On Euler's Solution of the simple Difference Equation

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Abstract

In this note we will discuss Euler's solution of the simple difference equation that he gave in his paper *"De serierum determinatione seu nova methodus inveniendi terminos generales serierum"* [6] (E189: "On the determination of series or a new method of finding the general terms of series") and also present a derivation for the values of the Riemann ζ-function at positive integer numbers based on Euler's ideas.

1 Introduction

In his paper *"De serierum determinatione seu nova methodus inveniendi terminos generales serierum"* [6] (E189: "On the determination of series or a new method of finding the general terms of series"), Euler, amongst other difference equations, gave a general solution of the simple difference equation:

$$f(x+1) - f(x) = g(x). \tag{1}$$

He had found a solution to (1) in form of the Euler-Maclaurin summation formula before, e.g., in his paper *"Inventio summae cuiusque seriei ex dato termino generali"* [2] (E47: "Finding of a sum of a series from the given general term"). But whereas the Euler-Maclaurin summation formula is a particular solution and leads to an asymptotic series for most choices of $g(x)$, his solution offered in [6] is the complete solution to (1) and contains the Euler-Maclaurin summation formula as a special case.

Therefore, in this note we will present Euler's solution of (1) (see section 2), address a conceptual error in Euler's approach (see section 3) and we will show how to correct it (see section 3.2). Furthermore, we argue that Euler could have corrected his formula himself applying results that he discovered after he wrote [6] (see section 3.3). Finally, we will present a derivation of the formula for the values of the Riemann ζ-function at positive integer numbers based on the solution to the simple difference equation (see section 4).

2 Euler's Solution of the Simple Difference Equation

Euler's general idea was to transform (1) into a differential equation of infinite order with constant coefficients and apply the procedure he had formulated for the finite order case earlier in his paper *"Methodus aequationes differentiales altiorum graduum integrandi ulterius promota"* [5] (E188: "The method to integrate differential equations of higher degrees expanded further"). In that paper he outlined the following procedure:

Given the differential equation:

$$\left(a_0 + a_1\frac{d}{dx} + a_2\frac{d^2}{dx^2} + \cdots + a_n\frac{d^n}{dx^n}\right) f(x) = g(x), \tag{2}$$

with complex coefficients $a_1, a_2, \cdots, a_n$, Euler told us to first find the zeros with their multiplicity of the following expression:

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n.$$

Next, assume $z = k$ is a solution of $P(z) = 0$. Then, if k is a simple zero[a] of $P(z)$, the solution of (2) is given by the sum of all functions of the form:

$$f_k(x) = \frac{e^{kx}}{P'(k)} \int e^{-kx} g(x) dx. \tag{3}$$

Note that the indefinite integral introduces a constant of integration.

Let's apply this to (1) by transforming it into a differential equation first. By Taylor's theorem we have:

$$f(x+1) = f(x) + \frac{d}{dx}f(x) + \frac{1}{2}\frac{d^2}{dx^2}f(x) + \frac{1}{3!}\frac{d^3}{dx^3}f(x) + \cdots$$

such that (1) can be rewritten as

$$\left(\frac{d}{dx} + \frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{3!}\frac{d^3}{dx^3} + \cdots\right) f(x) = g(x). \tag{4}$$

Thus, according to Euler's approach we need to find all zeros and their multiplicities of the expression:

$$P(z) = \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots = e^z - 1. \tag{5}$$

[a] In this note, we will only need the case of simple zeros and hence will only state the corresponding formula. In [5], Euler stated all cases from order 1 to 4 explicitly.

The general zero of this equation is $z = \log(1)$. But having established that the complex logarithm is a multivalued function in his work *"De la controverse entre Mrs. Leibnitz et Bernoulli sur les logarithmes des nombres négatifs et imaginaires"* [4] (E168: "On the controversy between Leibnitz and Bernoulli on logarithms of negative and imaginary number"), Euler knew that (5) has infinitely many solutions, namely - aside from the trivial $z = 0$ - the solutions are

$$\pm 2\pi i, \pm 4\pi i, \pm 6\pi i, \pm 8\pi i, \cdots .$$

Therefore, the formula (3) applied to (4) and hence the solution to (1) gives:

$$f(x) = \int g(x)dx + e^{-2\pi ix}\int g(x)e^{2\pi ix}dx + e^{2\pi ix}\int g(x)e^{-2\pi ix}dx \tag{6}$$
$$+e^{-4\pi ix}\int g(x)e^{4\pi ix}dx + e^{4\pi ix}\int g(x)e^{-4\pi ix}dx + \cdots$$

This is the solution Euler gave in [6]. Unfortunately, it is not quite correct. We will discuss this in the following section.

