Key varieties for prime $\mathbb{Q}$-Fano threefolds defined by Freudenthal triple systems

HIROMICHI TAKAGI

ABSTRACT. In this paper, we are concerned with the classification of complex prime $\mathbb{Q}$-Fano 3-folds of anti-canonical codimension 4 which are produced, as weighted complete intersections of appropriate weighted projectivizations of certain affine varieties related with $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$-fibrations. Such affine varieties or their appropriate weighted projectivizations are called key varieties for prime $\mathbb{Q}$-Fano 3-folds. We realize that the equations of the key varieties can be described conceptually by Freudenthal triple systems (FTS, for short). The paper consists of two parts. In Part 1, we revisit the general theory of FTS; the main purpose of Part 1 is to derive the conditions of so called strictly regular elements in FTS so as to fit with our description of key varieties. Then, in Part 2, we define several key varieties for prime $\mathbb{Q}$-Fano 3-folds from the conditions of strictly regular elements in FTS. Among other things obtained in Part 2, we show that there exists a 14-dimensional factorial affine variety $\mathfrak{U}^{14}_{\mathbb{A}}$ of codimension 4 in an affine 18-space with only Gorenstein terminal singularities, and we construct examples of prime $\mathbb{Q}$-Fano 3-folds of No.20544 in [GRDB] as weighted complete intersections of the weighted projectivization of $\mathfrak{U}^{14}_{\mathbb{A}}$ in the weighted projective space $\mathbb{P}(1^{15}, 2^2, 3)$. We also clarify in Part 2 a relation between $\mathfrak{U}^{14}_{\mathbb{A}}$ and the $G_2^{(4)}$-cluster variety, which is a key variety for prime $\mathbb{Q}$-Fano 3-folds constructed in [CD].

CONTENTS

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1. Introduction

1.1. **Classification of $\mathbb{Q}$-Fano threefolds.** A complex projective variety is called a $\mathbb{Q}$-*Fano variety* if it is a normal variety with only terminal singularities, and its anti-canonical divisor is ample. A $\mathbb{Q}$-Fano variety is called *prime* if its anti-canonical divisor generates the group of numerical equivalence classes of $\mathbb{Q}$-Cartier divisors. This paper concerns with the classification of prime $\mathbb{Q}$-Fano 3-folds and is a companion paper to [Tak3, Tak4, Tak6, Tak7], where we construct certain affine varieties and show that they produce, as weighted complete intersections of their appropriate weighted projectivizations, several examples of prime $\mathbb{Q}$-Fano 3-folds. We call such affine varieties or its appropriate weighted projectivizations *key varieties for* $\mathbb{Q}$-*Fano* 3-*folds*. The prime $\mathbb{Q}$-Fano 3-folds constructed in ibid. are of anti-canonical codimension 4, where, by the anti-canonical codimension of a $\mathbb{Q}$-Fano 3-fold X, we mean the codimension of X in the weighted projective space of the minimal dimension determined by the anti-canonical graded ring of X.

The affine varieties constructed in ibid. are related with $\mathbb{P}^2 \times \mathbb{P}^2$-fibration. In this paper, we construct several affine varieties related with $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$-fibration, and produce, as appropriate weighted complete intersections of their weighted projectivizations, certain examples of prime $\mathbb{Q}$-Fano 3-folds of anti-canonical codimension 4. Ahead of our study including ibid., Coughlan and Ducat did similar work in [CD] defining the C_2- and $G_2^{(4)}$-cluster varieties and using them as key varieties, where the former is related with $\mathbb{P}^2 \times \mathbb{P}^2$-fibration, and the latter is related with $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$-fibration.

Actually, further ahead of [CD] and ibid., Papadakis [P1, P2] constructed via the theory of unprojection more general affine varieties related with $\mathbb{P}^2 \times \mathbb{P}^2$- or $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$-fibration with nicknames Tom and Jerry respectively. They seem, however, experimentally too large as key varieties for prime $\mathbb{Q}$-Fano 3-folds; in [CD] and ibid., to produce prime $\mathbb{Q}$-Fano 3-fold, we extract appropriate subvarieties from Papadakis' affine varieties. It is non-trivial what kind of subvarieties are chosen. The cluster varieties was discovered in [CD] via mirror symmetry of log Calabi-Yau surfaces. One purpose of this paper is to reveal that the theory of *Freudenthal triple system* is a natural framework to define our key varieties and to describe their equations.

1.2. **Freudenthal triple system.** The first example of a Freudenthal triple system (FTS, in short) is given by Freudenthal in a series of works [Fr]; the example, which we denote by V_F, is a real 56-dimensional representation of the exceptional group of type E_7 and is constructed from a real 27-dimensional exceptional Jordan algebra of type E_6. This example V_F of an FTS is endowed with a tri-linear product and is associated with a symplectic form and a quartic form. The exceptional group of type E_7 is recovered from these two forms as the group of linear transformations of V_F leaving these two forms invariant.

In 1960s and 70s, axiomatic definitions of FTS were given in several researches, which start from a vector space with a symplectic form and a tri-linear product and a quartic form defined from them. Among such researches, we basically follow Ferrar's one [Fe] in this paper; the definition and the way of investigation of FTS given by him turn out to be quite suitable to describe our key varieties conceptually.

Ferrar's idea to investigate the structure of FTS is based upon the concept of *strictly regular element in FTS* (see Subsection 2.1 for the precise definition). He derives the Peirce decomposition of an FTS from a pair of supplementary strictly regular elements and then give a good coordinatization of the FTS. For us, it is important to state the conditions of strictly regular elements in a way suitable for our purpose (Theorem 2.18). Then, specializing to an 8-dimensional FTS, we derive another coordinatization of the FTS from the Ferrar's Peirce decomposition (Theorem 2.27).

1.3. **Key varieties for prime $\mathbb{Q}$-Fano threefolds.** For a general 8-dimensional FTS over the complex number field $\mathbb{C}$, the locus consisting of strictly regular elements is isomorphic to the affine cone over $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ (Corollary 2.31). From our coordinatization of an 8-dimensional FTS mentioned above, we may define an affine scheme $\mathfrak{F}^{22}_{\mathbb{A}}$ related with $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$-fibration (Definition 3.1). It is a subscheme of Papadakis' Jerry as mentioned above but is still too large to produce prime $\mathbb{Q}$-Fano 3-folds. We extract several subvarieties of $\mathfrak{F}^{22}_{\mathbb{A}}$ and obtain several results as for key varieties of prime $\mathbb{Q}$-Fano 3-folds. Among other things, we state the following as the main result of this paper, which summarizes Propositions 4.4, 4.5, 4.10, and Theorem 4.11:

Theorem 1.1. *The following assertions hold:*

(1) *There exists a 14-dimensional factorial affine variety $\mathfrak{U}^{14}_{\mathbb{A}}$ of codimension 4 in an affine 18-space with only Gorenstein terminal singularities.*

(2) *Examples of $\mathbb{Q}$-Fano 3-folds of No.20544 in* [GRDB] *are produced as weighted complete intersections of the weighted projectivization of $\mathfrak{U}^{14}_{\mathbb{A}}$ in the weighted projective space* $\mathbb{P}(1^{15}, 2^2, 3)$.

We remark that Examples of $\mathbb{Q}$-Fano 3-folds of No.20544 are constructed from the C_2-cluster variety but are not constructed from the $G_2^{(4)}$-cluster variety ([Table]).

1.4. **Structure of the paper.** The paper consists of two parts.

In Part 1, we revisit the theory of FTS in detail basically following [Fe] with emphasis on strictly regular elements (in some places, we also refer to [B] and [Kr]). The main result of Part 1 is Theorem 2.18, which states the conditions of strictly regular elements and leads to the equations of key varieties given in Part 2.

In [Fe], it is shown that an FTS with a nondegenerate skew form has a direct sum decomposition with two copies of a Jordan algebra of a cubic form as direct summands. For our purpose, however, we prefer not to identify Jordan algebras in FTS fully though several concepts for FTS are helpfully understood by Jordan algebraic considerations. For this reason, we decide to revisit the theory of FTS in detail in this paper.

In Part 2, we first construct an affine scheme $\mathfrak{F}^{22}_{\mathbb{A}}$ from a natural parameterization of an 8-dimensional FTS. In Sections 4–6, we extract three subvarieties $\mathfrak{U}^{14}_{\mathbb{A}}$,$\mathfrak{S}^{8}_{\mathbb{A}}$, $\mathfrak{Z}^{12}_{\mathbb{A}}$ respectively of $\mathfrak{F}^{22}_{\mathbb{A}}$ and show that they produce, as weighted complete intersections of their weighted projectivizations, certain prime $\mathbb{Q}$-Fano 3-folds. Several properties of $\mathfrak{U}^{14}_{\mathbb{A}}$,$\mathfrak{S}^{8}_{\mathbb{A}}$, $\mathfrak{Z}^{12}_{\mathbb{A}}$ are also obtained and they are important to construct prime $\mathbb{Q}$-Fano 3-folds. We describe $\mathfrak{U}^{14}_{\mathbb{A}}$ in detail in Section 4 (we refer to Theorem 1.1 for a summary). As is clarified in this paper, the affine variety $\mathfrak{U}^{14}_{\mathbb{A}}$ is also the cornerstone for studying other key varieties. Moreover, it will produce other prime $\mathbb{Q}$-Fano 3-folds in our future work. By these reasons, we devote many pages to the investigation of $\mathfrak{U}^{14}_{\mathbb{A}}$. As for the variety $\mathfrak{S}^{8}_{\mathbb{A}}$, we describe in detail its equations, an $\mathrm{SL}_2 \rtimes \mathrm{SL}_2$-action on it, and its relation with $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$-fibration. We revisit $\mathfrak{S}^{8}_{\mathbb{A}}$ in Section 7. As for the variety $\mathfrak{Z}^{12}_{\mathbb{A}}$, we show in Section 6 that a certain weighted projectivization of it is the 11-dimensional $\mathbb{Q}$-Fano variety constructed in the paper [Tak5].

In Section 7, we revisit the $G_2^{(4)}$-cluster variety defined in [CD], which we denote by $\mathfrak{Cl}^{10}_{\mathbb{A}}$. By the big table [Table] of prime $\mathbb{Q}$-Fano 3-folds obtained from $\mathfrak{Cl}^{10}_{\mathbb{A}}$, we observe that such prime $\mathbb{Q}$-Fano 3-folds are weighted complete intersections of some weighted projectivizations of $\mathfrak{Cl}^{10}_{\mathbb{A}}$ itself or its several subvarieties. In Subsections 7.3–7.7, we show that $\mathfrak{Cl}^{10}_{\mathbb{A}}$ itself and such subvarieties are actually isomorphic to subvarieties of $\mathfrak{U}^{14}_{\mathbb{A}}$ or $\mathfrak{Z}^{12}_{\mathbb{A}}$ weighted homogeneously with respect to some weights of coordinates. Hence it turns out that all prime $\mathbb{Q}$-Fano 3-folds obtained from $\mathfrak{Cl}^{10}_{\mathbb{A}}$ are also obtained from $\mathfrak{U}^{14}_{\mathbb{A}}$ or $\mathfrak{Z}^{12}_{\mathbb{A}}$. Among other things, we also show that the subvariety of $\mathfrak{Cl}^{10}_{\mathbb{A}}$ studied in Subsection 7.4 is weighted homogeneously isomorphic to $\mathfrak{S}^{8}_{\mathbb{A}}$, and the subvarieties $\mathfrak{T}^{8}_{\mathbb{A}}$ and $\mathfrak{B}^{6}_{\mathbb{A}}$ of $\mathfrak{Cl}^{10}_{\mathbb{A}}$ studied in Subsections 7.5 and 7.7 admit a nontrivial SL_2-action.

Finally, in Section 8, we see another aspect of the variety $\mathfrak{F}^{22}_{\mathbb{A}}$; we show that an open subset of the 23-dimensional affine variety $\mathfrak{P}^{23}_{\mathbb{A}}$ constructed by Papadakis using Type II_1 unprojection [P3, P4] is transformed to an open subset of the cone over $\mathfrak{F}^{22}_{\mathbb{A}}$.

This paper contains several assertions which can be proved by straightforward computations; we often omit such computations. Some computations are difficult by hand but are easy within a software package. In our computations, we use intensively the software systems Mathematica [W] and SINGULAR [DGPS].

1.5. **A future plan.** As is noted in Subsection 1.4, the affine variety $\mathfrak{U}^{14}_{\mathbb{A}}$ produces examples of prime $\mathbb{Q}$-Fano 3-folds which are also obtained from the $G_2^{(4)}$-cluster variety $\mathfrak{Cl}^{10}_{\mathbb{A}}$. By Theorem 1.1 (2), one more example is already added in this paper. Actually, we can verify that the affine variety $\mathfrak{U}^{14}_{\mathbb{A}}$ produce more examples of prime $\mathbb{Q}$-Fano 3-folds. Since the verification takes more pages (cf. [Tak3, Tak4]), we will publish it elsewhere.

Notation.

- $w(*)$: the weight of coordinate $*$ of a weighted projective space.
- If we put weights for the entries of a matrix A, we denote by $w(A)$ the set of the weights implemented in the matrix form corresponding to A.
- For a 2×3 matrix $M = \begin{pmatrix} f_1 & f_2 & f_3 \\ g_1 & g_2 & g_3 \end{pmatrix}$, we define

$$\left(\begin{vmatrix} f_2 & f_3 \\ g_2 & g_3 \end{vmatrix} \quad -\begin{vmatrix} f_1 & f_3 \\ g_1 & g_3 \end{vmatrix} \quad \begin{vmatrix} f_1 & f_2 \\ g_1 & g_2 \end{vmatrix} \right).$$

  For a 3×2 matrix, we have a similar definition.

Acknowledgment. I am grateful to Professor Shigeru Mukai; inspired by his articles [Mu1, Mu2], I was led to Jordan algebra and FTS to describe key varieties. Moreover, I was helped to find the key variety $\mathfrak{U}^{14}_{\mathbb{A}}$ by his inference as for dimensions of key varieties. I would like to dedicate this paper to him on his 70th birthday. This work is supported in part by Grant-in Aid for Scientific Research (C) 16K05090.

Part 1. **Freudenthal triple systems**

2. STRICTLY REGULAR ELEMENTS IN FREUDENTHAL TRIPLE SYSTEMS

In this section, we basically follow the treatment of Freudenthal triple system by Ferrar [Fe] with modifications in several coefficients of equalities after Brown [B].

Throughout this section, we assume that k is a field of characteristic $\neq 2, 3$.

2.1. **Basics of Freudenthal triple system.**

Definition 2.1 ([Fe, Sec.1])**.** A *Freudenthal triple system (FTS for short)* is a k-vector space V with a tri-linear product

$$V \times V \times V \ni (p_1, p_2, p_3) \mapsto p_1 \bullet p_2 \bullet p_3 \in V$$

and a skew bi-linear form

$$V \times V \ni (p_1, p_2) \mapsto \omega(p_1, p_2) \in \mathsf{k}$$

such that

(A1) the tri-linear product is symmetric in all arguments,

(A2)

$$\widetilde{F}(p_1, p_2, p_3, p_4) := \omega(p_1 \bullet p_2 \bullet p_3, p_4)$$

is a nonzero symmetric 4-linear form, and

(A3) the equality

$$6(p \bullet p \bullet p) \bullet p \bullet q = \omega(q,p)(p \bullet p \bullet p) + \omega\,(q, p \bullet p \bullet p)\,p$$

holds for any $p, q \in V$.

Remark 2.2. We note the coefficient 6 in the l.h.s. of the equality in (A3). This formulation is according to Brown [B] and this influences several coefficients in the equalities below although we will not mention one by one.

Linearizing the equality in (A3) completely, we obtain the following:

(2.1)

$$\begin{aligned}&6\left((p_1 \bullet p_2 \bullet p_3) \bullet p_4 + (p_1 \bullet p_2 \bullet p_4) \bullet p_3 + (p_1 \bullet p_3 \bullet p_4) \bullet p_2 + (p_2 \bullet p_3 \bullet p_4) \bullet p_1\right) \bullet q = \\ &\omega(q,p_4)(p_1 \bullet p_2 \bullet p_3) + \omega(q,p_3)(p_1 \bullet p_2 \bullet p_4) + \omega(q,p_2)(p_1 \bullet p_3 \bullet p_4) + \omega(q,p_1)(p_2 \bullet p_3 \bullet p_4) \\ &+\omega\,(q, p_1 \bullet p_2 \bullet p_3)\,p_4 + \omega\,(q, p_1 \bullet p_2 \bullet p_4)\,p_3 + \omega\,(q, p_1 \bullet p_3 \bullet p_4)\,p_2 + \omega\,(q, p_2 \bullet p_3 \bullet p_4)\,p_1.\end{aligned}$$

In this paper, we call this *the pentagram product formula*.

For $p \in V$, we denote by

$$L_{p,p} \colon V \to V$$

the linear map defined by

$$V \ni q \mapsto p \bullet p \bullet q \in V.$$

The following concept plays a central role in this paper.

Definition 2.3 ([Fe, Sec.3]). An element $p \in V$ is called *a strictly regular element* if it holds that

$$\mathrm{Image} L_{p,p} \subset \mathsf{k}p.$$

Proposition 2.4 ([Fe, p.317, (5) and Lem.3.1]). *An element $p \in V$ is strictly regular if and only if it holds that*

$$3L_{p,p}(q) + \omega(p,q)p = 0$$

for any $q \in V$.

2.2. **Jordan algebraic description of FTS.** In [Fe], it is shown that an FTS with nondegenerate ω has a direct sum decomposition with two copies of a Jordan algebra of cubic form as direct summands. In this subsection, we proceed slightly in a different way without identifying Jordan algebra fully, which is suitable for our purpose.

Proposition 2.5 ([Fe, Sec.4 (6)]). *Let $e_s, e_t \in V$ be supplementary, strictly regular elements (namely, e_s, e_t are strictly regular such that $\omega(e_s, e_t) = 1$). The following equality holds:*

$$L^2_{e_s,e_t} p = 1/12\,\omega(p,e_t)e_s - 1/12\,\omega(p,e_t)e_t + 1/36\,p.$$

Hereafter in this section, we assume that

(2.2) *the skew bilinear form ω is nondegenerate.*

Corollary 2.6 ([Fe, Sec.4]). *Let V_α be the L_{e_s,e_t}-eigenspace for the eigenvalue α. The vector space V has the following decomposition into the L_{e_s,e_t}-eigenspaces:*

$$V = V_{-1/3} \oplus V_{1/3} \oplus V_{1/6} \oplus V_{-1/6},$$

where $V_{-1/3} = \mathsf{k}e_s$ and $V_{1/3} = \mathsf{k}e_t$. This decomposition is called the Peirce decomposition.

Hereafter we use the following notation:

$$V_s := V_{-1/3},\, V_t := V_{1/3},\, V_x := V_{1/6},\, V_y := V_{-1/6}.$$

Now we begin Jordan algebraic treatments of FTS. We refer to [Mc2, Sec.4.2] for the subjects in the theory of Jordan algebra corresponding to those in the sequel.

