Uniqueness of steady states of Gorini-Kossakowski-Sudarshan-Lindblad equations: a simple proof

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We present a simple proof of a sufficient condition for the uniqueness of non-equilibrium steady states of Gorini-Kossakowski-Sudarshan-Lindblad equations. We demonstrate the applications of the sufficient condition using examples of the transverse-field Ising model, the XYZ model, and the tight-binding model with dephasing.

I. INTRODUCTION

Recent advances in quantum engineering have brought renewed interest in the effect of dissipation on quantum many-body systems. Under the Markov approximation, the dynamics of an open quantum system is described by the Gorini-Kossakowski-Sudarshan-Lindblad (GKSL) equation [1–3]. Throughout this paper, we consider quantum systems described by a d-dimensional Hilbert space $\mathcal{H}$. By writing the set of linear operators on $\mathcal{H}$ as $\mathcal{B}(\mathcal{H})$, a state of the system is described by a density operator $\rho \in \mathcal{B}(\mathcal{H})$ that is Hermitian, positive semidefinite, and $\mathrm{Tr}\,\rho = 1$. Then, the GKSL equation reads

$$\frac{d\rho}{d\tau} = \hat{\mathcal{L}}(\rho) = -i[H,\rho] + \sum_{m=1}^{M} \left(L_m \rho L_m^\dagger - \frac{1}{2}\left\{ L_m^\dagger L_m, \rho \right\} \right). \tag{1}$$

Here, τ is the time, H is the Hamiltonian, and L_m ($m = 1, \ldots, M$) are the Lindblad operators that act on $\mathcal{H}$.

We write the eigenvalues of $\hat{\mathcal{L}}$ as Γ_j and corresponding eigenmodes as ρ_j. Then, $\mathrm{Re}[\Gamma_j] \leq 0$ for all Γ_j. A non-equilibrium steady state (NESS) ρ_∞ is a density operator that is an eigenoperator of $\hat{\mathcal{L}}$ with eigenvalue 0. There always exists at least one NESS in a finite-dimensional system. However, whether the NESS is unique or not depends on the system. Frigerio [4, 5] gave an algebraic criterion for the uniqueness of the NESS with the assumption that there exists a positive definite (or full-rank) NESS. See Appendix A for the details of the result. See also related results by Spohn [6, 7] and Evans [8], Ref. [9, 10] for a review of these works, Ref. [11] for an application, and Ref. [12–20] for recent progress in understanding the degeneracy of the NESSs.

In this paper, we provide a proof of a sufficient condition for the uniqueness of NESS. Compared with Frigerio's theorem, our theorem does not require any prior information about the NESS. While a sufficient condition for general infinite-dimensional systems is presented in Ref. [21], the proof provided there requires knowledge of von Neumann algebras. In contrast, our paper focuses solely on finite-dimensional systems. The significant advantage of such a limitation is that our proof for the sufficient condition is much more concise compared to Ref. [21], and readers are only expected to possess an elementary knowledge of linear algebra to comprehend the proof. Next, we see that the sufficient condition can also be used to study the steady-state degeneracy of systems with strong symmetries. In the presence of a strong symmetry, there is at least one NESS in every symmetry sector [15, 19]. We give a sufficient condition for the uniqueness of the NESS in every symmetry sector. Finally, we demonstrate the applications of the sufficient condition using examples of the transverse-field Ising model, the XYZ model, and the tight-binding model with dephasing.

II. MAIN THEOREM

Theorem 1. *If the set of operators* $\left\{ H - \frac{i}{2}\sum_{m=1}^{M} L_m^\dagger L_m, L_1, \ldots, L_M \right\}$ *generates all the operators under multiplication, addition, and scalar multiplication, then* ρ_∞ *is unique and positive definite.*

To prove Theorem 1, we prove the following lemma.

Lemma 2. *Let* ρ *be a positive semidefinite operator that satisfies* $\hat{\mathcal{L}}(\rho) = 0$. *Under the same conditions as Theorem 1,* ρ *is positive definite or zero.*

The following proofs of Theorem 1 and Lemma 2 are inspired by the method of spin reflection positivity [22, 23].

