutions, without exploiting any specific structure of B. For what concerns local/global existence, several sharp results are already available in the literature, depending on the specific structure of the drift B in terms of, e.g., monotonicity [LR15], geometric conditions [GP20].

1.3. **Strategy and technical issues.** Let us begin by emphasising the main technical assumptions on the operators. The only technical requiremens on B are a local boundedness condition from $D(A^\alpha)$ to $D(A^{-\beta})$ as well as a weak continuity assumption of the form

$$B : D(A^\alpha) \to Z \quad \text{is strongly-weakly continuous,}$$

where Z is a Hilbert space containing $D(A^{-\beta})$. Since Z can be chosen to be arbitrarily large, and continuity is intended only in a weak sense on Z, this condition can be easily checked in practice according to the specific example under consideration. In particular, we stress that *no* Hölder-continuity assumption is required for B. This contrasts the above-mentioned literature where this assumption is often employed in order to exploit classical Schauder-type estimates. Furthermore, we stress that in the case where B is also bounded, the above continuity requirement is not needed, and only measurability of B suffices.

The only technical assumption on the covariance operator G is a compatibility condition with the semigroup generated by $-A$, as proposed by Da Prato and Zabczyk in [DPZ02, Prop. 6.4.2]. This is automatically satisfied in the Hilbert-Schmidt case, but it is significantly weaker. The only case where we need a Hilbert-Schmidt requirement on G is the critical case $\alpha = \frac{1}{2}$ and $\beta = 0$.

In terms of the strategy, roughly speaking, the idea is to write an Itô formula for $u(X)$, where X is a suitable weak solution to (1.1), and u is a suitable solution to the associated Kolmogorov equation given by:

$$\lambda u(x) - \frac{1}{2} \operatorname{Tr}(QD^2 u(x)) + (Ax, Du(x)) = f(x) + \langle B(x), Du(x) \rangle, \quad x \in D(A), \qquad (1.2)$$

where $Q := G^*G$, f is a given bounded continuous function on H, $\lambda > 0$ is a fixed coefficient, and $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $D(A^{-\beta})$ and $D(A^{\beta})$. Proceeding formally, if one is able to give sense to the Itô formula for $u(X)$, then direct computations show that

$$u(x) = \mathbb{E} \int_0^{+\infty} e^{-\lambda t} f(X(t)) \, \mathrm{d}t \, . \tag{1.3}$$

Since this holds for every weak solution X starting from x, we can infer the uniqueness of the law of X in the space of continuous trajectories with values in H.

In order to rigorously justify the aforementioned procedure, several technical issues need to be addressed. Firstly, it should be noted that X is generally only a mild solution to equation (1.1), and direct application of the Itô formula to X is not possible. Therefore, a suitable regularisation $(X_j := P_j X)_j$ must be introduced, where $(P_j)_j$ is a family of projections converging to the identity as $j \to \infty$. Secondly, a similar issue arises with the Kolmogorov equation (1.2), as one is only able to show existence of a suitable mild solution u that is only of class C^1 on H. Consequently, direct application of the Itô formula to u is not possible. Therefore, it becomes necessary to introduce an additional sequence of regularized functions $(u_\varepsilon)_\varepsilon$. These functions are solutions to the Kolmogorov equation, where the original functions f and B are replaced by smoothed counterparts $(f_\varepsilon)_\varepsilon$ and $(B_\varepsilon)_\varepsilon$ obtained through convolution with Gaussian measures on H. Eventually, one is able to justify rigorously the Itô formula for $u_\varepsilon(X_j)$, which reads, after taking expectations and doing standard computations, as follows

$$\mathbb{E} \left[u_\varepsilon(X_j(t)) \right] - u_\varepsilon(x_j) - \lambda \, \mathbb{E} \int_0^t u_\varepsilon(X_j(s)) \, \mathrm{d}s + \mathbb{E} \int_0^t f_\varepsilon(X_j(s)) \, \mathrm{d}s$$

$$= \underbrace{\mathbb{E} \int_0^t \left(P_j B(X(s)) - B_\varepsilon(X_j(s)), Du_\varepsilon(X_j(s)) \right) \, \mathrm{d}s}_{I_1(\varepsilon, j)} + \underbrace{\frac{1}{2} \mathbb{E} \int_0^t \mathrm{Tr} \left[(P_j - I) Q D^2 u_\varepsilon(X_j(s)) \right] \, \mathrm{d}s}_{I_2(\varepsilon, j)} ,$$

where I_1 and I_2 have to be interpreted as remainders of the limiting balance yielding (1.3). The novel technical idea is to pass to the limit jointly in j and ε by using a suitable scaling of the parameters in such a way that $I_1(\varepsilon, j), I_2(\varepsilon, j) \to 0$.

In order to pass to the limit in the Itô formula, for the terms on the left-hand side we need some convergence of $X_j \to X$ (which easily follows by the choice of $(P_j)_j$) and a convergence $u_\varepsilon \to u$ at least in some Hölder space $C_b^{0,\gamma}(H)$ for some $\gamma \in (0, 1]$. While for the term I_1 we only need a uniform bound on the gradients $(Du_\varepsilon)_\varepsilon$ at least in $C_b^0(H; D(A^\beta))$, the term I_2 is exactly the one that can be treated by choosing a suitable scaling of ε and j. Indeed, roughly speaking, the second derivatives $D^2 u_\varepsilon$ blow up as $\varepsilon \to 0$ while $P_j - I$ vanishes as $j \to \infty$. As a by product of this joint limiting procedure, we are able to avoid any Hölder regularity on B, and what really matters is that B_ε is Hölder for every ε.

The proof of convergence of u_ε to u and Du_ε to Du is based on compactness and density arguments in infinite dimension. More precisely, by using the regularising properties of the Kolmogorov operator, we first show uniform bounds of $(u_\varepsilon)_\varepsilon$ in $C_b^1(H)$ and of $(Du_\varepsilon)_\varepsilon$ in $C_b^0(H; D(A^\beta))$. By the Arzelà–Ascoli theorem and a localisation procedure on H, this allows to infer the convergence $u_\varepsilon \to \mathfrak{u}$ and $Du_\varepsilon \to D\mathfrak{u}$ pointwise in H for some candidate $\mathfrak{u} \in C_b^1(H)$. Therefore, the problem is reduced to the identification $u = \mathfrak{u}$. Indeed, since in general such convergences are only pointwise in H, and not with respect to the uniform topology, is it not possible to pass to the limit in the classical C^1-sense in the Kolmogorov equation. Nonetheless, the convergence in the Kolmogorov equation can be proved in $L^2(H, N_{Q_\infty})$, where N_{Q_∞} is the

nondegenerate Gaussian invariant measure of the associated Ornstein-Uhlenbeck semigroup. By uniqueness of the limiting Kolmogorov equation in $L^2(H, N_{Q_\infty})$, it follows that $u = \mathfrak{u}$, N_{Q_∞}-almost-everywhere on H. At this point, we exploit a lemma originally due Itô (see [Itô70]). In particular, this guarantees that, since N_{Q_∞} is nondegenerate, every Borel subset $E \subset H$ with full measure $N_{Q_\infty}(E) = 1$ is dense in H. By Lipschitz-continuity of u and $\mathfrak{u}$, this allows us to identify $u = \mathfrak{u}$ on the whole H.

Let us stress that the convergence of I_2 is not trivial since Q is not of trace-class. However, it is possible to exploit the regularising properties of the Kolmogorov operator to show that QD^2u_ε is continuous with values in the trace-class operators, even if Q is not trace-class itself. This allows us to deduce the convergence of I_2 to 0 anyway.

All the arguments described above are done in detail for the case where B is bounded from $D(A^\alpha)$ to $D(A^{-\beta})$. For the unbounded case, we use a localisation-extension procedure on the solutions to the SPDEs and exploit the available uniqueness result for the bounded case. More precisely, given a solution X to (1.1) in the unbounded case, we first employ some stochastic maximal regularity argument to show that X is actually continuous with values in $D(A^\alpha)$. This can be done thanks to the results contained in [AV22, vNVW12]. Since B is only defined in $D(A^\alpha)$, it is necessary to introduce a sequence of stopping times $(\tau_N)_N$, such that the stopped processes $(X^{\tau_N})_N$ solve the equation (1.1) locally on $[0, \tau_N]$ with B replaced by some truncation B_N. Eventually, we show that X^{τ_N} can be extended globally on $[0, +\infty)$ to a solution $\tilde{X}_N$ of (1.1) with B_N. Since B_N is bounded, the available uniqueness result in the bounded case yields, together with the arbitrariness of N, uniqueness in law also for X.

1.4. **Applications.** We briefly list below some examples of stochastic equations of the form (1.1) for which we can show weak uniqueness, and we refer to Section 5 for further details. For convenience, in the technical Proposition 5.2 we provide a list of the main properties of the covariance operator in the form $G = A^{-\delta}$, which can be used in the different settings.

We fix a smooth bounded domain $\mathcal{O} \subset \mathbb{R}^d$ ($d \geq 1$) and a time horizon $T > 0$ so that all the equations considered below are defined in $[0, T] \times \mathcal{O}$.

(a) We can deal with *heat equations with perturbation* in the form
$$\mathrm{d}X - \Delta X \,\mathrm{d}t = F(X)\,\mathrm{d}t + G\,\mathrm{d}W\,, \qquad X(0) = x_0\,,$$
with Dirichlet boundary conditions and where $F : \mathbb{R} \to \mathbb{R}$ is continuous and linearly bounded[4], $x_0 \in L^2(\mathcal{O})$, and $G \in \mathscr{L}(L^2(\mathcal{O}), L^2(\mathcal{O}))$. In the notation of the paper, we have that $\alpha = \beta = 0$. We are able to show that uniqueness in law holds for every $\delta \in [0, \frac{1}{2})$ in dimension $d = 1$, for every $\delta \in (0, \frac{1}{2})$ in dimension $d = 2$, and for every $\delta \in (\frac{1}{4}, \frac{1}{2})$ in dimension $d = 3$. This extends both the classical results [DPF10, DPFPR13] to higher dimensions $d = 2, 3$ and the work [AMP23] for what concerns the choice of the drift.

(b) We can consider *heat equations with polynomial perturbation* in the form
$$\mathrm{d}X - \Delta X \,\mathrm{d}t = F(X)\,\mathrm{d}t + G\,\mathrm{d}W\,, \qquad X(0) = x_0\,,$$
with Dirichlet boundary conditions and where $F : \mathbb{R} \to \mathbb{R}$ is a continuous function that behaves like a polynomial of order $p - 1$ with $p \geq 2$ (see Section 5 for precise details), $x_0 \in L^2(\mathcal{O})$, and $G \in \mathscr{L}(L^2(\mathcal{O}), L^2(\mathcal{O}))$. Here, it its possible to apply our setting under a specific choice of α, β and δ, depending on p and the space dimension. It turns out that uniqueness in law holds in dimensions $d = 1, 2$ for every growth $p \geq 2$ and in dimension $d = 3$ for every growth $p \in [2, 4)$.

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[4]In the whole paper, "linearly bounded" means that $|F(x)| \leq C(1 + |x|)$ for all $x \in \mathbb{R}$ for some constant $C > 0$.

(c) We can address *heat equations with perturbation in divergence-like form* that can be written as follows

$$\mathrm{d}X - \Delta X \, \mathrm{d}t = (-\Delta)^\beta F(X) \, \mathrm{d}t + G \, \mathrm{d}W \,, \qquad X(0) = x_0 \,,$$

with Dirichlet boundary conditions and where $\beta \in (0, \frac{1}{2})$, $F : \mathbb{R} \to \mathbb{R}$ is continuous and linearly bounded, $x_0 \in L^2(\mathcal{O})$, and $G \in \mathscr{L}(L^2(\mathcal{O}), L^2(\mathcal{O}))$. In this case, we have that $\alpha = 0$. In dimension $d = 1$, we can prove uniqueness in law for the choice of parameters $\beta \in (0, \frac{1}{2})$ and $\delta \in [0, \frac{1}{2} - \beta)$. We can also obtain uniqueness in law in dimensions $d = 2$ and $d = 3$ for the choices of parameters $\beta \in (0, \frac{1}{2})$ and $\delta \in (0, \frac{1}{2} - \beta)$, and $\beta \in (0, \frac{1}{4})$ and $\delta \in (\frac{1}{4}, \frac{1}{2} - \beta)$, respectively. We refer to [Pri21, Pri22] for the critical case $\beta = \frac{1}{2}$ and $\delta = 0$ (only in dimension $d = 1$).

(d) We can consider *heat equations with perturbation in non-divergence-like form* that can be written as follows

$$\mathrm{d}X - \Delta X \, \mathrm{d}t = F((-\Delta)^\alpha X) \, \mathrm{d}t + G \, \mathrm{d}W \,, \qquad X(0) = x_0 \,,$$

with Dirichlet boundary conditions and where $\alpha \in (0, 1)$, $F : \mathbb{R} \to \mathbb{R}$ is continuous and linearly bounded, $x_0 \in L^2(\mathcal{O})$, and $G \in \mathscr{L}(L^2(\mathcal{O}), L^2(\mathcal{O}))$. In this case, we have that $\beta = 0$. We can show that uniqueness in law holds in dimension $d = 1$ for the choice of parameters $\alpha \in (0, \frac{1}{2}]$ and $\delta \in [0, \frac{1}{2})$ (with also $\delta \in (\frac{1}{4}, \frac{1}{2})$ in the critical case $\alpha = \frac{1}{2}$), in dimension $d = 2$ with the choices $\alpha \in (0, \frac{1}{2})$ and $\delta \in (\alpha, \frac{1}{2})$, and in dimension $d = 3$ with the choices $\alpha \in (0, \frac{1}{4})$ and $\delta \in (\alpha + \frac{1}{4}, \frac{1}{2})$. Moreover, if F is also bounded, then uniqueness is law holds for every $\alpha \in (0, 1)$ and $\delta \in [0, \frac{1}{2})$ in every dimension $d = 1, 2, 3$. Up to our knowledge, this is one of the first contributions in the literature dealing with a fully-nonlinear drift.

(e) We can deal with *one-dimensional Burgers-type equations* in the form

$$\mathrm{d}X - \Delta X \, \mathrm{d}t = F(X) \cdot \nabla X \, \mathrm{d}t + G \, \mathrm{d}W \,, \qquad X(0) = x_0 \,,$$

with Dirichlet boundary conditions and where $F : \mathbb{R} \to \mathbb{R}$ is continuous and bounded, $x_0 \in H_0^1(\mathcal{O})$, and $G \in \mathscr{L}^2(L^2(\mathcal{O}), L^2(\mathcal{O}))$. Thanks to our results, we can prove that uniqueness in distribution holds for $\alpha = \frac{1}{2}$, $\beta = 0$, and $\delta \in (\frac{1}{4}, \frac{1}{2})$. This uniqueness result comes as a byproduct of our strategy, even though it is already available in the literature via other techniques (see [LR15]).

(f) Finally, we can also consider *Cahn–Hilliard equations with perturbation* in the form

$$\mathrm{d}X - \Delta(-\Delta X + F_1(X)) \, \mathrm{d}t = F_2(X, \nabla X, D^2 X) \, \mathrm{d}t + G \, \mathrm{d}W \,, \qquad X(0) = x_0 \,,$$

with Neumann-type boundary conditions and where $F_1 : \mathbb{R} \to \mathbb{R}$ is the derivative of the classical polynomial double-well potential as defined in (5.1), $F_2 : \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^{d \times d} \to \mathbb{R}$ is continuous and linearly bounded in every argument, and $G \in \mathscr{L}(L^2(\mathcal{O}), L^2(\mathcal{O}))$. In the notation of the paper, we have that $\alpha = \frac{1}{2}$ and $\beta = 0$. We can treat dimensions $d = 1, 2, 3$ and we can shows that uniqueness in law holds for every $\delta \in (\frac{d}{8}, \frac{1}{2})$. Again, this is the first contribution on uniqueness by noise for a Cahn-Hilliard equation with singular source possibly depending on higher space derivatives of the underlying process. We mention [Pri21, Pri22] for the case of linearly bounded potential and non-singular perturbation.

1.5. **Structure of the paper.** The remaining part of this article is organised as follows. In Section 2, we introduce the general setting of the paper and list the main uniqueness results. More precisely, Theorems 2.6-2.7 contain the results on weak uniqueness in the bounded and unbounded cases. In Section 3, we study the limit Kolmogorov equation in a mild sense and its approximation through suitably regularised equations. The proofs of the main results are given in Section 4, while applications to specific example of interest are thoroughly discussed in Section 5. Finally, in Appendix A in the bounded case we provide a sufficient condition for existence of global weak solutions to equation (1.1), which is used to extend local solutions in the uniqueness argument mentioned above.

2. SETTING AND MAIN RESULTS

In this section, we precisely state our assumptions and the main results of this article. Throughout the paper, we let H be a separable Hilbert space with scalar product $(\cdot, \cdot)$ and norm $\| \cdot \|$, which will always be identified to its dual space H^*. We denote by $\mathscr{L}^1(H, H)$ and $\mathscr{L}^2(H, H)$ the usual spaces of trace-class and Hilbert-Schmidt operators on H, respectively. Moreover, we also introduce the space

$$\mathscr{L}^4(H, H) := \left\{ L \in \mathscr{L}(H, H) : \sum_{k=0}^{\infty} \|Le_k\|_H^4 < +\infty \right\}$$

where $(e_k)_k$ is a complete orthonormal system of H, endowed with obvious choice of the norm. Eventually, for $k \in \mathbb{N}$, we use the symbols $C_b^k(H)$ and $UC_b(H)$ for the spaces of k-times bounded Fréchet-differentiable functions on H with continuous derivatives and uniformly continuous bounded functions on H. Analogously, we employ the notation $C_b^{k,\ell}(H)$ and $C_b^{k+\ell}(H)$ for the space of k-times bounded Fréchet-differentiable functions on H with ℓ-Hölder derivatives.

We assume the following condition.

(H1) (**Assumptions on** A) The operator A is a linear self-adjoint maximal monotone operator with compact resolvent. In particular, $(-A)$ generates a strongly continuous symmetric semigroup $(S(t))_{t \geq 0}$ of contractions on H.

Before stating the remaining conditions, we note that by assumption (H1), the fractional powers A^s, for $s \in (0, 1)$, are well-defined. Indeed, if $(e_k)_{k \in \mathbb{N}}$ is a complete orthonormal system of H made of eigenvectors of A and $(\lambda_k)_{k \in \mathbb{N}}$ are the corresponding eigenvalues, one can set

$$D(A^s) := \left\{ x \in H : \sum_{k=0}^{\infty} \lambda_k^{2s}(x, e_k)^2 < +\infty \right\}, \qquad A^s x := \sum_{k=0}^{\infty} \lambda_k^s (x, e_k) e_k, \quad x \in D(A^s).$$

For brevity of notation, we define

$$V_{2s} := D(A^s), \quad s \in (0, 1], \tag{2.1}$$

and we recall that V_{2s} is a separable Hilbert space with scalar product and norm given by

$$(x, y)_{2s} := (A^s x, A^s y), \quad \forall x, y \in V_{2s}, \qquad \|x\|_{2s} := \|A^s x\|, \quad \forall x \in V_{2s},$$

respectively. The duality pairing between $D(A^{-s})$ and $D(A^s)$ will be denoted by the symbol $\langle \cdot, \cdot \rangle$, without explicit notation for s. Moreover, the inclusion $V_{2s} \hookrightarrow H$ is dense for every $s \in (0, 1]$ and by definition it holds that

$$(Ax, x) = \|A^{1/2}x\|^2 = \|x\|_1^2, \quad \forall x \in D(A).$$

In particular, the semigroup S is also analytic and of negative type (see [DPZ14, Prop. A12]). Furthermore, for every $s, s' > 0$ with $s < s'$ the inclusion $V_{2s'} \hookrightarrow V_{2s}$ is compact.

For all $s \in (0, 1]$, we naturally set $D(A^{-s}) := D(A^s)^*$, endowed with its natural norm. Note that the semigroup S extends to a semigroup on $D(A^{-s})$, for all $s \in (0, 1]$, which will be denoted by the same symbol S for brevity. This follows by extrapolation from the fact that the restriction of S to $D(A^s)$ is a strongly continuous semigroup of contractions on $D(A^s)$. Indeed, for all $x \in H$, $y \in D(A^s)$, and $t \geq 0$, we have

$$(S(t)x, y)_H = (x, S(t)y)_H \leq \|x\|_{D(A^{-s})} \|S(t)y\|_{D(A^s)} \leq \|x\|_{D(A^{-s})} \|y\|_{D(A^s)},$$

and so the (unique) extension follows by density of H in $D(A^{-s})$.

