REAL BOTT TOWER

ADMI NAZRA

Abstract. In this paper we proved that there exists four distinct diffeomorphism classes of three dimensional real Bott tower $M(A) = (S^1)^3/(\mathbb{Z}_2)^3$, and 12 distinct diffeomorphism classes of four dimensional real Bott tower $M(A) = (S^1)^4/(\mathbb{Z}_2)^4$, where matrix A corresponds to the action of $(\mathbb{Z}_2)^n$ on $(S^1)^n$ for n=3,4.

Contents

1. Introduction

In this paper we study free actions of the product of cyclic groups of order 2, $(\mathbb{Z}_2)^n$ on the n-dimensional Torus $T^n = (S^1)^n$ for n=3,4. The action of $(\mathbb{Z}_2)^n$ can be expressed by an upper triangular matrix A where the diagonal entries are one and the other entries are either one or zero. We call such a matrix n-th *Bott matrix* and the n-dimensional orbit space $M(A) = T^n/(\mathbb{Z}_2)^n$ is said to be a **real Bott tower**. By the definition, Bott matrices consist of 8 for $n = 3$ and 64 for $n = 4$.

The main purpose of this paper is to determine the diffeomorphism classes of real Bott towers for dimension $n = 3$ and 4. The original conjecture is whether *A is conjugate to A' if and only if$M(A)$ and $M(A')$ are diffeomorphic.* However we found that there are non-diffeomorphic

Key words and phrases. Crystallographic group, Bieberbach group, Flat Riemannian Manifold.

Bott towers $M(A), M(A')$ with representative matrices A, A' are conjugate. So the necessary condition of the conjecture is not true. But the sufficient condition of the conjecture is not known so far. On the other hand the real Bott tower $M(A)$ turns out to be an n-dimensional compact euclidean space form (Riemannian flat manifold). Then we can apply the Bieberbach theorems to classify Bott towers.

If M is a compact flat Riemannian manifold of dimension n, then M is affinely diffeomorphic to $\mathbb{R}^n/\Gamma$, where Γ, the fundamental group of M, is a torsion-free discrete uniform subgroup of the Euclidean group $\mathbb{E}(n) = \mathbb{R}^n \rtimes O(n)$. Such groups Γ are called Bieberbach groups.

2. Preliminaries

Let us start with some definitions and theorems used throughout this paper.

2.1. **Groups Acting on Spaces.**

Definition 2.1. *Suppose that X is a topological space and G is a group. Then we say that X is a G–space if G acts on X such that the correspondence $G \times X \longrightarrow X$, $(g, x) \longrightarrow gx$, is continuous.*

Theorem 2.2. *If G is a finite group acting freely on a Hausdorff space X then $p : X \to X/G$ is a covering.*

Proof. Let $G = \{1 = g_0, g_1, g_2, ..., g_n\}$. Since X is Hausdorff there are open neighborhoods $U_0, U_1, ..., U_n$ of $g_0x, g_1x, g_2x, ..., g_nx$ respectively with $U_0 \cap U_j = \emptyset$ for $j = 1, 2, ..., n$. Let U be the intersection $\cap_{j=0}^{n} g_j^{-1}U_j$, which is clearly an open neighborhood of x (X is topological space).

Now $g_iU = \cap_{j=0}^{n} g_i(g_j^{-1}U_j) \subseteq U_i$ and

$$\begin{aligned} g_iU \cap g_jU &= g_j((g_j^{-1}g_iU) \cap U) \\ &= g_j(g_kU \cap U) \qquad \text{(for some k)} \\ &= \emptyset \end{aligned}$$

since $g_kU \subseteq U_k$ and $U \subseteq U_0$.

Next note that $p : X \longrightarrow X/G$ is a continuous surjective open map. Since p is an open map, $p(U)$ is an open neighborhood of $Gx = p(x)$ and $p^{-1}(p(U)) = \cup_{g \in G}\, gU$ with $\{gU; g \in G\}$ being disjoint open subsets of X. Furthermore $p|gU : gU \to p(U)$ is a continuous open bijective mapping and hence a homeomorphism. $\square$

We say G is *discontinuous* at $x \in X$ if, given sequence $\{\gamma_i\} \subset G$ of distinct elements, the sequence $\{\gamma_i x\} \subset X$ has no accumulation point. G is *discontinuous* on X if it is discontinuous at every point of X. The action of G is *properly discontinuous* on X if for every compact subset K of X the set of $\gamma \in G$ such that $\gamma K \cap K \neq \emptyset$ is finite. If $\{\gamma \in G | \gamma K \cap K \neq \emptyset\}$ is compact, then the action of G is called *proper* action.

Lemma 2.3. *If G acts properly discontinuously on X then G is discontinuous on X.*

Proof. Suppose G is not discontinuous on X. That is, given a sequence $\{\gamma_i\}$ of distinct elements of G such that $\{\gamma_i x : x \in X\}$ convergence to $y \in X$. Then $K = \{x, y, \gamma_1 x, \gamma_2 x, ...\}$ is compact subset of X. As $\gamma_i x \in K \cap \gamma_i K$ for each i, we have a contradiction. $\square$

Let G be a Hausdorff topological group. A *discrete subgroup* is a group with a countable number of elements and the discrete topology (every point is an open set).

Lemma 2.4. *Let Γ and K be subgroups of G with K compact and G locally compact. These conditions are equivalent:*

(i) Γ *is discontinuous at some point of* G/K,
(ii) Γ *is discontinuous on* G/K,
(iii) Γ *is properly discontinuous on* G/K,
(iv) Γ *is discrete in* G.

Proof. ($(i) \Rightarrow (iv)$). Let Γ be discontinuous at $y = gK \in G/K$, for some $g \in G$. Then y has a neighborhood UK, U open in G such that $\{\gamma \in \Gamma | \gamma(y) \in UK\}$ is finite. Now Ug^{-1} is a neighborhood of 1 in G which has finite intersection with Γ. For this, let $Ug^{-1} \cap \Gamma = \{\gamma_i = u_i g^{-1} | u_i \in U, \gamma_i \in \Gamma\}$. Then $\gamma_i g = u_i g^{-1} g = u_i$, $\gamma_i g K = u_i K \in UK$, so $\gamma_i y = u_i K \in UK$. Thus $Ug^{-1} \cap \Gamma$ is finite. As G is Hausdorff, a smaller neighborhood of 1 meets Γ only in 1. Let $Ug^{-1} = U'$. Then for any $\gamma \in \Gamma$, $V = \gamma U'$ contains only the element γ of Γ. Thus Γ is discrete in G.
($(iv) \Rightarrow (iii)$). Let Γ be discrete. Since G locally compact, we choose a neighborhood U of 1 in G with compact closure ($\bar{U}$ is compact). If $g \in G$, put $V_g = gUKU^{-1}g^{-1}$ has compact closure (i.e. $\bar{V}_g$ is compact). For if $\gamma \in V_g$, $\gamma g = g \Rightarrow \gamma g U = gUK$, so $\forall \gamma \in V_g$, $\gamma \in gUKU^{-1}g^{-1}$. Now $\gamma(p(gU)) \cap p(gU) \neq \emptyset \Leftrightarrow \gamma \in V_g$. Note that $\Gamma \cap \bar{V}_g \subset \bar{V}_g$. Since Γ is discrete, then $\Gamma \cap \bar{V}_g$ should be closed. We know that a closed subset of compact set is compact. Therefore $\Gamma \cap \bar{V}_g$ is compact. Hence $\Gamma \cap \bar{V}_g$ is finite. So, for any compact subset $C = p(g\bar{U}) \subset G/K$,

$\{\gamma \in \Gamma | \gamma C \cap C \neq \emptyset\}$ is finite. Thus Γ is properly discontinuous.
$((iii) \Rightarrow (ii))$ by Lemma 2.3, and $((ii) \Rightarrow (i))$ trivially. $\square$

2.2. **Group Extension and 2-cocycle.** A group extension is a short exact sequence

$$1 \longrightarrow \Delta \longrightarrow \pi \xrightarrow{p} Q \longrightarrow 1$$

where Δ is a normal subgroup of π. There is a homomorphism

$$\mu : \pi \longrightarrow Aut(\Delta) \quad \text{by } g \mapsto \mu_g;\ \mu_g(n) = gng^{-1}\ (g \in \pi,\, n \in \Delta).$$

It is easily checked that

$$\mu_{gh}(n) = \mu_g \mu_h(n). \tag{2.1}$$

Choose a section $s : Q \longrightarrow \pi$ such that $p \circ s = id$ and $s(1) = 1$. Consider the map $\phi : Q \longrightarrow Aut(\Delta)$ given by $\phi(\alpha) = \mu_{s(\alpha)}$ $(\alpha \in Q)$. Then suppose

$$1 \longrightarrow Inn(\Delta) \longrightarrow Aut(\Delta) \xrightarrow{\nu} Out(\Delta) \longrightarrow 1$$

is the natural exact sequence. Then ϕ induces a homomorphism

$$\varphi : Q \longrightarrow Out(\Delta) \quad \alpha \mapsto \nu \circ \mu_{s(\alpha)}. \tag{2.2}$$

Since $s(\alpha\beta)$, $s(\alpha)s(\beta)$ are mapped to $\alpha\beta \in Q$, i.e

$$\begin{aligned} ps(\alpha\beta) &= \alpha\beta = ps(\alpha)ps(\beta) \\ &= p(s(\alpha)s(\beta)), \end{aligned}$$

so there exist an element $f : Q \times Q \rightarrow \Delta$, $f(\alpha, \beta) \in \Delta$ such that

$$f(\alpha, \beta)s(\alpha\beta) = s(\alpha)s(\beta).$$

The function f satisfies the following conditions.

(i) $\phi(\alpha)(\phi(\beta)(n)) = f(\alpha, \beta)\phi(\alpha\beta)(n)f(\alpha, \beta)^{-1}$
(ii) $f(\alpha, 1) = f(1, \alpha) = 1$
(iii) $\phi(\alpha)(f(\beta, \gamma))f(\alpha, \beta\gamma) = f(\alpha, \beta)f(\alpha\beta, \gamma)$.

Conversely, given a function $\phi : Q \rightarrow Aut(\Delta)$ and a function $f : Q \times Q \rightarrow \Delta$ satisfying (i)-(iii), we define a group law in the product $\Delta \times Q$:

$$(n, \alpha)(m, \beta) = (n\phi(\alpha)(m)f(\alpha, \beta), \alpha\beta). \tag{2.3}$$

By this product (2.3), we have a group $\pi = \Delta \times Q$. Setting $p((n, \alpha)) = \alpha$, we obtain a group extension $1 \rightarrow \Delta \rightarrow \pi \xrightarrow{p} Q \rightarrow 1$.

Next let us consider

$$\begin{aligned} f(\alpha, \beta)s(\alpha\beta)ns(\alpha\beta)^{-1}f(\alpha, \beta)^{-1} &= s(\alpha)s(\beta)ns(\alpha\beta)^{-1}f(\alpha, \beta)^{-1} \\ \mu_{f(\alpha,\beta)s(\alpha\beta)}(n) &= s(\alpha)s(\beta)ns(\beta)^{-1}s(\alpha)^{-1} \\ \mu_{f(\alpha,\beta)}\mu_{s(\alpha\beta)}(n) &= \mu_{s(\alpha)s(\beta)}(n) \quad \text{by (2.1)} \\ &= \mu_{s(\alpha)}\mu_{s(\beta)}(n). \end{aligned}$$

Since $\nu(Inn(\Delta)) = 1$, and let $a' = f(\alpha,\beta) \in \Delta$, $\mu_{f(\alpha,\beta)}(n) = \mu_{a'}(n) = a'na'^{-1} \in Inn(\Delta)$, then for any $n \in \Delta$, $\nu(\mu_{f(\alpha,\beta)}) = 1$. So

$$\begin{aligned}
\nu(\mu_{f(\alpha,\beta)}\mu_{s(\alpha\beta)}) &= \nu(\mu_{s(\alpha)}\mu_{s(\beta)})\\
\nu(\mu_{f(\alpha,\beta)})\nu(\mu_{s(\alpha\beta)}) &= \nu(\mu_{s(\alpha)})\nu(\mu_{s(\beta)})\\
\nu(\mu_{s(\alpha\beta)}) &= \nu(\mu_{s(\alpha)})\nu(\mu_{s(\beta)})\\
\varphi(\alpha\beta) &= \varphi(\alpha)\varphi(\beta).
\end{aligned}$$

Hence φ is homomorphism. We call φ an *abstract kernel*.

Next, let us consider the case where $\Delta = A$ is abelian. In other words, we try to classify all group extensions of an abelian group A by Q with abstract kernel $\varphi : Q \longrightarrow Out(A)$. Since A is abelian, $Aut(A) = Out(A)$, so $\varphi = \phi : Q \longrightarrow Aut(A)$. Moreover the condition function f above, (i) holds automatically. A map $f : Q \times Q \longrightarrow A$ satisfying the two equalities (ii) and (iii) above is called a *2-cocycle* for the abstract kernel ϕ. The set of all 2-cocycles is denoted by $Z^2_\phi(Q;A)$ or simply, by $Z^2(Q;A)$.

Let $\lambda : Q \to A$ be any map such that $\lambda(1) = 1$. Define $\delta'\lambda : Q \times Q \to A$ by

$$\delta'\lambda(\alpha,\beta) = \phi(\alpha)(\lambda(\beta))\lambda(\alpha)\lambda(\alpha\beta)^{-1}. \tag{2.4}$$

It turns out that such a $\delta'\lambda$ is a 2-cocycle and is called a *2-coboundary*. The set of all 2-coboundaries is denoted by $B^2_\phi(Q;A)$. Clearly, $B^2_\phi(Q;A)$ is a subgroup of $Z^2_\phi(Q;A)$. Let $f_1, f_2 : Q \times Q \to A$ be 2-cocyles. We say f_1 is *cohomologous* to f_2 if there is a map λ such that $f_1(\alpha,\beta) = \delta'\lambda(\alpha,\beta)f_2(\alpha,\beta)$ $(\forall \alpha,\beta \in Q)$.

We define the second cohomology group as the quotient group

$$H^2_\phi(Q;A) = Z^2_\phi(Q;A)/B^2_\phi(Q;A).$$

From hypothesis 1 below, when $\mathcal{N} = \mathbb{R}^n$, $H^2_{\bar\phi}(Q;\mathcal{N}) = 0$ for any finite group Q. Then there is a function $\chi : Q \to \mathcal{N}$ (1-chain) such that $f = \delta'\chi$;

$$f(\alpha,\beta) = \bar\phi(\alpha)(\chi(\beta))\chi(\alpha)\chi(\alpha\beta)^{-1} \ (\alpha,\beta \in Q). \tag{2.5}$$

Now let's check the property of $\bar\phi$. As $\phi(\alpha)(n) = s(\alpha)ns(\alpha)^{-1}$, we see that

$$\begin{aligned}
\phi(\alpha)(\phi(\beta)(n)) &= s(\alpha)(s(\beta)ns(\beta)^{-1})s(\alpha)^{-1}\\
&= f(\alpha,\beta)s(\alpha\beta)ns(\alpha\beta)^{-1}f(\alpha,\beta)^{-1}\\
&= f(\alpha,\beta)\phi(\alpha\beta)(n)f(\alpha,\beta)^{-1}
\end{aligned}$$

By the uniqueness of $\bar\phi$, the following holds.

$$\bar{\phi}(\alpha)(\bar{\phi}(\beta)(x)) = f(\alpha,\beta)\bar{\phi}(\alpha\beta)(x)f(\alpha\beta)^{-1} \ (x \in \mathcal{N}). \tag{2.6}$$

Now, we give three **hypotheses** as follows:

1. Let $\mathcal{N}$ be connected Lie group contains Δ as a discrete uniform subgroup. $(\Delta, \mathcal{N})$ has the unique extension property; If $\phi(\alpha) : \Delta \longrightarrow \Delta$ is an automorphism, then ϕ extends to a unique automorphism $\bar{\phi}(\alpha) : \mathcal{N} \longrightarrow \mathcal{N}$.

2. $\mathcal{N}$ acts properly and freely on X such that there is a principal bundle

$$\mathcal{N} \longrightarrow X \longrightarrow W \ \text{ with } W = X/\mathcal{N} \text{ is manifold.}$$

3. Let (π, X) be a properly discontinuous action.

By hypothesis 1, we can consider the *pushout* $\pi\mathcal{N} = \{(n,\gamma) | n \in \mathcal{N}, \gamma \in Q\}$;

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Delta & \longrightarrow & \pi & \longrightarrow & Q & \longrightarrow & 1 \\ & & \big\downarrow & & \big\downarrow & & \| & & \\ 1 & \longrightarrow & \mathcal{N} & \longrightarrow & \pi\mathcal{N} & \longrightarrow & Q & \longrightarrow & 1, \end{array} \tag{2.7}$$

particularly, note that

$$\pi \cap \mathcal{N} = \Delta, \tag{2.8}$$

and

$$\pi \text{ normalizes } \mathcal{N}, \tag{2.9}$$

since for any $(x,\gamma) \in \pi$ and $(n,1) \in \mathcal{N}$, $(x,\gamma)(n,1)(x,\gamma)^{-1} = (x\bar{\phi}(\gamma)(n)x^{-1}, 1) \in \mathcal{N}$.

Theorem 2.5. *There is a proper action $(\pi\mathcal{N}, X)$ such that the following is equivariant fibration;*

$$(\mathcal{N}, \mathcal{N}) \longrightarrow (\pi\mathcal{N}, X) \xrightarrow{p} (Q, W).$$

Proof. We note that $\mathcal{N}$ acts properly and π acts properly discontinuous on X (hypothesis 2,3). Consider the natural projection $\mathcal{N} \longrightarrow \pi\mathcal{N}/\pi$ which induces a quotient diffeomorphism $\mathcal{N}/\mathcal{N} \cap \pi = \pi\mathcal{N}/\pi$. Since $\mathcal{N} \cap \pi = \Delta$ by (2.8) and $\mathcal{N}/\Delta$ is compact (Δ is an uniform subgroup of $\mathcal{N}$), we can choose a compact subset $K \subset \mathcal{N}$ such that $\pi\mathcal{N} = \pi K (\subset \pi\mathcal{N})$.

Now we prove $\pi\mathcal{N}$ acts properly. Let $(n_i, \gamma_i) \in \pi\mathcal{N}$ acts on $x_i \in X$ where $\lim x_i = x \in X$ $(i \rightarrow \infty)$. Suppose $(n_i, \gamma_i)x_i \rightarrow y \in X (i \rightarrow \infty)$. As above, $(n_i, \gamma_i) = s_i k_i$ ($s_i \in \pi$, $k_i \in K$). Then $(n_i, \gamma_i)x_i = s_i k_i x_i = s_i(k_i x_i)$. As K is compact, we may assume $k_i \to k \in K$ ($i \to \infty$) so that $k_i x_i \to kx$ ($i \to \infty$) and as $s_i(k_i x_i) \to y$ ($i \to \infty$), by properness of π, $\{s_i\}$ consist of finitely many elements, in fact $s_i \to s \in \pi$ for sufficiently larger i. Hence $(n_i, \gamma_i) = s_i k_i \to sk \in \pi\mathcal{N}$. $\square$

2.3. Rigid Motions.

Definition 2.6. *A rigid motion is an ordered pair (s, M) with $M \in O(n)$ and $s \in \mathbb{R}^n$. A rigid motion acts on $\mathbb{R}^n$ by*

$$(s, M)x = Mx + s \quad for\ x \in \mathbb{R}^n. \tag{2.10}$$

We multiply rigid motions by composing them, so

$$(s, M)(t, N) = (Mt + s, MN). \tag{2.11}$$

We denote the group of rigid motions (of $\mathbb{R}^n$) by $\mathbb{E}(n)$.

Definition 2.7. *An affine motion is an ordered pair (s, M) with $M \in GL(n, \mathbb{R})$ and $s \in \mathbb{R}^n$.*

Affine motions act on $\mathbb{R}^n$ via (2.10) and form a group, $\mathbb{A}(n)$, under (2.11).

Definition 2.8. *Define a homomorphism $L : \mathbb{E}(n) \to O(n)$ by*

$$L(s, M) = M.$$

If $\alpha \in \mathbb{E}(n)$, $L(\alpha)$ is called the rotational part of α. We can also define the translational part of α by

$$T(s, M) = s,$$

but the map $T : \mathbb{E}(n) \to \mathbb{R}^n$ is not a homomorphism.

We say (s, M) is a *pure translation* if $M = I$, the identity matrix. If Γ is a subgroup of $\mathbb{E}(n)$, $\Gamma \cap \mathbb{R}^n$ will denote the subgroup of Γ of pure translations.

Recall that such a group Γ is *torsionfree* if Γ has no elements of finite order.

Definition 2.9. *We say that a subgroup Γ of $\mathbb{E}(n)$ is uniform if $\mathbb{R}^n/\Gamma$ is compact where the orbit space, $\mathbb{R}^n/\Gamma$, is the set of orbits with the quotient (or identification) topology. We say Γ is reducible if we can find $\alpha \in \mathbb{A}(n)$, so that $T(\alpha\Gamma\alpha^{-1})$ does not span $\mathbb{R}^n$. In other words, Γ is reducible if after an affine change of coordinates, all the elements of Γ have translational parts in a proper subspace of $\mathbb{R}^n$. If Γ is not reducible, we say Γ is irreducible.*

Proposition 2.10. *Let Γ be a subgroup of $\mathbb{E}(n)$. Then if Γ acts freely, Γ is torsionfree. Furthermore, if Γ is discontinuous and torsionfree, Γ acts freely.*

Proof. Suppose $\alpha \in \Gamma$, $\alpha \neq (0, I)$, and $\alpha^k = (0, I)$. Let x be arbitrary in $\mathbb{R}^n$, and set

$$y = \sum_{i=0}^{k-1} \alpha^i x.$$

Then $\alpha y = y$, so Γ does not act freely.

If Γ is discontinuous on $\mathbb{R}^n$, then Γ acts properly discontinuously on $\mathbb{R}^n$. Then stabilizer Γ_x is finite $\forall x \in \mathbb{R}^n$. Hence if Γ is torsionfree, then $\Gamma_x = \{1\}$ $\forall x \in \mathbb{R}^n$. So Γ acts freely on $\mathbb{R}^n$. $\square$

In this paper we will be concerned with discrete subgroups Γ of $\mathbb{E}(n)$ which act freely on $\mathbb{R}^n$.

Definition 2.11. *We say that Γ is crystallographic if Γ is uniform and discrete. If Γ is crystallographic and torsionfree in $\mathbb{E}(n)$, we say Γ is a Bieberbach subgroup of $\mathbb{E}(n)$.*

Theorem 2.12. *Let Γ be a subgroup of the Euclidean group $\mathbb{E}(n) = \mathbb{R}^n \rtimes O(n)$. Then Γ is a torsionfree discrete uniform subgroup of $\mathbb{E}(n)$ if and only if Γ acts properly discontinuously and freely on $\mathbb{R}^n$ with compact quotient.*

Proof. Let $X = \mathbb{E}(n) = \mathbb{R}^n \rtimes O(n)$. Let $K = O(n)$, compact subgroup. Then $p : X \longrightarrow X/K = \mathbb{R}^n$ by $p(a, A) = a$.
($\Rightarrow$). Let Γ is a torsionfree discrete uniform subgroup of $\mathbb{E}(n)$. Then Γ is discontinuous by Lemma 2.4. Therefore Γ acts freely by Proposition 2.10. Also Γ acts properly discontinuously by Lemma 2.4. Finally, since Γ is uniform then $\mathbb{R}^n/\Gamma$ is compact.
($\Leftarrow$). Let Γ acts properly discontinuously and freely on $\mathbb{R}^n$ with compact quotient. By Proposition 2.10, Γ is torsionfree. Since Γ acts properly discontinuously then Γ is discrete in $\mathbb{E}(n)$ by Lemma 2.4. Γ is uniform, because $\mathbb{R}^n/\Gamma$ is compact. $\square$

We have defined the group of rigid motion $\mathbb{E}(n)$ which acts on $\mathbb{R}^n$. Suppose torsionfree crystallographic subgroup $\Gamma \subset \mathbb{E}(n) \subset \mathbb{A}(n)$ acts on $\mathbb{R}^n$, and by first Bieberbach theorem there is an exact sequence

$$1 \longrightarrow \mathbb{Z}^n \longrightarrow \Gamma \longrightarrow F \longrightarrow 1,$$

where F is a finite group.

Suppose $\mathcal{C}(\Gamma)$ the center of Γ is non trivial. If $\alpha = (s, M) \in \mathcal{C}(\Gamma)$, $\alpha n \alpha^{-1} = n \quad \forall n \in \mathbb{Z}^n (\mathbb{Z}^n \lhd \Gamma)$. Then $(n, I) = (s, M)(n, I)(s, M)^{-1} =$

$(Mn, I) \Rightarrow M = I$. Therefore $L(\mathcal{C}(\Gamma)) = I$, $\mathcal{C}(\Gamma) \subsetneqq \mathbb{Z}^n$. Since Γ is torsionfree, then $\mathcal{C}(\Gamma) \cong \mathbb{Z}^k \subset \mathbb{Z}^n$, $1 \leq k < n$. Hence we get

$$1 \longrightarrow \mathbb{Z}^k \longrightarrow \Gamma \longrightarrow \Gamma/\mathbb{Z}^k = Q \longrightarrow 1.$$

What is the form of Q which acts on $\mathbb{R}^{(n-k)}$ $(k < n)$?. Let

$$\gamma = \Big(\begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} A_{k\times k} & B_{k\times(n-k)} \\ C_{(n-k)\times k} & D_{(n-k)\times(n-k)} \end{pmatrix} \Big) \in \Gamma \text{ and}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^n = \begin{pmatrix} \mathbb{R}^k \\ \mathbb{R}^{n-k} \end{pmatrix} \text{ where } x \in \mathbb{R}^k,\ y \in \mathbb{R}^{n-k}.$$

Now consider $\phi : Q \longrightarrow Aut(\mathbb{Z}^k)$ by $\phi(\alpha)(x) = \gamma x \gamma^{-1}$ which is represented by matrix $A \in GL(k, \mathbb{Z})$ such that $\phi(\alpha)(x) = Ax$ for $\alpha \in Q$. Therefore $\phi(\alpha)$ naturally extends to $\bar{\phi}(\alpha) : \mathbb{R}^k \longrightarrow \mathbb{R}^k$ (Γ normalizes $\mathbb{R}^k$ also, $\gamma \mathbb{R}^k \gamma^{-1} = A\mathbb{R}^k$).

Then we have for any $x \in \mathbb{R}^k$,

$$\begin{aligned} \Big(\begin{pmatrix} x \\ 0 \end{pmatrix}, I \Big) =& \gamma \Big(\begin{pmatrix} x \\ 0 \end{pmatrix}, I \Big) \gamma^{-1} = \Big(\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix}, I \Big) \\ =& \Big(\begin{pmatrix} Ax \\ Cx \end{pmatrix}, I \Big) \end{aligned}$$

$\Rightarrow A = I,\ C = 0.$

By Theorem 2.5 there is an equivariant fibration

$$(\mathbb{Z}^k, \mathbb{R}^k) \longrightarrow (\Gamma, \mathbb{R}^n) \xrightarrow{(\nu,p)} (Q, W).$$

For $\alpha \in Q$, $y \in W = \mathbb{R}^{n-k}$

$$\begin{aligned} \alpha y = \alpha p \begin{pmatrix} x \\ y \end{pmatrix} &= \nu(\gamma) p \begin{pmatrix} x \\ y \end{pmatrix} = p\Big(\gamma \begin{pmatrix} x \\ y \end{pmatrix} \Big) \\ &= p\Big(\begin{pmatrix} a + x + By \\ b + Dy \end{pmatrix} \Big) = b + Dy \\ &= (b, D)y \end{aligned}$$

we get $\alpha = (b, D)$. If $\begin{pmatrix} I & B \\ 0 & D \end{pmatrix} \in O(n)$ then $D \in O(n-k)$. Thus $Q = \{(b, D) \subset \mathbb{E}(n-k)\}$.

2.4. **Bieberbach Theorems.** Bieberbach theorems are about discrete, uniform subgroups of $\mathbb{E}(n)$, but for geometric applications of his theorems (to flat manifolds), we require the subgroups to have an additional property.

Theorem 2.13. *(First Bieberbach) Let Γ be a crystallographic subgroup of $\mathbb{E}(n)$. Then*

(i) $L(\Gamma)$ *is finite,* $L : \mathbb{E}(n) \to O(n)$.
(ii) $\Gamma \cap \mathbb{R}^n$ *is a lattice (finitely generated free abelian group) which spans* $\mathbb{R}^n$. *(cf.* [1] *for proof)*

This means that for an n-dimensional crystallographic group Γ, the translation part $\Gamma^* = \Gamma \cap \mathbb{R}^n$ of Γ is a free abelian group isomorphic with $\mathbb{Z}^n$, such that the vector space spanned by Γ^* is the whole space $\mathbb{R}^n$. If $\Gamma^* = \Gamma$ then Γ is torsionfree and the manifold $\mathbb{R}^n/\Gamma$ is an n-dimensional torus.

Recall that a sequence of groups and homomorphisms is *exact* if the image of any of the homomorphisms is equal to the kernel of the next one. By the first Bieberbach theorem, if Γ is any crystallographic group, then Γ satisfies an exact sequence

$$0 \longrightarrow \Gamma^* \longrightarrow \Gamma \longrightarrow \Phi \longrightarrow 1 \tag{2.12}$$

where $\Gamma^* = \Gamma \cap \mathbb{R}^n$ is a lattice of rank n, and $\Phi = L(\Gamma)$ is a finite group. We call Φ, the *holonomy group* of Γ. We use the "0" at the left of (2.12) to indicate that we usually write Γ^* additively, while the "1" at the right indicates that we usually write Φ multiplicatively.

Proposition 2.14. *Let* Γ *be a crystallographic subgroup of* $\mathbb{E}(n)$. *Then* $\Gamma \cap \mathbb{R}^n$ *is the unique normal, maximal abelian subgroup of* Γ.

Proof. Let $\rho \subset \Gamma$ be a normal abelian subgroup. It suffices to show $\rho \subset \mathbb{R}^n$, i.e. $L(\rho) = I$. Let $(s, M) \in \rho$. We can assume Ms=s, because $(s, I)(s, M)(-s, I) = (s + s - Ms, M) = (s, M) \Leftrightarrow Ms = s$. Let (v, I) be any element of Γ^*. Then ρ normal implies

$$(v, I)(s, M)(-v, I) = (v - Mv + s, M) \in \rho.$$

But ρ abelian $\Rightarrow [(s, M), (v - Mv + s, M)] = (0, I)$ which yields

$$Mv - M^2 v - v + Mv = 0,$$

or $(M - I)^2 v = 0$.
Let $X = \{x \in \mathbb{R}^n : Mx = x\}$ and write $\mathbb{R}^n = X \oplus Y$ orthogonally. Write $v = v_x + v_y$ with $v_x \in X$ and $v_y \in Y$, then

$$0 = (M - I)^2 v = (M - I)^2 (v_x + v_y) = (M - I)^2 v_y.$$

Now on Y, $(M - I)$ is non-singular, so $(M - I)^2 v_y = 0 \Rightarrow v_y = 0$. $\Rightarrow v = v_x$. Hence $Mv = v$ $\forall v \in \Gamma^*$. Since Γ^* spans $\mathbb{R}^n$, $Mx = x$ $\forall x \in \mathbb{R}^n$, so $M = I$. Hence $(s, M) = (s, I) \in \mathbb{R}^n \Rightarrow \rho \subset \mathbb{R}^n$. □

Theorem 2.15. *(Second Bieberbach). Suppose that* $f : \Gamma_1 \longrightarrow \Gamma_2$ *is an isomorphism between crystallographic group. Then there exist a diffeomorphism* $h : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ *such that* $h(\gamma x) = f(\gamma)h(x)$. *Moreover,* *such a diffeomorphism can be taken as an affine transformation, i.e.,* $h = (a, A) \in \mathbb{A}(n)$ *and so* $f(\gamma) = h\gamma h^{-1}$ $(\gamma \in \Gamma_1)$.

Proof. $(\Rightarrow)$. Let $\mathbb{Z}^n$ be a maximal abelian subgroup of Γ_1 such that

$$1 \longrightarrow \mathbb{Z}^n \longrightarrow \Gamma_1 \longrightarrow Q_1 \longrightarrow 1$$

is a group extension. Since $\mathbb{Z}^n$ is characteristics, f maps isomorphism onto $\mathbb{Z}^n$ of Γ_2 so that we have a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z}^n & \xrightarrow{Z_1} & \Gamma_1 & \xrightarrow{p_1} & Q_1 & \longrightarrow & 1 \\ & & \downarrow g & & \downarrow f & & \downarrow \Phi & & \\ 1 & \longrightarrow & \mathbb{Z}^n & \xrightarrow{Z_2} & \Gamma_2 & \xrightarrow{p_2} & Q_2 & \longrightarrow & 1 \end{array} \tag{2.13}$$

where $g = f|_{\mathbb{Z}^n}$, Φ is an induced isomorphism. We write Γ_i as the set $\mathbb{Z}^n \times Q_i$ with group law

$$(n, \alpha)(m, \beta) = (n\phi_i(\alpha)(m)f_i(\alpha, \beta), \alpha\beta) \quad (i = 1, 2). \tag{2.14}$$

Here we write $\mathbb{Z}^n$ multiplicatively, and homomorphism

$$\phi_i : Q_i \longrightarrow Aut(\mathbb{Z}^n) = GL(n, \mathbb{Z})$$

is defined as follows; choose a section $s_i : Q_i \longrightarrow \Gamma_i$ so that $p_i \circ s_i = id$. Γ_i normalizes $\mathbb{Z}^n$, so

$$\phi_i(\alpha) : \mathbb{Z}^n \longrightarrow \mathbb{Z}^n$$

is defined by

$$\phi_i(\alpha)(n) = s_i(\alpha)ns_i(\alpha)^{-1}.$$

Then defined $f_i : Q_i \times Q_i \longrightarrow \mathbb{Z}^n$ by

$$f_i(\alpha, \beta)s_i(\alpha\beta) = s_i(\alpha)s_i(\beta).$$

As $p_2f(1, \alpha) = \Phi p_1(1, \alpha) = \Phi(\alpha)$ and $p_2f(n, 1) = \Phi p_1(n, 1) = \Phi(1) = 1$, we write

$$f(1, \alpha) = (\lambda(\alpha), \Phi(\alpha)) \quad , f(n, 1) = (g(n), 1) \tag{2.15}$$

for some function $\lambda : Q_1 \longrightarrow \mathbb{Z}^n$. Then

$$\begin{aligned} f(n, \alpha) &= f((n, 1)(1, \alpha)) = f(n, 1)f(1, \alpha) \\ &= (g(n), 1)(\lambda(\alpha), \Phi(\alpha)). \end{aligned}$$

Note we choose a normalized 2-cocycle f_i, that is

$$f_i(1, \alpha) = f_i(\beta, 1) = 1 \quad (\forall \alpha, \beta \in Q_i).$$

Therefore

(2.16)

$$f(n, \alpha) = (g(n)\phi_i(1)(\lambda(\alpha))f_i(1, \Phi(\alpha)), \Phi(\alpha)) = (g(n)\lambda(\alpha), \Phi(\alpha)).$$

Next consider the equalities

$$\begin{aligned} f(1,\alpha)f(g^{-1}(n),1) &= f(\phi_1(\alpha)(g^{-1}(n)),\alpha) \\ (\lambda(\alpha),\Phi(\alpha))(n,1) &= (g(\phi_1(\alpha)(g^{-1}(n)))\lambda(\alpha),\Phi(\alpha)) \\ (\lambda(\alpha)\phi_2(\Phi(\alpha))(n),\Phi(\alpha)) &= (g(\phi_1(\alpha)(g^{-1}(n)))\lambda(\alpha),\Phi(\alpha)) \end{aligned}$$

imply that

$$g\phi_1(\alpha)g^{-1}(n) = \lambda(\alpha)\phi_2(\Phi(\alpha))(n)\lambda(\alpha)^{-1}. \tag{2.17}$$

The equalities

$$\begin{aligned} f(1,\alpha)f(1,\beta) &= f((1,\alpha)(1,\beta)) \\ &= f(\phi_i(\alpha)(1)f_1(\alpha,\beta),\alpha\beta) \\ &= f(f_1(\alpha,\beta),\alpha\beta) \\ (\lambda(\alpha),\Phi(\alpha))(\lambda(\beta),\Phi(\beta)) &= (g(f_1(\alpha,\beta))\lambda(\alpha\beta),\Phi(\alpha\beta)) \quad \text{by (2.15), (2.16)} \end{aligned}$$

and

$$\begin{aligned} &(\lambda(\alpha),\Phi(\alpha))(\lambda(\beta),\Phi(\beta)) \\ &= (\lambda(\alpha)\phi_2(\Phi(\alpha))(\lambda(\beta))f_2(\Phi(\alpha),\Phi(\beta)),\Phi(\alpha)\Phi(\beta)) \\ &= (\lambda(\alpha)\phi_2(\Phi(\alpha))(\lambda(\beta))\lambda(\alpha)^{-1}\lambda(\alpha)f_2(\Phi(\alpha),\Phi(\beta)),\Phi(\alpha)\Phi(\beta)) \\ &= (g\phi_1(\alpha)g^{-1}(\lambda(\beta))\lambda(\alpha)f_2(\Phi(\alpha),\Phi(\beta)),\Phi(\alpha)\Phi(\beta)) \quad \text{by (2.17)}, \end{aligned}$$

then we have

$$g(f_1(\alpha,\beta)) = g\phi_1(\alpha)g^{-1}(\lambda(\beta))\lambda(\alpha)f_2(\Phi(\alpha),\Phi(\beta))\lambda(\alpha\beta)^{-1}. \tag{2.18}$$

Let us consider a group cohomology $H^2_\phi(Q;\mathbb{Z}^n)$, where cocycle $[f_i] \in H^2_{\phi_i}(Q;\mathbb{Z}^n)$ $(i=1,2)$. The exact sequence

$$1 \longrightarrow \mathbb{Z}^n \xrightarrow{i} \mathbb{R}^n \xrightarrow{j} T^n \longrightarrow 1$$

gives a cohomology exact sequence

$$\begin{aligned} ... \longrightarrow H^1_\phi(Q;\mathbb{R}^n) &\xrightarrow{j_*} H^1_\phi(Q;T^n) \xrightarrow{\delta} H^2_\phi(Q;\mathbb{Z}^n) \\ \xrightarrow{i_*} H^2_\phi(Q;\mathbb{R}^n) &\xrightarrow{j_*} H^2_\phi(Q;T^n) \longrightarrow \quad ... \end{aligned}$$

If Q is finite group, and $\mathbb{R}^n$ is the $Q-$module, then $H^i_\phi(Q;\mathbb{R}^n) = 0$ $(i \geq 1)$. Using this, we have

$$H^1_{\phi_i}(Q_i;T^n) \xrightarrow{\delta} H^2_{\phi_i}(Q_i;\mathbb{Z}^n)$$

which follows that $\delta[\hat{\chi}_i] = [f_i]$, $[\hat{\chi}_i] \in H^1_{\phi_i}(Q_i;T^n)$. The meaning of $\delta[\hat{\chi}_i] = [f_i]$ is explained as follows;

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C^1_\phi(Q,\mathbb{Z}^n) & \xrightarrow{i} & C^1_\phi(Q,\mathbb{R}^n) & \xrightarrow{j} & C^1_\phi(Q,T^n) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \delta'} & & \downarrow{\scriptstyle \delta'} & & \downarrow{\scriptstyle \delta'} & & \\
0 & \longrightarrow & C^2_\phi(Q,\mathbb{Z}^n) & \xrightarrow{i} & C^2_\phi(Q,\mathbb{R}^n) & \xrightarrow{j} & C^2_\phi(Q,T^n) & \longrightarrow & 0 \\
 & & & & \chi_i & \xrightarrow{j} & \hat{\chi}_i & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle \delta'} & & \downarrow{\scriptstyle \delta'} & & \\
 & & f_i & \xrightarrow{i} & \delta'\chi_i & \xrightarrow{j} & 0 & &
\end{array}$$

$f_i = if_i = \delta'\chi_i$ $(i = 1, 2)$. Thus

$$f_1(\alpha,\beta) = \delta'_{\bar{\phi}_1(\alpha)}\chi_1(\alpha,\beta) = \bar{\phi}_1(\alpha)(\chi_1(\beta))\chi_1(\alpha)\chi_1(\alpha\beta)^{-1} \tag{2.19}$$

where $\chi_i : Q_i \longrightarrow \mathbb{R}^n$ are functions, and $\delta'_{\bar{\phi}_i(\alpha)}\chi_i : Q_i \times Q_i \longrightarrow \mathbb{R}^n$ the coboundary of 1-cochains.