Proof of Lemma 2. Assume that ρ is positive semidefinite but not positive definite. Then, there exists a nonzero vector $|\psi\rangle \in \mathcal{H}$ such that $\rho\,|\psi\rangle = 0$. By expanding $\langle\psi|\,\hat{\mathcal{L}}(\rho)\,|\psi\rangle$, one finds

$$\langle\psi|\,\hat{\mathcal{L}}(\rho)\,|\psi\rangle = \sum_{m=1}^{M} \langle\psi|\, L_m \rho L_m^\dagger\,|\psi\rangle = \sum_{m=1}^{M} \|\sqrt{\rho} L_m^\dagger\,|\psi\rangle\,\|^2 = 0, \tag{2}$$

where $\sqrt{\rho}$ is a positive semidefinite operator such that $(\sqrt{\rho})^2 = \rho$. Since $\|\sqrt{\rho} L_m^\dagger\,|\psi\rangle\,\|^2 \geq 0$, we have $\sqrt{\rho} L_m^\dagger\,|\psi\rangle = 0$ for all m, which means that $\rho L_m^\dagger\,|\psi\rangle = 0$ for all m. Next, by expanding $\hat{\mathcal{L}}(\rho)\,|\psi\rangle$, one obtains

$$\hat{\mathcal{L}}(\rho)\,|\psi\rangle = i\rho \left(H + \frac{i}{2}\sum_{m=1}^{M} L_m^\dagger L_m \right) |\psi\rangle = 0. \tag{3}$$

Therefore, if $|\psi\rangle \in \operatorname{Ker}\rho$, then $L_m^\dagger|\psi\rangle \in \operatorname{Ker}\rho$ for all m and $\left(H+\frac{i}{2}\sum_{m=1}^M L_m^\dagger L_m\right)|\psi\rangle \in \operatorname{Ker}\rho$. By the assumption of Lemma 2, the set of operators $\left\{H+\frac{i}{2}\sum_{m=1}^M L_m^\dagger L_m,\ L_1^\dagger,\dots,\ L_M^\dagger\right\}$ generates $\mathcal{B}(\mathcal{H})$, and therefore $\operatorname{Ker}\rho = \mathcal{H}$, which means that $\rho = 0$. □

Proof of Theorem 1. Assume that ρ_1 and ρ_2 ($\rho_1 \neq \rho_2$) are NESSs. Since they are density operators, they are Hermitian, positive semidefinite, and $\operatorname{Tr}\rho_j = 1$ ($j = 1, 2$). Thus, by Lemma 2, they are positive definite. If we define [24]

$$\rho_{\mathrm{un}}(x) = (1-x)\rho_1 - x\rho_2 \quad (0 \leq x \leq 1), \tag{4}$$

$\rho_{\mathrm{un}}(0) = \rho_1$, $\rho_{\mathrm{un}}(1) = -\rho_2$, and $\hat{\mathcal{L}}(\rho_{\mathrm{un}}(x)) = (1-x)\hat{\mathcal{L}}(\rho_1) - x\hat{\mathcal{L}}(\rho_2) = 0$ for all x because $\hat{\mathcal{L}}(\rho_1) = \hat{\mathcal{L}}(\rho_2) = 0$ by definition. Since all the eigenvalues of $\rho_{\mathrm{un}}(0)$ [$\rho_{\mathrm{un}}(1)$] are positive [negative] and the spectrum of $\rho_{\mathrm{un}}(x)$ is continuous with respect to x, there exists a real number $0 \leq x_0 \leq 1$ such that the minimum eigenvalue of $\rho_{\mathrm{un}}(x_0)$ is zero. Namely, $\rho_{\mathrm{un}}(x_0)$ is positive semidefinite but not positive definite. Thus by Lemma 2, $\rho_{\mathrm{un}}(x_0) = 0$. Then $\operatorname{Tr}\rho_{\mathrm{un}}(x_0) = 1 - 2x_0 = 0$ and therefore $x_0 = 1/2$. Thus $\rho_{\mathrm{un}}(x_0) = (\rho_1 - \rho_2)/2 = 0$. However, this contradicts the assumption $\rho_1 \neq \rho_2$, so the NESS has to be unique. □

When all the Lindblad operators are Hermitian, the completely mixed state $\mathbb{I}_d/d$ is a NESS [25], where $\mathbb{I}_d$ is the identity matrix of size d. In this case, Theorem 1 boils down to the following corollary.