We can now proceed with the remaining assumptions.

(H2) (**Assumptions on** B) Let $\alpha \in [0, 1)$, $\beta \in [0, \frac{1}{2})$, and let $B : D(A^\alpha) \to D(A^{-\beta})$ be measurable and locally bounded.

(H3) (**Assumptions on** G) Let $\delta \in [0, \frac{1}{2} - \beta)$ and let $G \in \mathscr{L}(H, H)$ be positive self-adjoint such that G commutes with A, $\ker(G) = \{0\}$, and $G(H) = D(A^\delta)$. We set $Q := GG^*$ and

$$Q_t := \int_0^t S(s)QS(s)\, \mathrm{d}s = \frac{1}{2}A^{-1}Q(I - S(2t)), \quad t \geq 0. \tag{2.2}$$

We assume that there exist $\xi \in (0, \frac{1}{2})$, $\lambda > 0$, and $\vartheta \in (0, 1)$ such that

$$\int_0^T t^{-2\xi} \|S(t)G\|_{\mathscr{L}^2(H,H)}^2\, \mathrm{d}t < +\infty \quad \forall T > 0, \tag{2.3}$$

$$\int_0^{+\infty} e^{-\lambda t} \left\|Q_t^{-\frac{1}{2}}S(t)G\right\|_{\mathscr{L}^4(H,H)}^{2(1-\vartheta)} \left\|Q_t^{-\frac{1}{2}}S(2t)Q\right\|_{\mathscr{L}^2(H,H)}^{\vartheta}\, \mathrm{d}t < +\infty. \tag{2.4}$$

The prototypical choice of G satisfying assumption (H3) is given by $G = A^{-\delta}$, with suitable conditions on δ (see Section 5 for some examples).

Note that condition (2.3) is classical and guarantees both that $Q_t \in \mathscr{L}^1(H, H)$ for all $t \geq 0$ and the continuity in time with values in H for the stochastic convolution (see [DPZ14, Thm. 5.11]). Moreover, recalling that in our case the semigroup S is analytic, condition (2.3) implies (see [DPZ14, Thm. 5.15]) that the stochastic convolution also verifies

$$\int_0^{\cdot} S(\cdot - s)G\, \mathrm{d}W(s) \in C^0(\mathbb{R}_+; D(A^\zeta)) \quad \forall \zeta \in [0, \xi), \quad \mathbb{P}\text{-a.s.} \tag{2.5}$$

Furthermore, although condition (2.4) may look convoluted at first sight, it is satisfied in numerous well-known examples of stochastic equations in space dimension up to three (this will be shown in detail in Section 5). The reason why we require (2.4) is actually very natural, as it guarantees that strong solutions u to the Kolmogorov equation associated to the corresponding Ornstein-Uhlenbeck semigroup verify $QD^2u \in \mathscr{L}^1(H, H)$, even when $Q \notin \mathscr{L}^1(H, H)$. This is the only sufficient condition for such regularity that we are aware of. We refer to [DPZ02, Prop. 6.4.2] for more detail.

Remark 2.1. *We point out that if $G \in \mathscr{L}^2(H, H)$, then conditions (2.3) and (2.4) are satisfied for every $\xi \in (0, \frac{1}{2})$ and $\vartheta \in (\frac{2\delta}{2\delta+1}, 1)$. For a detailed proof see Lemma 5.1 in Section 5.*

We now state precisely what we mean by weak solution to problem (1.1).

Definition 2.2 (Global weak solution)**.** *Assume (H1)-(H2)-(H3) and let $x \in H$. A global weak solution to* (1.1) *is a sextuplet*

$$(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t \geq 0}, \mathbb{P}, W, X)$$

where $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t \geq 0}, \mathbb{P})$ is a filtered probability space satisfying the usual conditions, W is a H-cylindrical Wiener process, and X is a progressively measurable process such that

$$X \in C^0(\mathbb{R}_+; H) \cap L^1_{loc}(\mathbb{R}_+; D(A^\alpha)) \quad and \quad B(X) \in L^2_{loc}(\mathbb{R}_+; D(A^{-\beta})), \quad \mathbb{P}\text{-}a.s.,$$

satisfying the following equation

$$X(t) = S(t)x + \int_0^t S(t - s)B(X(s)) \, \mathrm{d}s + \int_0^t S(t - s)G \, \mathrm{d}W(s), \quad \forall t \geq 0, \quad \mathbb{P}\text{-}a.s. \qquad (2.6)$$

Definition 2.3 (Local weak solution)**.** *Assume (H1)-(H2)-(H3) and let $x \in H$. A local weak solution to* (1.1) *is a septuplet*

$$(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t \geq 0}, \mathbb{P}, W, X, \tau)$$

where $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t \geq 0}, \mathbb{P})$ is a filtered probability space satisfying the usual conditions, W is a H-cylindrical Wiener process, X is a progressively measurable process, and $\tau : \Omega \to [0, +\infty]$ is a stopping time such that the stopped process $X^\tau := X(\cdot \wedge \tau)$ satisfies

$$X^\tau \in C^0(\mathbb{R}_+; H) \cap L^1_{loc}(\mathbb{R}_+; D(A^\alpha)) \quad and \quad B(X^\tau) \in L^2_{loc}(\mathbb{R}_+; D(A^{-\beta})), \quad \mathbb{P}\text{-}a.s.,$$

and

$$X^\tau(t) = S(t \wedge \tau)x + \int_0^{t \wedge \tau} S((t \wedge \tau) - s)B(X^\tau(s)) \, \mathrm{d}s$$

$$+ \int_0^{t \wedge \tau} S((t \wedge \tau) - s)G \, \mathrm{d}W(s), \quad \forall t \geq 0, \quad \mathbb{P}\text{-}a.s. \qquad (2.7)$$

Note that if $\tau = +\infty$ then the Definitions 2.2 and 2.3 coincide.

Remark 2.4. *We recall that in Definition 2.2, we required that $B(X) \in L^2_{loc}(\mathbb{R}_+; D(A^{-\beta}))$. However, in order to give sense to* (2.6), *one could only require that $B(X) \in L^1_{loc}(\mathbb{R}_+; D(A^{-\beta}))$. However, the stronger condition $B(X) \in L^2_{loc}(\mathbb{R}_+; D(A^{-\beta}))$ is actually natural as it is always satisfied a posteriori by the assumptions made on B. Indeed, in the bounded case it trivially holds that $B(X) \in L^\infty(\mathbb{R}_+; D(A^{-\beta}))$ as soon as X takes values in $D(A^\alpha)$ almost everywhere. In the unbounded case, the requirement $B(X) \in L^2_{loc}(\mathbb{R}_+; D(A^{-\beta}))$ is needed in order to exploit a maximal regularity argument that guarantees*

$$\int_0^{\cdot} S(\cdot - s)B(X(s)) \, \mathrm{d}s \in C^0(\mathbb{R}_+; D(A^{\frac{1}{2} - \beta})), \quad \mathbb{P}\text{-}a.s. \qquad (2.8)$$

Nonetheless, since a posteriori the process X is in $L^\infty(0, T; D(A^\alpha))$, the locally boundedness assumption on B guarantees that actually $B(X) \in L^\infty(\mathbb{R}_+; D(A^{-\beta}))$ also in this case.

For completeness, we specify in the following definition what it means for two continuous processes taking values in H to have the same law.

Definition 2.5 (Uniqueness in law)**.** *Assume that X and Y are two progressively measurable processes defined on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t \geq 0}, \mathbb{P})$ such that $X, Y \in C^0(\mathbb{R}_+; H)$, $\mathbb{P}$-a.s. We say that X and Y have the same law on $C^0(\mathbb{R}_+; H)$ if*

$$\mathbb{E}\left[\psi(X)\right] = \mathbb{E}\left[\psi(Y)\right], \quad \forall \psi : C^0(\mathbb{R}_+; H) \to \mathbb{R} \text{ continuous and bounded}.$$

We are now ready to state our main results. The first result is a weak uniqueness property in the case of bounded drift for global weak solutions: considering global solutions here is very natural since the boundedness of the drift guarantees global existence under very reasonable assumptions (see Appendix A).

Theorem 2.6 (B bounded)**.** *Assume (H1)-(H2)-(H3), and that $B : D(A^\alpha) \to D(A^{-\beta})$ is measurable and bounded. Let $x \in H$ and let*

$$(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t \geq 0}, \mathbb{P}, W, X) , \qquad (\Omega, \mathscr{F}, (\mathscr{F}_t)_{t \geq 0}, \mathbb{P}, W, Y)$$

be two global weak solutions to (1.1) in the sense of Definition 2.2 with respect to the same initial datum x. Then, X and Y have the same law in the sense of Definition 2.5.

Our second result provides weak uniqueness for local solutions in the case of unbounded drift. Working with local solutions in this case is very natural since global existence may fail in general if the drift is unbounded. Here the unboundedness of the drift requires a restriction on the parameters and a weak continuity requirement on B.

Theorem 2.7 (B unbounded)**.** *Assume (H1)-(H2)-(H3) and let $\alpha \in [0, \frac{1}{2} - \beta] \cap [0, \xi)$. Suppose also that there exists a Hilbert space Z such that $D(A^{-\beta}) \hookrightarrow Z$ and that B is strongly-weakly continuous from $D(A^\alpha)$ to Z. Let $x \in D(A^\alpha)$ and*

$$(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t \geq 0}, \mathbb{P}, W, X, \tau_X) , \qquad (\Omega, \mathscr{F}, (\mathscr{F}_t)_{t \geq 0}, \mathbb{P}, W, Y, \tau_Y)$$

be two local weak solutions to (1.1) in the sense of Definition 2.3 with respect to the same initial datum x. Then, setting $\tau := \tau_X \wedge \tau_Y$, X^τ and Y^τ have the same law in the sense of Definition 2.5. Furthermore, under the stronger condition that $G \in \mathscr{L}^2(H, D(A^{\delta'}))$ for some $\delta' > 0$, then the same conclusion holds also in the limiting case $\alpha = \frac{1}{2}$ and $\beta = 0$.

Remark 2.8. *Let us point out that the choice of α and β is not intrinsic in B. Indeed, it is possible that the same differential operator B satisfies assumption (H2) for different choices of α and β, depending on whether one considers strong or weak formulations of it. What is actually intrinsic in the choice of B is the amplitude $\alpha + \beta$, which represents the differential order of B. More precisely, given a differential operator B of order q, one can frame the equation in several ways by choosing α, β such that $\alpha + \beta = q$: since the initial data are taken in $D(A^\alpha)$, the respective uniqueness result is more powerful the closer α is to zero. This will be shown in detail in Section 5.*

Remark 2.9. *Note that in Theorem 2.7, the requirement $\alpha \in [0, \frac{1}{2} - \beta] \cap [0, \xi)$ excludes a priori the limiting case $\alpha = \frac{1}{2}$ and $\beta = 0$. However, this issue can be overcome by enforcing the regularity of G in a Hilbert-Schmidt sense as specified in the theorem. This allows us to employ stochastic maximal regularity arguments in the localisation procedure.*

Note that the choice $\tau_X = \tau_Y = +\infty$ in Theorem 2.7 is allowed and corresponds to weak uniqueness for global solutions. Moreover, let us point out the particular case where $\alpha = \beta = \delta = 0$ is the setting considered in [DPFPR13, DPFPR15]. In our framework, when $\alpha = 0$ we are also able to cover the case $\beta \in [0, \frac{1}{2})$. The case $\alpha = 0$ and $\beta \in (\frac{1}{2}, 1)$, as well as the drop of the Hölder assumption in the case $(\alpha, \beta) = (0, \frac{1}{2})$, seems to be an open problem (see [Pri22, Rmk. 4]). The critical case $\alpha = \delta = 0$ and $\beta = \frac{1}{2}$ has been analysed in [Pri21] for a specific form of B satisfying some local Hölder condition.

3. The Kolmogorov equation

In this section, we assume that $B : D(A^\alpha) \to D(A^{-\beta})$ is bounded, and we work under assumptions (H1)-(H2)-(H3). This will be enough to prove uniqueness in law also for more general operators B by using a localisation argument at the SPDE-level (see [DPFPR15] for a similar technique).

We consider the Kolmogorov equation associated to (1.1), i.e.,

$$\lambda u(x) + Lu(x) = f(x) + \langle B(x), Du(x) \rangle \,, \quad x \in D(A) \,, \tag{3.1}$$

where $\lambda > 0$, $f \in UC_b^0(H)$, and the operator L is defined as

$$Lv(x) := -\frac{1}{2} \operatorname{Tr}(QD^2 v(x)) + (Ax, Dv(x)) \,, \quad x \in D(A) \,, \quad v \in C_b^2(H) \,.$$

We introduce the Ornstein-Uhlenbeck semigroup associated to L by letting

$$[R_t v](x) := \int_H v(y + S(t)x) \, N_{Q_t}(\mathrm{d}y) \,, \quad x \in H \,, \quad v \in \mathscr{B}_b(H) \,, \quad t \geq 0 \,,$$

where N_{Q_t} denotes the Gaussian measure with mean 0 and covariance operator Q_t defined in (2.2). We recall that the probability measure N_{Q_∞}, with

$$Q_\infty := \int_0^\infty S(s)QS(s)\,\mathrm{d}s = \frac{1}{2}A^{-1}Q \,, \tag{3.2}$$

is the unique invariant measure for the semigroup R, and that R uniquely extends to a semigroup of contractions also on $L^2(H, N_{Q_\infty})$.

We collect here some useful properties of the operator Q_t.

Lemma 3.1. *Assume (H1)-(H2)-(H3). Then, it holds that*

$$S(t)(H) \subset Q_t^{1/2}(H) \,, \quad \forall t > 0 \,, \tag{3.3}$$

and for all $\gamma \geq 0$ there exists a constant $C_\gamma > 0$, depending on G, such that

$$\left\| A^\gamma Q_t^{-1/2} S(t) \right\|_{\mathscr{L}(H,H)} \leq \frac{C_\gamma}{t^{\frac{1}{2}+\delta+\gamma}} \,, \quad \forall t > 0 \,. \tag{3.4}$$

Proof. Noting that $Q_t = \frac{1}{2}A^{-1}Q(I - S(2t))$ and $G = Q^{1/2}$, we have

$$A^\gamma Q_t^{-1/2} S(t) e_k = e^{-t\lambda_k} Q_t^{-1/2} A^\gamma e_k = e^{-t\lambda_k} \sqrt{2} \lambda_k^{\frac{1}{2}+\gamma} \frac{1}{\sqrt{1 - e^{-2\lambda_k t}}} G^{-1} e_k \,.$$

Since $G^{-1} : V_{2\delta} \to H$ is an isomorphism, we deduce that

$$\left\| A^\gamma Q_t^{-1/2} S(t) e_k \right\| \leq \left\| G^{-1} \right\|_{\mathscr{L}(V_{2\delta},H)} e^{-t\lambda_k} \sqrt{2} \lambda_k^{\frac{1}{2}+\gamma} \frac{1}{\sqrt{1 - e^{-2\lambda_k t}}} \|e_k\|_{2\delta}$$

$$= \sqrt{2} \left\| G^{-1} \right\|_{\mathscr{L}(V_{2\delta},H)} e^{-t\lambda_k} \lambda_k^{\frac{1}{2}+\delta+\gamma} \frac{1}{\sqrt{1 - e^{-2\lambda_k t}}} \,,$$

from which it follows that

$$\left\| A^\gamma Q_t^{-1/2} S(t) \right\|_{\mathscr{L}(H,H)} \leq \sqrt{2} \left\| G^{-1} \right\|_{\mathscr{L}(V_{2\delta},H)} \frac{1}{t^{\frac{1}{2}+\delta+\gamma}} \left(\max_{r \geq 0} \frac{r^{\frac{1}{2}+\delta+\gamma} e^{-r}}{\sqrt{1 - e^{-2r}}} \right) \,.$$

Therefore, the desired conclusion follows. $\qquad\square$

Thanks to the properties of the operator Q_t stated in the previous lemma, we can deduce the following results on the semigroup R.

Lemma 3.2. *Assume (H1)-(H2)-(H3). Then, the semigroup R is strong Feller. More specifically, there exists a constant $C_R > 0$ such that, for every $t > 0$ and $v \in \mathscr{B}_b(H)$, it holds that $R_t v \in C_b^\infty(H)$ and*

$$\sup_{x \in H} |(R_t v)(x)| \leq \sup_{x \in H} |v(x)|, \tag{3.5}$$

$$\sup_{x \in H} \|D(R_t v)(x)\|_H \leq \frac{C_R}{t^{\frac{1}{2} + \delta}} \sup_{x \in H} |v(x)|. \tag{3.6}$$

Moreover, for every $v \in C_b^1(H)$, it holds that

$$\sup_{x \in H} \|D^2(R_t v)(x)\|_{\mathscr{L}(H,H)} \leq \frac{C_R}{t^{\frac{1}{2} + \delta}} \sup_{x \in H} \|Dv(x)\|_H. \tag{3.7}$$

Furthermore, for every $\gamma \in (0, 1/2 - \delta)$ and $\eta \in (0, (1 - 2\delta)/(1 + 2\delta))$ there exists constants $C_{R,\gamma}, C_{R,\eta} > 0$ such that, for every $t > 0$ and $v \in UC_b^0(H)$, it holds that $D(R_t v)(H) \subset D(A^\gamma)$ and

$$\sup_{x \in H} \|A^\gamma D(R_t v)(x)\|_H \leq \frac{C_{R,\gamma}}{t^{\frac{1}{2} + \delta + \gamma}} \sup_{x \in H} |v(x)|, \tag{3.8}$$

$$\|R_t v\|_{C_b^{1,\eta}(H)} \leq \frac{C_{R,\eta}}{t^{(1+\eta)(\frac{1}{2}+\delta)}} \sup_{x \in H} |v(x)|. \tag{3.9}$$

Proof of Lemma 3.2. The proof of (3.5)–(3.7) follows from (3.3)–(3.4) and [DPF10, Thm. 4]. Moreover, (3.8) follows from [DPZ02, Thm. 6.2.2] and (3.4). Finally, (3.9) follows from [DPZ02, Lem. 6.4.1]. $\square$

The following preliminary result will be crucial in the sequel. Even if it is rather classical (see, e.g., [Itô70]), we present here a self contained proof for the reader's convenience.