We set

$$\beta(x,y) := \omega(x,y) \text{ for } x \in V_x \text{ and } y \in V_y. \tag{2.3}$$

The bi-linear form β corresponds to that for a Jordan algebra of a cubic form [Mc2, p.189]. As noted in [Fe, p.318, the 3rd line from the bottom],

$$\beta(x,y) \text{ is non-degenerate.} \tag{2.4}$$

Hence $\dim V_x = \dim V_y$, which we will denote by n;

$$n := \dim V_x = \dim V_y.$$

For $x \in V_x$ and $y \in V_y$, we set

$$N_x(x) := 1/2\, \omega(e_s, x \bullet x \bullet x),\; N_y(y) := 1/2\, \omega(e_t, y \bullet y \bullet y), \tag{2.5}$$

and

$$x^\sharp := 3/2\, x \bullet x \bullet e_s,\; y^\sharp := -3/2\, y \bullet y \bullet e_t. \tag{2.6}$$

By [Fe, Lem.4.1], we have $x^\sharp \in V_y$ and $y^\sharp \in V_x$. The cubic forms N_x and N_y correspond to the cubic form for a Jordan algebra of a cubic form.

For $x_1, x_2 \in V_x$ and $y_1, y_2 \in V_y$, we also set

$$x_1 \sharp x_2 := (x_1 + x_2)^\sharp - x_1^\sharp - x_2^\sharp,\; y_1 \sharp y_2 := (y_1 + y_2)^\sharp - y_1^\sharp - y_2^\sharp. \tag{2.7}$$

Then, by (2.6), we have

$$x_1 \sharp x_2 = 3 x_1 \bullet x_2 \bullet e_s,\; y_1 \sharp y_2 = -3 y_1 \bullet y_2 \bullet e_t. \tag{2.8}$$

Proposition 2.7. *For $x \in V_x$ and $y \in V_y$, it holds that*

$$N_x(x) = 1/3\, \beta\left(x, x^\sharp\right), N_y(y) = 1/3\, \beta\left(y^\sharp, y\right). \tag{2.9}$$

Proof. By (2.5), (A2) in Definition 2.1 and (2.6), we have the first equality as follows:

$$N_x(x) = 1/2\omega(e_s, x \bullet x \bullet x) = 1/2\omega(x, x \bullet x \bullet e_s) = 1/2\beta(x, 2/3x^\sharp) = 1/3\beta(x, x^\sharp).$$

The second one follows similarly. □

Corollary 2.8. *For $x, x' \in V_x$ and $y, y' \in V_y$, it holds that*

$$\partial_{x'} N_x(x) = \beta\left(x', x^\sharp\right), \partial_{y'} N_y(y) = \beta\left(y^\sharp, y'\right), \tag{2.10}$$

where $\partial_{x'} N_x(x)$ is the directional derivative of N_x in the direction x', evaluated at x, and $\partial_{y'} N_y(y)$ is similarly defined.

Proof. By (2.5), we have $\partial_{x'} N_x(x) = 1/2\, \omega(e_s, 3x \bullet x \bullet x')$, and the r.h.s. is equal to $3/2\, \omega(x', x \bullet x \bullet e_s) = 3/2\omega(x', 2/3x^\sharp) = \beta(x', x^\sharp)$ by (A2) in Definition 2.1 and (2.6). Therefore the first equality follows. The second one follows similarly. □

Corollary 2.8 corresponds to Trace-Sharp formula in [Mc2, p.189].

By [Fe], we have

Lemma 2.9. *The following equalities hold:*

$$(x \bullet x \bullet e_s) \bullet (x \bullet x \bullet e_s) \bullet e_t = 4/27\omega(x \bullet x \bullet x, e_s)x \tag{2.11}$$

$$x \bullet x \bullet y = -1/3\,\omega(x,y)x - 3(x \bullet x \bullet e_s) \bullet e_t \bullet y$$

Proposition 2.10. *For $x \in V_x$ and $y \in V_y$, it holds that*

$$(x^\sharp)^\sharp = N_x(x)x,\ (y^\sharp)^\sharp = N_y(y)y.$$

Proof. By (2.6), (2.11) and (2.5), we have the first equality as follows:

$$\begin{aligned}(x^\sharp)^\sharp &= (3/2\, x \bullet x \bullet e_s)^\sharp = (3/2)^2\,(-3/2\,(x \bullet x \bullet e_s) \bullet (x \bullet x \bullet e_s) \bullet e_t)\\ &= -\,(3/2)^3\,(4/27\,\omega(x \bullet x \bullet x, e_s)x) = -1/2\,\omega(x \bullet x \bullet x, e_s)x = N_x x.\end{aligned}$$

The second one follows similarly. □

Proposition 2.10 corresponds to Adjoint Identity in [Mc2, p.189].

The following two auxiliary results are needed to show Theorem 2.27.

Corollary 2.11. *Assume that $n \geq 3$ and $N_x(x)$ is not identically zero. Then $N_x(x)$ has no multiple factors. The similar statement for $N_y(y)$ also holds.*

Proof. Assume that $N_x(x)$ has a multiple factor $l(x)$, which must be a linear form since $N_x(x)$ is a cubic form. We choose coordinates $x_1, \ldots, x_n$ of V_x such that $l(x) = x_1$ and $N_x(x) = x_1^2 x_2$ (recall that $n \geq 2$). Then, for $a = {}^t(a_1, \ldots, a_n)$, $b = {}^t(b_1, \ldots, b_n) \in V_x$ (we consider a, b as column vectors), we have $\partial_a N_x(b) = 2a_1 b_1 b_2 + a_2 b_1^2$, which is equal to $\beta(a, b^\sharp)$ by Corollary 2.8. Note that the subset $\{b^\sharp \mid b \in V_x\} \subset V_y$ is dense in V_y by Proposition 2.10 and the assumption that $N_x(x)$ is not identically zero. Therefore, by the assumption that $n \geq 3$, the equality $\beta(a, b^\sharp) = 2a_1 b_1 b_2 + a_2 b_1^2$ implies that β is degenerate , a contradiction to (2.4).

We can show the claim for $N_y(y)$ similarly. □

Corollary 2.12. *Assume that $n \geq 3$ and $N_x(x)$ is not identically zero. The coordinates of $x^\sharp$ have no common factors and the similar statement holds for $y^\sharp$.*

Proof. If the coordinates of $x^\sharp$ have a common factor $F(x)$, we see that $F(x)^2$ divides $N_x(x)$ by Proposition 2.10 since the operation $\sharp$ is quadratic. This contradicts Corollary 2.11. We can show the claim for $y^\sharp$ similarly. □

The following formula is useful for calculations below.

Lemma 2.13. *For $x, \widetilde{x} \in V_x$ and $y \in V_y$, it holds that*

$$x \bullet x^\sharp \bullet \widetilde{x} = 1/6\,(\beta(\widetilde{x}, x^\sharp)x - N_x(x)\widetilde{x}), \tag{2.12}$$

$$x \bullet x^\sharp \bullet y = 1/6\,(-\beta(x,y)x^\sharp + N_x(x)y), \tag{2.13}$$

and

$$y \bullet y^\sharp \bullet x = 1/6\,(\beta(x,y)y^\sharp - N_y(y)x). \tag{2.14}$$

Proof. We only give a proof of the formula (2.12) since we can show the remaining two similarly. By (2.6), we have $x \bullet x^\sharp \bullet \widetilde{x} = 3/2(x \bullet x \bullet e_s) \bullet x \bullet \widetilde{x}$. Applying the pentagram product formula (2.1) with (2.6) and (2.9), we obtain (2.12). □

Hereafter we also follow the paper [Kr] with the above treatment in this subsection.

Cube Formula ([Kr, p.946,(48)]). For $p=(s,t,x,y)\in V=V_s\oplus V_t\oplus V_x\oplus V_y$, it holds that

$$p\bullet p\bullet p = (-s^2t+s\beta(x,y)-2N_y)e_s+(st^2-t\beta(x,y)+2N_x)e_t + (st-\beta(x,y))\,x+2x^\sharp\sharp y-2ty^\sharp - (st-\beta(x,y))\,y-2x\sharp y^\sharp+2sx^\sharp. \tag{2.15}$$

Proof. In [Kr], the proof is given for an FTS which contains two copies of a Jordan algebra of a cubic form. In any way, we may verify the assertion by a straightforward calculation using the facts obtained so far. □

2.3. **Condition on strict regularity.** Via partial linearization of (2.15) (with a minor correction), we arrive at the following condition on strict regularity by Proposition 2.4:

Proposition 2.14 (**Condition on strict regularity (**[Kr, p.947,(53)]**)**)**.**

For $p=(s,t,x,y)$, $q=(\widetilde{s},\widetilde{t},\widetilde{x},\widetilde{y})\in V$, it holds that

$$\begin{aligned}
&3p\bullet p\bullet q+\omega(p,q)p=\\
&\left(-\widetilde{s}\,(3st-\beta(x,y))+2\beta(sx-y^\sharp,\widetilde{y})\right)e_s\\
&+\left((3st-\beta(x,y))\,\widetilde{t}-2\beta(\widetilde{x},ty-x^\sharp)\right)e_t\\
&+(st-1/3\,\beta(x,y))\,\widetilde{x}+2\widetilde{t}\left(sx-y^\sharp\right)-2\widetilde{y}\sharp(ty-x^\sharp)+2\Delta_x(x,y;\widetilde{x})\\
&-(st-1/3\,\beta(x,y))\,\widetilde{y}-2\widetilde{s}\left(ty-x^\sharp\right)+2\widetilde{x}\sharp(sx-y^\sharp)-2\Delta_y(x,y;\widetilde{y}),
\end{aligned}$$

where we set

$$\begin{aligned}
\Delta_x(x,y;\widetilde{x})&:=-1/3\,\beta(x,y)\widetilde{x}-\beta(\widetilde{x},y)x+(x\sharp\widetilde{x})\sharp y,\\
\Delta_y(x,y;\widetilde{y})&:=-1/3\,\beta(x,y)\widetilde{y}-\beta(x,\widetilde{y})y+(y\sharp\widetilde{y})\sharp x.
\end{aligned}$$

Computing $(x\sharp\widetilde{x})\sharp y$ by the pentagram product formula (2.1), we see that

$$\Delta_x(x,y;\widetilde{x})=1/6\,\beta(x,y)\widetilde{x}-1/2\,\beta(\widetilde{x},y)x+3x\bullet\widetilde{x}\bullet y. \tag{2.16}$$

Similarly, we have

$$\Delta_y(x,y;\widetilde{y})=1/6\,\beta(x,y)\widetilde{y}-1/2\,\beta(x,\widetilde{y})y-3y\bullet\widetilde{y}\bullet x. \tag{2.17}$$

By Propositions 2.4 and 2.14, we have the following:

Corollary 2.15. *An element $p=(s,t,x,y)\in V$ is strictly regular if and only if it holds that*

$$sx=y^\sharp,\ ty=x^\sharp,\ st=1/3\,\beta(x,y),$$

and

$$\Delta_x(x,y;\widetilde{x})=\Delta_y(x,y;\widetilde{y})=0 \textit{ for any } \widetilde{x}\in V_x, \widetilde{y}\in V_y. \tag{2.18}$$

Now we will see that only the conditions on Δ_x in (2.18) is necessary. This is inspired by [Kr, Lem.23]. We give here a proof in a different flavor from the argument there.

Lemma 2.16. *For $x, \widetilde{x} \in V_x$ and $y, \widetilde{y} \in V_y$, the following two equalities hold:*

$$\beta\left(\Delta_x(x, y; \widetilde{x}), \widetilde{y}\right) = \beta\left(\widetilde{x}, \Delta_y(x, y; \widetilde{y})\right). \tag{2.19}$$

$$\beta\left(\Delta_x(x, y; \widetilde{x}), \widetilde{y}\right) = \beta\left(x, \Delta_y(\widetilde{x}, \widetilde{y}; y)\right). \tag{2.20}$$

Proof. The equation (2.19) follows from the following chain of the equalities:

$$\begin{aligned}
&\beta\left(\Delta_x(x, y; \widetilde{x}), \widetilde{y}\right)\\
&= \beta\left(1/6\, \beta(x, y)\widetilde{x} - 1/2\, \beta(\widetilde{x}, y)x + 3x \bullet \widetilde{x} \bullet y, \widetilde{y}\right) \text{ (by (2.16))}\\
&= 1/6\, \beta(x, y)\beta(\widetilde{x}, \widetilde{y}) - 1/2\, \beta(\widetilde{x}, y)\beta(x, \widetilde{y}) + \beta(3x \bullet \widetilde{x} \bullet y, \widetilde{y})\\
&= 1/6\, \beta(x, y)\beta(\widetilde{x}, \widetilde{y}) - 1/2\, \beta(\widetilde{x}, y)\beta(x, \widetilde{y}) + \omega(3x \bullet \widetilde{y} \bullet y, \widetilde{x}) \text{ (by (A2) in Def. 2.1)}\\
&= 1/6\, \beta(x, y)\beta(\widetilde{x}, \widetilde{y}) - 1/2\, \beta(\widetilde{x}, y)\beta(x, \widetilde{y}) - \beta(\widetilde{x}, 3x \bullet \widetilde{y} \bullet y)\\
&= \beta\left(\widetilde{x}, 1/6\, \beta(x, y)\widetilde{y} - 1/2\, \beta(x, \widetilde{y})y - 3x \bullet \widetilde{y} \bullet y\right)\\
&= \beta\left(\widetilde{x}, \Delta_y(x, y; \widetilde{y})\right) \text{ (by (2.17))}.
\end{aligned}$$

To obtain the equation (2.20), we have only to change the last 4 lines of the above chain of the equalities as follows:

$$\begin{aligned}
&= 1/6\, \beta(x, y)\beta(\widetilde{x}, \widetilde{y}) - 1/2\, \beta(\widetilde{x}, y)\beta(x, \widetilde{y}) + \omega(3\widetilde{x} \bullet \widetilde{y} \bullet y, x) \text{ (by (A2) in Def. 2.1)}\\
&= 1/6\, \beta(x, y)\beta(\widetilde{x}, \widetilde{y}) - 1/2\, \beta(\widetilde{x}, y)\beta(x, \widetilde{y}) - \beta(x, 3\widetilde{x} \bullet \widetilde{y} \bullet y)\\
&= \beta\left(x, 1/6\, \beta(\widetilde{x}, \widetilde{y})y - 1/2\, \beta(\widetilde{x}, y)\widetilde{y} - 3\widetilde{x} \bullet \widetilde{y} \bullet y\right)\\
&= \beta\left(x, \Delta_y(\widetilde{x}, \widetilde{y}; y)\right) \text{ (by (2.17))}.
\end{aligned}$$

□

Corollary 2.17. *The following conditions on $x \in V_x$ and $y \in V_y$ are equivalent:*

(1) $\Delta_x(x, y; \widetilde{x}) = 0$ *for any* $\widetilde{x} \in V_x$.
(2) $\Delta_y(x, y; \widetilde{y}) = 0$ *for any* $\widetilde{y} \in V_y$.

Proof. Assume that $\Delta_x(x, y; \widetilde{x}) = 0$ for any $\widetilde{x} \in V_x$. Then, by the equality (2.19) as in Lemma 2.16, it holds that $\beta\left(\widetilde{x}, \Delta_y(x, y; \widetilde{y})\right) = 0$ for any $\widetilde{x} \in V_x$ and $\widetilde{y} \in V_y$. Since β is nondegenerate, $\Delta_y(x, y; \widetilde{y}) = 0$ for any $\widetilde{y} \in V_y$. Thus (1) implies (2). Similarly we see that (2) implies (1). □

Now we arrive at the main result of Part 1 by Corollaries 2.15 and 2.17.

Theorem 2.18. *An element $p = (s, t, x, y)$ is strictly regular if and only if it holds that*

$$sx = y^{\sharp},\ ty = x^{\sharp}, \tag{2.21}$$

$$st = 1/3\, \beta(x, y), \tag{2.22}$$

and

$$\Delta_x(x, y, \widetilde{x}) = 0 \textit{ for any } \widetilde{x} \in V_x. \tag{2.23}$$

Definition 2.19. We denote by $\mathfrak{R}$ the affine scheme in V defined by the equations (2.21), (2.22), and (2.23) as in Proposition 2.18.

Remark 2.20. Here we mention a few background of Theorem 2.18. Our derivation of the defining equations of $\mathfrak{R}$ as in Theorem 2.18 is inspired by [Kr, Lem.23] and [YamA, p.253–254], and its origin is traced back to the series of the fundamental papers [Fr] (see also [Cl, Prop.6.2] and [Fa, (2.1)]). We refer to [Cl, Prop.6.2] and [KaYas, p.515, Section 0] for aspects of this equation of $\mathfrak{R}$ in a graded Lie algebra of contact type. In [KaYas], the projectivization of $\mathfrak{R}$ is called Freudenthal variety. In [Mu1, Section 5] and [Mu2], the projectivization of $\mathfrak{R}$ is studied in relation with smooth Fano threefolds and is called Legendre projective variety.

2.4. **More on Δ_x and Δ_y.** In this subsection, we examine the condition (2.23) more in detail. It turns out that we need this condition for several $\widetilde{x} \in V_x$ in general to define $\mathfrak{R}$.

By (2.9), Lemma 2.13, (2.16), and (2.17), we immediately obtain the following:

Lemma 2.21. *For any $x, \widetilde{x} \in V_x$ and any $y, \widetilde{y} \in V_y$, it holds that*

$$\Delta_x(x, x^\sharp, \widetilde{x}) = 0, \Delta_x(y^\sharp, y, \widetilde{x}) = 0,$$

and

$$\Delta_y(y^\sharp, y, \widetilde{y}) = 0, \Delta_y(x, x^\sharp, \widetilde{y}) = 0.$$

Lemma 2.22. *The following two conditions for $\widetilde{x} \in V_x$ with $N_x(\widetilde{x}) \neq 0$ and $\widetilde{y} \in V_y$ are equivalent:*

(1) It holds that

$$\beta\left(\Delta_x(x, y; \widetilde{x}), \widetilde{y}\right) = 0 \tag{2.24}$$

for any $x \in V_x$ and $y \in V_y$.

(2) $\widetilde{y} \in k \cdot \widetilde{x}^\sharp$.

Proof. By (2.20) as in Lemma 2.16, we have $\beta\left(\Delta_x(x, y; \widetilde{x}), \widetilde{y}\right) = \beta\left(x, \Delta_y(\widetilde{x}, y; \widetilde{y})\right)$. Therefore, (1) is equivalent to that $\beta\left(x, \Delta_y(\widetilde{x}, y; \widetilde{y})\right) = 0$ for any $x \in V_x$ and $y \in V_y$. By nondegeneracy of β, this is also equivalent to

$$\Delta_y(\widetilde{x}, \widetilde{y}; y) = 0 \tag{2.25}$$

for any $y \in V_y$.