We note that $\phi_i(\alpha) : \mathbb{Z}^n \longrightarrow \mathbb{Z}^n$ is represented by (an integral) matrix so that $\phi_i(\alpha)$ extends to $\mathbb{R}^n$ onto itself. We write it as $\bar{\phi}_i(\alpha)$ (= the same matrix). So $\phi_i : Q_i \longrightarrow Aut(\mathbb{Z}^n)$ extends uniquely to $\bar{\phi}_i : Q_i \longrightarrow Aut(\mathbb{R}^n) = GL(n,\mathbb{R})$.

Put $g(f_1(\alpha,\beta)) = g_*f_1(\alpha,\beta)$ and $(\bar{g}_*\chi_1)(\alpha) = \bar{g}(\chi_1(\alpha))$. Here the isomorphism $g : \mathbb{Z}^n \longrightarrow \mathbb{Z}^n$ extends uniquely to an isomorphism $\bar{g} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ by the same reason as before. Also we put $\Phi^*f_2(\alpha,\beta) = f_2(\Phi(\alpha),\Phi(\beta))$ and $\Phi^*\chi_2(\alpha) = \chi_2(\Phi(\alpha))$. Then (2.19) implies

$$\begin{aligned}
gf_1(\alpha,\beta) &= \bar{g}(\bar{\phi}_1(\alpha)(\chi_1(\beta))\chi_1(\alpha)\chi_1(\alpha\beta)^{-1}) \\
&= \bar{g}\bar{\phi}_1(\alpha)\bar{g}^{-1}(\bar{g}\chi_1(\beta))\bar{g}\chi_1(\alpha)\bar{g}\chi_1(\alpha\beta)^{-1} \\
&= \bar{g}\bar{\phi}_1(\alpha)\bar{g}^{-1}(\bar{g}_*\chi_1(\beta))\bar{g}_*\chi_1(\alpha)\bar{g}_*\chi_1(\alpha\beta)^{-1} \\
&= \delta'_{\bar{g}\bar{\phi}_1\bar{g}^{-1}}\bar{g}_*\chi_1(\alpha,\beta),
\end{aligned}$$

hence

$$g_*f_1 = \delta'_{\bar{g}\bar{\phi}_1\bar{g}^{-1}}\bar{g}_*\chi_1. \tag{2.20}$$

Here we can write $\overline{g\phi_1 g^{-1}} = \bar{g}\bar{\phi}_1(\alpha)\bar{g}^{-1}$. Also, $f_2 = \delta'\chi_2$ shows

$$\Phi^*f_2 = \delta'\Phi^*\chi_2. \tag{2.21}$$

Next we calculate

$$
\begin{aligned}
&\delta'_{\bar{g}\bar{\phi}_1\bar{g}^{-1}}(\lambda.\Phi^*\chi_2)(\alpha,\beta)\\
&= \bar{g}\bar{\phi}_1(\alpha)\bar{g}^{-1}(\lambda.\Phi^*\chi_2(\beta)).(\lambda.\Phi^*\chi_2(\alpha)).(\lambda.\Phi^*\chi_2(\alpha\beta))^{-1} \text{ by (2.19)}\\
&= \bar{g}\bar{\phi}_1(\alpha)\bar{g}^{-1}(\lambda(\beta).\chi_2(\Phi(\beta))).\lambda(\alpha)\chi_2(\Phi(\alpha)).\chi_2(\Phi(\alpha\beta))^{-1}\lambda(\alpha\beta)^{-1}\\
&= \bar{g}\bar{\phi}_1(\alpha)(\bar{g}^{-1}(\lambda(\beta)))\ \ \bar{g}\bar{\phi}_1(\alpha)\bar{g}^{-1}(\chi_2(\Phi(\beta)))\lambda(\alpha)\\
&\quad \chi_2(\Phi(\alpha))\chi_2(\Phi(\alpha\beta))^{-1}\lambda(\alpha\beta)^{-1}\\
&= \bar{g}\bar{\phi}_1(\alpha)(\bar{g}^{-1}(\lambda(\beta)))\ \ \lambda(\alpha)\bar{\phi}_2(\Phi(\alpha))(\chi_2(\Phi(\beta)))\\
&\quad \chi_2(\Phi(\alpha))\chi_2(\Phi(\alpha)\Phi(\beta))^{-1}\lambda(\alpha\beta)^{-1} \quad \text{by (2.17)}\\
&= \bar{g}\bar{\phi}_1(\alpha)(\bar{g}^{-1}(\lambda(\beta)))\ \ \lambda(\alpha)\ \delta'\chi_2(\Phi(\alpha),\Phi(\beta))\lambda(\alpha\beta)^{-1} \quad \text{by (2.19)}\\
&= \bar{g}\bar{\phi}_1(\alpha)(\bar{g}^{-1}(\lambda(\beta)))\ \ \lambda(\alpha)\ f_2(\Phi(\alpha),\Phi(\beta))\lambda(\alpha\beta)^{-1} \quad \text{by (2.19)}\\
&= g(f_1(\alpha,\beta)) \text{ by (2.18).}
\end{aligned}
$$

Hence

$$\delta'_{\bar{g}\bar{\phi}_1\bar{g}^{-1}}(\lambda.\Phi^*\chi_2) = g_*f_1. \tag{2.22}$$

Since $g_*f_1 = \delta'_{\bar{g}\bar{\phi}_1\bar{g}^{-1}}\bar{g}_*\chi_1$ by (2.20),

$$
\begin{aligned}
\delta'_{\bar{g}\bar{\phi}_1\bar{g}^{-1}}(\lambda.\Phi^*\chi_2) &= g_*f_1\\
&= \delta'_{\bar{g}\bar{\phi}_1\bar{g}^{-1}}(\bar{g}_*\chi_1)\\
[\bar{g}_*\chi_1(\lambda\Phi^*\chi_2)^{-1}] &\in H^1_{\bar{g}\bar{\phi}_1\bar{g}^{-1}}(Q;\mathbb{R}^n)
\end{aligned}
$$

On the other hand, using the fact that $H^1_{\bar{g}\bar{\phi}_1\bar{g}^{-1}}(Q;\mathbb{R}^n) = 0$, so there exist $\mu \in \mathbb{R}^n$ such that

$$
\begin{aligned}
\delta'_{\bar{g}\bar{\phi}_1\bar{g}^{-1}}\mu &= \bar{g}_*\chi_1.(\lambda\Phi^*\chi_2)^{-1}\\
\delta'_{\bar{g}\bar{\phi}_1\bar{g}^{-1}}\mu.(\lambda\Phi^*\chi_2) &= \bar{g}_*\chi_1\\
(\delta'_{\bar{g}\bar{\phi}_1\bar{g}^{-1}}\mu.\lambda\Phi^*\chi_2)(\alpha) &= \bar{g}_*\chi_1(\alpha)\\
(\delta'_{\bar{g}\bar{\phi}_1\bar{g}^{-1}}\mu(\alpha)).(\lambda\Phi^*\chi_2)(\alpha) &= \bar{g}\chi_1(\alpha)\\
\bar{g}\bar{\phi}_1(\alpha)\bar{g}^{-1}(\mu)\mu^{-1}\ ((\lambda.\Phi^*\chi_2)(\alpha)) &=\\
\bar{g}\bar{\phi}_1(\alpha)\bar{g}^{-1}(\mu)\,((\lambda.\Phi^*\chi_2)(\alpha))\,\mu^{-1} &=\\
\bar{g}\bar{\phi}_1(\alpha)\bar{g}^{-1}(\mu)\,(\lambda(\alpha).\chi_2(\Phi(\alpha)))\,\mu^{-1} &=\\
\bar{g}\bar{\phi}_1(\alpha)\bar{g}^{-1}(\mu)\lambda(\alpha)\chi_2(\Phi(\alpha))\,\mu^{-1} &=
\end{aligned}
$$

Hence by (2.17)

$$\bar{g}(\chi_1(\alpha)) = \lambda(\alpha)\bar{\phi}_2(\Phi(\alpha))(\mu)\chi_2(\Phi(\alpha))\,\mu^{-1}. \tag{2.23}$$

Next we consider a map

$$h:\mathbb{R}^n\longrightarrow\mathbb{R}^n \ \text{ by } \ h(x)=\bar{g}(x)\mu. \tag{2.24}$$

Since $\bar{g}$ is an isomorphism, so is h, then h is a diffeomorphism. Now we show that h is $(\Gamma_1,\Gamma_2)-$equivariant.

We first assume the rigid motion of Γ_i on $\mathbb{R}^n$ is the Seifert action. Note that Γ_1 acts on $\mathbb{R}^n$ as follows.

$$(n,\alpha)x=n\bar{\phi}_1(\alpha)(x)\chi_1(\alpha),\quad (n,\alpha)x=n\bar{\phi}_2(\alpha)(x)\chi_2(\alpha). \tag{2.25}$$

Then

$$\begin{aligned}
&h((n,\alpha)x)\\
&=h(n\bar{\phi}_1(\alpha)(x)\chi_1(\alpha))\\
&=\bar{g}(n\bar{\phi}_1(\alpha)(x)\chi_1(\alpha))\mu \quad \text{by (2.24)}\\
&=g(n)\bar{g}\bar{\phi}_1(\alpha)(x)\ \bar{g}\chi_1(\alpha)\ \mu \quad (\bar{g}\text{ is an isomorphism})\\
&=g(n)\bar{g}\bar{\phi}_1(\alpha)\bar{g}^{-1}(\bar{g}(x))\lambda(\alpha)\bar{\phi}_2(\Phi(\alpha))(\mu)\chi_2(\Phi(\alpha))\mu^{-1}\mu \quad \text{by (2.23)}\\
&=g(n)\lambda(\alpha)\bar{\phi}_2(\Phi(\alpha))(\bar{g}(x))\bar{\phi}_2(\Phi(\alpha))(\mu)\chi_2(\Phi(\alpha)) \quad \text{by (2.17)}\\
&=g(n)\lambda(\alpha).\ \bar{\phi}_2(\Phi(\alpha))(\bar{g}(x).\mu)\ \chi_2(\Phi(\alpha))\\
&=(g(n)\lambda(\alpha),\Phi(\alpha))\bar{g}(x)\mu \quad (\text{by (2.25) and note } g(n)\lambda(\alpha)\in\mathbb{Z}^n)\\
&=f(n,\alpha)\bar{g}(x)\mu \quad \text{by (2.16).}\\
&=f(n,\alpha)h(x).
\end{aligned}$$

Thus

$$h((n,\alpha)x)=f(n,\alpha)h(x). \tag{2.26}$$

Now, we shall see that the Seifert action coincides with the rigid motions.

Given a crystallographic group, we have the following extension:

$$1\to\mathbb{Z}^n\to\Gamma\to Q\to 1.$$

As usual an element γ of Γ is described as (n,α). We wrote the Seifert action on $\mathbb{R}^n$ additively as follows.

$$(n,\alpha)x=n+\bar{\phi}(\alpha)(x)+\chi(\alpha) \quad ((n,\alpha)\in\Gamma). \tag{2.27}$$

We may check it is a group action, namely it satisfies that

$$(n,\alpha)((m,\beta)x)=((n,\alpha)\cdot(m,\beta))x.$$

Suppose that Γ is a crystallographic group of $\mathbb{E}(n)$. Note that $(n,1)x=n+x$, that is, $\mathbb{Z}^n$ acts as translations. So its span $\mathbb{R}^n$, viewed as a group, acts on $\mathbb{R}^n$ as translations;

$$(y,1)x = y + x \ (\forall\ (y,1) \in \mathbb{R}^n).$$

Since Γ normalizes $\mathbb{Z}^n$, it normalizes its span, $\mathbb{R}^n$. Noting that $(1,\alpha)(n,1)(1,\alpha)^{-1} = (\phi(\alpha)(n),1)$, we can write

$$(1,\alpha)(x,1)(1,\alpha)^{-1} = (\bar{\phi}(\alpha)(x),1).$$

For the origin $0 \in \mathbb{R}^n$, we put

$$(1,\alpha)0 = \chi(\alpha) \in \mathbb{R}^n.$$

Noting $(x,1)0 = x$ as above, we can interpret the above Seifert action (2.27),

$$\begin{aligned}(n,\alpha)x &= ((n,1)(1,\alpha))x = (n,1)((1,\alpha)x)\\ &= (n,1)((1,\alpha)(x,1)0)\\ &= (n,1)(((1,\alpha)\cdot(x,1)\cdot(1,\alpha)^{-1})\cdot(1,\alpha)0)\\ &= (n,1)((\bar{\phi}(\alpha)(x),1)(1,\alpha)0)\\ &= (n,1)((\bar{\phi}(\alpha)(x),1)\chi(\alpha))\\ &= (n,1)(\bar{\phi}(\alpha)(x)+\chi(\alpha))\\ &= n + \bar{\phi}(\alpha)(x) + \chi(\alpha).\end{aligned}$$

On the other hand, when $\Gamma \subset \mathbb{E}(n)$, let

$$(n,\alpha) = \gamma = (a,A) \in \mathbb{E}(n).$$

As $\gamma x = a + Ax$ ($\forall\ x \in \mathbb{R}^n$), taking $x = 0$, compare the above equation so that

$$n + \chi(\alpha) = a.$$

Since $(n,\alpha)x = \gamma x = a + Ax$, it follows that $\bar{\phi}(\alpha)(x) = Ax$. This holds for all $x \in \mathbb{R}^n$, it implies that $\bar{\phi}(\alpha) = A$.

Hence identifying $(n+\chi(\alpha),\bar{\phi}(\alpha)) = (a,A)$, the Seifert action of Γ coincides with the rigid motions.

$$(n,\alpha) = (n+\chi(\alpha),\bar{\phi}(\alpha)) \in \mathbb{E}(n)\ (\forall (n,\alpha) \in \Gamma).$$

□

Theorem 2.16. *(Third Bieberbach). For any given n, there are only a finite number of n-dimensional crystallographic groups, up to affine equivalence.*

This implies that there are only finitely many n-dimensional flat Riemannian manifolds.

To prove the third Bieberbach theorem, first of all, we assume the theorem in [5] that

Theorem 2.17. *The number of non-isomorphic finite groups in $GL(n,\mathbb{Z})$ up to conjugation is finite.*

Proof. of Theorem 2.16. Given a finite group F and a faithful representation $\phi : F \to GL(n,\mathbb{Z})$, $\mathbb{Z}^n$ is viewed as a F-module through ϕ. Then each equivalence class of cocycle of the second cohomology $H^2_\phi(F,\mathbb{Z}^n)$ defines a group extension

$$1 \to \mathbb{Z}^n \to \Gamma \xrightarrow{L} F \to 1. \tag{2.28}$$

Then we have proved (see the proof of Theorem 2.15) that Γ acts on $\mathbb{R}^n$ by

$$(n,\alpha)x = n + \phi(\alpha)(x) + \chi(\alpha).$$

Then note that $\mathbb{Z}^n$ is a maximal abelian group of Γ. For this, let $\mathbb{Z}^n \subset \Delta$ be an abelian subgroup of Γ. Choose $(n,\alpha) \in \Delta$ so that

$$(n,\alpha)(m,1)(n,\alpha)^{-1} = (\phi(\alpha)(m),1).$$

Note

$$(n,\alpha)^{-1} = (\phi(\alpha^{-1})(n^{-1}\cdot f(\alpha,\alpha^{-1})^{-1}),\alpha^{-1}).$$

Since Δ is abelian, $(n,\alpha)(m,1)(n,\alpha)^{-1} = (m,1)$, $\phi(\alpha)(m) = m$ ($\forall\, m \in \mathbb{Z}^n$). It follows that $\phi(\alpha) = \mathrm{id}$. As our assumption is that $\phi : F \to GL(n,\mathbb{Z})$ is a faithful representation, this implies that $\alpha = 1$ which shows that $\Delta \ni (n,\alpha) = (n,1)$ so that $\Delta \subset \mathbb{Z}^n$.

Identify $(n,\alpha) = (n+\chi(\alpha),\phi(\alpha)) \in \mathbb{A}(n)$. Moreover since $\phi(F)$ is a finite group in $GL(n,\mathbb{Z}) \subset GL(n,\mathbb{R})$, there is an element $B \in GL(n,\mathbb{R})$ such that $B\cdot\phi(F)\cdot B^{-1} \subset O(n)$. Then the correspondence

$$(n,\alpha) \to (B(n+\chi(\alpha)), B\cdot\phi(\alpha)\cdot B^{-1})$$

is the injective homomorphism from Γ to $\mathbb{E}(n)$. Now $\Gamma \subset \mathbb{E}(n)$ where $\mathbb{Z}^n$ is a maximal abelian subgroup, by the definition Γ is a crystallographic group.

So, each equivalence class $((F,\phi),[f])$ of $\{(F,\phi), H^2_\phi(F,\mathbb{Z}^n)\}$ corresponds to the equivalence class of crystallographic groups.

Conversely, given a crystallographic group Γ of $\mathbb{E}(n)$, then there is a group extension as in (2.28) such that $\mathbb{Z}^n$ is the maximal free abelian group and F is a finite group. A group extension Γ defines a unique homomorphism $\phi : F \to \mathrm{Aut}(\mathbb{Z}^n) = GL(n,\mathbb{Z})$ induced by the conjugation of Γ. Note that this is a faithful homomorphism, that is, if $\phi(\alpha) = \mathrm{id}$, letting $\gamma = (a,A)$ such that $L(\gamma) = \alpha = A$,

$$An = \phi(\alpha)(n) = \mathrm{id}(n) = n.$$

Since $n \in \mathbb{Z}^n$ is the (integral) basis of $\mathbb{R}^n$, this implies that $A = \mathrm{I}$. So $\alpha = 1$. In particular, we have a pair (F,ϕ). Since $\mathbb{Z}^n$ is viewed as a

F-module through ϕ, Γ defines a 2-cocycle $[f] \in H^2_\phi(F, \mathbb{Z}^n)$. An equivalence class of crystallographic group Γ defines an element $((F, \phi), [f])$ which implies that $\{(F, \phi), H^2_\phi(F, \mathbb{Z}^n)\}$ is in one-to-one correspondence with the equivalence classes of crystallographic groups.

As $H^2_\phi(F, \mathbb{Z}^n)$ is of finite order (since F is finite), the finiteness of group extensions comes from that of (F, ϕ) where $\phi : F \rightarrow GL(n, \mathbb{Z})$ is a faithful representation. For each n fixed, this is the number of non-isomorphic conjugacy classes of a finite group $\phi(F)$ in $GL(n, \mathbb{Z})$. By Theorem 2.17, such (F, ϕ) is finite. This proves the theorem. $\square$

Bieberbach theorems imply many results in the theory of compact flat manifolds some of which we state below.

Theorem 2.18. *Let X and Y be compact flat manifolds. Then $\pi_1(X)$ is isomorphic to $\pi_1(Y)$ if and only if X and Y are diffeomorphic.*

Proof. ($\Rightarrow$) We consider $\pi_1(X)$ and $\pi_1(Y)$ as Bieberbach subgroup of $\mathbb{E}(n)$. By Theorem 2.15, there is $\alpha \in \mathbb{R}^n$ s.t if $F : \pi_1(X) \longrightarrow \pi_1(Y)$ is an isomorphism, $F(\beta) = \alpha\beta\alpha^{-1} \;\; \forall \beta \in \pi_1(X)$. Let

$$p_x : \mathbb{R}^n \longrightarrow X = \mathbb{R}^n/\pi_1(X) \text{ and } p_y : \mathbb{R}^n \longrightarrow Y = \mathbb{R}^n/\pi_1(Y)$$

be the projection (or covering maps). Define $f : X \longrightarrow Y$ by

$$f(x) = p_y \circ \alpha \circ p_x^{-1}(x), \text{ for } x \in X.$$

To see that this is well-defined, let $\bar{x} \in p_x^{-1}(x)$ and $\beta \in \pi_1(X)$. We must show $p_y \circ \alpha(\beta\bar{x}) = p_y \circ \alpha(\bar{x})$. But $\alpha\beta\alpha^{-1} = \gamma \in \pi_1(Y)$, so $\alpha\beta = \gamma\alpha$ and

$$\begin{aligned} p_y \circ \alpha(\beta\bar{x}) &= p_y(\alpha\beta\bar{x}) = p_y(\gamma\alpha\bar{x}) \\ &= p_y(\alpha\bar{x}) \\ &= p_y \circ \alpha(\bar{x}), \end{aligned}$$

so f is well-defined.

To show that f is diffeomorphism, we need merely to show f is bijective. Define $g : Y \longrightarrow X$ by

$$g(y) = p_x \circ \alpha^{-1} \circ p_y^{-1}(y), \text{ for } y \in Y.$$

Similarly, g is also well-defined. For $\bar{y} \in p_y^{-1}(y)$ and $\gamma \in \pi_1(Y)$. We show $p_x \circ \alpha^{-1}(\gamma\bar{y}) = p_x \circ \alpha^{-1}(\bar{y})$. But $\alpha\beta\alpha^{-1} = \gamma \in \pi_1(Y)$, so $\beta\alpha^{-1} = \alpha^{-1}\gamma$ and

$$\begin{aligned} p_x \circ \alpha^{-1}(\gamma\bar{y}) &= p_x(\alpha^{-1}\gamma\bar{y}) = p_x(\beta\alpha^{-1}\bar{y}) \\ &= p_x(\alpha^{-1}\bar{y}) \\ &= p_x \circ \alpha^{-1}(\bar{y}). \end{aligned}$$

Now,

$$\begin{aligned} f \circ g(y) &= f(p_x \circ \alpha^{-1} \circ p_y^{-1}(y)) \\ &= p_y \circ \alpha \circ p_x^{-1}(p_x \circ \alpha^{-1} \circ p_y^{-1}(y)) \\ &= y. \end{aligned}$$

Similarly, $g \circ f(x) = x$. So, $g = f^{-1}$, f is bijective.

($\Leftarrow$). Let $f : X \longrightarrow Y$ is a diffeomorphism. So f is a homeomorphism. We show that $F : \pi_1(X) \longrightarrow \pi_1(Y)$ is an isomorphism.

Since f is a homeomorphism, there exists the inverse map $f^{-1} : Y \longrightarrow X$ has the property that

$$f^{-1} \circ f = I_X, \quad f \circ f^{-1} = I_Y.$$

Let us consider the induced homomorphisms

$$f_* : \pi_1(X) \longrightarrow \pi_1(Y), \quad f_*^{-1} : \pi_1(Y) \longrightarrow \pi_1(X)$$

in the fundamental groups. Since $(f^{-1} \circ f)_* = f_*^{-1} \circ f_*$, and $(I_X)_* : \pi_1(X) \longrightarrow \pi_1(X)$, then $f_*^{-1} \circ f_*$ is the identity map on $\pi_1(X)$. Similarly, we can see that $f_* \circ f_*^{-1}$ is the identity map on $\pi_1(Y)$. Therefore, f_* and f_*^{-1} are inverses of each other, i.e., $F = f_* : \pi_1(X) \longrightarrow \pi_1(Y)$ is an isomorphism. □

Theorem 2.19. *Let Γ be a subgroup of $\mathbb{E}(n)$. Then the orbit space $\mathbb{R}^n/\Gamma$ is a compact flat n-dimensional manifold if and only if Γ is torsionfree, discrete and irreducible.*

Proof. ($\Rightarrow$) If $M = \mathbb{R}^n/\Gamma$ is a compact flat n-manifold, then $\pi_1(M) = \Gamma$ and hence Γ acts freely on $\mathbb{R}^n$, so Γ must be torsionfree (by Proposition 2.10). Since M is an n-manifold, Γ must be discrete. Since $\mathbb{R}^n/\Gamma$ is compact, by Lemma 7 [3], Γ is irreducible

($\Leftarrow$) Since Γ is a torsionfree discrete subgroup of $\mathbb{E}(n)$, then by Proposition 2.10 Γ acts freely on $\mathbb{R}^n$ and discrete subgroup of $\mathbb{E}(n)$ acts properly discontinuously on $\mathbb{R}^n$, so $\mathbb{R}^n/\Gamma$ is an n-manifold. By first Bieberbach theorem, $\mathbb{R}^n/\Gamma$ is covered by $\mathbb{R}^n/(\Gamma \cap \mathbb{R}^n)$ which is the n-torus. Hence $\mathbb{R}^n/\Gamma$ is compact. □

According to some theorems above, Bieberbach proved that, Γ and Γ' are two isomorphic discrete compact subgroups (crystallographic subgroups) of $\mathbb{E}(n)$ if and only if $\exists\ \gamma \in \mathbb{A}(n)$ such that $\gamma\Gamma\gamma^{-1} = \Gamma'$. This implies that, if M and M' are compact flat Riemannian manifolds with isomorphic fundamental groups Γ and Γ' respectively then, M is diffeomorphic to M' if and only if $\exists\ \gamma \in \mathbb{A}(n)$ such that $\gamma\Gamma\gamma^{-1} = \Gamma'$.

To show that $M(A) = (S^1)^n/(\mathbb{Z}_2)^n$ is aspherical, by Theorems 2.2 then $p : (S^1)^n \to (S^1)^n/(\mathbb{Z}_2)^n$ is covering. Since $q : \mathbb{R}^n \to (S^1)^n$ is a covering with $\mathbb{R}^n$ its universal covering space, then $p \circ q : \mathbb{R}^n \to (S^1)^n/(\mathbb{Z}_2)^n$ is covering. Hence $(S^1)^n/(\mathbb{Z}_2)^n$ is aspherical.

3. Three Dimensional Real Bott Tower

We begin this section with defining the action of cyclic group $(\mathbb{Z}_2)^3$ on $(S^1)^3$.

Let

$$A = \begin{pmatrix} 1 & a_{12} & a_{13} \\ 0 & 1 & a_{23} \\ 0 & 0 & 1 \end{pmatrix} \tag{3.1}$$

be one of the 3rd Bott matrices. Then each a_{ij} represents either 0 or 1. We use the following notation for $a \in \{0, 1\}$

$$\bar{z}^a = \begin{cases} \bar{z} & \text{if } a = 1 \\ z & \text{if } a = 0 \end{cases}$$

where $\bar{z}$ is the conjugate of the complex number z. If $(g_1, g_2, g_3) \in (\mathbb{Z}_2)^3$ are generators, then $(\mathbb{Z}_2)^3$ acts on T^3 as

$$\begin{aligned} g_1(z_1, z_2, z_3) &= (-z_1, \bar{z}_2^{a_{12}}, \bar{z}_3^{a_{13}}) \\ g_2(z_1, z_2, z_3) &= (z_1, -z_2, \bar{z}_3^{a_{23}}) \\ g_3(z_1, z_2, z_3) &= (z_1, z_2, -z_3). \end{aligned}$$

It is easy to see that $(\mathbb{Z}_2)^3$ acts freely on T^3 such that the orbit space $M(A) = T^3/(\mathbb{Z}_2)^3$ is a smooth compact manifold. In this way, taking an n-th Bott matrix A, we obtain a free action of $(\mathbb{Z}_2)^n$ on T^n.

Let $Pr(x, y, z) = (e^{2\pi \mathbf{i} x}, e^{2\pi \mathbf{i} y}, e^{2\pi \mathbf{i} z}) = (z_1, z_2, z_3)$ be the canonical covering map of $\mathbb{R}^3$ onto T^3. Given A as before, each generator g_i of $(\mathbb{Z}_2)^3$ acting on T^3 lifts to a map $\tilde{g}_i : \mathbb{R}^3 \to \mathbb{R}^3$ so that this diagram is commutative

$$\begin{tikzcd} \mathbb{R}^3 \arrow[r, "\tilde{g}_i"] \arrow[d, "Pr"'] & \mathbb{R}^3 \arrow[d, "Pr"] \\ T^3 \arrow[r, "g_i"] & T^3. \end{tikzcd}$$

Then the fundamental group of $M(A)$ is isomorphic to the group $\Gamma(A)$ generated by $\{\tilde{g}_1, \tilde{g}_2, \tilde{g}_3\}$. Moreover, it forms a torsion-free discrete uniform subgroup of the Euclidean group $\mathbb{E}(n) = \mathbb{R}^n \rtimes \mathrm{O}(n)$. $\Gamma(A)$ is called *Bieberbach group* (torsion free crystallographic group). We call $\Gamma(A)$ a Bott group associated to the Bott matrix A, for brevity.

For example, let

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.2}$$

and by definition we have

$$\begin{aligned} g_1(z_1, z_2, z_3) &= (-z_1, z_2, \bar{z}_3) \\ g_2(z_1, z_2, z_3) &= (z_1, -z_2, \bar{z}_3) \\ g_3(z_1, z_2, z_3) &= (z_1, z_2, -z_3). \end{aligned}$$

Thus we get $\tilde{g}_1 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \end{pmatrix}$. Therefore we may write the generator $\tilde{g}_1 = \left(\begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right) = (s_1, M_1) \in \Gamma(A)$ where $s_1 \in \mathbb{R}^3$, $M_1 \in O(3)$. Similarly, we have
$\tilde{g}_2 = \left(\begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right) = (s_2, M_2)$, $\tilde{g}_3 = \left(\begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right)$
$= (s_3, I_3)$, where $s_2, s_3 \in \mathbb{R}^3$ and $M_2 \in O(3)$. Hence $\Gamma(A) =< \tilde{g}_1, \tilde{g}_2, \tilde{g}_3 >$. By this explanation we see that action $(\mathbb{Z}_2)^3$ on $(S^1)^3$ coincides with action Γ on $\mathbb{R}^3$.

Given a Bott matrix A as (3.1), $M(A)$ admits a maximal T^k-action $(k = 1, 2, 3)$ which is obtained as follows.
Let $t_i \in T^3$, and define T^3-action on $(S^1)^3$ by

$$\begin{aligned} t_1(z_1, z_2, z_3) &= (e^{2\pi i\theta} z_1, z_2, z_3) \\ t_2(z_1, z_2, z_3) &= (z_1, e^{2\pi i\theta} z_2, z_3) \\ t_3(z_1, z_2, z_3) &= (z_1, z_2, e^{2\pi i\theta} z_3) \end{aligned}$$

where $0 \leq \theta \leq 1$.
If $t_i g(z_1, z_2, z_3) = g t_i(z_1, z_2, z_3)$ for $t_i \in T^k$ and $g \in (\mathbb{Z}_2)^3$ then $M(A)$ admits a T^k-action $(k = 1, 2, 3)$ which is maximal..

The above example, it is easy to check that $t_1 g_i = g_i t_1$, $t_2 g_i = g_i t_2$ for $i = 1, 2, 3$, but $t_3 g_1 \neq g_1 t_3$. Therefore $M(A)$ with representative matrix (3.2) admits T^2-action which is maximal.

According to Bieberbach theorem, fundamental group $\pi_1(M(A)) = \Gamma = \langle \tilde{g}_1, \tilde{g}_2, \tilde{g}_3 \rangle$ is isomorphic to $\pi_1(M(A')) = \Gamma' = \langle \tilde{h}_1, \tilde{h}_2, \tilde{h}_3 \rangle$ if and only if $\exists\, \gamma \in \mathbb{A}(3) = \mathbb{R}^3 \rtimes GL(3, \mathbb{R})$ such that $\gamma \Gamma \gamma^{-1} = \Gamma'$. So, if we can find $\tilde{\varphi} = \gamma \in \mathbb{A}(3)$ such that $\gamma \Gamma \gamma^{-1} = \Gamma'$, then $M(A)$ is diffeomorphic to $M(A')$. This means, we have to define map $\tilde{\varphi}$ or φ such that the diagrams below are commutative

$$\begin{array}{ccccccc}
\mathbb{R}^3 & \xrightarrow{\tilde{\varphi}} & \mathbb{R}^3 & \qquad & T^3 & \xrightarrow{\varphi} & T^3 \\
{\scriptstyle \tilde{g}_i}\downarrow & & \downarrow{\scriptstyle \tilde{h}_j} & & {\scriptstyle g_i}\downarrow & & \downarrow{\scriptstyle h_j} \\
\mathbb{R}^3 & \xrightarrow{\tilde{\varphi}} & \mathbb{R}^3 & & T^3 & \xrightarrow{\varphi} & T^3
\end{array}$$

for $i = j = 1, 2, 3$. This is a tool that we will use later to determine the diffeomorphism classes of $M(A)$.

If $M(A)$ is a 3-dimensional real Bott tower, then it is classified as follows.

Theorem 3.1. *Let $M(A)$ be a 3-dimensional real Bott tower. Then $M(A)$ admits a maximal T^k-action ($k = 1, 2, 3$). There exist exactly 4 distinct diffeomorphism classes of $M(A)$ in which all 8 Bott matrices fall into the following classes:*

(i) *(orientable)(T^3-action.) The Identity matrix I_3.*

(ii) *(nonorientable)(T^2-action.)*

$$A_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, A_2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, A_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

$$A_4 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(iii) *(nonorientable)(S^1-action.)*

$$A_5 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, A_6 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

(iv) *(orientable)(S^1-action.)* $A_7 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Moreover, Bott matrices in each class are conjugate.

Proof. For simplicity, notation "$\approx$" indicates diffeomorphic. The proof is organized by the following steps.

(1) Determine conjugacy classes of 8 Bott matrices (we used Maple to determine this).
(2) For each conjugacy class, we check whether $M(A), M(A')$ with representative matrices A, A' are diffeomorphic or not by using Bieberbach theorems.

A detailed proof is as follows:

a) $M(I_3)$ is not diffeomorphic to any $M(A_i)$ for $i = 1, 2, 3, 4, 5, 6, 7$.
b) $M(A_1) \approx M(A_2) \approx M(A_3) \approx M(A_4)$

c) $M(A_5) \approx M(A_6)$
d) $M(A_1)$ is not diffeomorphic to $M(A_5)$
e) $M(A_1)$ is not diffeomorphic to $M(A_7)$
f) $M(A_5)$ is not diffeomorphic to $M(A_7)$

a). This is clear, because holonomy group of I_3 is trivial.

b).
• $M(A_1) \approx M(A_2)$.
For

$$A_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned} g_1(z_1, z_2, z_3) &= (-z_1, \bar{z}_2, z_3) & h_1h_2(z_1, z_2, z_3) &= (-z_1, -z_2, z_3) \\ g_2(z_1, z_2, z_3) &= (z_1, -z_2, z_3) & h_2(z_1, z_2, z_3) &= (z_1, -z_2, \bar{z}_3) \\ g_3(z_1, z_2, z_3) &= (z_1, z_2, -z_3) & h_3(z_1, z_2, z_3) &= (z_1, z_2, -z_3) \end{aligned}$$

then

$$\Gamma_1 = \left\langle \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle$$

$$\Gamma_2 = \left\langle \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle.$$

Let $\varphi(z_1, z_2, z_3) = (z_3, z_1z_3, z_2)$, we get these commutative diagrams

$$\begin{array}{ccc} (z_1, z_2, z_3) & \xrightarrow{\varphi} & (z_3, z_1z_3, z_2) \\ {\scriptstyle g_1}\big\downarrow & & {\scriptstyle h_2}\big\downarrow \\ (-z_1, \bar{z}_2, z_3) & \xrightarrow{\varphi} & (z_3, -z_1z_3, \bar{z}_2) \end{array} \qquad \begin{array}{ccc} (z_1, z_2, z_3) & \xrightarrow{\varphi} & (z_3, z_1z_3, z_2) \\ {\scriptstyle g_3}\big\downarrow & & {\scriptstyle h_1h_2}\big\downarrow \\ (z_1, z_2, -z_3) & \xrightarrow{\varphi} & (-z_3, -z_1z_3, z_2) \end{array}$$

$$\begin{array}{ccc} (z_1, z_2, z_3) & \xrightarrow{\varphi} & (z_3, z_1z_3, z_2) \\ {\scriptstyle g_2}\big\downarrow & & {\scriptstyle h_3}\big\downarrow \\ (z_1, -z_2, z_3) & \xrightarrow{\varphi} & (z_3, z_1z_3, -z_2). \end{array}$$

Therefore $\exists\ \gamma = \left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right) \in \mathbb{A}(n)$ s.t. $\gamma\Gamma_1\gamma^{-1} = \Gamma_2$.

$\bullet M(A_1) \approx M(A_3)$.
For

$$A_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad A_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned} g_1(z_1,z_2,z_3) &= (-z_1,\bar{z}_2,z_3) & h_1(z_1,z_2,z_3) &= (-z_1,z_2,z_3) \\ g_2(z_1,z_2,z_3) &= (z_1,-z_2,z_3) & h_2(z_1,z_2,z_3) &= (z_1,-z_2,\bar{z}_3) \\ g_3(z_1,z_2,z_3) &= (z_1,z_2,-z_3) & h_3(z_1,z_2,z_3) &= (z_1,z_2,-z_3) \end{aligned}$$

then

$$\Gamma_1 = \left\langle \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle$$

$$\Gamma_3 = \left\langle \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle.$$

Let $\varphi(z_1,z_2,z_3) = (z_3,z_1,z_2)$, we get these commutative diagrams

$$\begin{array}{ccc} (z_1,z_2,z_3) & \xrightarrow{\varphi} & (z_3,z_1,z_2) \\ {\scriptstyle g_1}\big\downarrow & & {\scriptstyle h_2}\big\downarrow \\ (-z_1,\bar{z}_2,z_3) & \xrightarrow{\varphi} & (z_3,-z_1,\bar{z}_2) \end{array} \qquad \begin{array}{ccc} (z_1,z_2,z_3) & \xrightarrow{\varphi} & (z_3,z_1,z_2) \\ {\scriptstyle g_3}\big\downarrow & & {\scriptstyle h_1}\big\downarrow \\ (z_1,z_2,-z_3) & \xrightarrow{\varphi} & (-z_3,z_1,z_2) \end{array}$$

$$\begin{array}{ccc} (z_1,z_2,z_3) & \xrightarrow{\varphi} & (z_3,z_1,z_2) \\ {\scriptstyle g_2}\big\downarrow & & {\scriptstyle h_3}\big\downarrow \\ (z_1,-z_2,z_3) & \xrightarrow{\varphi} & (z_3,z_1,-z_2). \end{array}$$

Therefore $\exists\ \gamma = \left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right) \in \mathbb{A}(n)$ s.t. $\gamma\Gamma_1\gamma^{-1} = \Gamma_3$.

$\bullet M(A_1) \approx M(A_4)$.
For

$$A_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad A_4 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned} g_1(z_1,z_2,z_3) &= (-z_1,\bar{z}_2,z_3) & h_1(z_1,z_2,z_3) &= (-z_1,z_2,\bar{z}_3) \\ g_2(z_1,z_2,z_3) &= (z_1,-z_2,z_3) & h_2(z_1,z_2,z_3) &= (z_1,-z_2,z_3) \\ g_3(z_1,z_2,z_3) &= (z_1,z_2,-z_3) & h_3(z_1,z_2,z_3) &= (z_1,z_2,-z_3) \end{aligned}$$

then

$$\Gamma_1 = \left\langle \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle$$

$$\Gamma_4 = \left\langle \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle.$$

Let $\varphi(z_1, z_2, z_3) = (z_1, z_3, z_2)$, we get these commutative diagrams

$$\begin{tikzcd}
(z_1, z_2, z_3) \arrow[r, "\varphi"] \arrow[d, "g_1"'] & (z_1, z_3, z_2) \arrow[d, "h_1"'] \\
(-z_1, \bar{z}_2, z_3) \arrow[r, "\varphi"] & (-z_1, z_3, \bar{z}_2)
\end{tikzcd}
\qquad
\begin{tikzcd}
(z_1, z_2, z_3) \arrow[r, "\varphi"] \arrow[d, "g_3"'] & (z_1, z_3, z_2) \arrow[d, "h_2"'] \\
(z_1, z_2, -z_3) \arrow[r, "\varphi"] & (z_1, -z_3, z_2)
\end{tikzcd}$$

$$\begin{tikzcd}
(z_1, z_2, z_3) \arrow[r, "\varphi"] \arrow[d, "g_2"'] & (z_1, z_3, z_2) \arrow[d, "h_3"'] \\
(z_1, -z_2, z_3) \arrow[r, "\varphi"] & (z_1, z_3, -z_2).
\end{tikzcd}$$

Therefore $\exists\ \gamma = \left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right) \in \mathbb{A}(n)$ s.t. $\gamma \Gamma_1 \gamma^{-1} = \Gamma_4$.

c).

