BESSEL PERIODS ON $U(2,1) \times U(1,1)$, RELATIVE TRACE FORMULA AND NON-VANISHING OF CENTRAL L-VALUES

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ABSTRACT. In this paper we calculate the asymptotics of the second moment of the Bessel periods associated to certain holomorphic cuspidal representations (π, π') of $U(2,1) \times U(1,1)$ of regular infinity type (averaged over π). Using these, we obtain quantitative non-vanishing results for the Rankin-Selberg central L-values $L(1/2, \pi \times \pi')$, which are of degree twelve over $\mathbb{Q}$, with concomitant difficulty in applying standard methods, especially since we are in a 'conductor dropping' situation. We use the relative trace formula, and the orbital integrals are evaluated rather than compared with others. Besides their intrinsic interest, non-vanishing of these critical values also lead, by known results, to deducing certain associated Selmer groups have rank zero.

Contents

1. Introduction

Let $L(s, \pi)$ be an L-function admitting an Euler product factorisation

$$L(s,\pi) = \prod_p L_p(s,\pi), \ \mathrm{Re}(s) \gg 1$$

constructed out of some automorphic datum $\{\pi\}$; we assume that $L(s, \pi)$ is analytically normalized, self-dual and even so that its admit an analytic continuation to the whole s-plane with a functional equation relating $L(s, \pi)$ to $L(1-s, \pi)$ and root number ± 1. Under such hypothesis the central value of the finite part $L(1/2, \pi)$

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or its derivative $L'(1/2,\pi)$ (depending on the root number) is of great importance, both from the arithmetic and the analytic points of view.

A question of great interest is, given a family $\mathcal{F}$ of such similar automorphic data to exhibit some $\pi \in \mathcal{F}$ for which $L(\pi,1/2)$ or $L'(\pi,1/2)$ is non zero for N large enough. For instance, such questions occurs in problems related to the Birch-Swinnerton-Dyer conjecture for some motives; the existence of such non-vanishing is then be used to establish the non-triviality of some Euler system and ultimately conclude the expected value of its arithmetic rank and the finiteness of its Tate-Shafarevitch group.

A basic strategy to exhibit such non-vanishing is to consider some sequence of subfamilies $\mathcal{F}_N \subset \mathcal{F}$ indexed by some parameter N, growing with N, and to evaluate, for $N \to \infty$, some weight moment of the shape

$$M(\mathcal{F}_N) = \sum_{\pi \in \mathcal{F}_N} w(\pi) L(1/2,\pi)$$

(for some non-negative weights $(w(\pi))_{\pi \in \mathcal{F}_N}$) so as to show that this moment is non-zero for N large enough.

Some of the earliest examples were given in the (independent) works of Bump-Friedberg-Hoffsten and Murty-Murty [BFH90, MM91] regarding the non-vanishing of $L(1/2, E \times \chi)$ or its derivative for E a fixed elliptic curve (more generally a modular form) and χ varying over (odd) quadratic characters; both were motivated by the seminal work of Kolyvagin [Kol88]. Another very recent example is the work [RY23] by the third named author and M. Radziwill establishing the non-vanishing of $L(1/2, \pi_4 \times \chi)$ for π_4 a fixed GL_4 automorphic cuspidal representation and χ varying over (complex) Dirichlet characters; this result has important implications concerning the Birch-Swinnerton-Dyer conjecture for abelian surfaces in connection with the work of Loeffler-Zerbes and others [LZ23].

Another example relevant to the present paper is furnished by the family $\mathcal{F}_\chi(N) = \{\pi_E \otimes \chi\}$, where $E = \mathbb{Q}(\sqrt{D})$ is a fixed imaginary quadratic field, χ an ideal class character of E and π varies over the unitary cuspidal automorphic representation of $\mathrm{GL}(2)/\mathbb{Q}$ attached to a normalized newform φ of level N, weight 2 and trivial character, with π_E denoting te base change of π to $\mathrm{GL}(2)/E$. Suppose (for simplicity) that N is a prime which is inert in E so that the sign of the functional equation of $L(s,\pi_E)$ is $+1$. When $\chi = 1$ is the principal character, the L-function $L(s,\pi_E)$ factors as $L(s,\pi)L(s,\pi\otimes\eta)$, where $\eta = \eta_E$ is the quadratic Dirichlet character of $\mathbb{Q}$ attached to E. In [Duk95], W. Duke established the non-vanishing of $L(1/2,\pi_E)$ for $\gg N/\log N$ representations π; later, using the mollification method, Iwaniec and Sarnak established non-vanishing for a positive proportion of such forms [IS00]. If χ is not quadratic, $L(s,\pi_E \times \chi)$ is a Rankin-Selberg L-function $L(s,\pi_E \times \pi_\chi)$ where $\pi_\chi = \mathrm{Ind}_{\mathbb{Q}}^{E}(\chi)$ is the (cuspidal) automorphic induction of χ from $\mathrm{GL}(1)/E$ to $\mathrm{GL}(2)/\mathbb{Q}$; the existence of a positive proportion of π for which $L(1/2,\pi_E \times \pi_\chi) \neq 0$ was established by Kowalski and the first named author [KMV02]. The evaluation of these moments where based on the Petersson-Kuznetzov's formula; in [RR05], Rogawski and the second named author took a different route and used instead the Relative Trace Formula (RTF) for the pair $(\mathrm{GL}(2)/\mathbb{Q}, T_{\mathbb{Q}})$ for T the diagonal (split) torus (using the fact that for χ of order two, $L(s,\pi_E \times \chi)$ is the product of two $\mathrm{GL}(2)$ L-functions). A generalization to Hilbert modular forms over a totally real base field F and suitably general idele class characters χ was achieved by B. Feigon and D. Whitehouse in [FW09], using the RTF for anisotropic pairs (G,T), with G an inner form of $\mathrm{GL}(2)/F$ and $T \simeq \mathrm{Res}_{E/F}\,\mathbb{G}_{\mathrm{m}}$ a non-split torus attached to a totally imaginary quadratic extension E/F. In the present paper, we carry out this approach in a higher rank situation (for the base field $\mathbb{Q}$).

1.1. **First moment for Rankin-Selberg L-functions for $U(2,1)\times U(1,1)$.** In this paper, we consider families of L-functions attached to automorphic representations $\pi\times\pi'$ on $G\times G'$ for $G\simeq U(2,1)$ (resp. $G'\simeq U(1,1)$) be quasi-split unitary group in three (resp. two) variables, associated to an imaginary quadratic field $E/\mathbb{Q}$ of conductor D_E. These representations admits base changes π_E,π'_E to $\mathrm{GL}(3)/E$ and $\mathrm{GL}(2)/E$ whose existences are known: see Rogawski [Rog90] and Flicker [Fli82]. The L-functions we consider are the Rankin-Selberg L-functions $L(s,\pi_E\times\pi'_E)$ which for $\mathrm{Re}\, e(s)>1$ admit an Euler product factorisation

$$L(s,\pi_E\times\pi'_E)=\prod_p L_p(s,\pi_E\times\pi'_E)=\prod_{\mathfrak{p}|p}\prod_{\mathfrak{p}} L_{\mathfrak{p}}(s,\pi_E\times\pi'_E),\ \mathrm{Re}(s)>1$$

where $\mathfrak{p}$ runs over the primes of E above p. This L-function is completed by an archimedean local factor

$$L_\infty(s,\pi_E\times\pi'_E)=\prod_{w|\infty} L_w(s,\pi_E\times\pi'_E)$$

(a product of Gamma functions) and admits a functional equation of the shape

$$\Lambda(s,\pi_E\times\pi'_E)=\varepsilon(\pi_E\times\pi'_E)C_f(\pi_E\times\pi'_E)^s\Lambda(1-s,\pi_E\times\pi'_E) \tag{1.1}$$

where

$$\Lambda(s,\pi_E\times\pi'_E)=L_\infty(s,\pi_E\times\pi'_E)L(s,\pi_E\times\pi'_E)$$

is the "completed" L-function,

$$L_\infty(s,\pi_E\times\pi'_E)=\prod_{w|\infty} L_w(s,\pi_E\times\pi'_E),$$

$C_f(\pi_E\times\pi'_E)\geq 1$ is an integer (the arithmetic conductor) and $\varepsilon(\pi_E\times\pi'_E)\in\{\pm1\}$ is the root number.

1.1.1. *The main assumptions.* We will consider the families for which π', the form on the smaller group is *fixed*, while the form in the larger group π is varying.

More precisely (see §3 for greater details) let $k\geq 0$ be an integer and let $N,N'\geq 1$ be integers either equal to 1 or to prime numbers unramified in E . We assume that

– k is even and sufficently large:

$$k>32, \tag{1.2}$$

– If $N>1$, then N is *inert* in E, and

$$N'\geq 3 \tag{1.3}$$

– If $N'>1$, then N' is *split* in E and

$$N'\geq 10^6 \tag{1.4}$$

– The representation

$$\pi'\simeq\pi'_\infty\otimes{\bigotimes_p}'\pi'_p$$

is a cuspidal representation of $G'(\mathbb{A})$, with trivial central character, whose archimedean component π'_∞ is a holomorphic discrete series of weight k, which is unramified at every prime not dividing N' and, if N' is prime, that $\pi'_{N'}$ is the Steinberg representation .

– The representations

$$\pi\simeq\pi_\infty\otimes{\bigotimes_p}'\pi_p$$

are cuspidal automorphic representations of $G(\mathbb{A})$, with trivial central character whose archimedean component π_∞ is a holomorphic discrete series of weights $\Lambda=(-2k,k)$ (cf. [Wal76] and below) for the same value of k as above, which is unramified at every prime not dividing N and, if N is prime, that π_N is either unramified or the Steinberg representation.

We denote by $\mathcal{A}_k(N)$ the finite set of all such automorphic representations π and we denote by

$$\mathcal{A}_k^{\mathrm{n}}(N) \subset \mathcal{A}_k(N)$$

the subset of those representations which are ramified at N: if $N = 1$,

$$\mathcal{A}_k^{\mathrm{n}}(1) = \mathcal{A}_k(1)$$

and if N is prime, this is the set of π such that π_N is the Steinberg representation.

By a version of Weyl's law, one has[1]

$$|\mathcal{A}_k^{\mathrm{n}}(N)| \asymp k^3N^3 \text{ as } k+N \to \infty. \tag{1.5}$$

Remark 1.1. The condition (1.2), (1.3) and (1.4) are made either to avoid pathologies and technical difficulties in small weights or characteristic. They will insure the absolute convergence and possibly non-vanishing or various local and global integrals in our argument. The conditions (1.4) and (1.3) can perhaps be improved with more intensive combinatorial efforts while allowing very small weights (1.2) will certainly constitute a major technical challenge.

1.1.2. *Upper and lower bounds for the first moment.* Regarding the L-function $L(s, \pi_E \times \pi'_E)$, the assumptions above allow us to compute explicitly (see Propositions 5.8 and 5.9) the archimedean factor $L_\infty(s, \pi_E \times \pi'_E)$, the arithmetic conductor $C_f(\pi_E \times \pi'_E)$ and the root number which equals

$$\varepsilon(\pi_E \times \pi'_E) = +1.$$

Moreover the central value $L(1/2, \pi_E \times \pi'_E)$ is then non-negative (see below).

To state our first main result, we need the *Adjoint* L-functions of π_E and π'_E

$$L(s, \mathrm{Ad}, \pi_E),\ L(s, \mathrm{Ad}, \pi'_E)$$

whose analytic continuations around $s = 1$ are a consequence of Rankin-Selberg theory.

Theorem 1.1. *Let notations and assumptions be as in §1.1.1. Given N' there exists $C(N') \geq 1$ such that for any k, N as above (N split, $k > 32$ even) and such that*

- *$k + N \geq C(N')$ and*
- *either $N = 1$ or $N \geq C(N')$,*

one has

$$\frac{1}{|\mathcal{A}_k^{\mathrm{n}}(N)|} \sum_{\pi \in \mathcal{A}_k^{\mathrm{n}}(N)} \frac{L(1/2, \pi_E \times \pi'_E)}{L(1, \mathrm{Ad}, \pi_E) L(1, \mathrm{Ad}, \pi'_E)} \asymp 1, \tag{1.6}$$

where the implicit constants depend on E and N'.

In particular for any such (k, N, N') there exists $\pi \in \mathcal{A}_k^{\mathrm{n}}(N)$ such that

$$L(1/2, \pi_E \times \pi'_E) \neq 0. \tag{1.7}$$

Remark 1.2. Given $A, B : \mathcal{P} \to \mathbb{R}$ two real valued functions on a set $\mathcal{P}$, we use the notation

$$A \asymp B$$

to mean that there exists positive constants $0 < c < C$ such that

$$\forall p \in \mathcal{P},\ cA(p) \leq B(p) \leq CA(p).$$

[1]We would like to point out a seeming discrepancy in [Wal76, Lemma 9.4] between the formula computing the formal degree of π_λ and the original (correct) formula from Harish-Chandra [HC66, Remark 5.5]; the former would lead to the asymptotic in the k-aspect $|\mathcal{A}_k(N)| \asymp k^2N^3$ which is not correct; we are thankful to Paul Nelson for pointing to this error.

In particular, A and B have the same support and have the same sign (when non-zero). If $\mathcal{P}$ and A, B belong to families of sets $(\mathcal{P}_E)$ and functions $(A_E, B_E : \mathcal{P}_E \to \mathbb{R})$ indexed by some parameter E, we write

$$A \asymp_E B$$

to mean that there exists functions $E \to c_E, C_E \in \mathbb{R}_{>0}$ such that

$$\forall E, \forall p \in \mathcal{P}_E,\ c_E A_E(p) \leq B_E(p) \leq C_E A_E(p).$$

Remark 1.3. We have assumed that N and N' are prime to simplify the proof and leave it to the interested reader to extend these results to more general odd squarefree integers.

Remark 1.4. Theorem 1.1 gives the existence of at least *one* non vanishing central value $L(1/2, \pi_E \times \pi'_E)$ for $k+N$ large enough. In Theorem 1.3 below, we will deduce an infinitude of such pi (as $k+N \to \infty$), and in fact establish a weak lower bound on the number of π's such that (1.7) holds.

Remark 1.5. The problem of evaluating moments of L-functions involving families of automorphic forms on groups of higher rank is difficult and there are not many positive results. One may think of the work of X. Li [Li11] involving $\mathrm{GL}(3) \times \mathrm{GL}(2)$ Rankin-Selberg L-functions as well as the work of Blomer-Khan [BK15] and Qi [Qi20] in a similar context. However a common feature of these works is that the moments are on average over the automorphic forms of the smaller group GL_2. Closer to the spirit of this paper is the work of Blomer-Buttcane [BB20] who estimated the fourth moment of standard L-functions on average over families of GL_3-automorphic representations with large archimedean parameters and obtain subconvex bounds. Another is the work of Nelson-Venkatesh [NV21] who build on their substantial development of microlocal calculus on Lie groups, and obtain an asymptotic formula for the first moment of Rankin-Selberg L-functions $L(1/2, \pi_E \times \pi'_E)$ associated with pairs of unitary groups $U(n+1) \times U(n)$ (for any $n \geq 2$) on average over families of $U(n)$-automorphic forms with large archimedean parameters in general position. In [Nel23], these methods were expanded further, and Nelson succeeded in evaluating the first moment above this time on average over suitable families of $U(n+1)$-automorphic forms; finally in [Nel21], Nelson treated the degenerate case of *split* unitary groups (relative to $E = \mathbb{Q} \times \mathbb{Q}$, so that $G \times G' = \mathrm{GL}(n+1) \times \mathrm{GL}(n)$) and with π' being an Eisenstein series representation: this gave bounds for the n-th moment of standard $\mathrm{GL}(n+1)$ L-functions $L(1/2, \pi)$ with π having large archimedean parameters in generic position (so as to avoid the "conductor dropping phenomenon"). Further in that direction, Jana-Nunes and the third named author independently, have obtained recently, asymptotic formula for weighted moments of central values of products of $\mathrm{GL}(n+1) \times \mathrm{GL}(n)$ Rankin-Selberg L-functions $L(1/2, \pi \times \pi'_1)\overline{L(1/2, \pi \times \pi'_2)}$ for a pair of possibly varying cuspidal representations π'_1, π'_2 and on average over π's when their spectral parameters are in generic position (avoiding the conductor dropping phenomenon)[JN23, Yan23]. Let us point out that the situation we consider here is very non-generic and indeed the conductor of our degree 12 L-functions drop significantly (see §5.4).

1.2. **Galois representations, Ramanujan & the Bloch-Kato conjectures.** As pointed out earlier, the problem of exhibiting non-vanishing of central values within certain families of L-functions has important applications related to the Birch-Swinnerton-Dyer conjectures. In our cse the relevant context are the *Bloch-Kato conjectures*. We explain this connections here along with the fact that the π and π' we consider are tempered.

On temperedness. It is classical and due to Deligne that for any prime ℓ there is an ℓ-adic Galois representation, $V_\ell(\pi')$, associated to π' whose Frobenius eigenvalues at primes $v \nmid \ell N'$ equal (up to an appropriate twist) the Langlands parameters of π'_v. As $V_\ell(\pi')$ occurs in the cohomology of a certain Kuga-Sato modular variety, this implies, by Deligne's Weil II, the purity of the Frobenius Frob_v and that π'_v is tempered; varying v, that π' is tempered everywhere (the Ramanujan-Petersson conjecture).

For π, which is regular cohomological, the association of an ℓ-adic Galois representation $V_\ell(\pi)$ is due to the works of Rogawski, Kottwitz et al, and the proofs are assembled in [LR92] (see Chap. 7 Thms A and B). If π is stable, the associated 3-dimensional Galois representation $V_\ell(\pi)$ occurs in the cohomology in degree 2 of a modular Picard surface with locally constant coefficients: by the work of Deligne, this implies the purity of the Frobenius Frob_v, $v \nmid \ell N$ and eventually the temperedness of π at every place.

On the other hand, when π is endoscopic, the Galois representation occurring in the cohomology need not be 3-dimensional; however due to our infinity type, the archimedean parameter forces it to come from a representation $\pi_1 \times \xi$ of $U(1,1) \times U(1)$ with ξ unitary and π_1 in the discrete series at infinity, in fact of the same weight $2k$. So again π is tempered because π_1 and ξ are.

Note that regarding temperedness, π being cohomological is not sufficient and we need to use that π occurs in the middle degree cohomology of Picard modular surfaces. By contrast those occurring in degree 1 are always non-tempered.

On the Bloch-Kato conjecture. Since π and π' are cuspidal, the representations $V_\ell(\pi)$, $V_\ell(\pi')$ are irreducible and even absolutely irreducible as neither π nor π' admits self twists (because of the Steinberg components at N' and N). The same holds modulo ℓ for ℓ large enough.

The tensor product $V_\ell(\pi) \otimes V_\ell(\pi')$ is also absolutely irreducible : again the Steinberg components at the distinct N and N' prevent π from being a twist of the symmetric square of π'.

To the later representation is associated a Bloch-Kato Selmer group $H^1_f(V_\ell(\pi) \otimes V_\ell(\pi')(*))$ (here $(*)$ is a suitable Tate twist depending on k) and the Bloch-Kato conjecture predicts that this Selmer group is zero if $L(1/2, \pi_E \times \pi'_E)$ does not vanish.

We expect this conjecture to follow from the work of Y. Liu, Y. Tian , L. Xiao, W. Zhang and X. Zhu [LTX+22]. Indeed their results established the Bloch-Kato conjecture π and π' are regular algebraic (as we have here) but cohomological with trivial coefficients, for appropriate admissible ℓ's whenever $V_\ell(\pi) \otimes V_\ell(\pi')$ is absolutely irreducible and $V_\ell(\pi)$, $V_\ell(\pi')$ are residually irreducible (they also need to assume the presence of Steinberg components –which we have– and supercuspidal component).

In our situation the trivial coefficients condition forces k to be 2 which we do not consider to avoid some technical difficulties (that may be serious). We are happy on the other hand to hear from X. Zhu that their results extend to non-trivial coefficients (and without requiring supercuspidal components): this will be discussed in a forthcoming work.

1.3. **L-functions and Bessel periods.** Our proof of Theorem 1.1 follows along the lines of the earlier work of the second author and J. Rogawski [RR05] but with substancially more complicated calculations; it is a consequence of the asymptotic evaluation, using the *Relative Trace Formula*, of sums of *Bessel periods* of the shape

$$\mathcal{P}(\varphi, \varphi') := \int_{G'(\mathbb{Q})\backslash G'(\mathbb{A})} \varphi(g)\overline{\varphi'}(g')dg'$$

where $\varphi\in\pi$ and $\varphi'\in\pi'$ are suitable factorable automorphic forms (see [Liu16] for a detailed discussion of these periods).

1.3.1. *From periods to L-functions.* The derivation of Theorem 1.1 from 1.2 follows from Gan-Gross-Prasad type conjectures for unitary pairs $U(n+1)\times U(n)$, in their precise form given by Ichino-Ikeda: these redict a relation between the square of the Bessel period $|\mathcal{P}(\varphi,\varphi')|^2$ and the central L-value $L(1/2,\pi_E\times\pi'_E)$.

These conjectures have now been established for tempered representations (which is the case by the discussion in §1.2) due to the work of many people including Beuzart-Plessis, Chaudouard, Liu, Zhang, Zhu, and Zydor; we refer to Beuzart-Plessis' ICM lecture for a complete description of the conjectures and their resolution [BP23]. More precisely, Theorem 1.9 of [BPLZZ21] gives

$$\frac{\left|\mathcal{P}(\varphi,\varphi')\right|^2}{\langle\varphi,\varphi\rangle\langle\varphi',\varphi'\rangle}=\frac{\Lambda(1,\eta)\Lambda(2,\eta^2)\Lambda(3,\eta)}{2}\times \frac{\Lambda(1/2,\pi_E\times\pi'_E)}{\Lambda(1,\pi_E,\mathrm{Ad})\Lambda(1,\pi'_E,\mathrm{Ad})}\cdot\prod_v\mathcal{P}_v^\natural(\varphi,\varphi'),\tag{1.8}$$

where η is the quadratic character associated to $E/\mathbb{Q}$, $\Lambda(s,\cdot)$ denote the *completed L-function* (with the archimedean factor included) and $\prod_v\mathcal{P}_v^\natural(\varphi,\varphi')$ is a finite product of local periods.

In Section 5.8 we evaluate the local periods $\mathcal{P}_v^\natural(\varphi,\varphi')$ explicitly for a specific automorphic form φ' (see §4.5.1) and for φ varying over an orthogonal family $\mathcal{B}_k^{\widetilde{\mathfrak{n}}}(N)$ of factorable automorphic forms of level N and minimal weights $(-2k,k)$ belonging the various representations in $\mathcal{A}_k(N)$ (see §5.1 for precise definitions). We show that for such φ' and φ, the local periods are non-negative and that

$$\frac{L_\infty(1/2,\pi_E\times\pi'_E)}{L_\infty(1,\pi_E,\mathrm{Ad})L_\infty(1,\pi'_E,\mathrm{Ad})}\prod_v\mathcal{P}^\natural(\varphi,\varphi')\asymp_E\frac{1}{kNN'^2}\tag{1.9}$$

and by (1.8) one has

$$\frac{1}{kNN'^2}\frac{L(1/2,\pi_E\times\pi'_E)}{L(1,\mathrm{Ad},\pi_E)L(1,\mathrm{Ad},\pi'_E)}\asymp_E\frac{\left|\mathcal{P}(\varphi,\varphi')\right|^2}{\langle\varphi,\varphi\rangle\langle\varphi',\varphi'\rangle};\tag{1.10}$$

the central values $L(1/2,\pi_E\times\pi'_E)$ are thus non-negative.

Remark 1.6. Although the knowledge of the full Ichino-Ikeda conjecture seem to provide everything one needs, these explicit local computations are nevertheless necessary, first to infer positivity (since the test functions we use are not of positive type) and also to make sure that the (positive) constants implicit in the symbols $\asymp_E$ in (1.9) and (1.10), indeed do not depend on the varying parameters N,N',k.

1.3.2. *Averages of squares of Bessel periods.* With (1.10) established, (1.6) is then consequence (for $\ell=1$) of the following result which evaluate the average of the square of the Bessel periods along the family $\mathcal{B}_k^{\widetilde{\mathfrak{n}}}(N)$ (see Theorem 11.1 for a more precise version):

Theorem 1.2. *Let notations and assumptions be as in §1.1.1. Let $\varphi'\in\pi'$ be the (fixed) automorphic newform of level N' and minimal weight $k>32$, defined in §4.5.1 and let $\mathcal{B}_k^{\widetilde{\mathfrak{n}}}(N)$ be the finite family of automorphic forms defined in §5.1.*

Given $\ell\geq 1$ an integer coprime with N and divisible only by primes inert in E, we denote by $\lambda_\varphi(\ell)$ and $\lambda_{\varphi'}(\ell)$ the eigenvalues at φ and φ' of the Hecke operators $T(\ell)$ and $T'(\ell)$ described in §11.1.

There is an absolute constant $C\geq 1$ such that for any $\delta>0$ and any quadruple (k,ℓ,N,N') satisfying *either*

$$(\ell N')^2 \leq N^{1-\delta},\ \ N > 16, k \geq C(1+1/\delta) \tag{1.11}$$

or

$$(\ell N')^2 \leq k^{1-\delta},\ \ N \leq 2^{4k}, \tag{1.12}$$

we have as $k+\ell+N+N' \to \infty$*,*

$$\sum_{\varphi \in \mathcal{B}_k^{\tilde{\mathfrak{n}}}(N)} \lambda_\varphi(\ell) \frac{\left|\mathcal{P}(\varphi,\varphi')\right|^2}{\langle \varphi,\varphi\rangle\langle\varphi',\varphi'\rangle} = w_E \frac{d_\Lambda}{d_k}(\frac{N}{N'})^2 \Psi(N)\mathfrak{S}(N') \frac{\lambda_{\pi'}(\ell) + o_{\delta,E}(1)}{\ell}. \tag{1.13}$$

Here w_E *is the number of units of* E*,*

$$\Psi(N) = \prod_{p|N}\left(1 - \frac{1}{p} + \frac{1}{p^2}\right),\ \ \mathfrak{S}(N') = \prod_{p|N'}(1-\frac{1}{p^2})^{-1}$$

(possibly equal to 1 *if* N *or* N' *is equal to* 1*) and*

$$d_\Lambda = \frac{(2k-2)(k+2)(k-6)}{3},\ \ d_k = k-1$$

(the formal degrees of π_∞ *and* π'_∞ *respectively).*

Remark 1.7. We call the conditions (1.11) and (1.12), on the relative sizes of N', N and k, the *stable ranges*. As we will see these conditions imply that the error term $o_{\delta,E}(1)$ in (1.13) decay exponentially in k as $k \to \infty$ or by a positive power of N (with an exponent linear in k) as $N \to \infty$.

The first stable range (1.11) is reminiscent to the stable range present in [MR12] and subsequently in [FW09].

1.4. **Quantitative non-vanishing.** Theorem 1.1 show that the set of π's for which the corresponding central value $L(1/2, \pi_E \times \pi'_E)$ does not vanish is non empty. One may wonder on its size in terms of N or k. Using the *amplification method*, we prove that the size of this set has polynomial growth as $k+N \to \infty$.

Theorem 1.3. *Let notations and assumptions be as in Theorem 1.1.*

There exists an absoltute constant $\delta > 0$ *such that as* $k+N \to \infty$*, we have*

$$|\{\pi \in \mathcal{A}_k^{\mathfrak{n}}(N),\ L(1/2, \pi_E \times \pi'_E) \neq 0\}| \gg_{N'} (kN)^\delta. \tag{1.14}$$

The lower bound (1.14) is an immediate consequence of the following *pointwise* upper bound which we deduced from Theorem 1.2 using the *amplification method*:

Theorem 1.4. *Notations be as above; there exists an absolute constant* $\delta > 0$ *such that for any* $\pi \in \mathcal{A}_k(N)$ *one has*

$$\sum_{\varphi \in \mathcal{B}_{k,\pi}^{\tilde{\mathfrak{n}}}(N)} \frac{\left|\mathcal{P}(\varphi,\varphi')\right|^2}{\langle \varphi,\varphi\rangle\langle\varphi',\varphi'\rangle} \ll_{N'} (kN)^{2-\delta}. \tag{1.15}$$

This bound, together with (see (1.13))

$$\sum_{\varphi \in \mathcal{B}_k^{\tilde{\mathfrak{n}}}(N)} \frac{\left|\mathcal{P}(\varphi,\varphi')\right|^2}{\langle \varphi,\varphi\rangle\langle\varphi',\varphi'\rangle} \asymp (kN)^2$$

and (1.10) implies (1.14).