Lemma 3.3. *Assume (H1)-(H2)-(H3). Let $E \subset H$ be Borel-measurable with $N_{Q_\infty}(E) = 1$. Then E is dense in H.*

Proof. First, note that $Q_\infty \in \mathscr{L}^1(H, H)$ is a positive operator with $\ker(Q_\infty) = \{0\}$ thanks to (H3). Hence, there exists an orthonormal system $(q_k)_{k \in \mathbb{N}}$ of H and a sequence of eigenvalues $(\zeta_k)_k$ such that $Q_\infty q_k = \zeta_k q_k$ for every $k \in \mathbb{N}$. In particular, it holds (see [DPZ02, Thm. 1.2.1]) that N_{Q_∞} is the restriction to H (identified with ℓ^2) of the product measure on $(\mathbb{R}^\infty, \mathscr{B}(\mathbb{R}^\infty))$ given by

$$\bigotimes_{k \in \mathbb{N}} N_{\zeta_k},$$

where N_{ζ_k} denotes a Gaussian measure on $\mathbb{R}$ with variance ζ_k, for every $k \in \mathbb{N}$. Note that the nondegeneracy condition $\ker(Q_\infty) = \{0\}$ ensures that $\zeta_k > 0$ for every $k \in \mathbb{N}$, so that each N_{ζ_k} is absolutely continuous with respect to the Lebesgue measure. Now, by contradiction, we suppose that there exists $a \in H$ and $r > 0$ such that

$$\{x \in H : \|x - a\|_H < r\} \subset H \setminus \overline{E},$$

where $\overline{E}$ denotes the closure of E. Then, since $N_{Q_\infty}(E) = 1$, one has that

$$\|x - a\|_H \geq r, \quad \text{for } N_{Q_\infty}\text{-a.e. } x \in H.$$

Let now $\alpha > 0$ be fixed. It follows that

$$\int_H e^{-\frac{\alpha}{2}\|x-a\|_H^2}\, N_{Q_\infty}(\mathrm{d}x) \leq e^{-\frac{\alpha}{2}r^2}\,.$$

The left hand side can be explicitly computed by using the factorisation of N_{Q_∞} and the fact that $\zeta_k > 0$ for every $k \in \mathbb{N}$. More precisely, setting $x_k := (x, q_k)_H$ and $a_k := (a, q_k)$ for every $k \in \mathbb{N}$ and $x \in H$, after classical computations (see [Itô70] or [DPZ02, Prop. 1.2.8]), we have

$$\int_H e^{-\frac{\alpha}{2}\|x-a\|_H^2}\, N_{Q_\infty}(\,dx) = \int_H \prod_{k \in \mathbb{N}} e^{-\frac{\alpha}{2}(x_k-a_k)^2}\, N_{Q_\infty}(\,dx)$$

$$= \prod_{k \in \mathbb{N}} \int_{\mathbb{R}} e^{-\frac{\alpha}{2}(x_k-a_k)^2}\, N_{\zeta_k}(\,dx_k)$$

$$= \prod_{k \in \mathbb{N}} \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\zeta_k}} e^{-\frac{\alpha}{2}(x_k-a_k)^2 - \frac{1}{2\zeta_k}x_k^2}\, dx_k$$

$$= \prod_{k \in \mathbb{N}} (1+\alpha\zeta_k)^{-\frac{1}{2}} \exp\left(-\frac{\alpha}{2}\frac{a_k^2}{1+\alpha\zeta_k}\right)\,.$$

Putting everything together, we deduce that

$$\exp\left[\frac{1}{2}\sum_{k \in \mathbb{N}}\left(\ln(1+\alpha\zeta_k) + \frac{\alpha a_k^2}{1+\alpha\zeta_k}\right)\right] \geq e^{\frac{\alpha}{2}r^2}\,, \quad \forall \alpha > 0\,. \tag{3.10}$$

Now, for every $N \in \mathbb{N}$, one has

$$\exp\left[\frac{1}{2}\sum_{k \geq N}\left(\ln(1+\alpha\zeta_k) + \frac{\alpha a_k^2}{1+\alpha\zeta_k}\right)\right] \leq \exp\left[\frac{\alpha}{2}\sum_{k \geq N}\left(\zeta_k + a_k^2\right)\right]\,.$$

Since $(\zeta_k)_k \in \ell^1$ and $(a_k)_k \in \ell^2$, we can choose and fix $\bar{N} \in \mathbb{N}$, independent of α, such that

$$\sum_{k \geq \bar{N}}\left(\zeta_k + a_k^2\right) \leq \frac{r^2}{2}\,,$$

so that by using (3.10) we get

$$\exp\left[\frac{1}{2}\sum_{k < \bar{N}}\left(\ln(1+\alpha\zeta_k) + \frac{\alpha a_k^2}{1+\alpha\zeta_k}\right)\right] \geq e^{\frac{\alpha}{4}r^2}\,, \quad \forall \alpha > 0\,.$$

It follows that

$$1 \leq \left[e^{-\frac{\alpha}{4}r^2} \prod_{k < \bar{N}}(1+\alpha\zeta_k)^{\frac{1}{2}}\right] \exp\left(\frac{\alpha}{2}\sum_{k < \bar{N}}\frac{a_k^2}{1+\alpha\zeta_k}\right)$$

$$\leq \left[e^{-\frac{\alpha}{4}r^2} \prod_{k < \bar{N}}(1+\alpha\zeta_k)^{\frac{1}{2}}\right] \exp\left(\frac{1}{2}\sum_{k < \bar{N}}\frac{a_k^2}{\zeta_k}\right)\,, \quad \forall \alpha > 0\,,$$

so that letting $\alpha \to +\infty$ yields a contradiction, and this concludes the proof. $\qquad\square$

We are now ready to state the definition of mild solution for the Kolmogorov equation (3.1). For convenience, we set $\tilde{B} : H \to D(A^{-\beta})$ as

$$\tilde{B}(x) := \begin{cases} B(x) & \text{if } x \in V_{2\alpha}\,, \\ 0 & \text{if } x \in H \setminus V_{2\alpha}\,. \end{cases}$$

Definition 3.4. *A mild solution to the Kolmogorov equation* (3.1) *is a function* $u \in C_b^1(H)$ *such that* $Du \in C_b^0(H; D(A^\beta))$ *and*

$$u(x) = \int_0^\infty e^{-\lambda t} R_t[f + \langle \tilde{B}, Du \rangle](x)\,\mathrm{d}t \quad \forall\, x \in H\,.$$

Proposition 3.5. *Assume (H1)-(H2)-(H3). Then, there exists $\lambda_0 > 0$, only depending on A, B, G, such that, for every $\lambda > \lambda_0$, there exists a unique mild solution to the Kolmogorov equation* (3.1) *in the sense of Definition 3.4.*

Proof. We use a fixed point argument. To this end, we introduce the space

$$\mathcal{U} := \big\{ v \in C_b^1(H) : \quad Dv \in C_b^0(H; V_{2\beta}) \big\}$$

with norm

$$\|v\|_{\mathcal{U}} := \|v\|_{C_b^1(H)} + \|Dv\|_{C_b^0(H;V_{2\beta})}\,, \quad v \in \mathcal{U}\,,$$

and define the operator $\mathcal{T}_\lambda : \mathcal{U} \to C_b^0(H)$ as follows

$$\mathcal{T}_\lambda v(x) := \int_0^\infty e^{-\lambda t} R_t[f + \langle \tilde{B}, Dv \rangle](x)\,\mathrm{d}t\,, \quad v \in \mathcal{U}\,.$$

Since $\beta \in [0, \frac{1}{2} - \delta)$, by Lemma 3.2 we have

$$e^{-\lambda t}\|DR_t[f + \langle \tilde{B}, Dv \rangle](x)\|_{2\beta} = e^{-\lambda t}\|A^\beta DR_t[f + \langle \tilde{B}, Dv \rangle](x)\|_H$$

$$\leq C_R \frac{e^{-\lambda t}}{t^{\frac{1}{2}+\delta+\beta}} \sup_{y \in H} |f(y) + \langle \tilde{B}(y), Dv(y) \rangle|$$

$$\leq C_R \left(\|f\|_{C_b^0(H)} + \sup_{y \in V_{2\alpha}} \|B(y)\|_{-2\beta} \|v\|_{\mathcal{U}} \right) \frac{e^{-\lambda t}}{t^{\frac{1}{2}+\delta+\beta}}\,.$$

Hence, by the dominated convergence theorem, we can differentiate under integral sign, and we obtain that $\mathcal{T}_\lambda$ is well-defined as an operator from $\mathcal{U}$ to $\mathcal{U}$ and

$$D(\mathcal{T}_\lambda v)(x) = \int_0^\infty e^{-\lambda t} DR_t[f + \langle \tilde{B}, Dv \rangle](x)\,\mathrm{d}t\,, \quad v \in \mathcal{U}\,.$$

Moreover, by analogous computations, for every $v_1, v_2 \in \mathcal{U}$, we have

$$|\mathcal{T}_\lambda v_1(x) - \mathcal{T}_\lambda v_2(x)| \leq \frac{1}{\lambda} \sup_{y \in V_{2\alpha}} \|B(y)\|_{-2\beta} \|v_1 - v_2\|_{\mathcal{U}}\,,$$

and

$$\|D\mathcal{T}_\lambda v_1(x) - D\mathcal{T}_\lambda v_2(x)\|_{2\beta} \leq C_R \sup_{y \in V_{2\alpha}} \|B(y)\|_{-2\beta} \|v_1 - v_2\|_{\mathcal{U}} \int_0^{+\infty} e^{-\lambda t} \frac{1}{t^{\frac{1}{2}+\delta+\beta}}\,\mathrm{d}t$$

$$= \frac{C_R}{\lambda^{\frac{1}{2}-\delta-\beta}} \Gamma\left(\frac{1}{2} - \delta - \beta\right) \sup_{y \in V_{2\alpha}} \|B(y)\|_{-2\beta} \|v_1 - v_2\|_{\mathcal{U}}\,.$$

Hence, provided that λ is chosen sufficiently large (only depending on B, δ, β, R) we infer that $\mathcal{T}_\lambda : \mathcal{U} \to \mathcal{U}$ is a contraction. Hence, the thesis follows by the classical Banach fixed point theorem. $\qquad \square$

From now on, we fix $\lambda_0 > 0$ as in Proposition 3.5.

In order to prove uniqueness in law, we need a further regularisation of the solution u to the Kolmogorov equation, which is at least of class $C_b^2(H)$ so that a suitable Itô formula can be applied. The idea is to consider an approximated version of the Kolmogorov equation for which one is able to prove existence of strong solutions. To this end, we introduce a linear self-adjoint negative definite operator C with $C^{-1} \in \mathscr{L}^1(H,H)$ and denote by $(T(t))_{t\geq 0}$ the associated linear semigroup on H. Then, for every $\varepsilon > 0$, we define the regularised operator

$$B_\varepsilon : H \to H\,, \qquad B_\varepsilon(x) := \int_H T(\varepsilon)\tilde{B}(T(\varepsilon)x + y)\, N_{\frac{1}{2}C^{-1}(T(\varepsilon)-I)}(\mathrm{d}y)\,, \quad x \in H\,,$$

and we also define

$$f_\varepsilon : H \to \mathbb{R}\,, \qquad f_\varepsilon(x) := \int_H f(T(\varepsilon)x + y)\, N_{\frac{1}{2}C^{-1}(T(\varepsilon)-I)}(\mathrm{d}y)\,, \quad x \in H\,,$$

where, as usual, $N_{\frac{1}{2}C^{-1}(T(\varepsilon)-I)}$ denotes the Gaussian measure with mean 0 and covariance operator $\frac{1}{2}C^{-1}(T(\varepsilon) - I)$. Moreover, since $C^{-1} \in \mathscr{L}^1(H,H)$, it follows (see [DPZ02, Sec. 6.2]) that $B_\varepsilon \in C_b^\infty(H;H)$ and $f_\varepsilon \in C_b^\infty(H)$ for every $\varepsilon > 0$, and

$$\lim_{\varepsilon \to 0^+} \|B_\varepsilon(x) - \tilde{B}(x)\|_{-2\beta} = 0 \quad \forall\, x \in H\,, \qquad \lim_{\varepsilon \to 0^+} \|f_\varepsilon - f\|_{C_b^0(H)} = 0\,. \qquad (3.11)$$

Now, for every $\varepsilon > 0$, we consider the regularised Kolmogorov equation

$$\lambda u_\varepsilon(x) + L u_\varepsilon(x) = f_\varepsilon(x) + (B_\varepsilon(x), Du_\varepsilon(x))\,, \quad x \in D(A)\,. \qquad (3.12)$$

Definition 3.6. *Let $\varepsilon > 0$. A mild solution to the regularised Kolmogorov equation* (3.12) *is a function $u_\varepsilon \in C_b^1(H)$ such that*

$$u_\varepsilon(x) = \int_0^\infty e^{-\lambda t} R_t[f_\varepsilon + (B_\varepsilon, Du_\varepsilon)](x)\,\mathrm{d}t\,, \quad \forall\, x \in H\,.$$

A strong solution to the regularised Kolmogorov equation (3.12) *is a function $u_\varepsilon \in C_b^2(H)$ such that $QD^2 u_\varepsilon(x) \in \mathscr{L}^1(H,H)$ for all $x \in H$ and*

$$\lambda u_\varepsilon(x) - \frac{1}{2}\operatorname{Tr}(QD^2 u_\varepsilon(x)) + (Ax, Du_\varepsilon(x)) = f_\varepsilon(x) + (B_\varepsilon(x), Du_\varepsilon(x))\,, \quad \forall\, x \in D(A)\,.$$

Proposition 3.7. *Assume (H1)-(H2)-(H3) and let $\varepsilon > 0$. Then, for every $\lambda > \lambda_0$ there exists a unique strong solution u_ε to the regularised Kolmogorov equation* (3.12)*, which is also a mild solution in the sense of Definition 3.6, and it satisfies*

$$Du_\varepsilon \in C_b^0(H; D(A^\beta))\,, \qquad QD^2 u_\varepsilon \in C_b^0(H; \mathscr{L}^1(H,H))\,. \qquad (3.13)$$

Proof. First of all, we note that the same proof of Proposition 3.5 provides also a sequence $(u_\varepsilon)_\varepsilon \subset C_b^1(H)$ of mild solutions to the regularised Kolmogorov equation in the sense of Definition 3.6, with $(Du_\varepsilon)_\varepsilon \subset C_b^0(H; D(A^\beta))$. In particular, one has that

$$u_\varepsilon(x) = \int_0^\infty e^{-\lambda t} R_t[f_\varepsilon + (B_\varepsilon, Du_\varepsilon)](x)\,\mathrm{d}t\,, \quad \forall\, x \in H\,.$$

and

$$Du_\varepsilon(x) = \int_0^\infty e^{-\lambda t} DR_t[f_\varepsilon + (B_\varepsilon, Du_\varepsilon)](x)\,\mathrm{d}t\,, \quad \forall\, x \in H\,.$$

Now, note that for every $\eta \in (0, (1-2\delta)/(1+2\delta))$, thanks to (3.9) one has, for all $t > 0$, that

$$e^{-\lambda t}\,\|DR_t[f_\varepsilon + (B_\varepsilon, Du_\varepsilon)]\|_{C_b^{0,\eta}(H)}$$

$$\leq \frac{C_{R,\eta}}{t^{(1+\eta)(\frac{1}{2}+\delta)}}e^{-\lambda t}\left(\|f\|_{C_b^0(H)} + \|B_\varepsilon\|_{C_b^0(H;H)}\,\|Du_\varepsilon\|_{C_b^0(H;H)}\right)\,,$$

where $(1+\eta)(\frac{1}{2}+\delta) \in (0,1)$. Hence, the dominated convergence theorem yields $(u_\varepsilon)_\varepsilon \subset C_b^{1,\eta}(H)$. In order to show that $(u_\varepsilon)_\varepsilon$ are actually strong solutions in the sense of Definition 3.6, we use hypothesis (H3) and apply the result in [DPZ02, Prop. 6.4.2]. To this end, we only need to check that

$$x \mapsto f_\varepsilon(x) + (B_\varepsilon(x), Du_\varepsilon(x))$$

is Hölder continuous and bounded in H. Clearly, this is true for f_ε. As far as the second term is concerned, since $(u_\varepsilon)_\varepsilon \subset C^{1,\eta}(H)$ for all $\eta \in (0, (1-2\delta)/(1+2\delta))$, recalling that $B_\varepsilon \in C_b^1(H;H)$, for every $x, y \in H$ we have

$$|(B_\varepsilon(x), Du_\varepsilon(x)) - (B_\varepsilon(y), Du_\varepsilon(y))|$$

$$\leq |(B_\varepsilon(x), Du_\varepsilon(x) - Du_\varepsilon(y))| + |(B_\varepsilon(x) - B_\varepsilon(y), Du_\varepsilon(y))|$$

$$\leq \|B_\varepsilon\|_{C_b^0(H;H)}\,\|u_\varepsilon\|_{C^{1,\eta}(H)}\,\|x-y\|^\eta + \|B_\varepsilon\|_{C^{0,1}(H;H)}\,\|u_\varepsilon\|_{C_b^1(H)}\,\|x-y\|$$

$$\leq C_\varepsilon\left(\|x-y\|^\eta + \|x-y\|\right)\,.$$

This shows indeed that $x \mapsto f_\varepsilon(x) + (B_\varepsilon(x), Du_\varepsilon(x))$ is Hölder continuous for every $\varepsilon > 0$. Consequently, by [DPZ02, Prop. 6.4.2] we deduce that, for every $\varepsilon > 0$, it holds also that

$$u_\varepsilon \in C_b^{\eta + \frac{2}{1+2\delta}}(H)\,, \quad \forall\, \eta \in (0, (1-2\delta)/(1+2\delta))\,.$$

Since $\frac{1-2\delta}{1+2\delta} + \frac{2}{1+2\delta} = 1 + 2\frac{1-2\delta}{1+2\delta}$, this implies that

$$u_\varepsilon \in C_b^{1,\sigma}(H)\,, \quad \forall\, \sigma \in (0, 2(1-2\delta)/(1+2\delta))\,.$$

By a standard bootstrap argument, analogous computations show that

$$x \mapsto f_\varepsilon(x) + (B_\varepsilon(x), Du_\varepsilon(x)) \in C^{0,\sigma}(H)\,, \quad \forall\, \sigma \in (0, 2(1-2\delta)/(1+2\delta))\,,$$

and applying again [DPZ02, Prop. 6.4.2] we can refine the regularity of u_ε as

$$u_\varepsilon \in C_b^{\sigma + \frac{2}{1+2\delta}}(H)\,, \quad \forall\, \sigma \in (0, 2(1-2\delta)/(1+2\delta))\,.$$

By noticing that $2\frac{1-2\delta}{1+2\delta} + \frac{2}{1+2\delta} = \frac{4-4\delta}{1+2\delta} = 1 + 3\frac{1-2\delta}{1+2\delta}$, an iteration argument shows that

$$u_\varepsilon \in C_b^{\ell_k}(H)\,, \quad \forall\, k \in \mathbb{N}\,,$$

where

$$\ell_k = 1 + \min\left\{k\frac{1-2\delta}{1+2\delta}, \frac{2}{1+2\delta}\right\}\,, \quad k \in \mathbb{N}\,.$$

Hence, for $k = k(\delta)$ sufficiently large one obtains the optimal regularity

$$u_\varepsilon \in C^{1 + \frac{2}{1+2\delta}}(H)\,.$$

Noting that $1 + \frac{2}{1+2\delta} > 2$ for all $\delta \in (0, \frac{1}{2})$, this implies in particular that $u_\varepsilon \in C_b^2(H)$. Moreover, hypothesis (2.4) and the same result [DPZ02, Prop. 6.4.2] ensure also that $QD^2u_\varepsilon(x) \in \mathscr{L}^1(H,H)$ for all $x \in H$, so that u_ε is a strong solution to the regularised Kolmogorov equation in the sense of Definition 3.6.