$\bullet M(A_5) \approx M(A_6)$.

For

$$A_5 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \qquad A_6 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned} g_1(z_1, z_2, z_3) &= (-z_1, \bar{z}_2, z_3) & h_1h_2(z_1, z_2, z_3) &= (-z_1, -\bar{z}_2, z_3) \\ g_2(z_1, z_2, z_3) &= (z_1, -z_2, \bar{z}_3) & h_2(z_1, z_2, z_3) &= (z_1, -z_2, \bar{z}_3) \\ g_3(z_1, z_2, z_3) &= (z_1, z_2, -z_3) & h_3(z_1, z_2, z_3) &= (z_1, z_2, -z_3) \end{aligned}$$

then

$$\Gamma_5 = \left\langle \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle$$

$$\Gamma_6 = \left\langle \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle.$$

Let $\varphi(z_1, z_2, z_3) = (z_1, iz_2, z_3)$, we get these commutative diagrams

$$\begin{tikzcd}
(z_1, z_2, z_3) \arrow[r, "\varphi"] \arrow[d, "g_1"'] & (z_1, iz_2, z_3) \arrow[d, "h_1h_2"'] \\
(-z_1, \bar{z}_2, z_3) \arrow[r, "\varphi"] & (-z_1, -i\bar{z}_2, z_3)
\end{tikzcd}
\qquad
\begin{tikzcd}
(z_1, z_2, z_3) \arrow[r, "\varphi"] \arrow[d, "g_3"'] & (z_1, iz_2, z_3) \arrow[d, "h_3"'] \\
(z_1, z_2, -z_3) \arrow[r, "\varphi"] & (z_1, iz_2, -z_3)
\end{tikzcd}$$

$$\begin{tikzcd}
(z_1, z_2, z_3) \arrow[r, "\varphi"] \arrow[d, "g_2"'] & (z_1, iz_2, z_3) \arrow[d, "h_2"'] \\
(z_1, -z_2, \bar{z}_3) \arrow[r, "\varphi"] & (z_1, -iz_2, \bar{z}_3).
\end{tikzcd}$$

Therefore $\exists\ \gamma = \left(\begin{pmatrix} 0 \\ \frac{1}{4} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) \in \mathbb{A}(n)$ s.t. $\gamma\Gamma_5\gamma^{-1} = \Gamma_6$.

d). First, we determine the holonomy groups of Γ_1 and Γ_5. If $L : \mathbb{E}(3) \longrightarrow O(3)$ is the projection, then $L(\Gamma) = \Phi$ is called the holonomy group of Γ. Hence

$$\Phi_1 = \left\langle I, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle = \mathbb{Z}_2$$

$$\Phi_5 = \left\langle I, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\rangle = \mathbb{Z}_2 \times \mathbb{Z}_2.$$

If $M(A_1)$ is diffeomorphic to $M(A_5)$, by Bieberbach's theorem, then $\exists\ B \in GL(3, \mathbb{R})$ with $(b, B) = \gamma \in \mathbb{A}(3)$ such that $L(\gamma\Gamma_1\gamma^{-1}) = BL(\Gamma_1)B^{-1} = L(\Gamma_5)$. This is impossible because $L(\Gamma_1) = \mathbb{Z}_2$ and $L(\Gamma_5) = \mathbb{Z}_2 \times \mathbb{Z}_2$.

e). Let $\Phi_1 = \left\langle I, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = P \right\rangle \quad \Phi_7 = \left\langle I, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = Q \right\rangle$. If $M(A_1)$ is difffeomorphic to $M(A_7)$ then $\exists\ B$ such that $BPB^{-1} = Q$. Consider $|BPB^{-1}| = |Q|$ or $|B||P - \lambda I||B^{-1}| = |Q - \lambda I|$. It means the eigenvalues of P and Q are the same. This is a contradiction, because the eigenvalues of P are 1,1 and -1, but the eigenvalues of Q are -1,-1 and 1.

f). The argument is similar with d), because the holonomy of Γ_7 is $\mathbb{Z}_2$.

We observed that the sufficient condition of conjecture is true for $n = 3$. □

4. Four Dimensional Real Bott Tower

In this case there are 64 Bott matrices A that correspond to action of $(\mathbb{Z}_2)^4$ on $(S^1)^4$. Definition of the action is similar to the three dimensional real Bott tower.

Theorem 4.1. *Let $M(A)$ be a 4-dimensional real Bott tower. Then $M(A)$ admits a maximal T^k-action ($k = 1, 2, 3, 4$). There exists exactly 12-diffeomorphism classes of $M(A)$. The Bott matrices among* 64 *fall into the following classes*

(a) *(orientable) (T^4-action.) The Identity matrix I_4.*

(b) *(orientable) (T^2-action.)*

$$A_{11} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, A_{21} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, A_{31} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$A_{a5} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, A_{a17} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(c) *(orientable) (S^1-action.)*

$$A_{a15} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, A_{a27} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(d) *(nonorientable) (T^3-action.)*

$$A_{2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, A_{3} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, A_{4} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$A_{5} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, A_{6} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, A_{7} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$A_{8} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, A_{9} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, A_{17} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$A_{25} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, A_{a1} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(e) *(nonorientable) (S^1-action.)* $A_{a21} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$

(f) *(nonorientable) (T^2-action.)*

$$A_{a3}=\begin{pmatrix}1&1&0&0\\0&1&0&1\\0&0&1&0\\0&0&0&1\end{pmatrix},A_{a7}=\begin{pmatrix}1&1&0&1\\0&1&0&1\\0&0&1&0\\0&0&0&1\end{pmatrix},A_{10}=\begin{pmatrix}1&0&0&0\\0&1&1&0\\0&0&1&1\\0&0&0&1\end{pmatrix}$$
$$A_{12}=\begin{pmatrix}1&0&0&0\\0&1&1&1\\0&0&1&1\\0&0&0&1\end{pmatrix},A_{18}=\begin{pmatrix}1&0&1&0\\0&1&0&0\\0&0&1&1\\0&0&0&1\end{pmatrix},A_{22}=\begin{pmatrix}1&0&1&1\\0&1&0&0\\0&0&1&1\\0&0&0&1\end{pmatrix}$$
$$A_{26}=\begin{pmatrix}1&0&1&0\\0&1&1&0\\0&0&1&1\\0&0&0&1\end{pmatrix},A_{32}=\begin{pmatrix}1&0&1&1\\0&1&1&1\\0&0&1&1\\0&0&0&1\end{pmatrix},A_{a9}=\begin{pmatrix}1&1&0&0\\0&1&1&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$
$$A_{a25}=\begin{pmatrix}1&1&1&0\\0&1&1&0\\0&0&1&0\\0&0&0&1\end{pmatrix}.$$

(g) *(nonorientable)(T^2-action.)*

$$A_{a4}=\begin{pmatrix}1&1&0&0\\0&1&0&1\\0&0&1&1\\0&0&0&1\end{pmatrix},A_{a8}=\begin{pmatrix}1&1&0&1\\0&1&0&1\\0&0&1&1\\0&0&0&1\end{pmatrix},A_{14}=\begin{pmatrix}1&0&0&1\\0&1&1&0\\0&0&1&1\\0&0&0&1\end{pmatrix}$$
$$A_{16}=\begin{pmatrix}1&0&0&1\\0&1&1&1\\0&0&1&1\\0&0&0&1\end{pmatrix},A_{20}=\begin{pmatrix}1&0&1&0\\0&1&0&1\\0&0&1&1\\0&0&0&1\end{pmatrix},A_{24}=\begin{pmatrix}1&0&1&1\\0&1&0&1\\0&0&1&1\\0&0&0&1\end{pmatrix}$$
$$A_{28}=\begin{pmatrix}1&0&1&0\\0&1&1&1\\0&0&1&1\\0&0&0&1\end{pmatrix},A_{30}=\begin{pmatrix}1&0&1&1\\0&1&1&0\\0&0&1&1\\0&0&0&1\end{pmatrix}.$$

(h) *(nonorientable) (S^1-action.)*

$$A_{a13}=\begin{pmatrix}1&1&0&1\\0&1&1&0\\0&0&1&0\\0&0&0&1\end{pmatrix},A_{a29}=\begin{pmatrix}1&1&1&1\\0&1&1&0\\0&0&1&0\\0&0&0&1\end{pmatrix},A_{a18}=\begin{pmatrix}1&1&1&0\\0&1&0&0\\0&0&1&1\\0&0&0&1\end{pmatrix}$$
$$A_{a19}=\begin{pmatrix}1&1&1&0\\0&1&0&1\\0&0&1&0\\0&0&0&1\end{pmatrix},A_{a20}=\begin{pmatrix}1&1&1&0\\0&1&0&1\\0&0&1&1\\0&0&0&1\end{pmatrix},A_{a22}=\begin{pmatrix}1&1&1&1\\0&1&0&0\\0&0&1&1\\0&0&0&1\end{pmatrix}$$
$$A_{a23}=\begin{pmatrix}1&1&1&1\\0&1&0&1\\0&0&1&0\\0&0&0&1\end{pmatrix},A_{a24}=\begin{pmatrix}1&1&1&1\\0&1&0&1\\0&0&1&1\\0&0&0&1\end{pmatrix}.$$

(i) *(nonorientable)(S^1-action.)*

$$A_{a11} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, A_{a31} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(j) *(nonorientable) (T^2-action.)*

$$A_{13} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, A_{15} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, A_{19} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A_{23} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, A_{27} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, A_{29} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A_{a2} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, A_{a6} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(k) *(nonorientable) (S^1-action.)*

$$A_{a10} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, A_{a12} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, A_{a26} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A_{a32} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(l) *(nonorientable)(S^1-action.)*

$$A_{a14} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, A_{a16} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, A_{a28} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A_{a30} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Moreover, Bott matrices in each class are conjugate.

Proof. Similar with 3 dimensional real Bott tower, the proof is organized by the following steps.

(1) Determine conjugacy classes of 64 Bott matrices (we used Maple to determine this).
(2) For each conjugacy class, we check whether $M(A), M(A')$ with representative matrices A, A' are diffeomorphic or not by using Bieberbach theorems.

A detailed proof is as follows:

(1). I_4 is not diffeomorphic to any $M(A)$.
(2). $M(A_{11}) \approx M(A_{21}) \approx M(A_{31}) \approx M(A_{a5}) \approx M(A_{a17})$.
(3). $M(A_{a15}) \approx M(A_{a27})$.
(4). $M(A_{11})$ is not diffeomorphic to $M(A_{a15})$.
(5). $M(A_2) \approx M(A_3) \approx M(A_4) \approx M(A_5) \approx M(A_6) \approx M(A_7) \approx M(A_8) \approx M(A_9) \approx M(A_{17}) \approx M(A_{25}) \approx M(A_{a1})$.
(6). $M(A_{11})$ is not diffeomorphic to $M(A_2)$ and $M(A_{a21})$.
(7). $M(A_{a15})$ is not diffeomorphic to $M(A_2)$ and $M(A_{a21})$.
(8). $M(A_2)$ is not diffeomorphic to $M(A_{a21})$.
(9). $M(A_{a3}) \approx M(A_{a7}) \approx M(A_{10}) \approx M(A_{12}) \approx M(A_{18}) \approx M(A_{22}) \approx M(A_{26}) \approx M(A_{32}) \approx M(A_{a9}) \approx M(A_{a25})$.
(10). $M(A_{a3})$ is not diffeomorphic to $M(A_{11})$, $M(A_2)$ and $M(A_{a21})$.
(11). $M(A_{a3})$ is not diffeomorphic to $M(A_{a15})$.
(12). $M(A_{a4}) \approx M(A_{a8}) \approx M(A_{14}) \approx M(A_{16}) \approx M(A_{20}) \approx M(A_{24}) \approx M(A_{28}) \approx M(A_{30})$.
(13). $M(A_{a4})$ is not diffeomorphic to $M(A_{11})$, $M(A_2)$ and $M(A_{a21})$.
(14). $M(A_{a4})$ is not diffeomorphic to $M(A_{a15})$.
(15). $M(A_{14})$ is not diffeomorphic to $M(A_{a3})$.
(16). $M(A_{a13}) \approx M(A_{a29}) \approx M(A_{a18}) \approx M(A_{a19}) \approx M(A_{a20}) \approx M(A_{a22}) \approx M(A_{a23}) \approx M(A_{a24})$.
(17). $M(A_{a13})$ is not diffeomorphic to $M(A_{11})$, $M(A_2)$ and $M(A_{a21})$.
(18). $M(A_{a13})$ is not diffeomorphic to $M(A_{a15})$, $M(A_{a3})$ and $M(A_{a4})$.
(19). $M(A_{a11}) \approx M(A_{a31})$.
(20). $M(A_{a11})$ is not diffeomorphic to $M(A_{11})$, $M(A_2)$ and $M(A_{a21})$.
(21). $M(A_{a11})$ is not diffeomorphic to $M(A_{a15})$, $M(A_{a3})$ and $M(A_{a4})$.
(22). $M(A_{a11})$ is not diffeomorphic to $M(A_{a13})$.
(23). $M(A_{13}) \approx M(A_{15}) \approx M(A_{19}) \approx M(A_{23}) \approx M(A_{27}) \approx M(A_{29}) \approx M(A_{a2}) \approx M(A_{a6})$.
(24). $M(A_{13})$ is not diffeomorphic to $M(A_{11})$, $M(A_2)$ and $M(A_{a21})$.
(25). $M(A_{13})$ is not diffeomorphic to $M(A_{a15})$, $M(A_{a13})$ and $M(A_{a11})$.
(26). $M(A_{13})$ is not diffeomorphic to $M(A_{a3})$.
(27). $M(A_{13})$ is not diffeomorphic to $M(A_{a4})$.
(28). $M(A_{a10}) \approx M(A_{a12}) \approx M(A_{a26}) \approx M(A_{a32})$. and $M(A_{a14}) \approx M(A_{a16}) \approx M(A_{a28}) \approx M(A_{a30})$.
(29). $M(A_{a10})$ and $M(A_{a14})$ are not diffeomorphic to any $M(A)$ in classes **a)-j)** as written in theorem.
(30). $M(A_{a10})$ is not diffeomorphic to $M(A_{a14})$.

(1). This is clear because the holonomy of I_4 is trivial.

(2).
• $M(A_{11}) \approx M(A_{21})$.
For

$$A_{11} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad A_{21} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned} g_1(z_1,z_2,z_3,z_4) &= (-z_1,z_2,z_3,z_4). & h_1(z_1,z_2,z_3,z_4) &= (-z_1,z_2,\bar{z}_3,\bar{z}_4) \\ g_2(z_1,z_2,z_3,z_4) &= (z_1,-z_2,\bar{z}_3,\bar{z}_4). & h_2(z_1,z_2,z_3,z_4) &= (z_1,-z_2,z_3,z_4) \\ g_3(z_1,z_2,z_3,z_4) &= (z_1,z_2,-z_3,z_4). & h_3(z_1,z_2,z_3,z_4) &= (z_1,z_2,-z_3,z_4) \\ g_4(z_1,z_2,z_3,z_4) &= (z_1,z_2,z_3,-z_4). & h_4(z_1,z_2,z_3,z_4) &= (z_1,z_2,z_3,-z_4) \end{aligned}$$

then

$$\Gamma_{11} = \Big\langle \begin{pmatrix} \frac12 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac12 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$
$$\begin{pmatrix} 0 \\ 0 \\ \frac12 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac12 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Big\rangle$$

$$\Gamma_{21} = \Big\langle \begin{pmatrix} \frac12 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac12 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$
$$\begin{pmatrix} 0 \\ 0 \\ \frac12 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac12 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Big\rangle.$$

Let $\varphi(z_1,z_2,z_3,z_4) = (z_2,z_1,z_3,z_4)$, we get these commutative diagrams

$$\begin{array}{ccccccc} (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_2,z_1,z_3,z_4) & & (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_2,z_1,z_3,z_4) \\ {\scriptstyle g_1}\big\downarrow & & {\scriptstyle h_2}\big\downarrow & & {\scriptstyle g_3}\big\downarrow & & {\scriptstyle h_3}\big\downarrow \\ (-z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_2,-z_1,z_3,z_4) & & (z_1,z_2,-z_3,z_4) & \xrightarrow{\varphi} & (z_2,z_1,-z_3,z_4) \\ (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_2,z_1,z_3,z_4) & & (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_2,z_1,z_3,z_4) \\ {\scriptstyle g_2}\big\downarrow & & {\scriptstyle h_1}\big\downarrow & & {\scriptstyle g_4}\big\downarrow & & {\scriptstyle h_4}\big\downarrow \\ (z_1,-z_2,\bar{z}_3,\bar{z}_4) & \xrightarrow{\varphi} & (-z_2,z_1,\bar{z}_3,\bar{z}_4) & & (z_1,z_2,z_3,-z_4) & \xrightarrow{\varphi} & (z_2,z_1,z_3,-z_4). \end{array}$$

Therefore $\exists\ \gamma = \left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right) \in \mathbb{A}(n)$ s.t. $\gamma \Gamma_{11} \gamma^{-1} = \Gamma_{21}$.

$\bullet M(A_{11}) \approx M(A_{31})$.
For

$$A_{11} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad A_{31} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{array}{ll} g_1(z_1,z_2,z_3,z_4) = (-z_1,z_2,z_3,z_4). & h_1(z_1,z_2,z_3,z_4) = (-z_1,z_2,\bar{z}_3,\bar{z}_4) \\ g_2(z_1,z_2,z_3,z_4) = (z_1,-z_2,\bar{z}_3,\bar{z}_4). & h_2h_1(z_1,z_2,z_3,z_4) = (-z_1,-z_2,z_3,z_4) \\ g_3(z_1,z_2,z_3,z_4) = (z_1,z_2,-z_3,z_4). & h_3(z_1,z_2,z_3,z_4) = (z_1,z_2,-z_3,z_4) \\ g_4(z_1,z_2,z_3,z_4) = (z_1,z_2,z_3,-z_4). & h_4(z_1,z_2,z_3,z_4) = (z_1,z_2,z_3,-z_4) \end{array}$$

then

$$\Gamma_{11} = \Big\langle \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Big\rangle$$

$$\Gamma_{31} = \Big\langle \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Big\rangle.$$

Let $\varphi(z_1,z_2,z_3,z_4) = (z_1z_2,z_1,z_3,z_4)$, we get these commutative diagrams

$$\begin{array}{ccccccc} (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1z_2,z_1,z_3,z_4) & & (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1z_2,z_1,z_3,z_4) \\ {\scriptstyle g_1}\downarrow & & \downarrow{\scriptstyle h_2h_1} & & {\scriptstyle g_3}\downarrow & & \downarrow{\scriptstyle h_3} \\ (-z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (-z_1z_2,-z_1,z_3,z_4) & & (z_1,z_2,-z_3,z_4) & \xrightarrow{\varphi} & (z_1z_2,z_1,-z_3,z_4) \\ (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1z_2,z_1,z_3,z_4) & & (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1z_2,z_1,z_3,z_4) \\ {\scriptstyle g_2}\downarrow & & \downarrow{\scriptstyle h_1} & & {\scriptstyle g_4}\downarrow & & \downarrow{\scriptstyle h_4} \\ (z_1,-z_2,\bar{z}_3,\bar{z}_4) & \xrightarrow{\varphi} & (-z_1z_2,z_1,\bar{z}_3,\bar{z}_4) & & (z_1,z_2,z_3,-z_4) & \xrightarrow{\varphi} & (z_1z_2,z_1,z_3,-z_4). \end{array}$$

Therefore $\exists\ \gamma = \left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right) \in \mathbb{A}(n)$ s.t. $\gamma\Gamma_{11}\gamma^{-1} = \Gamma_{31}$.

$\bullet M(A_{11}) \approx M(A_{a5})$.
For

$$A_{11} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad A_{a5} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned} g_1(z_1,z_2,z_3,z_4) &= (-z_1,z_2,z_3,z_4). & h_1(z_1,z_2,z_3,z_4) &= (-z_1,\bar{z}_2,z_3,\bar{z}_4) \\ g_2(z_1,z_2,z_3,z_4) &= (z_1,-z_2,\bar{z}_3,\bar{z}_4). & h_2(z_1,z_2,z_3,z_4) &= (z_1,-z_2,z_3,z_4) \\ g_3(z_1,z_2,z_3,z_4) &= (z_1,z_2,-z_3,z_4). & h_3(z_1,z_2,z_3,z_4) &= (z_1,z_2,-z_3,z_4) \\ g_4(z_1,z_2,z_3,z_4) &= (z_1,z_2,z_3,-z_4). & h_4(z_1,z_2,z_3,z_4) &= (z_1,z_2,z_3,-z_4) \end{aligned}$$

then

$$\begin{aligned} \Gamma_{11} = \Big\langle & \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \\ & \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\Big\rangle \end{aligned}$$

$$\begin{aligned} \Gamma_{a5} = \Big\langle & \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\ & \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\Big\rangle. \end{aligned}$$

Let $\varphi(z_1,z_2,z_3,z_4) = (z_2,z_3,z_1,z_4)$, we get these commutative diagrams

$$\begin{array}{ccccccc} (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_2,z_3,z_1,z_4) & (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_2,z_3,z_1,z_4) \\ {\scriptstyle g_1}\big\downarrow & & {\scriptstyle h_3}\big\downarrow & {\scriptstyle g_3}\big\downarrow & & {\scriptstyle h_2}\big\downarrow \\ (-z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_2,z_3,-z_1,z_4) & (z_1,z_2,-z_3,z_4) & \xrightarrow{\varphi} & (z_2,-z_3,z_1,z_4) \\ (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_2,z_3,z_1,z_4) & (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_2,z_3,z_1,z_4) \\ {\scriptstyle g_2}\big\downarrow & & {\scriptstyle h_1}\big\downarrow & {\scriptstyle g_4}\big\downarrow & & {\scriptstyle h_4}\big\downarrow \\ (z_1,-z_2,\bar{z}_3,\bar{z}_4) & \xrightarrow{\varphi} & (-z_2,\bar{z}_3,z_1,\bar{z}_4) & (z_1,z_2,z_3,-z_4) & \xrightarrow{\varphi} & (z_2,z_3,z_1,-z_4). \end{array}$$

Therefore $\exists\ \gamma = \Big(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\Big) \in \mathbb{A}(n)$ s.t. $\gamma\Gamma_{11}\gamma^{-1} = \Gamma_{a5}$.

$\bullet M(A_{11}) \approx M(A_{a17})$.
For

$$A_{11} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad A_{a17} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{array}{ll} g_1(z_1,z_2,z_3,z_4) = (-z_1,z_2,z_3,z_4). & h_1(z_1,z_2,z_3,z_4) = (-z_1,\bar{z}_2,\bar{z}_3,z_4) \\ g_2(z_1,z_2,z_3,z_4) = (z_1,-z_2,\bar{z}_3,\bar{z}_4). & h_2(z_1,z_2,z_3,z_4) = (z_1,-z_2,z_3,z_4) \\ g_3(z_1,z_2,z_3,z_4) = (z_1,z_2,-z_3,z_4). & h_3(z_1,z_2,z_3,z_4) = (z_1,z_2,-z_3,z_4) \\ g_4(z_1,z_2,z_3,z_4) = (z_1,z_2,z_3,-z_4). & h_4(z_1,z_2,z_3,z_4) = (z_1,z_2,z_3,-z_4) \end{array}$$

then

$$\Gamma_{11} = \Big\langle \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Big\rangle$$

$$\Gamma_{a17} = \Big\langle \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Big\rangle.$$

Let $\varphi(z_1,z_2,z_3,z_4) = (z_2,z_3,z_4,z_1)$, we get these commutative diagrams

$$\begin{array}{ccccccc} (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_2,z_3,z_4,z_1) & & (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_2,z_3,z_4,z_1) \\ {\scriptstyle g_1}\downarrow & & {\scriptstyle h_4}\downarrow & & {\scriptstyle g_3}\downarrow & & {\scriptstyle h_2}\downarrow \\ (-z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_2,z_3,z_4,-z_1) & & (z_1,z_2,-z_3,z_4) & \xrightarrow{\varphi} & (z_2,-z_3,z_4,z_1) \\ (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_2,z_3,z_4,z_1) & & (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_2,z_3,z_4,z_1) \\ {\scriptstyle g_2}\downarrow & & {\scriptstyle h_1}\downarrow & & {\scriptstyle g_4}\downarrow & & {\scriptstyle h_3}\downarrow \\ (z_1,-z_2,\bar{z}_3,\bar{z}_4) & \xrightarrow{\varphi} & (-z_2,\bar{z}_3,\bar{z}_4,z_1) & & (z_1,z_2,z_3,-z_4) & \xrightarrow{\varphi} & (z_2,z_3,-z_4,z_1). \end{array}$$

Therefore $\exists\, \gamma = \Big(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \Big) \in \mathbb{A}(n)$ s.t. $\gamma \Gamma_{11} \gamma^{-1} = \Gamma_{a17}$.

(3).

$\bullet M(A_{a15}) \approx M(A_{a27})$.

For

$$A_{a15}=\begin{pmatrix}1&1&0&1\\0&1&1&1\\0&0&1&0\\0&0&0&1\end{pmatrix}\qquad A_{a27}=\begin{pmatrix}1&1&1&0\\0&1&1&1\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

$$\begin{aligned}
g_1(z_1,z_2,z_3,z_4)&=(-z_1,\bar z_2,z_3,\bar z_4). & h_1(z_1,z_2,z_3,z_4)&=(-z_1,\bar z_2,\bar z_3,z_4)\\
g_2(z_1,z_2,z_3,z_4)&=(z_1,-z_2,\bar z_3,\bar z_4). & h_2(z_1,z_2,z_3,z_4)&=(z_1,-z_2,\bar z_3,\bar z_4)\\
g_3(z_1,z_2,z_3,z_4)&=(z_1,z_2,-z_3,z_4). & h_3(z_1,z_2,z_3,z_4)&=(z_1,z_2,-z_3,z_4)\\
g_4(z_1,z_2,z_3,z_4)&=(z_1,z_2,z_3,-z_4). & h_4(z_1,z_2,z_3,z_4)&=(z_1,z_2,z_3,-z_4)
\end{aligned}$$

then

$$\Gamma_{a15}=\left\langle \begin{pmatrix}\frac12\\0\\0\\0\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}, \begin{pmatrix}0\\\frac12\\0\\0\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix},\right.$$
$$\left.\begin{pmatrix}0\\0\\\frac12\\0\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix}0\\0\\0\\\frac12\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\right\rangle$$

$$\Gamma_{a27}=\left\langle \begin{pmatrix}\frac12\\0\\0\\0\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix}0\\\frac12\\0\\0\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix},\right.$$
$$\left.\begin{pmatrix}0\\0\\\frac12\\0\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix}0\\0\\0\\\frac12\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\right\rangle.$$

Let $\varphi(z_1,z_2,z_3,z_4)=(z_1,z_2,z_4,z_3)$, we get these commutative diagrams

$$\begin{array}{ccccccc}
(z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1,z_2,z_4,z_3) & & (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1,z_2,z_4,z_3)\\
{\scriptstyle g_1}\big\downarrow & & \big\downarrow{\scriptstyle h_1} & & {\scriptstyle g_3}\big\downarrow & & \big\downarrow{\scriptstyle h_4}\\
(-z_1,\bar z_2,z_3,\bar z_4) & \xrightarrow{\varphi} & (-z_1,\bar z_2,\bar z_4,z_3) & & (z_1,z_2,-z_3,z_4) & \xrightarrow{\varphi} & (z_1,z_2,z_4,-z_3)
\end{array}$$

$$\begin{array}{ccccccc}
(z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1,z_2,z_4,z_3) & & (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1,z_2,z_4,z_3)\\
{\scriptstyle g_2}\big\downarrow & & \big\downarrow{\scriptstyle h_2} & & {\scriptstyle g_4}\big\downarrow & & \big\downarrow{\scriptstyle h_3}\\
(z_1,-z_2,\bar z_3,\bar z_4) & \xrightarrow{\varphi} & (z_1,-z_2,\bar z_4,\bar z_3) & & (z_1,z_2,z_3,-z_4) & \xrightarrow{\varphi} & (z_1,z_2,-z_4,z_3).
\end{array}$$

Therefore $\exists\ \gamma = \left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \right) \in \mathbb{A}(n)$ s.t. $\gamma \Gamma_{a15} \gamma^{-1} = \Gamma_{a27}$.

(4). They are not diffeomorphic, because the holonomy of Γ_{11} and Γ_{a15} are $\mathbb{Z}_2$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ respectively.

(5).
• $M(A_3) \approx M(A_2)$.
For

$$A_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{array}{ll} g_1(z_1, z_2, z_3, z_4) = (-z_1, z_2, z_3, z_4). & h_1(z_1, z_2, z_3, z_4) = (-z_1, z_2, z_3, z_4) \\ g_2(z_1, z_2, z_3, z_4) = (z_1, -z_2, z_3, \bar{z}_4). & h_2(z_1, z_2, z_3, z_4) = (z_1, -z_2, z_3, z_4) \\ g_3(z_1, z_2, z_3, z_4) = (z_1, z_2, -z_3, z_4). & h_3(z_1, z_2, z_3, z_4) = (z_1, z_2, -z_3, \bar{z}_4) \\ g_4(z_1, z_2, z_3, z_4) = (z_1, z_2, z_3, -z_4). & h_4(z_1, z_2, z_3, z_4) = (z_1, z_2, z_3, -z_4) \end{array}$$

then

$$\Gamma_3 = \left\langle \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\rangle$$

$$\Gamma_2 = \left\langle \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\rangle.$$

Let $\varphi(z_1, z_2, z_3, z_4) = (z_1, z_3, z_2, z_4)$, we get these commutative diagrams

$$\begin{tikzcd} (z_1, z_2, z_3, z_4) \arrow[r, "\varphi"] \arrow[d, "g_1"'] & (z_1, z_3, z_2, z_4) \arrow[d, "h_1"'] \\ (-z_1, z_2, z_3, z_4) \arrow[r, "\varphi"] & (-z_1, z_3, z_2, z_4) \end{tikzcd} \quad \begin{tikzcd} (z_1, z_2, z_3, z_4) \arrow[r, "\varphi"] \arrow[d, "g_3"'] & (z_1, z_3, z_2, z_4) \arrow[d, "h_2"'] \\ (z_1, z_2, -z_3, z_4) \arrow[r, "\varphi"] & (z_1, -z_3, z_2, z_4) \end{tikzcd}$$

$$\begin{array}{ccccccc}
(z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1,z_3,z_2,z_4) & & (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1,z_3,z_2,z_4) \\
{\scriptstyle g_2}\downarrow & & {\scriptstyle h_3}\downarrow & & {\scriptstyle g_4}\downarrow & & {\scriptstyle h_4}\downarrow \\
(z_1,-z_2,z_3,\bar{z}_4) & \xrightarrow{\varphi} & (z_1,z_3,-z_2,\bar{z}_4) & & (z_1,z_2,z_3,-z_4) & \xrightarrow{\varphi} & (z_1,z_3,z_2,-z_4).
\end{array}$$

Therefore $\exists\ \gamma = \Big(\begin{pmatrix}0\\0\\0\\0\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}\Big) \in \mathbb{A}(n)$ s.t. $\gamma\Gamma_3\gamma^{-1} = \Gamma_2$.

• $M(A_2) \approx M(A_4)$.
For

$$A_2 = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&1\\0&0&0&1\end{pmatrix} \qquad A_4 = \begin{pmatrix}1&0&0&0\\0&1&0&1\\0&0&1&1\\0&0&0&1\end{pmatrix}$$

$$\begin{array}{rr}
g_1(z_1,z_2,z_3,z_4) = (-z_1,z_2,z_3,z_4). & h_1(z_1,z_2,z_3,z_4) = (-z_1,z_2,z_3,z_4) \\
g_2(z_1,z_2,z_3,z_4) = (z_1,-z_2,z_3,z_4). & h_2h_3(z_1,z_2,z_3,z_4) = (z_1,-z_2,-z_3,z_4) \\
g_3(z_1,z_2,z_3,z_4) = (z_1,z_2,-z_3,\bar{z}_4). & h_3(z_1,z_2,z_3,z_4) = (z_1,z_2,-z_3,\bar{z}_4) \\
g_4(z_1,z_2,z_3,z_4) = (z_1,z_2,z_3,-z_4). & h_4(z_1,z_2,z_3,z_4) = (z_1,z_2,z_3,-z_4)
\end{array}$$

then

$$\Gamma_2 = \Big\langle \begin{pmatrix}\frac12\\0\\0\\0\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix}0\\\frac12\\0\\0\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},$$
$$\begin{pmatrix}0\\0\\\frac12\\0\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}, \begin{pmatrix}0\\0\\0\\\frac12\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\Big\rangle$$

$$\Gamma_4 = \Big\langle \begin{pmatrix}\frac12\\0\\0\\0\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix}0\\\frac12\\\frac12\\0\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},$$
$$\begin{pmatrix}0\\0\\\frac12\\0\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}, \begin{pmatrix}0\\0\\0\\\frac12\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\Big\rangle.$$

Let $\varphi(z_1,z_2,z_3,z_4) = (z_1,z_2,z_2z_3,z_4)$, we get these commutative diagrams

$$\begin{array}{ccccccc}
(z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1,z_2,z_2z_3,z_4) & & (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1,z_2,z_2z_3,z_4) \\
{\scriptstyle g_1}\downarrow & & {\scriptstyle h_1}\downarrow & & {\scriptstyle g_3}\downarrow & & {\scriptstyle h_3}\downarrow \\
(-z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (-z_1,z_2,z_2z_3,z_4) & & (z_1,z_2,-z_3,\bar{z}_4) & \xrightarrow{\varphi} & (z_1,z_2,-z_2z_3,\bar{z}_4)
\end{array}$$

$$\begin{array}{ccc} (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1,z_2,z_2z_3,z_4) \\ {\scriptstyle g_2}\downarrow & & \downarrow{\scriptstyle h_2h_3} \\ (z_1,-z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1,-z_2,-z_2z_3,z_4) \end{array} \quad \begin{array}{ccc} (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1,z_2,z_2z_3,z_4) \\ {\scriptstyle g_4}\downarrow & & \downarrow{\scriptstyle h_4} \\ (z_1,z_2,z_3,-z_4) & \xrightarrow{\varphi} & (z_1,z_2,z_2z_3,-z_4). \end{array}$$

Therefore $\exists\ \gamma = \left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right) \in \mathbb{A}(n)$ s.t. $\gamma\Gamma_2\gamma^{-1} = \Gamma_4$.

• $M(A_2) \approx M(A_5)$.
For

$$A_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad A_5 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{array}{ll} g_1(z_1,z_2,z_3,z_4) = (-z_1,z_2,z_3,z_4). & h_1(z_1,z_2,z_3,z_4) = (-z_1,z_2,z_3,\bar{z}_4) \\ g_2(z_1,z_2,z_3,z_4) = (z_1,-z_2,z_3,z_4). & h_2(z_1,z_2,z_3,z_4) = (z_1,-z_2,z_3,z_4) \\ g_3(z_1,z_2,z_3,z_4) = (z_1,z_2,-z_3,\bar{z}_4). & h_3(z_1,z_2,z_3,z_4) = (z_1,z_2,-z_3,z_4) \\ g_4(z_1,z_2,z_3,z_4) = (z_1,z_2,z_3,-z_4). & h_4(z_1,z_2,z_3,z_4) = (z_1,z_2,z_3,-z_4) \end{array}$$

then

$$\Gamma_2 = \Big\langle \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Big\rangle$$

$$\Gamma_5 = \Big\langle \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Big\rangle.$$

Let $\varphi(z_1,z_2,z_3,z_4) = (z_3,z_2,z_1,z_4)$, we get these commutative diagrams

$$\begin{array}{ccc} (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_3,z_2,z_1,z_4) \\ {\scriptstyle g_1}\downarrow & & \downarrow{\scriptstyle h_3} \\ (-z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_3,z_2,-z_1,z_4) \end{array} \quad \begin{array}{ccc} (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_3,z_2,z_1,z_4) \\ {\scriptstyle g_3}\downarrow & & \downarrow{\scriptstyle h_1} \\ (z_1,z_2,-z_3,\bar{z}_4) & \xrightarrow{\varphi} & (-z_3,z_2,z_1,\bar{z}_4) \end{array}$$

$$\begin{array}{ccccccc}
(z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_3,z_2,z_1,z_4) & (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_3,z_2,z_1,z_4) \\
{\scriptstyle g_2}\downarrow & & {\scriptstyle h_2}\downarrow & {\scriptstyle g_4}\downarrow & & {\scriptstyle h_4}\downarrow \\
(z_1,-z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_3,-z_2,z_1,z_4) & (z_1,z_2,z_3,-z_4) & \xrightarrow{\varphi} & (z_3,z_2,z_1,-z_4).
\end{array}$$

Therefore $\exists\ \gamma = \left(\begin{pmatrix} 0\\0\\0\\0 \end{pmatrix} \begin{pmatrix} 0&0&1&0\\0&1&0&0\\1&0&0&0\\0&0&0&1 \end{pmatrix} \right) \in \mathbb{A}(n)$ s.t. $\gamma\Gamma_2\gamma^{-1} = \Gamma_5$.

• $M(A_2) \approx M(A_6)$.
For

$$A_2 = \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&1\\0&0&0&1 \end{pmatrix} \qquad A_6 = \begin{pmatrix} 1&0&0&1\\0&1&0&0\\0&0&1&1\\0&0&0&1 \end{pmatrix}$$

$$\begin{aligned}
g_1(z_1,z_2,z_3,z_4) &= (-z_1,z_2,z_3,z_4). & h_1h_3(z_1,z_2,z_3,z_4) &= (-z_1,z_2,-z_3,z_4)\\
g_2(z_1,z_2,z_3,z_4) &= (z_1,-z_2,z_3,z_4). & h_2(z_1,z_2,z_3,z_4) &= (z_1,-z_2,z_3,z_4)\\
g_3(z_1,z_2,z_3,z_4) &= (z_1,z_2,-z_3,\bar z_4). & h_3(z_1,z_2,z_3,z_4) &= (z_1,z_2,-z_3,\bar z_4)\\
g_4(z_1,z_2,z_3,z_4) &= (z_1,z_2,z_3,-z_4). & h_4(z_1,z_2,z_3,z_4) &= (z_1,z_2,z_3,-z_4)
\end{aligned}$$

then

$$\begin{aligned}
\Gamma_2 = \Big\langle & \begin{pmatrix} \frac12\\0\\0\\0 \end{pmatrix} \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix}, \begin{pmatrix} 0\\\frac12\\0\\0 \end{pmatrix} \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix},\\
& \begin{pmatrix} 0\\0\\\frac12\\0 \end{pmatrix} \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1 \end{pmatrix}, \begin{pmatrix} 0\\0\\0\\\frac12 \end{pmatrix} \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix} \Big\rangle
\end{aligned}$$

$$\begin{aligned}
\Gamma_6 = \Big\langle & \begin{pmatrix} \frac12\\0\\\frac12\\0 \end{pmatrix} \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix}, \begin{pmatrix} 0\\\frac12\\0\\0 \end{pmatrix} \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix},\\
& \begin{pmatrix} 0\\0\\\frac12\\0 \end{pmatrix} \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1 \end{pmatrix}, \begin{pmatrix} 0\\0\\0\\\frac12 \end{pmatrix} \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix} \Big\rangle.
\end{aligned}$$

Let $\varphi(z_1,z_2,z_3,z_4) = (z_1,z_2,z_1z_3,z_4)$, we get these commutative diagrams

$$\begin{array}{ccccccc}
(z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1,z_2,z_1z_3,z_4) & (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1,z_2,z_1z_3,\bar z_4) \\
{\scriptstyle g_1}\downarrow & & {\scriptstyle h_1h_3}\downarrow & {\scriptstyle g_3}\downarrow & & {\scriptstyle h_3}\downarrow \\
(-z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (-z_1,z_2,-z_1z_3,z_4) & (z_1,z_2,-z_3,\bar z_4) & \xrightarrow{\varphi} & (z_1,z_2,-z_1z_3,\bar z_4)
\end{array}$$

$$\begin{array}{ccc} (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1,z_2,z_1z_3,z_4) \\ {\scriptstyle g_2}\downarrow & & {\scriptstyle h_2}\downarrow \\ (z_1,-z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1,-z_2,z_1z_3,z_4) \end{array} \quad \begin{array}{ccc} (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1,z_2,z_1z_3,z_4) \\ {\scriptstyle g_4}\downarrow & & {\scriptstyle h_4}\downarrow \\ (z_1,z_2,z_3,-z_4) & \xrightarrow{\varphi} & (z_1,z_2,z_1z_3,-z_4). \end{array}$$

Therefore $\exists\ \gamma = \left(\begin{pmatrix} 0\\0\\0\\0 \end{pmatrix} \begin{pmatrix} 1&0&0&0\\0&1&0&0\\1&0&1&0\\0&0&0&1 \end{pmatrix} \right) \in \mathbb{A}(n)$ s.t. $\gamma\Gamma_2\gamma^{-1} = \Gamma_6$.