Corollary 3. *If all L_m are Hermitian and the set of operators $\{H, L_1, \dots, L_M\}$ generates all the operators under multiplication, addition, and scalar multiplication, then ρ_∞ is unique and $\rho_\infty = \mathbb{I}_d/d$.*

III. STRONG SYMMETRY

Next, we consider systems with the strong symmetry [15, 17, 19].

Definition 4 (Strong symmetry)**.** The GKSL equation has a strong symmetry if there exists a unitary operator S on $\mathcal{H}$ such that

$$[S, H] = 0 \text{ and } [S, L_m] = 0 \text{ for all } m. \tag{5}$$

We write n_S different eigenvalues of S as $s_\alpha = e^{i\theta_\alpha}$ ($\alpha = 1, \dots, n_S$), and corresponding eigenspace as $\mathcal{H}_\alpha$. Then, the following theorem is proved in Ref. [15].

Theorem 5 (Buča and Prosen)**.** *If there is a unitary operator S that satisfies Eq.* (5)*, then we obtain the following 1. and 2.:*

1. *The space of operators $\mathcal{B}(\mathcal{H})$ can be decomposed into n_S^2 invariant subspaces of $\hat{\mathcal{L}}$:*

$$\hat{\mathcal{L}}(\mathcal{B}_{\alpha,\beta}) \subseteq \mathcal{B}_{\alpha,\beta},\ \mathcal{B}_{\alpha,\beta} = \{|\psi\rangle\langle\phi|\,;|\psi\rangle \in \mathcal{H}_\alpha, |\phi\rangle \in \mathcal{H}_\beta\} \tag{6}$$

for $\alpha, \beta = 1, \dots, n_S$.

2. *Every $\mathcal{B}_{\alpha,\alpha}$ contains at least one NESS:*

$$\rho_\infty^\alpha \in \mathcal{B}_{\alpha,\alpha} \text{ for } \alpha = 1, \dots, n_S. \tag{7}$$

This theorem states that in the presence of strong symmetry, NESSs are always degenerate. However, we can apply Theorem 1 to prove the uniqueness of the NESS in $\mathcal{B}_{\alpha,\alpha}$. When H and L_m commute with S, they can be decomposed as $H = \oplus_{\alpha=1}^{n_S} H|_{\mathcal{H}_\alpha}$ and $L_m = \oplus_{\alpha=1}^{n_S} L_m|_{\mathcal{H}_\alpha}$, where $H|_{\mathcal{H}_\alpha}$ and $L_m|_{\mathcal{H}_\alpha}$ are elements of $\mathcal{B}_{\alpha,\alpha}$. Then, if $\{H - \frac{i}{2}\sum_{m=1}^M L_m^\dagger L_m, L_1, \dots, L_M\}$ generates all the operators that commute with S, the set of operators $\{H - \frac{i}{2}\sum_{m=1}^M L_m^\dagger L_m|_{\mathcal{H}_\alpha}, L_1|_{\mathcal{H}_\alpha}, \dots, L_M|_{\mathcal{H}_\alpha}\}$ generates $\mathcal{B}_{\alpha,\alpha}$ for all α [26]. By applying Theorem 1 to $\mathcal{H}_\alpha$ and writing $\dim\mathcal{H}_\alpha = d_\alpha$, we have the following corollaries:

Corollary 6. *If the set of operators $\left\{H - \frac{i}{2}\sum_m L_m^\dagger L_m, L_1, \dots, L_M\right\}$ generates all the operators that commute with S under multiplication, addition, and scalar multiplication, then $\rho_\infty^\alpha|_{\mathcal{H}_\alpha}$ is unique and positive definite for all α.*

Corollary 7. *If all L_m are Hermitian and the set of operators $\{H, L_1, \dots, L_M\}$ generates all the operators that commute with S under multiplication, addition, and scalar multiplication, then $\rho_\infty^\alpha|_{\mathcal{H}_\alpha}$ is unique and $\rho_\infty^\alpha|_{\mathcal{H}_\alpha} = \mathbb{I}_{d_\alpha}/d_\alpha$ for all α.*

IV. EXAMPLES

In this section, we demonstrate the applications of Theorem 1, Corollaries 3 and 7. As the simplest example, we consider the two-level system with gain and loss. Next, we present an application of Corollary 3 to the transverse-field Ising model with boundary dephasing. Finally, we present applications of Corollary 7 to the XYZ model and the tight-binding model with bulk dephasing, as prototypical examples of models with $\mathbb{Z}_2$ and U(1) strong symmetries. We also note that Theorem 1 can be applied to the boundary-driven open XXZ chain [11, 27–29].