Eventually, we show that, for every $\varepsilon > 0$, it holds that $QD^2 u_\varepsilon \in C_b^0(H; \mathscr{L}^1(H, H))$. To this end, for every $\varphi \in \mathscr{B}_b(H)$ and $t > 0$, we introduce the linear bounded operator $\Psi_\varphi(t) \in \mathscr{L}(H, H)$ as

$$(\Psi_\varphi(t)h, k) := \int_H \left[(h, Q_t^{-\frac{1}{2}} y)(k, Q_t^{-\frac{1}{2}} y) - (h, k) \right] \varphi(y) \, N_{Q_t}(\mathrm{d}y), \quad h, k \in H.$$

By [DPZ02, Lem. 6.2.7] it holds that

$$\Psi_\varphi(t) \in \mathscr{L}^2(H, H), \qquad \|\Psi_\varphi(t)\|_{\mathscr{L}^2(H,H)} \leq 2 \sup_{x \in H} |\varphi(x)| \quad \forall \varphi \in \mathscr{B}_b(H), \quad \forall t > 0.$$

By linearity, this shows in particular that for every $t > 0$, $\varphi \mapsto \Psi_\varphi(t) \in \mathscr{L}(C_b^0(H); \mathscr{L}^2(H, H))$. Now, let us set for convenience $\varphi_\varepsilon := f_\varepsilon + (B_\varepsilon, Du_\varepsilon) \in C_b^0(H)$. By Proposition [DPZ02, Prop. 6.2.8] one has that, for every $t > 0$,

$$GD^2(R_t \varphi_\varepsilon)(x)G = Q_t^{-\frac{1}{2}} S(t) G \Psi_{\varphi_\varepsilon(S(t)x+\cdot)}(t) Q_t^{-\frac{1}{2}} S(t) G \quad \forall x \in H.$$

We note that for every $(x_n)_n, x \subset H$ such that $x_n \to x$ in H, since $\varphi_\varepsilon \in C_b^1(H)$ it holds that $\varphi_\varepsilon(S(t)x_n+\cdot) \to \varphi_\varepsilon(S(t)x+\cdot)$ in $C_b^0(H)$. Hence, recalling that $\varphi \mapsto \Psi_\varphi(t) \in \mathscr{L}(C_b^0(H); \mathscr{L}^2(H, H))$ one has that

$$x \mapsto \Psi_{\varphi_\varepsilon(S(t)x+\cdot)}(t) \in C^0(H; \mathscr{L}^2(H, H)),$$

which in turn implies, thanks to the fact that that $Q_t^{-\frac{1}{2}} S(t) G \in \mathscr{L}^4(H, H)$ for almost every $t > 0$ by (H3), that

$$x \mapsto Q_t^{-\frac{1}{2}} S(t) G \Psi_{\varphi_\varepsilon(S(t)x+\cdot)}(t) Q_t^{-\frac{1}{2}} S(t) G \in C^0(H; \mathscr{L}^1(H, H)) \quad \text{for a.e. } t > 0.$$

This implies that

$$x \mapsto GD^2(R_t \varphi_\varepsilon)(x)G \in C^0(H; \mathscr{L}^1(H, H)) \quad \text{for a.e. } t > 0.$$

Moreover, we note that

$$\left\| GD^2(R_t \varphi_\varepsilon)(x)G \right\|_{\mathscr{L}^1(H,H)} \leq 2 \left\| Q_t^{-\frac{1}{2}} S(t) G \right\|_{\mathscr{L}^4(H,H)}^2 \|\varphi_\varepsilon\|_{C_b^0(H)} \quad \text{for a.e. } t > 0.$$

Moreover, by [DPZ02, Prop. 6.2.9] and the fact that $\varphi_\varepsilon \in UC_b^1(H)$ one has also that

$$\left\| GD^2(R_t \varphi_\varepsilon)(x)G \right\|_{\mathscr{L}^1(H,H)} \leq \left\| Q_t^{-\frac{1}{2}} S(2t) Q \right\|_{\mathscr{L}^2(H,H)} \|\varphi_\varepsilon\|_{C_b^1(H)} \quad \text{for a.e. } t > 0.$$

By interpolation (see [DPZ02, Prop. 6.4.2]) we infer that

$$\left\| GD^2(R_t \varphi_\varepsilon)(x)G \right\|_{\mathscr{L}^1(H,H)} \leq 2^{1-\vartheta} \left\| Q_t^{-\frac{1}{2}} S(t) G \right\|_{\mathscr{L}^4(H,H)}^{2(1-\vartheta)} \left\| Q_t^{-\frac{1}{2}} S(2t) Q \right\|_{\mathscr{L}^2(H,H)}^{\vartheta} \|\varphi_\varepsilon\|_{C_b^{0,\vartheta}(H)}$$

for almost every $t > 0$. Thanks to assumption (H3), the dominated convergence theorem, and the fact that u_ε is a mild solution to the regularised Kolmogorov equation, this implies that

$$x \mapsto QD^2 u_\varepsilon(x) \in C_b^0(H; \mathscr{L}^1(H, H)),$$

and this concludes the proof. $\qquad \square$

Proposition 3.8. *Assume (H1)-(H2)-(H3). Let $\lambda > \lambda_0$, and let $(u_\varepsilon)_\varepsilon \subset C_b^2(H)$ be the strong solutions to the Kolmogorov equation (3.12). Then, there exists a constant $C > 0$, only depending on $\lambda, \beta, A, B, G, f$, such that*

$$\|u_\varepsilon\|_{C_b^1(H)} + \|Du_\varepsilon\|_{C_b^0(H;D(A^\beta))} \leq C. \tag{3.14}$$

Moreover, for every $\eta \in (0, (1 - 2\delta)/(1 + 2\delta))$, there exists a constant $C_\eta > 0$, only depending on $\lambda, \beta, A, B, G, f$, such that

$$\|u_\varepsilon\|_{C_b^{1,\eta}(H)} \le C_\eta\,. \tag{3.15}$$

Proof. We first note that

$$u_\varepsilon(x) = \int_0^\infty e^{-\lambda t} R_t[f_\varepsilon + (B_\varepsilon, Du_\varepsilon)](x)\,dt\,, \quad \forall\, x \in H\,.$$

Moreover, arguing as in the proof of Proposition 3.5, one has that

$$Du_\varepsilon(x) = \int_0^\infty e^{-\lambda t} DR_t[f_\varepsilon + (B_\varepsilon, Du_\varepsilon)](x)\,dt\,, \quad \forall\, x \in H\,. \tag{3.16}$$

Now, by (3.6), (3.8), the properties of convolution, the fact that T is a contraction on $V_{-2\beta}$, and the dominated convergence theorem, it follows that

$$\begin{aligned}
\left\|A^\beta Du_\varepsilon\right\|_{C_b^0(H;H)} &\le \int_0^\infty e^{-\lambda t} \frac{C_R}{t^{\frac{1}{2}+\delta+\beta}} \sup_{x \in H}\left(|f_\varepsilon(x)| + \|B_\varepsilon(x)\|_{-2\beta}\|Du_\varepsilon(x)\|_{2\beta}\right)dt \\
&\le C_R\left(\|f\|_{C_b^0(H)} + \sup_{y \in V_{2\alpha}}\|B(y)\|_{-2\beta}\|Du_\varepsilon\|_{C_b^0(H;V_{2\beta})}\right)\int_0^\infty e^{-\lambda t}\frac{1}{t^{\frac{1}{2}+\delta+\beta}}\,dt \\
&\le \frac{C_R}{\lambda^{\frac{1}{2}-\delta-\beta}}\Gamma\left(\frac{1}{2} - \delta - \beta\right)\left(\|f\|_{C_b^0(H)} + \sup_{y \in V_{2\alpha}}\|B(y)\|_{-2\beta}\|Du_\varepsilon\|_{C_b^0(H;V_{2\beta})}\right).
\end{aligned}$$

Now, noting that the choice $\lambda > \lambda_0$ yields

$$\frac{C_R}{\lambda^{\frac{1}{2}-\delta-\beta}}\Gamma\left(\frac{1}{2} - \delta - \beta\right)\sup_{y \in V_{2\alpha}}\|B(y)\|_{-2\beta} < 1\,,$$

the first estimate (3.14) follows. Eventually, for every $\eta \in (0, (1 - 2\delta)/(1 + 2\delta))$, by virtue of (3.9) one has, for all $t > 0$,

$$\begin{aligned}
&e^{-\lambda t}\left\|DR_t[f_\varepsilon + (B_\varepsilon, Du_\varepsilon)]\right\|_{C_b^{0,\eta}(H)} \\
&\qquad \le \frac{C_{R,\eta}}{t^{(1+\eta)(\frac{1}{2}+\delta)}}e^{-\lambda t}\left(\|f\|_{C_b^0(H)} + \sup_{y \in V_{2\alpha}}\|B(y)\|_{-2\beta}\|Du_\varepsilon\|_{C_b^0(H;V_{2\beta})}\right),
\end{aligned}$$

where $(1 + \eta)(\frac{1}{2} + \delta) \in (0, 1)$. Hence, the dominated convergence theorem and the estimate (3.14) yield that $u_\varepsilon \in C_b^{1,\eta}(H)$ and

$$\|u_\varepsilon\|_{C_b^{1,\eta}(H)} \le C_{R,\eta}\left(\|f\|_{C_b^0(H)} + C\sup_{y \in V_{2\alpha}}\|B(y)\|_{-2\beta}\right)\int_0^\infty \frac{1}{t^{(1+\eta)(\frac{1}{2}+\delta)}}e^{-\lambda t}\,dt\,,$$

which proves the estimate (3.15). □

Proposition 3.9. *Assume (H1)-(H2)-(H3). Let $\lambda > \lambda_0$, let $(u_\varepsilon)_\varepsilon \subset C_b^2(H)$ be the strong solutions to the Kolmogorov equation (3.12), and let $u \in C_b^1(H)$ be the mild solution to the Kolmogorov equation (3.1). Then, for all $\eta \in (0, (1 - 2\delta)/(1 + 2\delta))$ and $\gamma \in [0, \beta)$, it holds that $u \in C_b^{1,\eta}(H)$ and*

$$\lim_{\varepsilon \to 0^+}|u_\varepsilon(x) - u(x)| = 0\,, \quad \forall\, x \in H\,, \tag{3.17}$$

$$\lim_{\varepsilon \to 0^+}\|Du_\varepsilon(x) - Du(x)\|_{2\gamma} = 0\,, \quad \forall\, x \in H\,. \tag{3.18}$$

Proof. The main idea of the proof is to first extract a candidate limit $\tilde{u}$ for $(u_\varepsilon)_\varepsilon$ by using compactness methods. Secondly, exploiting the continuity of the semigroup R in $L^2(H, N_{Q_\infty})$ we show that $\tilde{u}$ coincides with u, N_{Q_∞}-almost surely. Finally, by the density result given in Lemma 3.3, we conclude that actually $u = \tilde{u}$ on H.

For every $\ell \in \mathbb{N}_+$ let $(\mathcal{B}_\ell)_{\ell>0}$ be a family of compact subsets of H such that $\mathcal{B}_{\ell_1} \subset \mathcal{B}_{\ell_2}$ for all $\ell_1 < \ell_2$ and with $\cup_{\ell>0}\mathcal{B}_\ell$ dense in H: for example, we can take

$$\mathcal{B}_\ell := \{x \in V_{2\alpha} : \|x\|_{2\alpha} \leq \ell\}$$

and note that $\mathcal{B}_\ell$ is a compact subset of H. Now, let us fix $\ell \in \mathbb{N}_+$ and set, for all $\varepsilon > 0$, $u_\varepsilon^\ell := (u_\varepsilon)_{|\mathcal{B}_\ell} : \mathcal{B}_\ell \to \mathbb{R}$. By the estimate (3.14) the sequence $(u_\varepsilon^\ell)_\varepsilon$ is uniformly equicontinuous and bounded. Hence, the Arzelà–Ascoli theorem yields the existence of a subsequence $(u_{\varepsilon_n}^\ell)_n$ and $u^\ell \in C_b^0(\mathcal{B}_\ell)$ such that

$$\lim_{n\to\infty} \sup_{x\in\mathcal{B}_\ell} |u_{\varepsilon_n}^\ell(x) - u^\ell(x)| = 0\,.$$

It is straightforward to check that the family $(u^\ell)_\ell$ is consistent on the family of balls $(\mathcal{B}_\ell)_\ell$ in the sense that $u^{\ell+1} = u^\ell$ on $\mathcal{B}_\ell$ for all $\ell \in \mathbb{N}_+$. Consequently, this procedure identifies an element $\tilde{u} : V_{2\alpha} \to \mathbb{R}$ such that, by possibly employing a diagonal argument,

$$\lim_{n\to\infty} \sup_{x\in\mathcal{B}_\ell} |u_{\varepsilon_n}(x) - \tilde{u}(x)| = 0 \quad \forall \ell \in \mathbb{N}_+\,, \tag{3.19}$$

hence, in particular,

$$\lim_{n\to\infty} |u_{\varepsilon_n}(x) - \tilde{u}(x)| = 0 \quad \forall x \in V_{2\alpha}\,.$$

We show now that actually $\tilde{u}$ can be uniquely extended to a Lipschitz function on the whole H. To this end, by (3.14), for every $x_1, x_2 \in V_{2\alpha}$ and $n \in \mathbb{N}$ we have

$$|\tilde{u}(x_1) - \tilde{u}(x_2)| \leq |\tilde{u}(x_1) - u_{\varepsilon_n}(x_1)| + |u_{\varepsilon_n}(x_1) - u_{\varepsilon_n}(x_2)| + |u_{\varepsilon_n}(x_2) - \tilde{u}(x_2)|$$
$$\leq C \|x_1 - x_2\| + |\tilde{u}(x_1) - u_{\varepsilon_n}(x_1)| + |u_{\varepsilon_n}(x_2) - \tilde{u}(x_2)|\,,$$

so that letting $n \to \infty$ yields

$$|\tilde{u}(x_1) - \tilde{u}(x_2)| \leq C \|x_1 - x_2\|\,, \quad \forall x_1, x_2 \in V_{2\alpha}\,.$$

Since $V_{2\alpha}$ is dense in H, this shows that actually $\tilde{u}$ uniquely extends to an element $\tilde{u} \in C^{0,1}(H)$. Moreover, by a similar argument one can show the pointwise convergence of $(u_{\varepsilon_n})_n$ to $\tilde{u}$ on the whole H. Indeed, for every $x \in H$, there is $(x_j)_j \subset V_{2\alpha}$ such that $x_j \to x$ in H as $j \to \infty$. Hence, by Lipschitz-continuity and (3.14), we get

$$|\tilde{u}(x) - u_{\varepsilon_n}(x)| \leq |\tilde{u}(x) - \tilde{u}(x_j)| + |\tilde{u}(x_j) - u_{\varepsilon_n}(x_j)| + |u_{\varepsilon_n}(x_j) - u_{\varepsilon_n}(x)|$$
$$\leq 2C \|x_j - x\| + |\tilde{u}(x_j) - u_{\varepsilon_n}(x_j)|\,,$$

which implies that

$$\lim_{n\to\infty} |u_{\varepsilon_n}(x) - \tilde{u}(x)| = 0\,, \quad \forall x \in H\,.$$

Let us consider now the sequence of derivatives $(Du_\varepsilon)_\varepsilon$. For all $\ell \in \mathbb{N}_+$ we set $\mathfrak{u}_\varepsilon^\ell := (Du_\varepsilon)_{|\mathcal{B}_\ell} : \mathcal{B}_\ell \to H$. Proceeding as before, given a fixed $\ell \in \mathbb{N}_+$, one has that $(\mathfrak{u}_\varepsilon^\ell)_\varepsilon$ is uniformly equicontinuous thanks to (3.15), and for all $x \in \mathcal{B}_\ell$ the sequence $(\mathfrak{u}_\varepsilon^\ell(x))_\varepsilon$ is relatively compact in $V_{2\gamma}$ thanks to (3.15). Hence, using again the Arzelà–Ascoli theorem and proceeding as above possibly employing a diagonal argument and suitably renominating the subsequence, this identifies an element $\mathfrak{u} : V_{2\alpha} \to H$ such that

$$\lim_{n\to\infty} \sup_{x\in\mathcal{B}_\ell} \|Du_{\varepsilon_n}(x) - \mathfrak{u}(x)\|_{2\gamma} = 0\,, \quad \forall \ell \in \mathbb{N}_+\,, \tag{3.20}$$

hence also

$$\lim_{n\to\infty} \|Du_{\varepsilon_n}(x) - \mathfrak{u}(x)\|_{2\gamma} = 0\,, \quad \forall\, x \in V_{2\alpha}\,.$$

Arguing on the line of the previous computations for $\tilde{u}$, we can show that $\mathfrak{u}$ uniquely extends to a function $\mathfrak{u} \in C^{0,\eta}(H;H)$, where η is arbitrary and fixed in $(0, (1-2\delta)/(1+2\delta))$. Indeed, for all $x_1, x_2 \in V_{2\alpha}$ and $n \in \mathbb{N}$, we have

$$\|\mathfrak{u}(x_1) - \mathfrak{u}(x_2)\| \leq \|\mathfrak{u}(x_1) - Du_{\varepsilon_n}(x_1)\| + \|Du_{\varepsilon_n}(x_1) - Du_{\varepsilon_n}(x_2)\| + \|Du_{\varepsilon_n}(x_2) - \mathfrak{u}(x_2)\|$$
$$\leq C\,\|x_1 - x_2\|^{\eta} + \|\mathfrak{u}(x_1) - Du_{\varepsilon_n}(x_1)\| + \|Du_{\varepsilon_n}(x_2) - \mathfrak{u}(x_2)\|\,,$$

so that letting $n \to \infty$ yields

$$\|\mathfrak{u}(x_1) - \mathfrak{u}(x_2)\| \leq C\,\|x_1 - x_2\|^{\eta} \quad \forall\, x_1, x_2 \in V_{2\alpha}\,,$$

and the desired extension follows by density of $V_{2\alpha}$ in H. Moreover, arguing as for $\tilde{u}$, it is immediate to show the pointwise convergence

$$\lim_{n\to\infty} \|Du_{\varepsilon_n}(x) - \mathfrak{u}(x)\|_{2\gamma} = 0\,, \quad \forall\, x \in H\,.$$

To conclude, it remains to prove that $\tilde{u} \in C_b^{1,\eta}(H)$ and $D\tilde{u} = \mathfrak{u}$. To this end, from (3.19) and (3.20) it follows that, for every $\ell \in \mathbb{N}_+$, the sequence $((u_{\varepsilon_n})_{|\mathcal{B}_\ell})_n$ is Cauchy in the space

$$\left\{ v \in C_b^1(\mathcal{B}_\ell) : Dv \in C_b^0(\mathcal{B}_\ell; H) \right\}.$$

Consequently, one has that $\tilde{u} \in C_b^1(\mathcal{B}_\ell)$ for every $\ell \in \mathbb{N}_+$ and $D\tilde{u}(x) = \mathfrak{u}(x)$ in H for every $x \in V_{2\alpha}$. Let now $x, h \in H$ and $(x_k)_k, (h_k)_k \subset V_{2\alpha}$ be such that $x_k \to x$ and $h_k \to h$ in H as $k \to \infty$. Then, we have

$$\tilde{u}(x_k + h_k) - \tilde{u}(x_k) = \int_0^1 (\mathfrak{u}(x_k + \sigma h_k), h_k)_H \, d\sigma\,, \quad \forall\, k \in \mathbb{N}\,,$$

so that letting $k \to \infty$ yields, by the dominated convergence theorem,

$$\tilde{u}(x + h) - \tilde{u}(x) = \int_0^1 (\mathfrak{u}(x + \sigma h), h)_H \, d\sigma\,, \qquad \forall\, x, h \in H\,,$$

showing that $\tilde{u} : H \to \mathbb{R}$ is Gâteaux differentiable with $D\tilde{u} = \mathfrak{u} \in C^{0,\eta}(H;H)$. This implies then that $\tilde{u} \in C^{1,\eta}(H)$, as required. The boundedness of $\tilde{u}$ and $D\tilde{u}$ follows from (3.14)–(3.15) and the pointwise convergences obtained above.

Finally, note that

$$u_{\varepsilon_n}(x) = \int_0^\infty e^{-\lambda t} R_t[f_{\varepsilon_n} + (B_{\varepsilon_n}, Du_{\varepsilon_n})](x)\, dt\,, \quad \forall\, x \in H\,,$$

and that $f_{\varepsilon_n} + (B_{\varepsilon_n}, Du_{\varepsilon_n}) \to f + \langle \tilde{B}, Du \rangle$ in $L^2(H, N_{Q_\infty})$ by the dominated convergence theorem. Hence, recalling that R is a strongly continuous semigroup of contractions on $L^2(H, N_{Q_\infty})$, by letting $n \to \infty$ one obtains

$$\tilde{u} = \int_0^\infty e^{-\lambda t} R_t[f + \langle \tilde{B}, D\tilde{u} \rangle]\, dt \quad \text{in } L^2(H, N_{Q_\infty})\,.$$

Now, by uniqueness of the mild solution u of the Kolmogorov equation (3.1) in $L^2(H, N_{Q_\infty})$, this implies that $\tilde{u} = u$ N_{Q_∞}-almost surely. By Lemma 3.3 and the fact that $\tilde{u} \in C^{0,1}(H)$, this implies that $\tilde{u}(x) = u(x)$ for all $x \in H$, hence in particular that $\tilde{u}$ is bounded and the convergences hold along the entire sequence ε. $\square$

4. Proofs of main theorems

4.1. Proof of Theorem 2.6. Let $x \in H$, and let

$$\left(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t \geq 0}, \mathbb{P}, W, X\right), \qquad \left(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t \geq 0}, \mathbb{P}, W, Y\right)$$

be two global weak solutions to (1.1) in the sense of Definition 2.2 with respect to the same initial datum x. Moreover, let $\lambda > \lambda_0$ (as specified in Proposition 3.5), let $(u_\varepsilon)_\varepsilon \subset C_b^2(H)$ be the strong solutions to the Kolmogorov equation (3.12), and let $u \in C_b^1(H)$ be the mild solution to the Kolmogorov equation (3.1).