• $M(A_3) \approx M(A_7)$.
For

$$A_3 = \begin{pmatrix} 1&0&0&0\\0&1&0&1\\0&0&1&0\\0&0&0&1 \end{pmatrix} \qquad A_7 = \begin{pmatrix} 1&0&0&1\\0&1&0&1\\0&0&1&0\\0&0&0&1 \end{pmatrix}$$

$$\begin{aligned} g_1(z_1,z_2,z_3,z_4) &= (-z_1,z_2,z_3,z_4). & h_1h_2(z_1,z_2,z_3,z_4) &= (-z_1,-z_2,z_3,z_4) \\ g_2(z_1,z_2,z_3,z_4) &= (z_1,-z_2,z_3,\bar{z}_4). & h_2(z_1,z_2,z_3,z_4) &= (z_1,-z_2,z_3,\bar{z}_4) \\ g_3(z_1,z_2,z_3,z_4) &= (z_1,z_2,-z_3,z_4). & h_3(z_1,z_2,z_3,z_4) &= (z_1,z_2,-z_3,z_4) \\ g_4(z_1,z_2,z_3,z_4) &= (z_1,z_2,z_3,-z_4). & h_4(z_1,z_2,z_3,z_4) &= (z_1,z_2,z_3,-z_4) \end{aligned}$$

then

$$\Gamma_3 = \Big\langle \begin{pmatrix} \frac12\\0\\0\\0 \end{pmatrix} \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix}, \begin{pmatrix} 0\\\frac12\\0\\0 \end{pmatrix} \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1 \end{pmatrix}, \begin{pmatrix} 0\\0\\\frac12\\0 \end{pmatrix} \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix}, \begin{pmatrix} 0\\0\\0\\\frac12 \end{pmatrix} \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix} \Big\rangle$$

$$\Gamma_7 = \Big\langle \begin{pmatrix} \frac12\\\frac12\\0\\0 \end{pmatrix} \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix}, \begin{pmatrix} 0\\\frac12\\0\\0 \end{pmatrix} \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1 \end{pmatrix}, \begin{pmatrix} 0\\0\\\frac12\\0 \end{pmatrix} \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix}, \begin{pmatrix} 0\\0\\0\\\frac12 \end{pmatrix} \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix} \Big\rangle.$$

Let $\varphi(z_1,z_2,z_3,z_4) = (z_1,z_1z_2,z_3,z_4)$, we get these commutative diagrams

$$\begin{array}{ccc} (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1,z_1z_2,z_3,z_4) \\ {\scriptstyle g_1}\downarrow & & {\scriptstyle h_1h_2}\downarrow \\ (-z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (-z_1,-z_1z_2,z_3,z_4) \end{array} \quad \begin{array}{ccc} (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1,z_1z_2,z_3,z_4) \\ {\scriptstyle g_3}\downarrow & & {\scriptstyle h_3}\downarrow \\ (z_1,z_2,-z_3,z_4) & \xrightarrow{\varphi} & (z_1,z_1z_2,-z_3,z_4) \end{array}$$

$$\begin{array}{ccccccc}
(z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1,z_1z_2,z_3,z_4) & & (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1,z_1z_2,z_3,z_4) \\
{\scriptstyle g_2}\downarrow & & {\scriptstyle h_2}\downarrow & & {\scriptstyle g_4}\downarrow & & {\scriptstyle h_4}\downarrow \\
(z_1,-z_2,z_3,\bar{z}_4) & \xrightarrow{\varphi} & (z_1,-z_1z_2,z_3,\bar{z}_4) & & (z_1,z_2,z_3,-z_4) & \xrightarrow{\varphi} & (z_1,z_1z_2,z_3,-z_4).
\end{array}$$

Therefore $\exists\ \gamma = \left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right) \in \mathbb{A}(n)$ s.t. $\gamma\Gamma_3\gamma^{-1} = \Gamma_7$.

• $M(A_2) \approx M(A_8)$.
For

$$A_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad A_8 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{array}{ll}
g_1(z_1,z_2,z_3,z_4) = (-z_1,z_2,z_3,z_4). & h_1h_2(z_1,z_2,z_3,z_4) = (-z_1,-z_2,z_3,z_4) \\
g_2(z_1,z_2,z_3,z_4) = (z_1,-z_2,z_3,z_4). & h_2h_3(z_1,z_2,z_3,z_4) = (z_1,-z_2,-z_3,z_4) \\
g_3(z_1,z_2,z_3,z_4) = (z_1,z_2,-z_3,\bar{z}_4). & h_3(z_1,z_2,z_3,z_4) = (z_1,z_2,-z_3,\bar{z}_4) \\
g_4(z_1,z_2,z_3,z_4) = (z_1,z_2,z_3,-z_4). & h_4(z_1,z_2,z_3,z_4) = (z_1,z_2,z_3,-z_4)
\end{array}$$

then

$$\Gamma_2 = \Big\langle \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$
$$\begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Big\rangle$$

$$\Gamma_8 = \Big\langle \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$
$$\begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Big\rangle.$$

Let $\varphi(z_1,z_2,z_3,z_4) = (z_1,z_1z_2,z_2z_3,z_4)$, we get these commutative diagrams

$$\begin{array}{ccccccc}
(z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1,z_1z_2,z_2z_3,z_4) & & (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1,z_1z_2,z_2z_3,z_4) \\
{\scriptstyle g_1}\downarrow & & {\scriptstyle h_1h_2}\downarrow & & {\scriptstyle g_3}\downarrow & & {\scriptstyle h_3}\downarrow \\
(-z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (-z_1,-z_1z_2,z_2z_3,z_4) & & (z_1,z_2,-z_3,\bar{z}_4) & \xrightarrow{\varphi} & (z_1,z_1z_2,-z_2z_3,\bar{z}_4)
\end{array}$$

$$\begin{array}{ccc} (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1,z_1z_2,z_2z_3,z_4) \\ {\scriptstyle g_2}\downarrow & & \downarrow{\scriptstyle h_2h_3} \\ (z_1,-z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1,-z_1z_2,-z_2z_3,z_4) \end{array} \quad \begin{array}{ccc} (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1,z_1z_2,z_2z_3,z_4) \\ {\scriptstyle g_4}\downarrow & & \downarrow{\scriptstyle h_4} \\ (z_1,z_2,z_3,-z_4) & \xrightarrow{\varphi} & (z_1,z_1z_2,z_2z_3,-z_4). \end{array}$$

Therefore $\exists\ \gamma = \left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right) \in \mathbb{A}(n)$ s.t. $\gamma\Gamma_2\gamma^{-1} = \Gamma_8$.

• $M(A_2) \approx M(A_9)$.
For

$$A_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad A_9 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{array}{ll} g_1(z_1,z_2,z_3,z_4) = (-z_1,z_2,z_3,z_4). & h_1(z_1,z_2,z_3,z_4) = (-z_1,z_2,z_3,z_4) \\ g_2(z_1,z_2,z_3,z_4) = (z_1,-z_2,z_3,z_4). & h_2(z_1,z_2,z_3,z_4) = (z_1,-z_2,\bar{z}_3,z_4) \\ g_3(z_1,z_2,z_3,z_4) = (z_1,z_2,-z_3,\bar{z}_4). & h_3(z_1,z_2,z_3,z_4) = (z_1,z_2,-z_3,z_4) \\ g_4(z_1,z_2,z_3,z_4) = (z_1,z_2,z_3,-z_4). & h_4(z_1,z_2,z_3,z_4) = (z_1,z_2,z_3,-z_4) \end{array}$$

then

$$\Gamma_2 = \Big\langle \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Big\rangle$$

$$\Gamma_9 = \Big\langle \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Big\rangle.$$

Let $\varphi(z_1,z_2,z_3,z_4) = (z_1,z_3,z_4,z_2)$, we get these commutative diagrams

$$\begin{array}{ccc} (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1,z_3,z_4,z_2) \\ {\scriptstyle g_1}\downarrow & & \downarrow{\scriptstyle h_1} \\ (-z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (-z_1,z_3,z_4,z_2) \end{array} \quad \begin{array}{ccc} (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1,z_3,z_4,z_2) \\ {\scriptstyle g_3}\downarrow & & \downarrow{\scriptstyle h_2} \\ (z_1,z_2,-z_3,\bar{z}_4) & \xrightarrow{\varphi} & (z_1,-z_3,\bar{z}_4,z_2) \end{array}$$

$$\begin{array}{ccccccc}
(z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1,z_3,z_4,z_2) & & (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1,z_3,z_4,z_2) \\
{\scriptstyle g_2}\downarrow & & {\scriptstyle h_4}\downarrow & & {\scriptstyle g_4}\downarrow & & {\scriptstyle h_3}\downarrow \\
(z_1,-z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1,z_3,z_4,-z_2) & & (z_1,z_2,z_3,-z_4) & \xrightarrow{\varphi} & (z_1,z_3,-z_4,z_2).
\end{array}$$

Therefore $\exists\ \gamma = \Big(\begin{pmatrix} 0\\0\\0\\0 \end{pmatrix} \begin{pmatrix} 1&0&0&0\\0&0&1&0\\0&0&0&1\\0&1&0&0 \end{pmatrix}\Big) \in \mathbb{A}(n)$ s.t. $\gamma\Gamma_2\gamma^{-1} = \Gamma_9$.

• $M(A_2) \approx M(A_{17})$.
For

$$A_2 = \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&1\\0&0&0&1 \end{pmatrix} \qquad A_{17} = \begin{pmatrix} 1&0&1&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix}$$

$$\begin{array}{ll}
g_1(z_1,z_2,z_3,z_4) = (-z_1,z_2,z_3,z_4). & h_1(z_1,z_2,z_3,z_4) = (-z_1,z_2,\bar{z}_3,z_4) \\
g_2(z_1,z_2,z_3,z_4) = (z_1,-z_2,z_3,z_4). & h_2(z_1,z_2,z_3,z_4) = (z_1,-z_2,z_3,z_4) \\
g_3(z_1,z_2,z_3,z_4) = (z_1,z_2,-z_3,\bar{z}_4). & h_3(z_1,z_2,z_3,z_4) = (z_1,z_2,-z_3,z_4) \\
g_4(z_1,z_2,z_3,z_4) = (z_1,z_2,z_3,-z_4). & h_4(z_1,z_2,z_3,z_4) = (z_1,z_2,z_3,-z_4)
\end{array}$$

then

$$\Gamma_2 = \Big\langle \begin{pmatrix} \frac12\\0\\0\\0 \end{pmatrix} \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix}, \begin{pmatrix} 0\\\frac12\\0\\0 \end{pmatrix} \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix},$$
$$\begin{pmatrix} 0\\0\\\frac12\\0 \end{pmatrix} \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1 \end{pmatrix}, \begin{pmatrix} 0\\0\\0\\\frac12 \end{pmatrix} \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix} \Big\rangle$$

$$\Gamma_{17} = \Big\langle \begin{pmatrix} \frac12\\0\\0\\0 \end{pmatrix} \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&1 \end{pmatrix}, \begin{pmatrix} 0\\\frac12\\0\\0 \end{pmatrix} \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix},$$
$$\begin{pmatrix} 0\\0\\\frac12\\0 \end{pmatrix} \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix}, \begin{pmatrix} 0\\0\\0\\\frac12 \end{pmatrix} \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix} \Big\rangle.$$

Let $\varphi(z_1,z_2,z_3,z_4) = (z_3,z_1,z_4,z_2)$, we get these commutative diagrams

$$\begin{array}{ccccccc}
(z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_3,z_1,z_4,z_2) & & (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_3,z_1,z_4,z_2) \\
{\scriptstyle g_1}\downarrow & & {\scriptstyle h_2}\downarrow & & {\scriptstyle g_3}\downarrow & & {\scriptstyle h_1}\downarrow \\
(-z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_3,-z_1,z_4,z_2) & & (z_1,z_2,-z_3,\bar{z}_4) & \xrightarrow{\varphi} & (-z_3,z_1,\bar{z}_4,z_2)
\end{array}$$

$$\begin{array}{ccc} (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_3,z_1,z_4,z_2) \\ g_2\big\downarrow & & h_4\big\downarrow \\ (z_1,-z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_3,z_1,z_4,-z_2) \end{array} \quad \begin{array}{ccc} (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_3,z_1,z_4,z_2) \\ g_4\big\downarrow & & h_3\big\downarrow \\ (z_1,z_2,z_3,-z_4) & \xrightarrow{\varphi} & (z_3,z_1,-z_4,z_2). \end{array}$$

Therefore $\exists\ \gamma = \left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \right) \in \mathbb{A}(n)$ s.t. $\gamma\Gamma_2\gamma^{-1} = \Gamma_{17}$.

• $M(A_2) \approx M(A_{25})$.
For

$$A_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad A_{25} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned} g_1(z_1,z_2,z_3,z_4) &= (-z_1,z_2,z_3,z_4). & h_1h_2(z_1,z_2,z_3,z_4) &= (-z_1,-z_2,z_3,z_4) \\ g_2(z_1,z_2,z_3,z_4) &= (z_1,-z_2,z_3,z_4). & h_2(z_1,z_2,z_3,z_4) &= (z_1,-z_2,\bar{z}_3,z_4) \\ g_3(z_1,z_2,z_3,z_4) &= (z_1,z_2,-z_3,\bar{z}_4). & h_3(z_1,z_2,z_3,z_4) &= (z_1,z_2,-z_3,z_4) \\ g_4(z_1,z_2,z_3,z_4) &= (z_1,z_2,z_3,-z_4). & h_4(z_1,z_2,z_3,z_4) &= (z_1,z_2,z_3,-z_4) \end{aligned}$$

then

$$\begin{aligned} \Gamma_2 = \Big\langle & \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\ & \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Big\rangle \end{aligned}$$

$$\begin{aligned} \Gamma_{25} = \Big\langle & \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\ & \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Big\rangle. \end{aligned}$$

Let $\varphi(z_1,z_2,z_3,z_4) = (z_2,z_2z_3,z_4,z_1)$, we get these commutative diagrams

$$\begin{array}{ccc} (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_2,z_2z_3,z_4,z_1) \\ g_1\big\downarrow & & h_4\big\downarrow \\ (-z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_2,z_2z_3,z_4,-z_1) \end{array} \quad \begin{array}{ccc} (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_2,z_2z_3,z_4,z_1) \\ g_3\big\downarrow & & h_2\big\downarrow \\ (z_1,z_2,-z_3,\bar{z}_4) & \xrightarrow{\varphi} & (z_2,-z_2z_3,\bar{z}_4,z_1) \end{array}$$

$$\begin{array}{ccccccc}
(z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_2,z_2z_3,z_4,z_1) & & (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (-z_2,-z_2z_3,z_4,z_1) \\
{\scriptstyle g_2}\big\downarrow & & \big\downarrow{\scriptstyle h_1h_2} & & {\scriptstyle g_4}\big\downarrow & & \big\downarrow{\scriptstyle h_3} \\
(z_1,-z_2,z_3,z_4) & \xrightarrow{\varphi} & (-z_2,-z_2z_3,z_4,z_1) & & (z_1,z_2,z_3,-z_4) & \xrightarrow{\varphi} & (z_2,z_2z_3,-z_4,z_1).
\end{array}$$

Therefore $\exists\ \gamma = \Big(\begin{pmatrix} 0\\0\\0\\0 \end{pmatrix} \begin{pmatrix} 0&1&0&0\\0&1&1&0\\0&0&0&1\\1&0&0&0 \end{pmatrix}\Big) \in \mathbb{A}(n)$ s.t. $\gamma\Gamma_2\gamma^{-1} = \Gamma_{25}$.

• $M(A_2) \approx M(A_{a1})$.
For

$$A_2 = \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&1\\0&0&0&1 \end{pmatrix} \qquad A_{a1} = \begin{pmatrix} 1&1&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix}$$

$$\begin{array}{ll}
g_1(z_1,z_2,z_3,z_4) = (-z_1,z_2,z_3,z_4). & h_1(z_1,z_2,z_3,z_4) = (-z_1,\bar z_2,z_3,z_4) \\
g_2(z_1,z_2,z_3,z_4) = (z_1,-z_2,z_3,z_4). & h_2(z_1,z_2,z_3,z_4) = (z_1,-z_2,z_3,z_4) \\
g_3(z_1,z_2,z_3,z_4) = (z_1,z_2,-z_3,\bar z_4). & h_3(z_1,z_2,z_3,z_4) = (z_1,z_2,-z_3,z_4) \\
g_4(z_1,z_2,z_3,z_4) = (z_1,z_2,z_3,-z_4). & h_4(z_1,z_2,z_3,z_4) = (z_1,z_2,z_3,-z_4)
\end{array}$$

then

$$\Gamma_2 = \Big\langle \begin{pmatrix} \frac12\\0\\0\\0 \end{pmatrix} \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix}, \begin{pmatrix} 0\\\frac12\\0\\0 \end{pmatrix} \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix},$$
$$\begin{pmatrix} 0\\0\\\frac12\\0 \end{pmatrix} \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1 \end{pmatrix}, \begin{pmatrix} 0\\0\\0\\\frac12 \end{pmatrix} \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix} \Big\rangle$$

$$\Gamma_{a1} = \Big\langle \begin{pmatrix} \frac12\\0\\0\\0 \end{pmatrix} \begin{pmatrix} 1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix}, \begin{pmatrix} 0\\\frac12\\0\\0 \end{pmatrix} \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix},$$
$$\begin{pmatrix} 0\\0\\\frac12\\0 \end{pmatrix} \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix}, \begin{pmatrix} 0\\0\\0\\\frac12 \end{pmatrix} \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix} \Big\rangle.$$

Let $\varphi(z_1,z_2,z_3,z_4) = (z_3,z_4,z_1,z_2)$, we get these commutative diagrams

$$\begin{array}{ccccccc}
(z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_3,z_4,z_1,z_2) & & (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_3,z_4,z_1,z_2) \\
{\scriptstyle g_1}\big\downarrow & & \big\downarrow{\scriptstyle h_3} & & {\scriptstyle g_3}\big\downarrow & & \big\downarrow{\scriptstyle h_1} \\
(-z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_3,z_4,-z_1,z_2) & & (z_1,z_2,-z_3,\bar z_4) & \xrightarrow{\varphi} & (-z_3,\bar z_4,z_1,z_2)
\end{array}$$

$$\begin{array}{ccccccc}
(z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_3,z_4,z_1,z_2) & (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_3,z_4,z_1,z_2) \\
{\scriptstyle g_2}\big\downarrow & & \big\downarrow{\scriptstyle h_4} & {\scriptstyle g_4}\big\downarrow & & \big\downarrow{\scriptstyle h_2} \\
(z_1,-z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_3,z_4,z_1,-z_2) & (z_1,z_2,z_3,-z_4) & \xrightarrow{\varphi} & (z_3,-z_4,z_1,z_2).
\end{array}$$

Therefore $\exists\ \gamma = \Big(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \Big) \in \mathbb{A}(n)$ s.t. $\gamma \Gamma_2 \gamma^{-1} = \Gamma_{a1}$.

(6). If $M(A_{11})$ is diffeomorphic to $M(A_2)$ and $M(A_{a21})$ respectively, then by Bieberbach's theorem $\exists B \in GL(4,\mathbb{R})$ such that $BPB^{-1} = Q$ and $BPB^{-1} = R$ respectively, where $\Phi_{11} = \Big\langle I, P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \Big\rangle$, $\Phi_2 = \Big\langle I, Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \Big\rangle$, $\Phi_{a21} = \Big\langle I, R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \Big\rangle$.
Similar to the proof of Theorem 3.1(e), by argument of eigenvalue, there is a contradiction, where the eigenvalues of P are 1,1,-1 and -1, but the eigenvalues of Q are 1,1,1 and -1, and the eigenvalues of R are 1,-1,-1 and -1.

(7). Similar to the proof of (4) above.

(8). Similar to the proof of (6) above.

(9).
• $M(A_{a3}) \approx M(A_{a7})$.
For

$$A_{a3} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad A_{a7} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{array}{ll}
g_1(z_1,z_2,z_3,z_4) = (-z_1,\bar{z}_2,z_3,z_4). & h_1h_2(z_1,z_2,z_3,z_4) = (-z_1,-\bar{z}_2,z_3,z_4) \\
g_2(z_1,z_2,z_3,z_4) = (z_1,-z_2,z_3,\bar{z}_4). & h_2(z_1,z_2,z_3,z_4) = (z_1,-z_2,z_3,\bar{z}_4) \\
g_3(z_1,z_2,z_3,z_4) = (z_1,z_2,-z_3,z_4). & h_3(z_1,z_2,z_3,z_4) = (z_1,z_2,-z_3,z_4) \\
g_4(z_1,z_2,z_3,z_4) = (z_1,z_2,z_3,-z_4). & h_4(z_1,z_2,z_3,z_4) = (z_1,z_2,z_3,-z_4)
\end{array}$$

then

$$\Gamma_{a3} = \left\langle \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\rangle$$

$$\Gamma_{a7} = \left\langle \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\rangle.$$

Let $\varphi(z_1, z_2, z_3, z_4) = (z_1, iz_2, z_3, z_4)$, we get these commutative diagrams

$$\begin{array}{ccc} (z_1, z_2, z_3, z_4) & \xrightarrow{\varphi} & (z_1, iz_2, z_3, z_4) \\ {\scriptstyle g_1}\big\downarrow & & \big\downarrow{\scriptstyle h_1h_2} \\ (-z_1, \bar{z}_2, z_3, z_4) & \xrightarrow{\varphi} & (-z_1, -\bar{iz}_2, z_3, z_4) \end{array} \qquad \begin{array}{ccc} (z_1, z_2, z_3, z_4) & \xrightarrow{\varphi} & (z_1, iz_2, z_3, z_4) \\ {\scriptstyle g_3}\big\downarrow & & \big\downarrow{\scriptstyle h_3} \\ (z_1, z_2, -z_3, z_4) & \xrightarrow{\varphi} & (z_1, iz_2, -z_3, z_4) \end{array}$$

$$\begin{array}{ccc} (z_1, z_2, z_3, z_4) & \xrightarrow{\varphi} & (z_1, iz_2, z_3, z_4) \\ {\scriptstyle g_2}\big\downarrow & & \big\downarrow{\scriptstyle h_2} \\ (z_1, -z_2, z_3, \bar{z}_4) & \xrightarrow{\varphi} & (z_1, -iz_2, z_3, \bar{z}_4) \end{array} \qquad \begin{array}{ccc} (z_1, z_2, z_3, z_4) & \xrightarrow{\varphi} & (z_1, iz_2, z_3, z_4) \\ {\scriptstyle g_4}\big\downarrow & & \big\downarrow{\scriptstyle h_4} \\ (z_1, z_2, z_3, -z_4) & \xrightarrow{\varphi} & (z_1, iz_2, z_3, -z_4). \end{array}$$

Therefore $\exists\ \gamma = \left(\begin{pmatrix} 0 \\ \frac{1}{4} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right) \in \mathbb{A}(n)$ s.t. $\gamma \Gamma_{a3} \gamma^{-1} = \Gamma_{a7}$.

• $M(A_{10}) \approx M(A_{a3})$.
For

$$A_{10} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad A_{a3} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{array}{ll} g_1(z_1, z_2, z_3, z_4) = (-z_1, z_2, z_3, z_4). & h_1(z_1, z_2, z_3, z_4) = (-z_1, \bar{z}_2, z_3, z_4) \\ g_2(z_1, z_2, z_3, z_4) = (z_1, -z_2, \bar{z}_3, z_4). & h_2(z_1, z_2, z_3, z_4) = (z_1, -z_2, z_3, \bar{z}_4) \\ g_3(z_1, z_2, z_3, z_4) = (z_1, z_2, -z_3, \bar{z}_4). & h_3(z_1, z_2, z_3, z_4) = (z_1, z_2, -z_3, z_4) \\ g_4(z_1, z_2, z_3, z_4) = (z_1, z_2, z_3, -z_4). & h_4(z_1, z_2, z_3, z_4) = (z_1, z_2, z_3, -z_4) \end{array}$$

then

$$\Gamma_{10} = \Big\langle \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Big\rangle$$

$$\Gamma_{a3} = \Big\langle \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Big\rangle.$$

Let $\varphi(z_1, z_2, z_3, z_4) = (z_2, z_3, z_1, z_4)$, we get these commutative diagrams

$$\begin{array}{ccccccc}
(z_1, z_2, z_3, z_4) & \xrightarrow{\varphi} & (z_2, z_3, z_1, z_4) & & (z_1, z_2, z_3, z_4) & \xrightarrow{\varphi} & (z_2, z_3, z_1, z_4) \\
{\scriptstyle g_1}\downarrow & & {\scriptstyle h_3}\downarrow & & {\scriptstyle g_3}\downarrow & & {\scriptstyle h_2}\downarrow \\
(-z_1, z_2, z_3, z_4) & \xrightarrow{\varphi} & (z_2, z_3, -z_1, z_4) & & (z_1, z_2, -z_3, \bar{z}_4) & \xrightarrow{\varphi} & (z_2, -z_3, z_1, \bar{z}_4) \\
(z_1, z_2, z_3, z_4) & \xrightarrow{\varphi} & (z_2, z_3, z_1, z_4) & & (z_1, z_2, z_3, z_4) & \xrightarrow{\varphi} & (z_2, z_3, z_1, z_4) \\
{\scriptstyle g_2}\downarrow & & {\scriptstyle h_1}\downarrow & & {\scriptstyle g_4}\downarrow & & {\scriptstyle h_4}\downarrow \\
(z_1, -z_2, \bar{z}_3, z_4) & \xrightarrow{\varphi} & (-z_2, \bar{z}_3, z_1, z_4) & & (z_1, z_2, z_3, -z_4) & \xrightarrow{\varphi} & (z_2, z_3, z_1, -z_4).
\end{array}$$

Therefore $\exists\ \gamma = \left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right) \in \mathbb{A}(n)$ s.t. $\gamma \Gamma_{10} \gamma^{-1} = \Gamma_{a3}$.

• $M(A_{a9}) \approx M(A_{a7})$.
For

$$A_{a9} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad A_{a7} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned}
g_1(z_1, z_2, z_3, z_4) &= (-z_1, \bar{z}_2, z_3, z_4). & h_1h_2(z_1, z_2, z_3, z_4) &= (-z_1, -\bar{z}_2, z_3, z_4) \\
g_2(z_1, z_2, z_3, z_4) &= (z_1, -z_2, \bar{z}_3, z_4). & h_2(z_1, z_2, z_3, z_4) &= (z_1, -z_2, z_3, \bar{z}_4) \\
g_3(z_1, z_2, z_3, z_4) &= (z_1, z_2, -z_3, z_4). & h_3(z_1, z_2, z_3, z_4) &= (z_1, z_2, -z_3, z_4) \\
g_4(z_1, z_2, z_3, z_4) &= (z_1, z_2, z_3, -z_4). & h_4(z_1, z_2, z_3, z_4) &= (z_1, z_2, z_3, -z_4)
\end{aligned}$$

then

$$\Gamma_{a9}=\left\langle \begin{pmatrix}\frac{1}{2}\\0\\0\\0\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\begin{pmatrix}0\\\frac{1}{2}\\0\\0\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&1\end{pmatrix},\right.$$
$$\left.\begin{pmatrix}0\\0\\\frac{1}{2}\\0\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\begin{pmatrix}0\\0\\0\\\frac{1}{2}\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\right\rangle$$

$$\Gamma_{a7}=\left\langle \begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\\0\\0\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\begin{pmatrix}0\\\frac{1}{2}\\0\\0\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix},\right.$$
$$\left.\begin{pmatrix}0\\0\\\frac{1}{2}\\0\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\begin{pmatrix}0\\0\\0\\\frac{1}{2}\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\right\rangle.$$

Let $\varphi(z_1,z_2,z_3,z_4)=(z_1,iz_2,z_4,z_3)$, we get these commutative diagrams

$$\begin{tikzcd}
(z_1,z_2,z_3,z_4) \arrow[r, "\varphi"] \arrow[d, "g_1"'] & (z_1,iz_2,z_4,z_3) \arrow[d, "h_1h_2"'] \\
(-z_1,\bar{z}_2,z_3,z_4) \arrow[r, "\varphi"] & (-z_1,-i\bar{z}_2,z_4,z_3)
\end{tikzcd}
\quad
\begin{tikzcd}
(z_1,z_2,z_3,z_4) \arrow[r, "\varphi"] \arrow[d, "g_3"'] & (z_1,iz_2,z_4,z_3) \arrow[d, "h_4"'] \\
(z_1,z_2,-z_3,z_4) \arrow[r, "\varphi"] & (z_1,iz_2,z_4,-z_3)
\end{tikzcd}$$

$$\begin{tikzcd}
(z_1,z_2,z_3,z_4) \arrow[r, "\varphi"] \arrow[d, "g_2"'] & (z_1,iz_2,z_4,z_3) \arrow[d, "h_2"'] \\
(z_1,-z_2,\bar{z}_3,z_4) \arrow[r, "\varphi"] & (z_1,-iz_2,z_4,\bar{z}_3)
\end{tikzcd}
\quad
\begin{tikzcd}
(z_1,z_2,z_3,z_4) \arrow[r, "\varphi"] \arrow[d, "g_4"'] & (z_1,iz_2,z_4,z_3) \arrow[d, "h_3"'] \\
(z_1,z_2,z_3,-z_4) \arrow[r, "\varphi"] & (z_1,iz_2,-z_4,z_3).
\end{tikzcd}$$

Therefore $\exists\ \gamma=\left(\begin{pmatrix}0\\\frac{1}{4}\\0\\0\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\right)\in\mathbb{A}(n)$ s.t. $\gamma\Gamma_{a9}\gamma^{-1}=\Gamma_{a7}$.

• $M(A_{a9})\approx M(A_{a25})$.

For

$$A_{a9}=\begin{pmatrix}1&1&0&0\\0&1&1&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\qquad A_{a25}=\begin{pmatrix}1&1&1&0\\0&1&1&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

$g_1(z_1,z_2,z_3,z_4)=(-z_1,\bar{z}_2,z_3,z_4)$. $\quad h_1h_2(z_1,z_2,z_3,z_4)=(-z_1,-\bar{z}_2,z_3,z_4)$

$g_2(z_1,z_2,z_3,z_4)=(z_1,-z_2,\bar{z}_3,z_4)$. $\quad h_2(z_1,z_2,z_3,z_4)=(z_1,-z_2,\bar{z}_3,z_4)$

$g_3(z_1,z_2,z_3,z_4)=(z_1,z_2,-z_3,z_4)$. $\quad h_3(z_1,z_2,z_3,z_4)=(z_1,z_2,-z_3,z_4)$

$g_4(z_1,z_2,z_3,z_4)=(z_1,z_2,z_3,-z_4)$. $\quad h_4(z_1,z_2,z_3,z_4)=(z_1,z_2,z_3,-z_4)$

then

$$\Gamma_{a9} = \left\langle \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\rangle$$

$$\Gamma_{a25} = \left\langle \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\rangle.$$

Let $\varphi(z_1, z_2, z_3, z_4) = (z_1, iz_2, z_3, z_4)$, we get these commutative diagrams

$$\begin{array}{ccccccc} (z_1, z_2, z_3, z_4) & \xrightarrow{\varphi} & (z_1, iz_2, z_3, z_4) & & (z_1, z_2, z_3, z_4) & \xrightarrow{\varphi} & (z_1, iz_2, z_3, z_4) \\ {\scriptstyle g_1}\Big\downarrow & & {\scriptstyle h_1 h_2}\Big\downarrow & & {\scriptstyle g_3}\Big\downarrow & & {\scriptstyle h_3}\Big\downarrow \\ (-z_1, \bar{z}_2, z_3, z_4) & \xrightarrow{\varphi} & (-z_1, -\bar{iz}_2, z_3, z_4) & & (z_1, z_2, -z_3, z_4) & \xrightarrow{\varphi} & (z_1, iz_2, -z_3, z_4) \end{array}$$

$$\begin{array}{ccccccc} (z_1, z_2, z_3, z_4) & \xrightarrow{\varphi} & (z_1, iz_2, z_3, z_4) & & (z_1, z_2, z_3, z_4) & \xrightarrow{\varphi} & (z_1, iz_2, z_3, z_4) \\ {\scriptstyle g_2}\Big\downarrow & & {\scriptstyle h_2}\Big\downarrow & & {\scriptstyle g_4}\Big\downarrow & & {\scriptstyle h_4}\Big\downarrow \\ (z_1, -z_2, \bar{z}_3, z_4) & \xrightarrow{\varphi} & (z_1, -iz_2, \bar{z}_3, z_4) & & (z_1, z_2, z_3, -z_4) & \xrightarrow{\varphi} & (z_1, iz_2, z_3, -z_4). \end{array}$$

Therefore $\exists\ \gamma = \left(\begin{pmatrix} 0 \\ \frac{1}{4} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right) \in \mathbb{A}(n)$ s.t. $\gamma \Gamma_{a9} \gamma^{-1} = \Gamma_{a25}$.

• $M(A_{10}) \approx M(A_{12})$.
For

$$A_{10} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad A_{12} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned} g_1(z_1,z_2,z_3,z_4) &= (-z_1,z_2,z_3,z_4). & h_1(z_1,z_2,z_3,z_4) &= (-z_1,z_2,z_3,z_4) \\ g_2(z_1,z_2,z_3,z_4) &= (z_1,-z_2,\bar{z}_3,z_4). & h_2h_3(z_1,z_2,z_3,z_4) &= (z_1,-z_2,-\bar{z}_3,z_4) \\ g_3(z_1,z_2,z_3,z_4) &= (z_1,z_2,-z_3,\bar{z}_4). & h_3(z_1,z_2,z_3,z_4) &= (z_1,z_2,-z_3,\bar{z}_4) \\ g_4(z_1,z_2,z_3,z_4) &= (z_1,z_2,z_3,-z_4). & h_4(z_1,z_2,z_3,z_4) &= (z_1,z_2,z_3,-z_4) \end{aligned}$$

then

$$\Gamma_{10} = \Big\langle \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\Big\rangle$$

$$\Gamma_{12} = \Big\langle \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\Big\rangle.$$

Let $\varphi(z_1,z_2,z_3,z_4) = (z_1,z_2,iz_3,z_4)$, we get these commutative diagrams

$$\begin{tikzcd} (z_1,z_2,z_3,z_4) \arrow[r, "\varphi"] \arrow[d, "g_1"'] & (z_1,z_2,iz_3,z_4) \arrow[d, "h_1"'] \\ (-z_1,z_2,z_3,z_4) \arrow[r, "\varphi"] & (-z_1,z_2,iz_3,z_4) \end{tikzcd} \quad \begin{tikzcd} (z_1,z_2,z_3,z_4) \arrow[r, "\varphi"] \arrow[d, "g_3"'] & (z_1,z_2,iz_3,z_4) \arrow[d, "h_3"'] \\ (z_1,z_2,-z_3,\bar{z}_4) \arrow[r, "\varphi"] & (z_1,z_2,-iz_3,\bar{z}_4) \end{tikzcd}$$

$$\begin{tikzcd} (z_1,z_2,z_3,z_4) \arrow[r, "\varphi"] \arrow[d, "g_2"'] & (z_1,z_2,iz_3,z_4) \arrow[d, "h_2h_3"'] \\ (z_1,-z_2,\bar{z}_3,z_4) \arrow[r, "\varphi"] & (z_1,-z_2,-\overline{iz_3},z_4) \end{tikzcd} \quad \begin{tikzcd} (z_1,z_2,z_3,z_4) \arrow[r, "\varphi"] \arrow[d, "g_4"'] & (z_1,z_2,iz_3,z_4) \arrow[d, "h_4"'] \\ (z_1,z_2,z_3,-z_4) \arrow[r, "\varphi"] & (z_1,z_2,iz_3,-z_4). \end{tikzcd}$$

Therefore $\exists\ \gamma = \Big(\begin{pmatrix} 0 \\ 0 \\ \frac{1}{4} \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\Big) \in \mathbb{A}(n)$ s.t. $\gamma\Gamma_{10}\gamma^{-1} = \Gamma_{12}$.

• $M(A_{10}) \approx M(A_{18})$.
For

$$A_{10} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad A_{18} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned}
g_1(z_1,z_2,z_3,z_4) &= (-z_1,z_2,z_3,z_4). & h_1(z_1,z_2,z_3,z_4) &= (-z_1,z_2,\bar{z}_3,z_4) \\
g_2(z_1,z_2,z_3,z_4) &= (z_1,-z_2,\bar{z}_3,z_4). & h_2(z_1,z_2,z_3,z_4) &= (z_1,-z_2,z_3,z_4) \\
g_3(z_1,z_2,z_3,z_4) &= (z_1,z_2,-z_3,\bar{z}_4). & h_3(z_1,z_2,z_3,z_4) &= (z_1,z_2,-z_3,\bar{z}_4) \\
g_4(z_1,z_2,z_3,z_4) &= (z_1,z_2,z_3,-z_4). & h_4(z_1,z_2,z_3,z_4) &= (z_1,z_2,z_3,-z_4)
\end{aligned}$$

then

$$\Gamma_{10} = \Big\langle \left(\begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right), \left(\begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right), \left(\begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\right), \left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right) \Big\rangle$$

$$\Gamma_{18} = \Big\langle \left(\begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right), \left(\begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right), \left(\begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\right), \left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right) \Big\rangle.$$

Let $\varphi(z_1,z_2,z_3,z_4) = (z_2,z_1,z_3,z_4)$, we get these commutative diagrams

$$\begin{array}{ccccccc}
(z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_2,z_1,z_3,z_4) & & (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_2,z_1,z_3,z_4) \\
{\scriptstyle g_1}\downarrow & & {\scriptstyle h_2}\downarrow & & {\scriptstyle g_3}\downarrow & & {\scriptstyle h_3}\downarrow \\
(-z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_2,-z_1,z_3,z_4) & & (z_1,z_2,-z_3,\bar{z}_4) & \xrightarrow{\varphi} & (z_2,z_1,-z_3,\bar{z}_4) \\
(z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_2,z_1,z_3,z_4) & & (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_2,z_1,z_3,z_4) \\
{\scriptstyle g_2}\downarrow & & {\scriptstyle h_1}\downarrow & & {\scriptstyle g_4}\downarrow & & {\scriptstyle h_4}\downarrow \\
(z_1,-z_2,\bar{z}_3,z_4) & \xrightarrow{\varphi} & (-z_2,z_1,\bar{z}_3,z_4) & & (z_1,z_2,z_3,-z_4) & \xrightarrow{\varphi} & (z_2,z_1,z_3,-z_4).
\end{array}$$

Therefore $\exists\ \gamma = \left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right) \in \mathbb{A}(n)$ s.t. $\gamma\Gamma_{10}\gamma^{-1} = \Gamma_{18}$.