A. Two-level system with gain and loss

As our first example, we consider a two-level system with gain and loss. We write an orthonormal basis of $\mathcal{H} = \mathbb{C}^2$ as $|\uparrow\rangle$ and $|\downarrow\rangle$. The Lindblad operators of gain and loss are $L_g = \sqrt{\gamma_g}\,|\uparrow\rangle\langle\downarrow|$ and $L_l = \sqrt{\gamma_l}\,|\downarrow\rangle\langle\uparrow|$. Then, $L_g L_l \propto |\uparrow\rangle\langle\uparrow|$ and $L_l L_g \propto |\downarrow\rangle\langle\downarrow|$, and therefore L_g, L_l, $L_g L_l$ and $L_l L_g$ form the basis of $\mathcal{B}(\mathbb{C}^2)$. From Theorem 1, the NESS ρ_∞ is unique and positive definite for an arbitrary Hamiltonian.

B. Transverse-field Ising model with boundary dephasing

Next, we consider the spin-1/2 transverse-field Ising chain under open boundary conditions [30]

$$H=\sum_{j=1}^{N-1}\sigma_j^z\sigma_{j+1}^z+h_x\sum_{j=1}^{N}\sigma_j^x \tag{8}$$

with dephasing noise $L_1=\sqrt{\gamma}\sigma_1^z$ at the first site of the lattice. Here, σ_j^α $(\alpha=x,y,z)$ are the Pauli operators at site $j=1,\ldots,N$ acting on $d=2^N$ dimensional Hilbert space $\mathcal{H}$, $h_x\neq 0$ is the external magnetic field, $\gamma>0$ is the dissipation strength parameter.

By using Corollary 3, we can prove that the NESS ρ_∞ is unique and written as $\rho_\infty=\mathbb{I}_{2^N}/2^N$.

Proof. We first note that $L_1\propto\sigma_1^z$, $[\sigma_1^z,H]\propto\sigma_1^y$, and then $\frac{1}{2}\sigma_1^y[\sigma_1^y\sigma_1^z,H]=\sigma_2^z$, $[\sigma_2^z,H]\propto\sigma_2^y$. Next, we observe the recurrence relation

$$\frac{1}{2}\sigma_j^y[\sigma_j^y\sigma_j^z,H]-\sigma_{j-1}^z=\sigma_{j+1}^z, \tag{9}$$

$$[\sigma_{j+1}^z,H]\propto\sigma_{j+1}^y \tag{10}$$

for $j=2,\ldots,N-1$, which generates σ_j^y and σ_j^z for all j from L_1 and H. Then, it is clear that they generate all the operators in $\mathcal{B}(\mathcal{H})$, and therefore the NESS is unique and written as $\rho_\infty=\mathbb{I}_{2^N}/2^N$ from Corollary 3. □

C. XYZ model with bulk dephasing

As a prototypical example of models with the $\mathbb{Z}_2$ strong symmetry, we consider the spin-1/2 XYZ chain under periodic boundary conditions [30, 31]

$$H=\sum_{j=1}^{N}(J_x\sigma_j^x\sigma_{j+1}^x+J_y\sigma_j^y\sigma_{j+1}^y+J_z\sigma_j^z\sigma_{j+1}^z)+h_z\sum_{j=1}^{N}\sigma_j^z \tag{11}$$

with dephasing strength $L_j=\sqrt{\gamma}\sigma_j^z$ at every site j. Here, $J_\alpha\in\mathbb{R}$ $(\alpha=x,y,z)$ are the exchange couplings, $h_z\in\mathbb{R}$ is the external magnetic field, $\gamma>0$ is the dissipation strength parameter, and N is the number of sites. We assume that $|J_x|\neq|J_y|$ [32]. By defining a unitary operator $S=\prod_{j=1}^N\sigma_j^z$, one finds