Note that $(e_k)_{k \in \mathbb{N}}$ is a complete orthonormal system of H made of eigenvectors of A, and $(\lambda_k)_{k \in \mathbb{N}}$ are the corresponding eigenvalues. By setting $H_n := \mathrm{span}\{e_1, \ldots, e_n\}$ and $P_n : H \to H_n$ as the orthogonal projection on H_n, for all $n \in \mathbb{N}$, we note that P_n can be extended to a linear continuous operator $P_n : D(A^{-\beta}) \to H_n$ as follows

$$P_n v := \sum_{k=1}^{n} \langle v, e_k \rangle e_k, \quad v \in D(A^{-\beta}).$$

Now, for every $j \in \mathbb{N}$, we set $X_j := P_j X$, $x_j := P_j x$, $B_j := P_j B$, $G_j := P_j G$, and $Q_j := G_j G_j^*$. Then, since $H_j \subset D(A)$ for every $j \in \mathbb{N}$, one has that X_j are analytically strong solutions to the equation

$$X_j(t) + \int_0^t A X_j(s) \, ds = x_j + \int_0^t B_j(X(s)) \, ds + \int_0^t G_j(s) \, dW(s) \quad \text{in } H, \quad \forall t \geq 0, \quad \mathbb{P}\text{-a.s.}$$

By Itô's formula and by adding and subtracting suitable terms one has that

$$\mathbb{E}\left[u_\varepsilon(X_j(t))\right] - \frac{1}{2} \mathbb{E} \int_0^t \mathrm{Tr}\left[Q D^2 u_\varepsilon(X_j(s))\right] \, ds + \mathbb{E} \int_0^t (A X_j(s), D u_\varepsilon(X_j(s))) \, ds$$

$$= u_\varepsilon(x_j) + \mathbb{E} \int_0^t (B_j(X(s)), D u_\varepsilon(X_j(s))) \, ds + \frac{1}{2} \mathbb{E} \int_0^t \mathrm{Tr}\left[(Q_j - Q) D^2 u_\varepsilon(X_j(s))\right] \, ds,$$

so that by the Kolmogorov equation (3.12) we get

$$\mathbb{E}\left[u_\varepsilon(X_j(t))\right] - u_\varepsilon(x_j) - \lambda \mathbb{E} \int_0^t u_\varepsilon(X_j(s)) \, ds + \mathbb{E} \int_0^t f_\varepsilon(X_j(s)) \, ds \tag{4.1}$$

$$= \underbrace{\mathbb{E} \int_0^t (B_j(X(s)) - B_\varepsilon(X_j(s)), D u_\varepsilon(X_j(s))) \, ds}_{I_1(\varepsilon, j)} + \underbrace{\frac{1}{2} \mathbb{E} \int_0^t \mathrm{Tr}\left[(Q_j - Q) D^2 u_\varepsilon(X_j(s))\right] \, ds}_{I_2(\varepsilon, j)}.$$

We now show that, by letting $\varepsilon \to 0^+$ and $j \to \infty$ under a suitable scaling, one obtains that $I_1(\varepsilon, j) \to 0$ and $I_2(\varepsilon, j) \to 0$. To this end, by (3.14) we have

$$|I_1(\varepsilon, j)|$$

$$\leq \mathbb{E} \int_0^t |\langle B_j(X(s)) - B(X(s)), D u_\varepsilon(X_j(s)) \rangle| \, ds + \mathbb{E} \int_0^t |\langle B(X(s)) - B_\varepsilon(X(s)), D u_\varepsilon(X_j(s)) \rangle| \, ds$$

$$+ \mathbb{E} \int_0^t |\langle B_\varepsilon(X(s)) - B_\varepsilon(X_j(s)), D u_\varepsilon(X_j(s)) \rangle| \, ds$$

$$\leq C \left(\|B_j(X) - B(X)\|_{L^2(\Omega; L^2(0,t; V_{-2\beta}))} + \|B_\varepsilon(X) - B(X)\|_{L^2(\Omega; L^2(0,t; V_{-2\beta}))} \right.$$

$$\left. + \|B_\varepsilon(X) - B_\varepsilon(X_j)\|_{L^2(\Omega; L^2(0,t; V_{-2\beta}))} \right).$$

As far as I_2 is concerned, note that since Q and A commute,

$$I_2(\varepsilon, j) = \frac{1}{2} \mathbb{E} \int_0^t \mathrm{Tr}\left[(Q_j - Q)D^2 u_\varepsilon(X_j(s))\right] \, \mathrm{d}s = \frac{1}{2} \mathbb{E} \int_0^t \mathrm{Tr}\left[(P_j - I)QD^2 u_\varepsilon(X_j(s))\right] \, \mathrm{d}s$$

$$= \frac{1}{2} \mathbb{E} \int_0^t \mathrm{Tr}\left[(P_j - I)GD^2 u_\varepsilon(X_j(s))G\right] \, \mathrm{d}s$$

Hence, since $\|P_j - I\|_{\mathscr{L}(H,H)} \leq 1$ one has

$$\mathbb{E} \int_0^t \mathrm{Tr}\left[(P_j - I)GD^2 u_\varepsilon(X_j(s))G\right] \, \mathrm{d}s$$

$$= \mathbb{E} \int_0^t \mathrm{Tr}\left[(P_j - I)GD^2 u_\varepsilon(X(s))G\right] \, \mathrm{d}s$$

$$+ \mathbb{E} \int_0^t \mathrm{Tr}\left[(P_j - I)\left(GD^2 u_\varepsilon(X_j(s))G - GD^2 u_\varepsilon(X(s))G\right)\right] \, \mathrm{d}s$$

$$\leq \mathbb{E} \int_0^t \mathrm{Tr}\left[(P_j - I)GD^2 u_\varepsilon(X(s))G\right] \, \mathrm{d}s$$

$$+ \mathbb{E} \int_0^t \left\|GD^2 u_\varepsilon(X_j(s))G - GD^2 u_\varepsilon(X(s))G\right\|_{\mathscr{L}^1(H,H)} \, \mathrm{d}s$$

Since $P_j \to I$ in the strong operator topology of $\mathscr{L}(H,H)$, i.e., $P_j x \to x$ for every $x \in H$ as $j \to \infty$, and since $QD^2 u_\varepsilon(x) \in \mathscr{L}^1(H,H)$ for all $x \in H$, we infer that

$$\mathrm{Tr}\left[(P_j - I)GD^2 u_\varepsilon(x)G\right] \to 0 \quad \text{in } \mathscr{L}^1(H,H) \quad \forall x \in H.$$

This implies by the dominated convergence theorem that

$$\lim_{j \to \infty} \mathbb{E} \int_0^t \mathrm{Tr}\left[(P_j - I)GD^2 u_\varepsilon(X(s))G\right] \, \mathrm{d}s = 0 \quad \forall \varepsilon > 0.$$

Moreover, by exploiting (3.13) and again the dominated convergence theorem we get

$$\lim_{j \to \infty} \mathbb{E} \int_0^t \left\|GD^2 u_\varepsilon(X_j(s))G - GD^2 u_\varepsilon(X(s))G\right\|_{\mathscr{L}^1(H,H)} \, \mathrm{d}s = 0 \quad \forall \varepsilon > 0.$$

Putting everything together we infer that

$$\lim_{j \to \infty} I_2(\varepsilon, j) = 0 \quad \forall \varepsilon > 0.$$

Note that we are only using that $u_\varepsilon \in C_b^2(H)$ for every $\varepsilon > 0$, not that $(u_\varepsilon)_\varepsilon$ is bounded in $C_b^2(H)$, which is not the case.

Now, by (3.11), the dominated convergence theorem, and the fact that $\tilde{B}(X) = B(X)$ in $V_{-2\beta}$ almost everywhere, for every $k \in \mathbb{N}$, we can find $\varepsilon_k > 0$ such that

$$\|B_{\varepsilon_k}(X) - B(X)\|_{L^2(\Omega; L^2(0,t; V_{-2\beta}))} \leq \frac{1}{k}.$$

Given such ε_k and noting that, by definition of P_j, the continuity of B_{ε_k}, and the computations above

$$\lim_{j \to \infty} \|B_j(X) - B(X)\|_{L^2(\Omega; L^2(0,t; V_{-2\beta}))} = 0,$$

$$\lim_{j \to \infty} \|B_{\varepsilon_k}(X) - B_{\varepsilon_k}(X_j)\|_{L^2(\Omega; L^2(0,t; V_{-2\beta}))} = 0,$$

$$\lim_{j \to \infty} I_2(\varepsilon_k, j) = 0,$$

we can find $j_k \in \mathbb{N}$ such that

$$\|B_{j_k}(X) - B(X)\|_{L^2(\Omega;L^2(0,t;V_{-2\beta}))} + \|B_{\varepsilon_k}(X) - B_{\varepsilon_k}(X_{j_k})\|_{L^2(\Omega;L^2(0,t;V_{-2\beta}))}$$

$$+ |I_2(\varepsilon_k, j_k)| \leq \frac{1}{k}.$$

Hence, one has that

$$|I_1(\varepsilon_k, j_k)| + |I_2(\varepsilon_k, j_k)| \leq \frac{C}{k}.$$

Then, the identity (4.1) evaluated at ε_k and j_k yields

$$\mathbb{E}\left[u_{\varepsilon_k}(X_{j_k}(t))\right] - u_{\varepsilon_k}(x_{j_k}) - \lambda \mathbb{E}\int_0^t u_{\varepsilon_k}(X_{j_k}(s))\,\mathrm{d}s + \mathbb{E}\int_0^t f_{\varepsilon_k}(X_{j_k}(s))\,\mathrm{d}s$$

$$= I_1(\varepsilon_k, j_k) + I_2(\varepsilon_k, j_k).$$

Now, we let $k \to \infty$. To this end, note that by the estimate (3.14), for every $r \geq 0$ and $\mathbb{P}$-a.s., we have

$$|u_{\varepsilon_k}(X_{j_k}(r)) - u(X(r))| \leq |u_{\varepsilon_k}(X_{j_k}(r)) - u_{\varepsilon_k}(X(r))| + |u_{\varepsilon_k}(X(r)) - u(X(r))|$$

$$\leq \|u_{\varepsilon_k}\|_{C_b^1(H)}\|X_{j_k}(r) - X(r)\| + |u_{\varepsilon_k}(X(r)) - u(X(r))|$$

$$\leq C\|X_{j_k}(r) - X(r)\| + |u_{\varepsilon_k}(X(r)) - u(X(r))|.$$

Consequently, by (3.14), (3.17), the properties of P_j, and the dominated convergence theorem, one has, as $k \to \infty$, that

$$\mathbb{E}\left[u_{\varepsilon_k}(X_{j_k}(t))\right] \to \mathbb{E}\left[u(X(t))\right],$$

$$\mathbb{E}\int_0^t u_{\varepsilon_k}(X_{j_k}(s))\,\mathrm{d}s \to \mathbb{E}\int_0^t u(X(s))\,\mathrm{d}s.$$

Analogously, since $x_{j_k} \to x$ in H, by the same argument it holds also, as $k \to \infty$, that

$$u_{\varepsilon_k}(x_{j_k}) \to u(x).$$

Eventually, for every $r \geq 0$ and $\mathbb{P}$-almost surely we have that

$$|f_{\varepsilon_k}(X_{j_k}(r)) - f(X(r))| \leq |f_{\varepsilon_k}(X_{j_k}(r)) - f(X_{j_k}(r))| + |f(X_{j_k}(r)) - f(X(r))|$$

$$\leq \|f_{\varepsilon_k} - f\|_{C_b^0(H)} + |f(X_{j_k}(r)) - f(X(r))|,$$

so that by (3.11), the continuity of f, the properties of P_j, and the dominated convergence theorem, it holds, as $k \to \infty$, that

$$\mathbb{E}\int_0^t f_{\varepsilon_k}(X_{j_k}(s))\,\mathrm{d}s \to \mathbb{E}\int_0^t f(X(s))\,\mathrm{d}s.$$

Hence, by letting $k \to \infty$ we are left with

$$\mathbb{E}\left[u(X(t))\right] - u(x) - \lambda\,\mathbb{E}\int_0^t u(X(s))\,\mathrm{d}s + \mathbb{E}\int_0^t f(X(s))\,\mathrm{d}s = 0 \quad \forall\, t \geq 0.$$

Equivalently, this can be written as

$$\mathbb{E}\left[e^{-\lambda t}u(X(t))\right] - u(x) + \mathbb{E}\int_0^t e^{-\lambda s}f(X(s))\,\mathrm{d}s = 0 \quad \forall\, t \geq 0,$$

so that letting $t \to +\infty$ yields, by the dominated convergence theorem, that

$$u(x) = \int_0^\infty e^{-\lambda s}\,\mathbb{E}\left[f(X(s))\right]\,\mathrm{d}s \quad \forall\, f \in UC_b(H). \tag{4.2}$$

Analogously, the same proof applied to the solution Y yields

$$u(x) = \int_0^\infty e^{-\lambda s}\, \mathbb{E}\left[f(Y(s))\right]\, \mathrm{d}s \quad \forall f \in UC_b(H)\,. \tag{4.3}$$

By the properties of the Laplace transform, the fact that any element $g \in C_b^0(H)$ can be approximated pointwise by a sequence in $UC_b(H)$, and by the dominated convergence theorem, this implies that

$$\mathbb{E}\left[g(X(t))\right] = \mathbb{E}\left[g(Y(t))\right] \quad \forall g \in C_b^0(H)\,, \quad \forall t \geq 0\,.$$

It follows that $X(t)$ and $Y(t)$ have the same law on H for every $t \geq 0$, hence also, by pathwise continuity of X and Y that the laws of X and Y coincide on $C^0(\mathbb{R}_+; H)$. For a more detailed argument we refer to [Pri15].

Remark 4.1. *Note that if G is also Hilbert-Schmidt, one can handle the term I_2 more directly as*

$$\begin{aligned}
|I_2(\varepsilon, j)| &\leq \frac{1}{2}\,\mathbb{E}\int_0^t \left|\mathrm{Tr}\left[(Q_j - Q)D^2 u_\varepsilon(X_j(s))\right]\right|\, \mathrm{d}s \\
&\leq \frac{t}{2}\,\|Q_j - Q\|_{\mathscr{L}^1(H,H)}\,\|D^2 u_\varepsilon\|_{C_b^0(H;\mathscr{L}(H,H))} \\
&\leq t\,\|G\|_{\mathscr{L}^2(H,H)}\,\|G_j - G\|_{\mathscr{L}^2(H,H)}\,\|D^2 u_\varepsilon\|_{C_b^0(H;\mathscr{L}(H,H))}\,.
\end{aligned}$$

4.2. **Proof of Theorem 2.7.** We remove here the boundedness assumption on B made in Section 4. Let $x \in D(A^\alpha)$, and let

$$(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P}, W, X, \tau_X)\,, \qquad (\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P}, W, Y, \tau_Y)$$

be two local weak solutions to (1.1) in the sense of Definition 2.3 with respect to the same initial datum x. Let us handle the three terms appearing in (2.7). Since $x \in D(A^\alpha)$ one has (see [vNVW12, Prop. 3.1]) that

$$S(\cdot)x \in C^0([0, T]; D(A^\alpha))\,, \quad \forall T > 0\,.$$

Moreover, since $B(X^\tau) \in L_{loc}^2(\mathbb{R}_+; V_{-2\beta})$, one has by [vNVW12, Thms. 3.3–3.5]

$$\int_0^{\cdot \wedge \tau} S((\cdot \wedge \tau) - s)B(X^\tau(s))\mathrm{d}s \in C^0([0, T]; D(A^{\frac{1}{2} - \beta}))\,, \quad \forall T > 0\,,$$

while assumption (H3) along with [DPZ14, Thm. 5.15]

$$\int_0^{\cdot \wedge \tau} S((\cdot \wedge \tau) - s)G\mathrm{d}W(s) \in \bigcap_{\zeta \in [0, \xi)} C^0([0, T]; D(A^\zeta))\,, \quad \forall T > 0\,.$$

Hence, since $\alpha + \beta \leq \frac{1}{2}$ and $\alpha < \xi$ by assumption, one obtains that

$$X^\tau, Y^\tau \in C^0(\mathbb{R}_+; D(A^\alpha))\,, \quad \mathbb{P}\text{-a.s.}$$

We introduce now, for every $N \in \mathbb{N}$, the stopping times

$$\tau_N^X := \tau \wedge \inf\left\{t \geq 0 : \|X(t)\|_{2\alpha} > N\right\}\,, \qquad \tau_N^Y := \tau \wedge \inf\left\{t \geq 0 : \|Y(t)\|_{2\alpha} > N\right\}\,,$$

with the usual convention that $\inf \emptyset = +\infty$, and set

$$\tau_N := \tau_N^X \wedge \tau_N^Y\,.$$

Analogously, we define for every $N \in \mathbb{N}$ the truncated operator

$$B_N : D(A^\alpha) \to D(A^{-\beta})\,, \qquad B_N(x) := B(\Pi_N x)\,, \quad x \in D(A^\alpha)\,,$$

where $\Pi_N : D(A^\alpha) \to D(A^\alpha)$ is the orthogonal projection on the closed convex set $\{z \in D(A^\alpha) : \|z\|_{2\alpha} \leq N\}$. Clearly, for every $N \in \mathbb{N}$, $B_N : D(A^\alpha) \to D(A^{-\beta})$ is measurable, bounded (since B is locally bounded), and strongly-weakly continuous from $D(A^\alpha)$ to Z by assumption (H2). Furthermore, letting $X^{\tau_N}(\cdot) := X(\tau_n \wedge \cdot)$ and $Y^{\tau_N}(\cdot) := Y(\tau_n \wedge \cdot)$, it holds for all $t \geq 0$ that

$$X^{\tau_N}(t) = S(t \wedge \tau_N)x + \int_0^{t \wedge \tau_N} S((t \wedge \tau_N) - s)B_N(X^{\tau_N}(s))\,\mathrm{d}s$$
$$+ \int_0^{t \wedge \tau_N} S((t \wedge \tau_N) - s)G\,\mathrm{d}W(s)\,, \tag{4.4}$$

$$Y^{\tau_N}(t) = S(t \wedge \tau_N)x + \int_0^{t \wedge \tau_N} S((t \wedge \tau_N) - s)B_N(Y^{\tau_N}(s))\,\mathrm{d}s$$
$$+ \int_0^{t \wedge \tau_N} S((t \wedge \tau_N) - s)G\,\mathrm{d}W(s)\,. \tag{4.5}$$

Note that in (4.4)–(4.5) we have used that $\tau_N \leq \tau$ and that for every $s \leq \tau_N$ one has $B(X^\tau(s)) = B_N(X^{\tau_N}(s))$ and $B(Y^\tau(s)) = B_N(Y^{\tau_N}(s))$. The idea is now to construct two processes $\tilde{X}_N$ and $\tilde{Y}_N$ which are global weak solutions to equation (1.1) with B_N in place of B and such that, setting

$$[\![0, \tau_N]\!] := \{(\omega, t) \in \Omega \times \mathbb{R}_+ : t \leq \tau_N(\omega)\}\,,$$

it holds $\tilde{X}_N \mathbb{1}_{[\![0,\tau_N]\!]} = X \mathbb{1}_{[\![0,\tau_N]\!]}$ and $\tilde{Y}_N \mathbb{1}_{[\![0,\tau_N]\!]} = Y \mathbb{1}_{[\![0,\tau_N]\!]}$ in distribution. To this end, let us consider the equations

$$\mathrm{d}X_N + AX_N\,\mathrm{d}t = B_N(X_N)\,\mathrm{d}t + G\mathrm{d}W\,, \qquad X_N(0) = X(\tau_N)\,,$$
$$\mathrm{d}Y_N + AY_N\,\mathrm{d}t = B_N(Y_N)\,\mathrm{d}t + G\mathrm{d}W\,, \qquad Y_N(0) = Y(\tau_N)\,.$$

As a consequence of Theorem A.1 (with the choice $\delta_0 = \alpha$), there exist two weak solutions

$$\left(\Omega'_N, \mathscr{F}'_N, (\mathscr{F}'_{N,t})_{t \geq 0}, \mathbb{P}'_N, W'_N, X_N\right)\,, \qquad \left(\Omega''_N, \mathscr{F}''_N, (\mathscr{F}''_{N,t})_{t \geq 0}, \mathbb{P}''_N, W''_N, Y_N\right)\,,$$

in the sense of Definition 2.2 (with B_N). At this point, we set

$$\tilde{\Omega}'_N := \Omega \times \Omega'_N\,, \qquad \tilde{\Omega}''_N := \Omega \times \Omega''_N\,,$$
$$\tilde{\mathscr{F}}'_N := \mathscr{F} \otimes \mathscr{F}'_N\,, \qquad \tilde{\mathscr{F}}''_N := \mathscr{F} \otimes \mathscr{F}''_N\,,$$
$$\tilde{\mathbb{P}}'_N := \mathbb{P} \otimes \mathbb{P}'_N\,, \qquad \tilde{\mathbb{P}}''_N := \mathbb{P} \otimes \mathbb{P}''_N\,.$$