Remark 1.8. As the proof will show, any fixed δ in the interval $(0, 1/82)$ would work. We have not tried to optimize this exponent as it is probably very far from the truth.

Remark 1.9. Notice that (1.15), (1.10) together with the upper bound

$$L(1,\mathrm{Ad},\pi_E)\ll_E (kN)^{o(1)} \tag{1.16}$$

(which is a consequence of temperedness), immediately imply the upper bound

$$L(1/2,\pi_E\times\pi'_E)\ll (kN)^{3-\delta+o(1)}.$$

Notice that this bound, is *weaker* that the *convexity bound* for $L(1/2,\pi_E\times\pi'_E)$: as we will see below in (5.32), the analytic conductor of $L(s,\pi_E\times\pi'_E)$ at $s=1/2$ is $\asymp_{N'} k^8N^4$ so that the convexity bound reads

$$L(1/2,\pi_E\times\pi'_E)\ll'_N (N^4k^8)^{1/4+o(1)}=(kN)^{o(1)}k^2N. \tag{1.17}$$

However we are unable to turn tables and use the, a priori stronger bound, (1.17) to improve Theorem 1.3. The reason is that we don't know how to obtain –unconditionally– a good *lower bound* for $L(1,\mathrm{Ad},\pi_E)$: one would expect that

$$L(1,\mathrm{Ad},\pi_E)=(kN)^{o(1)} \tag{1.18}$$

(see [HR95] for a discussion about this problem and [Bru06] for some unconditional, unfortunately not sufficient, lower bounds). If (1.18) were known it would give, by (1.10),

$$\frac{|\mathcal{P}(\varphi,\varphi')|^2}{\langle\varphi,\varphi\rangle\langle\varphi',\varphi'\rangle}\ll_{N'} (kN)^{o(1)}k \tag{1.19}$$

and by (1.6)

$$|\{\pi\in\mathcal{A}^{\mathrm{n}}_k(N),\ L(1/2,\pi_E\times\pi'_E)\neq 0\}|\gg_{N'} (kN)^{o(1)}kN^2\geq |\mathcal{A}^{\mathrm{n}}_k(N)|^{2/3-o(1)}.$$

It would be interesting to obtain the "convexity" bound (1.19) by a direct geometric analysis of the period integral $\mathcal{P}(\varphi,\varphi')$.

Remark 1.10. In view of the proof of the Bloch-Kato conjecture expected from the ongoing work of Y. Liu, Y. Tian , L. Xiao, W. Zhang and X. Zhu mentionned in §1.2, our result would imply the existence of infinitely many Selmer groups having rank 0.

1.5. **Idea of Proofs and Structure of the Paper.** The main ingredient of this work is the *relative trace formula of Jacquet and Rallis* for the pair (G,G') for a suitable choice of test functions (described in §4). As in [Nel23], our treatment differs from the traditional uses of the relative trace formula in functoriality (such as [BPLZZ21]) where one compares the geometric sides of two instances of the RTF to deduce consequences for the spectral sides) as we evaluate the geometric side by direct arguments. As pointed out above, such an approach was initiated by Rogawski and the second named author in [RR05] when they rederived, with a delineation of the underlying measure, the non-vanishing result of Duke.

The knowledgeable reader will have noted that the pair $(\mathrm{GL}(2),T)$ is not very far from to unitary group case $U(1,1)\times U(1)$ and we observe both similarities and discrepancies when passing to the $U(2,1)\times U(1,1)$ case. A similarity with [RR05] is that the main terms come from the contributions of the identity and unipotent cosets while the regular coset contribution is an error term[2]. It is worth noting, however that in the present case, the unipotent coset contribution is *significantly smaller* than the main term: by a factor which is at least a positive power of the size of the family $\mathcal{A}_k(N)$, while in [RR05] the difference is at most by a logarithmic factor. Another important difference is that the treatment of the regular orbital

[2]To be precise, in [RR05], the identity coset contribution vanishes identically but this is only because what was evaluated, was the average over a basis of GL(2) automorphic forms, of the product of two Hecke periods of twisted by two *distinct* characters; would these two characters have been equal this would have resulted been a main term

integrals is quite a bit more involved. We proceed by reducing the problem to bounding local integrals which we do by splitting into many subcases. In the present paper, we evaluate the average of the product $\mathcal{P}(\varphi,\varphi_1')\overline{\mathcal{P}(\varphi,\varphi_2')}$ for φ_1' and φ_2' belonging to the *same* representation π'; it turns out that, most of the time, the identity contribution is non-zero and in fact dominates the unipotent contribution. As in [RR05], if φ_1' and φ_2' belong to distinct representations, one can check easily that the identity contribution vanishes (because φ_1' and φ_2' are orthogonal). As for the unipotent contribution, we expect it to become the dominant term (proportional to $L(1,\pi_{1,E}'\times\pi_{2,E}')$); this will lead to simultanenous non-vanishing results analogous to those of [RR05, JN23, Yan23], namely the existence of π for which

$$L(1/2,\pi_E\times\pi_{1,E}')L(1/2,\pi_E\times\pi_{2,E}')\neq 0.$$

We will come back to this question in a forthcoming work.

Let us now provide a bit more details. We will allow ourselves, in this introduction, to be at time imprecise and write things which are only "morally" true. So the sketch should not be taken as a precise reflection of the details of our argument.

Let φ' be a primitive holomorphic cusp form on $G'(\mathbb{A})$ of weight k, level N' and trivial central character. Let π' be the corresponding cuspidal representation. We consider Jacquet's relative trace formula which takes the shape

$$\text{Spectral Side}=\int_{[G']}\int_{[G']}\mathrm{K}(x,y)\varphi'(x)\overline{\varphi'(y)}dxdy=\text{Geometric Side},\tag{1.20}$$

where $\mathrm{K}(x,y)=\mathrm{K}^f(x,y)$ is the kernel function of the Hecke operator $R(f)$ associated to a test function f, see Section 3 for details. Note that (1.20) can be thought as a 'section' of the Jacquet-Rallis trace formula [JR11]. In Sec. 4 we construct an explicit test function $f^{\mathfrak{n}}$ and use it into (1.20) to compute/estimate both sides.

The spectral side of (1.20) is handled in Sec. 5. We show that the operator $R(f^{\mathfrak{n}})$ eliminates the non-cuspidal spectrum so that the spectral side (1.13) becomes a (finite) second moment of Bessel periods relative to specific holomorphic cusp forms on G and G'.

For these automorphic forms, we use the recent work [BPLZZ21] to obtain an explicit Gan-Gross-Prasad formula of Ichino-Ikeda type for $G\times G'$ relating the central L-values $L(1/2,\pi_E\times\pi_E')$ to local and global period integrals. For this, we need compute explicitly several integrals of local matrix coefficients; this is done in §A.1 in the Appendix, and the main result in this section is Proposition 5.4. With this, one can write the spectral side of (1.20) as a weighted sum of central L-values $L(1/2,\pi_E\times\pi_E')$.

Next we evaluate the geometric side which is a sum of orbital integrals indexed by the double quotient $G'(\mathbb{Q})\backslash G(\mathbb{Q})/G'(\mathbb{Q})$. In Sec. 6 we decompose these orbital integrals into three subsets according to the properties of the classes in the quotient: the identity element, the unipotent type and regular type; a priori there also could be a term associated with an element of the shape $s.u$ with s,u non-trivial and respectively semisimple and unipotent but luckily, with our choice of global double coset representatives (Proposition 6.4) such a term does not occur. So (1.20) becomes

$$\text{Geometric Side}=\text{Identity Orb.}+\text{Unipotent Orb.}+\text{Regular Orb.},\tag{1.21}$$

where 'Orb.' refers to orbital integrals. The first term is made of a single orbital integral, the second is a finite sum of unipotent orbital integrals while the third term is an infinite sum of regular orbital integrals. They will be handled by different approaches in the subsequent sections.

The identity orbital integral is calculated in Section 7. Its contribution provides the main term in Theorem 1.2. In Section 8, we estimate the unipotent orbital integral by local computations. The contribution from this orbital integral gives a second main term on the right hand side of (1.13) which decays exponentially fast as k grows. Lastly, the more involved regular orbital integrals are investigated in Sections 9 and 10. The main result in this part is Theorem 10.1, which provides an upper bound for the infinite sum of the regular orbital integrals. A particular feature of this bound is that in the stable range (1.11) the contribution of the regular orbital integrals again decay exponentially fast with k.

Gathering these estimates in Section 11, we then prove Theorem 1.2 in its more precise form, Theorem 11.1.

In §12 we also interpret Theorem 11.1 as an horizontal Sato-Tate type equidistribution result for the Hecke eigenvalues of the π at a finite fixed set of inert primes weigthed by the periods $|\mathcal{P}(\varphi, \varphi')|^2$. This is inspired by the work of Royer [Roy00] who obtained vertical Sato-Tate type equidistribution results for Hecke eigenvalues of holomorphic modular forms of weight 2 and large level weigthed by the Hecke L-values $L(1/2, f)$.

Notice that Royer combined his results with a technique of Serre [Ser97] to exhibit irreducible factors of $\mathrm{Jac}(X_0(N))$ of dimension $\gg \log\log N$ and rank 0 (or of rank equal to the dimension). We expect that the ongoing work of X. Zhu and his collaborators will make it possible to obtain results of similar flavor.

Combining Theorem 11.1 with Proposition 5.4, one can deduce Theorem 1.1; however we need to be able to average only over *new forms* (if $N > 1$). In §13 we show that the old forms contribution is indeed smaller.

In §14 we prove Theorem 1.4 using Theorem 11.1 and the amplification method. Using again Proposition 5.4, we eventually prove Theorem 1.3.

2. Notations

2.1. **The quadratic field E.** Let $E = \mathbb{Q}(\sqrt{-D}) \hookrightarrow \mathbb{C}$ be an imaginary quadratic field; we denote by η the associated Legendre symbol which we view indifferently as a quadratic Dirichlet character, a cuarater on the group of idèles or a character on the Galois group of $\mathbb{Q}$. We denote the Galois involution by $\sigma \in \mathrm{Gal}(E/\mathbb{Q})$; it will also be useful to write

$$\sigma(z) = \overline{z}.$$

The trace and the norm are denoted by

$$z \mapsto \mathrm{tr}_{E/\mathbb{Q}}(z) = z + \overline{z}, z \mapsto \mathrm{Nr}_{E/\mathbb{Q}}(z) = z.\overline{z}$$

respectively.

We denote by $E^\times$ the multiplicative group of inversible elements and by

$$E^1 = \{z \in E^\times,\ z.\overline{z} = 1\} \subset E^\times$$

the subgroup of norm 1 elements; whenever useful we will denote in the same way the corresponding $\mathbb{Q}$-algebraic groups.

2.1.1. *Integers.* Let $\mathcal{O}_E$ be the ring of integers of E and

$$\mathcal{O}_E^\times = \mathcal{O}_E^1 = E^1 \cap \mathcal{O}_E$$

is group of units; set $w_E := \#\mathcal{O}_E^1$.

Let $D_E < 0$ be the discriminant of $\mathcal{O}_E$; we set

$$\Delta = i|D_E|^{1/2} \in \mathcal{O}_E.$$

The fractional ideal $\mathcal{D}_E^{-1} = \Delta^{-1}\mathcal{O}_E$ is the different: the $\mathbb{Z}$-dual of $\mathcal{O}_E$ with respect to the trace bilinear form

$$(z, z') \mapsto \mathrm{tr}_{E/\mathbb{Q}}(zz').$$

2.2. **The Hermitian space and its unitary group group.** Let V be a 3-dimensional vector space over E, with basis $\{e_1, e_0, e_{-11}\}$. Let $\langle \cdot, \cdot \rangle_J$ be a Hermitian form on V whose matrix with respect to $\{e_1, e_0, e_{-1}\}$ is

$$J = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}. \tag{2.1}$$

We denote by

$$G = U(V)$$

the unitary group preserving the form $\langle \cdot, \cdot \rangle_J$. this is an algebraic defined over $\mathbb{Q}$ and for any $\mathbb{Q}$-algebra R, the group of it R-points is

$$G(R) = \bigl\{ g \in \mathrm{GL}(V \otimes_{\mathbb{Q}} R),\ {}^{\mathrm{t}}\overline{g} J g = J \bigr\}.$$

The center of G is noted Z_G and made of the diagonal hermitian matrices

$$Z_G(R) = \Bigl\{ \begin{pmatrix} z & & \\ & z & \\ & & z \end{pmatrix},\ z \in E^1(R) \Bigr\}$$

so that

$$Z_G \simeq E^1 = U(1)$$

The special hermitian subgroup is noted $\mathrm{SU}(V)$ and its R point are given by

$$\mathrm{SU}(V)(R) = \{ g \in U(V)(R),\ \det g = 1 \}.$$

Also $n = p+q$ with $p, q \geq 0$ we denote by $U(p, q)$ the unitary group for the space E^n equipped with the hermitian form $\langle \cdot, \cdot \rangle_{p,q}$ with $n \times n$ matrix

$$J_{p,q} := \begin{pmatrix} \mathrm{Id}_p & \\ & -\mathrm{Id}_q \end{pmatrix}. \tag{2.2}$$

In other terms

$$U(p, q)(R) = \bigl\{ g \in \mathrm{GL}(V \otimes_{\mathbb{Q}} R),\ {}^{\mathrm{t}}\overline{g} J_{p,q} g = J_{p,q} \bigr\}.$$

We set $\mathrm{SU}(p, q)$ its special subgroup of elements of determinant 1. As usual we write $U(n)$ and $SU(n)$ for $U(n, 0)$ and $SU(n, 0)$. In particular we have

$$U(1) = E^1.$$

In fact, in this paper, excepted for $(p, q) = (1, 0)$, we will only need the $\mathbb{R}$-points of the groups $U(p, q)$ so to shorten notations, we will often write

$$U(p, q) \text{ for } U(p, q)(\mathbb{R}).$$

2.2.1. *The subgroup G'.* Let $G' \leq G$ be the stabilizer of the anisotropic line $\{e_0\}$. Then G' also preserves

$$W = \langle e_0 \rangle^{\perp} = \langle e_1, e_{-1} \rangle,$$

the orthocomplement of $\{e_0\}$. Note that W is a 2-dimensional Hermitian space, whose Hermitian form matrix is $\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$ with respect to the basis $\langle e_1, e_{-1} \rangle$. Hence we have an isomorphism of $\mathbb{Q}$-algebraic subgroups $U(W) \simeq G'$ via the embedding

$$i: \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & & b \\ & 1 & \\ c & & d \end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U(W \otimes_{\mathbb{Q}} R), \tag{2.3}$$

for any $\mathbb{Q}$-algebra R. We will identify $U(W)$ with G' henceforth. In particular we will sometimes represent an element of G' as a 2×2 matrix (its matrix in the above basis).

We also set

$$H := G' \times G' \subset G \times G. \tag{2.4}$$

2.2.2. *Convention regarding the split places.* Let p be a finite prime. For any $\mathbb{Q}$-algebra R we denote by R_p its completion with respect to the p-adic valuation. If p is split, we have a decomposition $p\mathcal{O}_E = \mathfrak{p}.\overline{\mathfrak{p}}$ into a product of distinct prime ideals of $\mathcal{O}_E$. Let us choose such a prime say $\mathfrak{p}$. The injection $\mathbb{Q} \hookrightarrow E$ of filed induces isomorphisms

$$E_{\mathfrak{p}} \simeq \mathbb{Q}_p,\ V_{\mathfrak{p}} = E_{\mathfrak{p}}.e_1 \oplus E_{\mathfrak{p}}.e_0 \oplus E_{\mathfrak{p}}.e_{-1} \simeq \mathbb{Q}_p.e_1 \oplus \mathbb{Q}_p.e_0 \oplus \mathbb{Q}_p.e_{-1}$$

and an isomorphism of linear groups

$$G(\mathbb{Q}_p) = U(V)(\mathbb{Q}_p) \simeq \mathrm{GL}(3, \mathbb{Q}_p). \tag{2.5}$$

For every split prime, we make a such choice, once and for all, and represent the elements of $G(\mathbb{Q}_p)$ as 3×3 matrices with coefficients in $\mathbb{Q}_p$. Likewise we represent the elements of $G'(\mathbb{Q}_p)$ either as 2×2 or 3×3 matrices with coefficients in $\mathbb{Q}_p$ (with a 1 as central coefficient for the later).

3. A Relative Trace Formula on $U(2,1)$

3.1. Recollection of the general principles of the trace Formula on $U(V)$.

Denote by $\mathbb{A}$ the adele ring of $\mathbb{Q}$. Let $\mathcal{A}_0(G)$ be the space of cuspidal representations on $G(\mathbb{A})$. In this section, we will introduce the framework of a relative trace formula on $U(V)$ for general test functions. We give the coarse geometric side of the trace formula, regardless of the convergence issue. In Sec. 4 we will specify our test function. Further careful analysis and computation towards the trace formula will be provided in following sections.

3.1.1. *Automorphic Kernel.* Let $K_\infty \subset G(\mathbb{R})$ be a maximal compact subgroup ($K_\infty \simeq U(2)(\mathbb{R}) \times U(1)(\mathbb{R})$). We consider a smooth function $h \in C_c^\infty(G(\mathbb{A}))$ which is left and right K_∞-finite, transforms by a unitary character ω of $Z_G(\mathbb{A})$. Denote by $\mathcal{H}(G(\mathbb{A}))$ the space of such functions. Then $h \in \mathcal{H}(G(\mathbb{A}))$ defines a convolution operator

$$R(h)\varphi = h * \varphi : x \mapsto \int_{G(\mathbb{A})} h(y)\varphi(xy)dy, \tag{3.1}$$

on the space $L^2\left(G(F)\backslash G(\mathbb{A}), \omega^{-1}\right)$ of functions on $G(F)\backslash G(\mathbb{A})$ which transform under $Z_G(\mathbb{A})$ by ω^{-1} and are square integrable on $G(F)\backslash G(\mathbb{A})$. This operator is represented by the kernel function

$$\mathrm{K}^h(x,y) = \sum_{\gamma \in G(\mathbb{Q})} h(x^{-1}\gamma y). \tag{3.2}$$

When the test function h is clear or fixed, we simply write $\mathrm{K}(x,y)$ for $\mathrm{K}^h(x,y)$.

It is well known that $L^2\left(G(\mathbb{Q})\backslash G(\mathbb{A}), \omega^{-1}\right)$ decomposes into the direct sums of the space $L_0^2\left(G(\mathbb{Q})\backslash G(\mathbb{A}), \omega^{-1}\right)$ of cusp forms and spaces $L_{\mathrm{Eis}}^2\left(G(\mathbb{Q})\backslash G(\mathbb{A}), \omega^{-1}\right)$ and $L_{\mathrm{Res}}^2\left(G(\mathbb{Q})\backslash G(\mathbb{A}), \omega^{-1}\right)$ defined using Eisenstein series and residues of Eisenstein series respectively and the operator K splits up as:

$$\mathrm{K} = \mathrm{K}_0 + \mathrm{K}_{\mathrm{Eis}} + \mathrm{K}_{\mathrm{Res}}\,.$$

Let notations be as before. Given φ' a cuspidal automorphic form on $G'(\mathbb{A})$ we consider the distribution

$$h \mapsto J(h) := \int_{H(\mathbb{Q})\backslash H(\mathbb{A})} \mathrm{K}(x_1, x_2)\varphi'(x_1)\overline{\varphi}'(x_2)dx_1dx_2, \tag{3.3}$$

where d refers the Tamagawa measure; more generally for $* \in \{0, \mathrm{Eis}, \mathrm{Res}\}$, we set

$$h \mapsto J_*(h) := \int_{H(\mathbb{Q})\backslash H(\mathbb{A})} \mathrm{K}_*(x_1, x_2)\varphi'(x_1)\overline{\varphi}'(x_2)dx_1dx_2. \tag{3.4}$$

Typically, because of convergence issue, one needs to introduce certain regularization into (3.4) to make these expressions well defined and then we have

$$J(h) = J_0(h) + J_{\mathrm{Eis}}(h) + J_{\mathrm{Res}}(h).$$

In our situation we have (§5.1)

$$J_{\mathrm{Eis}}(h) = J_{\mathrm{Res}}(h) = 0 \tag{3.5}$$

so that

$$J(h) = J_0(h).$$

Since K^h is the kernel of $R(h)$, $J(h)$ admit a spectral expansion, ie. is a weighted sum, over an orthogonal family $\varphi \in \mathcal{B}_k^{\tilde{\mathfrak{n}}}(N)$ of cuspforms of level N and weight k, of the periods squared $|\mathcal{P}(\varphi, \varphi')|^2$ (see Lemma 5.2). We refer to this expression as the spectral side of the relative trace formula.

3.1.2. *Geometric Reduction.* Assume now that $\omega = 1$.

Let Φ be a set of representatives of the double quotient $G'(\mathbb{Q})\backslash G(\mathbb{Q})/G'(\mathbb{Q})$. For each $\gamma \in \Phi$, we denote by

$$H_\gamma = \{(u, v) \in H = G' \times G',\ u^{-1}\gamma v = \gamma\}$$

its stabilizer in $G' \times G'$. Then one has (assuming that everything converges absolutely)

$$\begin{aligned} J(h) &= \int_{H(\mathbb{Q})\backslash H(\mathbb{A})} \sum_{\gamma \in G(\mathbb{Q})} h(x_1^{-1}\gamma x_2)\varphi'(x_1)\overline{\varphi}'(x_2)dx_1dx_2 \\ &= \int_{H(\mathbb{Q})\backslash H(\mathbb{A})} \sum_{\gamma \in \Phi} \sum_{\delta \in [\gamma]} h(x_1^{-1}\delta x_2)\varphi'(x_1)\overline{\varphi}'(x_2)dx_1dx_2. \end{aligned} \tag{3.6}$$

Therefore (assuming that everything converges absolutely) one can write $J(h)$ as

$$J(h) = \int_{H(\mathbb{Q})\backslash H(\mathbb{A})} \sum_{\gamma \in \Phi} \sum_{(\delta_1, \delta_2) \in H_\gamma(\mathbb{Q})\backslash H(\mathbb{Q})} h(x_1^{-1}\delta_1^{-1}\gamma\delta_2 x_2)\varphi'(x_1)\overline{\varphi}'(x_2)dx_1dx_2.$$

Then switching the sums and noticing the automorphy of φ', one then obtains

$$J(h) = \sum_{\gamma \in \Phi} \int_{H_\gamma(\mathbb{Q})\backslash H(\mathbb{A})} h(x_1^{-1}\gamma x_2)\varphi'(x_1)\overline{\varphi}'(x_2)dx_1dx_2. \tag{3.7}$$

In Sec. 4.4 (see (4.31)) we will choose $h = f^{\mathfrak{n}}$ precisely to make (3.6) converge absolutely so that (3.7) holds rigorously.

4. Choice of local and global data

In this section, we describe our choices of the test function h and the automorphic form φ' so that the relative trace formula captures the family of automorphic forms indicated above.

The test function $h \in \mathcal{C}_c^\infty(G(\mathbb{A}))$ will be a linear combination of factorable test functions of the shape

$$f^{\mathfrak{n}} = f_\infty \otimes \otimes'_p f_p^{\mathfrak{n}}. \tag{4.1}$$

The non-archimedean components $f_p^{\mathfrak{n}}$ are discussed in §4.2. As we will see these components also depend (in addition to N and N') on an integer $\ell \geq 1$ coprime to $NN'D$.

The archimedean component f_∞ is discussed in the next subsections. It is obtained from matrix coefficients of holomorphic discrete series of $U(2,1)$.

4.1. Holomorphic Discrete Series Representation of $U(2,1)$.

Let us recall that there are three types of discrete series of $U(2,1)$ which embed in the non-unitary principal series, namely the holomorphic, the antiholomorphic, and the nonholomorphic discrete series. A full description of these three discrete series and models for their respective representation spaces can be found in [Wal76]. In this paper, we will focus on holomorphic discrete series.

We recall that $U(2,1)$ is the unitary group of the hermitian space $\mathbb{C}^3$ with Hermitian form given by the matrix

$$J_{2,1} := \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}$$

Its maximal compact subgroup is

$$U(2,1) \cap U(3) = \begin{pmatrix} U(2) & \\ & U(1) \end{pmatrix} \simeq U(2) \times U(1)$$

This relates to the group $G(\mathbb{R})$ as follows: let

$$\mathrm{B} = \begin{pmatrix} 1/\sqrt{2} & & 1/\sqrt{2} \\ & 1 & \\ 1/\sqrt{2} & & -1/\sqrt{2} \end{pmatrix}. \tag{4.2}$$

Then we have $\mathrm{B} = \mathrm{B}^{-1}$, $J_{2,1} = \mathrm{B}J\mathrm{B}^{-1}$ and we have an isomorphism

$$\iota_{\mathrm{B}} : \ G(\mathbb{R}) \xrightarrow{\sim} G_{J_{2,1}}(\mathbb{R}), \qquad g \mapsto \mathrm{B}g\mathrm{B}^{-1}. \tag{4.3}$$

Consequently the maximal compact subgroup of $G(\mathbb{R})$ equals

$$K_\infty = \mathrm{B} \begin{pmatrix} U(2)(\mathbb{R}) & \\ & U(1)(\mathbb{R}) \end{pmatrix} \mathrm{B}^{-1}. \tag{4.4}$$

4.1.1. *Holomorphic Discrete Series on $SU(2,1)$.*

Let $SU(2,1) \subset U(2,1)$ be its special subgroup. It's maximal compact subgroup is noted

$$K_{\infty,2,1} = SU(2,1) \cap U(3) = \left\{ \begin{pmatrix} u & 0 \\ 0 & (\det u)^{-1} \end{pmatrix} : \ u \in U(2) \right\} \simeq U(2).$$

Since $\operatorname{rank} SU(2,1) = \operatorname{rank} K_{\infty,2,1} = 1$, $SU(2,1)$ has discrete series representations. In this section we recall that explicit description of holomorphic discrete series given by Wallach [Wal76].

Let S^3 be the unit sphere

$$S^3 = \left\{ z = {}^{\mathrm{t}}(z_1, z_2) \in \mathbb{C}^2 : \ |z_1|^2 + |z_2|^2 = 1 \right\}.$$

We have an homeomorphism $S^3 \simeq SU(2)(\mathbb{R})$

$$u : {}^{\mathrm{t}}(z_1, z_2) \in S^3 \mapsto u(z_1, z_2) = \begin{pmatrix} z_1 & -\overline{z}_2 \\ z_2 & \overline{z}_1 \end{pmatrix} \in SU(2). \tag{4.5}$$

The group $SU(2,1)(\mathbb{R})$ acts on S^3 via

$$g.z = \frac{Az + b}{\langle z, {}^{\mathrm{t}}\overline{c}\rangle + d}, \quad g = \begin{pmatrix} A & b \\ c & d \end{pmatrix} \in SU(2,1)(\mathbb{R}), \tag{4.6}$$

(here the $\langle \cdot, \cdot \rangle$ is the usual hermitian product on $\mathbb{C}^2$).

Let $\alpha_1 = (1, -1, 0)$ and $\alpha_2 = (0, 1, -1)$; this form a basis of simple roots in the root system of $\mathfrak{sl}(3, \mathbb{C})$ relative to the diagonal $\mathfrak{h}$. The basic highest weights for this order are

$$\Lambda_1 = (2/3, -1/3, -1/3) \text{ and } \Lambda_2 = (1/3, 1/3, -2/3).$$

For $(k_1, k_2) \in \mathbb{Z}^2$, let

$$\Lambda = k_1\Lambda_1 + k_2\Lambda_2.$$

For $h \in C^\infty(S^3)$ we define

$$(\pi_\Lambda(g)(h))(z) := a(g, z)^{k_1}\overline{a(g, z)}^{k_2} h(g^{-1}.z)$$

where

$$a(g, z) = \overline{d} - \langle z, b\rangle$$

for z and g as in (4.6). Then π_Λ extends to a bounded operator on $L^2(S^3)$ and $(\pi_\Lambda, L^2(S^3))$ defines a continuous representation of $SU(2,1)(\mathbb{R})$.