$$[S,H]=0 \text{ and } [S,L_j]=0 \text{ for all } j. \tag{12}$$

The eigenvalues of S are ±1, and we write the corresponding subspace of operators as $\mathcal{B}_{\alpha,\beta}$ $(\alpha,\beta=\pm)$. If we define $\rho^\pm$ as $\rho^\pm:=(\mathbb{I}_{2^N}\pm S)/2^N$, it can be checked that $\rho^+\in\mathcal{B}_{+,+}$, $\rho^-\in\mathcal{B}_{-,-}$, and $\hat{\mathcal{L}}(\rho^\pm)=0$. By using Corollary 7, we prove that they are the unique NESSs in $\mathcal{B}_{+,+}$ and $\mathcal{B}_{-,-}$, respectively.

Proof. First, we identify all the operators that commute with S. From the relations

$$S\sigma_j^x=-\sigma_j^xS,\ S\sigma_j^y=-\sigma_j^yS,\ S\sigma_j^z=\sigma_j^zS, \tag{13}$$

all the operators that commute with S are spanned by products of an even number of σ_j^x and σ_j^y. Thus it is sufficient to prove that $\sigma_j^\mu\sigma_k^\nu$ $(\mu,\nu=x,y)$ can be generated by H and L_j for all $1\leq j\leq k\leq N$. When $j=k$, it can be generated only by L_j, because $\sigma_j^x\sigma_j^y=-\sigma_j^y\sigma_j^x\propto L_j$ and $(\sigma_j^x)^2=(\sigma_j^y)^2\propto(L_j)^2$. Next, we consider the cases where $j\neq k$. First, one finds

$$A_1:=[\sigma_l^z,H]\propto\sum_{\sigma=\pm1}(J_x\sigma_l^y\sigma_{l+\sigma}^x-J_y\sigma_l^x\sigma_{l+\sigma}^y), \tag{14}$$

$$A_2:=[\sigma_{l+1}^z,A_1]\propto(J_x\sigma_l^y\sigma_{l+1}^y+J_y\sigma_l^x\sigma_{l+1}^x), \tag{15}$$

$$A_3:=[\sigma_l^z,A_2]\propto(J_x\sigma_l^x\sigma_{l+1}^y-J_y\sigma_l^y\sigma_{l+1}^x), \tag{16}$$

$$A_4:=[\sigma_{l+1}^z,A_3]\propto(J_x\sigma_l^x\sigma_{l+1}^x+J_y\sigma_l^y\sigma_{l+1}^y), \tag{17}$$

$$A_5:=[\sigma_l^z,A_4]\propto(J_x\sigma_l^y\sigma_{l+1}^x-J_y\sigma_l^x\sigma_{l+1}^y). \tag{18}$$

Noting that $|J_x|\neq|J_y|$, we obtain $\sigma_l^x\sigma_{l+1}^x$ and $\sigma_l^y\sigma_{l+1}^y$ by linear combinations of A_2 and A_4 and $\sigma_l^x\sigma_{l+1}^y$ and $\sigma_l^y\sigma_{l+1}^x$ by linear combinations of A_3 and A_5. Finally, since

$$\sigma_j^\mu\sigma_k^\nu=\sigma_j^\mu\sigma_{j+1}^x\left(\prod_{l=j+1}^{k-2}\sigma_l^x\sigma_{l+1}^x\right)\sigma_{k-1}^x\sigma_k^\nu, \tag{19}$$

we have $\sigma_j^\mu\sigma_k^\nu$ $(\mu,\nu=x,y)$ for all $1\leq j\leq k\leq N$, and thus we have all possible products of an even number of σ_j^x and σ_j^y $(j=1,2,\ldots,N)$. Therefore, from Corollary 7, the NESS is unique in $\mathcal{B}_{+,+}$ and $\mathcal{B}_{-,-}$, respectively. □