We show (1) implies (2). Assume that (1) holds. We set $y = \widetilde{x}^\sharp$ in the equality (2.25). Then we have $\beta(\widetilde{x}, y) = \beta(\widetilde{x}, \widetilde{x}^\sharp) = 3N_x(\widetilde{x})$ by (2.9). By (2.13), we also have

$$\widetilde{x} \bullet y \bullet \widetilde{y} = \widetilde{x} \bullet \widetilde{x}^\sharp \bullet \widetilde{y} = 1/6(-\beta(\widetilde{x}, \widetilde{y})\widetilde{x}^\sharp + N_x(\widetilde{x})\widetilde{y}).$$

Therefore the equality (2.25) with $y = \widetilde{x}^\sharp$ becomes

$$1/6\, \beta(\widetilde{x}, \widetilde{y})\widetilde{x}^\sharp - 3/2\, N_x(\widetilde{x})\widetilde{y} - 1/2\, (-\beta(\widetilde{x}, \widetilde{y})\widetilde{x}^\sharp + N_x(\widetilde{x})\widetilde{y}) = 0,$$

equivalently, we have $\widetilde{y} = \frac{\beta(\widetilde{x}, \widetilde{y})}{3N_x(\widetilde{x})}\widetilde{x}^\sharp$. Thus (2) follows.

Now we show (2) implies (1). Assume that (2) holds. We write $\widetilde{y} = \alpha \widetilde{x}^\sharp$ for some $\alpha \in \mathsf{k}$ (actually, computing $\beta(\widetilde{x}, \widetilde{y})$, we have $\alpha = \frac{\beta(\widetilde{x}, \widetilde{y})}{3N_x(\widetilde{x})}$). Inserting $\widetilde{y} = \alpha \widetilde{x}^\sharp$ in the l.h.s. of (2.25), we see that the equality (2.25) holds for any $y \in V_y$ by Lemma 2.21. $\square$

We recall that we set $n := \dim V_x = \dim V_y$. By non-degeneracy of β, we immediately obtain the following from Lemma 2.22:

Corollary 2.23. *For $\widetilde{x} \in V_x$ with $N_x(\widetilde{x}) \neq 0$, the entries of the bi-linear map*

$$\Delta_x(*, *, \widetilde{x})\colon V_x \times V_y \to V_x$$

generate an $(n-1)$-dimensional vector space.

Example 2.24. As mentioned in the beginning of Subsection 2.2, a basic example of an FTS is the one containing two copies of a Jordan algebra of a cubic form. For such an example, $V_x = V_y$ and this has the structure of the Jordan algebra of the cubic form compatible with the quartic form F and the skew-symmetric form ω. For the Jordan algebra $V := V_x = V_y$, the cubic form $N := N_x = N_y$ is the associated cubic form and the bilinear form β is the associated bi-linear trace (see [Fe, p.314] for more details).

(1) Let V be a 3-dimensional vector space with coordinates x_1, x_2, x_3. The cubic form $x_1x_2x_3$ define a Jordan algebra structure on V, whose $\sharp$-mapping is $V \ni (x_1, x_2, x_3) \mapsto (x_2x_3, x_1x_3, x_1x_2) \in V$, and the bi-linear trace is $\beta(x, y) = x_1y_1 + x_2y_2 + x_3y_3$ for $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. By an explicit calculation, we see that the condition (2.23) is reduced to the two equations: $2x_1y_1 - x_2y_2 - x_3y_3 = 0, x_1y_1 - 2x_2y_2 + x_3y_3 = 0$. Then we can write down the 9 equations of $\mathfrak{R}$ as follows:

$$\begin{aligned}
&sx_1 = y_2y_3,\ sx_2 = y_1y_3,\ sx_3 = y_1y_2,\\
&ty_1 = x_2x_3,\ ty_2 = x_1x_3,\ ty_3 = x_1x_2,\\
&st = 1/3(x_1y_1 + x_2y_2 + x_3y_3),\\
&2x_1y_1 - x_2y_2 - x_3y_3 = 0,\ x_1y_1 - 2x_2y_2 + x_3y_3 = 0.
\end{aligned}$$

Tidying up these equations, we see that these are derived from a $2 \times 2 \times 2$ hypermatrix and then $\mathfrak{R}$ is the affine cone over the Segre embedded $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

(2) The space of 3×3 matrix $M(3, \mathsf{k})$ is a Jordan algebra of a cubic form. The Jordan product $\cdot_J$ is defined as follows:

$$X \cdot_J Y := 1/2(XY + YX) \text{ for } X, Y \in M(3, \mathsf{k}).$$

The associated cubic form N is the determinant of a matrix, and for $X \in M(3, \mathsf{k})$, $X^\sharp$ is the adjoint matrix. The bi-linear trace β is defined as follows:

$$\beta(X, Y) = \mathrm{tr}(XY) \text{ for } X, Y \in M(3, \mathsf{k}).$$

In this setting, it is known that $\mathfrak{R}$ is the Grassmannian $\mathrm{G}(3, 6)$ (cf. [LM]) and we can compute the defining equation $\mathfrak{R}$ as follows:

$$sX = Y^\sharp,\ tY = X^\sharp,\ XY = YX,\ XY = stI,$$

where I is the 3×3 identity matrix.

(3) The space of 3×3 symmetric matrix $\mathrm{Sym}(3, \mathsf{k})$ is also a Jordan algebra of a cubic form, and is a Jordan subalgebra of $M(3, \mathsf{k})$. In this setting, $\mathfrak{R}$ is the 6-dimensional symplectic Grassmannian $\mathrm{Sp}(3, 6)$ and we can compute the defining equation of $\mathfrak{R}$ as follows:

$$sX = Y^\sharp,\ tY = X^\sharp,\ XY = stI$$

(cf.[I, p.32]).

By Example 2.24 (2) and (3), we observe that, for the defining equations of $\mathfrak{R}$, we need $\Delta_x(x, y; \widetilde{x})$ for *several* $\widetilde{x}$ in general (hence we realize that the equations given in [Cl, Prop.6.2] and [Fa, (2.1)] are not sufficient).

2.5. **More on FTS and the scheme $\mathfrak{R}$.** Finally in this section, we add further properties of FTS and the scheme $\mathfrak{R}$.

Explicit descriptions of the quartic form $\widetilde{F}$ and the symplectic form ω are given as follows (cf.[Cl, (21) in Sec.7, p.118–119],[Fe, p.321], [LM, Prop.5.5]). The proof is same as that given as in [Fe, p.321] with the above formulation.

Proposition 2.25. *For $p = (s, t, x, y)$ and $p' = (s', t', x', y') \in V$, it holds that*

$$\widetilde{F}(p) = 8\left(\beta\left(y^\sharp, x^\sharp\right) - sN_x(x) - tN_y(y)\right) - 2\left(st - \beta(x, y)\right)^2,$$
$$\omega(p, p') = \beta(x, y') - \beta(x', y) + st' - s't.$$

From this, we see that ω is nondegenerate if and only if β is so. We use this proposition to show Proposition 2.29 below.

We see that $\mathfrak{R}$ contains an open subset isomorphic to $\mathsf{k}^* \times \mathbb{A}^n$ as follows (cf.[Cl, p.114], [Mu2, (4.3)]):

Corollary 2.26. *The following two assertions hold:*

(1) *An element $p = (s, t, x, y)$ with $s \neq 0$ is strictly regular if and only if it holds that $x = s^{-1}y^\sharp, t = (3s)^{-1}\beta(x, y)$.*
(2) *An element $p = (s, t, x, y)$ with $t \neq 0$ is strictly regular if and only if it holds that $y = t^{-1}x^\sharp, s = (3t)^{-1}\beta(x, y)$.*

In particular, $\mathfrak{R} \cap \{s \neq 0\}$ and $\mathfrak{R} \cap \{t \neq 0\}$ are isomorphic to $\mathsf{k}^* \times \mathbb{A}^n$.

Proof. Since we can prove these assertions similarly, we only show (1). The only if part follows immediately by Corollary 2.15. To show the if part, we assume that $x = s^{-1}y^\sharp, t = (3s)^{-1}\beta(x, y)$ for $p = (s, t, x, y)$. By Corollary 2.15, it suffices to check that

$$\left((3s)^{-1}\beta(s^{-1}y^\sharp, y)\right) y - (s^{-1}y^\sharp)^\sharp = 0 \tag{2.26}$$

and

$$\Delta_x(s^{-1}y^\sharp, y, \widetilde{x}) = \Delta_y(s^{-1}y^\sharp, y, \widetilde{y}) = 0 \text{ for any } \widetilde{p} = (\widetilde{s}, \widetilde{t}, \widetilde{x}, \widetilde{y}). \tag{2.27}$$

The equality (2.26) follows from Propositions 2.7 and 2.10, and the equality (2.27) follows from Lemma 2.21. □

2.6. **A coordinatization of $\mathfrak{R}$ with parameters.** In this subsection, we further assume that

$$n = \dim V_x = \dim V_y = 3.$$

We show that the FTS V has a coordinatization with parameters as in the following theorem, which can be seen as a generalization of Example 2.24 (1):

Theorem 2.27. *Assume that N_x is not identically zero on V_x. The equations (2.23) as in Theorem 2.18 is reduced to the two equations*

$${}^t\!xPy = 0, {}^t\!xQy = 0,$$

where P, Q are 3×3 k-matrices, and x, y are considered as column vectors ${}^t(x_1, x_2, x_3)$ and ${}^t(y_1, y_2, y_3)$. Moreover, there exist nonzero constants $\mu, \nu \in \mathsf{k}$ such that

$$x^\sharp = \mu({}^tPx \times {}^tQx),\ y^\sharp = \nu(Py \times Qy), \tag{2.28}$$

and, by replacing s, t with $\mu s, \nu t$ respectively, the equation of $\mathfrak{R}$ is reduced to

$$\begin{aligned}&sx = Py \times Qy,\ ty = {}^tPx \times {}^tQx,\\ &st = \frac{1}{3\mu\nu}\beta(x,y),\\ &{}^txPy = 0, {}^txQy = 0,\end{aligned}$$

where ${}^tPx \times {}^tQx$ is the cross product of the two column vectors ${}^tPx$ and ${}^tQx$, and $Py \times Qy$ is similarly defined. In particular, the scheme $\mathfrak{R}$ in this case is defined by 9 quadratic forms.

Proof. Since N_x is not identically zero on V_x, we may take $\widetilde{x} \in V_x$ with $N_x(\widetilde{x}) \neq 0$. Since $\dim V_x = \dim V_y = 3$, the entries of the bi-linear map

$$\Delta_x(*, *, \widetilde{x}) \colon V_x \times V_y \to V_x$$

generate a 2-dimensional vector space by Corollary 2.23. We denote a basis of this vector space by ${}^txPy$ and ${}^txQy$ with some 3×3 matrices P and Q with entries in k. By Lemma 2.21, we have

$$ {}^txPx^\sharp = 0, {}^txQx^\sharp = 0 \text{ for any } x \in V_x, \tag{2.29}$$

$$ {}^ty^\sharp Py = 0, {}^ty^\sharp Qy = 0 \text{ for any } y \in V_y. \tag{2.30}$$

By (2.29), there exists a rational function $\mu(x)$ such that

$$x^\sharp = \mu(x)({}^tPx \times {}^tQx). \tag{2.31}$$

Since all the entries of both $x^\sharp$ and ${}^tPx \times {}^tQx$ are quadratic forms, the degree of $\mu(x)$ is 0. By Corollary 2.12, $\mu(x)$ is a nonzero constant, hence we will denote this by μ. Similarly, we can show that there exists a nonzero constant $\nu \in k$ such that

$$y^\sharp = \nu(Py \times Qy).$$

□

Under the situation of Theorem 2.27, we can write down several data of FTS using the entries of x, y, P, Q. Here we treat some of them as follows:

Recipe to write down $N_x(x), N_y(y)$, and $\beta(x, y)$.

(1) By (2.28), we have

$$\begin{aligned}(x^\sharp)^\sharp &= \mu^2\nu\left(P({}^tPx \times {}^tQx) \times Q({}^tPx \times {}^tQx)\right),\\ (y^\sharp)^\sharp &= \mu\nu^2\left({}^tP(Py \times Qy) \times {}^tQ(Py \times Qy)\right).\end{aligned}$$

From these, we may write down the cubic forms $N_x(x)$ and $N_y(y)$ by Proposition 2.10.

(2) By (1) and Corollary 2.8, we may write down the bi-linear trace $\beta(x, y)$ since $N_x(x) \neq 0$ for some $x \in V_x$ by the assumption, and then the subset $\{x^\sharp \mid x \in V_x\} \subset V_y$ is dense in V_y by Proposition 2.10.

Suggested by this recipe, we have a converse statement to Theorem 2.27. The verification of the following lemma and proposition is straightforward.

Lemma 2.28. *Let x, y be 3-dimensional column vectors and P, Q 3×3 matrices. We consider all the entries of x, y, P, Q as variables of a k-polynomial ring. The following assertions hold:*

(1) There exist the uniquely determined cubics forms cubic forms $N_x^{PQ}(x)$ and $N_y^{PQ}(y)$ such that

$$\big(P({}^tPx\times {}^tQx)\times Q({}^tPx\times {}^tQx)\big)=N_x^{PQ}(x)x,$$
$$\big({}^tP(Py\times Qy)\times {}^tQ(Py\times Qy)\big)=N_y^{PQ}(y)y.$$

(2) Let x', y' be 3-dimensional column vectors. There exists the uniquely determined bi-linear trace $\beta^{PQ}(x,y)$ such that

$$\partial_{x'}N_x^{PQ}(x)=\beta^{PQ}\left(x',{}^tPx\times {}^tQx\right),\partial_{y'}N_y^{PQ}(y)=\beta^{PQ}\left(Py\times Qy,y'\right). \tag{2.32}$$

Proposition 2.29. *Let V_x and V_y be 3-dimensional k-vector spaces. We write elements of V_x and V_y as column vectors. Let P,Q be 3×3 k-matrices. We define*

$$x^\sharp:={}^tPx\times {}^tQx,\ y^\sharp:=Py\times Qy \text{ for } x\in V_x,\ y\in V_y,$$

and the $\sharp$-product by (2.7). Let the cubic forms $N_x^{PQ}(x)$, $N_y^{PQ}(y)$ and the bi-linear form $\beta^{PQ}(x,y)$ ($x\in V_x$, $y\in V_y$) be as defined in Lemma 2.28. Let $V:=\mathsf{k}\oplus\mathsf{k}\oplus V_x\oplus V_y$. We further define a skew bi-linear form ω on V as follows: for $p=(s,t,x,y)$ and $p'=(s',t',x',y')\in V$,

$$\omega(p,p'):=\beta^{PQ}(x,y')-\beta^{PQ}(x',y)+st'-s't.$$

Finally, we define a tri-linear product on V by completely linearizing the formula (2.15) with $x^\sharp$, $y^\sharp$, $N_x^{PQ}(x)$, $N_y^{PQ}(y)$ and $\beta^{PQ}(x,y)$. The tri-linear product satisfies the axioms as in Definition 2.1 with respect to ω, and the symmetric 4-linear form as in $(A2)$ is the complete linearization of the following quartic form $\widetilde{F}^{PQ}$:

$$\widetilde{F}^{PQ}(p):=8\left(\beta^{PQ}\left(y^\sharp,x^\sharp\right)-sN_x^{PQ}(x)-tN_y^{PQ}(y)\right)-2\left(st-\beta^{PQ}(x,y)\right)^2.$$

In particular the vector space V is the FTS with respect to ω and this tri-linear product.

Remark 2.30. Let D_β be the determinant of the matrix defining the bi-linear form $\beta^{PQ}(x,y)$ as in Proposition 2.29; $\beta^{PQ}(x,y)$ is nondegenerate if and only if $D_\beta\neq 0$.

Corollary 2.31. *Under the situation as in Proposition 2.29, assume that the base field k is algebraically closed and $\beta^{PQ}(x,y)$ is nondegenerate, namely, $D_\beta\neq 0$. Let $\mathfrak{R}^{PQ}$ be the affine scheme defined by*

$$sx=Py\times Qy,\ ty={}^tPx\times {}^tQx, \tag{2.33}$$
$$st=1/3\,\beta^{PQ}(x,y), \tag{2.34}$$
$${}^txPy=0,{}^txQy=0. \tag{2.35}$$

The affine scheme $\mathfrak{R}^{PQ}$ is isomorphic to the affine cone over the Segre embedded $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$.

Proof. Since $\beta^{PQ}(x,y)$ is nondegenerate, V is simple as an FTS by [Fe, Thm.2.1]. Since $(1,0,0,0)$, $(0,1,0,0)\in V$ are supplementary strictly regular elements, V is reduced. Therefore, by [Fe, p.314 and Thm.5.1], V is constructed from a Jordan algebra J as in (i) or (ii) of [Fe, Thm.5.1].

We show (i) holds. Assume for a contradiction that (ii) holds. Then

$${}^tPx\times {}^tQx=0 \text{ for any } x\in V_x \tag{2.36}$$

by [Fe, p.314]. If $P=0$ or $Q=0$, then $\beta^{PQ}(x,y)$ is identically zero, a contradiction. Therefore, changing the coordinates of V_x and V_y, we may assume that

$P = \mathrm{diag}\,(1,1,1), \mathrm{diag}\,(1,1,0)$, or $\mathrm{diag}\,(1,0,0)$, where $\mathrm{diag}\,(a,b,c)$ is the diagonal matrix with a,b,c as the $(1,1)$, $(2,2)$, $(3,3)$-entries respectively. Then, from (2.36), we see that $Q = \alpha P$ for some $\alpha \in \mathsf{k}$ in the first or the second case, or $Q = \begin{pmatrix} * & 0 & 0 \\ * & 0 & 0 \\ * & 0 & 0 \end{pmatrix}$ in the third case. In any case, we see that $\beta^{PQ}(x,y)$ is identically zero, a contradiction. Therefore (i) must hold.

In this case, nondegeneracy of $\beta^{PQ}(x,y)$ implies that the Jordan algebra J does not contain an absolute zero divisor in the sense of [Ra, p.93] ([Mc2, Def.5.3.2, Prop.5.3.3, Ex.5.3.6]). Therefore, (i), (ii) or (iii) of [Ra, Thm.1] holds. The Jordan algebra J is not a division algebra since $N_x = 0$ for some nonzero $x \in V_x$. Thus (ii) or (iii) holds. Then, by the proof of ibid., the Jordan algebra J has the Peirce decomposition $J = J_1 \oplus J_{1/2} \oplus J_0$ with respect to a primitive idempotent. We have $\dim J_1 = 1$ and $\dim J_0 \geq 1$. If (iii) holds, then $\dim J_{1/2} \geq 2$ by [Ra, p.98, 5 and 6th line from the bottom], hence $\dim V_x = \dim J \geq 4$, a contradiction. Therefore (ii) holds, namely, $J = \mathsf{k} \oplus J(q)$, where $J(q)$ is a Jordan algebra of a quadratic form q. By [Mc1, p.506, Ex.2], q is nondegenerate since so is $\beta^{PQ}(x,y)$. Therefore, by ibid. and the assumption that k is algebraically closed, we see that N_x is a product of three linearly independent linear forms. Now, by Example 2.24 (1), $\mathfrak{R}^{PQ}$ is isomorphic to the affine cone over $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. $\square$

Part 2. Key varieties for $\mathbb{Q}$-Fano threefolds

Hereafter we work over $\mathbb{C}$, the complex number field, throughout the paper.