This representation restricted to the compact subgroup K_0 decomposes into irreducible as follows: let p, q be nonnegative integers and $\mathcal{H}^{p,q}$ be the space of polynomials $h \in \mathbb{C}[z_1, z_2, \overline{z}_1, \overline{z}_2]$ which are homogeneous of degree p in z_1, z_2, degree q in $\overline{z}_1, \overline{z}_2$; and harmonic, namely,

$$\Delta h = \left(\frac{\partial^2}{\partial z_1 \partial \overline{z}_1} + \frac{\partial^2}{\partial z_2 \partial \overline{z}_2}\right) h \equiv 0.$$

Denote by $\mathscr{H}^{p,q} = \mathcal{H}^{p,q} \mid_{S^3}$. Then $(\pi_\Lambda \mid_{K_0}, \mathscr{H}^{p,q})$ is irreducible and

$$L^2(S^3) = \oplus_{p \geq 0} \oplus_{q \geq 0} \mathscr{H}^{p,q}.$$

To describe the holomorphic discrete series, we also assume that $k_1 < 0$ and $k_2 \geq 0$. Let ρ be the half sum of positive roots, i.e.,

$$\rho = (1, 0, -1) = \Lambda_1 + \Lambda_2.$$

Given integers $p \geq 0$ and $0 \leq q \leq k_2$ we set

$$c_{p,q}(\Lambda) = \prod_{k=1}^{p} \frac{\langle \Lambda + (k+1)\rho, \alpha_2\rangle}{\langle -\Lambda + (k-1)\rho, \alpha_1\rangle} \cdot \prod_{j=1}^{q} \frac{\langle \Lambda + (j+1)\rho, \alpha_1\rangle}{\langle -\Lambda + (j-1)\rho, \alpha_2\rangle}$$

with each of the above products equal to 1 if p or $q = 0$.

A straightforward calculation shows that

$$c_{p,q}(\Lambda) = \prod_{k=1}^{p} \frac{k + k_2 + 1}{k - k_1 - 1} \cdot \prod_{j=1}^{q} \frac{j + k_1 + 1}{j - k_2 - 1}.$$

In particular $c_{p,q}(\Lambda)$ is well defined and if $k_2 + k_1 + 1 < 0$, it is nonvanishing. From these, we define an inner product $\langle \cdot, \cdot \rangle_\Lambda$ with respect to $\Lambda = k_1\Lambda_1 + k_2\Lambda_2$ via the one on $L^2(S^3)$ as follows: given $h_1, h_2 \in C^\infty(S^3)$, by spectral decomposition we can write

$$h_1 = \sum h_{1,p,q}, \quad h_2 = \sum h_{2,p,q}, \quad h_{1,p,q},\ h_{2,p,q} \in \mathscr{H}^{p,q}$$

and set

$$\langle h_1, h_2\rangle_\Lambda := \sum_p \sum_q c_{p,q}(\Lambda)\langle h_{1,p,q}, h_{2,p,q}\rangle. \tag{4.7}$$

The following parametrization of holomorphic discrete series of $SU(2,1)(\mathbb{R})$ is due to Wallach [Wal76] (see p. 183):

Proposition 4.1. *Let $(k_1, k_2) \in \mathbb{Z}^2$. Let $\Lambda = k_1\Lambda_1 + k_2\Lambda_2 \in \mathfrak{h}^*$. Assume*

$$\langle \Lambda + \rho, S_1S_2\alpha_i\rangle > 0,$$

where $1 \le i \le 2$, and

$$S_1 : (x, y, z) \mapsto (y, x, z), S_2 : (x, y, z) \mapsto (x, z, y)$$

are the simple Weyl reflections. Let

$$V_{k_2}^+ := \{h \in C^\infty(S^3) : \ h_{p,q} = 0 \text{ if } q > k_2\}.$$

Let V_+^Λ be the Hilbert space completion of $V_{k_2}^+$ relative to the inner product (4.7). Then $D_\Lambda^+ := \pi_\Lambda \mid_{V_+^\Lambda}$ is a unitary holomorphic discrete series representation of $SU(2,1)(\mathbb{R})$. Moreover, the holomorphic discrete series representations of $SU(2,1)(\mathbb{R})$ are of form D_Λ^+.

Remark 4.1. Note that the inner product $\langle\cdot,\cdot\rangle_\Lambda$ with respect to $\Lambda = k_1\Lambda_1 + k_2\Lambda_2$ is well defined on the space $V_{k_2}^+$.

Later we will need to compute explicitly some inner products $\langle h_{1,p,q}, h_{2,p,q}\rangle$: after decomposing $h_{1,p,q}$ (resp. $h_{2,p,q}$) into a finite linear combination of monomials of the form $z_1^a z_2^b \overline{z}_1^c \overline{z_2}^d$ one can use the following

Lemma 4.2. *Let notation be as above. Let a, b, c, d be nonnegative integers. Then*

$$\langle z_1^a z_2^b, z_1^c {z_2}^d\rangle = \frac{\delta_{a,c}\delta_{b,d}a!b!}{(a+b+1)!}. \tag{4.8}$$

Proof. The inner product

$$\langle z_1^a z_2^b, z_1^c {z_2}^d\rangle = \int_{S^3} z_1^a z_2^b \overline{z}_1^c \overline{z_2}^d d\mu(z_1, z_2)$$

where $\mu(z_1, z_2)$ denote the $SU(2,1)$-invariant probability measure on the sphere S^3 (which is also the Haar measure on $SU(2)(\mathbb{R})$ under (4.5)). Write

$$z_1 = e^{i(\alpha+\beta)}\cos\theta,\ \ z_2 = e^{i(\alpha-\beta)}\sin\theta,$$

where $\theta \in [0, \pi/2]$, $\alpha \in [-\pi, \pi]$, $\beta \in [0, \pi]$. In these polar coordinate system we have

$$d\mu(z, \overline{z}) = \frac{\sin\theta\cos\theta}{\pi^2} d\theta d\alpha d\beta.$$

Then the left hand side of (4.8) is equal to

$$\frac{1}{\pi^2}\int_0^{\frac{\pi}{2}}\int_0^{\pi}\int_{-\pi}^{\pi} e^{(a-c)(\alpha+\beta)i}\cos^{a+c+1}\theta e^{(b-d)(\alpha-\beta)i}\sin^{b+d+1}\theta d\alpha d\beta d\theta. \tag{4.9}$$

Appealing to orthogonality we then see that (4.9) is equal to

$$2\delta_{a,c}\delta_{b,d}\int_0^{\frac{\pi}{2}}\cos^{2a+1}\theta\sin^{2b+1}\theta d\theta = \frac{\delta_{a,c}\delta_{b,d}a!b!}{(a+b+1)!}.$$

Hence the formula (4.8) follows. ∎

4.1.2. *K-types.* Recall that the maximal compact subgroup K_0 of $SU_{J_{2,1}}(\mathbb{R})$ consisting of $SU_{J_{2,1}} \cap U(3)$ can be identified with $U(2)$.

Let $K_{0c} \simeq U(1)$ be the central part of K_0, and $K_{0s} \simeq SU(2)$ be the semisimple part. An irreducible unitary representation of K_0 is completely determined by its restriction to K_{0c} and K_{0s}. Therefore, such representations are parameterized by two integers m, n, such that $n \geq 0$ and $m - n$ even: m determines the character of K_{0c} and $n+1$ is the dimension of the irreducible representation of K_{0s}.

Write $z = {}^{\mathrm{t}}(z_1, z_2)$. For each integer N, the group K_0 acts on $\mathcal{H}^{p,q}$ via

$$\tau_{p,q}^N \begin{pmatrix} u & \\ & (\det u)^{-1} \end{pmatrix} h(z, \overline{z}) = (\det u)^{-N} h(uz, u^{-1}\overline{z}), \ u \in U(2).$$

Let T denote the Cartan subgroup of $SU_{J_{2,1}}(\mathbb{R})$:

$$T = \Big\{ \operatorname{diag}(z_1, z_2, z_3) : \ |z_1| = |z_2| = |z_3| = 1, \ z_1 z_2 z_3 = 1 \Big\}.$$

When restricted to K_{0s} we can take $\phi_{p,q}(z, \overline{z}) = z_1^p \overline{z}_2^q$ as a highest weight vector in $\mathcal{H}^{p,q}$. Observing that

$$\tau_{p,q}^N \begin{pmatrix} e^{i\alpha} & & \\ & e^{i\beta} & \\ & & e^{-i(\alpha+\beta)} \end{pmatrix} \phi_{p,q} = e^{pi\alpha} e^{-qi\beta} e^{-Ni(\alpha+\beta)} \phi_{p,q}.$$

Then the parametrization of irreducible unitary representations of K_0 becomes $(m, n) = (p - q - 2N, p + q)$. A straightforward computation shows that the highest weight of the representation $\tau_{p,q}^N$ is $(p+q)\Lambda_1 - (q+N)\Lambda_2$.

4.1.3. *Highest Weight Vector in Minimal K-types and Matrix Coefficients.* Let

$$\Lambda = k_1 \Lambda_1 + k_2 \Lambda_2$$

be as in Proposition 4.1. In this section, we will find a minimal K-type of the discrete series D_Λ^+. By definition and Theorem 9.20 in [Kna01] we see that the minimal K-type we are seeking is the Blattner parameter $\widetilde{\Lambda}$ of D_Λ^+.

Definition 4.1. Let $\Lambda = k_1\Lambda_1 + k_2\Lambda_2$ be a weight in $\mathfrak{h}^*$, with $k_1, k_2 \in \mathbb{Z}$. We say Λ is *holomorphic* if $\langle \Lambda + \rho, w_1 w_2 \alpha_i \rangle > 0$, where $1 \leq i \leq 2$, and w_i is the Weyl element.

Lemma 4.3. *Let notation be as before. Assume Λ is holomorphic. Then $k_2 \geq 0$, $k_1 + k_2 + 2 < 0$ and*

$$\widetilde{\Lambda} = k_2 \Lambda_1 + k_1 \Lambda_2. \tag{4.10}$$

Proof. Let $l = k_2 - k_1 \in \mathbb{Z}_{>0}$. By Lemma 7.9 in [Wal76], we have

$$(\pi_\Lambda |_K, \mathcal{H}^{p,q}) \equiv \tau_{p+q}^{2l+3(p-q)},$$

which is K-equivariant. Hence, for $p \geq 0$ and $0 \leq q \leq k_2$,

$$(D_\Lambda^+ |_K, \mathcal{H}^{p,q}) \equiv \tau_{p+q}^{2l+3(p-q)}.$$

We can describe the corresponding highest weight in terms of coordinates in $\mathbb{C}^3$. Choose the basis $\alpha_1 = (1, -1, 0)$, $\alpha_2 = (0, 1, -1)$ of simple roots in the root system of $\mathfrak{su}(2,1)^{\mathbb{C}} = \mathfrak{sl}(3, \mathbb{C})$ relative to the diagonal $\mathfrak{h}$. Under this choice, the basic weights are $\Lambda_1 = (2/3, -1/3, -1/3)$ and $\Lambda_2 = (1/3, 1/3, -2/3)$. So the highest weight is

$$H(p,q) = \left(\frac{3q - l}{3}, \frac{-3p - l}{3}, \frac{2l + 3p - 3q}{3} \right), \ p \geq 0, \ 0 \leq q \leq k_2.$$

Note that α_1 is the only positive compact root. Let

$$G(p,q) := \|H(p,q) + \alpha_1\|^2.$$

We then need to find pairs (p,q) such that $G(p,q)$ is minimal. Since $\frac{\partial G}{\partial p}(p,q)>0$ for all $p,q\geq 0$, we have $G(p,q)\geq G(0,q)$. Note that

$$G(0,q)=\frac{1}{9}\cdot\left[(3q-l+3)^2+(l+3)^2+(2l-3q)^2\right]=2\left(q-\frac{l-1}{2}\right)^2+\frac{(l+3)^2}{6}.$$

On the other hand, we have $\langle\Lambda+\rho,w_1w_2\alpha_i\rangle>0$, where $1\leq i\leq 2$, and w_i is the Weyl element. By definition, for a weight ν,

$$w_i\nu=\nu-\frac{2\langle\nu,\alpha_i\rangle}{\langle\alpha_i,\alpha_2\rangle}\cdot\alpha_i=\nu-\langle\nu,\alpha_i\rangle\alpha_i,\quad 1\leq i\leq 2.$$

Hence $\langle\Lambda+\rho,w_1w_2\alpha_i\rangle>0$ is equivalent to the conditions $k_2\geq 0$ and $k_1+k_2+2<0$. Therefore, $k_2<(l-1)/2$, implying that

$$G(p,q)\geq G(0,q)\geq G(0,k_2)$$

for all $p\geq 0$ and $0\leq q\leq k_2$. Then (4.10) follows. $\square$

From Lemma 4.3 we have a highest weight vector

$$\phi(z,\overline{z})=\overline{z}_2^{k_2}\tag{4.11}$$

for the minimal K-type of D_Λ^+. We then compute the corresponding matrix coefficient in Proposition 4.5. To prepare for the proof, we need the following auxiliary computation:

Lemma 4.4. *Let $A,B\in\mathbb{C}$ be such that $|A|\neq|B|$. Let $m,n\in\mathbb{Z}$ with $n>|m|$. Then*

$$I_{m,n}(A,B)=\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{-mi\alpha}}{(A+Be^{i\alpha})^n}d\alpha=\begin{cases}\frac{(-B)^m}{A^{n+m}}\binom{n+m-1}{m}, & \text{if } |A|>|B|;\\ 0, & \text{if } |A|<|B|.\end{cases}$$

Proof. Suppose $|A|>|B|\geq 0$. Let $C=B/A$. Then

$$\int_0^{2\pi}\frac{e^{-mi\alpha}}{(A+Be^{i\alpha})^n}d\alpha=\frac{1}{A^n}\int_0^{2\pi}\frac{e^{-mi\alpha}}{(1+Ce^{i\alpha})^n}d\alpha=\frac{1}{A^n}\sum_{k\geq 0}C_{n,k}\int_0^{2\pi}e^{(k-m)i\alpha}d\alpha,$$

which is vanishing if $m<0$. Suppose $m\geq 0$. Then

$$I_{m,n}(A,B)=\frac{C_{n,m}}{A^n}=\frac{C^m}{A^n}\binom{-n}{m}=\frac{(-B)^m}{A^{n+m}}\binom{n+m-1}{m}.$$

Now we suppose $|A|<|B|$. Let $D=A/B$. Then

$$I_{m,n}(A,B)=\frac{1}{2\pi B^n}\int_0^{2\pi}\frac{e^{-(m+n)i\alpha}}{(1+De^{-i\alpha})^n}d\alpha=\frac{1}{2\pi B^n}\sum_{k\geq 0}D_{n,k}\int_0^{2\pi}e^{-(k+m+n)i\alpha}d\alpha,$$

which is vanishing since $m+n>0$. $\square$

Proposition 4.5. *Let notation be as before. Let $g=(g_{i,j})_{1\leq i,j\leq 3}\in SU_{J_{2,1}}(\mathbb{R})$. Let*

$$\phi_\circ=\phi/\langle\phi,\phi\rangle_\Lambda^{1/2}\tag{4.12}$$

(for ϕ defined in (4.11)). Then

$$\langle D_+^\Lambda(g)\phi_\circ,\phi_\circ\rangle_\Lambda=g_{22}^{k_2}\overline{g}_{33}^{k_1}.\tag{4.13}$$

Proof. By definition (see Proposition 4.1) $D_\Lambda^+=\pi_\Lambda\mid_{V_+^\Lambda}$ and $\phi\in V_+^\Lambda$. Therefore,

$$\begin{aligned}\langle D_+^\Lambda(g)\phi,\phi\rangle_\Lambda=&\langle\pi_\Lambda(g)\phi,\phi\rangle_\Lambda=\sum_p\sum_q c_{p,q}(\Lambda)\langle(\pi_\Lambda(g)\phi)_{p,q},\phi_{p,q}\rangle\\ =&c_{0,k_2}(\Lambda)\langle(\pi_\Lambda(g)\phi)_{0,k_2},\phi\rangle.\end{aligned}$$

Write $g=(g_{ij})_{1\leq i\leq 3}\in SU_{J_{2,1}}\subset SL(3,\mathbb{C})$. By definition ${}^{\mathrm{t}}\overline{g}J_{2,1}g=J_{2,1}$. So

$$g^{-1}=J_{2,1}^{-1}\,{}^{\mathrm{t}}\overline{g}J_{2,1}=\begin{pmatrix}\overline{g}_{11} & \overline{g}_{21} & -\overline{g}_{31}\\ \overline{g}_{12} & \overline{g}_{22} & -\overline{g}_{32}\\ -\overline{g}_{13} & -\overline{g}_{23} & \overline{g}_{33}\end{pmatrix}. \tag{4.14}$$

According to the group action (4.6) we obtain

$$g^{-1}.z={}^{\mathrm{t}}\left(\frac{\overline{g}_{11}z_1+\overline{g}_{21}z_2-\overline{g}_{31}}{-\overline{g}_{13}z_1-\overline{g}_{23}z_2+\overline{g}_{33}},\frac{\overline{g}_{12}z_1+\overline{g}_{22}z_2-\overline{g}_{32}}{-\overline{g}_{13}z_1-\overline{g}_{23}z_2+\overline{g}_{33}}\right)\in S^3.$$

Thus $\pi_\Lambda(g)\phi(z,\overline{z})$ is equal to

$$(\overline{g}_{33}-\overline{g}_{13}z_1-\overline{g}_{23}z_2)^{k_1}\overline{(\overline{g}_{33}-\overline{g}_{13}z_1-\overline{g}_{23}z_2)}^{k_2}\cdot\overline{\left(\frac{\overline{g}_{12}z_1+\overline{g}_{22}z_2-\overline{g}_{32}}{-\overline{g}_{13}z_1-\overline{g}_{23}z_2+\overline{g}_{33}}\right)}^{k_2},$$

namely,

$$\pi_\Lambda(g)\phi(z,\overline{z})=(\overline{g}_{33}-\overline{g}_{13}z_1-\overline{g}_{23}z_2)^{k_1}\cdot(g_{12}\overline{z}_1+g_{22}\overline{z}_2-g_{32})^{k_2}.$$

Since

$$\phi_\circ=\frac{1}{c_{0,k_2}(\Lambda)^{1/2}}\frac{\phi}{\langle\phi,\phi\rangle^{1/2}}.$$

We have by Lemma 4.2

$$\begin{aligned}\langle D_+^\Lambda(g)\phi_\circ,\phi_\circ\rangle_\Lambda=&\frac{c_{0,k_2}(\Lambda)\cdot\int_{S^3}\frac{(g_{12}\overline{z}_1+g_{22}\overline{z}_2-g_{32})^{k_2}}{(\overline{g}_{33}-\overline{g}_{13}z_1-\overline{g}_{23}z_2)^{-k_1}}\cdot z_2^{k_2}d\mu(z,\overline{z})}{c_{0,k_2}(\Lambda)\cdot\int_{S^3}\overline{z}_2^{k_2}\cdot z_2^{k_2}d\mu(z,\overline{z})}\\ =&(k_2+1)\cdot\int_{S^3}\frac{(g_{12}\overline{z}_1+g_{22}\overline{z}_2-g_{32})^{k_2}}{(\overline{g}_{33}-\overline{g}_{13}z_1-\overline{g}_{23}z_2)^{|k_1|}}\cdot z_2^{k_2}d\mu(z,\overline{z})\end{aligned}$$

Since we have the parametrization

$$z_1=e^{i(\alpha+\beta)}\cos\theta,\ z_2=e^{i(\alpha-\beta)}\sin\theta,\ \theta\in[0,\pi/2],\ \alpha\in[-\pi,\pi],\ \beta\in[0,\pi],$$

and

$$d\mu(z,\overline{z})=\frac{\sin\theta\cos\theta}{\pi^2}d\theta d\alpha d\beta,$$

we have

$$\begin{aligned}&\pi^2\int_{S^3}\frac{(g_{12}\overline{z}_1+g_{22}\overline{z}_2-g_{32})^{k_2}}{(\overline{g}_{33}-\overline{g}_{13}z_1-\overline{g}_{23}z_2)^{|k_1|}}\cdot z_2^{k_2}d\mu(z,\overline{z})\\ =&\int_0^{\frac{\pi}{2}}\int_0^\pi\int_{-\pi}^\pi\frac{\left(g_{12}e^{-2i\beta}\cos\theta+g_{22}\sin\theta-g_{32}e^{i(\alpha-\beta)}\right)^{k_2}}{(\overline{g}_{33}-\overline{g}_{13}e^{i(\alpha+\beta)}\cos\theta-\overline{g}_{23}e^{i(\alpha-\beta)}\sin\theta)^{|k_1|}}\cdot\sin^{k_2+1}\theta\cos\theta d\alpha d\beta d\theta.\end{aligned}$$

Applying the expression (4.14) for g^{-1} into $g^{-1}g=\mathrm{Id}$ one has

$$|g|_{33}^2=|g_{13}|^2+|g_{23}|^2+1.$$

Then in conjunction with Cauchy inequality, we get

$$|\overline{g}_{13}e^{i\beta}\cos\theta-\overline{g}_{23}e^{-i\beta}\sin\theta|\leq\sqrt{|g_{13}|^2+|g_{23}|^2}=\sqrt{|g_{33}|^2-1}<|g_{33}|.$$

Therefore we can appeal to Lemma 4.4 to conclude that

$$\begin{aligned}\frac{\langle D_+^\Lambda(g)\phi_\circ,\phi_\circ\rangle_\Lambda}{k_2+1}=&\frac{2}{\pi\overline{g}_{33}^{|k_1|}}\int_0^{\frac{\pi}{2}}\int_0^\pi\left(g_{12}e^{-2i\beta}\cos\theta+g_{22}\sin\theta\right)^{k_2}\cdot\sin^{k_2+1}\theta\cos\theta d\beta d\theta\\ =&2g_{22}^{k_2}\overline{g}_{33}^{k_1}\int_0^{\frac{\pi}{2}}\sin^{2k_2+1}\theta\cos\theta d\theta=\frac{g_{22}^{k_2}\overline{g}_{33}^{k_1}}{k_2+1}.\end{aligned}$$

□

4.1.4. *Discrete Series on* $G(\mathbb{R})$. Recall that $\mathbf{B}=\mathbf{B}^{-1}$ and

$$G(\mathbb{R})=\mathbf{B}U(2,1)(\mathbb{R})\mathbf{B}^{-1}$$

and therefore, setting

$$G^1=\ker(\det: G\mapsto \mathbb{G}_{\mathrm{m}})=SU(W)$$

we have

$$G^1(\mathbb{R})=\mathbf{B}SU(2,1)\mathbf{B}^{-1}=\mathbf{B}SU(2,1)\mathbf{B}.$$

Setting for $g\in G^1(\mathbb{R})$

$$D^{\Lambda}_{+,\mathrm{B}}(g):=D^{\Lambda}_{+}(\mathrm{B}g\mathrm{B}),$$

we denote by $(D^{\Lambda}_{+,\mathrm{B}},V^{\Lambda}_{+})$ the discrete series representation on $G^1(\mathbb{R})$.

From the split exact sequence

$$1\longrightarrow G^1(\mathbb{R})\longrightarrow G(\mathbb{R})\longrightarrow Z_G(\mathbb{R})\longrightarrow 1,$$

(with $Z_G(\mathbb{R})\simeq U(1)(\mathbb{R})=\mathbb{C}^1$, we have the decomposition

$$G(\mathbb{R})=Z_G(\mathbb{R})^{+}G^1(\mathbb{R}). \tag{4.15}$$

where

$$Z^{+}_G(\mathbb{R})=\{\mathrm{diag}(e^{i\theta},e^{i\theta},e^{i\theta}):\ -\pi/3<\theta\le\pi/3\}.$$

Using (4.15) we extend the $G^1(\mathbb{R})$-action $(D^{\Lambda}_{+,\mathrm{B}},V^{\Lambda}_{+})$ to $G(\mathbb{R})$ by requiring $Z_G(\mathbb{R})^{+}$ to act trivially. Let $z=\mathrm{diag}(e^{i\theta},e^{i\theta},e^{i\theta})\in Z_G(\mathbb{R})$, $-\pi<\theta\le\pi$. Let $\theta_\circ\in(-\pi/3,\pi/3]$ be such that

$$\frac{3\theta}{2\pi}\equiv\frac{3\theta_\circ}{2\pi}\ (\mathrm{mod}\ 1). \tag{4.16}$$

Such a $\theta_\circ$ is uniquely determined by θ. Set

$$z_\circ=\mathrm{diag}(e^{i\theta_\circ},e^{i\theta_\circ},e^{i\theta_\circ})\in Z_G(\mathbb{R})^{+}.$$

Then by (4.16) there exists a unique $k_z\in\{-1,0,1\}$ be such that $z=z_\circ e^{2\pi k_z i/3}$. We have

$$z\cdot z_\circ^{-1}=\mathrm{diag}(e^{2\pi k_z i/3},e^{2\pi k_z i/3},e^{2\pi k_z i/3})\in Z_G(\mathbb{R})\cap G^1(\mathbb{R}),$$

$k_z\in\{-1,0,1\}$. Let $\rho(z):=e^{2\pi k_z i/3}$. Therefore, z acts by the scalar

$$\omega_\Lambda(z)=D^{\Lambda}_{+}(z\cdot z_\circ^{-1})=\overline{\rho(z)}^{k_1}\cdot\rho(z)^{k_2}=\rho(z)^{k_2-k_1}.$$

However, ω_Λ is typically not a homomorphism unless we assume that $k_2\equiv k_1\ (\mathrm{mod}\ 3)$ so that $\omega_\Lambda(z)\equiv 1$.

Under the assumption that $k_2\equiv k_1\ (\mathrm{mod}\ 3)$ we obtain a representation of $G(\mathbb{R})$ on V^{Λ}_{+} with trivial central character which we denote by D^{Λ} this representation. We have

Proposition 4.6. *Let $\Lambda=k_1\Lambda_1+k_2\Lambda_2$ be holomorphic with $k_1\equiv k_2\ (\mathrm{mod}\ 3)$. Let $(D^{\Lambda},V^{\Lambda}_{+})$ be the representation of $G(\mathbb{R})$ defined as above. Then D^{Λ} is irreducible and square-integrable. Furthermore, every square-integrable holomorphic representation of $G(\mathbb{R})$ is of the form $D^{\Lambda}\otimes\chi$ for some holomorphic Λ and a unitary character χ.*

Now we compute the matrix coefficient of $(D^{\Lambda},V^{\Lambda}_{+})$. For $g\in G(\mathbb{R})$ we set

$$\det g=e^{i\theta_g}\in U(1)$$

for $-\pi<\theta_g\le\pi$ uniquely defined.

Lemma 4.7. *Let $g=(g_{ij})_{1\le i,j\le 3}\in G(\mathbb{R})$ and ϕ° as in (4.12) we have*

$$\langle D^{\Lambda}(g)\phi_\circ,\phi_\circ\rangle_\Lambda=\frac{(\overline{g}_{11}-\overline{g}_{13}-\overline{g}_{31}+\overline{g}_{33})^{k_1}g_{22}^{k_2}}{2^{k_1}\cdot(\det g)^{\frac{k_2-k_1}{3}}}. \tag{4.17}$$

Proof. Denote by $z_g = \mathrm{diag}(e^{i\theta_g/3}, e^{i\theta_g/3}, e^{i\theta_g/3}) \in Z_G(\mathbb{R})^+$. Then $z_g^{-1} \cdot g \in G^1(\mathbb{R})$. We have

$$\langle D^\Lambda(g)\phi_\circ, \phi_\circ\rangle_\Lambda = \langle D_+^\Lambda(z_g^{-1} \cdot \mathrm{B}g\mathrm{B})\phi_\circ, \phi_\circ\rangle_\Lambda.$$

Note that

$$\mathrm{B}\begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix}\mathrm{B} = \begin{pmatrix} \frac{g_{11}+g_{13}+g_{31}+g_{33}}{2} & \frac{g_{12}+g_{32}}{\sqrt{2}} & \frac{g_{11}-g_{13}+g_{31}-g_{33}}{2} \\ \frac{g_{21}+g_{23}}{\sqrt{2}} & g_{22} & \frac{g_{21}-g_{23}}{\sqrt{2}} \\ \frac{g_{11}+g_{13}-g_{31}-g_{33}}{2} & \frac{g_{12}-g_{32}}{\sqrt{2}} & \frac{g_{11}-g_{13}-g_{31}+g_{33}}{2} \end{pmatrix}.$$

Hence it follows from Proposition 4.5 that

$$\langle D_+^\Lambda(z_g^{-1} \cdot \mathrm{B}g\mathrm{B})\phi_\circ, \phi_\circ\rangle_\Lambda = \frac{e^{-il\theta_g/3}(\overline{g}_{11} - \overline{g}_{13} - \overline{g}_{31} + \overline{g}_{33})^{k_1} g_{22}^{k_2}}{2^{k_1}}.$$

Then (4.17) follows. ☐

4.1.5. *Choice of the archimedean component.* Recall that $E = \mathbb{Q}(\sqrt{-D})$ is an imaginary quadratic extension. Let D_E be it fundamental discriminant. Let

$$g_E = \mathrm{diag}(|D_E|^{1/4}, 1, |D_E|^{-1/4}) \in G(\mathbb{R});$$

we set

$$f_\infty(g) := \langle D^\Lambda(g_E^{-1} g g_E)\phi_\circ, \phi_\circ\rangle_\Lambda = \langle D^\Lambda(g) D^\Lambda(g_E)\phi_\circ, D^\Lambda(g_E)\phi_\circ\rangle_\Lambda, \tag{4.18}$$

or more explicitly

$$f_\infty(g) = \frac{e^{-il\theta_g/3}(\overline{g}_{11} - \overline{g}_{13}|D_E|^{-1/2} - \overline{g}_{31}|D_E|^{1/2} + \overline{g}_{33})^{k_1} g_{22}^{k_2}}{2^{k_1}}, \tag{4.19}$$

where the notations are the same as those in Lemma 4.7 and $l = k_2 - k_1$.