• $M(A_{10}) \approx M(A_{22})$.
For

$$A_{10} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad A_{22} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned} g_1(z_1,z_2,z_3,z_4) &= (-z_1,z_2,z_3,z_4). & h_1h_3(z_1,z_2,z_3,z_4) &= (-z_1,z_2,-\bar{z}_3,z_4) \\ g_2(z_1,z_2,z_3,z_4) &= (z_1,-z_2,\bar{z}_3,z_4). & h_2(z_1,z_2,z_3,z_4) &= (z_1,-z_2,z_3,z_4) \\ g_3(z_1,z_2,z_3,z_4) &= (z_1,z_2,-z_3,\bar{z}_4). & h_3(z_1,z_2,z_3,z_4) &= (z_1,z_2,-z_3,\bar{z}_4) \\ g_4(z_1,z_2,z_3,z_4) &= (z_1,z_2,z_3,-z_4). & h_4(z_1,z_2,z_3,z_4) &= (z_1,z_2,z_3,-z_4) \end{aligned}$$

then

$$\begin{aligned} \Gamma_{10} = \Big\langle & \left(\begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right), \left(\begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right), \\ & \left(\begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\right), \left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right) \Big\rangle \end{aligned}$$

$$\begin{aligned} \Gamma_{22} = \Big\langle & \left(\begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right), \left(\begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right), \\ & \left(\begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\right), \left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right) \Big\rangle. \end{aligned}$$

Let $\varphi(z_1,z_2,z_3,z_4) = (z_2,z_1,iz_3,z_4)$, we get these commutative diagrams

$$\begin{tikzcd} (z_1,z_2,z_3,z_4) \arrow[r, "\varphi"] \arrow[d, "g_1"'] & (z_2,z_1,iz_3,z_4) \arrow[d, "h_2"'] \\ (-z_1,z_2,z_3,z_4) \arrow[r, "\varphi"] & (z_2,-z_1,iz_3,z_4) \end{tikzcd} \quad \begin{tikzcd} (z_1,z_2,z_3,z_4) \arrow[r, "\varphi"] \arrow[d, "g_3"'] & (z_2,z_1,iz_3,z_4) \arrow[d, "h_3"'] \\ (z_1,z_2,-z_3,\bar{z}_4) \arrow[r, "\varphi"] & (z_2,z_1,-iz_3,\bar{z}_4) \end{tikzcd}$$

$$\begin{tikzcd} (z_1,z_2,z_3,z_4) \arrow[r, "\varphi"] \arrow[d, "g_2"'] & (z_2,z_1,iz_3,z_4) \arrow[d, "h_1h_3"'] \\ (z_1,-z_2,\bar{z}_3,z_4) \arrow[r, "\varphi"] & (-z_2,z_1,-\overline{iz}_3,z_4) \end{tikzcd} \quad \begin{tikzcd} (z_1,z_2,z_3,z_4) \arrow[r, "\varphi"] \arrow[d, "g_4"'] & (z_2,z_1,iz_3,z_4) \arrow[d, "h_4"'] \\ (z_1,z_2,z_3,-z_4) \arrow[r, "\varphi"] & (z_2,z_1,iz_3,-z_4). \end{tikzcd}$$

Therefore $\exists\ \gamma = \left(\begin{pmatrix} 0 \\ 0 \\ \frac{1}{4} \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right) \in \mathbb{A}(n)$ s.t. $\gamma\Gamma_{10}\gamma^{-1} = \Gamma_{22}$.

• $M(A_{10}) \approx M(A_{26})$.
For

$$A_{10} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad A_{26} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned} g_1(z_1, z_2, z_3, z_4) &= (-z_1, z_2, z_3, z_4). & h_1h_2(z_1, z_2, z_3, z_4) &= (-z_1, -z_2, z_3, z_4) \\ g_2(z_1, z_2, z_3, z_4) &= (z_1, -z_2, \bar{z}_3, z_4). & h_2(z_1, z_2, z_3, z_4) &= (z_1, -z_2, \bar{z}_3, z_4) \\ g_3(z_1, z_2, z_3, z_4) &= (z_1, z_2, -z_3, \bar{z}_4). & h_3(z_1, z_2, z_3, z_4) &= (z_1, z_2, -z_3, \bar{z}_4) \\ g_4(z_1, z_2, z_3, z_4) &= (z_1, z_2, z_3, -z_4). & h_4(z_1, z_2, z_3, z_4) &= (z_1, z_2, z_3, -z_4) \end{aligned}$$

then

$$\Gamma_{10} = \Big\langle \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Big\rangle$$

$$\Gamma_{26} = \Big\langle \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Big\rangle.$$

Let $\varphi(z_1, z_2, z_3, z_4) = (z_1, z_1z_2, z_3, z_4)$, we get these commtative diagrams

$$\begin{tikzcd} (z_1, z_2, z_3, z_4) \arrow[r, "\varphi"] \arrow[d, "g_1"'] & (z_1, z_1z_2, z_3, z_4) \arrow[d, "h_1h_2"'] & (z_1, z_2, z_3, z_4) \arrow[r, "\varphi"] \arrow[d, "g_3"'] & (z_1, z_1z_2, z_3, z_4) \arrow[d, "h_3"'] \\ (-z_1, z_2, z_3, z_4) \arrow[r, "\varphi"] & (-z_1, -z_1z_2, z_3, z_4) & (z_1, z_2, -z_3, \bar{z}_4) \arrow[r, "\varphi"] & (z_1, z_1z_2, -z_3, \bar{z}_4) \end{tikzcd}$$

$$\begin{tikzcd} (z_1, z_2, z_3, z_4) \arrow[r, "\varphi"] \arrow[d, "g_2"'] & (z_1, z_1z_2, z_3, z_4) \arrow[d, "h_2"'] & (z_1, z_2, z_3, z_4) \arrow[r, "\varphi"] \arrow[d, "g_4"'] & (z_1, z_1z_2, z_3, z_4) \arrow[d, "h_4"'] \\ (z_1, -z_2, \bar{z}_3, z_4) \arrow[r, "\varphi"] & (z_1, -z_1z_2, \bar{z}_3, z_4) & (z_1, z_2, z_3, -z_4) \arrow[r, "\varphi"] & (z_1, z_1z_2, z_3, -z_4). \end{tikzcd}$$

Therefore $\exists\ \gamma = \Big(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Big) \in \mathbb{A}(n)$ s.t. $\gamma\Gamma_{10}\gamma^{-1} = \Gamma_{26}$.

• $M(A_{12}) \approx M(A_{32})$.
For

$$A_{12} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad A_{32} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned}
g_1(z_1,z_2,z_3,z_4) &= (-z_1,z_2,z_3,z_4). & h_1h_2(z_1,z_2,z_3,z_4) &= (-z_1,-z_2,z_3,z_4)\\
g_2(z_1,z_2,z_3,z_4) &= (z_1,-z_2,\bar{z}_3,\bar{z}_4). & h_2(z_1,z_2,z_3,z_4) &= (z_1,-z_2,\bar{z}_3,\bar{z}_4)\\
g_3(z_1,z_2,z_3,z_4) &= (z_1,z_2,-z_3,\bar{z}_4). & h_3(z_1,z_2,z_3,z_4) &= (z_1,z_2,-z_3,\bar{z}_4)\\
g_4(z_1,z_2,z_3,z_4) &= (z_1,z_2,z_3,-z_4). & h_4(z_1,z_2,z_3,z_4) &= (z_1,z_2,z_3,-z_4)
\end{aligned}$$

then

$$\Gamma_{12} = \Big\langle \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\Big\rangle$$

$$\Gamma_{32} = \Big\langle \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\Big\rangle.$$

Let $\varphi(z_1,z_2,z_3,z_4) = (z_1,z_1z_2,z_3,z_4)$, we get these commutative diagrams

$$\begin{array}{ccccccc}
(z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1,z_1z_2,z_3,z_4) & & (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1,z_1z_2,z_3,z_4) \\
{\scriptstyle g_1}\downarrow & & {\scriptstyle h_1h_2}\downarrow & & {\scriptstyle g_3}\downarrow & & {\scriptstyle h_3}\downarrow \\
(-z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (-z_1,-z_1z_2,z_3,z_4) & & (z_1,z_2,-z_3,\bar{z}_4) & \xrightarrow{\varphi} & (z_1,z_1z_2,-z_3,\bar{z}_4) \\
(z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1,z_1z_2,z_3,z_4) & & (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1,z_1z_2,z_3,z_4) \\
{\scriptstyle g_2}\downarrow & & {\scriptstyle h_2}\downarrow & & {\scriptstyle g_4}\downarrow & & {\scriptstyle h_4}\downarrow \\
(z_1,-z_2,\bar{z}_3,\bar{z}_4) & \xrightarrow{\varphi} & (z_1,-z_1z_2,\bar{z}_3,\bar{z}_4) & & (z_1,z_2,z_3,-z_4) & \xrightarrow{\varphi} & (z_1,z_1z_2,z_3,-z_4).
\end{array}$$

Therefore $\exists\ \gamma = \left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right) \in \mathbb{A}(n)$ s.t. $\gamma\Gamma_{12}\gamma^{-1} = \Gamma_{32}$.

(10). Similar to (4).

(11). We know that
$\Phi_{a15} = \left\langle I, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = P, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = Q \right\rangle$ and $\Phi_{a3} = \left\langle I, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = R, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = S \right\rangle$. If $M(A_{a15})$ is diffeomorphic to $M(A_{a3})$ then $\exists\ B \in GL(4, \mathbb{R})$ such that

$$BPB^{-1} = \begin{cases} R & (i) \\ S & (ii) \\ RS & (iii) \end{cases}$$

and

$$BQB^{-1} = \begin{cases} R & (iv) \\ S & (v) \\ RS & (vi). \end{cases}$$

By calculating the eigenvalues of cases (i), (ii),(iv) and (v), then we get a contradiction, and from the cases (iii) and (vi) we get $P = Q$ which is also a contradiction.

(12).
• $M(A_{a4}) \approx M(A_{a8})$.
For

$$A_{a4} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad A_{a8} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{array}{ll} g_1(z_1, z_2, z_3, z_4) = (-z_1, \bar{z}_2, z_3, z_4). & h_1h_2(z_1, z_2, z_3, z_4) = (-z_1, -\bar{z}_2, z_3, z_4) \\ g_2(z_1, z_2, z_3, z_4) = (z_1, -z_2, z_3, \bar{z}_4). & h_2(z_1, z_2, z_3, z_4) = (z_1, -z_2, z_3, \bar{z}_4) \\ g_3(z_1, z_2, z_3, z_4) = (z_1, z_2, -z_3, \bar{z}_4). & h_3(z_1, z_2, z_3, z_4) = (z_1, z_2, -z_3, \bar{z}_4) \\ g_4(z_1, z_2, z_3, z_4) = (z_1, z_2, z_3, -z_4). & h_4(z_1, z_2, z_3, z_4) = (z_1, z_2, z_3, -z_4) \end{array}$$

then

$$\Gamma_{a4} = \left\langle \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\rangle$$

$$\Gamma_{a8} = \left\langle \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\rangle.$$

Let $\varphi(z_1, z_2, z_3, z_4) = (z_1, iz_2, z_3, z_4)$, we get these commutative diagrams

$$\begin{array}{ccccccc}
(z_1, z_2, z_3, z_4) & \xrightarrow{\varphi} & (z_1, iz_2, z_3, z_4) & & (z_1, z_2, z_3, z_4) & \xrightarrow{\varphi} & (z_1, iz_2, z_3, z_4) \\
{\scriptstyle g_1}\downarrow & & {\scriptstyle h_1h_2}\downarrow & & {\scriptstyle g_3}\downarrow & & {\scriptstyle h_3}\downarrow \\
(-z_1, \bar{z}_2, z_3, z_4) & \xrightarrow{\varphi} & (-z_1, -\overline{iz}_2, z_3, z_4) & & (z_1, z_2, -z_3, \bar{z}_4) & \xrightarrow{\varphi} & (z_1, iz_2, -z_3, \bar{z}_4)
\end{array}$$

$$\begin{array}{ccccccc}
(z_1, z_2, z_3, z_4) & \xrightarrow{\varphi} & (z_1, iz_2, z_3, z_4) & & (z_1, z_2, z_3, z_4) & \xrightarrow{\varphi} & (z_1, iz_2, z_3, z_4) \\
{\scriptstyle g_2}\downarrow & & {\scriptstyle h_2}\downarrow & & {\scriptstyle g_4}\downarrow & & {\scriptstyle h_4}\downarrow \\
(z_1, -z_2, z_3, \bar{z}_4) & \xrightarrow{\varphi} & (z_1, -iz_2, z_3, \bar{z}_4) & & (z_1, z_2, z_3, -z_4) & \xrightarrow{\varphi} & (z_1, iz_2, z_3, -z_4).
\end{array}$$

Therefore $\exists\ \gamma = \left(\begin{pmatrix} 0 \\ \frac{1}{4} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right) \in \mathbb{A}(n)$ s.t. $\gamma \Gamma_{a4} \gamma^{-1} = \Gamma_{a8}$.

• $M(A_{a4}) \approx M(A_{14})$.
For

$$A_{a4} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad A_{14} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{array}{ll}
g_1(z_1, z_2, z_3, z_4) = (-z_1, \bar{z}_2, z_3, z_4). & h_1(z_1, z_2, z_3, z_4) = (-z_1, z_2, z_3, \bar{z}_4) \\
g_2(z_1, z_2, z_3, z_4) = (z_1, -z_2, z_3, \bar{z}_4). & h_2(z_1, z_2, z_3, z_4) = (z_1, -z_2, \bar{z}_3, z_4) \\
g_3(z_1, z_2, z_3, z_4) = (z_1, z_2, -z_3, \bar{z}_4). & h_3(z_1, z_2, z_3, z_4) = (z_1, z_2, -z_3, \bar{z}_4) \\
g_4(z_1, z_2, z_3, z_4) = (z_1, z_2, z_3, -z_4). & h_4(z_1, z_2, z_3, z_4) = (z_1, z_2, z_3, -z_4)
\end{array}$$

then

$$\Gamma_{a4} = \left\langle \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\rangle$$

$$\Gamma_{14} = \left\langle \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\rangle.$$

Let $\varphi(z_1, z_2, z_3, z_4) = (z_3, z_1, z_2, z_4)$, we get these commutative diagrams

$$\begin{array}{ccccccc}
(z_1, z_2, z_3, z_4) & \xrightarrow{\varphi} & (z_3, z_1, z_2, z_4) & & (z_1, z_2, z_3, z_4) & \xrightarrow{\varphi} & (z_3, z_1, z_2, z_4) \\
g_1 \downarrow & & h_2 \downarrow & & g_3 \downarrow & & h_1 \downarrow \\
(-z_1, \bar{z}_2, z_3, z_4) & \xrightarrow{\varphi} & (z_3, -z_1, \bar{z}_2, z_4) & & (z_1, z_2, -z_3, \bar{z}_4) & \xrightarrow{\varphi} & (-z_3, z_1, z_2, \bar{z}_4) \\
(z_1, z_2, z_3, z_4) & \xrightarrow{\varphi} & (z_3, z_1, z_2, z_4) & & (z_1, z_2, z_3, z_4) & \xrightarrow{\varphi} & (z_3, z_1, z_2, z_4) \\
g_2 \downarrow & & h_3 \downarrow & & g_4 \downarrow & & h_4 \downarrow \\
(z_1, -z_2, z_3, \bar{z}_4) & \xrightarrow{\varphi} & (z_3, z_1, -z_2, \bar{z}_4) & & (z_1, z_2, z_3, -z_4) & \xrightarrow{\varphi} & (z_3, z_1, z_2, -z_4).
\end{array}$$

Therefore $\exists\ \gamma = \left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right) \in \mathbb{A}(n)$ s.t. $\gamma \Gamma_{a4} \gamma^{-1} = \Gamma_{14}$.

• $M(A_{14}) \approx M(A_{16})$.
For

$$A_{14} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad A_{16} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{array}{rr}
g_1(z_1, z_2, z_3, z_4) = (-z_1, z_2, z_3, \bar{z}_4). & h_1(z_1, z_2, z_3, z_4) = (-z_1, z_2, z_3, \bar{z}_4) \\
g_2(z_1, z_2, z_3, z_4) = (z_1, -z_2, \bar{z}_3, z_4). & h_2 h_3(z_1, z_2, z_3, z_4) = (z_1, -z_2, -\bar{z}_3, z_4) \\
g_3(z_1, z_2, z_3, z_4) = (z_1, z_2, -z_3, \bar{z}_4). & h_3(z_1, z_2, z_3, z_4) = (z_1, z_2, -z_3, \bar{z}_4) \\
g_4(z_1, z_2, z_3, z_4) = (z_1, z_2, z_3, -z_4). & h_4(z_1, z_2, z_3, z_4) = (z_1, z_2, z_3, -z_4)
\end{array}$$

then

$$\Gamma_{14} = \Big\langle \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$
$$\begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Big\rangle$$

$$\Gamma_{16} = \Big\langle \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$
$$\begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Big\rangle.$$

Let $\varphi(z_1, z_2, z_3, z_4) = (z_1, z_2, iz_3, z_4)$, we get these commutative diagrams

$$\begin{array}{ccc} (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1,z_2,iz_3,z_4) \\ g_1\big\downarrow & & h_1\big\downarrow \\ (-z_1,z_2,z_3,\bar{z}_4) & \xrightarrow{\varphi} & (-z_1,z_2,iz_3,\bar{z}_4) \end{array} \quad \begin{array}{ccc} (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1,z_2,iz_3,z_4) \\ g_3\big\downarrow & & h_3\big\downarrow \\ (z_1,z_2,-z_3,\bar{z}_4) & \xrightarrow{\varphi} & (z_1,z_2,-iz_3,\bar{z}_4) \end{array}$$

$$\begin{array}{ccc} (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1,z_2,iz_3,z_4) \\ g_2\big\downarrow & & h_2h_3\big\downarrow \\ (z_1,-z_2,\bar{z}_3,z_4) & \xrightarrow{\varphi} & (z_1,-z_2,-i\bar{z}_3,z_4) \end{array} \quad \begin{array}{ccc} (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1,z_2,iz_3,z_4) \\ g_4\big\downarrow & & h_4\big\downarrow \\ (z_1,z_2,z_3,-z_4) & \xrightarrow{\varphi} & (z_1,z_2,iz_3,-z_4). \end{array}$$

Therefore $\exists\ \gamma = \left(\begin{pmatrix} 0 \\ 0 \\ \frac{1}{4} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right) \in \mathbb{A}(n)$ s.t. $\gamma\Gamma_{14}\gamma^{-1} = \Gamma_{16}$.

• $M(A_{14}) \approx M(A_{24})$.
For

$$A_{14} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad A_{24} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$g_1(z_1, z_2, z_3, z_4) = (-z_1, z_2, z_3, \bar{z}_4)$. $\quad h_1h_2(z_1, z_2, z_3, z_4) = (-z_1, -z_2, \bar{z}_3, z_4)$
$g_2(z_1, z_2, z_3, z_4) = (z_1, -z_2, \bar{z}_3, z_4)$. $\quad h_2(z_1, z_2, z_3, z_4) = (z_1, -z_2, z_3, \bar{z}_4)$
$g_3(z_1, z_2, z_3, z_4) = (z_1, z_2, -z_3, \bar{z}_4)$. $\quad h_3(z_1, z_2, z_3, z_4) = (z_1, z_2, -z_3, \bar{z}_4)$
$g_4(z_1, z_2, z_3, z_4) = (z_1, z_2, z_3, -z_4)$. $\quad h_4(z_1, z_2, z_3, z_4) = (z_1, z_2, z_3, -z_4)$

then

$$\Gamma_{14}=\Big\langle \begin{pmatrix}\frac{1}{2}\\0\\0\\0\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix},\begin{pmatrix}0\\\frac{1}{2}\\0\\0\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&1\end{pmatrix},$$
$$\begin{pmatrix}0\\0\\\frac{1}{2}\\0\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix},\begin{pmatrix}0\\0\\0\\\frac{1}{2}\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\Big\rangle$$

$$\Gamma_{24}=\Big\langle \begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\\0\\0\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&1\end{pmatrix},\begin{pmatrix}0\\\frac{1}{2}\\0\\0\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix},$$
$$\begin{pmatrix}0\\0\\\frac{1}{2}\\0\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix},\begin{pmatrix}0\\0\\0\\\frac{1}{2}\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\Big\rangle.$$

Let $\varphi(z_1,z_2,z_3,z_4)=(z_2,z_1z_2,z_3,z_4)$, we get these commutative diagrams

$$\begin{array}{ccccccc}
(z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_2,z_1z_2,z_3,z_4) & (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_2,z_1z_2,z_3,z_4)\\
g_1\big\downarrow & & h_2\big\downarrow & g_3\big\downarrow & & h_3\big\downarrow\\
(-z_1,z_2,z_3,\bar z_4) & \xrightarrow{\varphi} & (z_2,-z_1z_2,z_3,\bar z_4) & (z_1,z_2,-z_3,\bar z_4) & \xrightarrow{\varphi} & (z_2,z_1z_2,-z_3,\bar z_4)\\
(z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_2,z_1z_2,z_3,z_4) & (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_2,z_1z_2,z_3,z_4)\\
g_2\big\downarrow & & h_1h_2\big\downarrow & g_4\big\downarrow & & h_4\big\downarrow\\
(z_1,-z_2,\bar z_3,z_4) & \xrightarrow{\varphi} & (-z_2,-z_1z_2,\bar z_3,z_4) & (z_1,z_2,z_3,-z_4) & \xrightarrow{\varphi} & (z_2,z_1z_2,z_3,-z_4).
\end{array}$$

Therefore $\exists\ \gamma=\left(\begin{pmatrix}0\\0\\0\\0\end{pmatrix}\begin{pmatrix}0&1&0&0\\1&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\right)\in\mathbb{A}(n)$ s.t. $\gamma\Gamma_{14}\gamma^{-1}=\Gamma_{24}$.

• $M(A_{20})\approx M(A_{24})$.
For

$$A_{20}=\begin{pmatrix}1&0&1&0\\0&1&0&1\\0&0&1&1\\0&0&0&1\end{pmatrix}\qquad A_{24}=\begin{pmatrix}1&0&1&1\\0&1&0&1\\0&0&1&1\\0&0&0&1\end{pmatrix}$$

$g_1(z_1,z_2,z_3,z_4)=(-z_1,z_2,\bar z_3,z_4)$. $\quad h_1h_3(z_1,z_2,z_3,z_4)=(-z_1,z_2,-\bar z_3,z_4)$
$g_2(z_1,z_2,z_3,z_4)=(z_1,-z_2,z_3,\bar z_4)$. $\quad h_2(z_1,z_2,z_3,z_4)=(z_1,-z_2,z_3,\bar z_4)$
$g_3(z_1,z_2,z_3,z_4)=(z_1,z_2,-z_3,\bar z_4)$. $\quad h_3(z_1,z_2,z_3,z_4)=(z_1,z_2,-z_3,\bar z_4)$
$g_4(z_1,z_2,z_3,z_4)=(z_1,z_2,z_3,-z_4)$. $\quad h_4(z_1,z_2,z_3,z_4)=(z_1,z_2,z_3,-z_4)$

then

$$\Gamma_{20} = \left\langle \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\rangle$$

$$\Gamma_{24} = \left\langle \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\rangle.$$

Let $\varphi(z_1, z_2, z_3, z_4) = (z_1, z_2, iz_3, z_4)$, we get these commutative diagrams

$$\begin{array}{ccccccc}
(z_1, z_2, z_3, z_4) & \xrightarrow{\varphi} & (z_1, z_2, iz_3, z_4) & & (z_1, z_2, z_3, z_4) & \xrightarrow{\varphi} & (z_1, z_2, iz_3, z_4) \\
{\scriptstyle g_1}\downarrow & & {\scriptstyle h_1h_3}\downarrow & & {\scriptstyle g_3}\downarrow & & {\scriptstyle h_3}\downarrow \\
(-z_1, z_2, \bar{z}_3, z_4) & \xrightarrow{\varphi} & (-z_1, z_2, -i\bar{z}_3, z_4) & & (z_1, z_2, -z_3, \bar{z}_4) & \xrightarrow{\varphi} & (z_1, z_2, -iz_3, \bar{z}_4)
\end{array}$$

$$\begin{array}{ccccccc}
(z_1, z_2, z_3, z_4) & \xrightarrow{\varphi} & (z_1, z_2, iz_3, z_4) & & (z_1, z_2, z_3, z_4) & \xrightarrow{\varphi} & (z_1, z_2, iz_3, z_4) \\
{\scriptstyle g_2}\downarrow & & {\scriptstyle h_2}\downarrow & & {\scriptstyle g_4}\downarrow & & {\scriptstyle h_4}\downarrow \\
(z_1, -z_2, z_3, \bar{z}_4) & \xrightarrow{\varphi} & (z_1, -z_2, iz_3, \bar{z}_4) & & (z_1, z_2, z_3, -z_4) & \xrightarrow{\varphi} & (z_1, z_2, iz_3, -z_4).
\end{array}$$

Therefore $\exists\ \gamma = \left(\begin{pmatrix} 0 \\ 0 \\ \frac{1}{4} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right) \in \mathbb{A}(n)$ s.t. $\gamma\Gamma_{20}\gamma^{-1} = \Gamma_{24}$.

• $M(A_{14}) \approx M(A_{30})$.
For

$$A_{14} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad A_{30} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{array}{ll}
g_1(z_1, z_2, z_3, z_4) = (-z_1, z_2, z_3, \bar{z}_4). & h_1h_2(z_1, z_2, z_3, z_4) = (-z_1, -z_2, z_3, \bar{z}_4) \\
g_2(z_1, z_2, z_3, z_4) = (z_1, -z_2, \bar{z}_3, z_4). & h_2(z_1, z_2, z_3, z_4) = (z_1, -z_2, \bar{z}_3, z_4) \\
g_3(z_1, z_2, z_3, z_4) = (z_1, z_2, -z_3, \bar{z}_4). & h_3(z_1, z_2, z_3, z_4) = (z_1, z_2, -z_3, \bar{z}_4) \\
g_4(z_1, z_2, z_3, z_4) = (z_1, z_2, z_3, -z_4). & h_4(z_1, z_2, z_3, z_4) = (z_1, z_2, z_3, -z_4)
\end{array}$$

then

$$\Gamma_{14} = \left\langle \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\rangle$$

$$\Gamma_{30} = \left\langle \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\rangle.$$

Let $\varphi(z_1, z_2, z_3, z_4) = (z_1, z_1 z_2, z_3, z_4)$, we get these commutative diagrams

$$\begin{array}{ccccccc}
(z_1, z_2, z_3, z_4) & \xrightarrow{\varphi} & (z_1, z_1 z_2, z_3, z_4) & & (z_1, z_2, z_3, z_4) & \xrightarrow{\varphi} & (z_1, z_1 z_2, z_3, z_4) \\
{\scriptstyle g_1}\downarrow & & {\scriptstyle h_1 h_2}\downarrow & & {\scriptstyle g_3}\downarrow & & {\scriptstyle h_3}\downarrow \\
(-z_1, z_2, z_3, \bar{z}_4) & \xrightarrow{\varphi} & (-z_1, -z_1 z_2, z_3, \bar{z}_4) & & (z_1, z_2, -z_3, \bar{z}_4) & \xrightarrow{\varphi} & (z_1, z_1 z_2, -z_3, \bar{z}_4) \\
(z_1, z_2, z_3, z_4) & \xrightarrow{\varphi} & (z_1, z_1 z_2, z_3, z_4) & & (z_1, z_2, z_3, z_4) & \xrightarrow{\varphi} & (z_1, z_1 z_2, z_3, z_4) \\
{\scriptstyle g_2}\downarrow & & {\scriptstyle h_2}\downarrow & & {\scriptstyle g_4}\downarrow & & {\scriptstyle h_4}\downarrow \\
(z_1, -z_2, \bar{z}_3, z_4) & \xrightarrow{\varphi} & (z_1, -z_1 z_2, \bar{z}_3, z_4) & & (z_1, z_2, z_3, -z_4) & \xrightarrow{\varphi} & (z_1, z_1 z_2, z_3, -z_4).
\end{array}$$

Therefore $\exists\ \gamma = \left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right) \in \mathbb{A}(n)$ s.t. $\gamma \Gamma_{14} \gamma^{-1} = \Gamma_{30}$.

• $M(A_{30}) \approx M(A_{28})$.
For

$$A_{30} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad A_{28} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{array}{ll}
g_1(z_1, z_2, z_3, z_4) = (-z_1, z_2, \bar{z}_3, \bar{z}_4). & h_1(z_1, z_2, z_3, z_4) = (-z_1, z_2, \bar{z}_3, z_4) \\
g_2(z_1, z_2, z_3, z_4) = (z_1, -z_2, \bar{z}_3, z_4). & h_2(z_1, z_2, z_3, z_4) = (z_1, -z_2, \bar{z}_3, \bar{z}_4) \\
g_3(z_1, z_2, z_3, z_4) = (z_1, z_2, -z_3, \bar{z}_4). & h_3(z_1, z_2, z_3, z_4) = (z_1, z_2, -z_3, \bar{z}_4) \\
g_4(z_1, z_2, z_3, z_4) = (z_1, z_2, z_3, -z_4). & h_4(z_1, z_2, z_3, z_4) = (z_1, z_2, z_3, -z_4)
\end{array}$$

then

$$\Gamma_{30}=\Big\langle \begin{pmatrix}\frac{1}{2}\\0\\0\\0\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}, \begin{pmatrix}0\\\frac{1}{2}\\0\\0\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&1\end{pmatrix},$$
$$\begin{pmatrix}0\\0\\\frac{1}{2}\\0\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}, \begin{pmatrix}0\\0\\0\\\frac{1}{2}\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\Big\rangle$$

$$\Gamma_{28}=\Big\langle \begin{pmatrix}\frac{1}{2}\\0\\0\\0\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix}0\\\frac{1}{2}\\0\\0\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix},$$
$$\begin{pmatrix}0\\0\\\frac{1}{2}\\0\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}, \begin{pmatrix}0\\0\\0\\\frac{1}{2}\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\Big\rangle.$$

Let $\varphi(z_1,z_2,z_3,z_4)=(z_2,z_1,z_3,z_4)$, we get these commutative diagrams

$$\begin{array}{ccccccc}
(z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_2,z_1,z_3,z_4) & & (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_2,z_1,z_3,z_4)\\
{\scriptstyle g_1}\downarrow & & {\scriptstyle h_2}\downarrow & & {\scriptstyle g_3}\downarrow & & {\scriptstyle h_3}\downarrow\\
(-z_1,z_2,\bar z_3,\bar z_4) & \xrightarrow{\varphi} & (z_2,-z_1,\bar z_3,\bar z_4) & & (z_1,z_2,-z_3,\bar z_4) & \xrightarrow{\varphi} & (z_2,z_1,-z_3,\bar z_4)
\end{array}$$

$$\begin{array}{ccccccc}
(z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_2,z_1,z_3,z_4) & & (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_2,z_1,z_3,z_4)\\
{\scriptstyle g_2}\downarrow & & {\scriptstyle h_1}\downarrow & & {\scriptstyle g_4}\downarrow & & {\scriptstyle h_4}\downarrow\\
(z_1,-z_2,\bar z_3,z_4) & \xrightarrow{\varphi} & (-z_2,z_1,\bar z_3,z_4) & & (z_1,z_2,z_3,-z_4) & \xrightarrow{\varphi} & (z_2,z_1,z_3,-z_4).
\end{array}$$

Therefore $\exists\ \gamma=\Big(\begin{pmatrix}0\\0\\0\\0\end{pmatrix}\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\Big)\in\mathbb{A}(n)$ s.t. $\gamma\Gamma_{30}\gamma^{-1}=\Gamma_{28}$.

(13). Similar to (4)

(14). Similar to (11)

(15). $M(A_{14})$ is not diffeomorphic to $M(A_{a3})$.

Let

$$\Gamma_{a3} = <\tilde{g}_1, \tilde{g}_2, t_3, t_4>$$
$$= \left\langle \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\rangle$$

$$\Gamma_{14} = <\tilde{h}_1, \tilde{h}_2, \tilde{h}_3, t_4>$$
$$= \left\langle \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\rangle.$$

Since $\tilde{g}_1\tilde{g}_2\tilde{g}_1^{-1} = \tilde{g}_2^{-1}$, $\tilde{g}_1 t_4 \tilde{g}_1^{-1} = t_4$, $t_3\tilde{g}_2 t_3^{-1} = \tilde{g}_2$, and $t_3 t_4 t_3^{-1} = t_4$, then $\Gamma_{a3} = \langle \tilde{g}_2, t_4\rangle \rtimes \langle \tilde{g}_1, t_3\rangle$. Then the center $\mathcal{C}(\Gamma_{a3}) = \langle t_1, t_3\rangle$ where $t_1 = \tilde{g}_1^2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, t_3 = \tilde{g}_3 = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix}$. There is an central group extension:

$$1 \to \langle t_1, t_3\rangle \to \Gamma_{a3} \to \Delta_{a3} \to 1,$$

where

$$\Delta_{a3} = \left\langle \gamma_1 = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \gamma_2 = \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle \rtimes \langle \beta\rangle$$

with $\beta = \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right)$.

For $\Gamma_{14} = \langle \tilde{h}_1, \tilde{h}_2, \tilde{h}_3, t_4\rangle$, since $\tilde{h}_1\tilde{h}_3\tilde{h}_1^{-1} = \tilde{h}_3$, $\tilde{h}_1 t_4 \tilde{h}_1^{-1} = t_4^{-1}$, $\tilde{h}_2\tilde{h}_3\tilde{h}_2^{-1} = \tilde{h}_3^{-1}$, and $\tilde{h}_2 t_4 \tilde{h}_2^{-1} = t_4$ then $\Gamma_{14} = \langle \tilde{h}_3, t_4\rangle \rtimes \langle \tilde{h}_1, \tilde{h}_2\rangle$. The center $\mathcal{C}(\Gamma_{14}) = \langle t_1, t_2\rangle$ where $t_1 = \tilde{h}_1^2$, $t_2 = \tilde{h}_2^2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$. It induces an extension

$$1 \to \langle t_1, t_2\rangle \to \Gamma_{14} \to \Delta_{14} \to 1,$$

where

$$\Delta_{14} = \left\langle s_1 = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, s_2 = \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle \rtimes \langle \alpha, \beta \rangle$$

with $\alpha = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \beta = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

Suppose that Γ_{a3} is isomorphic to Γ_{14} with isomorphism $\varphi : \Gamma_{a3} \to \Gamma_{14}$. Then it induces an isomorphism $\hat{\varphi} : \Delta_{a3} \to \Delta_{14}$

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \langle t_1, t_3 \rangle & \longrightarrow & \Gamma_{a3} & \longrightarrow & \Delta_{a3} & \longrightarrow & 1 \\ & & & & {\scriptstyle\varphi}\big\downarrow & & {\scriptstyle\hat{\varphi}}\big\downarrow & & \\ 1 & \longrightarrow & \langle t_1, t_2 \rangle & \longrightarrow & \Gamma_{14} & \longrightarrow & \Delta_{14} & \longrightarrow & 1. \end{array}$$

Consider $\hat{\varphi}(\gamma_1^{a_1}\gamma_2^{a_2}\beta) = \alpha$ and $\hat{\varphi}(\gamma_1^{b_1}\gamma_2^{b_2}\beta) = \beta$ where $a_1, a_2, b_1, b_2 \in \mathbb{Z}$. Then

$$\begin{aligned} \alpha\beta =& \hat{\varphi}(\gamma_1^{a_1}\gamma_2^{a_2}\beta\gamma_1^{b_1}\gamma_2^{b_2}\beta) \\ =& \hat{\varphi}(\gamma_1^{a_1}\gamma_2^{a_2}\gamma_1^{-b_1}\beta\gamma_2^{b_2}\beta) \quad (\beta\gamma_1\beta^{-1} = \gamma_1^{-1}) \\ =& \hat{\varphi}(\gamma_1^{a_1}\gamma_2^{a_2}\gamma_1^{-b_1}\gamma_2^{b_2}) \qquad (\beta\gamma_2\beta^{-1} = \gamma_2). \end{aligned}$$

Since $\alpha\beta$ is a torsion element and $\hat{\varphi}$ is an isomorphism, then $\gamma_1^{a_1}\gamma_2^{a_2}\gamma_1^{-b_1}\gamma_2^{b_2}$ is also a torsion element. Let

$$\begin{aligned} g =& \gamma_1^{a_1}\gamma_2^{a_2}\gamma_1^{-b_1}\gamma_2^{b_2} \\ =& \left(\begin{pmatrix} \frac{1}{2}a_1 \\ (-1)^{a_1}\frac{1}{2}a_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & (-1)^{a_1} \end{pmatrix} \right) \left(\begin{pmatrix} -\frac{1}{2}b_1 \\ (-1)^{-b_1}\frac{1}{2}b_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & (-1)^{-b_1} \end{pmatrix} \right) \\ =& \left(\begin{pmatrix} \frac{1}{2}(a_1 - b_1) \\ \frac{1}{2}((-1)^{a_1}a_2 + (-1)^{(a_1-b_1)}b_2) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & (-1)^{a_1-b_1} \end{pmatrix} \right). \end{aligned}$$

We have

$$\begin{aligned} g^2 =& \left(\begin{pmatrix} a_1 - b_1 \\ (1 + (-1)^{(a_1-b_1)})\frac{1}{2}((-1)^{a_1}a_2 + (-1)^{(a_1-b_1)}b_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \\ =& \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right). \end{aligned}$$

Therefore we get $a_1 = b_1$ and $(-1)^{a_1}a_2 + b_2 = 0$. Hence $\hat{\varphi}(g) = \hat{\varphi}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) = \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right)$. This yields a contradiction.

(16).
• $M(A_{a13}) \approx M(A_{a29})$.
For

$$A_{a13} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad A_{a29} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned} g_1(z_1,z_2,z_3,z_4) &= (-z_1,\bar{z}_2,z_3,\bar{z}_4). & h_1h_2(z_1,z_2,z_3,z_4) &= (-z_1,-\bar{z}_2,z_3,\bar{z}_4) \\ g_2(z_1,z_2,z_3,z_4) &= (z_1,-z_2,\bar{z}_3,z_4). & h_2(z_1,z_2,z_3,z_4) &= (z_1,-z_2,\bar{z}_3,z_4) \\ g_3(z_1,z_2,z_3,z_4) &= (z_1,z_2,-z_3,z_4). & h_3(z_1,z_2,z_3,z_4) &= (z_1,z_2,-z_3,z_4) \\ g_4(z_1,z_2,z_3,z_4) &= (z_1,z_2,z_3,-z_4). & h_4(z_1,z_2,z_3,z_4) &= (z_1,z_2,z_3,-z_4) \end{aligned}$$

then

$$\Gamma_{a13} = \left\langle \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\rangle$$

$$\Gamma_{a29} = \left\langle \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\rangle.$$

Let $\varphi(z_1,z_2,z_3,z_4) = (z_1,iz_2,z_3,z_4)$, we get these commutative diagrams

$$\begin{tikzcd} (z_1,z_2,z_3,z_4) \arrow[r, "\varphi"] \arrow[d, "g_1"'] & (z_1,iz_2,z_3,z_4) \arrow[d, "h_1h_2"'] \\ (-z_1,\bar{z}_2,z_3,\bar{z}_4) \arrow[r, "\varphi"] & (-z_1,-i\bar{z}_2,z_3,\bar{z}_4) \end{tikzcd} \quad \begin{tikzcd} (z_1,z_2,z_3,z_4) \arrow[r, "\varphi"] \arrow[d, "g_3"'] & (z_1,iz_2,z_3,z_4) \arrow[d, "h_3"'] \\ (z_1,z_2,-z_3,z_4) \arrow[r, "\varphi"] & (z_1,iz_2,-z_3,z_4) \end{tikzcd}$$

$$\begin{tikzcd} (z_1,z_2,z_3,z_4) \arrow[r, "\varphi"] \arrow[d, "g_2"'] & (z_1,iz_2,z_3,z_4) \arrow[d, "h_2"'] \\ (z_1,-z_2,\bar{z}_3,z_4) \arrow[r, "\varphi"] & (z_1,-iz_2,\bar{z}_3,z_4) \end{tikzcd} \quad \begin{tikzcd} (z_1,z_2,z_3,z_4) \arrow[r, "\varphi"] \arrow[d, "g_4"'] & (z_1,iz_2,z_3,z_4) \arrow[d, "h_4"'] \\ (z_1,z_2,z_3,-z_4) \arrow[r, "\varphi"] & (z_1,iz_2,z_3,-z_4). \end{tikzcd}$$

Therefore $\exists\ \gamma = \left(\begin{pmatrix} 0 \\ \frac{1}{4} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right) \in \mathbb{A}(n)$ s.t. $\gamma\Gamma_{a13}\gamma^{-1} = \Gamma_{a29}$.