3. Affine scheme $\mathfrak{F}^{22}_{\mathbb{A}}$

Definition 3.1. In the affine 26-space whose coordinates are s, t, and the entries of x, y, P, Q, we define $\mathfrak{F}^{22}_{\mathbb{A}}$ to be the scheme with the equations (2.33), (2.34) and (2.35). We say that the entries of P, Q *parametric coordinates.*

Remark 3.2. The equations of $\mathfrak{F}^{22}_{\mathbb{A}}$ is a specialization of those given in [P2, Subsec.5.7]. These are also derived in [NP]. The main advantage here is that the meaning of the equations is quite clear in view of the theory of FTS; especially, the equation (2.34) is too complicated to write down fully but we can write it conceptually as above.

In the following Sections 4–7, setting some of the parametric coordinates of $\mathfrak{F}^{22}_{\mathbb{A}}$ to constants, or impose linear relations on them, we will obtain several affine varieties whose weighted projectivizations produce examples of prime $\mathbb{Q}$-Fano 3-folds of anti-canonical codimension 4. We refer to Section 8 for another aspect of $\mathfrak{F}^{22}_{\mathbb{A}}$.

We set

$$M_x := \begin{pmatrix} {}^t x P \\ {}^t x Q \end{pmatrix}, M_y := \begin{pmatrix} {}^t y {}^t P \\ {}^t y {}^t Q \end{pmatrix}.$$

4. A specialization of $\mathfrak{F}^{22}_{\mathbb{A}}$–an affine variety $\mathfrak{U}^{14}_{\mathbb{A}}$ with an $\mathrm{SL}_2 \times \mathrm{SL}_2$-action–

In this section, we define a closed subscheme $\mathfrak{U}^{14}_{\mathbb{A}}$ of the affine scheme $\mathfrak{F}^{22}_{\mathbb{A}}$ and investigate its properties. We show that a weighted projectivization of $\mathfrak{U}^{14}_{\mathbb{A}}$ give an example of a key variety for a prime $\mathbb{Q}$-Fano 3-fold of anticanonical codimension 4 which is not obtained from the $G_2^{(4)}$-cluster variety.

4.1. **Definition.**

Definition 4.1. We set

$$\mathfrak{U}^{14}_{\mathbb{A}} := \mathfrak{F}^{22}_{\mathbb{A}} \cap \{p_{13} = q_{23} = 1, p_{23} = p_{33} = q_{13} = q_{33} = 0, p_{11} = -q_{21}, p_{12} = -q_{22}\}.$$

We use the following notation for entries of the matrices P and Q, by which an group action on $\mathfrak{U}^{14}_{\mathbb{A}}$ will be more visible (see Subsection 4.2):

$$P = \begin{pmatrix} a_{11} & b_{11} & 1 \\ a_{12} & b_{12} & 0 \\ c_{11} & c_{12} & 0 \end{pmatrix}, Q = \begin{pmatrix} a_{21} & b_{21} & 0 \\ -a_{11} & -b_{11} & 1 \\ c_{21} & c_{22} & 0 \end{pmatrix}.$$

Setting further

$$A := \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & -a_{11} \end{pmatrix}, B := \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & -b_{11} \end{pmatrix}, C := \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix},$$
$$\widehat{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \widehat{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

we can write

$$M_x = \begin{pmatrix} {}^t\!xP \\ {}^t\!xQ \end{pmatrix} = \begin{pmatrix} A\widehat{x} & B\widehat{x} & \widehat{x} \end{pmatrix} + x_3 \begin{pmatrix} C & 0 \end{pmatrix}.$$

Explicitly,

$$M_x = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + c_{11}x_3 & b_{11}x_1 + b_{12}x_2 + c_{12}x_3 & x_1 \\ a_{21}x_1 - a_{11}x_2 + c_{21}x_3 & b_{21}x_1 - b_{11}x_2 + c_{22}x_3 & x_2 \end{pmatrix}.$$

We have also

$$M_y = \begin{pmatrix} {}^t\!y\,{}^t\!P \\ {}^t\!y\,{}^t\!Q \end{pmatrix} = \begin{pmatrix} a_{11}y_1 + b_{11}y_2 + y_3 & a_{12}y_1 + b_{12}y_2 & c_{11}y_1 + c_{12}y_2 \\ a_{21}y_1 + b_{21}y_2 & -a_{11}y_1 - b_{11}y_2 + y_3 & c_{21}y_1 + c_{22}y_2 \end{pmatrix}.$$

We denote by $\mathbb{A}^{18}_{\mathfrak{U}}$ the affine space whose coordinates are s, t and the entries of A, B, C, x, y.

4.2. $\mathrm{SL}_2 \times \mathrm{SL}_2$-**action.**

We see that $\mathfrak{U}^{14}_{\mathbb{A}}$ admits the following group action, which will be helpful to investigate properties of $\mathfrak{U}^{14}_{\mathbb{A}}$:

Proposition 4.2. *The scheme $\mathfrak{U}^{14}_{\mathbb{A}}$ is preserved by the following group actions on the affine space $\mathbb{A}^{18}_{\mathfrak{U}}$ of the two groups $(\mathrm{SL}_2)^x$, $(\mathrm{SL}_2)^y$ isomorphic to SL_2, and they define* *an action of the group* $\mathrm{SL}_2 \times \mathrm{SL}_2$ *on* $\mathfrak{U}^{14}_{\mathbb{A}}$*:*

$$\begin{aligned}
&(\mathrm{SL}_2)^x : \textit{For } g \in (\mathrm{SL}_2)^x,\ A \mapsto gAg^{-1},\ B \mapsto gBg^{-1}, \widehat{x} \mapsto g\widehat{x},\ C \mapsto gC,\\
&\qquad\qquad\qquad\qquad x_3 \mapsto x_3,\ y \mapsto y,\ s \mapsto s,\ t \mapsto t,\\
&(\mathrm{SL}_2)^y : \textit{For } h \in (\mathrm{SL}_2)^y,\ \begin{pmatrix} a_{11} & b_{11} \\ a_{21} & b_{21} \end{pmatrix} \mapsto \begin{pmatrix} a_{11} & b_{11} \\ a_{21} & b_{21} \end{pmatrix} h^{-1},\\
&\qquad\qquad\qquad\qquad \begin{pmatrix} a_{12} & b_{12} \\ -a_{11} & -b_{11} \end{pmatrix} \mapsto \begin{pmatrix} a_{12} & b_{12} \\ -a_{11} & -b_{11} \end{pmatrix} h^{-1},\\
&\qquad\qquad\qquad\qquad \widehat{y} \mapsto h\widehat{y},\ C \mapsto Ch^{-1},\\
&\qquad\qquad\qquad\qquad y_3 \mapsto y_3,\ x \mapsto x,\ s \mapsto s,\ t \mapsto t.
\end{aligned}$$

Proof. Note that $M_x y = {}^t(M_y x) = \begin{pmatrix} {}^t x P y \\ {}^t x Q y \end{pmatrix}$. By straightforward calculations, we see the following:

- M_x is mapped to gM_x by $g \in (\mathrm{SL}_2)^x$, and M_y is invariant for the action of $(\mathrm{SL}_2)^y$.
- $x^\sharp$ is invariant for the action of $(\mathrm{SL}_2)^x$ and is equivariant to y for the action of $(\mathrm{SL}_2)^y$. $y^\sharp$ is equivariant to x for the action of $(\mathrm{SL}_2)^x$ and is invariant for the action of $(\mathrm{SL}_2)^y$.
- $\beta(x, y)$ is invariant for the actions of $(\mathrm{SL}_2)^x$ and $(\mathrm{SL}_2)^y$.
- The actions of $(\mathrm{SL}_2)^x$ and $(\mathrm{SL}_2)^y$ are commutative.

Therefore we have the group action on $\mathfrak{U}^{14}_{\mathbb{A}}$ as in the statement. $\square$

4.3. **Weights for variables and equations .** We assign weights for variables of the polynomial ring $S_{\mathfrak{U}}$ such that all the 9 equations of $\mathfrak{U}^{14}_{\mathbb{A}}$ are homogeneous. Moreover, we assume that all the variables are not zero allowing some of them are constants. Then it is easy to derive the following relations between the weights of variables of $S_{\mathfrak{U}}$:

$$\begin{aligned}
&w(a_{11}) = -w(y_1) + w(y_3),\ w(a_{12}) = w(x_1y_3) - w(x_2y_1),\ w(a_{21}) = w(x_2y_3) - w(x_1y_1),\\
&w(b_{11}) = -w(y_2) + w(y_3),\ w(b_{12}) = w(x_1y_3) - w(x_2y_2),\ w(b_{21}) = w(x_2y_3) - w(x_1y_2),\\
&w(c_{11}) = w(x_1y_3) - w(x_3y_1),\ w(c_{12}) = w(x_1y_3) - w(x_3y_2),\\
&w(c_{21}) = w(x_2y_3) - w(x_3y_1),\ w(c_{22}) = w(x_2y_3) - w(x_3y_2),\\
&w(s) = -w(x_3) + 2w(y_3),\ w(t) = w(x_1x_2y_3) - w(y_1y_2).
\end{aligned}$$

Example 4.3. In Subsection 4.9, we use the following weights of coordinates:

$$\begin{aligned}
&w(A) = w(B) = w(C) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},\\
&w(x) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, w(y) = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix},\\
&w(s) = 3, w(t) = 2.
\end{aligned}$$

Let

$$\mathfrak{U}^{13}_{\mathbb{P}} \subset \mathbb{P}(1^{15}, 2^2, 3)$$

be the weighted projectivization of $\mathfrak{U}^{14}_{\mathbb{A}}$ by these weights of coordinates. In Subsection 4.9, we show that $\mathfrak{U}^{13}_{\mathbb{P}}$ is a key variety of a prime $\mathbb{Q}$-Fano 3-fold of anticanonical codimension 4 with No.20544. Existence of a positive grading of $\mathfrak{U}^{14}_{\mathbb{A}}$ also plays a role in the proof of Proposition 4.5.

4.4. **Charts and singular locus.** For a coordinate $*$, we call the open subset of $\mathfrak{U}^{14}_{\mathbb{A}}$ with $* \neq 0$ *the* $*$*-chart*. We describe the $*$-chart such that $*$ is one of the entries of x, y, or s, t.

$\underline{x_1}$**-chart**: We note that $\mathfrak{U}^{14}_{\mathbb{A}} \cap \{x_1 \neq 0\}$ is isomorphic to $(\mathfrak{U}^{14}_{\mathbb{A}} \cap \{x_1 = 1\}) \times \mathbb{A}^{1*}$ by the map $(x, y, s, t, A, B, C) \mapsto \big((x_1^{-1}x, x_1^{-1}y, x_1^{-1}s, x_1^{-1}t, A, B, C), x_1\big)$. This is because all the equations of $\mathfrak{U}^{14}_{\mathbb{A}}$ are quadratic when we consider the entries of A, B, C are constants. Therefore it suffices to describe $\mathfrak{U}^{14}_{\mathbb{A}} \cap \{x_1 = 1\}$. Solving regularly the 9 equations of $\mathfrak{U}^{14}_{\mathbb{A}}$ setting $x_1 = 1$, we see that the 9 equations are reduced to the following 4 equations:

$$
\begin{aligned}
y_3 =& -a_{11}y_1 - a_{12}x_2y_1 - c_{11}x_3y_1 - b_{11}y_2 - b_{12}x_2y_2 - c_{12}x_3y_2,\\
s =& 2a_{11}c_{11}y_1^2 + a_{12}c_{21}y_1^2 + a_{12}c_{11}x_2y_1^2 + c_{11}^2x_3y_1^2 + 2b_{11}c_{11}y_1y_2 +\\
& 2a_{11}c_{12}y_1y_2 + b_{12}c_{21}y_1y_2 + a_{12}c_{22}y_1y_2 + b_{12}c_{11}x_2y_1y_2 + a_{12}c_{12}x_2y_1y_2 +\\
& 2c_{11}c_{12}x_3y_1y_2 + 2b_{11}c_{12}y_2^2 + b_{12}c_{22}y_2^2 + b_{12}c_{12}x_2y_2^2 + c_{12}^2x_3y_2^2,\\
b_{21} =& 2b_{11}x_2 + b_{12}x_2^2 - c_{22}x_3 + c_{12}x_2x_3 - ty_1,\\
a_{21} =& 2a_{11}x_2 + a_{12}x_2^2 - c_{21}x_3 + c_{11}x_2x_3 + ty_2.
\end{aligned}
$$

$\underline{x_2}$**-chart**: Similarly to the x_1-chart, we have only to describe $\mathfrak{U}^{14}_{\mathbb{A}} \cap \{x_2 = 1\}$ as follows:

$$
\begin{aligned}
y_3 =& a_{11}y_1 - a_{21}x_1y_1 - c_{21}x_3y_1 + b_{11}y_2 - b_{21}x_1y_2 - c_{22}x_3y_2,\\
s =& a_{21}c_{11}y_1^2 - 2a_{11}c_{21}y_1^2 + a_{21}c_{21}x_1y_1^2 + c_{21}^2x_3y_1^2 + b_{21}c_{11}y_1y_2 +\\
& a_{21}c_{12}y_1y_2 - 2b_{11}c_{21}y_1y_2 - 2a_{11}c_{22}y_1y_2 + b_{21}c_{21}x_1y_1y_2 + a_{21}c_{22}x_1y_1y_2 +\\
& 2c_{21}c_{22}x_3y_1y_2 + b_{21}c_{12}y_2^2 - 2b_{11}c_{22}y_2^2 + b_{21}c_{22}x_1y_2^2 + c_{22}^2x_3y_2^2,\\
b_{12} =& -2b_{11}x_1 + b_{21}x_1^2 - c_{12}x_3 + c_{22}x_1x_3 + ty_1,\\
a_{12} =& -2a_{11}x_1 + a_{21}x_1^2 - c_{11}x_3 + c_{21}x_1x_3 - ty_2.
\end{aligned}
$$

$\underline{s}$**-chart**: Similarly to the x_1-chart, we have only to describe $\mathfrak{U}^{14}_{\mathbb{A}} \cap \{s = 1\}$ as follows:

$$
\begin{aligned}
&x = y^{\sharp},\\
&t = 1/3\,\beta(x, y).
\end{aligned}
$$

$\underline{t}$**-chart**: Similarly to the x_1-chart, we have only to describe $\mathfrak{U}^{14}_{\mathbb{A}} \cap \{t = 1\}$ as follows:

$$
\begin{aligned}
&y = x^{\sharp},\\
&s = 1/3\,\beta(x, y).
\end{aligned}
$$

The above descriptions of charts show that the x_1-, x_2-, s-, and t-charts of $\mathfrak{U}^{14}_{\mathbb{A}}$ are isomorphic to $\mathbb{A}^{13} \times \mathbb{A}^{1*}$. As for s- and t-charts, we also refer to Corollary 2.26.

$\underline{x_3}$**-chart**: Similarly to the x_1-chart, it suffices to describe $\mathfrak{U}^{14}_{\mathbb{A}} \cap \{x_3 = 1\}$. When $x_3 = 1$, we may eliminate the coordinate s by the 3rd entry of the equation $sx = y^{\sharp}$

with $x_3 = 1$. Then we may verify that $\mathfrak{U}^{14}_{\mathbb{A}} \cap \{x_3 = 1\}$ is defined by the five 4×4 Pfaffians of the following skew-symmetric matrix:

$$\begin{pmatrix} 0 & t & a_{11}x_1 + a_{12}x_2 + c_{11} & b_{11}x_1 + b_{12}x_2 + c_{12} & x_1 \\ & 0 & a_{21}x_1 - a_{11}x_2 + c_{21} & b_{21}x_1 - b_{11}x_2 + c_{22} & x_2 \\ & & 0 & y_3 & -y_2 \\ & & & 0 & y_1 \\ & & & & 0 \end{pmatrix}.$$

From this description, we see that

$$(\operatorname{Sing} \mathfrak{U}^{14}_{\mathbb{A}}) \cap \{x_3 \neq 0\} = \{x_3 \neq 0,\, x_1 = x_2 = 0,\, y = 0,\, t = 0,\, C = O\}.$$

We define the following skew-symmetric matrix:

$$A_y = \begin{pmatrix} 0 & s & a_{11}y_1 + b_{11}y_2 + y_3 & a_{12}y_1 + b_{12}y_2 & c_{11}y_1 + c_{12}y_2 \\ & 0 & a_{21}y_1 + b_{21}y_2 & -a_{11}y_1 - b_{11}y_2 + y_3 & c_{21}y_1 + c_{22}y_2 \\ & & 0 & x_3 & -x_2 \\ & & & 0 & x_1 \\ & & & & 0 \end{pmatrix}.$$

<u>y_1**-chart**</u>: Similarly to the x_3-chart, We may verify that $\mathfrak{U}^{14}_{\mathbb{A}} \cap \{y_1 = 1\}$ is defined by the five 4×4 Pfaffians of the skew-symmetric matrix A_y with $y_1 = 1$. From this description, we see that

$$\begin{aligned} (\operatorname{Sing} \mathfrak{U}^{14}_{\mathbb{A}}) \cap \{y_1 \neq 0\} &= \{y_1 \neq 0,\, x = 0,\, s = 0,\, M_y = O\} \\ &= \left\{ y_1 \neq 0,\, x = 0,\, s = y_3 = 0,\, \begin{pmatrix} a_{11} & b_{11} \\ a_{12} & b_{12} \\ c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \begin{pmatrix} 1 \\ y_2 \end{pmatrix} = \boldsymbol{o} \right\}. \end{aligned}$$

<u>y_2**-chart**</u>: Similarly to the x_3-chart, We may verify that $\mathfrak{U}^{14}_{\mathbb{A}} \cap \{y_2 = 1\}$ is defined by the five 4×4 Pfaffians of the following skew-symmetric matrix A_y with $y_2 = 1$. From this description, we see that

$$\begin{aligned} (\operatorname{Sing} \mathfrak{U}^{14}_{\mathbb{A}}) \cap \{y_2 \neq 0\} &= \{y_2 \neq 0,\, x = 0,\, s = 0,\, M_y = O\} \\ &= \left\{ y_2 \neq 0,\, x = 0,\, s = y_3 = 0,\, \begin{pmatrix} a_{11} & b_{11} \\ a_{12} & b_{12} \\ a_{21} & b_{21} \\ c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \begin{pmatrix} y_1 \\ 1 \end{pmatrix} = \boldsymbol{o} \right\}. \end{aligned}$$

<u>y_3**-chart**</u>: We may easily verify that the y_3-chart is contained in one of the above charts.

Let

$$\mathsf{S}_x := \{x_1 = x_2 = 0,\, y = 0,\, s = 0,\, t = 0,\, C = O\} \simeq \mathbb{A}^7,$$

and S_y be the closure of the locus

$$\left\{ x = 0,\, s = t = y_3 = 0,\, \begin{pmatrix} a_{11} & b_{11} \\ a_{12} & b_{12} \\ a_{21} & b_{21} \\ c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \boldsymbol{o} \right\} \setminus \{x = 0,\, y = 0,\, s = t = 0\}.$$

By a consideration with a little linear algebra, we see that

$$\mathsf{S}_y = \left\{ x=0,\ s=t=y_3=0,\ \mathrm{rank} \begin{pmatrix} a_{11} & b_{11} \\ a_{12} & b_{12} \\ a_{21} & b_{21} \\ c_{11} & c_{12} \\ c_{21} & c_{22} \\ -y_2 & y_1 \end{pmatrix} \leq 1 \right\},$$

which is the affine cone over $\mathbb{P}^1 \times \mathbb{P}^5$ and hence is 7-dimensional.