4.1.6. *Restriction of matrix coefficients.* As we explain below we have an isomorphism $\mathrm{SL}(2,\mathbb{R}) \xrightarrow{\sim} SU(W)(\mathbb{R})$ given by

$$\iota_E : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}) \mapsto g_E^{-1}\begin{pmatrix} a & & -b\Delta \\ & 1 & \\ -c\Delta^{-1} & & d \end{pmatrix} g_E \in SU(W)(\mathbb{R}). \tag{4.20}$$

where $\Delta = i|D_E|^{1/2}$. Under ι_E, the maximal compact subgroup $\mathrm{SO}_2(\mathbb{R})$ is mapped to the maximal compact subgroup K'_∞ whose matrices are given by

$$\kappa_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \xmapsto{\iota_E} \begin{pmatrix} \cos\theta & & -i\sin\theta \\ & 1 & \\ i\sin\theta & & \cos\theta \end{pmatrix} = \mathrm{B}\begin{pmatrix} e^{i\theta} & & \\ & 1 & \\ & & e^{-i\theta} \end{pmatrix}\mathrm{B},$$

where B is defined in (4.2), and $\theta \in [-\pi, \pi)$.

Lemma 4.8. *The restriction to $SU(W)(\mathbb{R})$ of the matrix coefficients $f_\infty(g)$ is, via the isomorphism (4.20), a matrix coefficient of the holomorphic discrete series on $\mathrm{SL}(2,\mathbb{R})$ of weight $-k_1$ (recall that $-k_1 \geq k_2 + 2$ and $k_2 \geq 0$)*

Proof. Let π_k^+ be the holomorphic discrete series of weight $k > 1$ on $\mathrm{SL}(2,\mathbb{R})$. Let F_k be the matrix coefficient of its normalized lowest weight vector v_k°. It is given explicitly for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})$ by (see [KL06, Prop. 14.1])

(4.21)

$$F_k(g) := \langle \pi_k^+(g)v_k^\circ, v_k^\circ\rangle = \int_{\mathbb{H}} \pi_k^+(g)v_k^\circ(z)\overline{v_k^\circ(z)}y^k \frac{dxdy}{y^2} = \frac{2^k}{(a+d-i(b-c))^k},$$

where $z = x + iy \in \mathbb{H}$ the upper half plane.

By Lemma 4.7 we have for $g\in \mathrm{SL}(2,\mathbb{R})$

$$f_\infty(\iota_E(g)) = F_{-k_1}(g) = \frac{2^{|k_1|}}{(a+d-ib+ic)^{|k_1|}}. \tag{4.22}$$

□

4.2. **Non-archimedean components.** Let $V = Ee_1 \oplus Ee_0 \oplus Ee_{-1}$. Note that V is a 3-dimensional Hermitian space with respect to J and $U(V) = G(\mathbb{Q})$. Let

$$L = \mathcal{O}_E e_1 \oplus \mathcal{O}_E e_0 \oplus \mathcal{D}_E^{-1} e_{-1} \subseteq V.$$

Then L is a lattice of V. Its $\mathbb{Z}$-dual lattice is

$$L^* = \left\{v\in V:\ \mathrm{tr}_{E/\mathbb{Q}}\langle v, L\rangle_J \subset \mathbb{Z}\right\} = \mathcal{O}_E e_1 \oplus \mathcal{D}_E^{-1} e_0 \oplus \mathcal{D}_E^{-1} e_{-1}.$$

One can verify that L is an $\mathcal{O}_E$-module of full rank equipped with $\langle\cdot,\cdot\rangle_J$. Since for $v, v'\in L$, $\langle v, v'\rangle_J \in \mathcal{D}_E^{-1}$, and $\langle v, v\rangle_J\in\mathbb{Z}$, the lattice L is integral and even.

Let

$$G(\mathbb{Z}) := U(L)$$

be the group of isometries preserving L Then $U(L)$ is an arithmetic subgroup of $G(\mathbb{Q})$. Explicitly, we have

$$G(\mathbb{Z}) = \{g\in G(\mathbb{Q}):\ g.L = L\} = G(\mathbb{Q})\cap \left\{ \begin{pmatrix} \mathcal{O}_E & \mathcal{O}_E & \mathcal{D}_E \\ \mathcal{O}_E & \mathcal{O}_E & \mathcal{D}_E \\ \mathcal{D}_E^{-1} & \mathcal{D}_E^{-1} & \mathcal{O}_E \end{pmatrix} \right\}. \tag{4.23}$$

Let

$$G(\widehat{\mathbb{Z}}) := \left\{g\in G(\mathbb{A}_f):\ g.L\otimes_{\mathbb{Z}}\widehat{\mathbb{Z}} = L\otimes_{\mathbb{Z}}\widehat{\mathbb{Z}}\right\} = \prod_p G(\mathbb{Z}_p).$$

Moreover, recall (cf §2.2.2) that for any split prime p, we have fixed a place $\mathfrak{p}$ above p so that

$$G(\mathbb{Q}_p)\simeq \mathrm{GL}(3,\mathbb{Q}_p).$$

Since $\mathcal{O}_{E,\mathfrak{p}}\simeq\mathbb{Z}_p$ we have

$$L_{\mathfrak{p}} \simeq \mathbb{Z}_p e_1 \oplus \mathbb{Z}_p.e_0 \oplus \mathbb{Z}_p.e_{-1}$$

and under the above isomorphism we have

$$G(\mathbb{Z}_p)\simeq \mathrm{GL}_3(\mathbb{Z}_p).$$

4.2.1. *The open compact $K_f(N)$.* Let N be either 1 or a positive odd prime that remains inert in E Let

$$K_f(N) := \prod_{p<\infty} K_p(N) \subset G(\mathbb{A}_{\mathrm{fin}}),$$

be the open compact subgroup whose local components $K_p(N)$ are defined as follows:

The case $p=2$. Let v be the place above 2 (when 2 is unramified, $v=2$). Choose $\lambda\in E_v$ such that

$$\mathrm{tr}_{E_v/\mathbb{Q}_2}(\lambda) = 1 \text{ and } \|\lambda\| = \min_{\substack{x\in E\\ \mathrm{tr}_{K_v/\mathbb{Q}_2}(x)=1}} \|x\|$$

with $\|x\| = 2^{-\nu_v(x)}$. Let K_2 be the stabilizer in $G(\mathbb{Q}_2)$ of the lattice

$$\left\{ae_1 + be_0 + c.e_{-1},\ \ a, b, \lambda c\in \mathcal{O}_{E_v}\right\}.$$

Then K_2 is a special maximal compact subgroup according to [Tit79] and we set

$$K_2(N) = K_2.$$

The generic case. If $p \nmid 2N$ (that is p is either ramified, split or inert but does not divide N) we set

$$K_p(N) = K_p := G(\mathbb{Z}_p).$$

The case $p = N$. The prime p is then inert and we set

$$K_p(N) := I_p = \left\{ g = (g_{i,j}) \in G(\mathbb{Z}_p),\ g \equiv \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \pmod p \right\} \tag{4.24}$$

the Iwahori subgroup of $K_p = G(\mathbb{Z}_p)$: the inverse image of the Borel subgroup $B(\mathbb{F}_p) \subset G(\mathbb{F}_p)$ under the reduction modulo p map.

4.3. **Definition of $f^{\mathfrak{n}}$.** We can now define the non-archimedean components $f_p^{\mathfrak{n}}$ of $f^{\mathfrak{n}}$ in (4.1). Let

$$\ell = \prod_p p^{r_p} \geq 1 \tag{4.25}$$

be an integer whose prime divisors (if any) p are inert primes and coprime with N.

4.3.1. *The generic case $p \nmid \ell N$.* We set

$$f_p := \frac{1}{\mu(K_p)} \cdot \mathbf{1}_{K_p}; \tag{4.26}$$

here $\mu(K_p)$ is the volume of K_p with respect to the Tamagawa measure.

4.3.2. *The case $p = N$.* Recall that $K_p(N) = I_p$ is the Iwahori subgroup and we choose the normalized characteristic function

$$f_p = \frac{1}{\mu(I_p)} \cdot \mathbf{1}_{I_p}$$

4.3.3. *If $p \mid \ell$.* If $p|\ell$ (in particular inert and coprime with N) we take f_p to be the compactly supported bi-$G(\mathbb{Z}_p)$ invariant characteristic function

$$f_p = \mathbf{1}_{G(\mathbb{Z}_p)A^{r_p}G(\mathbb{Z}_p)} \tag{4.27}$$

where for $r \geq 0$ any integer we have set

$$A^r := \begin{pmatrix} p^r & & \\ & 1 & \\ & & p^{-r} \end{pmatrix}.$$

4.3.4. *An extra twist for $p = N'$.* For $p = N'$ a prime split in E, we define $\mathfrak{n}_p \in G(\mathbb{Q}_p)$ to be the element corresponding to the matrix

$$\mathfrak{n}_p \simeq \begin{pmatrix} 1 & & p^{-1} \\ & 1 & \\ & & 1 \end{pmatrix} \in \mathrm{GL}(3, \mathbb{Q}_p)$$

under the isomorphism (2.5). Let w' be the Weyl element $w' = \begin{pmatrix} 1 & & \\ & & 1 \\ & 1 & \end{pmatrix}$ fixing e_1 and switching e_0 and e_{-1}. We then define for $p|N'$

$$\widetilde{\mathfrak{n}}_p = w' \mathfrak{n}_p w' \simeq \begin{pmatrix} 1 & p^{-1} & \\ & 1 & \\ & & 1 \end{pmatrix}$$

and set

$$f_p^{\mathfrak{n}_p} :\ x \in G(\mathbb{Q}_p) \mapsto f_p(\widetilde{\mathfrak{n}}_p^{-1} x \widetilde{\mathfrak{n}}_p). \tag{4.28}$$

Remark 4.2. The introduction of this extra twist is absolutely crucial: without it, the spectral size of the relative trace formula would select the spherical vector in π_p, and the local period integral at p would vanish. See Remark 5.8.

4.4. **Choice of the global test function.** We then set

$$f = f_\infty . \prod_p f_p,$$

$$f^{\mathfrak{n}} = f_\infty . \prod_{p|N'} f_p^{\mathfrak{n}_p} . \prod_{p\nmid N'} f_p. \tag{4.29}$$

Alternatively, if we set, for any place v

$$\widetilde{\mathfrak{n}}_v = \begin{cases} w'\mathfrak{n}_p w' \simeq \begin{pmatrix} 1 & p^{-1} & \\ & 1 & \\ & & 1 \end{pmatrix} & \text{if } v = p = N' \text{ and} \\ w'.w' = \mathrm{Id}_3 & \text{if } v = \infty \text{ or } v = p \nmid N' \end{cases} \tag{4.30}$$

and

$$\widetilde{\mathfrak{n}} = (\widetilde{\mathfrak{n}}_v)_v \in G(\mathbb{A}),$$

we have

$$f^{\mathfrak{n}} : x \mapsto f(\widetilde{\mathfrak{n}}^{-1} x \widetilde{\mathfrak{n}}). \tag{4.31}$$

4.5. **Cusp Forms on $U(W)$.** In this section, we discuss the cuspidal representation

$$\pi' \simeq \pi'_\infty \otimes {\bigotimes_p}' \pi'_p$$

of $G'(\mathbb{A}) = U(W)(\mathbb{A})$ and the associated cuspform $\varphi' \in \pi'$.

We recall that we want π' to have trivial central character, its archimedean component π'_∞ isomorphic to the holomorphic discrete series of weight k, for every $p|N'$ its p-component, π'_p is isomorphic to the Steinberg representation St_p and for any prime p not dividing N', π'_p is unramified.

Using the trace formula, one can show that such π' exists (see below); alternatively one can construct π' "explicitly" by functoriality: let π_1 be a cuspidal automorphic representation of $\mathrm{GL}(2,\mathbb{A})$ with trivial central character such that

- $\pi_{1\infty} \simeq \pi_k^+$ is the holomorphic discrete series of weight k,
- for every $p = N'$, π_{1p} is the Steinberg representation,
- for $p \nmid N'$, π_{1p} is unramified.

Let π'_E be the base change of π_1 to E; it follows from the works of Flicker and Rogawski ([Fli82, Rog90]) that $\pi_{1,E}$ descent to an automorphic cuspidal representation of $G'(\mathbb{A})$ with trivial central character which has the required local properties.

Indeed, because π_1 is selfdual and $\pi_{1,E}$ is by construction invariant under the complex conjugation σ_E, the representation $\pi_{1,E}$ is conjugate dual:

$$\pi_{1,E}^\vee \simeq \pi_{1,E} \simeq \pi_{1,E} \circ \sigma_E.$$

Since W is even dimensional, it is sufficient to see that $\pi_{1,E}$ is conjugate symplectic [GGP12] or in other terms that the automorphic induction $\mathrm{Ind}_E^{\mathbb{Q}}(\pi_{1,E})$ is symplectic. We can verify this by considering the global 2-dimensional Galois representation ρ' attached to π' and showing that the Galois representation $\mathrm{Ind}_E^{\mathbb{Q}}(\mathrm{res}_E(\rho'))$ is symplectic: its exterior square contains the trivial representation. We have

$$\mathrm{Ind}_E^{\mathbb{Q}}(\mathrm{res}_E(\rho')) \simeq \rho' \oplus \rho'.\eta$$

($\eta = \eta_E$ is the quadratic character corresponding to $E/\mathbb{Q}$) so that

$$\Lambda^2(\rho' \oplus \rho'.\eta) \simeq \Lambda^2(\rho') \oplus \Lambda^2(\rho'.\eta) \oplus \Lambda^2(\rho') \oplus (\rho' \otimes \rho'.\eta) = 1 \oplus 1 \oplus \mathrm{sym}_2(\rho').\eta \oplus \eta$$

since the determinant of ρ' is trivial.

Conversely one can show using [Rog90] that any representation π' can be obtained in that way.

4.5.1. *The automorphic form φ'.* We choose φ' to be the "newform" on $U(W)(\mathbb{A})$ corresponding to a pure tensor

$$\varphi' \simeq \otimes_v \xi'_v \in \bigotimes_v{}' \pi'_v$$

where

- $\xi'_\infty \in \pi'_\infty$ is a vector of lowest weight k (ie. is multiplied by $e^{-ik\theta}$ under the action of the matrix $\kappa(\theta) = \mathrm{diag}(e^{i\theta}, 1, e^{-i\theta}) \in SU(W)(\mathbb{R})$),
- for every prime p, ξ'_p is invariant under the open-compact subgroup $K'_p(N')$ defined below.

Regarding the last point we set

$$G'(\widehat{\mathbb{Z}}) = \prod_p G'(\mathbb{Z}_p) = G'(\mathbb{A}_{\mathrm{fin}}) \cap G(\widehat{\mathbb{Z}})$$

and define

$$K'_f(N') = \prod_p K'_p(N') \subset G'(\mathbb{Z}_p) \tag{4.32}$$

where

- If p split in E,

$$K'_p(N') := \Big\{ g = \begin{pmatrix} a & 0 & b \\ 0 & 1 & 0 \\ c & 0 & d \end{pmatrix} \in G'(\mathbb{Z}_p) : \ c \in N'\mathbb{Z}_p \Big\} \tag{4.33}$$

- If p is inert in E,

$$K'_p(N') := \Big\{ g = \begin{pmatrix} a & 0 & b \\ 0 & 1 & 0 \\ c & 0 & d \end{pmatrix} \in G'(\mathbb{Z}_p) : \ c \in N'\mathcal{O}_{E_p} \Big\} \tag{4.34}$$

- If p is ramified in E, $K'_p(N') = G'(\mathbb{Z}_p)$.

Remark 4.3. The subgroup $G'(\widehat{\mathbb{Z}})$ is the stabilizer in $G'(\mathbb{A}_{\mathrm{fin}})$ of the hermitian $\mathcal{O}_E$-submodule generated by e_1 and $\Delta^{-1}e_{-1}$,

$$L' = \mathcal{O}_E e_1 \oplus \mathcal{D}_E^{-1} e_{-1} \subseteq W;$$

In other terms $G'(\widehat{\mathbb{Z}})$ is the closure in $G'(\mathbb{A}_{\mathrm{fin}})$ of $U(L')$ the stabilizer of L' in $U(W)(\mathbb{Q})$.

4.5.2. *The exceptional isomorphism.* To be more concrete we recall that we have an exceptional isomorphism of $\mathbb{Q}$-algebraic group

$$\mathrm{SU}(W) \simeq \mathrm{SL}(2)$$

given by

$$\iota_E = \iota : \ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2) \mapsto \begin{pmatrix} a & 0 & -b\Delta \\ 0 & 1 & 0 \\ -c\Delta^{-1} & 0 & d \end{pmatrix} \in SU(W). \tag{4.35}$$

Under this isomorphism, the image of the full congruence subgroup $\mathrm{SL}(2,\mathbb{Z})$ is given by

$$\iota(\mathrm{SL}(2,\mathbb{Z})) = SU(L') = G'(\widehat{\mathbb{Z}}) \cap SU(W)(\mathbb{Q})$$

and more generally, if N' is a prime coprime with D_E, the image of the usual congruence subgroup of level N'

$$\Gamma_0(N')=\{\begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL(2,\mathbb{Z}),\ N'|c\},$$

is

$$\begin{aligned}\iota(\Gamma_0(N'))&=\Big\{\begin{pmatrix}a&0&-b\Delta\\0&1&0\\-c\Delta^{-1}&0&d\end{pmatrix},\ \begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL(2,\mathbb{Z}),\ N'|c\Big\}\\&=K'_f(N')\cap SU(W)(\mathbb{Q}).\end{aligned}$$

Given our automorphic form $\varphi':G'(\mathbb{A})\to\mathbb{C}$ as above let $\phi':\mathbb{H}\mapsto\mathbb{C}$ be the function on the upperhalf plane defined by

$$\phi'(z)=j(g_\infty,i)^k\varphi'(\iota(g_\infty)) \tag{4.36}$$

for $g_\infty\in\mathrm{SL}(2,\mathbb{R})$ such that $g_\infty.i=z$ and for

$$j(g,z)=(\det g)^{-1/2}(cz+d)$$

be the usual automorphy factor on $\mathrm{GL}(2,\mathbb{R})^+\times\mathbb{H}$. The invariance of φ' along with the strong approximation property for SL(2) imply that the function ϕ' is a well defined holomorphic cuspform of weight k with trivial nebentypus and a newform of level N'. Now using again the strong approximation property for SL(2) one can construct out of ϕ' an automorphic form which (abusing notations) we denote

$$\varphi:\mathrm{GL}(2,\mathbb{Q})\backslash\,\mathrm{GL}(2\mathbb{A})\mapsto\mathbb{C}$$

of weight k, ie. which multiplies by the character $e^{-ik\theta}$ under the left action of

$$\kappa_\theta=\begin{pmatrix}\cos\theta&-\sin\theta\\\sin\theta&\cos\theta\end{pmatrix}\in\mathrm{SO}_2(\mathbb{R})$$

and invariant under the center $Z_{\mathrm{GL}(2)}(\mathbb{A})$ and under the open compact congruence subgroup of level N'

$$K_{0,f}(N')=\{g=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\mathrm{GL}(2,\widehat{\mathbb{Z}}),\ c\in N'\widehat{\mathbb{Z}}\}.$$

The automorphic form φ' generates the representation denoted (abusing notations) π' and is a new, lowest weight vector of it.

We assume that φ' is L^2-normalized:

$$\langle\varphi',\varphi'\rangle=\int_{[G']}|\varphi'(g')|^2dg'=1 \tag{4.37}$$

where dg' denote the Tamagawa measure. With this normalization one has for the classical form ϕ'

$$\langle\phi',\phi'\rangle:=\frac{1}{\mathrm{vol}(\Gamma_0(N')\backslash\mathbb{H})}\int_{\Gamma_0(N')\backslash\mathbb{H}}y^k|\phi'(z)|^2\frac{dxdy}{y^2}=c_E \tag{4.38}$$

where

$$\mathrm{vol}(\Gamma_0(N')\backslash\mathbb{H})=\int_{\Gamma_0(N')\backslash\mathbb{H}}\frac{dxdy}{y^2}=\frac{\pi}{3}N'\prod_{p|N'}(1+\frac{1}{p})$$

and $c_E>0$ depends only on E.

Consider its Fourier expansion

$$\phi'(z)=\sum_{n\geq1}a_ne(nz)$$

where

$$a_n = a_n(\phi') := e^{2\pi n \operatorname{Im}\tau} \int_0^1 \phi'(\tau + x) e^{-2\pi i n x} dx. \tag{4.39}$$

Since ϕ' is a newform, we also have

$$a_n = a_1 . n^{\frac{k-1}{2}} \lambda_{\pi_1}(n); \tag{4.40}$$

here for any integer n, $\lambda_{\pi_1}(n)$ denote the n-th coefficient of the Hecke L-function $L(\pi_1, s)$ (normalized analytically); it satisfies Deligne's bound

$$|\lambda_{\pi_1}(n)| \leq d(n) = n^{o(1)} \tag{4.41}$$

($d(n)$ denote the divisor function) and its first coefficient a_1 satisfies (see [Nel11, (7)])

$$|a_1|^2 = c_E \frac{2\pi^3}{3} \prod_{p|N'} (1 + \frac{1}{p}) \frac{(4\pi)^{k-1}}{\Gamma(k) L(1, \mathrm{Ad}, \pi')} \tag{4.42}$$

where $L(s, \mathrm{Ad}, \pi')$ is the adjoint L-function. We have therefore

$$|a_n|^2 = c_E \frac{2\pi^3}{3} \frac{\prod_{p|N'}(1 + 1/p)}{L(1, \mathrm{Ad}, \pi')} \frac{(4\pi)^{k-1} n^{k-1}}{\Gamma(k)} |\lambda_{\pi'}(n)|^2 \tag{4.43}$$

Also, since N' is squarefree $L(s, \mathrm{Ad}, \pi')$ does not have a Landau-Siegel zero (see [HL94]) and one has

$$L(1, \mathrm{Ad}, \pi') = (kN')^{o(1)} \tag{4.44}$$

so that

$$|a_n|^2 \leq (kN')^{o(1)} \frac{(4\pi)^{k-1} n^{k-1}}{\Gamma(k)} n^{o(1)} \tag{4.45}$$

5. The spectral Side

Let $E/\mathbb{Q}$ be a quadratic extension and let $W \subset V$ be Hermitian spaces of dimensions n and $n+1$ over E respectively. Set $G = U(V)$ and $G' = U(W)$ be the corresponding unitary groups where G' is embedded into G in the obvious way (as the stabilizer of $W^\perp \subset V$).

Let (π, V_π) (resp. $(\pi', V_{\pi'})$) be cuspidal representations on $G(\mathbb{A})$ (resp $G'(\mathbb{A})$). Define the global Petersson pairing $\langle \cdot, \cdot \rangle$ on $G(\mathbb{A})$ by

$$\langle \varphi_1, \varphi_2 \rangle = \int_{G(\mathbb{Q})\backslash G(\mathbb{A})} \varphi_1(g) \overline{\varphi_2(g)} dg,$$

where dg denote the Tamagawa measure on $G(\mathbb{Q})\backslash G(\mathbb{A})$. Similarly one defines $\langle \varphi_1', \varphi_2' \rangle$ on $G'(\mathbb{A})$.

For any place v set $G_v = G(\mathbb{Q}_v)$, $G_v' = G'(\mathbb{Q}_v)$ and dg_v, dg_v''s are local Haar measures on G_v, G_v' such that $\prod dg_v = dg$, $\prod dg_v' = dg'$.

Under the decomposition $\pi = \otimes_v' \pi_v$ and $\pi' = \otimes_v' \pi_v'$ we fix a decomposition of either of the global inner products $\langle \cdot, \cdot \rangle$ into local ones as

$$\langle \cdot, \cdot \rangle = \prod_v \langle \cdot, \cdot \rangle_v.$$

5.1. **Spectral Expansion.** Given k_1, k_2 be integers such that $k_2 \geq 2$, $k_1+k_2+2 < 0$ and $k_1 \equiv k_2 \,(\mathrm{mod}\, 3)$, let

$$\Lambda = k_1\Lambda_1 + k_2\Lambda_2$$

as in §4.1 and let D^Λ be the corresponding holomorphic discrete series of $G(\mathbb{R})$.

Let

$$\mathcal{A}_\Lambda(N) = \{\pi = \pi_\infty \otimes \pi_f \in \mathcal{A}(G),\ \omega_\pi = \mathbf{1},\ \pi_\infty \simeq D^\Lambda,\ \pi_f^{K_f(N)} \neq \{0\}\}$$

the set of automorphic representations of $G(\mathbb{A})$ having trivial central character, whose archimedean component π_∞ is isomorphic to D_Λ and whose non-archimedean component π_f admits non-trivial $K_f(N)$-invariant vectors. The set $\mathcal{A}_\Lambda(N)$ is finite and contains only cuspidal representations.

For any $\pi \in \mathcal{A}_\Lambda(N)$, the subspace of lowest weight vectors of $\pi_\infty \simeq D_\Lambda$ is one dimensional (generated by $\phi_\circ$ say) and let $v_{\pi_\infty} \neq 0$ be such a non-zero vector. We define the finite dimensional vector spaces of automorphic forms

$$\mathcal{V}_{\pi,\Lambda}(N) := \mathrm{Im}(\mathbb{C}v_{\pi_\infty} \otimes \pi_f^{K_f(N)} \hookrightarrow L^2([G]))$$

$$\mathcal{V}_\Lambda(N) := \bigoplus_{\pi\in\mathcal{A}_\Lambda(N)} \mathcal{V}_{\pi,\Lambda}(N).$$

Let $\widetilde{\mathfrak{n}}$ be the matrix

$$\widetilde{\mathfrak{n}} = \prod_{p|N'} \begin{pmatrix} 1 & p^{-1} & \\ & 1 & \\ & & 1 \end{pmatrix},$$

which we view as an element in $G(\mathbb{A})$ (cf. § 4.2); let

$$\mathcal{V}^{\widetilde{\mathfrak{n}}}_{\pi,\Lambda}(N) := \pi(\widetilde{\mathfrak{n}}).\mathcal{V}_{\pi,\Lambda}(N) = \big\{\pi(\widetilde{\mathfrak{n}})\varphi :\ \varphi \in \mathcal{V}_{\pi,\Lambda}(N)\big\},$$

be their transforms under the action of $\widetilde{\mathfrak{n}}$ and

$$\mathcal{V}^{\widetilde{\mathfrak{n}}}_{\Lambda}(N) := \bigoplus_{\pi\in\mathcal{A}_\Lambda(N)} \mathcal{V}^{\widetilde{\mathfrak{n}}}_{\pi,\Lambda}(N).$$

We fix orthonormal bases of the $\mathcal{V}_{\pi,\Lambda}(N)$ and $\mathcal{V}^{\widetilde{\mathfrak{n}}}_{\pi,\Lambda}(N)$ by

$$\mathcal{B}_{\pi,\Lambda}(N) \text{ and } \mathcal{B}^{\widetilde{\mathfrak{n}}}_{\pi,\Lambda}(N) = \pi(\widetilde{\mathfrak{n}})\mathcal{B}_{\pi,\Lambda}(N)$$

and finally let

$$\mathcal{B}_{\pi,\Lambda}(N) \text{ and } \mathcal{B}^{\widetilde{\mathfrak{n}}}_{\Lambda}(N)$$

be their unions over the $\pi \in \mathcal{A}_\Lambda(N)$.