• $M(A_{a13}) \approx M(A_{a18})$.
For

$$A_{a13} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad A_{a18} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{array}{ll} g_1(z_1,z_2,z_3,z_4) = (-z_1,\bar{z}_2,z_3,\bar{z}_4). & h_1(z_1,z_2,z_3,z_4) = (-z_1,\bar{z}_2,\bar{z}_3,z_4) \\ g_2(z_1,z_2,z_3,z_4) = (z_1,-z_2,\bar{z}_3,z_4). & h_2(z_1,z_2,z_3,z_4) = (z_1,-z_2,z_3,z_4) \\ g_3(z_1,z_2,z_3,z_4) = (z_1,z_2,-z_3,z_4). & h_3(z_1,z_2,z_3,z_4) = (z_1,z_2,-z_3,\bar{z}_4) \\ g_4(z_1,z_2,z_3,z_4) = (z_1,z_2,z_3,-z_4). & h_4(z_1,z_2,z_3,z_4) = (z_1,z_2,z_3,-z_4) \end{array}$$

then

$$\Gamma_{a13} = \left\langle \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\rangle$$

$$\Gamma_{a18} = \left\langle \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\rangle.$$

Let $\varphi(z_1,z_2,z_3,z_4) = (z_1,z_4,z_2,z_3)$, we get these commutative diagrams

$$\begin{array}{ccccccc} (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1,z_4,z_2,z_3) & & (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1,z_4,z_2,z_3) \\ {\scriptstyle g_1}\downarrow & & \downarrow{\scriptstyle h_1} & & {\scriptstyle g_3}\downarrow & & \downarrow{\scriptstyle h_4} \\ (-z_1,\bar{z}_2,z_3,\bar{z}_4) & \xrightarrow{\varphi} & (-z_1,\bar{z}_4,\bar{z}_2,z_3) & & (z_1,z_2,-z_3,z_4) & \xrightarrow{\varphi} & (z_1,z_4,z_2,-z_3) \\ (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1,z_4,z_2,z_3) & & (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1,z_4,z_2,z_3) \\ {\scriptstyle g_2}\downarrow & & \downarrow{\scriptstyle h_3} & & {\scriptstyle g_4}\downarrow & & \downarrow{\scriptstyle h_2} \\ (z_1,-z_2,\bar{z}_3,z_4) & \xrightarrow{\varphi} & (z_1,z_4,-z_2,\bar{z}_3) & & (z_1,z_2,z_3,-z_4) & \xrightarrow{\varphi} & (z_1,-z_4,z_2,z_3). \end{array}$$

Therefore $\exists\ \gamma = \left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \right) \in \mathbb{A}(n)$ s.t. $\gamma \Gamma_{a13} \gamma^{-1} = \Gamma_{a18}$.

• $M(A_{a13}) \approx M(A_{a19})$.
For

$$A_{a13} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad A_{a19} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{array}{ll} g_1(z_1,z_2,z_3,z_4) = (-z_1,\bar{z}_2,z_3,\bar{z}_4). & h_1(z_1,z_2,z_3,z_4) = (-z_1,\bar{z}_2,\bar{z}_3,z_4) \\ g_2(z_1,z_2,z_3,z_4) = (z_1,-z_2,\bar{z}_3,z_4). & h_2(z_1,z_2,z_3,z_4) = (z_1,-z_2,z_3,\bar{z}_4) \\ g_3(z_1,z_2,z_3,z_4) = (z_1,z_2,-z_3,z_4). & h_3(z_1,z_2,z_3,z_4) = (z_1,z_2,-z_3,z_4) \\ g_4(z_1,z_2,z_3,z_4) = (z_1,z_2,z_3,-z_4). & h_4(z_1,z_2,z_3,z_4) = (z_1,z_2,z_3,-z_4) \end{array}$$

then

$$\Gamma_{a13} = \left\langle \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\rangle$$

$$\Gamma_{a19} = \left\langle \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\rangle.$$

Let $\varphi(z_1,z_2,z_3,z_4) = (z_1,z_2,z_4,z_3)$, we get these commutative diagrams

$$\begin{array}{ccccccc} (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1,z_2,z_4,z_3) & & (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1,z_2,z_4,z_3) \\ {\scriptstyle g_1}\big\downarrow & & {\scriptstyle h_1}\big\downarrow & & {\scriptstyle g_3}\big\downarrow & & {\scriptstyle h_4}\big\downarrow \\ (-z_1,\bar{z}_2,z_3,\bar{z}_4) & \xrightarrow{\varphi} & (-z_1,\bar{z}_2,\bar{z}_4,z_3) & & (z_1,z_2,-z_3,z_4) & \xrightarrow{\varphi} & (z_1,z_2,z_4,-z_3) \\ (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1,z_2,z_4,z_3) & & (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1,z_2,z_4,z_3) \\ {\scriptstyle g_2}\big\downarrow & & {\scriptstyle h_2}\big\downarrow & & {\scriptstyle g_4}\big\downarrow & & {\scriptstyle h_3}\big\downarrow \\ (z_1,-z_2,\bar{z}_3,z_4) & \xrightarrow{\varphi} & (z_1,-z_2,z_4,\bar{z}_3) & & (z_1,z_2,z_3,-z_4) & \xrightarrow{\varphi} & (z_1,z_2,-z_4,z_3). \end{array}$$

Therefore $\exists\ \gamma = \left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \right) \in \mathbb{A}(n)$ s.t. $\gamma \Gamma_{a13} \gamma^{-1} = \Gamma_{a19}$.

• $M(A_{a18}) \approx M(A_{a20})$.
For

$$A_{a18} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad A_{a20} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned} g_1(z_1, z_2, z_3, z_4) &= (-z_1, \bar{z}_2, \bar{z}_3, z_4) & h_1(z_1, z_2, z_3, z_4) &= (-z_1, \bar{z}_2, \bar{z}_3, z_4) \\ g_2(z_1, z_2, z_3, z_4) &= (z_1, -z_2, z_3, z_4) & h_2h_3(z_1, z_2, z_3, z_4) &= (z_1, -z_2, -z_3, z_4) \\ g_3(z_1, z_2, z_3, z_4) &= (z_1, z_2, -z_3, \bar{z}_4) & h_3(z_1, z_2, z_3, z_4) &= (z_1, z_2, -z_3, \bar{z}_4) \\ g_4(z_1, z_2, z_3, z_4) &= (z_1, z_2, z_3, -z_4) & h_4(z_1, z_2, z_3, z_4) &= (z_1, z_2, z_3, -z_4) \end{aligned}$$

then

$$\begin{aligned} \Gamma_{a18} = \Big\langle & \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\ & \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Big\rangle \end{aligned}$$

$$\begin{aligned} \Gamma_{a20} = \Big\langle & \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\ & \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Big\rangle. \end{aligned}$$

Let $\varphi(z_1, z_2, z_3, z_4) = (z_1, z_2, z_2 z_3, z_4)$, we get these commutative diagrams

$$\begin{tikzcd} (z_1, z_2, z_3, z_4) \arrow[r, "\varphi"] \arrow[d, "g_1"'] & (z_1, z_2, z_2 z_3, z_4) \arrow[d, "h_1"'] & (z_1, z_2, z_3, z_4) \arrow[r, "\varphi"] \arrow[d, "g_3"'] & (z_1, z_2, z_2 z_3, z_4) \arrow[d, "h_3"'] \\ (-z_1, \bar{z}_2, \bar{z}_3, z_4) \arrow[r, "\varphi"] & (-z_1, \bar{z}_2, \bar{z}_2 \bar{z}_3, z_4) & (z_1, z_2, -z_3, \bar{z}_4) \arrow[r, "\varphi"] & (z_1, z_2, -z_2 z_3, \bar{z}_4) \end{tikzcd}$$

$$\begin{array}{ccc} (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1,z_2,z_2z_3,z_4) \\ {\scriptstyle g_2}\downarrow & & \downarrow{\scriptstyle h_2h_3} \\ (z_1,-z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1,-z_2,-z_2z_3,z_4) \end{array} \quad \begin{array}{ccc} (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1,z_2,z_2z_3,z_4) \\ {\scriptstyle g_4}\downarrow & & \downarrow{\scriptstyle h_4} \\ (z_1,z_2,z_3,-z_4) & \xrightarrow{\varphi} & (z_1,z_2,z_2z_3,-z_4). \end{array}$$

Therefore $\exists\ \gamma = \left(\begin{pmatrix} 0\\0\\0\\0 \end{pmatrix} \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&1&1&0\\0&0&0&1 \end{pmatrix}\right) \in \mathbb{A}(n)$ s.t. $\gamma\Gamma_{a18}\gamma^{-1} = \Gamma_{a20}$.

• $M(A_{a22}) \approx M(A_{a23})$.
For

$$A_{a22} = \begin{pmatrix} 1&1&1&1\\0&1&0&0\\0&0&1&1\\0&0&0&1 \end{pmatrix} \qquad A_{a23} = \begin{pmatrix} 1&1&1&1\\0&1&0&1\\0&0&1&0\\0&0&0&1 \end{pmatrix}$$

$$\begin{array}{ll} g_1(z_1,z_2,z_3,z_4) = (-z_1,\bar{z}_2,\bar{z}_3,\bar{z}_4) & h_1(z_1,z_2,z_3,z_4) = (-z_1,\bar{z}_2,\bar{z}_3,\bar{z}_4) \\ g_2(z_1,z_2,z_3,z_4) = (z_1,-z_2,z_3,z_4) & h_2(z_1,z_2,z_3,z_4) = (z_1,-z_2,z_3,\bar{z}_4) \\ g_3(z_1,z_2,z_3,z_4) = (z_1,z_2,-z_3,\bar{z}_4) & h_3(z_1,z_2,z_3,z_4) = (z_1,z_2,-z_3,z_4) \\ g_4(z_1,z_2,z_3,z_4) = (z_1,z_2,z_3,-z_4) & h_4(z_1,z_2,z_3,z_4) = (z_1,z_2,z_3,-z_4) \end{array}$$

then

$$\Gamma_{a22} = \left\langle \begin{pmatrix} \frac12\\0\\0\\0 \end{pmatrix} \begin{pmatrix} 1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1 \end{pmatrix}, \begin{pmatrix} 0\\\frac12\\0\\0 \end{pmatrix} \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix}, \begin{pmatrix} 0\\0\\\frac12\\0 \end{pmatrix} \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1 \end{pmatrix}, \begin{pmatrix} 0\\0\\0\\\frac12 \end{pmatrix} \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix} \right\rangle$$

$$\Gamma_{a23} = \left\langle \begin{pmatrix} \frac12\\0\\0\\0 \end{pmatrix} \begin{pmatrix} 1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1 \end{pmatrix}, \begin{pmatrix} 0\\\frac12\\0\\0 \end{pmatrix} \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1 \end{pmatrix}, \begin{pmatrix} 0\\0\\\frac12\\0 \end{pmatrix} \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix}, \begin{pmatrix} 0\\0\\0\\\frac12 \end{pmatrix} \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix} \right\rangle.$$

Let $\varphi(z_1,z_2,z_3,z_4) = (z_1,z_3,z_2,z_4)$, we get these commutative diagrams

$$\begin{array}{ccc} (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1,z_3,z_2,z_4) \\ {\scriptstyle g_1}\downarrow & & \downarrow{\scriptstyle h_1} \\ (-z_1,\bar{z}_2,\bar{z}_3,\bar{z}_4) & \xrightarrow{\varphi} & (-z_1,\bar{z}_3,\bar{z}_2,\bar{z}_4) \end{array} \quad \begin{array}{ccc} (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1,z_3,z_2,z_4) \\ {\scriptstyle g_3}\downarrow & & \downarrow{\scriptstyle h_2} \\ (z_1,z_2,-z_3,\bar{z}_4) & \xrightarrow{\varphi} & (z_1,-z_3,z_2,\bar{z}_4) \end{array}$$

$$\begin{array}{ccccccc} (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1,z_3,z_2,z_4) & & (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1,z_3,z_2,z_4) \\ {\scriptstyle g_2}\downarrow & & \downarrow{\scriptstyle h_3} & & {\scriptstyle g_4}\downarrow & & \downarrow{\scriptstyle h_4} \\ (z_1,-z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1,z_3,-z_2,z_4) & & (z_1,z_2,z_3,-z_4) & \xrightarrow{\varphi} & (z_1,z_3,z_2,-z_4). \end{array}$$

Therefore $\exists\ \gamma = \left(\begin{pmatrix} 0\\0\\0\\0 \end{pmatrix} \begin{pmatrix} 1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1 \end{pmatrix}\right) \in \mathbb{A}(n)$ s.t. $\gamma\Gamma_{a22}\gamma^{-1} = \Gamma_{a23}$.

• $M(A_{a23}) \approx M(A_{a24})$.
For

$$A_{a23} = \begin{pmatrix} 1&1&1&1\\0&1&0&1\\0&0&1&0\\0&0&0&1 \end{pmatrix} \qquad A_{a24} = \begin{pmatrix} 1&1&1&1\\0&1&0&1\\0&0&1&1\\0&0&0&1 \end{pmatrix}$$

$$\begin{array}{rlrl} g_1(z_1,z_2,z_3,z_4) &= (-z_1,\bar z_2,\bar z_3,\bar z_4) & h_1(z_1,z_2,z_3,z_4) &= (-z_1,\bar z_2,\bar z_3,\bar z_4) \\ g_2(z_1,z_2,z_3,z_4) &= (z_1,-z_2,z_3,\bar z_4) & h_2h_3(z_1,z_2,z_3,z_4) &= (z_1,-z_2,-z_3,z_4) \\ g_3(z_1,z_2,z_3,z_4) &= (z_1,z_2,-z_3,z_4) & h_3(z_1,z_2,z_3,z_4) &= (z_1,z_2,-z_3,\bar z_4) \\ g_4(z_1,z_2,z_3,z_4) &= (z_1,z_2,z_3,-z_4) & h_4(z_1,z_2,z_3,z_4) &= (z_1,z_2,z_3,-z_4) \end{array}$$

then

$$\Gamma_{a23} = \Big\langle \begin{pmatrix} \frac12\\0\\0\\0 \end{pmatrix} \begin{pmatrix} 1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1 \end{pmatrix}, \begin{pmatrix} 0\\\frac12\\0\\0 \end{pmatrix} \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1 \end{pmatrix}, \begin{pmatrix} 0\\0\\\frac12\\0 \end{pmatrix} \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix}, \begin{pmatrix} 0\\0\\0\\\frac12 \end{pmatrix} \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix} \Big\rangle$$

$$\Gamma_{a24} = \Big\langle \begin{pmatrix} \frac12\\0\\0\\0 \end{pmatrix} \begin{pmatrix} 1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1 \end{pmatrix}, \begin{pmatrix} 0\\\frac12\\\frac12\\0 \end{pmatrix} \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix}, \begin{pmatrix} 0\\0\\\frac12\\0 \end{pmatrix} \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1 \end{pmatrix}, \begin{pmatrix} 0\\0\\0\\\frac12 \end{pmatrix} \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix} \Big\rangle.$$

Let $\varphi(z_1,z_2,z_3,z_4) = (z_1,z_3,z_2z_3,z_4)$, we get these commutative diagrams

$$\begin{tikzcd}
(z_1,z_2,z_3,z_4) \arrow[r, "\varphi"] \arrow[d, "g_1"'] & (z_1,z_3,z_2z_3,z_4) \arrow[d, "h_1"'] & (z_1,z_2,z_3,z_4) \arrow[r, "\varphi"] \arrow[d, "g_3"'] & (z_1,z_3,z_2z_3,z_4) \arrow[d, "h_2h_3"'] \\
(-z_1,\bar{z}_2,\bar{z}_3,\bar{z}_4) \arrow[r, "\varphi"] & (-z_1,\bar{z}_3,\bar{z}_2\bar{z}_3,\bar{z}_4) & (z_1,z_2,-z_3,z_4) \arrow[r, "\varphi"] & (z_1,-z_3,-z_2z_3,z_4)
\end{tikzcd}$$

$$\begin{tikzcd}
(z_1,z_2,z_3,z_4) \arrow[r, "\varphi"] \arrow[d, "g_2"'] & (z_1,z_3,z_2z_3,z_4) \arrow[d, "h_3"'] & (z_1,z_2,z_3,z_4) \arrow[r, "\varphi"] \arrow[d, "g_4"'] & (z_1,z_3,z_2z_3,z_4) \arrow[d, "h_4"'] \\
(z_1,-z_2,z_3,\bar{z}_4) \arrow[r, "\varphi"] & (z_1,z_3,-z_2z_3,\bar{z}_4) & (z_1,z_2,z_3,-z_4) \arrow[r, "\varphi"] & (z_1,z_3,z_2,-z_4).
\end{tikzcd}$$

Therefore $\exists\ \gamma = \left(\begin{pmatrix} 0\\0\\0\\0 \end{pmatrix} \begin{pmatrix} 1&0&0&0\\0&0&1&0\\0&1&1&0\\0&0&0&1 \end{pmatrix}\right) \in \mathbb{A}(n)$ s.t. $\gamma\Gamma_{a23}\gamma^{-1} = \Gamma_{a24}$.

• $M(A_{a23}) \approx M(A_{a29})$.
For

$$A_{a23} = \begin{pmatrix} 1&1&1&1\\0&1&0&1\\0&0&1&0\\0&0&0&1 \end{pmatrix} \qquad A_{a29} = \begin{pmatrix} 1&1&1&1\\0&1&1&0\\0&0&1&0\\0&0&0&1 \end{pmatrix}$$

$$\begin{aligned}
g_1(z_1,z_2,z_3,z_4) &= (-z_1,\bar{z}_2,\bar{z}_3,\bar{z}_4) & h_1(z_1,z_2,z_3,z_4) &= (-z_1,\bar{z}_2,\bar{z}_3,\bar{z}_4)\\
g_2(z_1,z_2,z_3,z_4) &= (z_1,-z_2,z_3,\bar{z}_4) & h_2(z_1,z_2,z_3,z_4) &= (z_1,-z_2,\bar{z}_3,z_4)\\
g_3(z_1,z_2,z_3,z_4) &= (z_1,z_2,-z_3,z_4) & h_3(z_1,z_2,z_3,z_4) &= (z_1,z_2,-z_3,z_4)\\
g_4(z_1,z_2,z_3,z_4) &= (z_1,z_2,z_3,-z_4) & h_4(z_1,z_2,z_3,z_4) &= (z_1,z_2,z_3,-z_4)
\end{aligned}$$

then

$$\begin{aligned}
\Gamma_{a23} = \Big\langle & \begin{pmatrix} \frac{1}{2}\\0\\0\\0 \end{pmatrix} \begin{pmatrix} 1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1 \end{pmatrix}, \begin{pmatrix} 0\\\frac{1}{2}\\0\\0 \end{pmatrix} \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1 \end{pmatrix},\\
& \begin{pmatrix} 0\\0\\\frac{1}{2}\\0 \end{pmatrix} \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix}, \begin{pmatrix} 0\\0\\0\\\frac{1}{2} \end{pmatrix} \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix}\Big\rangle
\end{aligned}$$

$$\begin{aligned}
\Gamma_{a29} = \Big\langle & \begin{pmatrix} \frac{1}{2}\\0\\0\\0 \end{pmatrix} \begin{pmatrix} 1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1 \end{pmatrix}, \begin{pmatrix} 0\\\frac{1}{2}\\0\\0 \end{pmatrix} \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&1 \end{pmatrix},\\
& \begin{pmatrix} 0\\0\\\frac{1}{2}\\0 \end{pmatrix} \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix}, \begin{pmatrix} 0\\0\\0\\\frac{1}{2} \end{pmatrix} \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix}\Big\rangle.
\end{aligned}$$

Let $\varphi(z_1,z_2,z_3,z_4)=(z_1,z_2,z_4,z_3)$, we get these commutative diagrams

$$\begin{array}{ccccccc}
(z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1,z_2,z_4,z_3) & & (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1,z_2,z_4,z_3) \\
{\scriptstyle g_1}\big\downarrow & & {\scriptstyle h_1}\big\downarrow & & {\scriptstyle g_3}\big\downarrow & & {\scriptstyle h_4}\big\downarrow \\
(-z_1,\bar z_2,\bar z_3,\bar z_4) & \xrightarrow{\varphi} & (-z_1,\bar z_2,\bar z_4,\bar z_3) & & (z_1,z_2,-z_3,z_4) & \xrightarrow{\varphi} & (z_1,z_2,z_4,-z_3)
\end{array}$$

$$\begin{array}{ccccccc}
(z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1,z_2,z_4,z_3) & & (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1,z_2,z_4,z_3) \\
{\scriptstyle g_2}\big\downarrow & & {\scriptstyle h_2}\big\downarrow & & {\scriptstyle g_4}\big\downarrow & & {\scriptstyle h_3}\big\downarrow \\
(z_1,-z_2,z_3,\bar z_4) & \xrightarrow{\varphi} & (z_1,-z_2,\bar z_4,z_3) & & (z_1,z_2,z_3,-z_4) & \xrightarrow{\varphi} & (z_1,z_2,-z_4,z_3).
\end{array}$$

Therefore $\exists\ \gamma = \left(\begin{pmatrix}0\\0\\0\\0\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\right)\in\mathbb{A}(n)$ s.t. $\gamma\Gamma_{a23}\gamma^{-1} = \Gamma_{a29}$.

(17). Similar to (4).

(18). Similar to (11).

(19).
• $M(A_{a11})\approx M(A_{a31})$.
For

$$A_{a11}=\begin{pmatrix}1&1&0&0\\0&1&1&1\\0&0&1&0\\0&0&0&1\end{pmatrix} \qquad A_{a31}=\begin{pmatrix}1&1&1&1\\0&1&1&1\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

$$\begin{array}{rr}
g_1(z_1,z_2,z_3,z_4)=(-z_1,\bar z_2,z_3,z_4). & h_1h_2(z_1,z_2,z_3,z_4)=(-z_1,-\bar z_2,z_3,z_4)\\
g_2(z_1,z_2,z_3,z_4)=(z_1,-z_2,\bar z_3,\bar z_4). & h_2(z_1,z_2,z_3,z_4)=(z_1,-z_2,\bar z_3,\bar z_4)\\
g_3(z_1,z_2,z_3,z_4)=(z_1,z_2,-z_3,z_4). & h_3(z_1,z_2,z_3,z_4)=(z_1,z_2,-z_3,z_4)\\
g_4(z_1,z_2,z_3,z_4)=(z_1,z_2,z_3,-z_4). & h_4(z_1,z_2,z_3,z_4)=(z_1,z_2,z_3,-z_4)
\end{array}$$

then

$$\Gamma_{a11}=\left\langle \begin{pmatrix}\frac12\\0\\0\\0\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix}0\\\frac12\\0\\0\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix},\right.$$
$$\left.\begin{pmatrix}0\\0\\\frac12\\0\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix}0\\0\\0\\\frac12\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\right\rangle$$

$$\Gamma_{a31} = \left\langle \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\rangle.$$

Let $\varphi(z_1, z_2, z_3, z_4) = (z_1, iz_2, z_3, z_4)$, we get these commutative diagrams

$$\begin{tikzcd}
(z_1, z_2, z_3, z_4) \arrow[r, "\varphi"] \arrow[d, "g_1"'] & (z_1, iz_2, z_3, z_4) \arrow[d, "h_1h_2"'] & (z_1, z_2, z_3, z_4) \arrow[r, "\varphi"] \arrow[d, "g_3"'] & (z_1, iz_2, z_3, z_4) \arrow[d, "h_3"'] \\
(-z_1, \bar{z}_2, z_3, z_4) \arrow[r, "\varphi"] & (-z_1, -i\bar{z}_2, z_3, z_4) & (z_1, z_2, -z_3, z_4) \arrow[r, "\varphi"] & (z_1, iz_2, -z_3, z_4)
\end{tikzcd}$$

$$\begin{tikzcd}
(z_1, z_2, z_3, z_4) \arrow[r, "\varphi"] \arrow[d, "g_2"'] & (z_1, iz_2, z_3, z_4) \arrow[d, "h_2"'] & (z_1, z_2, z_3, z_4) \arrow[r, "\varphi"] \arrow[d, "g_4"'] & (z_1, iz_2, z_3, z_4) \arrow[d, "h_4"'] \\
(z_1, -z_2, \bar{z}_3, \bar{z}_4) \arrow[r, "\varphi"] & (z_1, -iz_2, \bar{z}_3, \bar{z}_4) & (z_1, z_2, z_3, -z_4) \arrow[r, "\varphi"] & (z_1, iz_2, z_3, -z_4).
\end{tikzcd}$$

Therefore $\exists\ \gamma = \left(\begin{pmatrix} 0 \\ \frac{1}{4} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right) \in \mathbb{A}(n)$ s.t. $\gamma \Gamma_{a11} \gamma^{-1} = \Gamma_{a31}$.

(20). Similar to (4).

(21). Similar to (11).

(22). $M(A_{a11})$ is not diffeomorphic to $M(A_{a13})$.

Let

$$\begin{aligned}
\Gamma_{a13} &= \langle \tilde g_1, \tilde g_2, t_3, t_4\rangle \\
&= \Big\langle \begin{pmatrix} \frac12 \\ 0 \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac12 \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\
&\quad \begin{pmatrix} 0 \\ 0 \\ \frac12 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac12 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\Big\rangle \\
\Gamma_{a11} &= \langle \tilde h_1, \tilde h_2, t_3, t_4\rangle \\
&= \Big\langle \begin{pmatrix} \frac12 \\ 0 \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac12 \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \\
&\quad \begin{pmatrix} 0 \\ 0 \\ \frac12 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac12 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\Big\rangle.
\end{aligned}$$

Since $\tilde g_1 \tilde g_2 \tilde g_1^{-1} = \tilde g_2^{-1}, \tilde g_1 t_3 \tilde g_1^{-1} = t_3$, and $\tilde g_1 t_4 \tilde g_1^{-1} = t_4$ then $\Gamma_{a13} = \langle \tilde g_2, t_3, t_4\rangle \rtimes \langle \tilde g_1\rangle$ and the center $\mathcal{C}(\Gamma_{a13}) = \langle \tilde g_1^2\rangle = \langle t_1\rangle$, where $t_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

It induces an extension

$$1 \to \langle t_1\rangle \to \Gamma_{a13} \longrightarrow \Delta_{a13} \to 1,$$

where Δ_{a13} is isomorphic to

$$\begin{aligned}
&\Big\langle \begin{pmatrix} \frac12 \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac12 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
&\quad \begin{pmatrix} 0 \\ 0 \\ \frac12 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\Big\rangle \rtimes \Big\langle \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\Big\rangle.
\end{aligned}$$

Put

$$\begin{aligned}
\Delta_{a13}^f =& \langle p, q, r\rangle \\
=& \Big\langle \begin{pmatrix} \frac12 \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac12 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac12 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\Big\rangle
\end{aligned}$$

and

$$\beta = \Big\langle \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \Big\rangle$$

so that $\Delta_{a13} = \Delta^f_{a13} \rtimes \beta$.

Next, since $\tilde{h}_1\tilde{h}_2\tilde{h}_1^{-1} = \tilde{h}_2^{-1}, \tilde{h}_1 t_3 \tilde{h}_1^{-1} = t_3$, and $\tilde{h}_1 t_4 \tilde{h}_1^{-1} = t_4$ then $\Gamma_{a11} = \langle \tilde{h}_2, t_3, t_4 \rangle \rtimes \langle \tilde{h}_1 \rangle$ and the center $\mathcal{C}(\Gamma_{a11}) = \langle \tilde{h}_1^2 \rangle = \langle t_1 \rangle$. It induces an extension

$$1 \to \langle t_1 \rangle \to \Gamma_{a11} \longrightarrow \Delta_{a11} \to 1,$$

where Δ_{a11} is isomorphic to

$$\Big\langle \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Big\rangle \rtimes \langle \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rangle.$$

Put

$$\Delta^f_{a11} = \Big\langle \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Big\rangle$$

and

$$\alpha = \Big\langle \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Big\rangle$$

so that $\Delta_{a11} = \Delta^f_{a11} \rtimes \alpha$.

Suppose there is an isomorphism $\varphi : \Gamma_{a13} \to \Gamma_{a11}$ which induces an isomorphism $\hat{\varphi} : \Delta_{a13} \to \Delta_{a11}$

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \langle t_1 \rangle & \longrightarrow & \Gamma_{a13} & \longrightarrow & \Delta_{a13} & \longrightarrow & 1 \\ & & & & {\scriptstyle\varphi}\big\downarrow & & {\scriptstyle\hat{\varphi}}\big\downarrow & & \\ 1 & \longrightarrow & \langle t_1 \rangle & \longrightarrow & \Gamma_{a11} & \longrightarrow & \Delta_{a11} & \longrightarrow & 1. \end{array}$$

Consider $\alpha = \hat{\varphi}(p^{a_1}q^{a_2}r^{a_3}\beta) = \hat{\varphi}\Big(\begin{pmatrix} \frac{1}{2}a_1 \\ (-1)^{a_1}\frac{1}{2}a_2 \\ \frac{1}{2}a_3 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & (-1)^{a_1} & 0 \\ 0 & 0 & -1 \end{pmatrix} \Big)$.

Since α is a torsion element and $\hat{\varphi}$ is an isomorphism then $p^{a_1}q^{a_2}r^{a_3}\beta$

is also a torsion element. Therefore

$$\begin{aligned}(p^{a_1}q^{a_2}r^{a_3}\beta)^2 =&\left(\begin{pmatrix} 0 \\ (1+(-1)^{a_1})(-1)^{a_1}\frac{1}{2}a_2 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) \\ =&\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right).\end{aligned}$$

Hence $a_1 = odd$ or $a_2 = 0$. If $a_1 = odd$ then

$$p^{a_1}q^{a_2}r^{a_3}\beta = \left(\begin{pmatrix} \frac{1}{2}(2n+1) \\ -\frac{1}{2}a_2 \\ \frac{1}{2}a_3 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right).$$

By Bieberbach theorem, $\exists \gamma = (b, B) \in \mathbb{A}(4)$ such that $BL(p^{a_1}q^{a_2}r^{a_3}\beta)B^{-1} = L(\hat{\varphi}(p^{a_1}q^{a_2}r^{a_3}\beta))$ or

$$B \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} B^{-1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which is impossible. Also if $a_2 = 0$ then

$$p^{a_1}q^{a_2}r^{a_3}\beta = \left(\begin{pmatrix} \frac{1}{2}a_1 \\ 0 \\ \frac{1}{2}a_3 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & (-1)^{a_1} & 0 \\ 0 & 0 & -1 \end{pmatrix} \right).$$

By Bieberbach theorem, $\exists \gamma = (b, B) \in \mathbb{A}(4)$ such that $BL(p^{a_1}q^{a_2}r^{a_3}\beta)B^{-1} = L(\hat{\varphi}(p^{a_1}q^{a_2}r^{a_3}\beta))$ or

$$B \begin{pmatrix} -1 & 0 & 0 \\ 0 & (-1)^{a_1} & 0 \\ 0 & 0 & -1 \end{pmatrix} B^{-1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which is also impossible. Therefore Γ_{a11} is not isomorphic to Γ_{a13}.

(23).
• $M(A_{13}) \approx M(A_{15})$.
For

$$A_{13} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad A_{15} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned} g_1(z_1, z_2, z_3, z_4) &= (-z_1, z_2, z_3, \bar{z}_4). & h_1(z_1, z_2, z_3, z_4) &= (-z_1, z_2, z_3, \bar{z}_4) \\ g_2(z_1, z_2, z_3, z_4) &= (z_1, -z_2, \bar{z}_3, z_4). & h_2h_1(z_1, z_2, z_3, z_4) &= (-z_1, -z_2, \bar{z}_3, z_4) \\ g_3(z_1, z_2, z_3, z_4) &= (z_1, z_2, -z_3, z_4). & h_3(z_1, z_2, z_3, z_4) &= (z_1, z_2, -z_3, z_4) \\ g_4(z_1, z_2, z_3, z_4) &= (z_1, z_2, z_3, -z_4). & h_4(z_1, z_2, z_3, z_4) &= (z_1, z_2, z_3, -z_4) \end{aligned}$$

then

$$\Gamma_{13} = \Big\langle \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Big\rangle$$

$$\Gamma_{15} = \Big\langle \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Big\rangle.$$

Let $\varphi(z_1, z_2, z_3, z_4) = (z_1z_2, z_2, z_3, z_4)$, we get these commutative diagrams

$$\begin{array}{ccc|ccc} (z_1, z_2, z_3, z_4) & \xrightarrow{\varphi} & (z_1z_2, z_2, z_3, z_4) & (z_1, z_2, z_3, z_4) & \xrightarrow{\varphi} & (z_1z_2, z_2, z_3, z_4) \\ {\scriptstyle g_1}\downarrow & & \downarrow{\scriptstyle h_1} & {\scriptstyle g_3}\downarrow & & \downarrow{\scriptstyle h_3} \\ (-z_1, z_2, z_3, \bar{z}_4) & \xrightarrow{\varphi} & (-z_1z_2, z_2, z_3, \bar{z}_4) & (z_1, z_2, -z_3, z_4) & \xrightarrow{\varphi} & (z_1z_2, z_2, -z_3, z_4) \\ (z_1, z_2, z_3, z_4) & \xrightarrow{\varphi} & (z_1z_2, z_2, z_3, z_4) & (z_1, z_2, z_3, z_4) & \xrightarrow{\varphi} & (z_1z_2, z_2, z_3, z_4) \\ {\scriptstyle g_2}\downarrow & & \downarrow{\scriptstyle h_2h_1} & {\scriptstyle g_4}\downarrow & & \downarrow{\scriptstyle h_4} \\ (z_1, -z_2, \bar{z}_3, z_4) & \xrightarrow{\varphi} & (-z_1z_2, -z_2, \bar{z}_3, z_4) & (z_1, z_2, z_3, -z_4) & \xrightarrow{\varphi} & (z_1z_2, z_2, z_3, -z_4). \end{array}$$

Therefore $\exists\ \gamma = \left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right) \in \mathbb{A}(n)$ s.t. $\gamma\Gamma_{13}\gamma^{-1} = \Gamma_{15}$.

• $M(A_{13}) \approx M(A_{19})$.
For

$$A_{13} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad A_{19} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{array}{ll} g_1(z_1, z_2, z_3, z_4) = (-z_1, z_2, z_3, \bar{z}_4). & h_1(z_1, z_2, z_3, z_4) = (-z_1, z_2, \bar{z}_3, z_4) \\ g_2(z_1, z_2, z_3, z_4) = (z_1, -z_2, \bar{z}_3, z_4). & h_2(z_1, z_2, z_3, z_4) = (z_1, -z_2, z_3, \bar{z}_4) \\ g_3(z_1, z_2, z_3, z_4) = (z_1, z_2, -z_3, z_4). & h_3(z_1, z_2, z_3, z_4) = (z_1, z_2, -z_3, z_4) \\ g_4(z_1, z_2, z_3, z_4) = (z_1, z_2, z_3, -z_4). & h_4(z_1, z_2, z_3, z_4) = (z_1, z_2, z_3, -z_4) \end{array}$$

then

$$\Gamma_{13} = \left\langle \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\rangle$$

$$\Gamma_{19} = \left\langle \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\rangle.$$

Let $\varphi(z_1, z_2, z_3, z_4) = (z_1, z_2, z_4, z_3)$, we get these commutative diagrams

$$\begin{tikzcd}
(z_1, z_2, z_3, z_4) \arrow[r, "\varphi"] \arrow[d, "g_1"'] & (z_1, z_2, z_4, z_3) \arrow[d, "h_1"'] \\
(-z_1, z_2, z_3, \bar{z}_4) \arrow[r, "\varphi"] & (-z_1, z_2, \bar{z}_4, z_3)
\end{tikzcd}
\qquad
\begin{tikzcd}
(z_1, z_2, z_3, z_4) \arrow[r, "\varphi"] \arrow[d, "g_3"'] & (z_1, z_2, z_4, z_3) \arrow[d, "h_4"'] \\
(z_1, z_2, -z_3, z_4) \arrow[r, "\varphi"] & (z_1, z_2, z_4, -z_3)
\end{tikzcd}$$

$$\begin{tikzcd}
(z_1, z_2, z_3, z_4) \arrow[r, "\varphi"] \arrow[d, "g_2"'] & (z_1, z_2, z_4, z_3) \arrow[d, "h_2"'] \\
(z_1, -z_2, \bar{z}_3, z_4) \arrow[r, "\varphi"] & (z_1, -z_2, z_4, \bar{z}_3)
\end{tikzcd}
\qquad
\begin{tikzcd}
(z_1, z_2, z_3, z_4) \arrow[r, "\varphi"] \arrow[d, "g_4"'] & (z_1, z_2, z_4, z_3) \arrow[d, "h_3"'] \\
(z_1, z_2, z_3, -z_4) \arrow[r, "\varphi"] & (z_1, z_2, -z_4, z_3).
\end{tikzcd}$$

Therefore $\exists\ \gamma = \left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \right) \in \mathbb{A}(n)$ s.t. $\gamma \Gamma_{13} \gamma^{-1} = \Gamma_{19}$.

• $M(A_{13}) \approx M(A_{23})$.
For

$$A_{13} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad A_{23} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$g_1(z_1, z_2, z_3, z_4) = (-z_1, z_2, z_3, \bar{z}_4)$. $\quad h_1 h_2(z_1, z_2, z_3, z_4) = (-z_1, -z_2, \bar{z}_3, z_4)$
$g_2(z_1, z_2, z_3, z_4) = (z_1, -z_2, \bar{z}_3, z_4)$. $\quad h_2(z_1, z_2, z_3, z_4) = (z_1, -z_2, z_3, \bar{z}_4)$
$g_3(z_1, z_2, z_3, z_4) = (z_1, z_2, -z_3, z_4)$. $\quad h_3(z_1, z_2, z_3, z_4) = (z_1, z_2, -z_3, z_4)$
$g_4(z_1, z_2, z_3, z_4) = (z_1, z_2, z_3, -z_4)$. $\quad h_4(z_1, z_2, z_3, z_4) = (z_1, z_2, z_3, -z_4)$

then

$$\Gamma_{13} = \left\langle \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\rangle$$

$$\Gamma_{23} = \left\langle \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\rangle.$$

Let $\varphi(z_1, z_2, z_3, z_4) = (z_1, z_1 z_2, z_4, z_3)$, we get these commutative diagrams

$$\begin{tikzcd}
(z_1, z_2, z_3, z_4) \arrow[r, "\varphi"] \arrow[d, "g_1"'] & (z_1, z_1 z_2, z_4, z_3) \arrow[d, "h_1 h_2"'] \\
(-z_1, z_2, z_3, \bar{z}_4) \arrow[r, "\varphi"] & (-z_1, -z_1 z_2, \bar{z}_4, z_3)
\end{tikzcd}
\quad
\begin{tikzcd}
(z_1, z_2, z_3, z_4) \arrow[r, "\varphi"] \arrow[d, "g_3"'] & (z_1, z_1 z_2, z_4, z_3) \arrow[d, "h_4"'] \\
(z_1, z_2, -z_3, z_4) \arrow[r, "\varphi"] & (z_1, z_1 z_2, z_4, -z_3)
\end{tikzcd}$$

$$\begin{tikzcd}
(z_1, z_2, z_3, z_4) \arrow[r, "\varphi"] \arrow[d, "g_2"'] & (z_1, z_1 z_2, z_4, z_3) \arrow[d, "h_2"'] \\
(z_1, -z_2, \bar{z}_3, z_4) \arrow[r, "\varphi"] & (z_1, -z_1 z_2, z_4, \bar{z}_3)
\end{tikzcd}
\quad
\begin{tikzcd}
(z_1, z_2, z_3, z_4) \arrow[r, "\varphi"] \arrow[d, "g_4"'] & (z_1, z_1 z_2, z_4, z_3) \arrow[d, "h_3"'] \\
(z_1, z_2, z_3, -z_4) \arrow[r, "\varphi"] & (z_1, z_2, -z_4, z_3).
\end{tikzcd}$$

Therefore $\exists\ \gamma = \left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \right) \in \mathbb{A}(n)$ s.t. $\gamma \Gamma_{13} \gamma^{-1} = \Gamma_{23}$.