Investigating the 9 equations of $\mathfrak{U}^{14}_{\mathbb{A}}$, we see that

$$\mathsf{S}_x \cap \{x_3 \neq 0\} = (\mathrm{Sing}\, \mathfrak{U}^{14}_{\mathbb{A}}) \cap \{x_3 \neq 0\},$$
$$\mathsf{S}_y \cap \{y_i \neq 0\} = (\mathrm{Sing}\, \mathfrak{U}^{14}_{\mathbb{A}}) \cap \{y_i \neq 0\}\ (i=1,2),$$

Thus, from the above descriptions of the charts, we can describe the singularities of $\mathfrak{U}^{14}_{\mathbb{A}}$ as follows:

Proposition 4.4. *The open subset $\mathfrak{U}^{14}_{\mathbb{A}} \setminus \{x=y=0, s=t=0\}$ is 14-dimensional and irreducible. Its singular locus is equal to the 7-dimensional locus*

$$\Delta := (\mathsf{S}_x \cup \mathsf{S}_y) \setminus \{x=y=0, s=t=0\},$$

and it has $c(\mathrm{G}(2,5))$-singularities along Δ, where we call a singularity isomorphic to the vertex of the cone over $\mathrm{G}(2,5)$ a $c(\mathrm{G}(2,5))$-singularity.

We will show that $\mathfrak{U}^{14}_{\mathbb{A}}$ itself is 14-dimensional and irreducible in Proposition 4.5, and determine the singularities of $\mathfrak{U}^{14}_{\mathbb{A}}$ along $\{x=0, y=0, s=t=0\}$ in Proposition 4.10.

4.5. **Gorensteinness and 9×16 graded minimal free resolution of the ideal of $\mathfrak{U}^{14}_{\mathbb{A}}$.**

Proposition 4.5. *Let $S_{\mathfrak{U}}$ be the polynomial ring over $\mathbb{C}$ whose variables are s, t and the entries of A, B, C, x, y. Let $I_{\mathfrak{U}}$ be the ideal of the polynomial ring $S_{\mathfrak{U}}$ generated by the 9 equations of $\mathfrak{U}^{14}_{\mathbb{A}}$. Set $R_{\mathfrak{U}} := S_{\mathfrak{U}}/I_{\mathfrak{U}}$. The following assertions hold:*

(1) *We give nonnegative weights for coordinates of $S_{\mathfrak{U}}$ such that all the equations of $\mathfrak{U}^{14}_{\mathbb{A}}$ are weighted homogeneous, and we denote by $w(*)$ the weight of the monomial $*$. We denote by $\mathbb{P}$ the corresponding weighted projective space, and by $\mathfrak{U}_{\mathbb{P}} \subset \mathbb{P}$ the weighted projectivization of $\mathfrak{U}^{14}_{\mathbb{A}}$, where we allow some coordinates being nonzero constants (thus $\dim \mathfrak{U}_{\mathbb{P}}$ could be less than 13). We set*

$$\delta = 2w(x_1x_2) - w(x_3) - w(y_1y_2) + 5w(y_3).$$

(1-1) *It holds that*

$$\omega_{\mathbb{P}} = \mathcal{O}_{\mathbb{P}}(-4w(x_1x_2) + 5w(y_1y_2) + 4w(x_3) - 14w(y_3)).$$

(1-2) *The ideal $I_{\mathcal{H}}$ has the following graded minimal $S_{\mathcal{H}}$-free resolution*

$$0 \leftarrow P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow P_3 \leftarrow P_4 \leftarrow 0, \text{ where} \tag{4.1}$$

$$\begin{aligned}
P_0 =& S,\\
P_1 =& S(-w(x_1y_3)) \oplus S(-w(x_2y_3))\\
&\oplus S(-w(sx_1)) \oplus S(-w(sx_2)) \oplus S(-w(sx_3))\\
&\oplus S(-w(ty_1)) \oplus S(-w(ty_2)) \oplus S(-w(ty_3))\\
&\oplus S(-w(st)),\\
P_2 =& S(-w(sx_1y_3)) \oplus S(-w(sx_2y_3)) \oplus S(-w(tx_1y_3)) \oplus S(-w(tx_2y_3))\\
&\oplus S(-w(stx_1)) \oplus S(-w(stx_2)) \oplus S(-w(stx_3))\\
&\oplus S(-w(sty_1)) \oplus S(-w(sty_2)) \oplus S(-w(sty_3))\\
&\oplus S(-w(sx_1x_2)) \oplus S(-w(sx_1x_3)) \oplus S(-w(sx_2x_3))\\
&\oplus S(-w(ty_1y_2)) \oplus S(-w(ty_1y_3)) \oplus S(-w(ty_2y_3))\\
P_3 =& S(-(\delta - w(x_1y_3))) \oplus S(-(\delta - w(x_2y_3)))\\
&\oplus S(-(\delta - w(sx_1))) \oplus S(-(\delta - w(sx_2))) \oplus S(-(\delta - w(sx_3)))\\
&\oplus S(-(\delta - w(ty_1))) \oplus S(-(\delta - w(ty_2))) \oplus S(-(\delta - w(ty_3)))\\
&\oplus S(-(\delta - w(st))),\\
P_4 =& S(-\delta).
\end{aligned}$$

(1-3) *It holds that*

$$\omega_{\mathfrak{U}_{\mathbb{P}}} = \mathcal{O}_{\mathfrak{U}_{\mathbb{P}}}(-2w(x_1x_2) + 4w(y_1y_2) + 3w(x_3) - 9w(y_3)). \tag{4.2}$$

(2) $I_{\mathfrak{U}}$ *is a Gorenstein ideal of codimension* 4.
(3) $\mathfrak{U}^{14}_{\mathbb{A}}$ *is irreducible and reduced, thus* $I_{\mathfrak{U}}$ *is a prime ideal.*
(4) $\mathfrak{U}^{14}_{\mathbb{A}}$ *is normal.*

Proof. We may compute the $S_{\mathfrak{U}}$-free resolution (4.1) of $I_{\mathfrak{U}}$ by SINGULAR [DGPS]. For the remaining assertions, the proof of [Tak6, Prop.4.8] works verbatim. □

4.6. **Factoriality of $\mathfrak{U}^{14}_{\mathbb{A}}$.** We denote by $\bar{x_1}$ the image of x_1 in $R_{\mathfrak{U}}$.

Lemma 4.6. $\mathfrak{U}^{14}_{\mathbb{A}} \cap \{x_1 = 0\}$ *is irreducible and reduced, and is normal. In particular, the element* $\bar{x_1} \in R_{\mathfrak{U}}$ *is a prime element.*

Proof. By [Tak7, Lem.3.6], it suffices to show that $\mathrm{Sing}\,(\mathfrak{U}^{14}_{\mathbb{A}} \cap \{x_1 = 0\})$ has codimension ≥ 2 in $\mathfrak{U}^{14}_{\mathbb{A}} \cap \{x_1 = 0\}$. By the descriptions of charts as in Subsection 4.4, we see that $\mathfrak{U}^{14}_{\mathbb{A}} \cap \{x_1 = 0\}$ is smooth on the x_2- and the t-charts, and is isomorphic to a hypersurface

$$\begin{aligned}
\{&-a_{11}c_{11}y_1^2 - a_{12}c_{21}y_1^2 - b_{11}c_{11}y_1y_2 - a_{11}c_{12}y_1y_2 - b_{12}c_{21}y_1y_2\\
&- a_{12}c_{22}y_1y_2 - b_{11}c_{12}y_2^2 - b_{12}c_{22}y_2^2 + c_{11}y_1y_3 + c_{12}y_2y_3 = 0\}
\end{aligned}$$

on the s-chart, whose singular locus has codimension ≥ 2 in $\mathfrak{U}^{14}_{\mathbb{A}} \cap \{x_1 = 0\}$. The locus $\mathfrak{U}^{14}_{\mathbb{A}} \cap \{x_1 = x_2 = 0, s = t = 0\}$ is the union of the following two loci (i) and (ii):

(i) $\{x = 0, s = t = 0, y^\sharp = 0\}$, which is a fibration of relative dimension 7 over the affine cone of $\mathbb{P}^1 \times \mathbb{P}^2$, hence is 11-dimensional.
(ii) $\{\mathrm{rank} \begin{pmatrix} c_{11} & c_{21} & -y_2 \\ c_{12} & c_{22} & y_1 \end{pmatrix} \leq 1, F_3 = 0\}$, which is easily to be seen of codimension 2 in $\mathfrak{U}^{14}_{\mathbb{A}} \cap \{x_1 = 0\}$ (F_3 is defined in the proof of (1)).

Therefore we have shown that $\mathrm{Sing}\,(\mathfrak{U}^{14}_{\mathbb{A}} \cap \{x_1 = 0\})$ has codimension ≥ 2 in $\mathfrak{U}^{14}_{\mathbb{A}} \cap \{x_1 = 0\}$. □

Proposition 4.7. *The affine coordinate ring $R_{\mathfrak{U}}$ of $\mathfrak{U}^{14}_{\mathbb{A}}$ is a UFD.*

Proof. Using the description of the x_1-chart as in Subsection 4.4 and Lemma 4.6, the proof of [Tak6, Prop.4.9] work verbatim. □

The following corollary can be proved in the same way as the proof of [Tak6, Cor.4.10]:

Corollary 4.8. *Let $\mathfrak{U}^{13}_{\mathbb{P}}$ be the weighted projectivization of $\mathfrak{U}^{14}_{\mathbb{A}}$ with some positive weights of coordinates. The following assertions hold:*

(1) Any prime Weil divisor on $\mathfrak{U}^{13}_{\mathbb{P}}$ is the intersection between $\mathfrak{U}^{13}_{\mathbb{P}}$ and a weighted hypersurface. In particular, $\mathfrak{U}^{13}_{\mathbb{P}}$ is $\mathbb{Q}$-factorial and has Picard number one.

(2) Let X be a quasi-smooth threefold such that X is a codimension 10 weighted complete intersection in $\mathfrak{U}^{13}_{\mathbb{P}}$, i.e., there exist ten weighted homogeneous polynomials $G_1, \dots, G_{10}$ such that $X = \mathfrak{U}^{13}_{\mathbb{P}} \cap \{G_1 = 0\} \cap \cdots \cap \{G_{10} = 0\}$. Assume moreover that $\{x_1 = 0\} \cap X$ is a prime divisor. Then any prime Weil divisor on X is the intersection between X and a weighted hypersurface. In particular, X is $\mathbb{Q}$-factorial and has Picard number one.

4.7. $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$**-fibration .** We denote by $\mathbb{A}_{\mathrm{P}} \simeq \mathbb{A}^{10}$ the affine space whose coordinates are the entries of the matrices A, B, C. Note that any equation of $\mathfrak{U}^{14}_{\mathbb{A}}$ is of degree two if we regard the entries of A, B, C as constants. Therefore, considering the variables of the equations of $\mathfrak{U}^{14}_{\mathbb{A}}$ except the entries of A, B, C as projective coordinates, we obtain a 13-dimensional quasi-projective variety with the same equation as $\mathfrak{U}^{14}_{\mathbb{A}}$. We denote this 13-dimensional variety by $\widehat{\mathfrak{U}}$, and by $\rho_{\mathfrak{U}} \colon \widehat{\mathfrak{U}} \to \mathbb{A}_{\mathrm{P}}$ the natural projection. Note that the $\mathrm{SL}_2 \times \mathrm{SL}_2$-action on $\mathfrak{U}^{14}_{\mathbb{A}}$ defined as in Subsection 4.2 induces those on $\widehat{\mathfrak{U}}$ and those on $\mathbb{A}_{\mathrm{P}}$, and the natural projection $\rho_{\mathfrak{U}} \colon \widehat{\mathfrak{U}} \to \mathbb{A}_{\mathrm{P}}$ is equivariant with respect to these actions. Using these group actions, detail analysis of $\rho_{\mathfrak{U}} \colon \widehat{\mathfrak{U}} \to \mathbb{A}_{\mathrm{P}}$ is possible. We, however, omit details since it is lengthy. Instead, we will describe the restriction of $\rho_{\mathfrak{U}}$ to the subvariety $\mathfrak{S}^8_{\mathbb{A}}$ defined in Section 5. Here we only state the following, which can be shown immediately by Corollary 2.31:

Proposition 4.9. *Let $\Delta_{\rho_{\mathfrak{U}}}$ be the closed subset of $\mathbb{A}_{\mathrm{P}}$ defined by D_β as in Remark 2.30. The $\rho_{\mathfrak{U}}$-fiber over a point outside $\Delta_{\rho_{\mathfrak{U}}}$ is isomorphic to $\mathbb{P} \times \mathbb{P}^1 \times \mathbb{P}^1$.*

4.8. **Terminal singularities.** By Proposition 4.4 and the result in Subsection 4.7, we can show the following in the same way as [Tak6, Prop.4.11]:

Proposition 4.10. *The variety $\mathfrak{U}^{14}_{\mathbb{A}}$ has only terminal singularities with the following descriptions:*

(1) The singularities along the 7-dimensional locus $(\mathsf{S}_x \cup \mathsf{S}_y) \setminus \{x = 0, y = 0, s = t = 0\}$ are $c(G(2,5))$-singularities.

(2) There exists a primitive K-negative divisorial extraction $f \colon \widetilde{\mathfrak{U}} \to \mathfrak{U}^{14}_{\mathbb{A}}$ such that

   (a) singularities of $\widetilde{\mathfrak{U}}$ are only $c(G(2,5))$-singularities along the strict transforms of $\mathsf{S}_x \cup \mathsf{S}_y$, and

   (b) for the f-exceptional divisor $E_{\mathfrak{U}}$, the morphism $f|_{E_{\mathfrak{U}}}$ can be identified with $\rho_{\mathfrak{U}} \colon \widehat{\mathfrak{U}}^{13} \to \mathbb{A}^{10}_{\mathrm{B}}$ as in Subsection 4.7.

4.9. **An example of a $\mathbb{Q}$-Fano threefold.** We consider the weighted projectivization of $\mathfrak{U}^{14}_{\mathbb{A}}$

$$\mathfrak{U}^{13}_{\mathbb{P}} \subset \mathbb{P}(1^{15}, 2^2, 3)$$

by the weights of coordinates as in Example 4.3. In this subsection, we show the following:

Theorem 4.11. *Let $L_1, \dots, L_{10}$ be general forms of weight 1 in $\mathbb{P}(1^{15}, 2^2, 3)$. The subscheme*

$$X := \mathfrak{U}^{13}_{\mathbb{P}} \cap L_1 \cap \cdots \cap L_{10}$$

is a prime $\mathbb{Q}$-Fano 3-fold of No.20544 and of anticanonical codimension 4.

Proof. In this proof, we denote $\mathbb{P}(1^{15}, 2^2, 3)$ by $\mathbb{P}$ for simplicity. Using the equations of $\mathfrak{U}^{13}_{\mathbb{P}}$, we may easily verify that $\mathrm{Bs}|\mathcal{O}_{\mathbb{P}}(1)| \cap \mathfrak{U}^{13}_{\mathbb{P}}$ consists of the s-point and the t-point, where the $*$-point for a coordinate $*$ of $\mathbb{P}$ means the point of $\mathbb{P}$ such that all the coordinates except $*$ are zero. Note that $\dim \mathrm{Sing}\, \mathfrak{U}^{14}_{\mathbb{A}}$ is less than the codimension of X in $\mathfrak{U}^{13}_{\mathbb{P}}$. Therefore, by the Bertini theorem, we see that X is a smooth 3-fold outside the s-point and the t-point. Computing the linear parts of the equations of X at each of the s-point and the t-point (cf. LPC in [Tak3, Subsec. 5.1]), we see that X has a $1/3(1,1,2)$-singularity at the s-point and a $1/2(1,1,1)$-singularity at the t-point. By Proposition 4.5 (1-2), we have the following $S_{\mathfrak{U}}$-free minimal resolution of $R_{\mathfrak{U}}$ which is graded with respect to the weights of variables given in Example 4.3:

$$\begin{aligned} &0 \leftarrow R_{\mathfrak{U}} \leftarrow S_{\mathfrak{U}} \leftarrow S_{\mathfrak{U}}(-3)^{\oplus 4} \oplus S_{\mathfrak{U}}(-4)^{\oplus 4} \oplus S_{\mathfrak{U}}(-5) \\ &\leftarrow S_{\mathfrak{U}}(-4) \oplus S_{\mathfrak{U}}(-5)^{\oplus 7} \oplus S_{\mathfrak{U}}(-6)^{\oplus 7} \oplus S_{\mathfrak{U}}(-7) \\ &\leftarrow S_{\mathfrak{U}}(-6) \oplus S_{\mathfrak{U}}(-7)^{\oplus 4} \oplus S_{\mathfrak{U}}(-8)^{\oplus 4} \leftarrow S_{\mathfrak{U}}(-11) \leftarrow 0. \end{aligned}$$

From this, we see that $-K_X = \mathcal{O}_X(1)$ and $(-K_X)^3 = 31/6$. It remains to show that $\rho(X) = 1$. By Corollary 4.8, it suffices to check that $X \cap \{x_1 = 0\}$ is a prime divisor. This follows by the Bertini theorem since $\mathfrak{U}^{13}_{\mathbb{P}} \cap \{x_1 = 0\}$ is normal by Proposition 4.5 (3), and the fact that $\mathrm{Bs}|\mathcal{O}_{\mathbb{P}}(1)| \cap \mathfrak{U}^{13}_{\mathbb{P}}$ consists of the s-point and the t-point. Hence X is a prime $\mathbb{Q}$-Fano 3-fold of No.20544, and by [Tak6, Proof of Thm.1.2 (1)], X is of anti-canonical codimension 4. □

Remark 4.12. In a future work, we describe via $\mathfrak{U}^{13}_{\mathbb{P}}$ the Sarkisov link for a prime $\mathbb{Q}$-Fano threefold of No.20544 starting from the weighted blow-up at the unique 1/3(1,1,2)-singularity.

5. A specialization of $\mathfrak{U}^{14}_{\mathbb{A}}$–an affine variety $\mathfrak{S}^{8}_{\mathbb{A}}$ with an SL_2-action–

In this section, we define a closed subvariety $\mathfrak{S}^{8}_{\mathbb{A}}$ of the affine variety $\mathfrak{U}^{14}_{\mathbb{A}}$ and investigate its properties. The whole story in this section is very similar to that of Section 4, hence we will not write down it fully.

As we will see in Subsection 7.4, $\mathfrak{S}^{8}_{\mathbb{A}}$ is isomorphic to a subvariety of the $G_2^{(4)}$-cluster variety. We will discuss there a prime $\mathbb{Q}$-Fano threefold of anti-canonical codimension 4 which is obtained as a weighted complete intersection of a weighted projectivization of $\mathfrak{S}^{8}_{\mathbb{A}}$.