Remark 8. For simplicity, we assumed that the Hamiltonian is one-dimensional and translationally invariant. However, these assumptions are not necessary. To illustrate this, we write the set of sites as Λ and the set of bonds as B, and consider the following Hamiltonian on a general lattice (Λ,B):

$$H=\sum_{j,k\in\Lambda}(J_{j,k}^x\sigma_j^x\sigma_k^x+J_{j,k}^y\sigma_j^y\sigma_k^y+J_{j,k}^z\sigma_j^z\sigma_k^z)+\sum_{j\in\Lambda}h_j^z\sigma_j^z \tag{20}$$

with dephasing noise $L_j=\sqrt{\gamma_j}\sigma_j^z$, where $\gamma_j>0$ for all $j\in\Lambda$. We assume that $|J_{j,k}^x|\neq|J_{j,k}^y|$ when $(j,k)\in B$ and $J_{j,k}^x=J_{j,k}^y=0$ when $(j,k)\notin B$. If the lattice (Λ,B) is connected [33], one can prove that the NESS is unique in $\mathcal{B}_{+,+}$ and $\mathcal{B}_{-,-}$, respectively. For example, equation (20) includes the one-dimensional quantum compass model

$$H=-\sum_{j=1}^{N/2}J_x\sigma_{2j-1}^x\sigma_{2j}^x-\sum_{j=1}^{N/2-1}J_y\sigma_{2j}^y\sigma_{2j+1}^y \tag{21}$$

with dephasing noise $L_j=\sqrt{\gamma}\sigma_j^z$ discussed in Ref. [34].

D. Tight-binding model with bulk dephasing

Finally, we consider the tight-binding chain under the periodic boundary conditions [35]

$$H = t\sum_{j=1}^{N}(c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j) + \delta\mathbb{I}_{2^N} \tag{22}$$

with dephasing noise $L_j = \sqrt{\gamma}n_j$ at every site j. Here, $c_j^\dagger$ and c_j are the creation and annihilation operators, respectively, of a fermion at site $j = 1,\ldots,N$, $n_j = c_j^\dagger c_j$ is the number operator, $t \neq 0$ is the hopping amplitude, $\gamma > 0$ is the dephasing strength, and δ is a real constant. The eigenvalues and eigenmodes of $\hat{\mathcal{L}}$ do not depend on the constant δ, but we assume that $\delta \neq 0$ to simplify the proof. If we write the vacuum state annihilated by all c_j as $|0\rangle$, then the Hilbert space $\mathcal{H}$ is spanned by states of the form $\{\prod_{j=1}^{L}(c_j^\dagger)^{m_j}\}\,|0\rangle$ $(m_j = 0,1)$.

Next, we write the total number operator as $N_{\mathrm{tot}} = \sum_{j=1}^{N} n_j$ and define a unitary operator $S = e^{iN_{\mathrm{tot}}}$. Then, the eigenvalues of S are $e^{i\alpha}$ $(\alpha = 0,1,\ldots,N)$. Since S commutes with H and all L_j, $\mathcal{B}(\mathcal{H})$ can be decomposed into invariant subspaces of $\hat{\mathcal{L}}$ and we write them as $\mathcal{B}_{\alpha,\beta}$ $(\alpha,\beta = 0,1,\ldots,N)$. Then, we prove that the NESS is unique in every $\mathcal{B}_{\alpha,\alpha}$.

Proof. Since all the operators that commute with S are written as a sum of monomials that are products of the same number of creation and annihilation operators, it is sufficient to prove that $\mathbb{I}_{2^N}$ and $c_j^\dagger c_k$ can be generated by H and L_j for all $1 \leq j,k \leq N$. When $j = k$, $c_j^\dagger c_j$ is proportional to L_j, so we concentrate on the case $j \neq k$. Without loss of generality, we can assume that $j < k$. First, we see that the following commutation relations hold:

$$[n_l, H] = t\sum_{\sigma=\pm1}(c_l^\dagger c_{l+\sigma} - c_{l+\sigma}^\dagger c_l), \tag{23}$$

$$[n_{l+1},[n_l,H]] = -t(c_l^\dagger c_{l+1} + c_{l+1}^\dagger c_l), \tag{24}$$

$$[n_l,[n_{l+1},[n_l,H]]] = -t(c_l^\dagger c_{l+1} - c_{l+1}^\dagger c_l). \tag{25}$$