• $M(A_{13}) \approx M(A_{27})$.
For

$$A_{13} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad A_{27} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned}
g_1(z_1, z_2, z_3, z_4) &= (-z_1, z_2, z_3, \bar{z}_4). & h_1(z_1, z_2, z_3, z_4) &= (-z_1, z_2, \bar{z}_3, z_4) \\
g_2(z_1, z_2, z_3, z_4) &= (z_1, -z_2, \bar{z}_3, z_4). & h_2 h_1(z_1, z_2, z_3, z_4) &= (-z_1, -z_2, z_3, \bar{z}_4) \\
g_3(z_1, z_2, z_3, z_4) &= (z_1, z_2, -z_3, z_4). & h_3(z_1, z_2, z_3, z_4) &= (z_1, z_2, -z_3, z_4) \\
g_4(z_1, z_2, z_3, z_4) &= (z_1, z_2, z_3, -z_4). & h_4(z_1, z_2, z_3, z_4) &= (z_1, z_2, z_3, -z_4)
\end{aligned}$$

then

$$\Gamma_{13} = \left\langle \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\rangle$$

$$\Gamma_{27} = \left\langle \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\rangle.$$

Let $\varphi(z_1, z_2, z_3, z_4) = (z_1z_2, z_2, z_4, z_3)$, we get these commutative diagrams

$$\begin{array}{ccccccc}
(z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1z_2,z_2,z_4,z_3) & & (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1z_2,z_2,z_4,z_3) \\
{\scriptstyle g_1}\big\downarrow & & \big\downarrow{\scriptstyle h_1} & & {\scriptstyle g_3}\big\downarrow & & \big\downarrow{\scriptstyle h_4} \\
(-z_1,z_2,z_3,\bar{z}_4) & \xrightarrow{\varphi} & (-z_1z_2,z_2,\bar{z}_4,z_3) & & (z_1,z_2,-z_3,z_4) & \xrightarrow{\varphi} & (z_1z_2,z_2,z_4,-z_3)
\end{array}$$

$$\begin{array}{ccccccc}
(z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1z_2,z_2,z_4,z_3) & & (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1z_2,z_2,z_4,z_3) \\
{\scriptstyle g_2}\big\downarrow & & \big\downarrow{\scriptstyle h_2h_1} & & {\scriptstyle g_4}\big\downarrow & & \big\downarrow{\scriptstyle h_3} \\
(z_1,-z_2,\bar{z}_3,z_4) & \xrightarrow{\varphi} & (-z_1z_2,-z_2,z_4,\bar{z}_3) & & (z_1,z_2,z_3,-z_4) & \xrightarrow{\varphi} & (z_1z_2,z_2,-z_4,z_3).
\end{array}$$

Therefore $\exists\ \gamma = \left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \right) \in \mathbb{A}(n)$ s.t. $\gamma \Gamma_{13} \gamma^{-1} = \Gamma_{27}$.

• $M(A_{13}) \approx M(A_{29})$.

For

$$A_{13} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad A_{29} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned}
g_1(z_1,z_2,z_3,z_4) &= (-z_1,z_2,z_3,\bar{z}_4). & h_1h_2(z_1,z_2,z_3,z_4) &= (-z_1,-z_2,z_3,\bar{z}_4) \\
g_2(z_1,z_2,z_3,z_4) &= (z_1,-z_2,\bar{z}_3,z_4). & h_2(z_1,z_2,z_3,z_4) &= (z_1,-z_2,\bar{z}_3,z_4) \\
g_3(z_1,z_2,z_3,z_4) &= (z_1,z_2,-z_3,z_4). & h_3(z_1,z_2,z_3,z_4) &= (z_1,z_2,-z_3,z_4) \\
g_4(z_1,z_2,z_3,z_4) &= (z_1,z_2,z_3,-z_4). & h_4(z_1,z_2,z_3,z_4) &= (z_1,z_2,z_3,-z_4)
\end{aligned}$$

then

$$\Gamma_{13} = \left\langle \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\rangle$$

$$\Gamma_{29} = \left\langle \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\rangle.$$

Let $\varphi(z_1, z_2, z_3, z_4) = (z_1, z_1 z_2, z_3, z_4)$, we get these commutative diagrams

$$\begin{array}{ccccccc}
(z_1, z_2, z_3, z_4) & \xrightarrow{\varphi} & (z_1, z_1 z_2, z_3, z_4) & & (z_1, z_2, z_3, z_4) & \xrightarrow{\varphi} & (z_1, z_1 z_2, z_3, z_4) \\
{\scriptstyle g_1}\downarrow & & {\scriptstyle h_1 h_2}\downarrow & & {\scriptstyle g_3}\downarrow & & {\scriptstyle h_3}\downarrow \\
(-z_1, z_2, z_3, \bar{z}_4) & \xrightarrow{\varphi} & (-z_1, -z_1 z_2, z_3, \bar{z}_4) & & (z_1, z_2, -z_3, z_4) & \xrightarrow{\varphi} & (z_1, z_1 z_2, -z_3, z_4) \\
(z_1, z_2, z_3, z_4) & \xrightarrow{\varphi} & (z_1, z_1 z_2, z_3, z_4) & & (z_1, z_2, z_3, z_4) & \xrightarrow{\varphi} & (z_1, z_1 z_2, z_3, z_4) \\
{\scriptstyle g_2}\downarrow & & {\scriptstyle h_2}\downarrow & & {\scriptstyle g_4}\downarrow & & {\scriptstyle h_4}\downarrow \\
(z_1, -z_2, \bar{z}_3, z_4) & \xrightarrow{\varphi} & (z_1, -z_1 z_2, \bar{z}_3, z_4) & & (z_1, z_2, z_3, -z_4) & \xrightarrow{\varphi} & (z_1, z_1 z_2, z_3, -z_4).
\end{array}$$

Therefore $\exists\ \gamma = \left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right) \in \mathbb{A}(n)$ s.t. $\gamma \Gamma_{13} \gamma^{-1} = \Gamma_{29}$.

• $M(A_{13}) \approx M(A_{a2})$.
For

$$A_{13} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad A_{a2} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{array}{ll}
g_1(z_1, z_2, z_3, z_4) = (-z_1, z_2, z_3, \bar{z}_4). & h_1(z_1, z_2, z_3, z_4) = (-z_1, \bar{z}_2, z_3, z_4) \\
g_2(z_1, z_2, z_3, z_4) = (z_1, -z_2, \bar{z}_3, z_4). & h_2(z_1, z_2, z_3, z_4) = (z_1, -z_2, z_3, z_4) \\
g_3(z_1, z_2, z_3, z_4) = (z_1, z_2, -z_3, z_4). & h_3(z_1, z_2, z_3, z_4) = (z_1, z_2, -z_3, \bar{z}_4) \\
g_4(z_1, z_2, z_3, z_4) = (z_1, z_2, z_3, -z_4). & h_4(z_1, z_2, z_3, z_4) = (z_1, z_2, z_3, -z_4)
\end{array}$$

then

$$\Gamma_{13} = \left\langle \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\rangle$$

$$\Gamma_{a2} = \left\langle \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\rangle.$$

Let $\varphi(z_1, z_2, z_3, z_4) = (z_2, z_3, z_1, z_4)$, we get these commutative diagrams

$$\begin{tikzcd}
(z_1, z_2, z_3, z_4) \arrow[r, "\varphi"] \arrow[d, "g_1"'] & (z_2, z_3, z_1, z_4) \arrow[d, "h_3"'] & (z_1, z_2, z_3, z_4) \arrow[r, "\varphi"] \arrow[d, "g_3"'] & (z_2, z_3, z_1, z_4) \arrow[d, "h_2"'] \\
(-z_1, z_2, z_3, \bar{z}_4) \arrow[r, "\varphi"] & (z_2, z_3, -z_1, \bar{z}_4) & (z_1, z_2, -z_3, z_4) \arrow[r, "\varphi"] & (z_2, -z_3, z_1, z_4)
\end{tikzcd}$$

$$\begin{tikzcd}
(z_1, z_2, z_3, z_4) \arrow[r, "\varphi"] \arrow[d, "g_2"'] & (z_2, z_3, z_1, z_4) \arrow[d, "h_1"'] & (z_1, z_2, z_3, z_4) \arrow[r, "\varphi"] \arrow[d, "g_4"'] & (z_2, z_3, z_1, z_4) \arrow[d, "h_4"'] \\
(z_1, -z_2, \bar{z}_3, z_4) \arrow[r, "\varphi"] & (-z_2, \bar{z}_3, z_1, z_4) & (z_1, z_2, z_3, -z_4) \arrow[r, "\varphi"] & (z_2, z_3, z_1, -z_4).
\end{tikzcd}$$

Therefore $\exists\ \gamma = \left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right) \in \mathbb{A}(n)$ s.t. $\gamma \Gamma_{13} \gamma^{-1} = \Gamma_{a2}$.

• $M(A_{a2}) \approx M(A_{a6})$.
For

$$A_{a2} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad A_{a6} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned}
g_1(z_1, z_2, z_3, z_4) &= (-z_1, \bar{z}_2, z_3, z_4). & h_1 h_3(z_1, z_2, z_3, z_4) &= (-z_1, \bar{z}_2, -z_3, z_4) \\
g_2(z_1, z_2, z_3, z_4) &= (z_1, -z_2, z_3, z_4). & h_2(z_1, z_2, z_3, z_4) &= (z_1, -z_2, z_3, z_4) \\
g_3(z_1, z_2, z_3, z_4) &= (z_1, z_2, -z_3, \bar{z}_4). & h_3(z_1, z_2, z_3, z_4) &= (z_1, z_2, -z_3, \bar{z}_4) \\
g_4(z_1, z_2, z_3, z_4) &= (z_1, z_2, z_3, -z_4). & h_4(z_1, z_2, z_3, z_4) &= (z_1, z_2, z_3, -z_4)
\end{aligned}$$

then

$$\Gamma_{a2} = \Big\langle \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Big\rangle$$

$$\Gamma_{a6} = \Big\langle \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Big\rangle.$$

Let $\varphi(z_1, z_2, z_3, z_4) = (z_1, z_2, z_1z_3, z_4)$, we get these commutative diagrams

$$\begin{array}{ccccccc} (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1,z_2,z_1z_3,z_4) & & (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1,z_2,z_1z_3,z_4) \\ {\scriptstyle g_1}\big\downarrow & & \big\downarrow{\scriptstyle h_1h_3} & & {\scriptstyle g_3}\big\downarrow & & \big\downarrow{\scriptstyle h_3} \\ (-z_1,\bar{z}_2,z_3,z_4) & \xrightarrow{\varphi} & (-z_1,\bar{z}_2,-z_1z_3,z_4) & & (z_1,z_2,-z_3,\bar{z}_4) & \xrightarrow{\varphi} & (z_1,z_2,-z_1z_3,\bar{z}_4) \end{array}$$

$$\begin{array}{ccccccc} (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1,z_2,z_1z_3,z_4) & & (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1,z_2,z_1z_3,z_4) \\ {\scriptstyle g_2}\big\downarrow & & \big\downarrow{\scriptstyle h_2} & & {\scriptstyle g_4}\big\downarrow & & \big\downarrow{\scriptstyle h_4} \\ (z_1,-z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1,-z_2,z_1z_3,z_4) & & (z_1,z_2,z_3,-z_4) & \xrightarrow{\varphi} & (z_1,z_2,z_1z_3,-z_4). \end{array}$$

Therefore $\exists\ \gamma = \Big(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Big) \in \mathbb{A}(n)$ s.t. $\gamma\Gamma_{a2}\gamma^{-1} = \Gamma_{a6}$.

(24). Similar to (4).

(25). Similar to (11).

(26). $M(A_{13})$ is not diffeomorphic to $M(A_{a3})$.

Let

$$\Gamma_{a3} = <\tilde{g}_1, \tilde{g}_2, t_3, t_4>$$
$$= \Big\langle \left(\begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right), \left(\begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\right),$$
$$\left(\begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right), \left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right)\Big\rangle$$

$$\Gamma_{13} = <\tilde{h}_1, \tilde{h}_2, t_3, t_4>$$
$$= \Big\langle \left(\begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\right), \left(\begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right),$$
$$\left(\begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right), \left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right)\Big\rangle.$$

We know from (15) above, that the center $\mathcal{C}(\Gamma_{a3}) = \langle t_1, t_3\rangle$ where $t_1 = \tilde{g}_1^2$. There is an extension

$$1 \to \langle t_1, t_3\rangle \to \Gamma_{a3} \to \Delta_{a3} \to 1,$$

where Δ_{a3} is isomorphic to

$$\Big\langle \gamma_1 = \left(\begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right), \gamma_2 = \left(\begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\Big\rangle \rtimes \langle\beta\rangle$$

where $\beta = \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\right)$.

For Γ_{13}, since $\tilde{h}_1 t_3 \tilde{h}_1^{-1} = t_3$, $\tilde{h}_1 t_4 \tilde{h}_1^{-1} = t_4^{-1}$, $h_2 t_3 h_2^{-1} = t_3^{-1}$ and $h_2 t_4 h_2^{-1} = t_4$, then $\Gamma_{13} = \langle t_3, t_4\rangle \rtimes \langle \tilde{h}_1, \tilde{h}_2\rangle$. Then center $\mathcal{C}(\Gamma_{13}) = \langle t_1, t_2\rangle$. It induces an extension

$$1 \to \langle t_1, t_2\rangle \to \Gamma_{13} \to \Delta_{13} \to 1,$$

where Δ_{13} is isomorphic to

$$\Big\langle s_1 = \left(\begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right), s_2 = \left(\begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\Big\rangle \rtimes \langle\alpha, \beta\rangle$$

with $\alpha = \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right), \beta = \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\right)$.

Suppose that Γ_{a3} is isomorphic to Γ_{13} with isomorphism $\varphi : \Gamma_{a3} \to \Gamma_{13}$ which induces an isomorphism $\hat{\varphi} : \Delta_{a3} \to \Delta_{13}$

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \langle t_1, t_3\rangle & \longrightarrow & \Gamma_{a3} & \longrightarrow & \Delta_{a3} & \longrightarrow & 1 \\
 & & & & {\scriptstyle\varphi}\big\downarrow & & {\scriptstyle\hat{\varphi}}\big\downarrow & & \\
1 & \longrightarrow & \langle t_1, t_2\rangle & \longrightarrow & \Gamma_{13} & \longrightarrow & \Delta_{13} & \longrightarrow & 1.
\end{array}$$

By the same argument as (15) above, consider $\hat{\varphi}(\gamma_1^{a_1}\gamma_2^{a_2}\beta) = \alpha$ and $\hat{\varphi}(\gamma_1^{b_1}\gamma_2^{b_2}\beta) = \beta$ where $a_1, a_2, b_1, b_2 \in \mathbb{Z}$. Then

$$\begin{aligned}
\alpha\beta =& \hat{\varphi}(\gamma_1^{a_1}\gamma_2^{a_2}\beta\gamma_1^{b_1}\gamma_2^{b_2}\beta) \\
=& \hat{\varphi}(\gamma_1^{a_1}\gamma_2^{a_2}\gamma_1^{-b_1}\beta\gamma_2^{b_2}\beta) \quad (\beta\gamma_1\beta^{-1} = \gamma_1^{-1}) \\
=& \hat{\varphi}(\gamma_1^{a_1}\gamma_2^{a_2}\gamma_1^{-b_1}\gamma_2^{b_2}) \qquad (\beta\gamma_2\beta^{-1} = \gamma_2).
\end{aligned}$$

Since $\alpha\beta$ is a torsion element and $\hat{\varphi}$ is an isomorphism, then $\gamma_1^{a_1}\gamma_2^{a_2}\gamma_1^{-b_1}\gamma_2^{b_2}$ is also a torsion element. Let

$$\begin{aligned}
g =& \gamma_1^{a_1}\gamma_2^{a_2}\gamma_1^{-b_1}\gamma_2^{b_2} \\
=& \Big(\begin{pmatrix} \frac{1}{2}a_1 \\ (-1)^{a_1}\frac{1}{2}a_2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & (-1)^{a_1} \end{pmatrix}\Big)\Big(\begin{pmatrix} -\frac{1}{2}b_1 \\ (-1)^{-b_1}\frac{1}{2}b_2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & (-1)^{-b_1} \end{pmatrix}\Big) \\
=& \Big(\begin{pmatrix} \frac{1}{2}(a_1 - b_1) \\ \frac{1}{2}((-1)^{a_1}a_2 + (-1)^{(a_1-b_1)}b_2) \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & (-1)^{a_1-b_1} \end{pmatrix}\Big).
\end{aligned}$$

We have

$$\begin{aligned}
g^2 =& \Big(\begin{pmatrix} a_1 - b_1 \\ (1 + (-1)^{(a_1-b_1)})\frac{1}{2}((-1)^{a_1}a_2 + (-1)^{(a_1-b_1)}b_2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\Big) \\
=& \Big(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\Big).
\end{aligned}$$

Therefore we get $a_1 = b_1$ and $(-1)^{a_1}a_2 + b_2 = 0$. Hence $\hat{\varphi}(g) = \hat{\varphi}\Big(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\Big) = \Big(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\Big)$. This yields a contradiction.

(27). $M(A_{13})$ is not diffeomorphic to $M(A_{a4})$.

Let

$$\begin{aligned}
\Gamma_{a4} =& <\tilde{g}_1, \tilde{g}_2, \tilde{g}_3, t_4> \\
=& \Big\langle \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \\
& \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Big\rangle
\end{aligned}$$

$$\begin{aligned}
\Gamma_{13} =& <\tilde{h}_1, \tilde{h}_2, t_3, t_4> \\
=& \Big\langle \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\
& \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Big\rangle.
\end{aligned}$$

Since $\tilde{g}_1\tilde{g}_2\tilde{g}_1^{-1} = \tilde{g}_2^{-1}$, $\tilde{g}_1 t_4 \tilde{g}_1^{-1} = t_4$, $\tilde{g}_3\tilde{g}_2\tilde{g}_3^{-1} = \tilde{g}_2$, $\tilde{g}_3 t_4 \tilde{g}_3^{-1} = t_4^{-1}$, then $\Gamma_{a4} = \langle \tilde{g}_2, t_4 \rangle \rtimes \langle \tilde{g}_1, \tilde{g}_3 \rangle$. Then the center $\mathcal{C}(\Gamma_{a4}) = \langle t_1, t_3 \rangle$ where $t_1 = \tilde{g}_1^2$ and $t_3 = \tilde{g}_3^2$. There is an extension

$$1 \to \langle t_1, t_3 \rangle \to \Gamma_{a4} \to \Delta_{a4} \to 1,$$

where Δ_{a4} is isomorphic to

$$\Big\langle \gamma_1 = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \gamma_2 = \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Big\rangle \rtimes \langle \alpha_1, \alpha_2 \rangle$$

with $\alpha_1 = \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right)$ and $\alpha_2 = \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right)$.

For Γ_{13}, the center $\mathcal{C}(\Gamma_{13}) = \langle t_1, t_2 \rangle$. It induces an extension

$$1 \to \langle t_1, t_2 \rangle \to \Gamma_{13} \to \Delta_{13} \to 1,$$

where Δ_{13} is isomorphic to

$$\Big\langle s_1 = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, s_2 = \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Big\rangle \rtimes \langle \alpha, \beta \rangle$$

with $\alpha = \alpha_1$ and $\beta = \alpha_2$.

Suppose that Γ_{13} is isomorphic to Γ_{a4} with isomorphism $\varphi : \Gamma_{13} \to \Gamma_{a4}$ which induces an isomorphism $\hat{\varphi} : \Delta_{13} \to \Delta_{a4}$

$$\begin{tikzcd}
1 \arrow[r] & \langle t_1, t_3 \rangle \arrow[r] & \Gamma_{a4} \arrow[r] & \Delta_{a4} \arrow[r] & 1 \\
1 \arrow[r] & \langle t_1, t_2 \rangle \arrow[r] & \Gamma_{13} \arrow[r] \arrow[u, "\varphi"] & \Delta_{13} \arrow[r] \arrow[u, "\hat{\varphi}"] & 1.
\end{tikzcd}$$

Since $\tilde{g}_2\tilde{g}_3 = \tilde{g}_3\tilde{g}_2$ then

$$\begin{aligned}
\tilde{g}_2^{-1}\tilde{g}_3 &= \tilde{g}_3^{-1}\tilde{g}_3\tilde{g}_2^{-1}\tilde{g}_3 \\
&= \tilde{g}_3^{-1}\tilde{g}_2^{-1}\tilde{g}_3^2 \\
&= (\tilde{g}_2\tilde{g}_3)^{-1}\tilde{g}_3^2 \\
\tilde{g}_2^{-1}\tilde{g}_1\tilde{g}_3\tilde{g}_1^{-1} &= (\tilde{g}_2\tilde{g}_3)^{-1}\tilde{g}_3^2.
\end{aligned}$$

We get

$$\tilde{g}_1(\tilde{g}_2\tilde{g}_3)\tilde{g}_1{}^{-1} = (\tilde{g}_2\tilde{g}_3)^{-1}\tilde{g}_3^2. \tag{4.1}$$

Now we want to find the elements $h, h' \in \Gamma_{13}$ such that $\varphi(h) = \tilde{g}_1$ and $\varphi(h') = \tilde{g}_2\tilde{g}_3$. According to the above diagram, let us consider the following diagram

$$\begin{tikzcd}
\left(\begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}, I\right) = \tilde{g}_2\tilde{g}_3 \arrow[r] & \gamma_1\alpha_2 = \left(\begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}, I\right) \\
\left(\begin{pmatrix} c \\ d \\ \frac{1}{2}a \\ \frac{1}{2}b \end{pmatrix}, I\right) = t_3{}^a t_4{}^b t_1{}^c t_2{}^d \arrow[r] \arrow[u, "\varphi"] & s_1{}^a s_2{}^b = \left(\begin{pmatrix} \frac{1}{2}a \\ \frac{1}{2}b \end{pmatrix}, I\right) \arrow[u, "\hat{\varphi}"]
\end{tikzcd}$$

so,

$$\tilde{g}_2\tilde{g}_3 = \varphi(t_3{}^a t_4{}^b t_1{}^c t_2{}^d), \tag{4.2}$$

and from the diagrams

$$\begin{tikzcd}
\left(\begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right) = \tilde{g}_1 \arrow[r] & \alpha_1 = \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\right) \\
\begin{matrix} \tilde{h}_1 {t_3}^{a'} {t_4}^{b'} {t_1}^{c'} {t_2}^{d'} = \\ \left(\begin{pmatrix} c' + \frac{1}{2} \\ d' \\ \frac{1}{2}a' \\ -\frac{1}{2}b' \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\right) \end{matrix} \arrow[u, "\varphi"] \arrow[r] & \begin{matrix} \beta {s_1}^{a'} {s_2}^{b'} = \\ \left(\begin{pmatrix} \frac{1}{2}a' \\ -\frac{1}{2}b' \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) \end{matrix} \arrow[u, "\hat{\varphi}"]
\end{tikzcd}$$

or

$$\begin{tikzcd}
\left(\begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right) = \tilde{g}_1 \arrow[r] & \alpha_1 = \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\right) \\
\begin{matrix} \tilde{h}_2 {t_3}^{a'} {t_4}^{b'} {t_1}^{c'} {t_2}^{d'} = \\ \left(\begin{pmatrix} c' \\ d' + \frac{1}{2} \\ -\frac{1}{2}a' \\ \frac{1}{2}b' \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right) \end{matrix} \arrow[u, "\varphi"] \arrow[r] & \begin{matrix} \alpha {s_1}^{a'} {s_2}^{b'} = \\ \left(\begin{pmatrix} -\frac{1}{2}a' \\ \frac{1}{2}b' \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\right) \end{matrix} \arrow[u, "\hat{\varphi}"]
\end{tikzcd}$$

we get

$$\tilde{g}_1 = \varphi(\tilde{h}_1 {t_3}^{a'} {t_4}^{b'} {t_1}^{c'} {t_2}^{d'}) = \varphi(\tilde{h}_1 t), \tag{4.3}$$

or

$$\tilde{g}_1 = \varphi(\tilde{h}_2 {t_3}^{a'} {t_4}^{b'} {t_1}^{c'} {t_2}^{d'}) = \varphi(\tilde{h}_2 t), \tag{4.4}$$

where $t = {t_3}^{a'} {t_4}^{b'} {t_1}^{c'} {t_2}^{d'}$.

Next we get

$$\begin{aligned}
\tilde{g}_1(\tilde{g}_2\tilde{g}_3)\tilde{g}_1{}^{-1} =&\varphi(\tilde{h}_1 t)\varphi({t_3}^a{t_4}^b{t_1}^c{t_2}^d)\varphi(\tilde{h}_1 t)^{-1} \quad \text{(by (4.2), (4.3))}\\
(\tilde{g}_2\tilde{g}_3)^{-1}t_3 =&\varphi(\tilde{h}_1 t({t_3}^a{t_4}^b{t_1}^c{t_2}^d)t^{-1}\tilde{h}_1^{-1}) \quad \text{(by (4.1))}\\
=&\varphi(\tilde{h}_1({t_3}^a{t_4}^b{t_1}^c{t_2}^d)\tilde{h}_1^{-1})\\
=&\varphi(\tilde{h}_1{t_3}^a{t_4}^b\tilde{h}_1^{-1}{t_1}^c{t_2}^d)\\
=&\varphi(({t_3}^a{t_4}^b)^{-1}t_3^{2a}{t_1}^c{t_2}^d) \quad (h_1{t_3}^a{t_4}^b h_1^{-1} = {t_3}^a{t_4}^{-b})\\
=&\varphi(({t_3}^a{t_4}^b)^{-1}({t_1}^c{t_2}^d)t_3^{2a})\\
=&\varphi(({t_3}^a{t_4}^b)^{-1}({t_1}^c{t_2}^d)^{-1}(({t_1}^c{t_2}^d)t_3^a)^2)\\
=&\varphi(({t_3}^a{t_4}^b{t_1}^c{t_2}^d)^{-1}({t_1}^c{t_2}^d t_3^a)^2)\\
=&(\tilde{g}_2\tilde{g}_3)^{-1}\varphi(({t_1}^c{t_2}^d t_3^a)^2),
\end{aligned}$$

or

$$\begin{aligned}
\tilde{g}_1(\tilde{g}_2\tilde{g}_3)\tilde{g}_1{}^{-1} =&\varphi(\tilde{h}_2 t)\varphi({t_3}^a{t_4}^b{t_1}^c{t_2}^d)\varphi(\tilde{h}_2 t)^{-1} \quad \text{(by (4.2), (4.4))}\\
(\tilde{g}_2\tilde{g}_3)^{-1}t_3 =&\varphi(\tilde{h}_2 t({t_3}^a{t_4}^b{t_1}^c{t_2}^d)t^{-1}\tilde{h}_2^{-1}) \quad \text{(by (4.1))}\\
=&\varphi(\tilde{h}_2({t_3}^a{t_4}^b{t_1}^c{t_2}^d)\tilde{h}_2^{-1})\\
=&\varphi(\tilde{h}_2{t_3}^a{t_4}^b\tilde{h}_1^{-1}{t_1}^c{t_2}^d)\\
=&\varphi(({t_3}^a{t_4}^b)^{-1}t_4^{2b}{t_1}^c{t_2}^d) \quad (h_2{t_3}^a{t_4}^b h_2^{-1} = {t_3}^{-a}{t_4}^{b})\\
=&\varphi(({t_3}^a{t_4}^b)^{-1}({t_1}^c{t_2}^d)t_4^{2b})\\
=&\varphi(({t_3}^a{t_4}^b)^{-1}({t_1}^c{t_2}^d)^{-1}(({t_1}^c{t_2}^d)t_4^b)^2)\\
=&\varphi(({t_3}^a{t_4}^b{t_1}^c{t_2}^d)^{-1}({t_1}^c{t_2}^d t_4^b)^2)\\
=&(\tilde{g}_2\tilde{g}_3)^{-1}\varphi(({t_1}^c{t_2}^d t_4^b)^2).
\end{aligned}$$

Hence

$$t_3 = \varphi(({t_1}^c{t_2}^d t_3^a)^2), \tag{4.5}$$

or

$$t_3 = \varphi(({t_1}^c{t_2}^d t_4^b)^2). \tag{4.6}$$

Let us consider the diagrams below

$$\begin{tikzcd}
\left(\begin{pmatrix} p \\ \frac{1}{2}m \\ \frac{1}{2}m+l \\ \frac{1}{2}q \end{pmatrix}, I\right) = t_3^l(\tilde{g}_2\tilde{g}_3)^m(t_1^pt_4^q) \arrow[r] & (\gamma_1\alpha_2)^m\gamma_2^q = \left(\begin{pmatrix} \frac{1}{2}m \\ \frac{1}{2}q \end{pmatrix}, I\right) \\
\left(\begin{pmatrix} c \\ d \\ \frac{1}{2}a \\ 0 \end{pmatrix}, I\right) = {t_1}^c{t_2}^dt_3^a \arrow[r] \arrow[u, "\varphi"] & {s_1}^a = \left(\begin{pmatrix} \frac{1}{2}a \\ 0 \end{pmatrix}, I\right) \arrow[u, "\hat{\varphi}"]
\end{tikzcd}$$

or

$$\begin{tikzcd}
\left(\begin{pmatrix} p \\ \frac{1}{2}m \\ \frac{1}{2}m+l \\ \frac{1}{2}q \end{pmatrix}, I\right) = t_3^l(\tilde{g}_2\tilde{g}_3)^m(t_1^pt_4^q) \arrow[r] & (\gamma_1\alpha_2)^m\gamma_2^q = \left(\begin{pmatrix} \frac{1}{2}m \\ \frac{1}{2}q \end{pmatrix}, I\right) \\
\left(\begin{pmatrix} c \\ d \\ 0 \\ \frac{1}{2}b \end{pmatrix}, I\right) = {t_1}^c{t_2}^dt_4^b \arrow[r] \arrow[u, "\varphi"] & {s_2}^b = \left(\begin{pmatrix} 0 \\ \frac{1}{2}b \end{pmatrix}, I\right), \arrow[u, "\hat{\varphi}"]
\end{tikzcd}$$

we have

$$\begin{aligned}
\varphi({t_1}^c{t_2}^dt_3^a) =& t_3^l(\tilde{g}_2\tilde{g}_3)^m(t_1^pt_4^q) \\
\varphi(({t_1}^c{t_2}^dt_3^a)^2) =& t_3^{2l}(\tilde{g}_2\tilde{g}_3)^{2m}(t_1^{2p}t_4^{2q}) \\
=& t_3^{2l}(t_2t_3)^m(t_1^{2p}t_4^{2q}) \qquad ((\tilde{g}_2\tilde{g}_3)^2 = t_2t_3) \\
=& t_1^{2p}t_2^mt_3^{m+2l}t_4^{2q},
\end{aligned}$$

or

$$\begin{aligned}
\varphi({t_1}^c{t_2}^dt_4^b) =& t_3^l(\tilde{g}_2\tilde{g}_3)^m(t_1^pt_4^q) \\
\varphi(({t_1}^c{t_2}^dt_4^b)^2) =& t_3^{2l}(\tilde{g}_2\tilde{g}_3)^{2m}(t_1^{2p}t_4^{2q}) \\
=& t_3^{2l}(t_2t_3)^m(t_1^{2p}t_4^{2q}) \qquad ((\tilde{g}_2\tilde{g}_3)^2 = t_2t_3) \\
=& t_1^{2p}t_2^mt_3^{m+2l}t_4^{2q}.
\end{aligned}$$

Therefore from (4.5) or (4.6) $t_3 = t_1^{2p}t_2^mt_3^{m+2l}t_4^{2q}$. Hence $p = m = q = 0$ and $m + 2l = 1$ or $2l = 1$. This yields a contradiction.

(28).
- $M(A_{a10}) \approx M(A_{a12})$.

For

$$A_{a10} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad A_{a12} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned} g_1(z_1, z_2, z_3, z_4) &= (-z_1, \bar{z}_2, z_3, z_4). & h_1(z_1, z_2, z_3, z_4) &= (-z_1, \bar{z}_2, z_3, z_4) \\ g_2(z_1, z_2, z_3, z_4) &= (z_1, -z_2, \bar{z}_3, z_4). & h_2h_3(z_1, z_2, z_3, z_4) &= (z_1, -z_2, -\bar{z}_3, z_4) \\ g_3(z_1, z_2, z_3, z_4) &= (z_1, z_2, -z_3, \bar{z}_4). & h_3(z_1, z_2, z_3, z_4) &= (z_1, z_2, -z_3, \bar{z}_4) \\ g_4(z_1, z_2, z_3, z_4) &= (z_1, z_2, z_3, -z_4). & h_4(z_1, z_2, z_3, z_4) &= (z_1, z_2, z_3, -z_4) \end{aligned}$$

then

$$\begin{aligned} \Gamma_{a10} = \Big\langle & \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\ & \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Big\rangle \end{aligned}$$

$$\begin{aligned} \Gamma_{a12} = \Big\langle & \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\ & \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Big\rangle. \end{aligned}$$

Let $\varphi(z_1, z_2, z_3, z_4) = (z_1, z_2, iz_3, z_4)$, we get these commutative diagrams

$$\begin{tikzcd} (z_1, z_2, z_3, z_4) \arrow[r, "\varphi"] \arrow[d, "g_1"'] & (z_1, z_2, iz_3, z_4) \arrow[d, "h_1"'] & (z_1, z_2, z_3, z_4) \arrow[r, "\varphi"] \arrow[d, "g_3"'] & (z_1, z_2, iz_3, z_4) \arrow[d, "h_3"'] \\ (-z_1, \bar{z}_2, z_3, z_4) \arrow[r, "\varphi"] & (-z_1, \bar{z}_2, iz_3, z_4) & (z_1, z_2, -z_3, \bar{z}_4) \arrow[r, "\varphi"] & (z_1, z_2, -iz_3, \bar{z}_4) \end{tikzcd}$$

$$\begin{tikzcd} (z_1, z_2, z_3, z_4) \arrow[r, "\varphi"] \arrow[d, "g_2"'] & (z_1, z_2, iz_3, z_4) \arrow[d, "h_2h_3"'] & (z_1, z_2, z_3, z_4) \arrow[r, "\varphi"] \arrow[d, "g_4"'] & (z_1, z_2, iz_3, z_4) \arrow[d, "h_4"'] \\ (z_1, -z_2, \bar{z}_3, z_4) \arrow[r, "\varphi"] & (z_1, -z_2, -\bar{i}z_3, z_4) & (z_1, z_2, z_3, -z_4) \arrow[r, "\varphi"] & (z_1, z_2, iz_3, -z_4). \end{tikzcd}$$

Therefore $\exists\ \gamma = \left(\begin{pmatrix} 0 \\ 0 \\ \frac{1}{4} \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right) \in \mathbb{A}(n)$ s.t. $\gamma\Gamma_{a10}\gamma^{-1} = \Gamma_{a12}$.

• $M(A_{a10}) \approx M(A_{a26})$.
For

$$A_{a10} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad A_{a26} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned} g_1(z_1,z_2,z_3,z_4) &= (-z_1,\bar{z}_2,z_3,z_4). & h_1h_2(z_1,z_2,z_3,z_4) &= (-z_1,-\bar{z}_2,z_3,z_4) \\ g_2(z_1,z_2,z_3,z_4) &= (z_1,-z_2,\bar{z}_3,z_4). & h_2(z_1,z_2,z_3,z_4) &= (z_1,-z_2,\bar{z}_3,z_4) \\ g_3(z_1,z_2,z_3,z_4) &= (z_1,z_2,-z_3,\bar{z}_4). & h_3(z_1,z_2,z_3,z_4) &= (z_1,z_2,-z_3,\bar{z}_4) \\ g_4(z_1,z_2,z_3,z_4) &= (z_1,z_2,z_3,-z_4). & h_4(z_1,z_2,z_3,z_4) &= (z_1,z_2,z_3,-z_4) \end{aligned}$$

then

$$\begin{aligned} \Gamma_{a10} = \Big\langle & \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\ & \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\Big\rangle \end{aligned}$$

$$\begin{aligned} \Gamma_{a26} = \Big\langle & \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\ & \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\Big\rangle. \end{aligned}$$

Let $\varphi(z_1,z_2,z_3,z_4) = (z_1,iz_2,z_3,z_4)$, we get these commutative diagrams

$$\begin{array}{ccc} (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1,iz_2,z_3,z_4) \\ {\scriptstyle g_1}\downarrow & & {\scriptstyle h_1h_2}\downarrow \\ (-z_1,\bar{z}_2,z_3,z_4) & \xrightarrow{\varphi} & (-z_1,-i\bar{z}_2,z_3,z_4) \end{array} \qquad \begin{array}{ccc} (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1,iz_2,z_3,z_4) \\ {\scriptstyle g_3}\downarrow & & {\scriptstyle h_3}\downarrow \\ (z_1,z_2,-z_3,\bar{z}_4) & \xrightarrow{\varphi} & (z_1,iz_2,-z_3,\bar{z}_4) \end{array}$$

$$\begin{array}{ccccccc}
(z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1,iz_2,z_3,z_4) & (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1,iz_2,z_3,z_4) \\
{\scriptstyle g_2}\downarrow & & {\scriptstyle h_2}\downarrow & {\scriptstyle g_4}\downarrow & & {\scriptstyle h_4}\downarrow \\
(z_1,-z_2,\bar{z}_3,z_4) & \xrightarrow{\varphi} & (z_1,-iz_2,\bar{z}_3,z_4) & (z_1,z_2,z_3,-z_4) & \xrightarrow{\varphi} & (z_1,iz_2,z_3,-z_4).
\end{array}$$

Therefore $\exists\ \gamma = \left(\begin{pmatrix} 0 \\ \frac{1}{4} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right) \in \mathbb{A}(n)$ s.t. $\gamma \Gamma_{a10} \gamma^{-1} = \Gamma_{a26}$.

• $M(A_{a12}) \approx M(A_{a32})$.
For

$$A_{a12} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad A_{a32} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{array}{rr}
g_1(z_1,z_2,z_3,z_4) = (-z_1,\bar{z}_2,z_3,z_4). & h_1h_2(z_1,z_2,z_3,z_4) = (-z_1,-\bar{z}_2,z_3,z_4) \\
g_2(z_1,z_2,z_3,z_4) = (z_1,-z_2,\bar{z}_3,\bar{z}_4). & h_2(z_1,z_2,z_3,z_4) = (z_1,-z_2,\bar{z}_3,\bar{z}_4) \\
g_3(z_1,z_2,z_3,z_4) = (z_1,z_2,-z_3,\bar{z}_4). & h_3(z_1,z_2,z_3,z_4) = (z_1,z_2,-z_3,\bar{z}_4) \\
g_4(z_1,z_2,z_3,z_4) = (z_1,z_2,z_3,-z_4). & h_4(z_1,z_2,z_3,z_4) = (z_1,z_2,z_3,-z_4)
\end{array}$$

then

$$\Gamma_{a12} = \left\langle \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\rangle$$

$$\Gamma_{a32} = \left\langle \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\rangle.$$

Let $\varphi(z_1,z_2,z_3,z_4) = (z_1,iz_2,z_3,z_4)$, we get these commutative diagrams

$$\begin{array}{ccccccc}
(z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1,iz_2,z_3,z_4) & & (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1,iz_2,z_3,z_4) \\
{\scriptstyle g_1}\downarrow & & {\scriptstyle h_1h_2}\downarrow & & {\scriptstyle g_3}\downarrow & & {\scriptstyle h_3}\downarrow \\
(-z_1,\bar{z}_2,z_3,z_4) & \xrightarrow{\varphi} & (-z_1,-\bar{i}z_2,z_3,z_4) & & (z_1,z_2,-z_3,\bar{z}_4) & \xrightarrow{\varphi} & (z_1,iz_2,-z_3,\bar{z}_4) \\
(z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1,iz_2,z_3,z_4) & & (z_1,z_2,z_3,z_4) & \xrightarrow{\varphi} & (z_1,iz_2,z_3,z_4) \\
{\scriptstyle g_2}\downarrow & & {\scriptstyle h_2}\downarrow & & {\scriptstyle g_4}\downarrow & & {\scriptstyle h_4}\downarrow \\
(z_1,-z_2,\bar{z}_3,\bar{z}_4) & \xrightarrow{\varphi} & (z_1,-iz_2,\bar{z}_3,\bar{z}_4) & & (z_1,z_2,z_3,-z_4) & \xrightarrow{\varphi} & (z_1,iz_2,z_3,-z_4).
\end{array}$$

Therefore $\exists\ \gamma = \left(\begin{pmatrix} 0 \\ \frac{1}{4} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right) \in \mathbb{A}(n)$ s.t. $\gamma\Gamma_{a12}\gamma^{-1} = \Gamma_{a32}$.