5.1. **Definition.**

Definition 5.1. We set

$$\mathfrak{S}^8_{\mathbb{A}} := \mathfrak{U}^{14}_{\mathbb{A}} \cap \left\{ b_{11} = a_{12}, b_{21} = -a_{11}, \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

Remark 5.2. We note that the condition on $\mathfrak{S}^8_{\mathbb{A}}$ is equivalent to that M_x and M_y as in Subsection 4.1 have the same form.

Now we introduce different notation for coordinates of $\mathfrak{S}^8_{\mathbb{A}}$ as follows, with which we derive another presentation of the equations of $\mathfrak{S}^8_{\mathbb{A}}$ (Proposition 5.3) such that an group action on $\mathfrak{S}^8_{\mathbb{A}}$ will be more visible (Subsection 5.2):

$$P = \begin{pmatrix} d_2 & d_1 & 1 \\ d_1 & d_0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, Q = \begin{pmatrix} -d_3 & -d_2 & 0 \\ -d_2 & -d_1 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$U := \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix} := \begin{pmatrix} y_1 & -x_1 \\ y_2 & -x_2 \end{pmatrix},$$

$$V := \begin{pmatrix} v_2 & v_1 & v_0 \\ -v_3 & -v_2 & -v_1 \end{pmatrix} := \begin{pmatrix} y_3 & x_3 & t \\ -s & -y_3 & -x_3 \end{pmatrix},$$

We also set

$$\widehat{U} := \begin{pmatrix} u_1^2 & u_1u_2 & u_2^2 \\ 2u_1u_3 & u_1u_4 + u_2u_3 & 2u_2u_4 \\ u_3^2 & u_3u_4 & u_4^2 \end{pmatrix}, \widehat{U}^{\dagger} := \begin{pmatrix} u_4^2 & -u_2u_4 & u_2^2 \\ -2u_3u_4 & u_1u_4 + u_2u_3 & -2u_1u_2 \\ u_3^2 & -u_1u_3 & u_1^2 \end{pmatrix},$$

$$D := \begin{pmatrix} d_2 & d_1 & d_0 \\ -d_3 & -d_2 & -d_1 \end{pmatrix}.$$

By a straightforward calculation, we can check the following:

Proposition 5.3. *The affine variety $\mathfrak{S}^8_{\mathbb{A}}$ is defined by the following equations:*

$$UV = D\widehat{U},\ \wedge^2 V = (\wedge^2 D)\,{}^t\widehat{U}^{\dagger}. \tag{5.1}$$

Remark 5.4. If $\det U \neq 0$, then $\wedge^2 V = (\wedge^2 D)\,{}^t\widehat{U}^{\dagger}$ can be derived from $UV = D\widehat{U}$ by the Cauchy-Binet formula.

5.2. $((\mathrm{SL}_2 \rtimes \mathrm{SL}_2) \times (\mathbb{C}^*)^2)$**-action.**

We denote by $S_{\mathfrak{S}}$ the polynomial ring over the field $\mathbb{C}$ with the entries of U, V, D as the variables. We consider the affine space $\mathbb{A}^{12}_{\mathfrak{S}}$ with the coordinate ring $S_{\mathfrak{S}}$. We denote by $\mathbb{A}^4_D$ the affine space with the entries of D as the coordinates.

From the presentation of the equation of $\mathfrak{S}^8_{\mathbb{A}}$ as in Proposition 5.3, we can immediately read off an $\big((\mathrm{SL}_2 \rtimes \mathrm{SL}_2) \times (\mathbb{C}^*)^2\big)$-action on $\mathfrak{S}^8_{\mathbb{A}}$ as follows:

Proposition 5.5. *The following assertions hold:*

(1) (1-1) *The variety $\mathfrak{S}^8_{\mathbb{A}}$ is preserved by the following actions of the two groups $\mathrm{SL}_2^{\mathrm{I}}$, $\mathrm{SL}_2^{\mathrm{II}}$ isomorphic to SL_2, and they define an action of the group $\mathrm{SL}_2^{\mathrm{II}} \rtimes \mathrm{SL}_2^{\mathrm{I}}$ on $\mathfrak{S}^8_{\mathbb{A}}$:*

$$\text{For } g \in \mathrm{SL}_2^{\mathrm{I}}, U \mapsto gUg^{-1},\ V \mapsto gV\widehat{g}^{-1},\ D \mapsto gD\widehat{g}^{-1},$$

$$\text{For } h \in \mathrm{SL}_2^{\mathrm{II}}, U \mapsto Uh,\ V \mapsto h^{-1}V\widehat{h},\ D \mapsto D,$$

where the definitions of $\widehat{g}$ and $\widehat{h}$ for g and h respectively are similar to that of $\widehat{U}$ for U.

(1-2) The affine space $\mathbb{A}^{12}_{\mathfrak{S}}$ has the $(\mathbb{C}^*)^2$-action preserving $\mathfrak{S}^8_{\mathbb{A}}$ defined by

$$U \mapsto \alpha U,\ D \mapsto \beta D,\ V \mapsto \alpha\beta V,$$

where $\alpha, \beta \in \mathbb{C}^*$.

These induce an $\big((\mathrm{SL}_2^{\mathrm{II}} \rtimes \mathrm{SL}_2^{\mathrm{I}}) \times (\mathbb{C}^*)^2\big)$-action on $\mathfrak{S}^8_{\mathbb{A}}$.

(2) The induced $(\mathrm{SL}_2^{\mathrm{I}} \times (\mathbb{C}^*)^2)$-action on $\mathbb{A}^4_D$ has the following orbits:

(a) $\{0\}$.

(b) The complement of $\{0\}$ in the cone over the twisted cubic γ_D defined by $\wedge^2 D = \mathbf{0}$.

(c) The complement of the orbits as described in (a) and (b) in the cone over the tangential scroll of the twisted cubic γ_D, where the equation of the tangential scroll is

$$3d_1^2d_2^2 - 4d_1^3d_3 - 4d_0d_2^3 + 6d_0d_1d_2d_3 - d_0^2d_3^2 = 0. \tag{5.2}$$

(d) The complement of the orbits as described in (a), (b) and (c) in $\mathbb{A}^4_D$.

Proof. For (1), we only note that $\widehat{U}$ is mapped to $\widehat{g}\widehat{U}\widehat{g}^{-1}$ by the $\mathrm{SL}_2^{\mathrm{I}}$-action. The assertion (2) is well-known to be true. □

Remark 5.6. By an explicit calculation, we may easily verify that the rank of the matrix associated to the bi-linear trace β for $\mathfrak{S}^8_{\mathbb{A}}$ is $0, 1, 2,$ or 3 if and only if the Proposition 5.5 (2) (a), (b), (c), or (d) holds respectively.

5.3. $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$**-fibration.** We denote by $\mathbb{P}^7_{\mathtt{F}}$ the projective space whose coordinates are the entries of the matrices U, V. Note that any equation of $\mathfrak{S}^8_{\mathbb{A}}$ is of degree two if we regard the entries of D as constants. Therefore, considering the variables of the equations of $\mathfrak{S}^8_{\mathbb{A}}$ except the entries of D as projective coordinates, we obtain a 7-dimensional variety in $\mathbb{A}^4_{\mathrm{D}} \times \mathbb{P}^7_{\mathtt{F}}$. We denote this 7-dimensional variety by $\widehat{\mathfrak{S}}^7$, and by $\rho_{\mathfrak{S}} \colon \widehat{\mathfrak{S}}^7 \to \mathbb{A}^4_D$ the natural projection. Note that the $(\mathrm{SL}_2 \times (\mathbb{C}^*)^2)$-action on $\mathfrak{S}^8_{\mathbb{A}}$ defined as in Subsection 5.2 induces those on $\widehat{\mathfrak{S}}^7$ and those on $\mathbb{A}_{\mathrm{D}}$, and the natural projection $\rho_{\mathfrak{S}} \colon \widehat{\mathfrak{S}}^7 \to \mathbb{A}^4_{\mathrm{D}}$ is equivariant with respect to these actions. Using these group actions, we will describe $\rho_{\mathfrak{S}} \colon \widehat{\mathfrak{S}}^7 \to \mathbb{A}^4_D$.

Lemma 5.7. *The following two assertions hold:*

(1) In the projective 7-space with coordinates x_{ij} $(i = 0, 1,\ 0 \le j \le 3)$, let

$$\mathsf{S} := \left\{ \mathrm{rank} \begin{pmatrix} x_{00} & x_{01} & x_{02} & x_{03} \\ x_{10} & x_{11} & x_{12} & x_{13} \end{pmatrix} \le 1 \right\},$$

which is nothing but the image of the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^3$. Let $\mathsf{Q} \subset \mathbb{P}^3$ be a cone over a smooth conic. The 3-fold $\mathbb{P}^1 \times \mathsf{Q}$ which is embedded in $\mathbb{P}^7$ by the restriction of the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^3$ is projectively equivalent to

$$\left\{ \mathrm{rank} \begin{pmatrix} x_{00} & x_{01} & x_{02} & x_{03} \\ x_{10} & x_{11} & x_{12} & x_{13} \end{pmatrix} \le 1,\ x_{03}^2 = x_{01}x_{02},\ x_{13}^2 = x_{11}x_{12},\ x_{03}x_{13} = x_{02}x_{11} \right\}. \tag{5.3}$$

The projective variety defined by the equation (5.3) is a sextic del Pezzo 3-fold with A_1 singularities along the $(x_{00} : x_{10})$-line.

(2) In the projective 7-space with coordinates $u_i\,(1\le i\le 4), v_j\,(0\le j\le 3)$, the variety defined by

(5.4)
$$\left\{\mathrm{rk}\begin{pmatrix} u_3 & v_3 & v_2 & v_1\\ u_4 & v_2 & v_1 & v_0\end{pmatrix}\le 1, u_1v_2-u_2v_3=u_3^2, u_1v_1-u_2v_2=u_3u_4, u_1v_0-u_2v_1=u_4^2\right\}$$

is a sextic del Pezzo 3-fold with A_2 singularities along the $(u_1:u_2)$-line. This variety is isomorphic to the projective 3-fold $\mathbb{P}^{1,1,1}$ defined in [Fuk, Def.6.6].

Proof. (1). Let $s\colon \mathbb{P}^1\times\mathbb{P}^3\to \mathsf{S}$ be the Segre embedding defined by $x_{ij}=p_iq_j$, where $p_i\,(i=0,1)$ and $q_j\,(0\le j\le 3)$ are the coordinates of $\mathbb{P}^1$ and $\mathbb{P}^3$ respectively. Then the inverse image of (5.3) is $\{q_3^2=q_1q_2\}\subset\mathbb{P}^1\times\mathbb{P}^3$, which is the product of $\mathbb{P}^1$ and the cone over a smooth conic in $\mathbb{P}^3$. Therefore we have shown the former assertion.

Now we show the latter assertion. By the equation (5.3), we see that the projective variety defined by the equation (5.3) has A_1 singularities along the $(x_{00}:x_{10})$-line. We can compute the minimal free resolution of the structure sheaf of the variety defined by (5.3) as follows:

$$\mathcal{O}(-2)^{\oplus 9}\leftarrow\mathcal{O}(-3)^{\oplus 16}\leftarrow\mathcal{O}(-4)^{\oplus 9}\leftarrow\mathcal{O}(-6).$$

From this, we see that the projective variety defined by the equation (5.3) is a sextic del Pezzo 3-fold.

(2). The former assertion can be shown similarly to the proof of the latter assertion of (1). The latter assertion follows since $\mathbb{P}^{1,1,1}$ is also a sextic del Pezzo 3-fold with A_2 singularities along a line by [Fuk, Rem.6.7] and there is only one such a sextic del Pezzo 3-fold (see [Fuj, (si31i)]). □

Proposition 5.8. *Let p be a point of $\mathbb{A}^4_D$ and F_{p} the $\rho_{\mathfrak{S}}$-fiber over p. We use the descriptions of the $(\mathrm{SL}_2^{\mathrm{I}}\times(\mathbb{C}^*)^2)$-action on $\mathbb{A}^4_D$ as in Proposition 5.5 (a)–(d).*

If $\mathsf{p}=0$, then $F_{\mathsf{p}}=\{UV=\mathbf{0},\wedge^2V=\mathbf{0}\}$.

If p belongs to the orbit as in (b), then $F_{\mathsf{p}}\simeq\mathbb{P}^{1,1,1}$.

If p belongs to the orbit as in (c), then $F_{\mathsf{p}}\simeq\mathbb{P}^1\times\mathsf{Q}$.

If p belongs to the orbit as in (d), then $F_{\mathsf{p}}\simeq\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$.

Proof. If $\mathsf{p}=0$, then the description of F_{p} follows from the equation of $\mathfrak{S}^8_{\mathbb{A}}$.

By the $(\mathrm{SL}_2^{\mathrm{I}}\times(\mathbb{C}^*)^2)$-action, we may choose the point p as a special point as follows according to the orbit to which p belongs:

If p belongs to the orbit as in (b), we may assume p is the d_0-point. Then the equation of F_{p} is

$$\left\{\mathrm{rk}\begin{pmatrix} u_3 & v_3 & v_2 & v_1\\ u_4 & v_2 & v_1 & v_0\end{pmatrix}\le 1, u_1v_2-u_2v_3=u_3^2, u_1v_1-u_2v_2=u_3u_4, u_1v_0-u_2v_1=u_4^2\right\},$$

which is nothing but $\mathbb{P}^{1,1,1}$.

If p belongs to the orbit as in (c), we may assume p is the d_1-point. Then the equation of F_{p} coincides with (5.3) by setting

$$\begin{pmatrix} x_{00} & x_{01} & x_{02} & x_{03}\\ x_{10} & x_{11} & x_{12} & x_{13}\end{pmatrix}=\begin{pmatrix} u_1 & v_1+2u_4 & v_3 & v_2+u_3\\ u_2 & v_0 & v_2-2u_3 & v_1-u_4\end{pmatrix},$$

hence F_{p} is isomorphic to $\mathbb{P}^1\times\mathsf{Q}$.

Now assume that p belongs to the orbit as in (d). Note that the cone over the tangential scroll of the twisted cubic γ_D as in (5.2) is the restriction of $D_\beta=0$

as in Remark 2.30, which follows by a straightforward calculation. Therefore, by Corollary 2.31, we see that $F_{\mathsf{p}} \simeq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. $\square$

5.4. **Summary of properties of $\mathfrak{S}^8_{\mathbb{A}}$.** In this subsection, we sum up the properties of $\mathfrak{S}^8_{\mathbb{A}}$ corresponding to those of $\mathfrak{U}^{14}_{\mathbb{A}}$ as in Subsections 4.5, 4.6 and 4.8 as follows:

Proposition 5.9. *The following assertions hold:*

(1) *Let $S_{\mathfrak{S}}$ be the polynomial ring over $\mathbb{C}$ whose variables are the entries of U, V, D. Let $I_{\mathfrak{S}}$ be the ideal of the polynomial ring $S_{\mathfrak{S}}$ generated by the 9 equations of $\mathfrak{S}^8_{\mathbb{A}}$. Set $R_{\mathfrak{S}} := S_{\mathfrak{S}}/I_{\mathfrak{S}}$.*
   - *(1-1) $I_{\mathfrak{S}}$ is a prime, and Gorenstein ideal of codimension four.*
   - *(1-2) $\mathfrak{S}^8_{\mathbb{A}}$ and $\mathfrak{S}^8_{\mathbb{A}} \cap \{u_2 = 0\}$ are normal.*
   - *(1-3) $R_{\mathfrak{S}}$ is a UFD.*

(2) *The weighted projectivization $\mathfrak{S}^7_{\mathbb{P}}$ of $\mathfrak{S}^8_{\mathbb{A}}$ with some positive weights of coordinates is $\mathbb{Q}$-factorial and has Picard number one (we refer to Example 7.9 for an example of such a set of weights).*

(3) *There exists a primitive K-negative divisorial extraction $f\colon \widetilde{\mathfrak{S}} \to \mathfrak{S}^8_{\mathbb{A}}$ such that $\widetilde{\mathfrak{S}}$ is smooth, and, for the f-exceptional divisor $E_{\mathfrak{S}}$, the morphism $f|_{E_{\mathfrak{S}}}$ can be identified with $\rho_{\mathfrak{S}}\colon \widehat{\mathfrak{S}}^7 \to \mathbb{A}^4_{\mathrm{D}}$ as in Subsection 5.3. In particular, the variety $\mathfrak{S}^8_{\mathbb{A}}$ has only terminal singularities.*

We omit the proof since we can prove this in the same (and simpler) way as for the variety $\mathfrak{U}^{14}_{\mathbb{A}}$.

As for examples of prime $\mathbb{Q}$-Fano 3-folds obtained from $\mathfrak{S}^8_{\mathbb{A}}$, we see in Subsection 7.4 that they are actually obtained from the $G_2^{(4)}$-variety.

6. Another specialization of $\mathfrak{F}^{22}_{\mathbb{A}}$–an affine variety $\mathfrak{Z}^{12}_{\mathbb{A}}$ with an SL_3-action–

In this section, we consider one more specialization of $\mathfrak{F}^{22}_{\mathbb{A}}$ as follows:

Definition 6.1. We set

$$\mathfrak{Z}^{12}_{\mathbb{A}} := \mathfrak{F}^{22}_{\mathbb{A}} \cap \{P = E,\ \mathrm{Tr}\, Q = 0\},$$

where E is the 3×3 identity matrix, and $\mathrm{Tr}\, Q$ is the trace of the matrix $Q = (q_{ij})$.

Actually, the equations of $\mathfrak{Z}^{12}_{\mathbb{A}}$ is originally obtained in [Re, Ex.6.8]. Indeed, we note that the equation $y_1 y_2 = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} A^{\dagger} \begin{pmatrix} x_4 \\ x_5 \\ x_6 \end{pmatrix}$ given in [Re, p.24] corresponds to

$$st = -{}^t x Q^{\dagger} y,$$

where $A^{\dagger}$ and $Q^{\dagger}$ are the adjoint matrices of A and Q, respectively, and A, y_1, y_2, $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $\begin{pmatrix} x_4 \\ x_5 \\ x_6 \end{pmatrix}$ in ibid. correspond to $Q, -t, s, x, y$ here, respectively. For, we can verify that

$$\begin{aligned} \beta(x,y) = &-3\,{}^t x Q^{\dagger} y \\ &- 2(q_{12}q_{21} + q_{13}q_{31} + q_{23}q_{32} + q_{22}^2 + q_{22}q_{33} + q_{33}^2)\,{}^t x y. \end{aligned}$$

Moreover, we can also verify that all the other equations of $\mathfrak{Z}^{12}_{\mathbb{A}}$ can be identified with the eight 4×4 Pfaffians in [Re, p.24, 25].