As far as the Wiener process is concerned, we define

$$\tilde{W}'_N(\omega, \omega', t) := W(\omega, t \wedge \tau_N) + W'_N(\omega', t - \tau_N(\omega))\mathbb{1}_{\{t > \tau_N(\omega)\}}\,, \quad (\omega, \omega', t) \in \tilde{\Omega}'_N \times \mathbb{R}_+\,,$$
$$\tilde{W}''_N(\omega, \omega'', t) := W(\omega, t \wedge \tau_N) + W''_N(\omega'', t - \tau_N(\omega))\mathbb{1}_{\{t > \tau_N(\omega)\}}\,, \quad (\omega, \omega'', t) \in \tilde{\Omega}''_N \times \mathbb{R}_+\,.$$

Analogously, we define

$$\tilde{X}_N(\omega, \omega', t) := X(\omega, t)\mathbb{1}_{\{t \leq \tau_N(\omega)\}} + X_N(\omega', t - \tau_N(\omega))\mathbb{1}_{\{t > \tau_N(\omega)\}}\,, \quad (\omega, \omega', t) \in \tilde{\Omega}'_N \times \mathbb{R}_+\,,$$
$$\tilde{Y}_N(\omega, \omega'', t) := Y(\omega, t)\mathbb{1}_{\{t \leq \tau_N(\omega)\}} + Y_N(\omega'', t - \tau_N(\omega))\mathbb{1}_{\{t > \tau_N(\omega)\}}\,, \quad (\omega, \omega'', t) \in \tilde{\Omega}''_N \times \mathbb{R}_+\,.$$

Let us consider the filtrations

$$\tilde{\mathscr{F}}'_{N,t} := \sigma(\tilde{W}'_N(s), \tilde{X}_N(s) : s \leq t)\,, \quad \tilde{\mathscr{F}}''_{N,t} := \sigma(\tilde{W}''_N(s), \tilde{Y}_N(s) : s \leq t)\,, \qquad t \geq 0\,.$$

Then, the families

$$\left(\tilde{\Omega}'_N, \tilde{\mathscr{F}}'_N, (\tilde{\mathscr{F}}'_{N,t})_{t \geq 0}, \tilde{\mathbb{P}}'_N, \tilde{W}'_N, \tilde{X}_N\right)\,, \qquad \left(\tilde{\Omega}''_N, \tilde{\mathscr{F}}''_N, (\tilde{\mathscr{F}}''_{N,t})_{t \geq 0}, \tilde{\mathbb{P}}''_N, \tilde{W}''_N, \tilde{Y}_N\right)\,, \tag{4.6}$$

are weak solutions to equation (1.1) with B_N, globally on $\mathbb{R}_+$. Indeed, since $\tilde{X}_N = X$ and $\tilde{Y}_N = Y$ on $[\![0, \tau_N]\!]$, as a consequence of (4.4)-(4.5) one has

$$\tilde{X}_N(t) = S(t)x + \int_0^t S(t-s)B_N(\tilde{X}_N(s))\,\mathrm{d}s + \int_0^t S(t-s)G\,\mathrm{d}W(s)\,, \quad \forall\, t \in [0, \tau_N]\,,$$

$$\tilde{Y}_N(t) = S(t)x + \int_0^t S(t-s)B_N(\tilde{Y}_N(s))\,\mathrm{d}s + \int_0^t S(t-s)G\,\mathrm{d}W(s)\,, \quad \forall\, t \in [0, \tau_N]\,.$$

Furthermore, for $t > \tau_N$, from the definitions of $\tilde{X}_N$, $\tilde{W}'_N$, and from the equations satisfied by X and by X_N, it follows that

$$\tilde{X}_N(t) = X_N(t - \tau_N)$$

$$= S(t - \tau_N)X(\tau_N) + \int_0^{t-\tau_N} S(t - \tau_N - s)B_N(X_N(s))\,\mathrm{d}s + \int_0^{t-\tau_N} S(t - \tau_N - s)G\,\mathrm{d}W'_N(s)$$

$$= S(t - \tau_N)S(\tau_N)x + S(t - \tau_N)\int_0^{\tau_N} S(\tau_N - s)B_N(\tilde{X}_N(s))\,\mathrm{d}s$$

$$+ S(t - \tau_N)\int_0^{\tau_N} S(\tau_N - s)G\,\mathrm{d}W(s) + \int_0^{t-\tau_N} S(t - \tau_N - s)B_N(X_N(s))\,\mathrm{d}s$$

$$+ \int_0^{t-\tau_N} S(t - \tau_N - s)G\,\mathrm{d}W(s)$$

$$= S(t)x + \int_0^{\tau_N} S(t - s)B_N(\tilde{X}_N(s))\,\mathrm{d}s + \int_0^{\tau_N} S(t - s)G\,\mathrm{d}W(s)$$

$$+ \int_{\tau_N}^t S(t - s)B_N(X_N(s - \tau_N))\,\mathrm{d}s + \int_{\tau_N}^t S(t - s)G\,\mathrm{d}W'_N(s)$$

$$= S(t)x + \int_0^t S(t - s)B_N(\tilde{X}_N(s))\,\mathrm{d}s + \int_0^t S(t - s)G\,\mathrm{d}\tilde{W}'_N(s)\,.$$

Arguing analogously on $\tilde{Y}_N$, this shows indeed that (4.6) are weak solutions to equation (1.1) with B_N, globally on $\mathbb{R}_+$. Since B_N is bounded, the uniqueness in law result proved in Section 4 ensures that $\tilde{X}_N$ and $\tilde{Y}_N$ have the same distribution $C^0(\mathbb{R}_+; H)$. Consequently, it follows that $\tilde{X}_N^{\tau_N}$ and $\tilde{Y}_N^{\tau_N}$ have the same distribution on $C^0(\mathbb{R}_+; H)$. Recalling that $X = \tilde{X}_N$ and $Y = \tilde{Y}_N$ on $[\![0, \tau_N]\!]$, we deduce also that X^{τ_N} and Y^{τ_N} have the same distribution on $C^0(\mathbb{R}_+; H)$. Noting that $\tau_N \nearrow \tau$ almost surely as $N \to \infty$, we conclude that X^τ and Y^τ have the same distribution on $C^0(\mathbb{R}_+; H)$.

Finally, it remains to show that the conclusion holds also when $G \in \mathscr{L}^2(H, D(A^{\delta'}))$ for some $\delta' > 0$, and $\alpha = \frac{1}{2}$ and $\beta = 0$. The only difference concerns the estimate on the stochastic convolution. To this end, one has by [vNVW12, Thms. 3.3–3.5] that

$$\int_0^{\cdot \wedge \tau} S((\cdot \wedge \tau) - s)G\mathrm{d}W(s) \in \bigcap_{\theta > 0} C^0([0, T]; D(A^{\frac{1}{2} + \delta' - \theta}))\,, \quad \forall\, T > 0\,.$$

In the case $\alpha = \frac{1}{2}$ and $\beta = 0$, since $\delta' > 0$ one has in particular that

$$\int_0^{\cdot \wedge \tau} S((\cdot \wedge \tau) - s)G\mathrm{d}W(s) \in C^0([0, T]; D(A^{\frac{1}{2}}))\,, \quad \forall\, T > 0\,,$$

and the same argument as above allows to conclude that

$$X^\tau, Y^\tau \in C^0(\mathbb{R}_+; D(A^{\frac{1}{2}}))\,, \quad \mathbb{P}\text{-a.s.}$$

The argument follows then as before, and this concludes the proof of Theorem 2.7.

5. Examples and applications

We collect here some class of equations for which we can prove weak uniqueness by noise as an application of Theorems 2.6-2.7.

First of all, as it was pointed out in Remark 2.1, let us state the following lemma.

Lemma 5.1. *Assume that $G \in \mathscr{L}^2(H, H)$. Then, conditions (2.3) and (2.4) are satisfied for every $\xi \in (0, \frac{1}{2})$ and $\vartheta \in (\frac{2\delta}{2\delta+1}, 1)$*

Proof. It holds that

$$\int_0^T t^{-2\xi} \|S(t)G\|^2_{\mathscr{L}^2(H,H)} \, \mathrm{d}t \leq \|G\|^2_{\mathscr{L}^2(H,H)} \int_0^T t^{-2\xi} \, \mathrm{d}t < +\infty \quad \forall \xi \in (0, 1/2) \,.$$

Moreover, we have

$$Q_t^{-\frac{1}{2}} S(t)G = \sqrt{2} A^{\frac{1}{2}} (I - S(2t))^{-\frac{1}{2}} S(t) \,,$$

so that, for every $\varepsilon > 0$,

$$\sum_{k=0}^\infty \left\| Q_t^{-\frac{1}{2}} S(t)Ge_k \right\|_H^4 = 4 \sum_{k=0}^\infty \lambda_k^2 \frac{e^{-4t\lambda_k}}{(1 - e^{-2t\lambda_k})^2}$$

$$= 4 \sum_{k=0}^\infty \lambda_k^2 \frac{1}{(t\lambda_k)^{2+\varepsilon}} \frac{(t\lambda_k)^{2+\varepsilon} e^{-4t\lambda_k}}{(1 - e^{-2t\lambda_k})^2}$$

$$\leq 4 \left(\max_{r \geq 0} \frac{r^{2+\varepsilon} e^{-4r}}{(1 - e^{-2r})^2} \right) \frac{1}{t^{2+\varepsilon}} \sum_{k=0}^\infty \frac{1}{\lambda_k^\varepsilon} \,.$$

Recalling that $G : V_{2\delta} \to H$ is an isomorphism, one has

$$\sum_{k=0}^\infty \lambda_k^{-2\delta} = \sum_{k=0}^\infty \lambda_k^{-2\delta} \|e_k\|_H^2 = \sum_{k=0}^\infty \|G^{-1} A^{-\delta} Ge_k\|_H^2$$

$$\leq \|G^{-1}\|_{\mathscr{L}(V_{2\delta}, H)} \|A^{-\delta}\|_{\mathscr{L}(H, V_{2\delta})} \sum_{k=0}^\infty \|Ge_k\|_H^2 < +\infty$$

since $G \in \mathscr{L}^2(H, H)$. This shows that, for all $t > 0$, $\varepsilon \geq 2\delta$, and $\vartheta \in (0, 1)$,

$$Q_t^{-\frac{1}{2}} S(t)G \in \mathscr{L}^4(H, H) \qquad \text{and} \qquad \left\| Q_t^{-\frac{1}{2}} S(t)G \right\|_{\mathscr{L}^4(H,H)}^{2(1-\vartheta)} \leq C_{\varepsilon,\vartheta} t^{-(1+\frac{\varepsilon}{2})(1-\vartheta)}$$

for some constant $C_{\varepsilon,\vartheta} > 0$ independent of t. Similarly, it holds that

$$Q_t^{-\frac{1}{2}} S(2t)Q = \sqrt{2} A^{\frac{1}{2}} (I - S(2t))^{-\frac{1}{2}} S(2t)G$$

so that, for every $\sigma > 0$,

$$\sum_{k=0}^\infty \left\| Q_t^{-\frac{1}{2}} S(2t)Qe_k \right\|_H^2 = 2 \sum_{k=0}^\infty \lambda_k \frac{e^{-4t\lambda_k}}{1 - e^{-2t\lambda_k}} \|Ge_k\|_H^2$$

$$\leq 2 \left(\max_{r \geq 0} \frac{r e^{-4r}}{1 - e^{-2r}} \right) \frac{1}{t} \|G\|^2_{\mathscr{L}^2(H,H)} \,.$$

This shows that, for all $t > 0$ and $\vartheta \in (0, 1)$, one has

$$Q_t^{-\frac{1}{2}} S(2t)Q \in \mathscr{L}^2(H, H) \qquad \text{and} \qquad \left\| Q_t^{-\frac{1}{2}} S(2t)Q \right\|_{\mathscr{L}^2(H,H)}^\vartheta \leq C_{\sigma,\vartheta} t^{-\frac{\vartheta}{2}}$$

for some constant $C_{\sigma,\vartheta} > 0$ independent of t. Putting everything together, condition (2.4) is satisfied provided that $\varepsilon \geq 2\delta$ is chosen with $(1 + \frac{\varepsilon}{2})(1 - \vartheta) + \frac{\vartheta}{2} < 1$, i.e. $\varepsilon(1 - \vartheta) < \vartheta$. It is easy to see that this is possible if $\vartheta \in (\frac{2\delta}{2\delta+1}, 1)$. $\qquad\square$

It what follows, $\mathcal{O} \subset \mathbb{R}^d$ $(d \geq 1)$ is a smooth bounded domain, $L_0^2(\mathcal{O})$ denotes the closed subspace of $L^2(\mathcal{O})$ of elements with null space average, and we use the notation

$$A_D : L^2(\mathcal{O}) \to L^2(\mathcal{O}), \qquad A_D v := -\Delta v, \quad v \in D(A_D) := H^2(\mathcal{O}) \cap H_0^1(\mathcal{O}),$$

and

$$A_N : L_0^2(\mathcal{O}) \to L_0^2(\mathcal{O}), \qquad A_N v := -\Delta v, \quad v \in D(A_N) := \{v \in L_0^2(\mathcal{O}) \cap H^2(\mathcal{O}) : \partial_{\mathbf{n}} v = 0\},$$

for the Laplace operator with Dirichlet and Neumann boundary conditions, respectively. Note that both A_D and A_N satisfy (H1). Before moving to specific examples, we collect here some useful properties of the Laplace operator and its fractional powers. These will be useful in order to check assumptions (H2)-(H3). In particular, in the notation of the paper, we are interested in the choice $G = A^{-\delta}$ for some $\delta \in [0, 1)$, where A is either A_D or A_N.

Proposition 5.2. *Let $\delta \in [0, 1)$ be fixed, let $A = A_D$ or $A = A_N$, let S be the C^0-semigroup generated by A, and let Q_t be defined as in (2.2) with $G = A^{-\delta}$ and $Q = A^{-2\delta}$, for $t \geq 0$. Then, it holds that:*

(a) *$G \in \mathscr{L}^2(L^2(\mathcal{O}), L^2(\mathcal{O}))$ if and only if $\delta > \frac{d}{4}$;*

(b) *$G \in \mathscr{L}^2(L^2(\mathcal{O}), D(A^{\delta'}))$ if and only if $\delta > \frac{d}{4} + \delta'$;*

(c) *$Q_t \in \mathscr{L}^1(L^2(\mathcal{O}), L^2(\mathcal{O}))$ if and only if $\delta > \frac{d}{4} - \frac{1}{2}$, for all $t > 0$;*

(d) *if $\delta > \frac{d}{4} - \frac{1}{2}$, condition (2.3) is satisfied for all $\xi \in (0, \delta + \frac{1}{2} - \frac{d}{4})$;*

(e) *$Q_t^{-\frac{1}{2}} S(t) G \in \mathscr{L}^4(L^2(\mathcal{O}), L^2(\mathcal{O}))$ for all $t > 0$. Moreover, for all $\varepsilon > \frac{d}{2}$ there exists a constant $C_\varepsilon > 0$ such that*

$$\left\| Q_t^{-\frac{1}{2}} S(t) G \right\|_{\mathscr{L}^4(L^2(\mathcal{O}), L^2(\mathcal{O}))}^2 \leq C_\varepsilon t^{-(1+\frac{\varepsilon}{2})} \quad \forall t > 0;$$

(f) *$Q_t^{-\frac{1}{2}} S(2t) Q \in \mathscr{L}^2(L^2(\mathcal{O}), L^2(\mathcal{O}))$ for all $t > 0$. Moreover, for all $\sigma > \frac{d}{2} - 2\delta$ there exists a constant $C_\sigma > 0$ such that*

$$\left\| Q_t^{-\frac{1}{2}} S(2t) Q \right\|_{\mathscr{L}^2(L^2(\mathcal{O}), L^2(\mathcal{O}))} \leq C_\sigma t^{-\frac{1}{2}(1+\sigma)} \quad \forall t > 0;$$

(g) *if $\delta > \frac{d}{4} - \frac{1}{2}$, there exists $\vartheta \in (0, 1)$ such that condition (2.4) is satisfied.*

As a direct consequence, we have the following corollary.

Corollary 5.3. *In the setting of Proposition 5.2, for every $t > 0$ the following conditions hold.*

(i) *If $d = 1$: $G \in \mathscr{L}^2(L^2(\mathcal{O}), L^2(\mathcal{O}))$ if and only if $\delta > \frac{1}{4}$, $Q_t \in \mathscr{L}^1(L^2(\mathcal{O}), L^2(\mathcal{O}))$ for every $\delta \geq 0$, and conditions (2.3) and (2.4) are satisfied for every $\delta \geq 0$ and $\xi \in (0, \frac{1}{2})$.*

(ii) *If $d = 2$: $G \notin \mathscr{L}^2(L^2(\mathcal{O}), L^2(\mathcal{O}))$, $Q_t \in \mathscr{L}^1(L^2(\mathcal{O}), L^2(\mathcal{O}))$ for every $\delta > 0$, and conditions (2.3) and (2.4) are satisfied for every $\delta > 0$ and $\xi \in (0, \delta)$.*

(iii) If $d = 3$: $G \notin \mathscr{L}^2(L^2(\mathcal{O}), L^2(\mathcal{O}))$, $Q_t \in \mathscr{L}^1(L^2(\mathcal{O}), L^2(\mathcal{O}))$ for every $\delta > \frac{1}{4}$, and conditions (2.3) and (2.4) is satisfied for every $\delta > \frac{1}{4}$ and $\xi \in (0, \delta - \frac{1}{4})$.

Proof of Proposition 5.2. Let $(e_k)_k$ be a complete orthonormal system of H made of eigenvectors of A, and let $(\lambda_k)_k$ be the corresponding eigenvalues. Recall that $\lambda_k \sim k^{\frac{2}{d}}$ as $k \to \infty$.

POINTS (a)-(b). We have that

$$\sum_{k=0}^{\infty} \|Ge_k\|_{D(A^{\delta'})}^2 = \sum_{k=0}^{\infty} \left\|A^{\delta'}Ge_k\right\|_{L^2(\mathcal{O})}^2 = \sum_{k=0}^{\infty} \left\|A^{-(\delta-\delta')}e_k\right\|_{L^2(\mathcal{O})}^2 = \sum_{k=0}^{\infty} \lambda_k^{-2(\delta-\delta')},$$

where $\lambda_k^{-2(\delta-\delta')} \sim k^{-\frac{4}{d}(\delta-\delta')}$ as $k \to \infty$, so the series converges if and only if $\delta - \delta' > \frac{d}{4}$.

POINT (c). For every $t \geq 0$, one has

$$Q_t = \frac{1}{2}A^{-1}Q(I - S(2t)) = \frac{1}{2}A^{-(1+2\delta)}(I - S(2t))$$

so that by (a) we have that $Q_t \in \mathscr{L}^1(L^2(\mathcal{O}), L^2(\mathcal{O}))$ if and only if $1 + 2\delta > \frac{d}{2}$, i.e., $\delta > \frac{d}{4} - \frac{1}{2}$.

POINT (d). For every $t > 0$ and $\rho > 0$ we have

$$\|S(t)G\|_{\mathscr{L}^2(H,H)}^2 = \sum_{k=0}^{\infty} \|S(t)A^{-\delta}e_k\|_H^2 = \sum_{k=0}^{\infty} e^{-2t\lambda_k}\lambda_k^{-2\delta} = \frac{1}{t^\rho}\sum_{k=0}^{\infty}\frac{e^{-2t\lambda_k}(t\lambda_k)^\rho}{\lambda_k^{2\delta+\rho}}$$

$$\leq \max_{r \geq 0}\left(e^{-2r}r^\rho\right)\frac{1}{t^\rho}\sum_{k=0}^{\infty}\frac{1}{\lambda_k^{2\delta+\rho}}.$$

Since $\lambda_k \sim k^{\frac{2}{d}}$ as $k \to \infty$, the series on the right-hand side converges if and only if $\frac{2}{d}(2\delta+\rho) > 1$, hence $\rho > \frac{d}{2} - 2\delta$. Now, condition (2.3) is satisfied for some $\xi \in (0, \frac{1}{2})$ provided that $2\xi + \rho < 1$. These two conditions are possible if and only if

$$\frac{d}{2} - 2\delta < 1 - 2\xi.$$

Consequently, if $\delta > \frac{d}{4} - \frac{1}{2}$, there exists $\xi \in (0, \frac{1}{2})$ such that $\delta > \frac{d}{4} + \xi - \frac{1}{2}$ (e.g. $\xi \in (0, \delta + \frac{1}{2} - \frac{d}{4})$): hence, it holds also that $\frac{d}{2} - 2\delta < 1 - 2\xi$, and one can choose $\rho \in (\frac{d}{2} - 2\delta, 1 - 2\xi)$. The computations above imply then that condition (2.3) is satisfied.