3 Discussion of Euler's Solution

3.1 Example of linearly increasing Differences

Applying Euler's formula (6) to certain examples, we quickly discover that it does not give the correct results. For the purpose of illustration, let us take $g(x) = x$ such that we want to solve:

$$f(x+1) - f(x) = x. \tag{7}$$

The general solution to this equation is easily seen to be given as

$$f(x) = \frac{1}{2}x(x-1) + h(x), \tag{8}$$

where $h(x)$ satisfies $h(x+1) = h(x)$. Now let us apply (6). For this, we need to evaluate:

$$e^{-2k\pi ix}\int xe^{2k\pi ix}dx = \frac{1-2k\pi ix}{4\pi^2k^2} + C_k e^{-2k\pi ix},$$

where C_k is a constant of integration. For $k = 0$, we have $\int xdx = \frac{x^2}{2} + C_0$, where C_0 is the constant of integration. Inserting all this into (6), we find:

$$f(x) = \frac{x^2}{2} + C_0 + \sum_{k\in\mathbb{Z}\setminus\{0\}} \left(\frac{1}{4\pi^2k^2} - \frac{x}{2k\pi i}\right) + C_k e^{-2k\pi i x}.$$

Calling $C_0 + \sum_{k\in\mathbb{Z}\setminus\{0\}} C_k e^{-2k\pi i x} = h(x)$, we see that $h(x) = h(x+1)$. Furthermore,

$$\sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{1}{2k\pi i} = 0,$$

because all terms cancel. Finally,

$$\sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{1}{4\pi^2k^2} = \frac{2}{4\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{1}{2\pi^2}\cdot\frac{\pi^2}{6} = \frac{1}{12},$$

where we used the result $\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6}$ that Euler had discovered in his paper *"De summis serierum reciprocarum"* [1] (E41: "On the sums of series of reciprocals") in the last step.

Thus, Euler's formula (6) gives the following solution to (7):

$$g(x) = \frac{x^2}{2} + h(x),$$

where we absorbed the value $\frac{1}{12}$ in the formula in the periodic function. Comparing this result to (8), we see that the solution from Euler's formula is off by the term $-\frac{x}{2}$. In the next sections we will elaborate on why (6) is wrong and how to correct it.

3.2 Correction of Euler's Formula

Euler's formula (6) is actually almost correct. Indeed, the correct formula reads:

$$\begin{gathered} f(x) = -\frac{1}{2}g(x) + \int g(x)dx + e^{-2\pi i x}\int g(x)e^{2\pi i x}dx + e^{2\pi i x}\int g(x)e^{-2\pi i x}dx \\ e^{-4k\pi i x}\int g(x)e^{4\pi i x}dx + e^{4\pi i x}\int g(x)e^{-4\pi i x}dx + \cdots \end{gathered} \tag{9}$$

such that Euler's formula is off by just the term $-\frac{1}{2}g(x)$. As we mentioned in the introduction, Euler missed this term since the method of construction the solution to a differential equation from the zeros of the characteristic polynomial does not carry over smoothly from the finite to the infinite order case. Indeed, we have to construct the solution from the reciprocal of the characteristic polynomial, if we want the method to be applicable in the infinite order case. For, setting $z = \frac{d}{dx}$, we can rewrite (4) as:

$$f(x) = \frac{1}{P(z)} g(x) \tag{10}$$

with $P(z) = e^z - 1$. In order to apply the operator $\frac{1}{P(z)}$ to $g(x)$, we need to rewrite it in integer powers of z. There are many ways to achieve this task. The one we will need to prove (9) is the following partial fraction decomposition that can be proved, e.g., by using complex analysis[b]:

$$\frac{1}{e^z - 1} = -\frac{1}{2} + \sum_{k \in \mathbb{Z} \setminus \{0\}}^{\infty} \frac{1}{z - 2k\pi i}. \tag{11}$$

Thus, next we have to evaluate:

$$\frac{1}{z - 2k\pi i} g(x). \tag{12}$$

Writing $2k\pi i = \alpha$ we have:

$$\frac{1}{z - \alpha} g(x) = \frac{1}{z\left(1 - \frac{\alpha}{z}\right)} g(x) = \sum_{n=0}^{\infty} \frac{\alpha^n}{z^{n+1}} g(x).$$

Since $z = \dfrac{d}{dx}$, we can interpret $\frac{1}{z}$ as an integral and hence $\dfrac{1}{z^n}$ is an n-times iterated integral. Writing $\int^n$ for a n-times iterated integral, the following formula holds:

$$\int^n g(x) dx = \int^x \frac{(x-t)^{n-1}}{(n-1)!} g(t) dt. \tag{13}$$

Inserting this into (12), we have:

$$\frac{1}{z - \alpha} g(x) = \sum_{n=0}^{\infty} \alpha^n \int^x \frac{(x-t)^n}{n!} g(t) dt = \int^x e^{\alpha(x-t)} g(t) dt.$$