Lemma 6.2. *The following assertions hold:*

(1) *The scheme $\mathfrak{Z}^{12}_{\mathbb{A}}$ is Gorenstein of codimension 4, and is normal.*
(2) *The affine coordinate ring R_3 of $\mathfrak{Z}^{12}_{\mathbb{A}}$ is a UFD.*

Proof. (1). We can check that the x_i-, y_i-, s-, and t-charts $(i=1,2,3)$ of $\mathfrak{Z}^{12}_{\mathbb{A}}$ are smooth and the union of them is irreducible and is of codimension 4 in the ambient affine space. Using [DGPS], we can show that the ideal of $\mathfrak{Z}^{12}_{\mathbb{A}}$ has the 9×16 graded minimal free resolution of length 4. Therefore, by the proofs of [Tak6, Prop. 4.8], we obtain (1).

(2). We can check that $(R_3)_{x_1}$ is a localization of a polynomial ring. We can also check that $\mathfrak{Z}^{12}_{\mathbb{A}}\cap\{x_1=0\}$ is normal. Indeed, on the x_2-, x_3-, y_2- and y_3-charts, $\mathfrak{Z}^{12}_{\mathbb{A}}\cap\{x_1=0\}$ is smooth, and $\mathfrak{Z}^{12}_{\mathbb{A}}\cap\{x_1=x_2=x_3=y_2=y_3=0\}$ is of codimension 2 in $\mathfrak{Z}^{12}_{\mathbb{A}}\cap\{x_1=0\}$. Therefore The proofs of [Tak6, Prop. 4.9] work verbatim for (2). $\square$

The main purpose of this section is to interpret $\mathfrak{Z}^{12}_{\mathbb{A}}$ in the context of [Tak5] as follows:

Proposition 6.3. *We define $\mathfrak{Z}^{11}_{\mathbb{P}}$ to be the projective variety obtained from $\mathfrak{Z}^{12}_{\mathbb{A}}$ by setting $w(s)=w(t)=2$ and all the other weights of coordinates as 1 (Note that $\mathfrak{Z}^{11}_{\mathbb{P}}$ is contained in $\mathbb{P}(1^{14},2^2)$). The variety $\mathfrak{Z}^{11}_{\mathbb{P}}$ is isomorphic to the $\mathbb{Q}$-Fano variety Σ as in* [Tak5, Thm.1.1] *associated to prime $\mathbb{Q}$-Fano 3-folds of No.1.1 in* [Tak1].

Proof. Note that the projective variety $\overline{\mathfrak{Z}}:=\{{}^t xPy=0,{}^t xQy=0\}\subset\mathbb{P}^{13}$ with $P=E$ and $\operatorname{Tr}Q=0$ can be identified with $\overline{\Sigma}$ as in [Tak5, Subsec.4.1.1] with the obvious correspondence between the coordinates of them.

By the construction of Σ from $\overline{\Sigma}$ as in [Tak5, Sec.5], Σ is isomorphic to $\overline{\Sigma}$ in codimension 1 except the image $\overline{\Gamma}$ in $\overline{\Sigma}$ of the exceptional divisor for the blow-up of two $1/2(1^{11})$-singularities.

We show that a similar fact holds for $\mathfrak{Z}^{11}_{\mathbb{P}}$ and $\overline{\mathfrak{Z}}$. We consider the rational map $\pi\colon\mathfrak{Z}^{11}_{\mathbb{P}}\dashrightarrow\overline{\mathfrak{Z}}$ which is the restriction of the projection from the $(s:t)$-line. Note that π is defined on $U:=\mathfrak{Z}^{11}_{\mathbb{P}}\setminus\{\text{the } s\text{-}, t\text{-points}\}$. We set $\overline{\Delta}:=\big\{x=0,y^\sharp=0\big\}\cup\big\{y=0,x^\sharp=0\big\}\subset\overline{\mathfrak{Z}}$. From the equation of $\mathfrak{Z}^{11}_{\mathbb{P}}$, we see that the closure of the inverse image of $\overline{\Delta}|_{\pi(U)}$ by $\pi|_U$ is $\Delta:=\big\{x=0,y^\sharp=0,t=0\big\}\cup\big\{y=0,x^\sharp=0,s=0\big\}$. By the equations of $\mathfrak{Z}^{11}_{\mathbb{P}}$ and $\overline{\mathfrak{Z}}$, we see that, if $x\neq 0$ and $y\neq 0$ on $\overline{\mathfrak{Z}}$, then both s and t are recovered by the equations of $\mathfrak{Z}^{11}_{\mathbb{P}}$. Moreover, we see that points of $\{x=0\}\cup\{y=0\}$ outside $\overline{\Delta}$ are not the π-images of points of U, and the fiber of $\pi|_U$ over a point $\mathrm{p}\in\overline{\Delta}$ is 1-dimensional if $\mathrm{p}\not\in\{x=y=0\}$, and 2-dimensional if $\mathrm{p}\in\{x=y=0\}$. Therefore the $\pi|_U$-exceptional locus is $\Delta|_U$. By the equation of $\mathfrak{Z}^{11}_{\mathbb{P}}$ and [Tak5, Prop.4.4], the locus $\overline{\Delta}$ coincides with $\operatorname{Sing}\overline{\mathfrak{Z}}$, which is of codimension 3 in $\overline{\mathfrak{Z}}$. Therefore the exceptional locus of $\pi|_U$ is of codimension 2 in $\mathfrak{Z}^{11}_{\mathbb{P}}$, which implies $\mathfrak{Z}^{11}_{\mathbb{P}}$ is isomorphic to $\overline{\mathfrak{Z}}$ in codimension 1 except $\{x=0\}\cup\{y=0\}$. Note that $\{x=0\}\cup\{y=0\}$ is identified with $\overline{\Gamma}$. Therefore, we see that Σ and $\mathfrak{Z}^{11}_{\mathbb{P}}$ are isomorphic in codimension 1 identifying $\overline{\Sigma}$ and $\overline{\mathfrak{Z}}$. By [Tak5, Thm.1.1], Σ is a $\mathbb{Q}$-factorial $\mathbb{Q}$-Fano variety with Picard numbers 1, and, by Lemma 6.2 and the proof of [Tak6, Cor.4.10], so is $\mathfrak{Z}^{11}_{\mathbb{P}}$. Therefore, Σ and $\mathfrak{Z}^{11}_{\mathbb{P}}$ are actually isomorphic by [Tak2, Lem.5.5] as desired. $\square$

By Proposition 6.3 and [Tak5, Thm.1.1], we obtain the following result of Gushel´-Mukai type:

Corollary 6.4. *Any prime $\mathbb{Q}$-Fano 3-fold of No.1.1 in* [Tak1] *is a weighted complete intersection of $\mathfrak{Z}^{11}_{\mathbb{P}}$ as in Proposition 6.3 with respect to hypersurfaces of weight* 1.

Finally, we describe an SL_3 on $\mathfrak{Z}^{12}_{\mathbb{A}}$ as follows:

Proposition 6.5. *Let $\mathbb{A}^{16}_3$ be the affine subspace $\{\mathrm{Tr}\, Q = 0\}$ in the affine space whose coordinates are s, t and the entries of x, y, Q. The affine space $\mathbb{A}^{16}_3$ has the SL_3-action preserving $\mathfrak{Z}^{12}_{\mathbb{A}}$ defined by*

$$x \mapsto {}^t g x,\ y \mapsto g^{-1} y,\ Q \mapsto g^{-1} Q g, s \mapsto s, t \mapsto t$$

for $g \in \mathrm{SL}_3$.

7. The $G_2^{(4)}$-Cluster variety as a specializations of $\mathfrak{U}^{14}_{\mathbb{A}}$

In this section, we clarify the relationship between the $G_2^{(4)}$-cluster variety constructed in [CD], and the affine varieties $\mathfrak{F}^{14}_{\mathbb{A}}$ and $\mathfrak{Z}^{12}_{\mathbb{A}}$. We review in our context the prime $\mathbb{Q}$-Fano 3-folds constructed in ibid. from the $G_2^{(4)}$-cluster variety.

7.1. **The $G_2^{(4)}$-cluster variety $\mathfrak{Cl}^{10}_{\mathbb{A}}$.** In [CD], the $G_2^{(4)}$-cluster variety is defined in the affine 16-space with coordinates

$$\begin{aligned}&\theta_i\,(1 \le i \le 4),\ \theta_{23},\ \theta_{41},\\ &A_j\,(1 \le j \le 4), A_{kl}\,((k,l) = (1,2),\ (2,3),\ (3,4),\ (4,1)),\\ &\lambda_{13},\ \lambda_{24}.\end{aligned}$$

In this paper, however, we call a smaller variety as the $G_2^{(4)}$-cluster variety. By the big table [Table], we observe that A_{12} and A_{34} is always nonzero constant when a $\mathbb{Q}$-Fano 3-fold is constructed from a weighted projectivization of the $G_2^{(4)}$-cluster variety. Then, replacing A_1, A_2, A_3, A_4, λ_{13}, λ_{24} with $A_{12}^{-1}A_1$, $A_{12}^{-1}A_2$, $A_{34}^{-1}A_3$, $A_{34}^{-1}A_4$, $A_{12}^{-1}A_{34}^{-1}\lambda_{13}$, $A_{12}^{-1}A_{34}^{-1}\lambda_{24}$, we see that it is possible to set $A_{12} = A_{34} = 1$.

Definition 7.1. *The $G_2^{(4)}$-cluster variety $\mathfrak{Cl}^{10}_{\mathbb{A}}$* is defined as a subvariety of the affine 14-space with coordinates

$$\begin{aligned}&\theta_i\,(1 \le i \le 4), \theta_{23},\ \theta_{41},\\ &A_j\,(1 \le j \le 4), A_{kl}\,((k,l) = (2,3), (4,1)),\\ &\lambda_{13},\ \lambda_{24}\end{aligned}$$

by setting for the equations of $\mathfrak{F}^{22}_{\mathbb{A}}$ as follows:

$$\begin{aligned}&x = {}^t\begin{pmatrix}\theta_4 & \theta_1 & A_{23}\end{pmatrix}, y = {}^t\begin{pmatrix}A_{41} & \theta_2 & \theta_3\end{pmatrix},\\ &P = \begin{pmatrix}-A_4 & 0 & 0\\ 0 & 0 & 1\\ -\lambda_{24} & -A_2 & 0\end{pmatrix}, Q = \begin{pmatrix}0 & 1 & 0\\ -A_1 & 0 & 0\\ -\lambda_{13} & 0 & -A_3\end{pmatrix},\\ &s = -\theta_{23}, t = -\theta_{41}.\end{aligned}$$

We can immediately check that this definition of the $G_2^{(4)}$-cluster variety coincides with that in [CD, Subset. 1.2.2] when $A_{12} = A_{34} = 1$.

By an elementary calculation, we have the following:

Proposition 7.2. *All the equations of $\mathfrak{Cl}^{10}_{\mathbb{A}}$ are weighted homogeneous if and only if the following conditions on the weights of coordinates hold:*

$$\begin{aligned}
&w(A_1) = -2w(\theta_1) + w(\theta_2) + w(\theta_{41}),\ w(A_2) = w(\theta_1) - 2w(\theta_2) + w(\theta_{23}),\\
&w(A_3) = -2w(\theta_3) + w(\theta_4) + w(\theta_{23}),\ w(A_4) = w(\theta_3) - 2w(\theta_4) + w(\theta_{41}),\\
&w(A_{23}) = w(\theta_2) + w(\theta_3) - w(\theta_{23}),\ w(A_{41}) = w(\theta_1) + w(\theta_4) - w(\theta_{41}),\\
&w(\lambda_{13}) = -w(\theta_1) - w(\theta_3) + w(\theta_{23}) + w(\theta_{41}),\\
&w(\lambda_{24}) = -w(\theta_2) - w(\theta_4) + w(\theta_{23}) + w(\theta_{41}).
\end{aligned}$$

In the following subsections, we only consider the weights of coordinates as in Proposition 7.2 such that the weighted projectivizations of $\mathfrak{Cl}^{10}_{\mathbb{A}}$ itself or its subvarieties associated to these produce prime $\mathbb{Q}$-Fano 3-folds. The list of such weights are presented in [Table].

7.2. **The maximal case.** By the big table [Table], we observe that all the weights of the coordinates of the $G_2^{(4)}$-cluster variety $\mathfrak{Cl}^{10}_{\mathbb{A}}$ is positive only in the two cases No.5530 and No.11455 of [GRDB]. We may verify the following by a straightforward calculation.

Proposition 7.3. *The equations of the affine variety $\mathfrak{Cl}^{10}_{\mathbb{A}}$ is presented in the format of the equations of $\mathfrak{U}^{14}_{\mathbb{A}}$ by setting*

$$x = {}^t\begin{pmatrix} \theta_1 - A_2A_{23} & \theta_4 - A_3A_{23} & A_{23} \end{pmatrix}, y = {}^t\begin{pmatrix} A_{41} & 1/2(\theta_3-\theta_2) & 1/2(\theta_3+\theta_2) \end{pmatrix},$$
$$P = \begin{pmatrix} 0 & 1 & 1 \\ -A_4 & 0 & 0 \\ -\lambda_{24} - A_3A_4 & 2A_2 & 0 \end{pmatrix}, Q = \begin{pmatrix} -A_1 & 0 & 0 \\ 0 & -1 & 1 \\ -\lambda_{13} - A_1A_2 & -2A_3 & 0 \end{pmatrix},$$
$$s = \theta_{23}, t = 2\theta_{41}.$$

By Proposition 7.2 and the weights of coordinates of $\mathfrak{Cl}^{10}_{\mathbb{A}}$ for No.5530 and No.11455, we see that all the entries of x, y, P, Q as in Proposition 7.3 are weighted homogeneous. Therefore we have the following:

Corollary 7.4. *The affine variety $\mathfrak{Cl}^{10}_{\mathbb{A}}$ is isomorphic to the subvariety of $\mathfrak{U}^{14}_{\mathbb{A}}$*

$$\mathfrak{U}^{14}_{\mathbb{A}} \cap \{a_{11} = 0, b_{11} = 1, b_{12} = 0, b_{21} = 0\}. \tag{7.1}$$

Moreover, the isomorphism is weighted homogeneous for No.5530 and No.11455 with the corresponding weights of coordinates of (7.1).

7.3. **The case only one coordinate is a nonzero constant.** By [Table], there are 12 classes of prime $\mathbb{Q}$-Fano 3-folds for which only one coordinate of $\mathfrak{Cl}^{10}_{\mathbb{A}}$ (actually A_1,A_3 or A_4) is a nonzero constant. By symmetry, we may identify the three cases, so we have only to consider the case that A_4 is a nonzero constant; this is the case for No.1169, 1182, 5860, 5870, 16228. We denote by $\mathfrak{Cl}^{9}_{\mathbb{A},A_4}$ the subvariety of $\mathfrak{Cl}^{10}_{\mathbb{A}}$ with $A_4 = -1$. By suitable changes of scales of coordinates, we may assume that $A_4 = -1$.

We may verify the following by a straightforward calculation.

Proposition 7.5. *The equations of the affine variety $\mathfrak{Cl}^9_{\mathbb{A},A_4}$ is presented in the format of the equations of $\mathfrak{U}^{14}_{\mathbb{A}}$ by setting*

$$x = {}^t\begin{pmatrix} A_{41} & \theta_2 & \theta_3 \end{pmatrix}, y = {}^t\begin{pmatrix} A_{23} & \theta_1 & \theta_4 - 1/2\lambda_{24}A_{23} \end{pmatrix},$$
$$P = \begin{pmatrix} -1/2\lambda_{24} & 0 & 1 \\ -A_2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, Q = \begin{pmatrix} -\lambda_{13} & -A_1 & 0 \\ 1/2\lambda_{24} & 0 & 1 \\ -A_3 & 0 & 0 \end{pmatrix},$$
$$s = -\theta_{41}, t = \theta_{23}.$$

By Proposition 7.2 and the assumption that $A_4 = -1$, we see that the entry $\theta_4 - 1/2\lambda_{24}A_{23}$ of y is weighted homogeneous. Therefore we have the following:

Corollary 7.6. *The affine variety $\mathfrak{Cl}^9_{\mathbb{A},A_4}$ is isomorphic to the subvariety of $\mathfrak{U}^{14}_{\mathbb{A}}$*

$$\mathfrak{U}^{14}_{\mathbb{A}} \cap \{b_{11} = b_{12} = c_{11} = c_{22} = 0, c_{12} = 1\}. \tag{7.2}$$

Moreover, the isomorphism is weighted homogeneous with the corresponding weights of coordinates of (7.2).

In the following subsections, we consider the cases in which exactly two of the coordinates are nonzero constants. There are 4 cases; the set of the two nonzero constant coordinates is $\{A_3, A_4\}$, $\{A_1, A_4\}$, $\{A_2, A_3\}$, or $\{A_1, A_3\}$. For them, we may assume that the nonzero constants are -1. We denote by $\mathfrak{Cl}^8_{\mathbb{A},A_iA_j}$ the subvariety of $\mathfrak{Cl}^{10}_{\mathbb{A}}$ with $A_i = A_j = -1$. We will treat the 4 cases separately in the sequel.

7.4. **The case A_3 and A_4 are nonzero constants –another appearance of $\mathfrak{S}^8_{\mathbb{A}}$–.** This is the case for 12 classes in [GRDB]. In this subsection, we see another appearance of the affine variety $\mathfrak{S}^8_{\mathbb{A}}$.

We may verify the following by a straightforward calculation.

Proposition 7.7. *The equations of the affine variety $\mathfrak{Cl}^8_{\mathbb{A},A_3A_4}$ is presented in the format of the equations of $\mathfrak{S}^8_{\mathbb{A}}$ by setting*

$$x = {}^t\begin{pmatrix} A_{41} & \theta_2 & \theta_3 - 2/3\lambda_{13}A_{41} + 1/3\lambda_{24}\theta_2 \end{pmatrix},$$
$$y = {}^t\begin{pmatrix} \theta_1 & A_{23} & \theta_4 + 1/3\lambda_{13}\theta_1 - 2/3\lambda_{24}A_{23} \end{pmatrix},$$
$$P = \begin{pmatrix} 1/3\lambda_{13} & -1/3\lambda_{24} & 1 \\ -1/3\lambda_{24} & -A_2 & 0 \\ 1 & 0 & 0 \end{pmatrix}, Q = \begin{pmatrix} -A_1 & -1/3\lambda_{13} & 0 \\ -1/3\lambda_{13} & 1/3\lambda_{24} & 1 \\ 0 & 1 & 0 \end{pmatrix},$$
$$s = -\theta_{41}, t = -\theta_{23}.$$

By Proposition 7.2 and the assumption that $A_3 = A_4 = -1$, we see that the entries $\theta_3 - 2/3\lambda_{13}A_{41} + 1/3\lambda_{24}\theta_2$ and $\theta_4 + 1/3\lambda_{13}\theta_1 - 2/3\lambda_{24}A_{23}$ of x and y are weighted homogeneous. Therefore we have the following:

Corollary 7.8. *The affine variety $\mathfrak{Cl}^8_{\mathbb{A},A_3A_4}$ is isomorphic to $\mathfrak{S}^8_{\mathbb{A}}$. Moreover, the isomorphism is weighted homogeneous with the corresponding weights of coordinates of the equations as in Proposition 5.3.*

Example 7.9. We consider the weighted projectivization $\mathfrak{S}^7_{\mathbb{P}}$ of $\mathfrak{S}^8_{\mathbb{A}}$ in $\mathbb{P}(1^8, 2^4)$ by putting the weights of coordinates as follows:

$$w(u_i) = 1\,(1 \le i \le 4), w(v_j) = 2\,(0 \le j \le 3), w(d_k) = 1\,(0 \le k \le 3).$$

By [CD], this produces a prime $\mathbb{Q}$-Fano 3-fold X of No.20652 as its weighted complete intersection by one hypersurface of weight 2 and three hypersurfaces of weight 1. It is easy to verify that the the set of three $1/2\,(1,1,1)$-singularities of X is the intersection between the twisted cubic curve $\gamma := \{U = O, D = 0, \wedge^2 V = \boldsymbol{o}\}$ and X.