Lemma 5.1. *Notations be as above. Let $\ell \geq 1$ be an integer whose prime divisors are all inert and corpime with N and let $f^{\mathfrak{n}}$ be the function defined in §4.4, (4.29). The image of $R(f^{\mathfrak{n}})$ is contained in $\mathcal{V}^{\widetilde{\mathfrak{n}}}_{\Lambda}(N)$: let $\tilde{\varphi}$ be an automorphic form on $G(\mathbb{Q})\backslash G(\mathbb{A})$, then $R(f^{\mathfrak{n}})\varphi = 0$ unless $\tilde{\varphi} \in \mathcal{V}^{\widetilde{\mathfrak{n}}}_{\Lambda}(N)$. Moreover, for $\pi \in \mathcal{A}_\Lambda(N)$ and $\tilde{\varphi} \in \mathrm{Im}(\mathbb{C}v_{\pi_\infty} \otimes \pi_f^{K_f(N)} \hookrightarrow L^2([G]))$, we have*

$$R(f^{\mathfrak{n}})\varphi = \frac{1}{d_\Lambda}\lambda_\pi(f_\ell).\varphi, \tag{5.1}$$

where d_Λ is the formal degree of D^Λ and

$$\lambda_\pi(f_\ell) = \prod_{p|\ell} \lambda_{\pi_p}(f_p) \in \mathbb{C}$$

is a scalar depending on $\pi_\ell = \otimes_{p|\ell}\pi_p$ and on the test functions $(f_p)_{p|\ell}$.

Proof. Denote by $R^{\widetilde{\mathfrak{n}}}(f)=\pi(\widetilde{\mathfrak{n}})^{-1}R(f^{\mathfrak{n}})\pi(\widetilde{\mathfrak{n}})$. Then

$$R^{\widetilde{\mathfrak{n}}}(f)\varphi(x)=\int_{\overline{G}(\mathbb{A})}f(\widetilde{\mathfrak{n}}^{-1}y\widetilde{\mathfrak{n}})\varphi(x\widetilde{\mathfrak{n}}^{-1}y\widetilde{\mathfrak{n}})dy=\int_{\overline{G}(\mathbb{A})}f(y)\varphi(xy)dy=\pi(f)\varphi(x).$$

From the definition of f :

$$f=f_\infty\otimes\bigotimes_p f_p\in C^\infty(G(\mathbb{A})),$$

where f_∞ is a matrix coefficient of D^Λ and each f_p is right $K_p(N)$-invariant, we see that the image of $R^{\widetilde{\mathfrak{n}}}(f)$ is contained in $\mathcal{V}_\Lambda(N)$ (in particular $R^{\widetilde{\mathfrak{n}}}(f)$ is zero on the Eisenstein or one the cuspidal spectrum: this comes from the choice of f_∞ to be the matrix coefficient of the holomorphic discrete series D_Λ). For the same reason we have

$$\pi_\infty(f_\infty).v_{\pi_\infty}=\frac{1}{d_\Lambda}v_{\pi_\infty}.$$

Moreover, for $p\mid\ell$, π_p is unramified and $\pi_p^{K_p(N)}$ is one dimensional; since f_p is bi-$K_p(N)$ invariant, for any $v_p\in\pi_p^{K_p(N)}$, one has

$$\pi_p(f_p).v_p=\lambda_{\pi_p}(f_p).v_p.$$

It follows that $\mathcal{V}_{\pi,\Lambda}(N)$ is one dimensional made of factorable vectors and that for any $\varphi\in\mathcal{V}_\Lambda(N)$ one has

$$R(f)\varphi=\frac{1}{d_\Lambda}\prod_{p\mid\ell}\lambda_{\pi_p}(f_p)\varphi=\frac{1}{d_\Lambda}\lambda_\pi(f_\ell)\varphi$$

and that

$$R(f^{\mathfrak{n}})\pi(\widetilde{\mathfrak{n}})\varphi=\frac{1}{d_\Lambda}\lambda_\pi(f_\ell)\pi(\widetilde{\mathfrak{n}})\varphi,$$

proving (5.1). □

By the spectral decomposition of the automorphic kernel and the above lemma, one has

$$\sum_{\gamma\in G(\mathbb{Q})}f^{\mathfrak{n}}(x^{-1}\gamma y) \tag{5.2}$$

Applying the expansion (5.2) into $J(f^{\mathfrak{n}})$ (see (3.3)) we then conclude that

$$\begin{aligned}J(f^{\mathfrak{n}})&=\frac{1}{d_\Lambda}\sum_{\pi\in\mathcal{A}_\Lambda(N)}\lambda_\pi(f_\ell)\sum_{\tilde{\varphi}\in\mathcal{B}^{\widetilde{\mathfrak{n}}}_{\pi,\Lambda}(N)}\frac{\mathcal{P}(\tilde{\varphi},\varphi')\overline{\mathcal{P}(\tilde{\varphi},\varphi')}}{\langle\tilde{\varphi},\tilde{\varphi}\rangle}\\&=\frac{1}{d_\Lambda}\sum_{\pi\in\mathcal{A}_\Lambda(N)}\lambda_\pi(f_\ell)\sum_{\tilde{\varphi}\in\mathcal{B}^{\widetilde{\mathfrak{n}}}_{\pi,\Lambda}(N)}\frac{\left|\mathcal{P}(\tilde{\varphi},\varphi')\right|^2}{\langle\tilde{\varphi},\tilde{\varphi}\rangle}.\end{aligned} \tag{5.3}$$

where $\mathcal{P}(\varphi,\varphi')$ denote the automorphic period integral

$$\mathcal{P}(\tilde{\varphi},\varphi')=\int_{G'(\mathbb{Q})\backslash G'(\mathbb{A})}\tilde{\varphi}(g)\varphi'(g)dg. \tag{5.4}$$

Since $\tilde{\varphi}$ and φ' are cusp forms, $\mathcal{P}(\tilde{\varphi},\varphi')$ converges absolutely and since $\mathcal{R}^{\widetilde{\mathfrak{n}}}_\Lambda(N)$ is finite dimensional, the right hand side of (5.3) is absolutely converging.

Setting

$$\mathcal{B}^{\widetilde{\mathfrak{n}}}_\Lambda(N)=\bigsqcup_{\pi\in\mathcal{A}_\Lambda(N)}\mathcal{B}^{\widetilde{\mathfrak{n}}}_{\pi,\Lambda}(N)$$

and for $\tilde{\varphi}\in\mathcal{B}^{\widetilde{\mathfrak{n}}}_{\pi,\Lambda}(N)$

$$\lambda_{\tilde{\varphi}}(f_\ell)=\lambda_\pi(f_\ell),$$

(5.3) becomes

Lemma 5.2. *Let notations be as above. We have*

$$J(f^{\mathfrak{n}})=\frac{1}{d_\Lambda}\sum_{\tilde\varphi\in\mathcal{B}^{\tilde{\mathfrak{n}}}_\Lambda(N)}\lambda_{\tilde\varphi}(f_\ell)\frac{\left|\mathcal{P}(\tilde\varphi,\varphi')\right|^2}{\langle\tilde\varphi,\tilde\varphi\rangle}. \tag{5.5}$$

5.2. **The Ichino-Ikeda Conjecture for Unitary Groups.** In this section we review the global Ichino-Ikeda conjecture for automorphic forms of $G\times G'=U(V)\times U(W)$ (e.g., see [Har14], Conjecture 1.3), which is a refinement of the Gan-Gross-Prasad conjecture [GGP12] by giving an explicit formula relating the periods and central values of Rankin-Selberg L-functions. We now recall the definition of the local analogs of the global period integral (5.4) (see the beginning of §5 for the notations)

We recall that $\pi\simeq\otimes_v\pi_v$ and $\pi'\simeq\otimes_v\pi'_v$ denote suitable automorphic representations of $U(V)$ and $U(W)$ which we assume are everywhere tempered. Their respective base changes to $\mathrm{GL}_{3,E}$ and $\mathrm{GL}_{2,E}$ are noted $\pi_E\simeq\otimes_v\pi_{E_v}$ and $\pi'_E\simeq\otimes_v\pi'_{E_v}$. We denote by

$$L(s,\pi_E\times\pi'_E)=\prod_p L_p(s,\pi_E\times\pi'_E)=\prod_{\mathfrak{p}|p}\prod_{\mathfrak{p}}L_{\mathfrak{p}}(s,\pi_E\times\pi'_E)$$

the finite part of their Rankin-Selberg L-function and

$$\Lambda(s,\pi_E\times\pi'_E)=L_\infty(s,\pi_E\times\pi'_E)L(s,\pi_E\times\pi'_E)$$

its completed version (see Prop. 5.8 for the exact expresion of $L_\infty(s,\pi_E\times\pi'_E)$). As recalled in the introduction, it admits analytic continuation to $\mathbb{C}$ and a functional equation

$$\Lambda(s,\pi_E\times\pi'_E)=\varepsilon(\pi_E\times\pi'_E)C_f(\pi_E\times\pi'_E)^s\Lambda(1-s,\pi_E\times\pi'_E) \tag{5.6}$$

In Proposition 5.9) below we prove that

$$C_f(\pi_E\times\pi'_E)=N^4N'^6$$

and that the root number equals

$$\varepsilon(\pi_E\times\pi'_E)=+1.$$

Let

$$\Delta_G=\Lambda(M_G^\vee(1),0)$$

be the special (complete) L-value where $M_G^\vee(1)$ is the twisted dual of the motive M_G associated to G by Gross [Gro97]. Locally, for v any place, we set

$$\Delta_{G,v}=L_v(M_G^\vee(1),0).$$

Explicitely, let $\eta=\prod_v\eta_v$ denote the quadratic character of $\mathbb{Q}^\times\backslash\mathbb{A}^\times$ associated to $E/\mathbb{Q}$ by class field theory. We have

$$\Delta_{G,v}=\prod_{j=1}^3L_v(j,\eta^j)=L_v(1,\eta)L_v(2,\mathbf{1})L_v(3,\eta^3),$$

and

$$\Delta_G=\Lambda(1,\eta)\Lambda(2,\mathbf{1})\Lambda(3,\eta^3).$$

We set

$$\Lambda(\pi,\pi'):=\Delta_G\frac{\Lambda(1/2,\pi_E\times\pi'_E)}{\Lambda(1,\mathrm{Ad},\pi_E)\Lambda(1,\mathrm{Ad},\pi'_E)} \tag{5.7}$$

and for any place v we set

$$L_v(\pi_v,\pi'_v):=\Delta_{G,v}\frac{L_v(1/2,\pi_{E_v}\times\pi'_{E_v})}{L_v(1,\mathrm{Ad},\pi_{E_v})L_v(1,\mathrm{Ad},\pi')}. \tag{5.8}$$

Note that, by temperedness, we have for any prime p

$$L_p(\pi_p,\pi'_p)=1+O(p^{-1/2})$$

where the implicit constant is absolute; moreover there exists an absolute constant $C\geq 1$ such that we have for any prime p

$$C^{-1}\leq L_p(\pi_p,\pi'_p)\leq C. \tag{5.9}$$

We also denote by

$$L(\pi,\pi'):=\frac{\Lambda(\pi,\pi')}{L_\infty(\pi_\infty,\pi'_\infty)} \tag{5.10}$$

the "finite part" of the complete Euler product $\Lambda(\pi,\pi')$.

Given any place v of $\mathbb{Q}$ and any tuple of local vectors

$$(\xi_{1,v},\xi_{2,v},\xi'_{1,v},\xi'_{2,v})\in\pi_v\times\pi_v\times\pi'_v\times\pi'_v,$$

the local period is defined formally by

$$\mathcal{P}_v(\xi_{1,v},\xi_{2,v};\xi'_{1,v},\xi'_{2,v}):=\int_{G'_v}\langle\pi_v(g_v)\xi_{1,v},\xi_{2,v}\rangle_v\cdot\overline{\langle\pi'_v(g_v)\xi'_{1,v},\xi'_{2,v}\rangle_v}dg_v;$$

by a result of Harris [Har14] the integral $\mathcal{P}_v(\xi_{1,v},\xi_{2,v};\xi'_{1,v},\xi'_{2,v})$ converges absolutely when both π_v and π'_v are tempered and

$$\mathcal{P}_v(\xi_{1,v},\xi_{1,v};\xi'_{1,v},\xi'_{1,v})\geq 0. \tag{5.11}$$

One then defines the unitarily and arithmetically normalized local period as

$$\mathcal{P}^*_v(\xi_{1,v},\xi_{2,v};\xi'_{1,v},\xi'_{2,v}):=\frac{\mathcal{P}_v(\xi_{1,v},\xi_{2,v};\xi'_{1,v},\xi'_{2,v})}{\langle\xi_{1,v},\xi_{2,v}\rangle_v\langle\xi'_{1,v},\xi'_{2,v}\rangle_v}, \tag{5.12}$$

$$\mathcal{P}^\natural_v(\xi_{1,v},\xi_{2,v};\xi'_{1,v},\xi'_{2,v}):=\frac{\mathcal{P}^*_v(\xi_{1,v},\xi_{2,v};\xi'_{1,v},\xi'_{2,v})}{L_v(\pi_v,\pi'_v)}. \tag{5.13}$$

According to Theorem 2.12 in [Har14], we have

$$\mathcal{P}^\natural_v=1$$

for almost all places and

$$\prod_v\mathcal{P}^\natural_v:\ (V_\pi\boxtimes V_\pi)\otimes(V_{\pi'}\boxtimes V_{\pi'})\longrightarrow\mathbb{C}.$$

is a well defined $G(\mathbb{A})\times G'(\mathbb{A})$-invariant functional.

The global Ichino-Ikeda conjecture for the unitary groups $G\times G'$ then provides an explicit constant of proportionality between $|\mathcal{P}|^2$ and $\prod\mathcal{P}^\natural_v$. It is now a theorem due to the recent work [BPLZZ21]:

Theorem 5.3 ([BPLZZ21], Theorem 1.9)**.** *Let notation be as before. Assume π and π' are tempered. Let $\varphi_1,\varphi_2\in V_\pi$, $\varphi'_1,\varphi'_2\in V_{\pi'}$ be factorable vectors. We have*

$$\frac{\mathcal{P}(\varphi_1,\varphi'_1)\overline{\mathcal{P}(\varphi_2,\varphi'_2)}}{\langle\varphi_1,\varphi_2\rangle\langle\varphi'_1,\varphi'_2\rangle}=\frac{1}{2}\cdot\Lambda(\pi,\pi')\cdot\prod_v\mathcal{P}^\natural_v, \tag{5.14}$$

where $\mathcal{P}^\natural_v$'s are defined by (5.13).

5.2.1. *Explicitation of the Ichino-Ikeda formula.* In the next subsection, we explicitate the right-hand side of the formula (5.14) for the pairs

$$(\varphi_1,\varphi'_1)=(\varphi_2,\varphi'_2)=(\tilde\varphi,\varphi')\in\pi\times\pi'$$

for $\tilde\varphi=\pi(\tilde{\mathfrak{n}})\varphi$ appearing in (5.5) and φ' discussed in §4.

Let us recall that

- the representation $\pi=\otimes'_{v\leq\infty}\pi_v$ is a cuspidal representation of $G(\mathbb{A})$ with trivial central character, of level N equal to 1 or to a prime inert in E such that $\pi_\infty\simeq D^\Lambda$, with
$$\Lambda=k_1\Lambda_1+k_2\Lambda_2,\ k_1+k_2+2<0,\ k_2\geq 0$$
and $k_1\equiv k_2\,(\mathrm{mod}\,3)$ namely, D^Λ is a holomorphic discrete series.
- the representation $\pi'=\otimes'_{v\leq\infty}\pi'_v$ is a cuspidal representation of $G'(\mathbb{A})$ with trivial central character, which at finite places is everywhere unramified excepted for at most one place N', split in E, where it is Steinberg and at the finite place has restriction, under the isomorphism $SU(W)\simeq\mathrm{SL}_2$, such that
$$\pi'_\infty\simeq\pi'_k$$
the holomorphic discrete series representation of even weight $k\geq 2$.

Let us now recall (this is a slight generalization of the discussion in §1.2) why the representations π and π' are tempered. For π' this is classical and due to Deligne ([Del71]). For π, if N is a prime (in which case π_N is the Steinberg representation whose parameters are 3-dimensional and indecomposable), π cannot be globally endoscopic. If π has prime level 1 it could a priori be endoscopic but then come from a representation $\tau=\tau_1\otimes\tau_2$ on $U(1,1)\times U(1)$ as the parameter of τ_∞ but match the parameter of π_∞, $\tau_{1,\infty}$ must in the discrete series and hence τ_1 must be tempered by Deligne and therefore τ is tempered.

By the works of Kottwitz, Milne, Rogawski et al, as assembled in [LR92] (see Theorems A p. 291 and B p. 293) the L-function $L(s-1,\pi_E)$ is a factor of $L^{(2)}(s,\widetilde{S}^K,V_\ell)$, the L-function defined by the $\mathrm{Gal}(\overline{E}/E)$-action on $\mathrm{IH}^2_{et}(\overline{S}^K_{\overline{\mathbb{Q}}},V_\ell)$, the intersection cohomology in degree 2 of the Baily-Borel-Satake compactification $\overline{S}^K_{\overline{\mathbb{Q}}}$ with coefficient in a local system V depending on the weight of π_∞, of the associated Picard modular surface S^K for $K=K_0(N)$. By the work of Gabber, one knows that the intersection etale cohomology is pure and by the Beilinson-Bernstein-Deligne decomposition theorem, $\mathrm{IH}^*_{et}(\overline{S}^K_{\overline{\mathbb{Q}}},V_\ell)$ is a direct summand of $H^*_{et}(\widetilde{S}^K_{\overline{\mathbb{Q}}},V_\ell)$ for any smooth toroidal compactification $\widetilde{S}^K$ of S^K relative to $\widetilde{S}^K\mapsto\overline{S}^K$. Now by Deligne's proof of the Weil conjectures [Del74], the eigenvalues of $\mathrm{Frob}_{\mathfrak{p}}$ at any prime $\mathfrak{p}|p\nmid ND$ acting on $H^*_{et}(\widetilde{S}^K_{\overline{\mathbb{Q}}},\mathbb{Q}_\ell)$ have absolute value $\mathrm{Nr}_{E/\mathbb{Q}}(\mathfrak{p})^{j/2}$ (for j depending on the degree and V_ℓ) from which it follows that $\pi_{E,p}$ is tempered.

Remark 5.1. If π weren't stable, it could be that only a factor of $L(s-1,\pi_E)$ divide $L^{(2)}(s,\widetilde{S}^K,V_\ell)$.

Let us recall that the automorphic forms $\varphi,\tilde\varphi=\pi(\tilde{\mathfrak{n}})\varphi$ and φ' are factorable vectors and correspond to pure tensors which we denote by

$$\varphi\simeq\otimes'_v\xi_v,\ \tilde\varphi\simeq\otimes'_v\tilde\xi_v,\ \varphi'\simeq\otimes'_v\xi'_v. \tag{5.15}$$

and that the local vectors

$$\xi_v,\ \tilde\xi_v=\pi_v(\tilde{\mathfrak{n}}_v)\xi_v,\ \xi'_v$$

have the following properties and are uniquely defined up to scalars:

- $\tilde\xi_v=\xi_v$ unless $v=p=N'$.

– If $v=\infty$, $\tilde{\xi}_\infty=\xi_\infty$ is an highest weight vector of the minimal K-type of D^Λ and ξ'_∞ is of minimal weight k (see §4.5.1).
– For every p, ξ_p is $K_p(N)$-invariant and ξ'_p is $K'_p(N')$-invariant.
– In particular if p does not divide $D_E NN'$, then $\xi_p=\tilde{\xi}_p$ and ξ'_p are invariant under the maximal compact subgroups $G(\mathbb{Z}_p)$ and $G'(\mathbb{Z}_p)$ respectively and π_p, π'_p are unramified principal series representations.

Since π' and π are everywhere tempered, formula (5.14) holds and in the next subsections we will evaluate the local period integrals and will provide for

$$\Lambda=k_1\Lambda_1+k_2\Lambda_2,\quad k_1=-k,\quad k_2=-k/2$$

an explicit approximation of the central value $L(1/2,\pi_E\times\pi'_E)$ in terms of the square of the period $|\mathcal{P}(\varphi,\varphi')|^2$. Our main objective in this section is the following

Proposition 5.4. *Let notations and assumptions be as in §1.1.1 and as above. We have*

$$\frac{\left|\mathcal{P}(\tilde{\varphi},\varphi')\right|^2}{\langle\tilde{\varphi},\tilde{\varphi}\rangle\langle\varphi',\varphi'\rangle}\asymp\frac{1}{d_kNN'^2}\frac{L(1/2,\pi_E\times\pi'_E)}{L(1,\pi_E,\mathrm{Ad})L(1,\pi'_E,\mathrm{Ad})}\tag{5.16}$$

where $L(\cdot)$ refers to the finite part of the L-functions, and the implicit (positive) constants in $\asymp$ depends at most on the absolute discriminant of E.

Remark 5.2. In particular, this implies

$$L(1/2,\pi_E\times\pi'_E)\geq 0\tag{5.17}$$

whenever the local periods $\mathcal{P}^\natural_v$ are non-zero. This follows immediately from Theorem 5.14 and (5.11) and was likely known to experts. However, what we achieve here is an effective dependency between the sizes of the global period and the central L-value for our explicit test vectors. This will crucial for our forthcoming argument (see the proof of Theorem 1.1 in §13.1).

The proof is a consequence of Theorem 5.3 and of the following proposition which evaluate the local period (5.13) for each place v:

Theorem 5.5. *Let π and π' be the automorphic representations of $G(\mathbb{A})$ and $G'(\mathbb{A})$ described in §5.2.1 with $\Lambda=-k\Lambda_1+\frac{k}{2}\Lambda_2$, and let φ,φ' be the factorable automorphic forms described in (5.15) and below. Let v be a place of $\mathbb{Q}$. We have*

– Archimedean case: If $v=\infty$ we have

$$L_\infty(\pi_\infty,\pi'_\infty)\mathcal{P}^\natural_\infty(\xi_\infty,\xi_\infty;\xi'_\infty,\xi'_\infty)=\frac{1}{d_k}\tag{5.18}$$

where

$$d_k=k-1\tag{5.19}$$

is the formal degree of π_k^+.

– Unramified case: If $v=p$ does not divide $2D_ENN'$, one has

$$\mathcal{P}^\natural_p(\xi_p,\xi_p;\xi'_p,\xi'_p)=1.\tag{5.20}$$

– The case $v=p|D_E$: we have

$$\left|L_p(\pi_p,\pi'_p)\mathcal{P}^\natural_p(\xi_p,\xi_p;\xi'_p,\xi'_p)-1\right|\leq\frac{71}{18}\frac{1}{p^2}.\tag{5.21}$$

In particular, we have

$$C^{-1}\leq\mathcal{P}^\natural_p(\xi_p,\xi_p;\xi'_p,\xi'_p)\leq C,\tag{5.22}$$

for some absolute constant C.

– *The case $v=p=N$: we have*

$$L_p(\pi_p,\pi'_p)\mathcal{P}_p^\natural(\xi_p,\xi_p;\xi'_p,\xi'_p)=\frac{1}{p}(1-\frac{1}{p})+\frac{O}{p^2} \tag{5.23}$$

with $|O|\leq 4$. In particular (since $p\geq 3$) we have

$$\frac{C^{-1}}{p}\leq \mathcal{P}_p^\natural(\xi_p,\xi_p;\xi'_p,\xi'_p)\leq \frac{C}{p} \tag{5.24}$$

for some absolute constant $C\geq 1$.

– *The case $v=p=N'$: we have*

$$L_p(\pi_p,\pi'_p)\mathcal{P}_p^\natural(\tilde{\xi}_p,\tilde{\xi}_p;\xi'_p,\xi'_p)=\frac{1}{p^2-1}(1+\frac{O}{p}) \tag{5.25}$$

with $|O|\leq 10^6$. In particular for $p>10^6$ we have

$$\frac{C^{-1}}{p^2}\leq \mathcal{P}_p^\natural(\tilde{\xi}_p,\tilde{\xi}_p;\xi'_p,\xi'_p)\leq \frac{C}{p^2} \tag{5.26}$$

for some absolute constant $C\geq 1$.

5.3. **The archimedean local period.** In this subsection we discuss (5.18). We start with the following proposition which justifies our choice (k_1,k_2):

Proposition 5.6. *Let notations and assumption be as above. The global period (5.4) vanishes unless $k_1=-k$ and $k_2=k/2$.*

Proof. For $\theta\in[-\pi,\pi)$, let

$$z'(\theta)=\mathrm{diag}(e^{i\theta},1,e^{i\theta})\in Z_{G'}(\mathbb{R}),\ \kappa(\theta)=\mathrm{diag}(e^{i\theta},1,e^{-i\theta})\in G'(\mathbb{R}),$$
$$\widetilde{z}(\theta)=\mathrm{diag}(e^{i\theta/3},e^{-2i\theta/3},e^{i\theta/3})\in SU(V)(\mathbb{R}).$$

Then an explicit computation (as in the proof of Proposition 4.5) shows that for

$$\forall g\in G'(\mathbb{A}),\ \varphi(g\widetilde{z}(\theta))=e^{-(k_1+2k_2)i\theta}\varphi(g).$$

One has for any θ

$$\varphi'(gz'(\theta))=\varphi'(g),\ \varphi'(g\kappa(\theta))=e^{-ik\theta}\varphi'(g). \tag{5.27}$$

The first equality implies that

$$\varphi(gz'(\theta))=\varphi(g)$$

(or the period integral would be 0). Since

$$z'(\theta)\tilde{z}(\theta)^{-1}=\mathrm{diag}(e^{2i\theta/3},e^{2i\theta/3},e^{2i\theta/3})\in Z_G(\mathbb{R})$$

and φ is invariant under the center one has

$$\varphi(g)=\varphi(gz'(\theta))=\varphi(g\tilde{z}(\theta))=e^{-(k_1+2k_2)i\theta}\varphi(g)$$

and

$$k_1+2k_2=0.$$

The computation in Lemma 9.6 shows that

$$\varphi(g\kappa(\theta))=e^{ik_1\theta}\varphi(g)$$

and by the second equality in (5.27) we have $k+k_1=0$ or otherwise $\mathcal{P}(\varphi,\varphi')=0$. ∎

Remark 5.3. One could also obtain Lemma 5.6 through the relative trace formula by computing the geometric side, i.e., orbital integrals: as a consequence of Lemma 4.7 we have $f_\infty(gz(\theta))=e^{-(k_1+2k_2)i\theta}f_\infty(g)$, for all $g\in G'(\mathbb{R})$. Hence, if $k_1+2k_2\neq 0$, the geometric side vanishes. Then the sum of $|\mathcal{P}(\varphi,\varphi')|^2$ is zero. Since each term is nonnegative, then each periods is vanishing.

Remark 5.4. Due to Lemma 5.6, we will take from now on $(k_1, k_2) = (-k, k/2)$, i.e.

$$\Lambda = -k\Lambda_1 + \frac{k}{2}\Lambda_2. \tag{5.28}$$

To insure absolute convergence of various integrals later we will moreover assume that

$$k \geq 32$$

an even integer. In the sequel, to simplify notations and since Λ is defined in terms of k, we will sometimes replace the indice Λ by k and write $\mathcal{V}_k(N)$ for $\mathcal{V}_\Lambda(N)$, $\mathcal{B}_{\pi,k}(N)$ for $\mathcal{B}_{\pi,\Lambda}(N)$, etc...

The next lemma provides the value of d_Λ:

Lemma 5.7. *Let*

$$\Lambda = k_1\Lambda_1 + k_2\Lambda_2 = -k\Lambda_1 + \frac{k}{2}\Lambda_2$$

defined in (5.28) *and* D^Λ *be the corresponding holomorphic discrete series. When* dg *is the Euler-Poincaré measure, its formal degree equals*

$$d_\Lambda = \frac{(k_1+1)(k_2+1)(k_1+k_2+2)}{6} = \frac{(2k-2)(k+2)(k-6)}{3} \asymp \frac{2}{3}k^3. \tag{5.29}$$

Proof. We recall that the simple positive root in this case are $e_1 - e_2$ and $e_2 - e_3$ where e_i are the standard basis vectors in $\mathbb{R}^3$ and the root space is the hyperplane $\{(x, y, z) \in \mathbb{R}^3, x + y + z = 0\}$. The $\Lambda_j,\ j = 1, 2$ are given by

$$\Lambda_1 = \frac{1}{3}(2, 1, -1),\ \Lambda_2 = \frac{1}{3}(1, 1, -2).$$

Let ρ be half the sum of the positive roots, it is given by

$$\rho = \frac{1}{2}(e_1 - e_2 + e_2 - e_3 + e_1 - e_3) = e_1 - e_3 = (1, 0, -1).$$

Consider the Weyl reflections

$$S_1 : (x, y, z) \mapsto (y, x, z),\ S_2 : (x, y, z) \mapsto (x, z, y)$$

and let Λ' be such that

$$k_1\Lambda_1 + k_2\Lambda_2 = S_1 \circ S_2(\Lambda' + \rho) - \rho.$$

Remark 5.5. In [Wal76, Lemma 9.4] $\langle\lambda + \rho, \alpha\rangle/\langle\alpha, \alpha\rangle$ should be $\langle\lambda + \rho, \alpha\rangle/\langle\rho, \alpha\rangle$.