• $M(A_{a14}) \approx M(A_{a16})$.
For

$$A_{a14} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad A_{a16} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{array}{rr}
g_1(z_1,z_2,z_3,z_4) = (-z_1,\bar{z}_2,z_3,\bar{z}_4). & h_1(z_1,z_2,z_3,z_4) = (-z_1,\bar{z}_2,z_3,\bar{z}_4) \\
g_2(z_1,z_2,z_3,z_4) = (z_1,-z_2,\bar{z}_3,z_4). & h_2h_3(z_1,z_2,z_3,z_4) = (z_1,-z_2,-\bar{z}_3,z_4) \\
g_3(z_1,z_2,z_3,z_4) = (z_1,z_2,-z_3,\bar{z}_4). & h_3(z_1,z_2,z_3,z_4) = (z_1,z_2,-z_3,\bar{z}_4) \\
g_4(z_1,z_2,z_3,z_4) = (z_1,z_2,z_3,-z_4). & h_4(z_1,z_2,z_3,z_4) = (z_1,z_2,z_3,-z_4)
\end{array}$$

then

$$\Gamma_{a14} = \Big\langle \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$
$$\begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Big\rangle$$

$$\Gamma_{a16} = \Big\langle \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$
$$\begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Big\rangle.$$

Let $\varphi(z_1, z_2, z_3, z_4) = (z_1, z_2, iz_3, z_4)$, we get these commutative diagrams

$$\begin{tikzcd}
(z_1, z_2, z_3, z_4) \arrow[r, "\varphi"] \arrow[d, "g_1"'] & (z_1, z_2, iz_3, z_4) \arrow[d, "h_1"'] & (z_1, z_2, z_3, z_4) \arrow[r, "\varphi"] \arrow[d, "g_3"'] & (z_1, z_2, iz_3, z_4) \arrow[d, "h_3"'] \\
(-z_1, \bar{z}_2, z_3, \bar{z}_4) \arrow[r, "\varphi"] & (-z_1, \bar{z}_2, iz_3, \bar{z}_4) & (z_1, z_2, -z_3, \bar{z}_4) \arrow[r, "\varphi"] & (z_1, z_2, -iz_3, \bar{z}_4)
\end{tikzcd}$$

$$\begin{tikzcd}
(z_1, z_2, z_3, z_4) \arrow[r, "\varphi"] \arrow[d, "g_2"'] & (z_1, z_2, iz_3, z_4) \arrow[d, "h_2h_3"'] & (z_1, z_2, z_3, z_4) \arrow[r, "\varphi"] \arrow[d, "g_4"'] & (z_1, z_2, iz_3, z_4) \arrow[d, "h_4"'] \\
(z_1, -z_2, \bar{z}_3, z_4) \arrow[r, "\varphi"] & (z_1, -z_2, -i\bar{z}_3, z_4) & (z_1, z_2, z_3, -z_4) \arrow[r, "\varphi"] & (z_1, z_2, iz_3, -z_4).
\end{tikzcd}$$

Therefore $\exists\ \gamma = \left(\begin{pmatrix} 0 \\ 0 \\ \frac{1}{4} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right) \in \mathbb{A}(n)$ s.t. $\gamma \Gamma_{a14} \gamma^{-1} = \Gamma_{a16}$.

• $M(A_{a14}) \approx M(A_{a30})$.
For

$$A_{a14} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad A_{a30} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned}
g_1(z_1, z_2, z_3, z_4) &= (-z_1, \bar{z}_2, z_3, \bar{z}_4). & h_1h_2(z_1, z_2, z_3, z_4) &= (-z_1, -\bar{z}_2, z_3, \bar{z}_4) \\
g_2(z_1, z_2, z_3, z_4) &= (z_1, -z_2, \bar{z}_3, z_4). & h_2(z_1, z_2, z_3, z_4) &= (z_1, -z_2, \bar{z}_3, z_4) \\
g_3(z_1, z_2, z_3, z_4) &= (z_1, z_2, -z_3, \bar{z}_4). & h_3(z_1, z_2, z_3, z_4) &= (z_1, z_2, -z_3, \bar{z}_4) \\
g_4(z_1, z_2, z_3, z_4) &= (z_1, z_2, z_3, -z_4). & h_4(z_1, z_2, z_3, z_4) &= (z_1, z_2, z_3, -z_4)
\end{aligned}$$

then

$$\Gamma_{a14} = \left\langle \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\rangle$$

$$\Gamma_{a30} = \left\langle \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\rangle.$$

Let $\varphi(z_1, z_2, z_3, z_4) = (z_1, iz_2, z_3, z_4)$, we get these commutative diagrams

$$\begin{tikzcd}
(z_1, z_2, z_3, z_4) \arrow[r, "\varphi"] \arrow[d, "g_1"'] & (z_1, iz_2, z_3, z_4) \arrow[d, "h_1 h_2"'] \\
(-z_1, \bar{z}_2, z_3, \bar{z}_4) \arrow[r, "\varphi"] & (-z_1, -i\bar{z}_2, z_3, \bar{z}_4)
\end{tikzcd}
\quad
\begin{tikzcd}
(z_1, z_2, z_3, z_4) \arrow[r, "\varphi"] \arrow[d, "g_3"'] & (z_1, iz_2, z_3, z_4) \arrow[d, "h_3"'] \\
(z_1, z_2, -z_3, \bar{z}_4) \arrow[r, "\varphi"] & (z_1, iz_2, -z_3, \bar{z}_4)
\end{tikzcd}$$

$$\begin{tikzcd}
(z_1, z_2, z_3, z_4) \arrow[r, "\varphi"] \arrow[d, "g_2"'] & (z_1, iz_2, z_3, z_4) \arrow[d, "h_2"'] \\
(z_1, -z_2, \bar{z}_3, z_4) \arrow[r, "\varphi"] & (z_1, -iz_2, \bar{z}_3, z_4)
\end{tikzcd}
\quad
\begin{tikzcd}
(z_1, z_2, z_3, z_4) \arrow[r, "\varphi"] \arrow[d, "g_4"'] & (z_1, iz_2, z_3, z_4) \arrow[d, "h_4"'] \\
(z_1, z_2, z_3, -z_4) \arrow[r, "\varphi"] & (z_1, iz_2, z_3, -z_4).
\end{tikzcd}$$

Therefore $\exists\ \gamma = \left(\begin{pmatrix} 0 \\ \frac{1}{4} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right) \in \mathbb{A}(n)$ s.t. $\gamma \Gamma_{a14} \gamma^{-1} = \Gamma_{a30}$.

• $M(A_{a28}) \approx M(A_{a30})$.
For

$$A_{a28} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad A_{a30} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned}
g_1(z_1, z_2, z_3, z_4) &= (-z_1, \bar{z}_2, \bar{z}_3, z_4). & h_1 h_3(z_1, z_2, z_3, z_4) &= (-z_1, \bar{z}_2, -\bar{z}_3, z_4) \\
g_2(z_1, z_2, z_3, z_4) &= (z_1, -z_2, \bar{z}_3, \bar{z}_4). & h_2 h_3(z_1, z_2, z_3, z_4) &= (z_1, -z_2, -\bar{z}_3, \bar{z}_4) \\
g_3(z_1, z_2, z_3, z_4) &= (z_1, z_2, -z_3, \bar{z}_4). & h_3(z_1, z_2, z_3, z_4) &= (z_1, z_2, -z_3, \bar{z}_4) \\
g_4(z_1, z_2, z_3, z_4) &= (z_1, z_2, z_3, -z_4). & h_4(z_1, z_2, z_3, z_4) &= (z_1, z_2, z_3, -z_4)
\end{aligned}$$

then

$$\Gamma_{a28} = \left\langle \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\rangle$$

$$\Gamma_{a30} = \left\langle \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\rangle.$$

Let $\varphi(z_1, z_2, z_3, z_4) = (z_1, z_2, iz_3, z_4)$, we get these commutative diagrams

$$\begin{tikzcd} (z_1, z_2, z_3, z_4) \arrow[r, "\varphi"] \arrow[d, "g_1"'] & (z_1, z_2, iz_3, z_4) \arrow[d, "h_1h_3"'] \\ (-z_1, \bar{z}_2, \bar{z}_3, z_4) \arrow[r, "\varphi"] & (-z_1, \bar{z}_2, -\bar{iz}_3, z_4) \end{tikzcd} \quad \begin{tikzcd} (z_1, z_2, z_3, z_4) \arrow[r, "\varphi"] \arrow[d, "g_3"'] & (z_1, z_2, iz_3, z_4) \arrow[d, "h_3"'] \\ (z_1, z_2, -z_3, \bar{z}_4) \arrow[r, "\varphi"] & (z_1, z_2, -iz_3, \bar{z}_4) \end{tikzcd}$$

$$\begin{tikzcd} (z_1, z_2, z_3, z_4) \arrow[r, "\varphi"] \arrow[d, "g_2"'] & (z_1, z_2, iz_3, z_4) \arrow[d, "h_2h_3"'] \\ (z_1, -z_2, \bar{z}_3, \bar{z}_4) \arrow[r, "\varphi"] & (z_1, -z_2, -\bar{iz}_3, \bar{z}_4) \end{tikzcd} \quad \begin{tikzcd} (z_1, z_2, z_3, z_4) \arrow[r, "\varphi"] \arrow[d, "g_4"'] & (z_1, z_2, iz_3, z_4) \arrow[d, "h_4"'] \\ (z_1, z_2, z_3, -z_4) \arrow[r, "\varphi"] & (z_1, z_2, iz_3, -z_4). \end{tikzcd}$$

Therefore $\exists\ \gamma = \left(\begin{pmatrix} 0 \\ 0 \\ \frac{1}{4} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right) \in \mathbb{A}(n)$ s.t. $\gamma \Gamma_{a28} \gamma^{-1} = \Gamma_{a30}$.

(29). Similar to (4), because $\Phi_{a10} = \Phi_{a14} = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

(30). $M(A_{a10})$ is not diffeomorphic to $M(A_{a14})$.
For

$$A_{a10} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad A_{a14} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned} g_1(z_1,z_2,z_3,z_4) &= (-z_1,\bar z_2,z_3,z_4). & h_1(z_1,z_2,z_3,z_4) &= (-z_1,\bar z_2,z_3,\bar z_4) \\ g_2(z_1,z_2,z_3,z_4) &= (z_1,-z_2,\bar z_3,z_4). & h_2(z_1,z_2,z_3,z_4) &= (z_1,-z_2,\bar z_3,z_4) \\ g_3(z_1,z_2,z_3,z_4) &= (z_1,z_2,-z_3,\bar z_4). & h_3(z_1,z_2,z_3,z_4) &= (z_1,z_2,-z_3,\bar z_4) \\ g_4(z_1,z_2,z_3,z_4) &= (z_1,z_2,z_3,-z_4). & h_4(z_1,z_2,z_3,z_4) &= (z_1,z_2,z_3,-z_4) \end{aligned}$$

then

$$\begin{aligned} \Gamma_{a10} =& \langle \tilde g_1, \tilde g_2, \tilde g_3, t_4\rangle \\ & \left\langle \begin{pmatrix} \frac12 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac12 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \right. \\ & \left. \begin{pmatrix} 0 \\ 0 \\ \frac12 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac12 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\rangle \end{aligned}$$

$$\begin{aligned} \Gamma_{a14} =& \langle \tilde h_1, \tilde h_2, \tilde h_3, t_4\rangle \\ & \left\langle \begin{pmatrix} \frac12 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac12 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \right. \\ & \left. \begin{pmatrix} 0 \\ 0 \\ \frac12 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac12 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\rangle. \end{aligned}$$

Since $\tilde g_1\tilde g_2\tilde g_1^{-1} = \tilde g_2^{-1}$, $\tilde g_1\tilde g_3\tilde g_1^{-1} = \tilde g_3$, and $\tilde g_1 t_4\tilde g_1^{-1} = t_4$, then $\Gamma_{a10} = \langle \tilde g_2, \tilde g_3, t_4\rangle \rtimes \langle \tilde g_1\rangle$. Then the center $\mathcal{C}(\Gamma_{a10}) = \langle t_1\rangle$ where $t_1 = \tilde g_1^2$. There is an extension

$$1 \to \langle t_1\rangle \to \Gamma_{a10} \to \Delta_{a10} \to 1,$$

where $\Delta_{a10} = \Delta_{a10}^f \rtimes \langle\alpha\rangle = \langle p,q,r\rangle \rtimes \langle\alpha\rangle$ is isomorphic to

$$\left\langle \begin{pmatrix} \frac12 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac12 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac12 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle \rtimes \left\langle \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle.$$

Then, since $\tilde{h}_1\tilde{h}_2\tilde{h}_1^{-1} = \tilde{h}_2^{-1}$, $\tilde{h}_1\tilde{h}_3\tilde{h}_1^{-1} = \tilde{h}_3$, and $\tilde{h}_1 t_4\tilde{h}_1^{-1} = t_4^{-1}$, then $\Gamma_{a14} = \langle \tilde{h}_2, \tilde{h}_3, t_4\rangle \rtimes \langle \tilde{h}_1\rangle$. Then the center $\mathcal{C}(\Gamma_{a14}) = \langle t_1\rangle$ where $t_1 = \tilde{h}_1^2$. There is an extension

$$1 \to \langle t_1\rangle \to \Gamma_{a14} \to \Delta_{a14} \to 1,$$

where $\Delta_{a14} = \Delta_{a14}^f \rtimes \langle\beta\rangle = \langle s,t,u\rangle \rtimes \langle\beta\rangle$ is isomorphic to

$$\left\langle \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\rangle \rtimes \left\langle \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\right\rangle.$$

Suppose that Γ_{a10} is isomorphic to Γ_{a14} with isomorphism $\varphi : \Gamma_{a10} \to \Gamma_{a14}$. Then it induces an isomorphism $\hat{\varphi} : \Delta_{a10} \to \Delta_{a14}$

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \langle t_1\rangle & \longrightarrow & \Gamma_{a10} & \longrightarrow & \Delta_{a10} & \longrightarrow & 1 \\ & & & & \varphi\big\downarrow & & \hat{\varphi}\big\downarrow & & \\ 1 & \longrightarrow & \langle t_1\rangle & \longrightarrow & \Gamma_{a14} & \longrightarrow & \Delta_{a14} & \longrightarrow & 1. \end{array}$$

Claim 1:
Given an isomorphism $\hat{\varphi} : \Delta_{a10} \to \Delta_{a14}$, then $\hat{\varphi}(\Delta_{a10}^f) = \Delta_{a14}^f$, that is, $\hat{\varphi}$ maps isomorphically.

Since $\hat{\varphi}(\alpha)$ is a torsion element, we may write $\hat{\varphi}(\alpha) = h\beta$ for some $h \in \Delta_{a14}^f$. Since $\hat{\varphi}$ is an isomorphism, there exist $g \in \Delta_{a10}^f$ such that $\hat{\varphi}(g\alpha) = \beta$. Hence $\beta = \hat{\varphi}(g)\hat{\varphi}(\alpha) = \hat{\varphi}(g)h\beta$. So $\hat{\varphi}(g)h = 1$ or $h = \hat{\varphi}(g)^{-1} = \hat{\varphi}(g^{-1})$. Hence

$$\hat{\varphi}(\alpha) = \hat{\varphi}(g^{-1})\beta. \tag{4.7}$$

By using (4.7),

$$\begin{aligned} \hat{\varphi}(\Delta_{a10}) =& \hat{\varphi}(\Delta_{a10}^f) \rtimes \hat{\varphi}(\alpha) \\ =& \hat{\varphi}(\Delta_{a10}^f) \rtimes \hat{\varphi}(g^{-1})\beta \\ =& \hat{\varphi}(\Delta_{a10}^f) \rtimes \beta \qquad (\hat{\varphi}(g^{-1}) \in \hat{\varphi}(\Delta_{a10}^f)) \end{aligned}$$

while $\hat{\varphi}(\Delta_{a10}) = \Delta_{a14}$. Therefore

$$\hat{\varphi}(\Delta_{a10}^f) \rtimes \beta = \Delta_{a14}^f \rtimes \beta. \tag{4.8}$$

Let $s \in \Delta^f_{a14}$, then $s\beta = \left(\begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right)$ which is a torsion element. By using (4.8) $\hat{\varphi}(g'_2)\beta = s\beta$ for some $g'_2 \in \Delta^f_{a10}$. Hence

$$\hat{\varphi}(g'_2) = s. \tag{4.9}$$

Let $t \in \Delta^f_{a14}$, then $st\beta = \left(\begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right)$ which is a torsion element. By using (4.8) there exist $g'_3 \in \Delta^f_{a10}$ such that $\hat{\varphi}(g'_3)\beta = st\beta$. So, $\hat{\varphi}(g'_3) = st$ or $s^{-1}\hat{\varphi}(g'_3) = t$, and by (4.9)

$$\hat{\varphi}((g'_2)^{-1}g'_3) = t \tag{4.10}$$

where $(g'_2)^{-1}g'_3 \in \Delta^f_{a10}$.

As $u\beta = \left(\begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right)$ is a torsion element. By using (4.8) there exist $g'' \in \Delta^f_{a10}$ such that $\hat{\varphi}(g'')\beta = u\beta$. Hence

$$\hat{\varphi}(g'') = u. \tag{4.11}$$

If $g \in \Delta^f_{a10}$, we have $\hat{\varphi}(g) = h \in \Delta^f_{a14}$. If this is not true, then $\hat{\varphi}(g) = h\beta$. But $h = \hat{\varphi}(g')$ for some $g' \in \Delta^f_{a10}$. So, the above implies $h^{-1}\hat{\varphi}(g) = \beta$ or $\hat{\varphi}(g'^{-1}g) = \beta$ where $g'^{-1}g \in \Delta^f_{a10}$. Since Δ^f_{a10} is torsion free, it is impossible. Therefore, since for any $g \in \Delta^f_{a10}$ $\hat{\varphi}(g) = h \in \Delta^f_{a14}$ then $\hat{\varphi}(\Delta^f_{a10}) \subset \Delta^f_{a14}$, and as (4.9) (4.10) and (4.11) imply surjectivity, then $\hat{\varphi}(\Delta^f_{a10}) = \Delta^f_{a14}$.

Claim 2:
Suppose Δ_{a10} and Δ_{a14} are isomorphic by $\hat{\varphi}$. Since Δ_{a10} and Δ_{a14} are crystallographics groups (rigid motion of $\mathbb{R}^3$), there are affinely conjugate, i.e. there exist an equivariant diffeomorphism (=affine transformation)

$$(\hat{\varphi}, S) : (\Delta_{a10}, \mathbb{R}^3) \longrightarrow (\Delta_{a14}, \mathbb{R}^3) \tag{4.12}$$

such that $S(\gamma v) = \hat{\varphi}(\gamma)S(v)$ or equivalently $S\gamma = \hat{\varphi}(\gamma)S$, $S\gamma S^{-1} = \hat{\varphi}(\gamma)$.

First we note that $\hat{\varphi}$ induces an isomorphism

$$\bar{\varphi} : \Delta_{a10}/\Delta^f_{a10} = \langle\bar{\alpha}\rangle \longrightarrow \Delta_{a14}/\Delta^f_{a14} = \langle\bar{\beta}\rangle$$

by $\bar{\varphi}(\bar{\gamma}) = \overline{\hat{\varphi}(\gamma)}$. For $\bar{\gamma} = \bar{\gamma}'$ $(\gamma, \gamma' \in \Delta_{a10})$, then $\gamma = r\gamma'$ $(r \in \Delta^f_{a10})$. Then $\hat{\varphi}(\gamma) = \hat{\varphi}(r)\hat{\varphi}(\gamma') = r'\hat{\varphi}(\gamma')$ for some $r' \in \Delta^f_{a14}(\because \hat{\varphi}(\Delta^f_{a10}) = \Delta^f_{a14})$. Hence $\overline{\hat{\varphi}(\gamma)} = \overline{\hat{\varphi}(\gamma')}$ which show that it is well defined.

Note that, if $\gamma = \tilde{g}\alpha$, $\tilde{g} \in \Delta^f_{a10}$, then $\bar{\gamma} = \overline{\tilde{g}\alpha} = \bar{\alpha}$ and

$$\begin{aligned}\hat{\varphi}(\gamma) =& \hat{\varphi}(\tilde{g})\hat{\varphi}(\alpha)\\ =& \hat{\varphi}(\tilde{g})\hat{\varphi}(g^{-1})\beta \qquad \text{by } (4.7)\\ =& \hat{\varphi}(\tilde{g}g^{-1})\beta.\end{aligned}$$

Since $g, \tilde{g} \in \Delta^f_{a10}$ then $\hat{\varphi}(\tilde{g}g^{-1}) \in \Delta^f_{a14}$ and $\hat{\varphi}(\gamma) \in \Delta_{a14}$ by assumption. Hence $\overline{\hat{\varphi}(\gamma)} = \bar{\beta}$, and by definition

$$\bar{\varphi}(\bar{\alpha}) = \bar{\beta}. \tag{4.13}$$

Note also, S induces a diffeomorphism

$$\bar{S} : M(\Delta^f_{a10}) = \mathbb{R}^3/\Delta^f_{a10} \longrightarrow M(\Delta^f_{a14}) = \mathbb{R}^3/\Delta^f_{a14}$$

by $\bar{S}(\bar{v}) = \overline{S(v)}$. For this, if $\bar{v} = \bar{v}'$ then $v' = rv$ ($r \in \Delta^f_{a10}$). $S(v') = S(rv) = \hat{\varphi}(r)S(v)$ by (4.12) where $\hat{\varphi}(r) \in \Delta^f_{a14}$ (because of Claim 1). Thus $\overline{S(v')} = \overline{S(v)}$.

Moreover $\bar{S}$ is an equivariant diffeomorphism

$$(\bar{\varphi}, \bar{S}) : (\langle\bar{\alpha}\rangle, M(\Delta^f_{a10})) \longrightarrow (\langle\bar{\beta}\rangle, M(\Delta^f_{a14})).$$

For this,

$$\begin{aligned}\bar{S}(\bar{\alpha}\bar{v}) =& \bar{S}(\overline{\alpha v}) = \overline{S(\alpha v)}\\ =& \overline{\hat{\varphi}(\alpha)S(v)} \qquad (\text{by} \quad (4.12))\\ =& \overline{\hat{\varphi}(\alpha)} \quad \overline{S(v)} \quad (\text{by} \quad (*) \quad \text{below})\\ =& \bar{\beta}\bar{S}(\bar{v})\end{aligned}$$

(*)
The action of $\Delta_{a10}/\Delta^f_{a10} = \langle\bar{\alpha}\rangle$ on $M(\Delta^f_{a10})$ is obtained: $\bar{\gamma}\bar{v} = \overline{\gamma v}$ where $\bar{\gamma} \in \Delta_{a10}/\Delta^f_{a10}$ and $\bar{v} \in \mathbb{R}^3/\Delta^f_{a10}$. If $\bar{\gamma} = \bar{\gamma'}$ then $\gamma' = g\gamma$, where $\gamma', \gamma \in \Delta_{a10}$ and $g \in \Delta^f_{a10}$. Therefore $\overline{\gamma' v} = \overline{g\gamma v} = \overline{g(\gamma v)} = \overline{\gamma v}$. If $\bar{v} = \bar{v'}$ then $v' = g'v$ where $g' \in \Delta^f_{a10}$. Therefore $\overline{\gamma' v'} = \overline{\gamma' g' v} = \overline{(\gamma' g' \gamma'^{-1})\gamma' v} = \overline{g''\gamma' v} = \overline{\gamma' v}$, where $g'' \in \Delta^f_{a10}$ normal subgroup. So, when $\bar{\gamma} = \bar{\gamma'}$, $\bar{v} = \bar{v'}$ then $\bar{\gamma'}\bar{v'} = \overline{\gamma' v'} = \overline{\gamma' v} = \overline{\gamma v} = \bar{\gamma}\bar{v}$.

We have this diagram:

$$\begin{array}{ccc}
\Delta^f_{a10} & \xrightarrow{\varphi} & \Delta^f_{a14} \\
\downarrow & & \downarrow \\
(\Delta_{a10}, \mathbb{R}^3) & \xrightarrow{(\varphi,S)} & (\Delta_{a14}, \mathbb{R}^3) \\
\downarrow & & \downarrow \\
(\Delta_{a10}/\Delta^f_{a10}, \mathbb{R}^3/\Delta^f_{a10}) & \xrightarrow{(\bar\varphi,\bar S)} & (\Delta_{a14}/\Delta^f_{a14}, \mathbb{R}^3/\Delta^f_{a14})
\end{array}$$

where $\bar S(\bar\alpha\bar v) = \bar\beta\bar S(\bar v)$ as $\bar S$ is an equivariant diffeomorphism.

We note that, the fixed point set of $\bar\alpha$, $Fix(\bar\alpha) = \{\bar v \in M(\Delta^f_{a10}) | \bar\alpha\bar v = \bar v\}$ mapped diffeomorphically onto $Fix(\bar\beta)$.

Now, to get $Fix(\bar\alpha)$ and $Fix(\bar\beta)$, the argument is as follows; By considering the exact sequence

$$\mathbb{Z}^3 \longrightarrow \Delta^f_{a10} \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2,$$

the action of $\mathbb{Z}_2 \times \mathbb{Z}_2 =< \tau, \mu >$ on T^3 by $\tau\overline{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}} = \overline{\begin{pmatrix} \frac{1}{2}+x_1 \\ -x_2 \\ x_3 \end{pmatrix}} = \begin{pmatrix} -z_1 \\ \overline{z_2} \\ z_3 \end{pmatrix}$, $\mu\overline{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}} = \overline{\begin{pmatrix} x_1 \\ \frac{1}{2}+x_2 \\ -x_3 \end{pmatrix}} = \begin{pmatrix} z_1 \\ -z_2 \\ \overline{z_3} \end{pmatrix}$, and the action of $< \bar\alpha >$ on $\mathbb{R}^3/\Delta^f_{a10}$ by $\bar\alpha\overline{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}} = \overline{\begin{pmatrix} -x_1 \\ x_2 \\ x_3 \end{pmatrix}}$, we have the commutative diagram

$$\begin{array}{ccc}
\mathbb{Z}_2 \times \mathbb{Z}_2 & & \mathbb{Z}_2 \times \mathbb{Z}_2 \\
\downarrow & & \downarrow \\
T^3 = \mathbb{R}^3/\mathbb{Z}^3 & \xrightarrow{\tilde\alpha} & \mathbb{R}^3/\mathbb{Z}^3 \\
{\scriptstyle p}\downarrow & & {\scriptstyle p}\downarrow \\
\mathbb{R}^3/\Delta^f_{a10} & \xrightarrow{\bar\alpha} & \mathbb{R}^3/\Delta^f_{a10}
\end{array}$$

where $\tilde\alpha$ is a lift of $\bar\alpha$ and p is defined by $p(\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}) = \overline{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}$ such that $p \circ \tilde\alpha = \bar\alpha \circ p$.

Note that $\tilde{\alpha}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} \bar{z}_1 \\ z_2 \\ z_3 \end{pmatrix} \in T^3$, since $\tilde{\alpha}\overline{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}} = \overline{\begin{pmatrix} -x_1 \\ x_2 \\ x_3 \end{pmatrix}} = \begin{pmatrix} \bar{z}_1 \\ z_2 \\ z_3 \end{pmatrix} \in T^3$.

Let $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and $z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$. If $z \in Fix(\tilde{\alpha})$, $\tilde{\alpha}z = z$, then $p(\tilde{\alpha}z) = p(z)$. This implies

$$\begin{pmatrix} \bar{z}_1 \\ z_2 \\ z_3 \end{pmatrix} = \tilde{\alpha}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \tau^m \mu^n \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \quad m, n \in \{0, 1\}.$$

So, the possibilities of $Fix(\tilde{\alpha})$ is obtained from

$$\begin{pmatrix} \bar{z}_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{cases} z \\ \tau z \\ \mu z \\ \tau\mu z. \end{cases} \tag{4.14}$$

Before determining $Fix(\tilde{\alpha})$, note that $\tilde{\alpha}(\tau z) = \tau(\tilde{\alpha}z)$ and $\tilde{\alpha}(\mu z) = \mu(\tilde{\alpha}z)$, since $-\bar{z}_i = \overline{-z_i}$. Hence if $z \in Fix(\tilde{\alpha})$, $\tau z \in Fix(\tilde{\alpha})$ and $\mu z \in Fix(\tilde{\alpha})$.

Now we calculate $Fix(\tilde{\alpha})$ for any case in (4.14).

If $\begin{pmatrix} \bar{z}_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$, $\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} \pm 1 \\ z_2 \\ z_3 \end{pmatrix}$. Since $\tau\begin{pmatrix} 1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} -1 \\ \bar{z}_2 \\ z_3 \end{pmatrix}$, so $Fix(\tilde{\alpha}) = \left\{\begin{pmatrix} 1 \\ z_2 \\ z_3 \end{pmatrix}\right\}$.

If $\begin{pmatrix} \bar{z}_1 \\ z_2 \\ z_3 \end{pmatrix} = \tau\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} -z_1 \\ \bar{z}_2 \\ z_3 \end{pmatrix}$, $\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} \pm i \\ \pm 1 \\ z_3 \end{pmatrix}$. Since $\tau\begin{pmatrix} i \\ \pm 1 \\ z_3 \end{pmatrix} = \begin{pmatrix} -i \\ \pm 1 \\ z_3 \end{pmatrix}$, and $\mu\begin{pmatrix} i \\ 1 \\ z_3 \end{pmatrix} = \begin{pmatrix} i \\ -1 \\ \bar{z}_3 \end{pmatrix}$ so $Fix(\tilde{\alpha}) = \left\{\begin{pmatrix} i \\ 1 \\ z_3 \end{pmatrix}\right\}$.

If $\begin{pmatrix} \bar{z}_1 \\ z_2 \\ z_3 \end{pmatrix} = \mu\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} z_1 \\ -z_2 \\ \bar{z}_3 \end{pmatrix}$, there is no $z_2 = -z_2$. So $Fix(\tilde{\alpha}) = \emptyset$.

If $\begin{pmatrix} \bar{z}_1 \\ z_2 \\ z_3 \end{pmatrix} = \tau\mu \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} -z_1 \\ -\bar{z}_2 \\ \bar{z}_3 \end{pmatrix}$, $\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} \pm i \\ \pm i \\ \pm 1 \end{pmatrix}$. Since

$$\mu \begin{pmatrix} i \\ i \\ 1 \end{pmatrix} = \begin{pmatrix} i \\ -i \\ 1 \end{pmatrix}, \qquad \mu \begin{pmatrix} i \\ i \\ -1 \end{pmatrix} = \begin{pmatrix} i \\ -i \\ -1 \end{pmatrix}$$

$$\mu \begin{pmatrix} -i \\ i \\ 1 \end{pmatrix} = \begin{pmatrix} -i \\ -i \\ 1 \end{pmatrix}, \qquad \mu \begin{pmatrix} -i \\ i \\ -1 \end{pmatrix} = \begin{pmatrix} -i \\ -i \\ -1 \end{pmatrix}$$

$$\tau\mu \begin{pmatrix} i \\ -i \\ 1 \end{pmatrix} = \begin{pmatrix} -i \\ -i \\ 1 \end{pmatrix}, \qquad \tau\mu \begin{pmatrix} i \\ -i \\ -1 \end{pmatrix} = \begin{pmatrix} -i \\ -i \\ -1 \end{pmatrix}$$

so $Fix(\tilde{\alpha}) = \left\{ \begin{pmatrix} i \\ -i \\ 1 \end{pmatrix}, \begin{pmatrix} i \\ -i \\ -1 \end{pmatrix} \right\}$.

To determine $Fix(\bar{\alpha})$, the argument is as follows. If $z \in Fix(\tilde{\alpha})$,

$$\begin{aligned} \bar{\alpha}\bar{x} &= \bar{\alpha}p(z) = p(\tilde{\alpha}z) = p(z) \\ &= \bar{x}, \end{aligned}$$

hence $p(Fix(\tilde{\alpha})) = Fix(\bar{\alpha})$. Now we get

$$Fix(\bar{\alpha}) = \begin{cases} \left\{ p \begin{pmatrix} 1 \\ z_2 \\ z_3 \end{pmatrix} \right\} = \left\{ \overline{\begin{pmatrix} 1 \\ x_2 \\ x_3 \end{pmatrix}} \right\} = T^2 \\ \left\{ p \begin{pmatrix} i \\ 1 \\ z_3 \end{pmatrix} \right\} = \left\{ \overline{\begin{pmatrix} \frac{1}{4} \\ 1 \\ x_3 \end{pmatrix}} \right\} = S^1 \\ \left\{ p \begin{pmatrix} i \\ i \\ 1 \end{pmatrix}, p \begin{pmatrix} i \\ i \\ -1 \end{pmatrix} \right\} = \left\{ \overline{\begin{pmatrix} \frac{1}{4} \\ \frac{1}{4} \\ 1 \end{pmatrix}}, \overline{\begin{pmatrix} \frac{1}{4} \\ \frac{1}{4} \\ -1 \end{pmatrix}} \right\}. \end{cases}$$

Similarly, we can find $Fix(\bar{\beta})$ as follows. If $z \in Fix(\tilde{\beta})$, $\tilde{\beta}z = z$, then $p(\tilde{\beta}z) = p(z)$. This implies

$$\begin{pmatrix} \bar{z}_1 \\ z_2 \\ \bar{z}_3 \end{pmatrix} = \tilde{\beta} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \tau^m \mu^n \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \quad m, n \in \{0, 1\}.$$

So, the possibilities of $Fix(\tilde{\beta})$ is obtained from

$$\begin{pmatrix} \bar{z}_1 \\ z_2 \\ \bar{z}_3 \end{pmatrix} = \begin{cases} z \\ \tau z \\ \mu z \\ \tau\mu z. \end{cases} \tag{4.15}$$

Note that $\tilde{\beta}(\tau z) = \tau(\tilde{\beta} z)$ and $\tilde{\beta}(\mu z) = \mu(\tilde{\beta} z)$. Hence if $z \in Fix(\tilde{\beta})$, $\tau z \in Fix(\tilde{\beta})$ and $\mu z \in Fix(\tilde{\beta})$.

Now we calculate $Fix(\tilde{\beta})$ for any case in (4.15).

If $\begin{pmatrix} \bar{z}_1 \\ z_2 \\ \bar{z}_3 \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$, $\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} \pm 1 \\ z_2 \\ \pm 1 \end{pmatrix}$. Since $\tau \begin{pmatrix} 1 \\ z_2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ \bar{z}_2 \\ 1 \end{pmatrix}$, $\tau \begin{pmatrix} 1 \\ z_2 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ \bar{z}_2 \\ -1 \end{pmatrix}$ so $Fix(\tilde{\beta}) = \left\{ \begin{pmatrix} 1 \\ z_2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ z_2 \\ -1 \end{pmatrix} \right\}$.

If $\begin{pmatrix} \bar{z}_1 \\ z_2 \\ \bar{z}_3 \end{pmatrix} = \tau \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} -z_1 \\ \bar{z}_2 \\ z_3 \end{pmatrix}$, $\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} \pm i \\ \pm 1 \\ \pm 1 \end{pmatrix}$. Since

$$\mu \begin{pmatrix} i \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} i \\ -1 \\ 1 \end{pmatrix}, \qquad \mu \begin{pmatrix} i \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} i \\ -1 \\ -1 \end{pmatrix}$$

$$\mu \begin{pmatrix} -i \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -i \\ -1 \\ 1 \end{pmatrix}, \qquad \mu \begin{pmatrix} -i \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -i \\ -1 \\ -1 \end{pmatrix}$$

$$\tau\mu \begin{pmatrix} i \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -i \\ 1 \\ 1 \end{pmatrix}, \qquad \tau\mu \begin{pmatrix} i \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} -i \\ 1 \\ -1 \end{pmatrix}$$

so $Fix(\tilde{\beta}) = \left\{ \begin{pmatrix} i \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} i \\ 1 \\ -1 \end{pmatrix} \right\}$.

If $\begin{pmatrix} \bar{z}_1 \\ z_2 \\ \bar{z}_3 \end{pmatrix} = \mu \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} z_1 \\ -z_2 \\ \bar{z}_3 \end{pmatrix}$, there is no $z_2 = -z_2$. So $Fix(\tilde{\beta}) = \emptyset$.

If $\begin{pmatrix} \bar{z}_1 \\ z_2 \\ \bar{z}_3 \end{pmatrix} = \tau\mu \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} -z_1 \\ -\bar{z}_2 \\ \bar{z}_3 \end{pmatrix}$, $\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} \pm i \\ \pm i \\ z_3 \end{pmatrix}$. Since

$$\mu \begin{pmatrix} i \\ i \\ z_3 \end{pmatrix} = \begin{pmatrix} i \\ -i \\ z_3 \end{pmatrix}, \qquad \mu \begin{pmatrix} -i \\ i \\ z_3 \end{pmatrix} = \begin{pmatrix} -i \\ -i \\ z_3 \end{pmatrix}$$

$$\tau \begin{pmatrix} i \\ i \\ z_3 \end{pmatrix} = \begin{pmatrix} -i \\ -i \\ z_3 \end{pmatrix}$$

so $Fix(\tilde{\beta}) = \left\{ \begin{pmatrix} i \\ i \\ z_3 \end{pmatrix} \right\}$.

To determine $Fix(\bar{\beta})$, the argument is as follows. If $z \in Fix(\tilde{\beta})$,

$$\begin{aligned} \bar{\beta}\bar{x} &= \bar{\beta}p(z) = p(\tilde{\beta}z) = p(z) \\ &= \bar{x}, \end{aligned}$$

hence $p(Fix(\tilde{\beta})) = Fix(\bar{\beta})$. Now we get

$$Fix(\bar{\beta}) = \begin{cases} \left\{ p \begin{pmatrix} 1 \\ z_2 \\ -1 \end{pmatrix}, p \begin{pmatrix} 1 \\ z_2 \\ 1 \end{pmatrix} \right\} = \left\{ \overline{\begin{pmatrix} 1 \\ x_2 \\ -1 \end{pmatrix}}, \overline{\begin{pmatrix} 1 \\ x_2 \\ 1 \end{pmatrix}} \right\} = 2S^1 \\ \left\{ p \begin{pmatrix} i \\ 1 \\ 1 \end{pmatrix}, p \begin{pmatrix} i \\ 1 \\ -1 \end{pmatrix} \right\} = \left\{ \overline{\begin{pmatrix} \frac{1}{4} \\ 1 \\ 1 \end{pmatrix}}, \overline{\begin{pmatrix} \frac{1}{4} \\ 1 \\ -1 \end{pmatrix}} \right\} \\ \left\{ p \begin{pmatrix} i \\ i \\ z_3 \end{pmatrix} \right\} = \left\{ \overline{\begin{pmatrix} \frac{1}{4} \\ \frac{1}{4} \\ x_3 \end{pmatrix}} \right\} = S^1. \end{cases}$$

This yields a contradiction.

We observed that the sufficient condition of conjecture is true for $n = 4$.

□

References

[1] Charlap, L.S., *Bieberbach Groups and Flat Manifolds*, Springer-Verlag, 1985.

[2] Charlap, L.S., Compact Flat Riemannian Manifolds I, *Annal of Math.* 81, (1965), 15.

[3] Kobayashi, S., and K. Nomizu, *Foundation of Differential Geometry I*, Wiley-Interscience, New York, 1963.

[4] Kosniowski, C., *A First Course in Algebraic Topology*, Cambridge University Press, 1980.

[5] Thurston, W.P., *Three-Dimensional Geometry and Topology*, vol 1, Princeton University Press, 1997.
[6] Wolf, J.A., *Spaces of Constant Curvature*, McGraw-Hill Book Company, 1967.

Department of Mathematics, Tokyo Metropolitan University.
Email address: nazra-admi@ed.tmu.ac.jp