Therefore, by (11) our equation (10) reads:

[b] In section 3.3 we will present a proof that uses only method that were available to Euler.

$$f(x) = -\frac{1}{2}g(x) + \sum_{k\in\mathbb{Z}} \int^{x} e^{2k\pi i(x-t)} g(t)dt = -\frac{1}{2}g(x) + \sum_{k\in\mathbb{Z}} e^{2k\pi i x} \int^{x} e^{-2k\pi i t} g(t)dt,$$

which is (9). It is the same solution as in [8], which derived (9) using complex analysis.

3.3 Discussion

Although we operated on a purely formal basis in our derivation of (9), the procedure can be justified applying the Fourier transform which allows to consider (4) as an algebraic equation in the new variable, say, p. To find (12) we then need the inverse Fourier transform, which we can either calculate using complex analysis or look up in a table.

But Fourier analysis was not available to Euler, of course. Nevertheless, we argue that Euler could have given the proof we presented himself. The proof hinges essentially on the proof of (11). Later in his career, in his paper *"De resolutione fractionum transcendentium in infinitas fractiones simplices"* [7] (E592: "On the resolution of transcendental fractions into infinitely many simple fractions"), Euler indeed considered partial fraction decompositions of transcendental functions. The method outlined there would have given him the formula:

$$\frac{1}{e^z - 1} = R(z) + \sum_{k\in\mathbb{Z}} \frac{1}{z - 2k\pi i}, \tag{14}$$

where $R(z)$ is a function to be determined. Next, one could expand the sum into a Laurent series around $z = 0$ by expanding each geometric series and compare it to the Laurent series obtained by direct expansion. The direct expansion reads:

$$\frac{1}{e^z - 1} = -\frac{1}{2} + \frac{1}{z} + \sum_{k=0}^{\infty} B_n \frac{z^n}{n!}, \tag{15}$$

where B_n are the Bernoulli numbers. Since Euler considered a similar function, namely $\frac{z}{1-e^{-z}}$, and its series expansion around $z = 0$ in his work *"De seriebus quibusdam considerationes"* [3] (E130: "Considerations about certain series"), the previous formula could definitely also been found by him. Finally, comparing the Laurent series obtained from (14) to (15), we can infer that $R(z) = -\frac{1}{2}$.

4 An Application of the Solution to the simple Difference Equation

In this section, we want to consider the choice $g(x) = x^n$ for $n \in \mathbb{N}$ in (1), since it is one of the few cases in which (9) can be evaluated explicitly. As it will turn out, we will be led to the values $\zeta(2n)$, i.e., the sums

$$\zeta(2n) := \sum_{k=1}^{\infty} \frac{1}{k^{2n}} \tag{16}$$

in the process. Euler evaluated these sums on many occasions using a large number of different methods. We mention his papers [1] and [3] as examples, but the way we will arrive at those values seems to be different from all methods used by Euler.

4.1 Preparation

Considering (9) we need to evaluate the expression: $e^{ax}\int e^{-ax}x^n dx$. This can be done as follows: First, we note that

$$\int e^{-ax}dx = -\frac{e^{ax}}{a} = -e^{ax}\cdot a^{-1}, \tag{17}$$

where we omitted the constant of integration, since it will not be necessary in the following. Next, we differentiate (17) with respect to a exactly n times. The left-hand side gives:

$$\frac{d^n}{da^n}\int e^{-ax}dx = \int \frac{d^n}{da^n}e^{-ax}dx = (-1)^n \int e^{-ax}x^n dx,$$

whereas the right-hand side gives:

$$\begin{aligned}
-\frac{d^n}{da^n}e^{-ax}\cdot a^{-1} &= -\sum_{k=0}^{n}\binom{n}{k}\frac{d^k}{da^k}e^{-ax}\cdot\frac{d^{n-k}}{da^{n-k}}a^{-1} \\
&= -e^{-ax}\cdot\frac{(-1)^n}{a^{n+1}}\sum_{k=0}^{n}\frac{n!}{k!}(ax)^k,
\end{aligned}$$

where we used Leibniz' rule for the differentiation of products in the first step. Thus, combining both results we arrive at:

$$e^{ax}\int e^{-ax}x^n dx = -\frac{1}{a^{n+1}}\sum_{k=0}^{n}\frac{n!}{k!}a^k x^k.$$

Inserting this into (9) for the special case $g(x) = x^n$ we get:

$$f(x) = \frac{x^{n+1}}{n+1} - \frac{x^n}{2} - \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{1}{(2k\pi i)^{n+1}} \cdot \sum_{j=0}^{n} \frac{n!}{j!}(2k\pi i)^j x^j + h(x), \quad (18)$$

where $h(x)$ satisfies $h(x+1) = h(x)$.