Let us compute the Langlands parameter $\Lambda' = (a, b, c)$:

$$S_1 \circ S_2(\Lambda' + \rho) - \rho = (c - 2, a + 1, b + 1) = \frac{k_1}{3}(2, -1, -1) + \frac{k_2}{3}(1, 1, -2)$$

so that

$$\Lambda' = (\frac{-k_1 + k_2}{3} - 1, -\frac{k_1 + 2k_2}{3} - 1, \frac{2k_1 + k_2}{3} + 2). \tag{5.30}$$

We have the Blattner parameter

$$\Lambda' + \rho = (\frac{-k_1 + k_2}{3}, -\frac{k_1 + 2k_2}{3} - 1, \frac{2k_1 + k_2}{3} + 1)$$

and

$$\langle\Lambda' + \rho, \alpha\rangle = \begin{cases} k_2 + 1 & \alpha = e_1 - e_2 \\ -k_1 - k_2 - 2 & \alpha = e_2 - e_3 \\ -k_1 - 1 & \alpha = e_1 - e_3. \end{cases}$$

Then (5.29) follows from Harish-Chandra's formula [HC66] that

$$d(D^\Lambda) = \frac{1}{3}\prod_{\alpha>0}\frac{\langle\Lambda' + \rho, \alpha\rangle}{\langle\rho, \alpha\rangle},$$

with the product over all the positive roots $\alpha\in\{e_1-e_2,e_2-e_3,e_1-e_3\}$. $\square$

We can now evaluate explicitly the archimedean local period integral:

Proposition 5.8. *Let π and π' be the automorphic representations of $G(\mathbb{A})$ and $G'(\mathbb{A})$ described in §5.2.1. We have (with $k_1=-k,\ k_2=k/2$)*

$$L_\infty(s,\pi_E\times\pi'_E)=\Gamma_{\mathbb{C}}\left(s+\frac{1}{2}\right)\Gamma_{\mathbb{C}}\left(s+\frac{3}{2}\right)\Gamma_{\mathbb{C}}\left(s+\frac{k}{2}-\frac{3}{2}\right)\Gamma_{\mathbb{C}}\left(s+\frac{k}{2}+\frac{1}{2}\right)\Gamma_{\mathbb{C}}\left(s+k-\frac{5}{2}\right)\Gamma_{\mathbb{C}}\left(s+k-\frac{3}{2}\right). \tag{5.31}$$

where

$$\Gamma_{\mathbb{C}}(s)=2(2\pi)^{-s}\Gamma(s)=\Gamma_{\mathbb{R}}(s+1)\Gamma_{\mathbb{R}}(s),\ \Gamma_{\mathbb{R}}(s)=\pi^{-s/2}\Gamma(s/2).$$

Proof. Recall that the Langlands parameter of the base changed representation of the holomorphic discrete series of weight k is $(\frac{k-1}{2},-\frac{k-1}{2})$, i.e.,

$$z\mapsto \mathrm{diag}((\overline{z}/z)^{\frac{k-1}{2}},(\overline{z}/z)^{-\frac{k-1}{2}}).$$

Therefore, the archimedean parameter of the base changed representation π'_E is given by

$$(\frac{k-1}{2},\frac{1-k}{2}):\ z\mapsto \mathrm{diag}((\bar{z}/z)^{\frac{k-1}{2}},(\bar{z}/z)^{-\frac{k-1}{2}}),$$

Recall that (cf. (5.30)) the Langlands parameter of π_∞ is

$$\Lambda'=(\frac{-k_1+k_2}{3}-1,-\frac{k_1+2k_2}{3}-1,\frac{2k_1+k_2}{3}+2)=(\frac{k}{2}-1,-1,-\frac{k}{2}+2).$$

Let $\pi_{\infty,\mathbb{C}}$ be the base change of π_∞ to $\mathrm{GL}_3(\mathbb{C})$. Then the archimedean parameter of $\pi_{\infty,\mathbb{C}}\otimes\pi'_{\infty,\mathbb{C}}$ is given by

$$(\frac{k}{2}-1,-1,-\frac{k}{2}+2)\otimes(\frac{k-1}{2},\frac{1-k}{2})=(k-\frac{3}{2},\frac{k}{2}-\frac{3}{2},\frac{3}{2},-\frac{1}{2},-\frac{k}{2}-\frac{1}{2},-k+\frac{5}{2}).$$

and the result follows since for $r\in\frac{1}{2}\mathbb{Z}$

$$L_\infty(z\mapsto(\overline{z}/z)^r,s)=L_\infty((z\overline{z})^s(\overline{z}/z)^r)=\Gamma_{\mathbb{C}}(s+|r|)$$

$\square$

5.3.1. *Proof of* (5.18). By definition we have

$$\mathcal{P}_\infty(\xi_\infty,\xi_\infty;\xi'_\infty,\xi'_\infty)=\int_{G'(\mathbb{R})}\langle g_\infty.\xi_\infty,\xi_\infty\rangle_\infty\overline{\langle\pi'_\infty(g_\infty)\xi'_\infty,\xi'_\infty\rangle}_\infty dg_\infty.$$

Note that by Lemma 4.7,

$$\frac{\langle\pi_\infty(g_\infty)\xi_\infty,\xi_\infty\rangle_\infty}{\langle\xi_\infty,\xi_\infty\rangle_\infty}=f_\infty(g_\infty).$$

Then by Lemma 4.8 and the Schur orthogonality relations we obtain

$$\frac{\mathcal{P}_\infty(\xi_\infty,\xi_\infty;\xi'_\infty,\xi'_\infty)}{\langle\xi_\infty,\xi_\infty\rangle_\infty\langle\xi'_\infty,\xi'_\infty\rangle_\infty}=\int_{G'(\mathbb{R})}f_\infty(g_\infty)\frac{\langle\pi'_\infty(g_\infty)\xi'_\infty,\xi'_\infty\rangle_\infty}{\langle\xi'_\infty,\xi'_\infty\rangle_\infty}dg_\infty=\frac{1}{d_{\pi'_\infty}}.$$

where $d_{\pi'_\infty}=d_k$ is the formal degree of π_k^+ Then (5.18) follows. $\square$

5.4. **The root number and the conductor of** $L(s,\pi_E\times\pi'_E)$**.** In this section we compute the root number and the arithmetic conductor of $L(s,\pi_E\times\pi'_E)$. First we observe that π_E and π'_E are conjugate self-dual, ie. if $c\in\mathrm{Aut}_{\mathbb{Q}}(E)$ denote the non-trivial automorphism, we have

$$\pi_E\circ c\simeq\pi_E^\vee\simeq\overline{\pi}_E,\ \ {\pi'}_E\circ c\simeq{\pi'}_E^\vee\simeq\overline{\pi'}_E$$

(and since the representations are unitary $\pi_E^\vee\simeq\overline{\pi}_E,\ {\pi'}_E^\vee\simeq\overline{\pi'}_E$). In particular, the functional equation indeed relates $\Lambda(s,\pi_E\times\pi'_E)$ to $\Lambda(1-s,\pi_E\times\pi'_E)$ and $\varepsilon(\pi_E\times\pi'_E)\in\{\pm1\}$.

Proposition 5.9. *Let π and π' be the automorphic representations of $G(\mathbb{A})$ and $G'(\mathbb{A})$ described in §5.2.1 ; let π_E, π'_E be the corresponding base change to $\mathrm{GL}(3,\mathbb{A}_E)$ and $\mathrm{GL}(2,\mathbb{A}_E)$ and $L(s,\pi_E\times\pi'_E)$ be (the finite part of) its associated Rankin-Selberg L-function. Its arithmetic conductor equal*

$$C_f(\pi_E\times\pi'_E)=N^4{N'}^6$$

and its root number equals

$$\varepsilon(\pi_E\times\pi'_E)=+1.$$

Consequently, its analytic conductor $C(s,\pi_E\times\pi'_E)$ satisfies (for $\mathrm{Re}\,s=1/2$)

$$C(s,\pi_E\times\pi'_E)\asymp N^4{N'}^6|s|^4(|s|+k)^8. \tag{5.32}$$

In particular for $s=1/2$ one has the convexity bound

$$L(1/2,\pi_E\times\pi'_E)\ll(kNN')^{o(1)}C(1/2,\pi_E\times\pi'_E)^{1/4}\ll(kNN')^{o(1)}k^3N{N'}^{3/2}.$$

Proof. Since $(N,N')=1$, the arithmetic conductor of $L(s,\pi_E\times\pi'_E)$ is simply

$$C_f(\pi_E\times\pi'_E)=(N^2)^2({N'}^2)^3=N^4{N'}^6;$$

indeed for N a prime inert in E, the conductor of the Steinberg representation for $\mathrm{GL}_3(E_N)$ is N^2 (the norm of the ideal $N\mathcal{O}_{E_N}$) and for N' a prime split in E, $\mathrm{GL}_2(E_{N'})\simeq\mathrm{GL}_2(\mathbb{Q}_{N'})^2$ and $\pi_{N'}\simeq\mathrm{St}_{N'}\otimes\mathrm{St}_{N'}$ has conductor ${N'}^2$.

By (5.31) the archimedean conductor is (for $\mathrm{Re}\,s=1/2$),

$$\begin{aligned}C_\infty(s,\pi_E\times\pi'_E)&\asymp|s+1/2|^2|s+3/2|^2|s+k/2-3/2|^2\\&\quad\times|s+k/2+1/2|^2|s+k-5/2|^2|s+k-3/2|^2\\&\asymp|s|^4(|s|+k)^8.\end{aligned}$$

and the analytic conductor

$$C(s,\pi_E\times\pi'_E)=C_\infty(s,\pi_E\times\pi'_E)C_f(\pi_E\times\pi'_E)$$

satisfies (5.32).

The convexity bound follows from apply the approximate functional equation for $L(1/2,\pi_E\times\pi'_E)$ and from the following bounds for the coefficients of $L(s,\pi_E\times\pi'_E)$

$$\lambda_{\pi_E\times\pi'_E}(n)\ll_\varepsilon n^\varepsilon$$

for any $\varepsilon>0$ (the later is a consequence of the temperedness of π and π').

Let us turn to the computation of the root number: ita decomposes as a product of local root numbers (along the places of $\mathbb{Q}$ and, say, relative to the usual unramified additive character)

$$\varepsilon(\pi_E\times\pi'_E)=\varepsilon_\infty(\pi_E\times\pi'_E)\prod_p\varepsilon_p(\pi_E\times\pi'_E)$$

and for any such place v we have the further factorisation

$$\varepsilon_v(\pi_E\times\pi'_E)=\prod_{w|v}\varepsilon_w(\pi_E\times\pi'_E)$$

If $v=p\not|NN'$ then π_E and π'_E are un ramified at any place $w|p$ and

$$\varepsilon_w(\pi_E\times\pi'_E)=1.$$

If $p=N$ which is inert in E, that π'_E is unramified and

$$\varepsilon_p(\pi_E\times\pi'_E)=\varepsilon_p(\pi_E)^2=1$$

($\pi_{E,p}=\mathrm{St}_p$ and $\varepsilon_p(\pi_E)=\pm1$).

If $p|N'$ then

$$\varepsilon_p(\pi_E\times\pi'_E)=\varepsilon_{\mathfrak{p}}(\pi_E\times\pi'_E)\varepsilon_{\overline{\mathfrak{p}}}(\pi_E\times\pi'_E)=(\varepsilon_{\mathfrak{p}}(\pi'_E)\varepsilon_{\overline{\mathfrak{p}}}(\pi'_E))^3$$

since π_E is unramified at these places. Also since $\pi'_{E_{\mathfrak{p}}}$ is the base change of the Steinberg representation, $\pi'_{E_{\mathfrak{p}}}\simeq\pi'_{E_{\overline{\mathfrak{p}}}}$ and $\varepsilon_{\mathfrak{p}}(\pi'_E)\varepsilon_{\overline{\mathfrak{p}}}(\pi'_E)=1$.

Finally for $v=\infty$ the unique archimedean place we have seen above that $\pi_E\times\pi'_E$ has parameters

$$(k-\frac{3}{2},\frac{k}{2}-\frac{3}{2},\frac{3}{2},-\frac{1}{2},-\frac{k}{2}-\frac{1}{2},-k+\frac{5}{2}).$$

By [Tat79, §3.2] for $r\in\frac{1}{2}\mathbb{Z}$

$$\varepsilon_\infty(z\to(\overline{z}/z)^r)=i^{2r}$$

and in this case we have

$$k-\frac{3}{2}+\frac{k}{2}-\frac{3}{2}+\frac{3}{2}-\frac{1}{2}-\frac{k}{2}-\frac{1}{2}-k+\frac{5}{2}=0$$

hence

$$\varepsilon_\infty(\pi_E\times\pi'_E)=1.$$

□

Remark 5.6. Notice that by [GGP12], π_E is conjugate orthogonal and π'_E conjugate symplectic so with this information only the sign $\varepsilon(\pi_E\times\pi'_E)$ coudl be $+1$ or -1.

5.5. **The local periods at unramified primes.** Let now $v=p$ be a prime that does not divide $NN'D_E$ then (see the discussion in §5.2.1) ξ_p,ξ'_p are unramified vectors and the 7 conditions on page 308 of [Har14] (see also [II10], (U1)-(U6), p.5) are satisfied for π and π'. Consequently, by Theorem 2.12 of [Har14],

$$\mathcal{P}_p^\natural(\xi_p,\xi_p;\xi'_p,\xi'_p)=1$$

which is (5.20).

5.6. **The local periods at primes ramified in $E/\mathbb{Q}$.** In this section we establish (5.21). Let $p=\mathfrak{p}^2$ be a ramified prime and ϖ be a uniformizer of $\mathfrak{p}$. Let

$$A_n=\mathrm{diag}(\varpi^n,1,\overline{\varpi}^{-n}),\ n\geq0.$$

Since π_p and π'_p are tempered principal series, we may assume $\pi_p=\mathrm{Ind}\,\chi_p$ and $\pi'_p=\mathrm{Ind}\,\chi'_p$, for some unramified unitary characters χ_p and χ'_p of the respective diagonal tori. Denote by $\gamma_p=\chi_p(A_1)$ and $\gamma'_p=\chi'_p(A'_1)$.

By Macdonald's formula (cf. [Mac71] or [Cas80]) we have for $n\geq0$,

$$\frac{\langle\pi'_p(A_n)\xi'_p,\xi'_p\rangle_p}{\langle\xi'_p,\xi'_p\rangle_p}=\frac{(1-p^{-1}\gamma_p'^{-1})\gamma_p'^{n}-(1-p^{-1}\gamma'_p)\gamma_p'^{-n-1}}{p^n\big[(1-p^{-1}\gamma_p'^{-1})-(1-p^{-1}\gamma'_p)\gamma_p'^{-1}\big]},\tag{5.33}$$

$$\frac{\langle\pi_p(A_n)\xi_p,\xi_p\rangle_p}{\langle\xi_p,\xi_p\rangle_p}=\frac{1}{p^{2n}}\Big[\frac{(\gamma_p-p^{-2})(1+p^{-1}\gamma_p^{-1})\gamma_p^n-(\gamma_p^{-1}-p^{-2})(1+p^{-1}\gamma_p)\gamma_p^{-n}}{(\gamma_p-p^{-2})(1+p^{-1}\gamma_p^{-1})\gamma_p-(\gamma_p^{-1}-p^{-2})(1+p^{-1}\gamma_p)\gamma_p^{-1}}\Big]\tag{5.34}$$

and these inner product vanish for $n<0$.

Therefore, if the Haar measure on $G'(\mathbb{Q}_p)$ is such that $G'(\mathbb{Z}_p)$ has measure 1, by the Cartan decomposition we have

$$\int\frac{\langle\pi_p(g_p)\xi_p,\xi_p\rangle_p\langle\pi_p'(g_p)\xi_p',\xi_p'\rangle_p}{\langle\xi_p,\xi_p\rangle_p\langle\xi_p',\xi_p'\rangle_p}dg_p=\sum_{n\geq 0}\frac{\langle\pi_p(A_n)\xi_p,\xi_p\rangle_p\langle\pi_p'(A_n)\xi_p',\xi_p'\rangle_p}{\langle\xi_p,\xi_p\rangle_p\langle\xi_p',\xi_p'\rangle_p}, \tag{5.35}$$

where the integral on the left hand side is taken over $G'(\mathbb{Q}_p)$.

A straightforward calculation shows that

$$\left|\frac{\langle\pi_p'(A_n)\xi_p',\xi_p'\rangle_p}{\langle\xi_p',\xi_p'\rangle_p}\right|=\left|\frac{1-\gamma_p'^{2n+1}-p^{-1}\gamma_p'(1-\gamma_p'^{2n-1})}{(1+p^{-1})(1-\gamma_p')p^n}\right|.$$

Then expanding the fractions into geometric series and appealing to triangle inequality, we obtain, when $n\geq 1$, that

$$L_n':=\left|\frac{\langle\pi_p'(A_n)\xi_p',\xi_p'\rangle_p}{\langle\xi_p',\xi_p'\rangle_p}\right|\leq\frac{2n+1+p^{-1}(2n-1)}{(1+p^{-1})p^n}. \tag{5.36}$$

Similarly, after a straightforward calculation we have

$$L_n:=\left|\frac{\langle\pi_p(A_n)\xi_p,\xi_p\rangle_p}{\langle\xi_p,\xi_p\rangle_p}\right|=\left|\frac{\widetilde{\gamma}_p^{n+1}+p^{-1}(1-p^{-1})\widetilde{\gamma}_p^{n}-p^{-3}\widetilde{\gamma}_p^{n-1}}{p^{2n}(1+p^{-3})(\gamma_p-\gamma_p^{-1})}\right|,$$

where

$$\widetilde{\gamma}_p^{m}:=\gamma_p^m-\gamma_p^{-m},\ m\in\mathbb{Z}.$$

Expanding the fractions into geometric series and appealing to triangle inequality we then obtain

$$L_n\leq\frac{2+(1-p^{-3})\left[2\left\lfloor\frac{n-1}{2}\right\rfloor+\frac{(-1)^n+1}{2}\right]+p^{-1}(1-p^{-1})\left[2\left\lfloor\frac{n}{2}\right\rfloor+\frac{(-1)^{n-1}+1}{2}\right]}{(1+p^{-3})p^{2n}} \tag{5.37}$$

Therefore, we have from (5.36) and (5.37) that

$$\left|\frac{\langle\pi_p(A_1)\xi_p,\xi_p\rangle_p\cdot\langle\pi_p'(A_1')\xi_p',\xi_p'\rangle_p}{\langle\xi_p,\xi_p\rangle_p\cdot\langle\xi_p',\xi_p'\rangle_p}\right|\leq\frac{3+p^{-1}}{p+1}\cdot\frac{2+p^{-1}(1-p^{-1})}{(1+p^{-3})p^2}<\frac{6-p^{-1}}{p^3+1}; \tag{5.38}$$

and

$$\left|\frac{\langle\pi_p(A_2)\xi_p,\xi_p\rangle_p\cdot\langle\pi_p'(A_2')\xi_p',\xi_p'\rangle_p}{\langle\xi_p,\xi_p\rangle_p\cdot\langle\xi_p',\xi_p'\rangle_p}\right|\leq\frac{(5+3p^{-1})(3+2p^{-1})}{(1+p^{-1})p^6}; \tag{5.39}$$

moreover, when $n\geq 3$,

$$\left|\frac{\langle\pi_p(A_n)\xi_p,\xi_p\rangle_p\cdot\langle\pi_p'(A_n)\xi_p',\xi_p'\rangle_p}{\langle\xi_p,\xi_p\rangle_p\cdot\langle\xi_p',\xi_p'\rangle_p}\right|<\frac{2n^2+5n+2}{p^{3n}}. \tag{5.40}$$

Combining (5.38), (5.39) with (5.40) we then conclude that

$$\sum_{n\geq 1}L_n'L_n\leq\frac{6-p^{-1}}{p^3+1}+\frac{(5+3p^{-1})(3+2p^{-1})}{(1+p^{-1})p^6}+\sum_{n\geq 3}\frac{2n^2+5n+2}{p^{3n}}. \tag{5.41}$$

By induction one has $2n^2+5n+2\leq 5\cdot 2^n$ for $n\geq 2$. Therefore, substituting this estimate into (5.41) and computing the geometric series we then obtain

$$\sum_{n\geq 1}\left|\frac{\langle\pi_p(A_1)\xi_p,\xi_p\rangle_p\cdot\langle\pi_p'(A_1')\xi_p',\xi_p'\rangle_p}{\langle\xi_p,\xi_p\rangle_p\cdot\langle\xi_p',\xi_p'\rangle_p}\right|\leq H(p), \tag{5.42}$$

where the auxiliary arithmetic function $H(\cdot)$ is defined by

$$H(n) := \frac{6-n^{-1}}{n^3+1} + \frac{(5+3n^{-1})(3+2n^{-1})}{(1+n^{-1})n^6} + \frac{40}{n^6}\cdot\frac{1}{n^3-2},\ n\geq 2.$$

For $n\geq 2$, one has $n^2H(n)\geq (n+1)^2H(n+1)$ and $H(p)\leq \frac{4H(2)}{p^2}=\frac{71}{18p^2}$. Hence, combining (5.35) with (5.42) we then obtain

$$\left|\int_{G'(\mathbb{Q}_p)}\frac{\langle\pi_p(g_p)\xi_p,\xi_p\rangle_p\langle\pi'_p(g_p)\xi'_p,\xi'_p\rangle_p}{\langle\xi_p,\xi_p\rangle_p\langle\xi'_p,\xi'_p\rangle_p}dg_p-1\right|\leq\frac{4H(2)}{p^2}=\frac{71}{18p^2}$$

and (5.21) follows. □

5.7. **The matrix coefficient of the Steinberg representation for $U(V)$.** We continue to assume that p is inert in E. Let $\mu=\mu_p$ be a Haar measure on $G(\mathbb{Q}_p)$. Denote by $W_0=\{\mathbf{1}_p,J\}$. For $n\geq 1$, set

$$W_n=\{A_n,JA_n,A_nJ,JA_nJ\}.$$

Let

$$W:=\bigsqcup_{n\geq 0}W_n.$$

By Lemma A.6 (see the Appendix) and the Cartan decomposition we then obtain

$$G(\mathbb{Q}_p)=\bigsqcup_{n\geq 0}G(\mathbb{Z}_p)A_nG(\mathbb{Z}_p)=\bigsqcup_{w\in W}I_pwI_p. \tag{5.43}$$

By Lemma A.4 one has $\mu(I_pJI_p)=p^3\mu(I_p)$. Moreover, by Lemma A.5, for $n\geq 1$,

$$\mu(I_pA_nI_p)=p^{4n}\mu(I_p),\ \ \mu(I_pJA_nI_p)=p^{4n-3}\mu(I_p),$$
$$\mu(I_pA_nJI_p)=p^{4n+3}\mu(I_p),\ \ \mu(I_pJA_nJI_p)=p^{4n}\mu(I_p).$$

Then for $w\in W$, there exists a unique integer $\lambda(w)\in\mathbb{Z}_{\geq 0}$ such that

$$\mu(I_pwI_p)=p^{\lambda(w)}\mu(I_p).$$

In particular, $\lambda(\mathbf{1}_p)=0$, $\lambda(J)=3$; and for $n\geq 1$,

$$\lambda(A_n)=4n,\ \lambda(JA_n)=4n-3,\ \lambda(A_nJ)=4n+3 \text{ and } \lambda(JA_nJ)=4n.$$

Thus, the Poincaré series

$$\sum_{w\in W}p^{-2\lambda(w)}=1+p^{-6}+(2+p^6+p^{-6})\cdot\sum_{n\geq 1}p^{-8n} \tag{5.44}$$

converges absolutely.

Let Ξ_p be the I_p-bi-invariant function on $G(\mathbb{Q}_p)$ defined by

$$\Xi_p(w)=(-p)^{-\lambda(w)}\mu(I_p),\ w\in W.$$

Lemma 5.10. *Let f be a function on $G(\mathbb{Q}_p)$ defined by*

$$f(pk)=\delta_P(p)\Xi_p(k),$$

for $p\in P(\mathbb{Q}_p)$ and $k\in G(\mathbb{Z}_p)$ and δ_P the module character of the Borel subgroup P. Then

$$\varphi_f(g):=\int_{I_p}f(\kappa g)d\kappa=\Xi_p(g)\Xi_p(\mathbf{1}_p),\quad \forall g\in G(\mathbb{Q}_p). \tag{5.45}$$

Proof. Let $g = pk$ be the Iwasawa decomposition, then by the Iwahori decomposition $G(\mathbb{Z}_p) = I_p \bigsqcup I_p J I_p$ we obtain

$$\begin{aligned}\int_{I_p \bigsqcup I_p J I_p} f(g\kappa)d\kappa = \int_{G(\mathbb{Z}_p)} f(p\kappa)d\kappa \\ = \delta_P(p)\int_{G(\mathbb{Z}_p)} \Xi(\kappa)d\kappa = \delta_P(p)(1+p^3\Xi_p(J))\mu(I_p),\end{aligned}$$

where the last equality comes from Lemma A.4. Notice that $1 + p^3\Xi_p(J) = 0$, so that

$$\int_{I_p \bigsqcup I_p J I_p} f(g\kappa)d\kappa = 0, \quad \forall g \in G(\mathbb{Q}_p). \tag{5.46}$$

Let $g = pk$ with $k \in I_p$. Then we have by definition

$$\begin{aligned}\int_{I_p \bigsqcup I_p J A_1 I_p} f(g\kappa)d\kappa =& \delta_P(p)\cdot\left[\int_{I_p} \Xi_p(\kappa)d\kappa + \int_{I_p J A_1 I_p} \Xi_p(\kappa)d\kappa\right] \\ =& \delta_P(p)\cdot(\mu(I_p) + \Xi_p(JA_1)\mu(I_pJA_1I_p)) = 0.\end{aligned}$$

When $g = pk$ and $k \in I_pJI_p$, we can write $k = n(\delta,\tau)J\kappa_1$ by (A.11). So

$$\begin{aligned}\int_{I_p \bigsqcup I_p J A_1 I_p} f(g\kappa)d\kappa =& \int_{I_p} f(pn(\delta,\tau)J\kappa)d\kappa + \int_{I_p J A_1 I_p} f(pn(\delta,\tau)J\kappa)d\kappa \\ =& \delta_P(p)\cdot\left[\int_{I_p} \Xi_p(J\kappa)d\kappa + \int_{I_p J A_1 I_p} \Xi_p(J\kappa)d\kappa\right].\end{aligned}$$

By Lemma A.5 we have

$$\begin{aligned}\int_{I_p} \Xi_p(J\kappa)d\kappa + \int_{I_p J A_1 I_p} \Xi_p(J\kappa)d\kappa =& \Big[\Xi_p(J) + \sum_{\substack{\tau\in p\mathcal{O}_p/p^2\mathcal{O}_p \\ \tau+\overline{\tau}=0}} \Xi_p(n(0,\tau)A_1)\Big]\mu(I_p) \\ =& \Xi_p(J)\mu(I_p) + p\Xi_p(A_1)\mu(I_p).\end{aligned}$$

Note that $\Xi_p(J) + p\Xi_p(A_1) = -p^{-3} + p\cdot(-p)^{-4} = 0$. Hence, we have

$$\int_{I_p \bigsqcup I_p J A_1 I_p} f(g\kappa)d\kappa = 0, \quad \forall g \in G(\mathbb{Q}_p). \tag{5.47}$$

Note that the function φ_f is I_p-bi-invariant. So we only need to verify (5.45) for all $g \in W$. Clearly, (5.45) holds for $g = \mathbf{1}_p \in W$. Moreover, by (5.46) and (5.47), (5.45) holds for $g \in \{J, JA_1\} \subset W$. Hence, we have

$$\int_{I_p \bigsqcup I_p w I_p} \varphi_f(g\kappa)d\kappa = 0,\ \forall w \in \big\{J, JA_1\big\},\ g \in G(\mathbb{Q}_p). \tag{5.48}$$

Hence, expanding (5.48) we then see $\varphi_f(w) = \Xi_p(w)$ holds for all $w \in W$ with $\lambda(w) \leq 3$.