POINT (e). We have

$$Q_t^{-\frac{1}{2}}S(t)G = \sqrt{2}A^{\frac{1}{2}+\delta}(I - S(2t))^{-\frac{1}{2}}S(t)A^{-\delta}$$

$$= \sqrt{2}A^{\frac{1}{2}}(I - S(2t))^{-\frac{1}{2}}S(t),$$

so that, for every $\varepsilon > 0$,

$$\sum_{k=0}^{\infty}\left\|Q_t^{-\frac{1}{2}}S(t)Ge_k\right\|_{L^2(\mathcal{O})}^4 = 4\sum_{k=0}^{\infty}\lambda_k^2\frac{e^{-4t\lambda_k}}{(1 - e^{-2t\lambda_k})^2}$$

$$= 4\sum_{k=0}^{\infty}\lambda_k^2\frac{1}{(t\lambda_k)^{2+\varepsilon}}\frac{(t\lambda_k)^{2+\varepsilon}e^{-4t\lambda_k}}{(1 - e^{-2t\lambda_k})^2}$$

$$\leq 4\left(\max_{r \geq 0}\frac{r^{2+\varepsilon}e^{-4r}}{(1 - e^{-2r})^2}\right)\frac{1}{t^{2+\varepsilon}}\sum_{k=0}^{\infty}\frac{1}{\lambda_k^\varepsilon}.$$

Recalling that $\lambda_k \sim k^{\frac{2}{d}}$ as $k \to \infty$, the series on the right-hand side converges if and only if $\frac{2\varepsilon}{d} > 1$, i.e., $\varepsilon > \frac{d}{2}$. This shows that, for all $t \geq 0$, $\varepsilon > \frac{d}{2}$, and $\vartheta > 0$,

$$Q_t^{-\frac{1}{2}}S(t)G \in \mathscr{L}^4(L^2(\mathcal{O}), L^2(\mathcal{O})) \qquad \text{and} \qquad \left\|Q_t^{-\frac{1}{2}}S(t)G\right\|_{\mathscr{L}^4(L^2(\mathcal{O}), L^2(\mathcal{O}))}^{2(1-\vartheta)} \leq C_{\varepsilon,\vartheta} t^{-(1+\frac{\varepsilon}{2})(1-\vartheta)}$$

for some constant $C_{\varepsilon,\vartheta} > 0$ independent of t.

POINT (f). It holds that

$$Q_t^{-\frac{1}{2}}S(2t)Q = \sqrt{2}A^{\frac{1}{2}+\delta}(I - S(2t))^{-\frac{1}{2}}S(2t)A^{-2\delta}$$
$$= \sqrt{2}A^{\frac{1}{2}-\delta}(I - S(2t))^{-\frac{1}{2}}S(2t),$$

so that, for every $\sigma > 0$,

$$\sum_{k=0}^{\infty} \left\|Q_t^{-\frac{1}{2}}S(2t)Qe_k\right\|_{L^2(\mathcal{O})}^2 = 2\sum_{k=0}^{\infty} \lambda_k^{1-2\delta} \frac{e^{-4t\lambda_k}}{1 - e^{-2t\lambda_k}}$$

$$= 2\sum_{k=0}^{\infty} \lambda_k^{1-2\delta} \frac{1}{(t\lambda_k)^{1+\sigma}} \frac{(t\lambda_k)^{1+\sigma} e^{-4t\lambda_k}}{1 - e^{-2t\lambda_k}}$$

$$\leq 2\left(\max_{r\geq 0} \frac{r^{1+\sigma} e^{-4r}}{1 - e^{-2r}}\right) \frac{1}{t^{1+\sigma}} \sum_{k=0}^{\infty} \frac{1}{\lambda_k^{2\delta+\sigma}} .$$

Recalling again that $\lambda_k \sim k^{\frac{2}{d}}$ as $k \to \infty$, the series on the right-hand side converges if and only if $\frac{2}{d}(2\delta + \sigma) > 1$, i.e., $\sigma > \frac{d}{2} - 2\delta$. This shows that, for all $t \geq 0$, $\sigma > \frac{d}{2} - 2\delta$, and $\vartheta > 0$,

$$Q_t^{-\frac{1}{2}}S(2t)Q \in \mathscr{L}^2(L^2(\mathcal{O}), L^2(\mathcal{O})) \qquad \text{and} \qquad \left\|Q_t^{-\frac{1}{2}}S(2t)Q\right\|_{\mathscr{L}^2(L^2(\mathcal{O}), L^2(\mathcal{O}))}^{\vartheta} \leq C_{\sigma,\vartheta} t^{-\frac{\vartheta}{2}(1+\sigma)}$$

for some constant $C_{\sigma,\vartheta} > 0$ independent of t.

POINT (g). Putting everything together, one has for every $\lambda > 0$, $\varepsilon > \frac{d}{2}$, and $\sigma > \frac{d}{2} - 2\delta$ that

$$\int_0^{+\infty} e^{-\lambda t} \left\|Q_t^{-\frac{1}{2}}S(t)G\right\|_{\mathscr{L}^4(L^2(\mathcal{O}), L^2(\mathcal{O}))}^{2(1-\vartheta)} \left\|Q_t^{-\frac{1}{2}}S(2t)Q\right\|_{\mathscr{L}^2(L^2(\mathcal{O}), L^2(\mathcal{O}))}^{\vartheta} \, dt$$

$$\leq C_{\varepsilon,\vartheta} C_{\sigma,\vartheta} \int_0^{+\infty} e^{-\lambda t} \frac{1}{t^{(1+\frac{\varepsilon}{2})(1-\vartheta)+\frac{\vartheta}{2}(1+\sigma)}} \, dt .$$

The integral on the right-hand side converges if and only if $(1 + \frac{\varepsilon}{2})(1 - \vartheta) + \frac{\vartheta}{2}(1 + \sigma) < 1$, i.e., $\varepsilon(1 - \vartheta) + \sigma\vartheta < \vartheta$. In order to guarantee also the conditions $\varepsilon > \frac{d}{2}$ and $\sigma > \frac{d}{2} - 2\delta$, δ and ϑ have to be chosen such that

$$\frac{d}{2}(1 - \vartheta) + \vartheta\left(\frac{d}{2} - 2\delta\right) < \vartheta .$$

Direct computations show that this is equivalent to

$$\vartheta > \frac{d}{2(1 + 2\delta)} .$$

We can choose then $\vartheta \in (0, 1)$ satisfying such inequality if and only if

$$\frac{d}{2(1 + 2\delta)} < 1 ,$$

i.e., $\delta > \frac{d}{4} - \frac{1}{2}$, as required. □

5.1. **Heat equations with perturbation.** We consider SPDEs in the form

$$\mathrm{d}X - \Delta X \,\mathrm{d}t = F(X)\,\mathrm{d}t + G\,\mathrm{d}W \quad \text{in } (0,T)\times\mathcal{O}\,,$$
$$X = 0 \quad \text{in } (0,T)\times\partial\mathcal{O}\,,$$
$$X(0) = x_0 \quad \text{in } \mathcal{O}\,,$$

where $\mathcal{O} \subset \mathbb{R}^d$ is a smooth bounded domain, Δ and ∇ denote the Laplacian and the gradient with respect to the space-variables, $F : \mathbb{R} \to \mathbb{R}$ is continuous and linearly bounded, $x_0 \in L^2(\mathcal{O})$, and $G \in \mathscr{L}(L^2(\mathcal{O}), L^2(\mathcal{O}))$. We define the Nemytskii-type operator

$$B : L^2(\mathcal{O}) \to L^2(\mathcal{O})\,, \qquad B(\varphi)(\xi) := F(\varphi(\xi))\,, \quad \xi \in \mathcal{O}\,, \quad \varphi \in L^2(\mathcal{O})\,,$$

which is measurable, linearly bounded, and strongly continuous (by continuity and linear boundedness of F). Then, setting $H := L^2(\mathcal{O})$ and $G := A_D^{-\delta}$ for some $\delta \in [0,1)$, the abstract formulation of the problem reads as the following evolution equation on H:

$$\mathrm{d}X + A_D X \,\mathrm{d}t = B(X)\,\mathrm{d}t + G\,\mathrm{d}W\,, \quad X(0) = x_0\,.$$

In the notation of the paper, we have then $\alpha = \beta = 0$. By Corollary 5.3 we deduce that uniqueness in distribution holds up to dimension $d = 3$ included, with the choices $\delta \in [0, \frac{1}{2})$ in dimension $d = 1$, $\delta \in (0, \frac{1}{2})$ in dimension $d = 2$, and $\delta \in (\frac{1}{4}, \frac{1}{2})$ in dimension $d = 3$.

5.2. **Heat equations with polynomial perturbation.** We consider SPDEs in the form

$$\mathrm{d}X - \Delta X \,\mathrm{d}t = F(X)\,\mathrm{d}t + G\,\mathrm{d}W \quad \text{in } (0,T)\times\mathcal{O}\,,$$
$$X = 0 \quad \text{in } (0,T)\times\partial\mathcal{O}\,,$$
$$X(0) = x_0 \quad \text{in } \mathcal{O}\,,$$

where $\mathcal{O} \subset \mathbb{R}^d$ is a smooth bounded domain, $F : \mathbb{R} \to \mathbb{R}$ is a continuous function, $x_0 \in L^2(\mathcal{O})$, and $G \in \mathscr{L}(L^2(\mathcal{O}), L^2(\mathcal{O}))$. We further suppose that F behaves like a polynomial, in the sense that there exist $p \geq 2$ and $C > 0$ such that

$$|F(r)| \leq C(1 + |r|^{p-1}) \quad \forall r \in \mathbb{R}\,.$$

Given $r \in [\max\{2, p-1\}, 2(p-1)]$, setting $s := \frac{r}{p-1} \in [1,2]$ one can view B as an operator

$$B : L^r(\mathcal{O}) \to L^s(\mathcal{O})$$

as

$$B(\varphi)(\xi) := F(\varphi(\xi))\,, \quad \varphi \in L^r(\mathcal{O})\,, \quad \xi \in \mathcal{O}\,.$$

The idea is to choose $\alpha \in [0, \frac{1}{2}]$ and $\beta \in [0, \frac{1}{2})$ such that $\alpha + \beta \in [0, \frac{1}{2}]$, $D(A_D^\alpha) \hookrightarrow L^r(\mathcal{O})$, and $L^s(\mathcal{O}) \hookrightarrow D(A_D^{-\beta})$, so that B can be considered as

$$B : D(A_D^\alpha) \to D(A_D^{-\beta})$$

given by

$$\langle B(\varphi), \psi \rangle_{V_{-2\beta}, V_{2\beta}} := \int_{\mathcal{O}} F(\varphi(\xi))\psi(\xi)\,\mathrm{d}\xi\,, \quad \varphi \in D(A_D^\alpha)\,, \psi \in D(A_D^\beta)\,.$$

Note that such a choice yields a well-defined, measurable, and strongly-weakly continuous operator (by continuity and polynomial growth of F). Then, setting $H := L^2(\mathcal{O})$ and $G := A_D^{-\delta}$ for some $\delta \in [0,1)$, the abstract formulation of the problem reads as the following evolution equation on H:

$$\mathrm{d}X + A_D X \,\mathrm{d}t = B(X)\,\mathrm{d}t + G\,\mathrm{d}W\,, \quad X(0) = x_0\,.$$

Global existence of solutions follows by classical variational arguments along with coercivity-type conditions in the form

$$-F(r)r \geq C^{-1}|r|^p - C \quad \forall r \in \mathbb{R}\,.$$

If coercivity fails, or more generally if F is not necessarily nonincreasing nor polynomially-bounded, then only local existence can be proved.

As far as uniqueness is concerned, we have to carefully tune the parameters α and β in order to achieve the best possible uniqueness result. To this end, by the classical Sobolev embedding, the inclusions $D(A_D^\alpha) \hookrightarrow L^r(\mathcal{O})$ and $L^s(\mathcal{O}) \hookrightarrow D(A_D^{-\beta})$ are satisfied if $\alpha = \frac{d}{2}(\frac{1}{2} - \frac{1}{r})$ and $\beta = \frac{d}{2}(\frac{1}{s} - \frac{1}{2})$. With such a choice, it is immediate to see that $\alpha \geq 0$ and $\beta \geq 0$. Moreover, note that $\alpha + \beta = \frac{d}{2}(\frac{1}{s} - \frac{1}{r}) = \frac{d(p-2)}{2r}$, so that $\alpha + \beta \leq \frac{1}{2}$ provided that $r \geq d(p-2)$. Hence, it is possible to choose r such that $r \in [\max\{2, p-1\}, 2(p-1)]$ and $r \geq d(p-2)$ if and only if $d(p-2) \leq 2(p-1)$. In dimensions $d = 1, 2$ this is always true, while in dimension $d = 3$ this requires the limitation $p \leq 4$. Clearly, the optimal choice of r is the one that minimises α. By definition of α, this is the minimal admissible value of r, i.e. $r_{opt} = \max\{2, p-1, d(p-2)\}$ (with the constraint $p \leq 4$ in $d = 3$). To summarise, we have then the following uniqueness results.

DIMENSION $d = 1$. We can handle every growth $p \in [2, +\infty)$. The optimal choice of r is $r_{opt} = \max\{2, p-1, p-2\} = \max\{2, p-1\}$, yielding $\alpha_{opt} = \frac{1}{2}(\frac{1}{2} - \frac{1}{\max\{2, p-1\}})$. Thanks to Corollary 5.3, we can apply Theorem 2.7 and infer that weak uniqueness holds for all $\delta \in [0, \frac{1}{2})$.

DIMENSION $d = 2$. We can handle every growth $p \in [2, +\infty)$. The optimal choice of r is $r_{opt} = \max\{2, p-1, 2(p-2)\}$, yielding $\alpha_{opt} = \frac{1}{2} - \frac{1}{\max\{2, p-1, 2(p-2)\}}$. Thanks to Corollary 5.3, we can apply Theorem 2.7 and infer that weak uniqueness holds for all $\delta \in (\alpha_{opt}, \frac{1}{2})$.

DIMENSION $d = 3$. We can handle every growth $p \in [2, 4)$. The optimal choice of r is $r_{opt} = \max\{2, p-1, 3(p-2)\}$, yielding $\alpha_{opt} = \frac{3}{2}(\frac{1}{2} - \frac{1}{\max\{2, p-1, 3(p-2)\}})$. Note that $\alpha_{opt} \in [0, \frac{1}{2})$ since $p \in [2, 4)$. Hence, thanks to Corollary 5.3, we can apply Theorem 2.7 and infer that weak uniqueness holds for all $\delta \in (\alpha_{opt}, \frac{1}{2})$. Let us point out that here the case $p = 4$ is excluded as it leads to the choice $\alpha_{opt} = \frac{1}{2}$. In particular, this would require a Hilbert-Schmidt condition on G in Theorem 2.7, which is not true in dimension $d = 3$.

5.3. **Heat equations with perturbation: divergence-like case.** We consider SPDEs in the form

$$\mathrm{d}X - \Delta X\,\mathrm{d}t = (-\Delta)^\beta F(X)\,\mathrm{d}t + G\,\mathrm{d}W \quad \text{in } (0, T) \times \mathcal{O}\,,$$
$$X = 0 \quad \text{in } (0, T) \times \partial\mathcal{O}\,,$$
$$X(0) = x_0 \quad \text{in } \mathcal{O}\,,$$

where $\mathcal{O} \subset \mathbb{R}^d$ is a smooth bounded domain, $\beta \in (0, \frac{1}{2})$, $F : \mathbb{R} \to \mathbb{R}$ is continuous and linearly bounded, $x_0 \in L^2(\mathcal{O})$, and $G \in \mathscr{L}(L^2(\mathcal{O}), L^2(\mathcal{O}))$. We define the operator

$$B : L^2(\mathcal{O}) \to D(A_D^{-\beta})$$

as

$$\langle B(\varphi), \psi \rangle_{V_{-2\beta}, V_{2\beta}} := \int_\mathcal{O} F(\varphi(\xi))(A_D^\beta \psi)(\xi)\,\mathrm{d}\xi\,, \quad \varphi \in L^2(\mathcal{O})\,, \quad \psi \in D(A_D^\beta)\,,$$

which is measurable, linearly bounded, and strongly continuous (by continuity and linear boundedness of F). Then, setting $H := L^2(\mathcal{O})$ and $G := A_D^{-\delta}$ for some $\delta \in [0, 1)$, the abstract formulation of the problem reads as the following evolution equation on H:

$$\mathrm{d}X + A_D X\,\mathrm{d}t = B(X)\,\mathrm{d}t + G\,\mathrm{d}W\,, \quad X(0) = x_0\,.$$

In the notation of the paper, we have then $\alpha = 0$ and $\beta > 0$. Now, thanks to Corollary 5.3 we deduce that uniqueness in distribution holds up to dimension $d = 3$ included, with the choices:

- dimension $d = 1$: $\beta \in (0, \frac{1}{2})$ and $\delta \in [0, \frac{1}{2} - \beta)$,

- dimension $d = 2$: $\beta \in (0, \frac{1}{2})$ and $\delta \in (0, \frac{1}{2} - \beta)$,

- dimension $d = 3$: $\beta \in (0, \frac{1}{4})$ and $\delta \in (\frac{1}{4}, \frac{1}{2} - \beta)$.

5.4. **Heat equations with perturbation: non-divergence-like case.** We consider SPDEs in the form

$$\mathrm{d}X - \Delta X \,\mathrm{d}t = F((-\Delta)^{\alpha}X) \,\mathrm{d}t + G \,\mathrm{d}W \quad \text{in } (0,T) \times \mathcal{O} \,,$$
$$X = 0 \quad \text{in } (0,T) \times \partial\mathcal{O} \,,$$
$$X(0) = x_0 \quad \text{in } \mathcal{O} \,,$$

where $\mathcal{O} \subset \mathbb{R}^d$ is a smooth bounded domain, $\alpha \in (0,1)$, $F : \mathbb{R} \to \mathbb{R}$ is continuous and linearly bounded, $x_0 \in L^2(\mathcal{O})$, and $G \in \mathscr{L}(L^2(\mathcal{O}), L^2(\mathcal{O}))$. We define the operator

$$B : D(A_D^{\alpha}) \to L^2(\mathcal{O})$$

as

$$B(\varphi)(\xi) := F((A_D^{\alpha}\varphi)(\xi)) \,, \quad \varphi \in D(A_D^{\alpha}) \,,$$

which is measurable, linearly bounded, and strongly continuous (by continuity and linear boundedness of F). Then, setting $H := L^2(\mathcal{O})$ and $G := A_D^{-\delta}$ for some $\delta \in [0,1)$, the abstract formulation of the problem reads as the following evolution equation on H:

$$\mathrm{d}X + A_D X \,\mathrm{d}t = B(X) \,\mathrm{d}t + G \,\mathrm{d}W \,, \quad X(0) = x_0 \,.$$

In the notation of the paper, we have then $\alpha > 0$ and $\beta = 0$. We note straightaway that if F is bounded, thanks to Corollary 5.3, we can apply Theorem 2.6 and deduce that uniqueness in distribution holds for every $\alpha \in (0,1)$, $\delta \in [0, \frac{1}{2})$, and initial datum in H, in every dimension $d = 1, 2, 3$. Otherwise, if F is unbounded, depending on the dimension, we can proceed as follows.

DIMENSION $d = 1$. Thanks to Corollary 5.3, we can apply Theorem 2.7 to deduce that uniqueness in distribution holds for every $\alpha \in (0, \frac{1}{2})$, $\delta \in [0, \frac{1}{2})$, and initial datum in $D(A_D^{\alpha})$. Note also that if F is unbounded and $\alpha = \frac{1}{2}$, we can still infer uniqueness in distribution by the last statement of Theorem 2.7 for every $\delta \in (\frac{1}{4}, \frac{1}{2})$.

DIMENSION $d = 2$. Thanks to Corollary 5.3, we can apply Theorem 2.7 to deduce that uniqueness in distribution holds for every $\alpha \in (0, \frac{1}{2})$, $\delta \in (\alpha, \frac{1}{2})$, and initial datum in $D(A_D^{\alpha})$.

DIMENSION $d = 3$. Thanks to Corollary 5.3, we can apply Theorem 2.7 to deduce that uniqueness in distribution holds for every $\alpha \in (0, \frac{1}{4})$, $\delta \in (\alpha + \frac{1}{4}, \frac{1}{2})$, and initial datum in $D(A_D^{\alpha})$.