We refine Example 7.9. In the paper [Tak1], we obtain two different classes No.4.1 and No.5.4 of prime $\mathbb{Q}$-Fano threefolds of No.20652. In the following proposition, we determine which class a prime $\mathbb{Q}$-Fano 3-fold X constructed in Example 7.9 belongs to:

Proposition 7.10. *A prime $\mathbb{Q}$-Fano 3-fold X constructed in Example 7.9 belongs to the class No.4.1 of* [Tak1] *for any $1/2\,(1,1,1)$-singularity of X.*

Proof. We choose a $1/2\,(1,1,1)$-singularity p and construct the Sarkisov link starting from the blow-up at p. By [Tak1, Tables 4 and 5], we can distinguish between prime $\mathbb{Q}$-Fano threefolds of No.4.1 and No.5.4 by the dimension of forms of weight 2 vanishing at p with weighted multiplicity ≥ 3; the dimension is 3 for No.4.1, and is 4 for No.5.4, which can be verified by looking at the rational map $X \dashrightarrow X'$ in the Sarkisov link (note that this map is defined by the linear system $|-2K_X - 3\mathsf{p}|$). By the group action on $\mathfrak{S}^8_{\mathbb{A}}$ described in Subsection 5.2, we may assume that p is the v_0-point (the point whose coordinates except v_0 are zero). Then, by the equation (5.1), we see that 4 forms of weight 2, $u_1^2, u_1u_3, u_3^2, v_3$ are all such forms and they are linearly independent on $\mathfrak{S}^8_{\mathbb{A}}$. We have only to show that they are still linearly independent on X. Assume the contrary. Then we may assume that the hypersurface Q of weight 2 cutting X from $\mathfrak{S}^7_{\mathbb{P}}$ is $\{av_3 + bu_1^2 + cu_1u_3 + du_3^2 = 0\}$ with some $a, b, c, d \in \mathbb{C}$. Then Q intersects the twisted cubic curve γ only at the v_0-point, or contains γ. This is a contradiction since Q must intersects γ at three distinct points for X to have three $1/2\,(1,1,1)$-singularities. □

Remark 7.11. In [Tak6], we will construct a prime $\mathbb{Q}$-Fano threefold of No.5.4 via another key variety $\mathcal{H}^{13}_{\mathbb{A}}$.

7.5. **The case A_1 and A_4, or A_2 and A_3 are nonzero constants –the affine variety $\mathfrak{T}^8_{\mathbb{A}}$ with an SL_2-action–.** This is the case for 16 classes in [GRDB]. It is easy to see that we may identify $\mathfrak{Cl}^8_{\mathbb{A},A_1A_4}$ and $\mathfrak{Cl}^8_{\mathbb{A},A_2A_3}$ changing the suffixes of the coordinates. Hence we only consider $\mathfrak{Cl}^8_{\mathbb{A},A_1A_4}$ in this subsection. This is the case for 12 classes.

We may verify the following by a straightforward calculation.

Proposition 7.12. *The equations of the affine variety $\mathfrak{Cl}^8_{\mathbb{A},A_1A_4}$ is presented in the format of the equations of $\mathfrak{U}^{14}_{\mathbb{A}}$ by the setting as in Proposition 7.5 with $A_1 = -1$.*

Now we arrive at a new interpretation of $\mathfrak{Cl}^8_{\mathbb{A},A_1A_4}$; we set

$$\begin{aligned}
&w_1 := \theta_1 - 1/3\lambda_{13}A_{23},\ w_2 := -\theta_4 + 1/3\lambda_{24}A_{23},\\
&z_1 := A_{41},\ z_2 := \theta_2,\ z_3 := \theta_3,\\
&f_0 := -A_3,\ f_1 := 1/3\lambda_{13},\ f_2 := 1/3\lambda_{24}, f_3 := -A_2,\\
&s := -\theta_{41}, t := \theta_{23}, u := A_{23}.
\end{aligned}$$

By Proposition 7.2 and the assumption that $A_1 = A_4 = -1$, we see that the entries $\theta_1 - 1/3\lambda_{13}A_{23}$ and $-\theta_4 + 1/3\lambda_{24}A_{23}$ are weighted homogeneous. Moreover, we set

$$w := \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, Z := \begin{pmatrix} z_1 & -z_2 \\ z_3 & -z_1 \end{pmatrix}, z := \begin{pmatrix} z_2 \\ -2z_1 \\ z_3 \end{pmatrix},$$
$$F := \begin{pmatrix} f_2 & f_1 & f_0 \\ f_3 & f_2 & f_1 \end{pmatrix}, F^\dagger := \begin{pmatrix} -f_1 & f_0 \\ 2f_2 & -2f_1 \\ -f_3 & f_2 \end{pmatrix}.$$

Proposition 7.13. *The affine variety $\mathfrak{Cl}^8_{\mathbb{A},A_1A_4}$ is isomorphic to the affine subvariety $\mathfrak{T}^8_{\mathbb{A}}$ in the affine 8-space $\mathbb{A}^{12}_{\mathfrak{T}}$ with the coordinates*

$$w_1, w_2, z_1, z_2, z_3, s, t, u, f_0, f_1, f_2, f_3$$

defined by the following equations:

$$\begin{aligned}
&Zw + uFz = \boldsymbol{o},\\
&tw = ZFz,\\
&tu = \det Z,\\
&sz = -2u^2 \wedge^2 ({}^tF) + uF^\dagger w + \begin{pmatrix} w_1^2 \\ -2w_1w_2 \\ w_2^2 \end{pmatrix},\\
&st = -1/2\, u(\wedge^2 F^\dagger)z + \begin{pmatrix} w_2 & -w_1 \end{pmatrix} Fz.
\end{aligned}$$

Moreover, the isomorphism is weighted homogeneous with the corresponding weights of coordinates of these equations.

Though the equations of the affine variety $\mathfrak{T}^8_{\mathbb{A}}$ look complicated, it turn out to be suitable to see an SL_2-action on $\mathfrak{T}^8_{\mathbb{A}}$ as follows:

Proposition 7.14. *The affine space $\mathbb{A}^{12}_{\mathfrak{T}}$ has the SL_2-action preserving $\mathfrak{T}^8_{\mathbb{A}}$ defined by*

$$Z \mapsto gZg^{-1}, w \mapsto gw, F \mapsto gF\widehat{g}^{-1}, s \mapsto s,\ t \mapsto t,\ u \mapsto u,$$

where $g \in \mathrm{SL}_2$, and the definition of $\widehat{g}$ for $g \in \mathrm{SL}_2$ is as in Proposition 5.5.

Proof. We only note that z is mapped to $\widehat{g}z$ by the SL_2-action. □

7.6. **The case A_1 and A_3 are nonzero constants– a subvariety of $\mathfrak{Z}^{12}_{\mathbb{A}}$–.** We may verify the following by a straightforward calculation.

Proposition 7.15. *The equations of the affine variety $\mathfrak{Cl}^{10}_{\mathbb{A}}$ is presented in the format of the equations of $\mathfrak{Z}^{12}_{\mathbb{A}}$ by setting*

$$\begin{aligned}
&x = {}^t\begin{pmatrix} \theta_4 & \theta_1 & A_{23} \end{pmatrix}, y = {}^t\begin{pmatrix} \theta_2 & A_{41} & \theta_3 - \lambda_{13}A_{41} \end{pmatrix},\\
&P = E,\, Q = \begin{pmatrix} -1/3\lambda_{13} & -A_4 & 0 \\ 0 & 2/3\lambda_{13} & 1 \\ -A_2 & -\lambda_{24} & -1/3\lambda_{13} \end{pmatrix},\\
&s = \theta_{23}, t = -\theta_{41}.
\end{aligned}$$

By Proposition 7.2 and the assumption that $A_1 = A_3 = -1$, we see that the entry $\theta_3 - \lambda_{13}A_{41}$ of y is weighted homogeneous. Therefore we have the following:

Corollary 7.16. *The affine variety $\mathfrak{Cl}^8_{\mathbb{A},A_1A_3}$ is isomorphic to the subvariety*

$$\{q_{22} = -2q_{11}, q_{33} = q_{11}, q_{13} = q_{21} = 0, q_{23} = 1\}$$

of $\mathfrak{Z}^{12}_{\mathbb{A}}$. Moreover, the isomorphism is weighted homogeneous with the corresponding weights of coordinates of the equations as in Definition 6.1.

This is the case for 16 classes in [GRDB].

7.7. **The case with the most number of nonzero constant coordinates –the affine variety $\mathfrak{B}^6_{\mathbb{A}}$ with an SL_2-action– .** By [Table], the number of nonzero constant coordinates is at most 3, and, if the number is 3, then the nonzero constant coordinates are always A_1, A_3 and A_4. This is the case for 35 classes in [GRDB]. In this case, we may assume that $A_1 = A_3 = A_4 = -1$. Then the subvariety of $\mathfrak{Cl}^{10}_{\mathbb{A}}$ with $A_1 = A_3 = A_4 = -1$ is a subvariety of $\mathfrak{Cl}^8_{\mathbb{A},A_1A_3}$, $\mathfrak{Cl}^8_{\mathbb{A},A_1A_4}$ and $\mathfrak{Cl}^8_{\mathbb{A},A_3A_4}$.

Here we consider that this is a subvariety of $\mathfrak{Cl}^8_{\mathbb{A},A_3A_4}$ and then identifying $\mathfrak{Cl}^8_{\mathbb{A},A_3A_4}$ with $\mathfrak{S}^8_{\mathbb{A}}$, we describe $\mathfrak{Cl}^8_{\mathbb{A},A_3A_4}$ as the subvariety $\mathfrak{S}^8_{\mathbb{A}} \cap \{d_3 = -1\}$ of $\mathfrak{S}^8_{\mathbb{A}}$. Moreover, defining the new coordinates D_0, D_1, U_1, U_2 corresponding to d_0, d_1, u_1, u_2 by

$$D_0 = d_0 + 3d_1d_2 + 2d_2^3, D_1 = d_1 + d_2^2, U_1 = u_1 - d_2u_3, U_2 = u_2 - d_2u_4,$$

we see by a straightforward calculation that $\mathfrak{S}^8_{\mathbb{A}} \cap \{d_3 = -1\}$ is isomorphic to the cone over $\mathfrak{S}^8_{\mathbb{A}} \cap \{d_2 = 0,\, d_3 = -1\}$. The isomorphism is weighted homogeneous with the corresponding weights of coordinates since D_0, D_1, U_1, U_2 are weighted homogeneous if $A_1 = A_3 = A_4 = -1$ by Proposition 7.2.

We set

$$\mathfrak{B}^6_{\mathbb{A}} := \mathfrak{S}^8_{\mathbb{A}} \cap \{d_2 = 0,\, d_3 = -1\}.$$

Let $\rho_{\mathfrak{B}} : \widehat{\mathfrak{B}} \to \{d_2 = 0, d_3 = -1\}$ be the base change of $\rho_{\mathfrak{S}}$ as in Subsection 5.3 by the inclusion map $\{d_2 = 0, d_3 = -1\} \hookrightarrow \mathbb{A}^4_D$. By Proposition 5.8, we have immediately the following noting that the cone over the tangential scroll (5.2) of the twisted cubic restricts to the affine cuspidal cubic curve $\{d_2 = 0, d_3 = -1, d_0^2 = 4d_1^3\}$:

Proposition 7.17. *Let p be a point of $\{d_2 = 0, d_3 = -1\}$ and F_{p} the $\rho_{\mathfrak{B}}$-fiber over p. We identify $\{d_2 = 0, d_3 = -1\}$ with the affine 2-space with d_0, d_1 as the coordinates.*

If $\mathsf{p} = 0$, then $F_{\mathsf{p}} \simeq \mathbb{P}^{1,1,1}$.

If p belongs to $\{d_0^2 = 4d_1^3\} \setminus \{0\}$, then $F_{\mathsf{p}} \simeq \mathbb{P}^1 \times \mathtt{Q}$.

If p does not belong to $\{d_0^2 = 4d_1^3\}$, then $F_{\mathsf{p}} \simeq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

We can also easily check the following:

Proposition 7.18. *The variety $\mathfrak{B}^6_{\mathbb{A}}$ has an SL_2-action which is the restriction of the $\mathrm{SL}_2^{\mathrm{II}}$-action of $\mathfrak{S}^8_{\mathbb{A}}$.*

7.8. **Summary of the results.** In the following table, we summarize the numbers of classes of prime $\mathbb{Q}$-Fano 3-folds which are obtained from the $G_2^{(4)}$-cluster variety $\mathfrak{Cl}^{10}_{\mathbb{A}}$ and its subvarieties:

Subsection	constant coordinates	$\sharp$ of classes
7.2	None	2
7.3	A_1, A_3, or A_4	12
7.4	A_3 and A_4	12
7.5	A_1 and A_4, or A_2 and A_3	16
7.6	A_1 and A_3	16
7.7	A_1, A_3 and A_4	35

TABLE 1. prime $\mathbb{Q}$-Fano 3-fold obtained from $\mathfrak{Cl}^{10}_{\mathbb{A}}$

8. AFFINE VARIETY $\mathfrak{P}^{23}_{\mathbb{A}}$ DEFINED BY TYPE II_2 UNPROJECTION DUE TO S. PAPADAKIS

In the paper [P3], Papadakis constructs affine varieties via type II_2 unprojection and proves they are Gorenstein (see [P3, Thm.2.15]). Moreover, in [P4], he provides explicit descriptions of parts of their equations, and a full description of the equation in one particular case (see [P4, Sect.4]), for which we denote by $\mathfrak{P}^{23}_{\mathbb{A}}$ the affine variety he constructs. The affine variety $\mathfrak{P}^{23}_{\mathbb{A}}$ is a subvariety of codimension 4 in the affine 27-space with the coordinates implemented in the following format:

$$A^k = \begin{pmatrix} 0 & a^k_{12} & a^k_{13} \\ -a^k_{12} & 0 & a^k_{23} \\ -a^k_{13} & -a^k_{23} & 0 \end{pmatrix},\ B^k = \begin{pmatrix} b^k_{11} & b^k_{12} & b^k_{13} \\ b^k_{12} & b^k_{22} & b^k_{23} \\ b^k_{13} & b^k_{23} & b^k_{33} \end{pmatrix}\ (k=1,2),$$
$${}^tX = \begin{pmatrix} X_1 & X_2 & X_3 \end{pmatrix},\ {}^tY = \begin{pmatrix} Y_1 & Y_2 & Y_3 \end{pmatrix},$$
$$s_0,\ s_1,\ z$$

(here we use a slightly different notation from Papadakis' one; we denote by X_i, Y_i, a^k_{ij} and b^k_{ij} his x_i, y_i, A^k_{ij} and B^k_{ij} respectively). In the following proposition, we clarify that the variety $\mathfrak{P}^{23}_{\mathbb{A}}$ is closely related with the variety $\mathfrak{F}^{22}_{\mathbb{A}}$. The assertion follows by a straightforward calculation.

We set

$$D_X = \det \begin{pmatrix} X_1 & X_2 & X_3 \\ a^1_{23} & -a^1_{13} & a^1_{12} \\ a^2_{23} & -a^2_{13} & a^2_{12} \end{pmatrix},\ D_Y = \det \begin{pmatrix} Y_1 & Y_2 & Y_3 \\ a^1_{23} & -a^1_{13} & a^1_{12} \\ a^2_{23} & -a^2_{13} & a^2_{12} \end{pmatrix},$$
$${}^t\boldsymbol{a}_1 = \begin{pmatrix} -a^1_{23} & a^1_{13} & -a^1_{12} \end{pmatrix},\ {}^t\boldsymbol{a}_2 = \begin{pmatrix} -a^2_{23} & a^2_{13} & -a^2_{12} \end{pmatrix}.$$

Proposition 8.1. *The affine variety $\mathfrak{P}^{23}_{\mathbb{A}}$ is transformed over the locus $\{z \neq 0\}$ to the cone over $\mathfrak{F}^{22}_{\mathbb{A}}$ by the following correspondence between coordinates:*

$$x = \sqrt{z}X + Y,\ y = -\sqrt{z}X + Y,$$
$$P = \frac{1}{2\sqrt{z}}A_1 + B_1,\ Q = \frac{1}{2\sqrt{z}}A_2 + B_2,$$
$$s = -\sqrt{z}s_0 - s_1 - \frac{3}{4\sqrt{z}}D_X + \frac{1}{4z}D_Y + \frac{1}{2\sqrt{z}}({}^t\boldsymbol{a}_1B^2Y - {}^t\boldsymbol{a}_2B^1Y) - \frac{1}{2}({}^t\boldsymbol{a}_1B^2X - {}^t\boldsymbol{a}_2B^1X),$$
$$t = \sqrt{z}s_0 - s_1 + \frac{3}{4\sqrt{z}}D_X + \frac{1}{4z}D_Y - \frac{1}{2\sqrt{z}}({}^t\boldsymbol{a}_1B^2Y - {}^t\boldsymbol{a}_2B^1Y) - \frac{1}{2}({}^t\boldsymbol{a}_1B^2X - {}^t\boldsymbol{a}_2B^1X),$$

where the l.h.s. and the r.h.s. of the equalities correspond to $\mathfrak{F}_{\mathbb{A}}^{22}$ and $\mathfrak{P}_{\mathbb{A}}^{23}$ respectively, and z is also the free coordinate of the cone over $\mathfrak{F}_{\mathbb{A}}^{22}$.

Remark 8.2. (1) The definition of b in [P4, p.2203, (4.2)] is slightly incorrect; the coefficients of $B_{ij}^1 \mathrm{ad} B_{ij}^2$ and $B_{ij}^2 \mathrm{ad} B_{ij}^1$ in the last part of r.h.s. of (4.2) should be 2 when $i \neq j$.

(2) In [P4, Sec.5], two candidates of prime $\mathbb{Q}$-Fano 3-folds of anti-canonical codimension 4 with type II_1 projections are constructed from $\mathfrak{P}_{\mathbb{A}}^{23}$. In her phd thesis submitted to Warwick University [Tay], Taylor constructs more such candidates. The remaining problem is to show their examples have Picard number 1.

(3) By Proposition 8.1 and Corollary 2.31, the suitable partial projectivization of $\mathfrak{P}_{\mathbb{A}}^{23}$ has a $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$-fibration.

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DEPARTMENT OF MATHEMATICS, GAKUSHUIN UNIVERSITY, MEJIRO, TOSHIMA-KU, TOKYO 171-8588, JAPAN

Email address: `hiromici@math.gakushuin.ac.jp`