Let $n \geq 3$. Suppose $\varphi_f(w) = \Xi_p(w)$ holds for all $w \in W$ such that $\lambda(w) \leq n$. Let $w' \in W$ be such that $\lambda(w') = n+1$. Then there exists $w_1 \in \big\{J, JA_1\big\}$, and $w_2 \in W - \big\{\mathbf{1}_p\big\}$ with $w_2w_1 = w'$ and $\lambda(w_2) + \lambda(w_1) = n+1$. Explicitly, suppose $w' \in W_m$, $m \geq 1$. When $w' = A_mJ$, then $w_1 = J$ and $w_2 = A_m$; when $w' = JA_mJ$, then $w_1 = J$ and $w_2 = JA_m$; when $w' = A_m$, then $w_1 = JA_1$ and $w_2 = A_{m-1}J$; when $w' = JA_m$, then $w_1 = JA_1$ and $w_2 = JA_{m-1}J$.

We have $1 \leq \lambda(w_1) \leq 3$ and $\lambda(w_2) \leq n$. Also, by Lemma A.5 one has

$$I_pw_2I_pw_1I_p = I_pwI_p.$$

Hence, by our assumption, and taking $g=w_1$, one then has

$$\int_{I_p}\varphi_f(w_2\kappa)d\kappa+\int_{I_pw_1I_p}\varphi_f(w_2\kappa)d\kappa=\int_{I_p\bigsqcup I_pw_1I_p}\varphi_f(w_2\kappa)d\kappa=0. \tag{5.49}$$

Since φ_f is bi-invariant under I_p and $I_pw_1I_pw_2I_p=I_pwI_p$, (5.49) then becomes

$$\varphi_f(w_2)+\frac{\mu(I_pw_1I_p)}{\mu(I_p)}\varphi_f(w)=0,\ i.e.,\ \varphi_f(w_2)+q^{\lambda(w_1)}\varphi_f(w')=0. \tag{5.50}$$

By assumption, $\varphi_f(w_2)=\Xi_p(w_2)$. Hence by (5.50),

$$\varphi_f(w')=-p^{-\lambda(w_1)}\Xi_p(w_2)=(-p)^{-\lambda(w_1)-\lambda(w_2)}\mu(I_p)=\Xi_p(w').$$

Thus (5.45) follows by induction. □

Proposition 5.11. *Let p be a prime inert in E and St_p be the Steinberg representation of $G(\mathbb{Q}_p)$; the function Ξ_p is a matrix coefficient of St_p. Precisely, let $\xi_p\neq 0$ be a local new vector in St_p (a generator of the one dimensional space of I_p-invariant vectors), then*

$$\frac{\langle \mathrm{St}_p(g)\xi_p,\xi_p\rangle_p}{\langle\xi_p,\xi_p\rangle_p}=\frac{\Xi_p(g)}{\Xi_p(\boldsymbol{1}_p)},\quad \forall\ g\in G(\mathbb{Q}_p). \tag{5.51}$$

Proof. Let $\mathcal{V}$ be the vector space spanned by the right translates of the function Ξ_p. Then $\mathcal{V}$ is a smooth representation of $G(\mathbb{Q}_p)$ which we denote by $(\pi_p,\mathcal{V})$. Suppose that π_p is reducible: there exists some nonzero $G(\mathbb{Q}_p)$-invariant subspace $\mathcal{V}'\subsetneq\mathcal{V}$ and some $g_0\in G(\mathbb{Q}_p)$ such that $\pi_p(g_0)\Xi_p\in\mathcal{V}'$ and

$$\Xi_p(g_0)=\Xi_p(\mathbf{1}_p\cdot g_0)=\pi_p(g_0)\Xi_p(\mathbf{1}_p)\neq 0.$$

Let $g_1\in G(\mathbb{Q}_p)$. Denote by

$$\varphi(g):=\int_{I_p}\Xi_p(g_1\kappa g)d\kappa. \tag{5.52}$$

Then φ is a function of $g\in G(\mathbb{Q}_p)$ bi-invariant under I_p. Moreover, similar computation as in (5.46) and (5.47) shows that

$$\int_{I_p\bigsqcup I_pwI_p}\Xi_p(g'\kappa)d\kappa=0,\ \forall w\in\big\{J,JA_1\big\},\ g'\in G(\mathbb{Q}_p). \tag{5.53}$$

Take g to be the form of $g_1\kappa' g$ in (5.53) and integrate over $\kappa'\in I_p$, one then has

$$\int_{I_p\bigsqcup I_pwI_p}\varphi(g\kappa)d\kappa=\int_{I_p\bigsqcup I_pwI_p}\int_{I_p}\Xi_p(g_1\kappa' g\kappa)d\kappa' d\kappa=0,\ \forall w\in\big\{J,JA_1\big\}.$$

Then one can apply a similar induction argument to the proof of Lemma 5.10 to deduce that

$$\varphi(g)=\Xi_p(g_1)\Xi_p(g). \tag{5.54}$$

We then take $g=g_0$ and let g_1 vary, obtaining

$$\Xi_p(g_0)\Xi_p(g_1)=\varphi(g_0)=\int_{I_p}\Xi_p(g_1\kappa g_0)d\kappa=\int_{I_p}(\pi_p(\kappa g_0))\Xi_p(g_1)d\kappa. \tag{5.55}$$

Note by our assumption, $\Xi_p(g_0)\neq 0$. Hence, from (5.55) we obtain

$$\Xi_p(\bullet)=\frac{1}{\Xi_p(g_0)}\int_{I_p}\pi_p(\kappa g_0)\Xi_p(\bullet)d\kappa\in\mathcal{V}'.$$

Therefore, $\mathcal{V}\subseteq\mathcal{V}'$, a contradiction! So $(\pi_p,\mathcal{V})$ is irreducible. Let

$$\Pi_p=\mathrm{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}(|\cdot|_p^1,1,|\cdot|_p^{-1})$$

be the induced representation. Then the functions f on $G(\mathbb{Q}_p)$ which belong to the space of Π_p are precisely the functions $G(\mathbb{Z}_p)$-finite on the right which satisfy

$$f(pg) = \delta_P(p)f(g).$$

Hence the function φ_f defined in Lemma 5.10 belongs to the space of Π_p and therefore π_p is an irreducible component of Π_p . The classification of irreducible admissible representations on $G(\mathbb{Q}_p)$, implies that π_p "is" the Steinberg representation : $\pi_p \simeq \mathrm{St}_p$.

Since Ξ_p is I_p-invariant and the space of I_p-invariant vectors (the space of local new-vectors) in the Steinberg representation has dimension 1, there exists a nonzero constant λ such that $\xi_p = \lambda \cdot \Xi_p$. It remains to compute the matrix coefficient of Ξ_p. Let $g \in G(\mathbb{Q}_p)$. Then

$$\langle \pi_p(g)\Xi_p, \Xi_p\rangle_p = \int_{\overline{G}(\mathbb{Q}_p)} \Xi_p(g_1 g)\overline{\Xi}_p(g_1)dg_1 = \frac{1}{\Xi_p(\mathbf{1}_p)}\int_{\overline{G}(\mathbb{Q}_p)} \overline{\Xi}_p(g_1)dg_1 \int_{I_p} \Xi_p(g_1\kappa g)d\kappa.$$

Then by (5.52) and (5.54) (noting that Ξ_p is L^2-integrable by (5.44)), we deduce that

$$\langle \pi_p(g)\Xi_p, \Xi_p\rangle_p = \frac{\Xi_p(g)}{\Xi_p(\mathbf{1}_p)}\int_{\overline{G}(\mathbb{Q}_p)} \Xi_p(g_1)\overline{\Xi}_p(g_1)dg_1 = \frac{\langle \xi_p, \xi_p\rangle_p}{\Xi_p(\mathbf{1}_p)}\cdot \Xi_p(g), \tag{5.56}$$

which proves (5.51). □

5.8. **The local period at $p = N$.** In this section we use the results of the previous section to establish (5.23).

We recall that $p = N$ is inert in E, that π_p is the Steinberg representation and that π_p' is a (tempered) unramified principal series representation.

Let ξ_p and ξ_p' be local new vectors, of π_p and π_p' respectively. We will show that

$$\left| L_p(\pi_p, \pi_p')\cdot \mathcal{P}_p^\natural(\xi_p, \xi_p; \xi_p', \xi_p') - \frac{p-1}{p^2}\right| \leq \frac{3(1-p^{-2})}{p}\cdot\frac{p-1}{p^2}. \tag{5.57}$$

This implies (5.23) as well as (5.24) since the local factors at 1/2 and 1 do not vanish ((5.9)) and are of the shape $1 + o(1)$ as p becomes large.

Write $\pi_p' = \mathrm{Ind}\,\chi_p'$ and let

$$\gamma_p' = \chi_p'(A_1')$$

where

$$A_n = \mathrm{diag}(p^n, p^{-n}),\ n \geq 0.$$

By Macdonald's formula (cf. [Mac71] or [Cas80]) and Lemma A.1, we have

$$\frac{\langle \pi_p'(A_n)\xi_p', \xi_p'\rangle_p}{\langle \xi_p', \xi_p'\rangle_p} = \frac{(1-p^{-1}\gamma_p'^{-1})\gamma_p'^{n} - (1-p^{-1}\gamma_p')\gamma_p'^{-n-1}}{p^n\left[(1-p^{-1}\gamma_p'^{-1}) - (1-p^{-1}\gamma_p')\gamma_p'^{-1}\right]}. \tag{5.58}$$

By Lemma A.5, Lemma A.6, Proposition 5.11, Lemma A.2 and Lemma A.3,

$$\begin{aligned}\frac{\mathcal{P}_p(\xi_p, \xi_p; \xi_p', \xi_p')}{\langle \xi_p, \xi_p\rangle_p\cdot \langle \xi_p', \xi_p'\rangle_p} =& \sum_{n\geq 0}\sum_{w'\in W_n'} (-p)^{-\lambda'(i(w'))}p^{\lambda'(w')}\cdot \frac{\langle \pi_p'(A_n)\xi_p', \xi_p'\rangle_p}{\langle \xi_p', \xi_p'\rangle_p}\cdot \mu(I_p')\\ =& \frac{1-p^{-2}}{p+1} + \sum_{n\geq 1}\frac{\langle \pi_p'(A_n)\xi_p', \xi_p'\rangle_p}{\langle \xi_p', \xi_p'\rangle_p}\cdot\frac{2p^{-2n} - p^{-2n+2} - p^{-2n-2}}{p+1},\end{aligned}$$

where the last equality follows from the fact that ξ_p' is spherical. Therefore,

$$\frac{\mathcal{P}_p(\xi_p, \xi_p; \xi_p', \xi_p')}{\langle \xi_p, \xi_p\rangle_p\cdot \langle \xi_p', \xi_p'\rangle_p} = \frac{p-1}{p^2} - \sum_{n\geq 1}\frac{\langle \pi_p'(A_n)\xi_p', \xi_p'\rangle_p}{\langle \xi_p', \xi_p'\rangle_p}\cdot\frac{(p-1)^2(p+1)}{p^{2n+2}}. \tag{5.59}$$

Since π_p' is tempered, then $|\gamma_p'|=1$. Hence, we obtain

$$\sum_{n\geq 1}\frac{(1-p^{-1}\overline{\gamma_p'})\gamma_p'^n-(1-p^{-1}\gamma_p')\overline{\gamma_p'}^{n+1}}{p^{3n}\big[(1-p^{-1}\overline{\gamma_p})-(1-p^{-1}\gamma_p')\gamma_p'^{-1}\big]}=\frac{\frac{(1-p^{-1}\gamma_p'^{-1})\gamma_p'}{p^3-\gamma_p'}-\frac{(1-p^{-1}\gamma_p')\overline{\gamma_p'}^2}{p^3-\overline{\gamma_p'}}}{(1+p^{-1})(1-\overline{\gamma_p'})}. \tag{5.60}$$

Substituting (5.58) and (5.60) into (5.59) one then obtains

$$\frac{\mathcal{P}_p(\xi_p,\xi_p;\xi_p',\xi_p')}{\langle\xi_p,\xi_p\rangle_p\cdot\langle\xi_p',\xi_p'\rangle_p}=\frac{p-1}{p^2}-\frac{(p-1)^2(p+1)}{p^2}\cdot\frac{\frac{(1-p^{-1}\gamma_p'^{-1})\gamma_p'}{p^3-\gamma_p'}-\frac{(1-p^{-1}\gamma_p')\gamma_p'^{-2}}{p^3-\gamma_p'^{-1}}}{(1+p^{-1})(1-\gamma_p'^{-1})}.$$

A straightforward simplification shows that

$$\frac{\frac{(1-p^{-1}\gamma_p'^{-1})\gamma_p'}{p^3-\gamma_p'}-\frac{(1-p^{-1}\gamma_p')\gamma_p'^{-2}}{p^3-\gamma_p'^{-1}}}{(1+p^{-1})(1-\gamma_p'^{-1})}=\frac{p^3\gamma_p'(1+\gamma_p'^{-1}+\gamma_p'^{-2})-p^2-1-p^{-1}}{(p^3-\gamma_p')(p^3-\gamma_p'^{-1})(1+p^{-1})}.$$

In conjunction with $|\gamma_p'|=1$ we then conclude, when $p\geq 3$, that

$$\left|\frac{\frac{(1-p^{-1}\gamma_p'^{-1})\gamma_p'}{p^3-\gamma_p'}-\frac{(1-p^{-1}\gamma_p')\gamma_p'^{-2}}{p^3-\gamma_p'^{-1}}}{(1+p^{-1})(1-\gamma_p'^{-1})}\right|\leq\frac{3p^3+p^2+1+p^{-1}}{(p^3-1)^2(1+p^{-1})}\leq\frac{3}{p^3}. \tag{5.61}$$

Therefore, we have by (5.61) that

$$\left|\frac{\mathcal{P}_p(\xi_p,\xi_p;\xi_p',\xi_p')}{\langle\xi_p,\xi_p\rangle_p\cdot\langle\xi_p',\xi_p'\rangle_p}-\frac{p-1}{p^2}\right|\leq\frac{3(p-1)^2(p+1)}{p^5}=\frac{3(1-p^{-2})}{p}\cdot\frac{p-1}{p^2}.$$

Thus, (5.57) follows. □

5.9. **The matrix coefficient of the Steinberg representation for $U(W)$.** In this section, we assume that $p=N'$ is *split*. Let μ' be a Haar measure on $G'(\mathbb{Q}_p)$. Denote by $W_0'=\{\mathbf{1}_p',J'\}$, where $\mathbf{1}_p'$ is the identity in $G'(\mathbb{Q}_p)$. For $n\geq 1$, set

$$W_n'=\{A_n,J'A_n,A_nJ',J'A_nJ'\}.$$

Let

$$W':=\bigsqcup_{n\geq 0}W_n'.$$

By Lemma A.3 and the Cartan decomposition we have

$$G'(\mathbb{Q}_p)=\bigsqcup_{n\geq 0}G'(\mathbb{Z}_p)A_nG'(\mathbb{Z}_p)=\bigsqcup_{w'\in W'}I_p'w'I_p'.$$

By Lemma A.1 one has

$$\mu'(I_p'J'I_p')=p\mu'(I_p').$$

Moreover, by Lemma A.2, for $n\geq 1$, one has

$$\mu'(I_p'A_nI_p')=p^{2n}\mu'(I_p'),\ \ \mu'(I_p'J'A_nI_p')=p^{2n-1}\mu'(I_p'),$$
$$\mu'(I_p'A_nJ'I_p')=p^{2n+1}\mu'(I_p'),\ \text{and}\ \mu'(I_p'J'A_nJ'I_p')=p^{2n}\mu'(I_p').$$

Then for $w'\in W'$, there exists a unique integer $\lambda'(w')\in\mathbb{Z}_{\geq 0}$ such that

$$\mu'(I_p'w'I_p')=p^{\lambda'(w')}\mu(I_p').$$

In particular, $\lambda'(\mathbf{1}_p')=0$, $\lambda'(J')=1$; and for $n\geq 1$,

$$\lambda'(A_n)=2n,\ \lambda'(J'A_n)=2n-1,\ \lambda'(A_nJ')=2n+1\ \text{and}\ \lambda'(J'A_nJ')=2n.$$

Let Ξ_p' be the I_p'-bi-invariant function on $G'(\mathbb{Q}_p)$ defined by

$$\Xi_p'(w')=(-p)^{-\lambda'(w')}\mu'(I_p'),\ w'\in W'.$$

Then by similar analysis as that in §5.7 we have a counterpart of Proposition 5.11:

Proposition 5.12. *Let notation be as before. Let* St'_p *be the Steinberg representation of* $G'(\mathbb{Q}_p)$*. The function* Ξ'_p *is a matrix coefficient of* St'_p*. Precisely, let* $\xi'_p \neq 0$ *be a local new vector in* St'_p *then*

$$\frac{\langle \pi'_p(g)\xi'_p, \xi'_p\rangle_p}{\langle \xi'_p, \xi'_p\rangle_p} = \frac{\Xi'_p(g)}{\Xi'_p(\mathbf{1}'_p)}, \quad \forall\, g \in G'(\mathbb{Q}_p). \tag{5.62}$$

5.10. **The local period at $p = N'$.** In this section we deal with the case $v = p = N'$ is a split prime and establish (5.25) and (5.26).

We recall that in this case, we have the identifications

$$G(\mathbb{Q}_p) \simeq \mathrm{GL}_3(\mathbb{Q}_p),\ K_p \simeq \mathrm{GL}_3(\mathbb{Z}_p)$$

and

$$G'(\mathbb{Q}_p) \simeq \mathrm{GL}_2(\mathbb{Q}_p),\ K'_p \simeq \mathrm{GL}_2(\mathbb{Z}_p).$$

Moreover, the subgroup $G' \subset G$ is identified with the subgroup of GL_3 leaving invariant the second element of the canonical basis:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & & b \\ & 1 & \\ c & & d \end{pmatrix} \tag{5.63}$$

We recall that $\pi'_p \simeq \mathrm{St}'_p$ is the Steinberg representation and we denote its new vector by ξ'_p; the representation

$$\pi_p \simeq \mathrm{Ind}\,\chi_p$$

is a tempered unramified principal series induced from a unitary character $\chi_p = \chi$ of the diagonal torus. We denote by ξ_p is a nonzero spherical vector and by Ξ_p its associated matrix coefficient:

$$\frac{\langle \pi_p(g_p)\xi_p, \xi_p\rangle}{\langle \xi_p, \xi_p\rangle} = \frac{\Xi_p(g_p)}{\Xi_p(\mathbf{1}_p)}.$$

Finally we set

$$\tilde{\xi}_p := \pi_p(\tilde{\mathfrak{n}}_p)\xi_p$$

where we recall that

$$\tilde{\mathfrak{n}}_p \simeq \begin{pmatrix} 1 & p^{-1} & \\ & 1 & \\ & & 1 \end{pmatrix} = w'.\mathfrak{n}_p.w',\ \mathfrak{n}_p = \begin{pmatrix} 1 & & p^{-1} \\ & 1 & \\ & & 1 \end{pmatrix},\ w' = \begin{pmatrix} 1 & & \\ & & 1 \\ & 1 & \end{pmatrix}$$

Our aim is to compute the normalized period

$$\mathcal{P}^*(\widetilde{\xi}_p, \widetilde{\xi}_p; \xi'_p, \xi'_p) := \int_{G'(\mathbb{Q}_p)} \frac{\langle \pi_p(g_p)\widetilde{\xi}_p, \widetilde{\xi}_p\rangle_p \langle \pi'_p(g_p)\xi'_p, \xi'_p\rangle_p}{\langle \widetilde{\xi}_p, \widetilde{\xi}_p\rangle_p \langle \xi'_p, \xi'_p\rangle_p} dg_p \tag{5.64}$$

We have

$$\frac{\langle \pi_p(g_p)\widetilde{\xi}_p, \widetilde{\xi}_p\rangle}{\langle \widetilde{\xi}_p, \widetilde{\xi}_p\rangle} = \frac{\langle \pi_p(\tilde{\mathfrak{n}}_p^{-1} g_p \tilde{\mathfrak{n}}_p)\xi_p, \xi_p\rangle}{\langle \xi_p, \xi_p\rangle} = \frac{\Xi_p(\tilde{\mathfrak{n}}_p^{-1} g_p \tilde{\mathfrak{n}}_p)}{\Xi_p(\mathbf{1}_p)}$$

so that

$$\mathcal{P}^*(\widetilde{\xi}_p, \widetilde{\xi}_p; \xi'_p, \xi'_p) = \int_{G'(\mathbb{Q}_p)} \frac{\Xi_p(\tilde{\mathfrak{n}}_p^{-1} g_p \tilde{\mathfrak{n}}_p)\Xi'_p(g_p)}{\Xi_p(\mathbf{1}_p)\Xi'_p(\mathbf{1}'_p)} dg_p \tag{5.65}$$

by Proposition 5.12.

Remark 5.7. Observe that since $w'\in K_p$ we have

$$\Xi_p(\tilde{\mathfrak{n}}_p^{-1}g_p\tilde{\mathfrak{n}}_p)=\Xi_p(w'.\mathfrak{n}_p^{-1}.w'.g_p.w'.\mathfrak{n}_p.w')=\Xi_p(\mathfrak{n}_p^{-1}.w'.g_p.w'.\mathfrak{n}_p)$$

and for $g_p\in G'(\mathbb{Q}_p)$

$$w'.g_p.w'=g_p'=\begin{pmatrix}a&b&\\c&d&\\&&1\end{pmatrix}$$

which the usual embedding of $\mathrm{GL}_2\hookrightarrow\mathrm{GL}_3$. For the rest of this section and to simplify notations, we will use this later embedding in place of (5.63). This will allow us to replace $\tilde{\mathfrak{n}}_p$ by $\mathfrak{n}_p$ in all our forthcoming computation.

Let

$$w=\begin{pmatrix}&1\\1&\end{pmatrix}\in K_p' \text{ and } I_p'\simeq\left\{\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\mathrm{GL}_2(\mathbb{Z}_p),\ c\in p\mathbb{Z}_p\right\}$$

be the Iwahori subgroup of $G'(\mathbb{Q}_p)$.

For $(m,n)\in\mathbb{Z}^2$, we set

$$A_{m,n}=\begin{pmatrix}p^m&\\&p^n\end{pmatrix}\in G'(\mathbb{Q}_p).$$

By the Iwahori-Cartan decomposition one has

$$G'(\mathbb{Q}_p)=G_1'\bigsqcup G_2',$$

where

$$G_1':=\bigsqcup_{n\in\mathbb{Z}}\left(I_p'A_{n,n}\bigsqcup I_p'A_{n,n}wI_p'\right)\tag{5.66}$$

and

$$G_2':=\bigsqcup_{m\geq n+1}\left(I_p'A_{m,n}I_p'\bigsqcup I_p'wA_{m,n}I_p'\bigsqcup I_p'A_{m,n}wI_p'\bigsqcup I_p'wA_{m,n}I_p'\right).\tag{5.67}$$

From (5.64), we have

$$\mathcal{P}^*(\widetilde{\xi}_p,\widetilde{\xi}_p;\xi_p',\xi_p')=\sum_{j=1}^{2}\int_{G_j'}\frac{\Xi_p(\mathfrak{n}_p^{-1}g_p\mathfrak{n}_p)\Xi_p'(g_p)}{\Xi_p(\mathbf{1}_p)\Xi_p'(\mathbf{1}_p')}dg_p.\tag{5.68}$$

Let

$$I_p'(1)\simeq\left\{\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\mathrm{GL}_2(\mathbb{Z}_p):\ c,a-1\in p\mathbb{Z}_p\right\}\subset I_p'.\tag{5.69}$$

We have

$$\mathfrak{n}_p^{-1}\begin{pmatrix}a&b&\\c&d&\\&&1\end{pmatrix}\mathfrak{n}_p=\begin{pmatrix}a&b&(a-1)/p\\c&d&c/p\\&&1\end{pmatrix}$$

so that

$$\mathfrak{n}_p^{-1}I_p'(1)\mathfrak{n}_p\subseteq K_p.$$

It follows that

$$\int_{I_p'A_{n,n}}\frac{\Xi_p(\mathfrak{n}_p^{-1}g_p\mathfrak{n}_p)\Xi_p'(g_p)}{\Xi_p(\mathbf{1}_p)\Xi_p'(\mathbf{1}_p')}dg_p=\frac{\mu(I_p'(1))}{\Xi_p(\mathbf{1}_p)}\sum_{\delta}\Xi_p\left(\mathfrak{n}_p^{-1}A_{n,n}\begin{pmatrix}\delta&&\\&1&\\&&1\end{pmatrix}\mathfrak{n}_p\right),$$

where δ runs over $(\mathbb{Z}_p/p\mathbb{Z}_p)^{\times}$.

We will compute the above integral depending on n: for this we notice that

1. For $n \geq 1$, we have

$$\mathfrak{n}_p^{-1} A_{n,n} \begin{pmatrix} \delta & & \\ & 1 & \\ & & 1 \end{pmatrix} \mathfrak{n}_p = \begin{pmatrix} \delta.p^n & & (\delta p^n - 1)/p \\ & p^n & \\ & & 1 \end{pmatrix}$$
$$\in K_p \begin{pmatrix} p^{n+1} & & \\ & p^n & \\ & & p^{-1} \end{pmatrix} K_p.$$

2. For $n \leq -1$ we have similarly

$$\mathfrak{n}_p^{-1} A_{n,n} \begin{pmatrix} \delta & & \\ & 1 & \\ & & 1 \end{pmatrix} \mathfrak{n}_p \in K_p \begin{pmatrix} p & & \\ & p^n & \\ & & p^{n-1} \end{pmatrix} K_p.$$

3. For $n = 0$ and $\delta \neq 1 \in (\mathbb{Z}_p/p\mathbb{Z}_p)^\times$ we have

$$\mathfrak{n}_p^{-1} A_{n,n} \begin{pmatrix} \delta & & \\ & 1 & \\ & & 1 \end{pmatrix} \mathfrak{n}_p \in K_p \begin{pmatrix} p & & \\ & 1 & \\ & & p^{-1} \end{pmatrix} K_p.$$

From the above discussion we obtain that

$$\int_{I'_p A_{n,n}} \frac{\Xi_p(\mathfrak{n}_p^{-1} g_p \mathfrak{n}_p) \Xi'_p(g_p)}{\Xi_p(\mathbf{1}_p) \Xi'_p(\mathbf{1}'_p)} dg_p = \Sigma_{01} + \Sigma_{02} + \Sigma_{03}, \tag{5.70}$$

where

$$\Sigma_{01} := \frac{\mu'(I'_p(1))}{\Xi_p(\mathbf{1}_p)} \left[\Xi_p(\mathbf{1}_p) + (p-2) \Xi_p \begin{pmatrix} p & & \\ & 1 & \\ & & p^{-1} \end{pmatrix} \right],$$
$$\Sigma_{02} := \frac{(p-1)\mu'(I'_p(1))}{\Xi_p(\mathbf{1}_p)} \sum_{n \geq 1} \Xi_p \begin{pmatrix} p^{n+1} & & \\ & p^n & \\ & & p^{-1} \end{pmatrix},$$
$$\Sigma_{03} := \frac{(p-1)\mu'(I'_p(1))}{\Xi_p(\mathbf{1}_p)} \sum_{n \leq -1} \Xi_p \begin{pmatrix} p^{n+1} & & \\ & p^n & \\ & & p^{-1} \end{pmatrix}.$$

Regarding the integral along $I'_p A_{n,n} w I'_p$ we have

$$\int_{I'_p A_{n,n} w' I'_p} \frac{\Xi_p(\mathfrak{n}_p^{-1} g_p \mathfrak{n}_p) \Xi'_p(g_p)}{\Xi_p(\mathbf{1}_p) \Xi'_p(\mathbf{1}'_p)} dg_p = -\frac{\mu(I'_p(1))}{\Xi_p(\mathbf{1}_p)} \sum_\delta \Xi_p \left(\mathfrak{n}_p^{-1} w \begin{pmatrix} \delta & & \\ & 1 & \\ & & 1 \end{pmatrix} \mathfrak{n}_p \right),$$

where δ runs over $(\mathbb{Z}_p/p\mathbb{Z}_p)^\times$.