5.5. **One-dimensional Burgers-type equations.** We consider SPDEs in the form

$$\mathrm{d}X - \Delta X \,\mathrm{d}t = F(X) \cdot \nabla X \,\mathrm{d}t + G \,\mathrm{d}W \quad \text{in } (0,T) \times \mathcal{O} \,,$$
$$X = 0 \quad \text{in } (0,T) \times \partial\mathcal{O} \,,$$
$$X(0) = x_0 \quad \text{in } \mathcal{O} \,,$$

where $\mathcal{O} \subset \mathbb{R}$ is a smooth bounded domain, $F : \mathbb{R} \to \mathbb{R}$ is continuous and bounded, $x_0 \in H_0^1(\mathcal{O}) = D(A_D^{\frac{1}{2}})$, and $G \in \mathscr{L}^2(L^2(\mathcal{O}), L^2(\mathcal{O}))$. We define the operator

$$B : D(A_D^{\frac{1}{2}}) \to L^2(\mathcal{O}) \,, \qquad B(\varphi)(\xi) := F(\varphi(\xi)) \cdot \nabla\varphi(\xi) \,, \quad \xi \in \mathcal{O} \,, \quad \varphi \in D(A_D^{\frac{1}{2}}) \,,$$

which is measurable, linearly bounded, and strongly continuous (by continuity and boundedness of F). Moreover, for every $\delta \in (\frac{1}{4}, \frac{1}{2})$ and for every $\delta' \in (0, \delta - \frac{1}{4}) \subset (0, \delta) \subset (0, \frac{1}{2})$, we have that that $G := A_D^{-\delta} \in \mathscr{L}^2(H, D(A^{\delta'}))$ and the abstract formulation of the problem reads

$$\mathrm{d}X + A_D X \, \mathrm{d}t = B(X) \, \mathrm{d}t + G \, \mathrm{d}W \,, \quad X(0) = x_0 \,.$$

Hence, Theorem 2.7 applies with the choices $\alpha = \frac{1}{2}$ and $\beta = 0$, so that uniqueness in distribution holds for initial data in $H_0^1(\mathcal{O})$. Note that this result is sub-optimal. Indeed, also strong uniqueness is guaranteed even under more general conditions (see e.g. [Gyö98]) by using direct computational techniques on the SPDE. In this case, our approach only provides an alternative proof of weak uniqueness. Existence of solutions is also well-understood, even in higher dimensions (see for example [LR15, Ex. 5.1.8]).

5.6. **Cahn–Hilliard equations with perturbation.** We consider SPDEs in the form

$$\mathrm{d}X - \Delta(-\Delta X + F_1(X)) \, \mathrm{d}t = F_2(X, \nabla X, D^2 X) \, \mathrm{d}t + G \, \mathrm{d}W \quad \text{in } (0, T) \times \mathcal{O} \,,$$
$$\partial_{\mathbf{n}} X = \partial_{\mathbf{n}} \Delta X = 0 \quad \text{in } (0, T) \times \partial\mathcal{O} \,,$$
$$X(0) = x_0 \quad \text{in } \mathcal{O} \,,$$

where $\mathcal{O} \subset \mathbb{R}^d$ is a smooth bounded domain with $d \in \{1, 2, 3\}$, and $F_1 : \mathbb{R} \to \mathbb{R}$ is of class C^2 with locally Lipschitz-continuous derivatives F_1', F_1''. Typically, F_1 is the derivative of a double-well potential: the classical example is given by the quartic double-well potential

$$F_1(r) = \frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{4}(r^2 - 1)^2\right] = r(r^2 - 1)\,, \quad r \in \mathbb{R}\,. \tag{5.1}$$

We stress that in our framework no assumption on the growth of the nonlinear term F_1 is required: for example, we can cover polynomial potentials of every order, as well as exponential-growth potentials. The term $F_2 : \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^{d \times d} \to \mathbb{R}$ is only continuous and linearly bounded in every argument, representing some singular source term possibly depending on all derivatives up to second order, $x_0 \in L^2(\mathcal{O})$, and $G \in \mathscr{L}(L^2(\mathcal{O}), L^2(\mathcal{O}))$. Noting that

$$-\Delta F_1(\varphi) = -\operatorname{div}[F_1'(\varphi)\nabla\varphi] = -F_1''(\varphi)|\nabla\varphi|^2 - F_1'(\varphi)\Delta\varphi \quad \forall\, \varphi \in H^2(\mathcal{O})\,,$$

we define the operator

$$B_1 : H^2(\mathcal{O}) \to L^2(\mathcal{O})$$

as

$$B_1(\varphi)(\xi) := -F_1''(\varphi(\xi))|\nabla\varphi(\xi)|^2 - F_1'(\varphi(\xi))\Delta\varphi(\xi)\,, \quad \varphi \in H^2(\mathcal{O})\,, \quad \xi \in \mathcal{O}\,,$$

which is well-defined, measurable, locally bounded (by the inclusion $H^2(\mathcal{O}) \hookrightarrow L^\infty(\mathcal{O})$ and continuity of F_1'' and F_1'), and strongly continuous (by local Lipschitz-continuity of F_1' and F_1''). Analogously, we define

$$B_2 : H^2(\mathcal{O}) \to L^2(\mathcal{O})$$

as

$$B_2(\varphi)(\xi) := F_2(\varphi(\xi), \nabla\varphi(\xi), D^2\varphi(\xi))\,, \quad \varphi \in L^2(\mathcal{O})\,, \quad \xi \in \mathcal{O}\,,$$

which is measurable, locally bounded, and strongly continuous by the hypothesis on F_2. Then, setting $H := L^2(\mathcal{O})$ and noting that $D((A_N^2)^{\frac{1}{2}}) = D(A_N) = \{\varphi \in H^2(\mathcal{O}) : \partial_{\mathbf{n}}\varphi = 0\}$, we consider the operator

$$B : D(A_N) \to H\,, \qquad B(\varphi) = B_2(\varphi) - B_1(\varphi)\,, \quad \varphi \in D(A_N)\,.$$

With this notation, by setting $G := (A_N^2)^{-\delta} = A_N^{-2\delta}$ for some $\delta \in [0, 1)$, the abstract formulation of the problem reads

$$\mathrm{d}X + A_N^2 X \, \mathrm{d}t = B(X) \, \mathrm{d}t + G \, \mathrm{d}W \,, \quad X(0) = x_0 \,.$$

In the notation of the paper, we have then $\alpha = \frac{1}{2}$, $\beta = 0$, and $A = A_N^2$.

DIMENSION $d = 1,2,3$. For every $\delta \in (\frac{d}{8}, \frac{1}{2})$ and for every $\delta' \in (0, \delta - \frac{d}{8}) \subset (0, \delta) \subset (0, \frac{1}{2})$, by standard computations it holds that $G \in \mathscr{L}^2(H, D(A^{\delta'}))$. Hence, assumption (H3) is satisfied, Theorem 2.7 applies and uniqueness in distribution holds with such choice of G and β for every initial data $x \in D(A_N)$.

PARTICULAR CASE: THE BI-LAPLACE EQUATION. In the special case where $F_1 = 0$, one recovers the analogous uniqueness results for perturbed bi-Laplace equations. Here, under the assumption that F_2 is bounded in its second and third arguments, the corresponding uniqueness results hold also for arbitrary initial data in H.

APPENDIX A. EXISTENCE OF WEAK SOLUTIONS

In this appendix, we provide a weak existence result for equation (1.1) for the case where B is bounded. This result follows essentially by a stochastic maximal regularity argument.

We emphasise that here we do not aim for optimality. This is just a sufficient condition for existence in a general setting. More refined results can be found in the literature according to the specific example of SPDE in consideration. In general, if B is unbounded, local existence results can be obtained with classical techniques in several cases, e.g. under locally Lipschitz-continuity assumptions or the so-called structure condition between B and G (see [AMP23]).

Let us stress that even if B is bounded, existence of weak solution is not a direct consequence of Girsanov theorem due to the presence of a coloured noise (which is indeed necessary to recover the $V_{2\alpha}$-regularity). In the following result we give a sufficient condition for global existence in the case of a bounded B.

Theorem A.1. *Assume (H1)-(H2)-(H3), and that there exists a Hilbert space $Z \supseteq D(A^{-\beta})$ such that B is bounded from $D(A^\alpha)$ to $D(A^{-\beta})$ and strongly-weakly continuous from $D(A^\alpha)$ to Z. Let $\delta_0 \in [0,1]$ and $x \in D(A^{\delta_0})$. Then:*

(i) if $\alpha < \xi$, there exists a weak solution to (1.1) in the sense of Definition 2.2 such that

$$X \in C^0(\mathbb{R}_+; D(A^{\delta_0 \wedge \zeta})) \cap L^1_{loc}(\mathbb{R}_+; D(A^\alpha)) \quad \forall \zeta \in [0, \xi), \qquad B(X) \in L^\infty(\mathbb{R}_+; H), \qquad \mathbb{P}\text{-}a.s.$$

(ii) if $G \in \mathscr{L}^2(H, D(A^{\delta'}))$ for some $\delta' \in (0, \delta)$ and $\alpha < (\delta_0 + 1) \wedge (\delta' + \frac{1}{2})$, there exists a weak solution to (1.1) in the sense of Definition 2.2 such that

$$X \in C^0(\mathbb{R}_+; D(A^{\delta_0 \wedge \delta'})) \cap L^1_{loc}(\mathbb{R}_+; D(A^\alpha)), \qquad B(X) \in L^\infty(\mathbb{R}_+; H), \qquad \mathbb{P}\text{-}a.s.$$

Proof. We recall that $(e_k)_{k \in \mathbb{N}}$ is a complete orthonormal system of H made of eigenvectors of A, and $(\lambda_k)_{k \in \mathbb{N}}$ are the corresponding eigenvalues. We define $H_n := \text{span}\{e_1, \ldots, e_n\}$ and we introduce $P_n : H \to H_n$ as the orthogonal projection on H_n, for all $n \in \mathbb{N}$. Note that P_n can be extended to a linear continuous operator $P_n : D(A^{-\beta}) \to H_n$ as follows

$$P_n v := \sum_{k=1}^n \langle v, e_k \rangle e_k, \quad v \in D(A^{-\beta}).$$

We consider the approximated problem

$$dX_n + AX_n \, dt = B_n(X_n) \, dt + G_n \, dW_n, \qquad X(0) = x_n, \tag{A.1}$$

where

$$B_n := P_n \circ B \circ P_n : H \to H_n, \qquad G_n := P_n \circ G \in \mathscr{L}^2(H, H_n),$$

W_n is an n-dimensional standard Brownian motion, and $x_n := P_n x \in H_n$. If we look for a solution X_n in the form

$$X_n(t) = \sum_{k=1}^{n} a_k(t) e_k \,, \quad t \geq 0 \,,$$

plugging it into equation (A.1) and testing by arbitrary elements in $\{e_1, \ldots, e_n\}$ yields the following system of SDEs for the coefficients $\{a_k\}_{k=1}^{n}$:

$$\mathrm{d}a_j + \lambda_j a_j \,\mathrm{d}t = \left(B_n \left(\sum_{k=1}^{n} a_k e_k \right), e_j \right) + (G_n \,\mathrm{d}W_n, e_j) \,, \quad a_j(0) = (x_n, e_j) \,, \qquad j = 1, \ldots, n \,.$$

Since in $\mathbb{R}^n$ all norms are equivalent and the restriction $B_n : H_n \to H_n$ is continuous and bounded, by classical results based on Girsanov arguments, it follows that (A.1) admits a weak solution $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t \geq 0}, \mathbb{P}, X_n, W_n)$ with

$$X_n \in L^2(\Omega; C^0(\mathbb{R}_+; H_n)) \,.$$

From (A.1) it follows that

$$X_n(t) = S(t) x_n + \int_0^t S(t-s) B_n(X_n(s)) \,\mathrm{d}s + \int_0^t S(t-s) G_n \,\mathrm{d}W_n(s) \,, \quad \forall t \geq 0 \,, \quad \mathbb{P}\text{-a.s.}$$

Let us now show uniform estimates on $(X_n)_n$, in the two cases (i) and (ii).

Case (i). First of all, for $\zeta \in [0, \xi)$, using the regularisation properties of S (see [Lun95, Prop. 2.2.9]), we have

$$\|X_n(t)\|_{2(\delta_0 \wedge \zeta)}$$

$$\leq \|S(t) x_n\|_{2(\delta_0 \wedge \zeta)} + \int_0^t \|S(t-s) B_n(X_n(s))\|_{2(\delta_0 \wedge \zeta)} \,\mathrm{d}s + \left\| \int_0^t S(t-s) G_n \,\mathrm{d}W_n(s) \right\|_{2(\delta_0 \wedge \zeta)}$$

$$\leq \|x_n\|_{2\delta_0} + \int_0^t \frac{1}{(t-s)^{(\delta_0 \wedge \zeta) + \beta}} \|B_n(X_n(s))\|_{-2\beta} \,\mathrm{d}s + \left\| \int_0^t S(t-s) G_n \,\mathrm{d}W_n(s) \right\|_{2\zeta} \,.$$

Hence, taking supremum in time and expectations, by [DPZ14, Thm 5.15] we deduce, for all $T > 0$, that

$$\mathbb{E} \sup_{t \in [0,T]} \|X_n(t)\|_{2(\delta_0 \wedge \zeta)}$$

$$\leq \|x\|_{2\delta_0} + \sup_{y \in V_{2\alpha}} \|B(y)\|_{-2\beta} \frac{T^{1-(\delta_0 \wedge \zeta) - \beta}}{1 - (\delta_0 \wedge \zeta) - \beta} + C \,\mathbb{E} \left(\int_0^T t^{-2\xi} \|S(t) G\|_{\mathscr{L}^2(H,H)}^2 \,\mathrm{d}t \right)^{1/2} \,,$$

where we have used again that S contracts in $V_{2\delta_0}$ and $V_{2\zeta}$, that $(\delta_0 \wedge \zeta) + \beta \in [0, 1)$, and the analyticity of the semigroup S. This shows that there exists a constant $C_T > 0$ independent of n such that

$$\|X_n\|_{L^2(\Omega; C^0([0,T]; V_{2(\delta_0 \wedge \zeta)}))} \leq C_T \,, \quad \forall T > 0 \,. \tag{A.2}$$

Furthermore, deterministic maximal L^2-regularity (see [vNVW12, Thms. 3.3]) and the analyticity of the semigroup S (see [DPZ14, Thm 5.15]) ensure also that

$$\left\| \int_0^{\cdot} S(\cdot - s) B_n(X_n(s)) \,\mathrm{d}s \right\|_{L^2(\Omega; L^2(0,T; V_{2-2\beta}))} \leq C_T \,, \quad \forall T > 0$$

and

$$\left\| \int_0^{\cdot} S(\cdot - s) G_n \,\mathrm{d}W_n(s) \right\|_{L^2(\Omega; C^0([0,T]; V_{2\zeta}))} \leq C_T \,, \quad \forall T > 0 \,.$$

Moreover, since $\alpha < \xi$, we can fix $\zeta \in (\alpha, \xi)$: with such a choice, let $\bar{s} \in (\alpha, \zeta)$ and note that by the regularising properties of S one has

$$\|S(t)x_n\|_{2\bar{s}} \leq C \frac{1}{t^{\bar{s}-\delta_0}} \|x\|_{2\delta_0} \quad \forall t > 0.$$

Recalling that $\zeta < 1 - \beta$ by assumptions (H2)-(H3), as well as $\bar{s} - \delta_0 < 1$, we infer that there exists $q > 1$ such that

$$\|X_n\|_{L^2(\Omega;L^q(0,T;V_{2\bar{s}}))} \leq C_T, \quad \forall T > 0, \tag{A.3}$$

In particular, since the embedding $V_{2\bar{s}} \hookrightarrow V_{2\alpha}$ is compact, it follows that the laws of $(X_n)_n$ are tight on the space

$$L^q(0,T;V_{2\alpha}) \cap C^0([0,T];V_{2(\delta_0 \wedge \zeta)}).$$

Exploiting the fact that $B : V_{2\alpha} \to Z$ is strongly-weakly continuous, by letting $n \to \infty$ in (A.1) yields the existence of a weak solution to (1.1) by classical stochastic compactness techniques.

Case (ii). The proof is similar to the previous case, the only difference being the treatment of the stochastic convolution. Indeed, one can estimate the terms directly in the space $D(A^{\delta_0 \wedge \delta'})$ instead of $D(A^{\delta_0 \wedge \zeta})$ as

$$\mathbb{E} \sup_{t \in [0,T]} \|X_n(t)\|_{2(\delta_0 \wedge \delta')}$$

$$\leq \|x\|_{2\delta_0} + \sup_{y \in V_{2\alpha}} \|B(y)\|_{-2\beta} \frac{T^{1-(\delta_0 \wedge \delta')-\beta}}{1-(\delta_0 \wedge \delta') - \beta} + \mathbb{E}\left(\int_0^T \|G\|_{\mathscr{L}^2(H,V_{2\delta'})}^2 \, \mathrm{d}s\right)^{1/2},$$

where we have used that S contracts in $V_{2\delta_0}$ and $V_{2\delta'}$, maximal regularity, the Burkholder-Davis-Gundy inequality, and that $(\delta_0 \wedge \delta') + \beta \in [0,1)$. Moreover, by [DPZ14, Thm. 6.12, Prop. 6.18]) one has also that

$$\left\|\int_0^{\cdot} S(\cdot - s)G_n \, \mathrm{d}W_n(s)\right\|_{L^2(\Omega;L^2(0,T;V_{2\delta'+1}))} \leq C_T, \quad \forall T > 0.$$

Moreover, since $\alpha < (\delta_0 + 1) \wedge (\delta' + \frac{1}{2})$, we can fix $\bar{s} \in (\alpha, (\delta_0 + 1) \wedge (\delta' + \frac{1}{2}))$ and note that by the regularising properties of S one has

$$\|S(t)x_n\|_{2\bar{s}} \leq C \frac{1}{t^{\bar{s}-\delta_0}} \|x\|_{2\delta_0} \quad \forall t > 0.$$

Recalling that $2\bar{s} < 2\delta' + 1 < 2 - 2\beta$ by assumption, as well as $\bar{s} - \delta_0 < 1$, we infer that there exists $q > 1$ such that

$$\|X_n\|_{L^2(\Omega;L^q(0,T;V_{2\bar{s}}))} \leq C_T, \quad \forall T > 0, \tag{A.4}$$

In particular, since the embedding $V_{2\bar{s}} \hookrightarrow V_{2\alpha}$ is compact, it follows that the laws of $(X_n)_n$ are tight on the space

$$L^q(0,T;V_{2\alpha}) \cap C^0([0,T];V_{2(\delta_0 \wedge \delta')}).$$

The proof follows then as in the previous case. $\qquad\square$

Remark A.2. *If B is not bounded, one can obtain existence of local solutions by classical stopping arguments. Instead, existence of global solutions in the unbounded case requires appropriate conditions on B, according to the specific equation in consideration. Typically, these can be assumptions of monotonicity type or assumption on the growth: for example, one can suppose that there exists a constant $C_B > 0$ such that*

$$\|B(x)\|_{-2\beta} \leq C_B (1 + \|x\|_{2\alpha}) \quad \forall x \in D(A^{\alpha}).$$

In this framework, the existence proof of Case (i) can be adapted with the additional condition that $\alpha \leq \delta_0$: indeed, one can estimate

$$\int_0^t \frac{1}{(t-s)^{(\delta_0 \wedge \zeta)+\beta}} \left\| B_n(X_n(s)) \right\|_{-2\beta} \, \mathrm{d}s \leq C_B \int_0^t \frac{1}{(t-s)^{(\delta_0 \wedge \zeta)+\beta}} \left(1 + \|X_n(s)\|_{2\alpha}\right) \, \mathrm{d}s$$

and conclude by using the Gronwall's lemma together with the relation $\alpha \leq \delta_0 \wedge \zeta$. Analogously, the existence proof of Case (ii) can be adapted with the additional condition that $\alpha \leq \delta_0 \wedge \delta'$: indeed, one can write

$$\int_0^t \frac{1}{(t-s)^{(\delta_0 \wedge \delta')+\beta}} \left\| B_n(X_n(s)) \right\|_{-2\beta} \, \mathrm{d}s \leq C_B \int_0^t \frac{1}{(t-s)^{(\delta_0 \wedge \delta')+\beta}} \left(1 + \|X_n(s)\|_{2\alpha}\right) \, \mathrm{d}s$$

and conclude by using the Gronwall's lemma together with the relation $\alpha \leq \delta_0 \wedge \delta'$.

Acknowledgement

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