4.2 The Application

(18) is the general solution to (1) for the particular choice $g(x) = x^n$. But we can also easily find a particular solution to (1) by noting that for integer x:

$$f(x) = \sum_{k=1}^{x-1} g(k) = \sum_{k=1}^{x} g(k) - g(x)$$

satisfies the equation. Therefore, for the particular choice $g(x) = x^n$ we also have the solution:

$$\begin{aligned} f(x) = \sum_{k=1}^{x-1} k^n &= \sum_{k=1}^{x} k^n - x^n = \frac{x^{n+1}}{n+1} + \frac{x^n}{2} + \frac{1}{n+1} \sum_{j=2}^{n} \binom{n+1}{j} B_j x^{n+1-j} - x^n \\ &= \frac{x^{n+1}}{n+1} - \frac{x^n}{2} + \frac{1}{n+1} \sum_{j=2}^{n} \binom{n+1}{j} B_j x^{n+1-j}, \end{aligned} \quad (19)$$

where we used Faulhaber's formula for the sums of integer powers and B_n is the n-th Bernoulli number as above. (19) is a polynomial in x and hence x is not restricted to integer values in this form. Let us transform (18) into a similar form. Ignoring the periodic function we have:

$$f(x) = \frac{x^{n+1}}{n+1} - \frac{x^n}{2} - \sum_{j=0}^{n} \frac{n!}{j!} \sum_{k\in\mathbb{Z}\setminus\{0\}} (2k\pi i)^{j-(n+1)} x^j. \quad (20)$$

Since (19) and (20) differ only by a periodic function, we can compare coefficients of respective powers of x. Let us call

$$B(n,j) = \frac{1}{n+1}\binom{n+1}{j} B_j \quad \text{for} \quad j \geq 2, \quad (21)$$

for all other values of j we set $B(j,n) = 0$; furthermore, we set

$$A(n,j) = -\frac{n!}{j!} \sum_{k\in\mathbb{Z}\setminus\{0\}} (2k\pi i)^{j-(n+1)}. \tag{22}$$

Then, comparing coefficients from (19) and (20) gives:

$$A(n, n+1-j) = B(n,j).$$

Thus, substituting the values form (22) and (21), respectively:

$$-\frac{n!}{(n+1-j)!} \sum_{k\in\mathbb{Z}\setminus\{0\}} (2k\pi i)^{-j} = \frac{1}{n+1}\binom{n+1}{j} B_j.$$

Finally, solving for the sum:

$$\sum_{k\in\mathbb{Z}\setminus\{0\}} (2k\pi i)^{-j} = -\frac{(n+1-j)!}{n!} \cdot \frac{1}{n+1} \cdot \frac{(n+1)!}{j!(n+1-j)!} B_j = -\frac{B_j}{j!}.$$

Thus,

$$\sum_{k\in\mathbb{Z}\setminus\{0\}} k^{-j} = -(2\pi i)^j \frac{B_j}{j!}.$$

But due canceling terms, the sum vanishes for odd j such that we arrive at:

$$\zeta(2j) = \sum_{k=1}^{\infty} \frac{1}{k^{2j}} = \frac{(-1)^{j-1}(2\pi)^{2j} B_{2j}}{2(2j)!}. \tag{23}$$

This is Euler's famous formula for the even values of the ζ-function that he gave, e.g., in [3].

5 Conclusion

In this note we considered Euler's general solution to the simple difference equation (1) that he gave in [6]. His final formula (6) is slightly incorrect due to unjustified application of his solution (3) to differential equations of infinite order. Nevertheless, we discussed how to fix Euler's derivation (see section 3.2) and also argued that Euler could have done so himself, if he just reconsidered the same subject later in his career (see section 3.3). Furthermore, we used the correct solution (9) to (1) to give a proof of Euler's famous formula for the values of the Riemann ζ-function at even positive integers (see section 4). The method of derivation of (23) that we presented seems to be not to have been used by Euler in any of his other papers. Finally, we mention that our approach allowed to derive the exact values of (16) from the corresponding *finite* sums of natural powers (19). Although this is clear, since (19) and (23) are connected via the Bernoulli numbers – as Euler also pointed out, e.g., in [3] –, the deeper explanation for this connection is provided by (9).

Despite the minor mistake, [6] is an interesting paper and contains subjects and approaches that are not found in any other of Euler's papers.

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