Therefore, we obtain $c_l^\dagger c_{l+1}$ and $c_{l+1}^\dagger c_l$ from multiplication, addition, and scalar multiplication of H, L_l, and L_{l+1}. Since $[c_l^\dagger c_m, c_m^\dagger c_n] = c_l^\dagger c_n$ when $l \neq n$, we can generate $c_j^\dagger c_k$ for any $1 \leq j < k \leq N$ with $c_j^\dagger c_{j+1}$, $\ldots$, $c_{k-1}^\dagger c_k$. Finally, $\mathbb{I}_{2^N}$ can be obtained by $[H - t\sum_{j=1}^{N}(c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j)]/\delta$. Therefore, from Corollary 7, the NESS is unique in every $\mathcal{B}_{\alpha,\alpha}$. ☐

Remark 9. The result can be generalized to the tight-binding model on a general lattice (Λ, B) with N sites:

$$H = \sum_{j,k\in\Lambda} t_{j,k}c_j^\dagger c_k + \delta\mathbb{I}_{2^N}, \quad L_j = \sqrt{\gamma_j}n_j, \tag{26}$$

where H is Hermitian, i.e., $t_{j,k} = t_{k,j}^*$, $\gamma_j > 0$ for all $j \in \Lambda$, and δ is a real constant. We also assume that $t_{j,k} \neq 0$ when $(j,k) \in B$ and $t_{j,k} = 0$ when $(j,k) \notin B$. When the lattice (Λ, B) is connected, one can prove that the NESS is unique in every $\mathcal{B}_{\alpha,\alpha}$.

V. CONCLUSION

We presented a simple proof of a sufficient condition for the uniqueness of NESSs of GKSL equations. We also presented applications of the sufficient condition to the transverse-field Ising model, the XYZ model, and the tight-binding model with dephasing. Our results here open many interesting questions. The most important direction for future study is to generalize our proof to the sufficient and necessary condition for the uniqueness of the NESS. Another direction is to apply the sufficient condition to clarify the degeneracy of the NESS in the presence of the non-abelian strong symmetries [19] or the hidden strong symmetries in the form of quasi-local charges [36].

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Appendix A: Frigerio's theorem

In this section, we briefly review Frigerio's theorem on the uniqueness of the NESS. While his result applies to general infinite-dimensional systems, here we state the theorem in the d-dimensional case. For a set of operators $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$, let us denote by $\mathcal{A}'$ the commutant of the set $\mathcal{A}$, i.e., the set of operators that commute with all the elements of $\mathcal{A}$. Frigerio [4, 5] proved that if there exists a positive definite NESS ρ_∞, then ρ_∞ is the unique NESS iff $\left\{H, L_1,\ldots,L_M, L_1^\dagger,\ldots,L_M^\dagger\right\}' = \{c\mathbb{I}_d | c \in \mathbb{C}\}$, where $\mathbb{I}_d$ is the identity matrix of size d. This condition is equivalent to the condition that the set of operators $\left\{H, L_1,\ldots,L_M, L_1^\dagger,\ldots,L_M^\dagger\right\}$ generates all the operators under multiplication, addition, and scalar multiplication. Note that the assumption of the existence of a positive definite NESS is necessary, and without this assumption, several counterexamples can be found [37].

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[25] More generally, if there exists an $M \times M$ unitary matrix u such that $L_m^\dagger = \sum_{n=1}^{M} u_{mn} L_n$, the completely mixed state $\mathbb{I}_d/d$ is a NESS.

[26] Note that an operator $A \in \mathcal{B}(\mathcal{H})$ commutes with S if and only if A is expressed as $A = \oplus_{\alpha=1}^{n_S} A_\alpha$, where $A_\alpha \in \mathcal{B}_{\alpha,\alpha}$.

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[32] If $J_x = -J_y$ with even N or $J_x = J_y$, the system has the U(1) strong symmetry [38]. Then we can prove that the NESS is unique in every $\mathcal{B}_{\alpha,\alpha}$ in the same manner as in Sec. IV D.

[33] A lattice (Λ, B) is connected if for any $j, k \in \Lambda$ such that $j \neq k$, there exists a finite sequence $l_1, \ldots, l_n \in \Lambda$ such that $l_1 = j$, $l_n = k$, and $(l_i, l_{i+1}) \in B$ for $i = 1, \ldots, n-1$.

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