We will compute the above integral depending on the value of n: for this we observe that

1. For $n \geq 1$, we have

$$\mathfrak{n}_p^{-1} A_{n,n} w \begin{pmatrix} \delta & & \\ & 1 & \\ & & 1 \end{pmatrix} \mathfrak{n}_p = \begin{pmatrix} \delta p^n & & \delta p^{n-1} \\ & p^n & -p^{-1} \\ & & 1 \end{pmatrix}$$
$$\in K_p \begin{pmatrix} p^{n+1} & & \\ & p^n & \\ & & p^{-1} \end{pmatrix} K_p.$$

2. For $n \leq -1$ we have

$$\mathfrak{n}_p^{-1} A_{n,n} w \begin{pmatrix} \delta & & \\ & 1 & \\ & & 1 \end{pmatrix} \mathfrak{n}_p \in K_p \begin{pmatrix} p & & \\ & p^n & \\ & & p^{n-1} \end{pmatrix} K_p.$$

3. For $n=0$ we have
$$\mathfrak{n}_p^{-1} w \begin{pmatrix} \delta & & \\ & 1 & \\ & & 1 \end{pmatrix} \mathfrak{n}_p \in K_p \begin{pmatrix} p & & \\ & 1 & \\ & & p^{-1} \end{pmatrix} K_p.$$

Thus from the above discussion we then obtain
$$\int_{I_p' A_{n,n} w' I_p'} \frac{\Xi_p(\mathfrak{n}_p^{-1} g_p \mathfrak{n}_p)\Xi_p'(g_p)}{\Xi_p(\mathbf{1}_p)\Xi_p'(\mathbf{1}_p')} dg_p = \Sigma_{01}' + \Sigma_{02}' + \Sigma_{03}', \tag{5.71}$$
where
$$\Sigma_{01}' := -\frac{(p-1)\mu(I_p'(1))}{\Xi_p(\mathbf{1}_p)} \Xi_p \begin{pmatrix} p & & \\ & 1 & \\ & & p^{-1} \end{pmatrix},$$
$$\Sigma_{02}' := -\frac{(p-1)\mu(I_p'(1))}{\Xi_p(\mathbf{1}_p)} \sum_{n\geq 1} \Xi_p \begin{pmatrix} p^{n+1} & & \\ & p^n & \\ & & p^{-1} \end{pmatrix},$$
$$\Sigma_{03}' := -\frac{(p-1)\mu(I_p'(1))}{\Xi_p(\mathbf{1}_p)} \sum_{n\leq -1} \Xi_p \begin{pmatrix} p^{n+1} & & \\ & p^n & \\ & & p^{-1} \end{pmatrix}.$$

Then we have by (5.70) and (5.71) that
$$\int_{G_1'} \frac{\Xi_p(\mathfrak{n}_p^{-1} g_p \mathfrak{n}_p)\Xi_p'(g_p)}{\Xi_p(\mathbf{1}_p)\Xi_p'(\mathbf{1}_p')} dg_p = \mu(I_p'(1)) - \frac{\mu'(I_p'(1))}{\Xi_p(\mathbf{1}_p)} \Xi_p \begin{pmatrix} p & & \\ & 1 & \\ & & p^{-1} \end{pmatrix}. \tag{5.72}$$

Let $n\in\mathbb{Z}$. Let $m\geq n+1$. We have, similar to Lemma A.2, that
$$I_p' A_{m,n} I_p' = \bigsqcup_{\tau\in\mathbb{Z}_p/p^{m-n}\mathbb{Z}_p} \begin{pmatrix} 1 & \tau \\ & 1 \end{pmatrix} A_{m,n} I_p'.$$

A straightforward calculation shows that
$$\mathfrak{n}_p^{-1} \begin{pmatrix} 1 & \tau & \\ & 1 & \\ & & 1 \end{pmatrix} A_{m,n} \begin{pmatrix} \delta & & \\ & 1 & \\ & & 1 \end{pmatrix} \mathfrak{n}_p \in K_p \begin{pmatrix} p^m & & (\delta p^m - 1)p^{-1} \\ & p^n & \\ & & 1 \end{pmatrix} K_p.$$

1. Suppose $n\geq 0$. Then $m\geq n+1\geq 1$. Then
$$\mathfrak{n}_p^{-1} \begin{pmatrix} 1 & \tau & \\ & 1 & \\ & & 1 \end{pmatrix} A_{m,n} \begin{pmatrix} \delta & & \\ & 1 & \\ & & 1 \end{pmatrix} \mathfrak{n}_p \in K_p \begin{pmatrix} p^{m+1} & & \\ & p^n & \\ & & p^{-1} \end{pmatrix} K_p.$$
2. Suppose $n\leq -1$ and $m\geq 1$. Then
$$\mathfrak{n}_p^{-1} \begin{pmatrix} 1 & \tau & \\ & 1 & \\ & & 1 \end{pmatrix} A_{m,n} \begin{pmatrix} \delta & & \\ & 1 & \\ & & 1 \end{pmatrix} \mathfrak{n}_p \in K_p \begin{pmatrix} p^{m+1} & & \\ & p^{-1} & \\ & & p^n \end{pmatrix} K_p.$$
3. Suppose $n\leq -1$ and $m=0$. If $\delta\neq 1\in\mathbb{F}_p$, then
$$\mathfrak{n}_p^{-1} \begin{pmatrix} 1 & \tau & \\ & 1 & \\ & & 1 \end{pmatrix} A_{m,n} \begin{pmatrix} \delta & & \\ & 1 & \\ & & 1 \end{pmatrix} \mathfrak{n}_p \in K_p \begin{pmatrix} p^{m+1} & & \\ & p^{-1} & \\ & & p^n \end{pmatrix} K_p.$$
4. Suppose $n\leq -1$ and $m\leq -1$. Note that $m-1\geq n$. Then
$$\mathfrak{n}_p^{-1} \begin{pmatrix} 1 & \tau & \\ & 1 & \\ & & 1 \end{pmatrix} A_{m,n} \begin{pmatrix} \delta & & \\ & 1 & \\ & & 1 \end{pmatrix} \mathfrak{n}_p \in K_p \begin{pmatrix} p & & \\ & p^{m-1} & \\ & & p^n \end{pmatrix} K_p.$$

Therefore, similar to (5.70) and (5.71) we have that

$$\begin{aligned}\Sigma_1 &:= \sum_{n\in\mathbb{Z}}\sum_{m\geq n+1}\int_{I_p' A_{m,n} I_p'} \frac{\Xi_p(\mathfrak{n}_p^{-1} g_p \mathfrak{n}_p)\Xi_p'(g_p)}{\Xi_p(\mathbf{1}_p)\Xi_p'(\mathbf{1}_p')} dg_p y \\ &= \Sigma_{11}+\Sigma_{12}+\Sigma_{13}+\Sigma_{14}, \end{aligned} \tag{5.73}$$

where

$$\Sigma_{11} := \frac{(p-1)\mu(I_p'(1))}{\Xi_p(\mathbf{1}_p)} \sum_{n\geq 0}\sum_{m\geq n+1} \Xi_p\begin{pmatrix} p^{m+1} & & \\ & p^n & \\ & & p^{-1}\end{pmatrix},$$

$$\Sigma_{12} := \frac{(p-1)\mu(I_p'(1))}{\Xi_p(\mathbf{1}_p)} \sum_{n\leq -1}\sum_{m\geq 1} \Xi_p\begin{pmatrix} p^{m+1} & & \\ & p^{-1} & \\ & & p^{n}\end{pmatrix},$$

$$\Sigma_{13} := \frac{\mu(I_p'(1))}{\Xi_p(\mathbf{1}_p)} \sum_{n\leq -1}\left[\Xi_p\begin{pmatrix} 1 & & \\ & 1 & \\ & & p^{n}\end{pmatrix} + (p-2)\Xi_p\begin{pmatrix} p & & \\ & p^{-1} & \\ & & p^{n}\end{pmatrix}\right],$$

$$\Sigma_{14} := \frac{(p-1)\mu(I_p'(1))}{\Xi_p(\mathbf{1}_p)} \sum_{n\leq -2}\sum_{n+1\leq m\leq -1} \Xi_p\begin{pmatrix} p & & \\ & p^{m-1} & \\ & & p^{n}\end{pmatrix}.$$

Let $m\geq n+1$. We have, similar to Lemma A.2, that

$$I_p' w' A_{m,n} I_p' = \bigsqcup_{\tau\in p\mathbb{Z}_p/p^{m-n}\mathbb{Z}_p} \begin{pmatrix} 1 & \\ \tau & 1\end{pmatrix} w' A_{m,n} I_p'.$$

$$\mathfrak{n}_p^{-1}\begin{pmatrix} 1 & & \\ \tau & 1 & \\ & & 1\end{pmatrix} w' A_{m,n}\begin{pmatrix} \delta & & \\ & 1 & \\ & & 1\end{pmatrix}\mathfrak{n}_p \in K_p\begin{pmatrix} p^m & & p^{m-1} \\ & p^n & -p^{-1} \\ & & 1\end{pmatrix}K_p.$$

1. Suppose $n\geq 0$. Then $m\geq n+1\geq 1$. Then

$$\mathfrak{n}_p^{-1}\begin{pmatrix} 1 & & \\ \tau & 1 & \\ & & 1\end{pmatrix} w' A_{m,n}\begin{pmatrix} \delta & & \\ & 1 & \\ & & 1\end{pmatrix}\mathfrak{n}_p \in K_p\begin{pmatrix} p^m & & \\ & p^{n+1} & \\ & & p^{-1}\end{pmatrix}K_p.$$

2. Suppose $n\leq -1$ and $m\geq 1$. Then

$$\mathfrak{n}_p^{-1}\begin{pmatrix} 1 & & \\ \tau & 1 & \\ & & 1\end{pmatrix} w' A_{m,n}\begin{pmatrix} \delta & & \\ & 1 & \\ & & 1\end{pmatrix}\mathfrak{n}_p \in K_p\begin{pmatrix} p^m & & \\ & 1 & \\ & & p^{n}\end{pmatrix}K_p.$$

3. Suppose $n\leq -1$ and $m=0$. Then

$$\mathfrak{n}_p^{-1}\begin{pmatrix} 1 & & \\ \tau & 1 & \\ & & 1\end{pmatrix} w' A_{m,n}\begin{pmatrix} \delta & & \\ & 1 & \\ & & 1\end{pmatrix}\mathfrak{n}_p \in K_p\begin{pmatrix} p & & \\ & p^{-1} & \\ & & p^{n}\end{pmatrix}K_p.$$

4. Suppose $n\leq -1$ and $m\leq -1$. Note that $m-1\geq n$. Then

$$\mathfrak{n}_p^{-1}\begin{pmatrix} 1 & & \\ \tau & 1 & \\ & & 1\end{pmatrix} w' A_{m,n}\begin{pmatrix} \delta & & \\ & 1 & \\ & & 1\end{pmatrix}\mathfrak{n}_p \in K_p\begin{pmatrix} p & & \\ & p^{m-1} & \\ & & p^{n}\end{pmatrix}K_p.$$

Therefore,similar to (5.70) and (5.73) we have that

$$\begin{aligned}\Sigma_2 &:= \sum_{n\in\mathbb{Z}}\sum_{m\geq n+1}\int_{I_p' w' A_{m,n} I_p'} \frac{\Xi_p(\mathfrak{n}_p^{-1} g_p \mathfrak{n}_p)\Xi_p'(g_p)}{\Xi_p(\mathbf{1}_p)\Xi_p'(\mathbf{1}_p')} dg_p \\ &= \Sigma_{21}+\Sigma_{22}+\Sigma_{23}+\Sigma_{24}, \end{aligned} \tag{5.74}$$

$$\Sigma_{21} := -\frac{(p-1)\mu(I_p'(1))}{\Xi_p(\mathbf{1}_p)}\sum_{n\geq 0}\sum_{m\geq n+1}\Xi_p\begin{pmatrix} p^m & & \\ & p^{n+1} & \\ & & p^{-1}\end{pmatrix},$$
$$\Sigma_{22} := -\frac{(p-1)\mu(I_p'(1))}{\Xi_p(\mathbf{1}_p)}\sum_{n\leq -1}\sum_{m\geq 1}\Xi_p\begin{pmatrix} p^m & & \\ & 1 & \\ & & p^{n}\end{pmatrix},$$
$$\Sigma_{23} := -\frac{(p-1)\mu(I_p'(1))}{\Xi_p(\mathbf{1}_p)}\sum_{n\leq -1}\Xi_p\begin{pmatrix} p & & \\ & p^{-1} & \\ & & p^{n}\end{pmatrix},$$
$$\Sigma_{24} := -\frac{(p-1)\mu(I_p'(1))}{\Xi_p(\mathbf{1}_p)}\sum_{n\leq -2}\sum_{n+1\leq m\leq -1}\Xi_p\begin{pmatrix} p & & \\ & p^{m-1} & \\ & & p^{n}\end{pmatrix}.$$

Let $m\geq n+1$. We have, similar to Lemma A.2, that

$$I_p'A_{m,n}w'I_p' = \bigsqcup_{\tau\in\mathbb{Z}_p/p^{m-n+1}\mathbb{Z}_p}\begin{pmatrix}1 & \tau\\ & 1\end{pmatrix}A_{m,n}w'I_p'.$$

$$\mathfrak{n}_p^{-1}\begin{pmatrix}1 & \tau & \\ & 1 & \\ & & 1\end{pmatrix}A_{m,n}w'\begin{pmatrix}\delta & & \\ & 1 & \\ & & 1\end{pmatrix}\mathfrak{n}_p\in K_p\begin{pmatrix}p^m & & -p^{-1}\\ & p^n & p^{n-1}\\ & & 1\end{pmatrix}K_p.$$

1. Suppose $n\geq 0$. Then $m\geq n+1\geq 1$. Then
$$\mathfrak{n}_p^{-1}\begin{pmatrix}1 & \tau & \\ & 1 & \\ & & 1\end{pmatrix}A_{m,n}w'\begin{pmatrix}\delta & & \\ & 1 & \\ & & 1\end{pmatrix}\mathfrak{n}_p\in K_p\begin{pmatrix}p^{m+1} & & \\ & p^n & \\ & & p^{-1}\end{pmatrix}K_p.$$
2. Suppose $n\leq -1$ and $m\geq 1$. Then
$$\mathfrak{n}_p^{-1}\begin{pmatrix}1 & \tau & \\ & 1 & \\ & & 1\end{pmatrix}A_{m,n}w'\begin{pmatrix}\delta & & \\ & 1 & \\ & & 1\end{pmatrix}\mathfrak{n}_p\in K_p\begin{pmatrix}p^{m+1} & & \\ & 1 & \\ & & p^{n-1}\end{pmatrix}K_p.$$
3. Suppose $n\leq -1$ and $m=0$. Then
$$\mathfrak{n}_p^{-1}\begin{pmatrix}1 & \tau & \\ & 1 & \\ & & 1\end{pmatrix}A_{m,n}w'\begin{pmatrix}\delta & & \\ & 1 & \\ & & 1\end{pmatrix}\mathfrak{n}_p\in K_p\begin{pmatrix}p & & \\ & 1 & \\ & & p^{n-1}\end{pmatrix}K_p.$$
4. Suppose $n\leq -1$ and $m\leq -1$. Note that $m-1\geq n$. Then
$$\mathfrak{n}_p^{-1}\begin{pmatrix}1 & \tau & \\ & 1 & \\ & & 1\end{pmatrix}A_{m,n}w'\begin{pmatrix}\delta & & \\ & 1 & \\ & & 1\end{pmatrix}\mathfrak{n}_p\in K_p\begin{pmatrix}p & & \\ & p^m & \\ & & p^{n-1}\end{pmatrix}K_p.$$

Therefore, similar to (5.70) and (5.74) we have that

$$\begin{aligned}\Sigma_3 &:= \sum_{n\in\mathbb{Z}}\sum_{m\geq n+1}\int_{I_p'w'A_{m,n}I_p'}\frac{\Xi_p(\mathfrak{n}_p^{-1}g_p\mathfrak{n}_p)\Xi_p'(g_p)}{\Xi_p(\mathbf{1}_p)\Xi_p'(\mathbf{1}_p')}dg_p\\ &= \Sigma_{31}+\Sigma_{32}+\Sigma_{33}+\Sigma_{34},\end{aligned}\tag{5.75}$$

where

$$\Sigma_{31} := -\frac{(p-1)\mu(I_p'(1))}{\Xi_p(\mathbf{1}_p)}\sum_{n\geq 0}\sum_{m\geq n+1}\Xi_p\begin{pmatrix}p^{m+1}&&\\&p^n&\\&&p^{-1}\end{pmatrix},$$

$$\Sigma_{32} := -\frac{(p-1)\mu(I_p'(1))}{\Xi_p(\mathbf{1}_p)}\sum_{n\leq -1}\sum_{m\geq 1}\Xi_p\begin{pmatrix}p^{m+1}&&\\&1&\\&&p^{n-1}\end{pmatrix},$$

$$\Sigma_{33} := -\frac{(p-1)\mu(I_p'(1))}{\Xi_p(\mathbf{1}_p)}\sum_{n\leq -1}\Xi_p\begin{pmatrix}p&&\\&1&\\&&p^{n-1}\end{pmatrix},$$

$$\Sigma_{34} := -\frac{(p-1)\mu(I_p'(1))}{\Xi_p(\mathbf{1}_p)}\sum_{n\leq -2}\sum_{n+1\leq m\leq -1}\Xi_p\begin{pmatrix}p&&\\&p^m&\\&&p^{n-1}\end{pmatrix}.$$

Let $m\geq n+1$. We have, similar to Lemma A.2, that

$$I_p'w'A_{m,n}w'I_p' = \bigsqcup_{\tau\in p\mathbb{Z}_p/p^{m-n+1}\mathbb{Z}_p}\begin{pmatrix}1&\\ \tau&1\end{pmatrix}w'A_{m,n}I_p'.$$

$$\mathfrak{n}_p^{-1}\begin{pmatrix}1&&\\ \tau&1&\\&&1\end{pmatrix}w'A_{m,n}w'\begin{pmatrix}\delta&&\\&1&\\&&1\end{pmatrix}\mathfrak{n}_p\in K_p\begin{pmatrix}p^m&&\\&p^n&(\delta p^n-1)p^{-1}\\&&1\end{pmatrix}K_p.$$

1. Suppose $n\geq 1$. Then $m\geq n+1\geq 2$. Then
$$\mathfrak{n}_p^{-1}\begin{pmatrix}1&&\\ \tau&1&\\&&1\end{pmatrix}w'A_{m,n}w'\begin{pmatrix}\delta&&\\&1&\\&&1\end{pmatrix}\mathfrak{n}_p\in K_p\begin{pmatrix}p^m&&\\&p^{n+1}&\\&&p^{-1}\end{pmatrix}K_p.$$
2. Suppose $n=0$. Let $\delta\neq 1\in\mathbb{F}_p$. Then
$$\mathfrak{n}_p^{-1}\begin{pmatrix}1&&\\ \tau&1&\\&&1\end{pmatrix}w'A_{m,n}w'\begin{pmatrix}\delta&&\\&1&\\&&1\end{pmatrix}\mathfrak{n}_p\in K_p\begin{pmatrix}p^m&&\\&p&\\&&p^{-1}\end{pmatrix}K_p.$$
3. Suppose $n\leq -1$ and $m\geq 1$. Then
$$\mathfrak{n}_p^{-1}\begin{pmatrix}1&&\\ \tau&1&\\&&1\end{pmatrix}w'A_{m,n}w'\begin{pmatrix}\delta&&\\&1&\\&&1\end{pmatrix}\mathfrak{n}_p\in K_p\begin{pmatrix}p^m&&\\&p&\\&&p^{n-1}\end{pmatrix}K_p.$$
4. Suppose $n\leq -1$ and $m\leq 0$. Then
$$\mathfrak{n}_p^{-1}\begin{pmatrix}1&&\\ \tau&1&\\&&1\end{pmatrix}w'A_{m,n}w'\begin{pmatrix}\delta&&\\&1&\\&&1\end{pmatrix}\mathfrak{n}_p\in K_p\begin{pmatrix}p&&\\&p^m&\\&&p^{n-1}\end{pmatrix}K_p.$$

Therefore, similar to (5.70) and (5.75) we have that

$$\Sigma_4 := \sum_{n\in\mathbb{Z}}\sum_{m\geq n+1}\int_{X_{m,n}}\frac{\Xi_p(\mathfrak{n}_p^{-1}g_p\mathfrak{n}_p)\Xi_p'(g_p)}{\Xi_p(\mathbf{1}_p)\Xi_p'(\mathbf{1}_p')}dg_p \tag{5.76}$$
$$= \Sigma_{41}+\Sigma_{42}+\Sigma_{43}+\Sigma_{44},$$

where

$$X_{m,n} = I_p'w'A_{m,n}w'I_p'$$

and

$$\Sigma_{41} := \frac{(p-1)\mu(I_p'(1))}{\Xi_p(\mathbf{1}_p)} \sum_{n\geq 1}\sum_{m\geq n+1} \Xi_p\begin{pmatrix} p^m & & \\ & p^{n+1} & \\ & & p^{-1}\end{pmatrix},$$

$$\Sigma_{42} := \frac{\mu(I_p'(1))}{\Xi_p(\mathbf{1}_p)} \sum_{m\geq 1}\left[\Xi_p\begin{pmatrix} p^m & & \\ & 1 & \\ & & 1\end{pmatrix} + (p-2)\Xi_p\begin{pmatrix} p^m & & \\ & p & \\ & & p^{-1}\end{pmatrix}\right],$$

$$\Sigma_{43} := \frac{(p-1)\mu(I_p'(1))}{\Xi_p(\mathbf{1}_p)} \sum_{n\leq -1}\sum_{m\geq 1} \Xi_p\begin{pmatrix} p^m & & \\ & p & \\ & & p^{n-1}\end{pmatrix},$$

$$\Sigma_{44} := \frac{(p-1)\mu(I_p'(1))}{\Xi_p(\mathbf{1}_p)} \sum_{n\leq -1}\sum_{n+1\leq m\leq 0} \Xi_p\begin{pmatrix} p & & \\ & p^m & \\ & & p^{n-1}\end{pmatrix}.$$

Recall that $\mathcal{P}^*(\widetilde{\xi}_p, \widetilde{\xi}_p; \xi_p', \xi_p')$ was defined in (5.64). Then combining (5.72), (5.73), (5.74), (5.75) with (5.76) we obtain

$$\mathcal{P}^*(\widetilde{\xi}_p, \widetilde{\xi}_p; \xi_p', \xi_p') - \mu(I_p'(1)) = -\frac{\mu(I_p'(1))}{\Xi_p(\mathbf{1}_p)}\Xi_p\begin{pmatrix} p & & \\ & 1 & \\ & & p^{-1}\end{pmatrix} + \sum_{i=1}^{4}\sum_{j=1}^{4}\Sigma_{ij}. \tag{5.77}$$

Denote by RHS the right hand side of (5.77). Then substituting definitions of Σ_{ij}'s one finds that $\Xi_p(\mathbf{1}_p)\cdot\mu(I_p'(1))^{-1}\cdot$ RHS is equal to

$$\begin{aligned}
&-\Xi_p\begin{pmatrix} p & & \\ & 1 & \\ & & p^{-1}\end{pmatrix} + (p-1)\sum_{n\leq -1}\sum_{m\geq 1}\Xi_p\begin{pmatrix} p^{m+1} & & \\ & p^{-1} & \\ & & p^n\end{pmatrix}\\
&+\sum_{n\leq -1}\left[\Xi_p\begin{pmatrix} 1 & & \\ & 1 & \\ & & p^n\end{pmatrix} - \Xi_p\begin{pmatrix} p & & \\ & p^{-1} & \\ & & p^n\end{pmatrix}\right]\\
&-\sum_{m\geq 1}\Xi_p\begin{pmatrix} p^m & & \\ & p & \\ & & p^{-1}\end{pmatrix} - (p-1)\sum_{n\leq -1}\sum_{m\geq 1}\Xi_p\begin{pmatrix} p^m & & \\ & 1 & \\ & & p^n\end{pmatrix}\\
&-(p-1)\sum_{n\leq -1}\sum_{m\geq 1}\Xi_p\begin{pmatrix} p^{m+1} & & \\ & 1 & \\ & & p^{n-1}\end{pmatrix} + \sum_{m\geq 1}\Xi_p\begin{pmatrix} p^m & & \\ & 1 & \\ & & 1\end{pmatrix}\\
&+(p-1)\sum_{n\leq -1}\sum_{m\geq 1}\Xi_p\begin{pmatrix} p^m & & \\ & p & \\ & & p^{n-1}\end{pmatrix}.
\end{aligned}$$

To bound this last sum, we recall Macdonald's formula for $\mathrm{GL}(3,\mathbb{Q}_p)$ (cf. [Mac71]). Let $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$ a dominant coweight (ie. $\lambda_1 \geq \lambda_2 \geq \lambda_3$) and

$$p^{\boldsymbol{\lambda}} := \mathrm{diag}(p^{\lambda_1}, p^{\lambda_2}, p^{\lambda_3}).$$

Let $\mathcal{C}_\rho$ be the set of weights of the irreducible representation of highest weight ρ. By [Cas17, Theorem 5.5, p. 31],

$$\frac{\Xi_p(p^{\boldsymbol{\lambda}})}{\Xi_p(\mathbf{1}_p)} = \frac{\delta_B(p^{\boldsymbol{\lambda}})^{\frac{1}{2}}}{|K_p p^{\boldsymbol{\lambda}} K_p/K_p|\sum_w q^{-l(w)}} \sum_{w\in W_{\boldsymbol{\lambda}}}\mathrm{sgn}(w)\sum_{\boldsymbol{\mu}\in[W_{\boldsymbol{\lambda}}\backslash\mathcal{C}_\rho]}\sum_{S\subseteq\Sigma^+\ \sum_{\gamma\in S}\gamma=\rho-w\boldsymbol{\mu}}(-1)^{|S|}q^{-|S|}\tau_{\boldsymbol{\lambda}+\boldsymbol{\mu}-\rho}(\chi),$$

where $W_{\boldsymbol{\lambda}}$ is the group generated by the simple roots α such that $\langle \boldsymbol{\lambda}, \alpha^\vee\rangle = 0$, Σ^+ is the set of positive roots, and $\tau_{\boldsymbol{\lambda}+\boldsymbol{\mu}-\rho}$ is the character of the irreducible representation $\sigma_{\boldsymbol{\lambda}+\boldsymbol{\mu}-\rho}$ of $\mathrm{GL}_3(\mathbb{C})$ with highest weight $\boldsymbol{\lambda}+\boldsymbol{\mu}-\rho$.

Note that
$$|C_\rho| = \sum_{S\subseteq\Sigma^+} 1 = 2^3 = 8.$$
So
$$\begin{aligned}\left|\frac{\Xi_p(p^{\boldsymbol{\lambda}})}{\Xi_p(\mathbf{1}_p)}\right| \leq & \frac{\delta_B(p^{\boldsymbol{\lambda}})^{\frac12}}{|K_pp^{\boldsymbol{\lambda}}K_p/K_p|\sum_w q^{-l(w)}}\sum_w\sum_{\boldsymbol{\mu}}\sum_{\substack{S\subseteq\Sigma^+\\ \sum_{\gamma\in S}\gamma=\rho-w\boldsymbol{\mu}}} q^{-|S|}\cdot\dim\sigma_{\boldsymbol{\lambda}+\boldsymbol{\mu}-\rho}\\ \leq & \frac{\delta_B(p^{\boldsymbol{\lambda}})^{\frac12}|C_\rho|}{|K_pp^{\boldsymbol{\lambda}}K_p/K_p|}\cdot\max_{\boldsymbol{\mu}}\dim\sigma_{\boldsymbol{\lambda}+\boldsymbol{\mu}-\rho}\end{aligned}$$

By definition as $\boldsymbol{\mu}$ ranges through C_ρ, $\boldsymbol{\mu}-\rho$ ranges over negative roots. Let Λ_1 and Λ_2 be basic weights. Then the possible values of $\boldsymbol{\mu}-\rho$ is
$$0,\ -2\Lambda_1+\Lambda_2,\ \Lambda_1-2\Lambda_2,\ -\Lambda_1-\Lambda_2,\ -3\Lambda_1,\ -3\Lambda_2,\ -2\Lambda_1-2\Lambda_2.$$

Note that $\boldsymbol{\lambda}\equiv(\lambda_1-\lambda_2)\Lambda_1+(\lambda_2-\lambda_3)\Lambda_2$ modulo the center. Hence, by [Hal15, Example 10.23, p. 288], we have
$$\max_{\boldsymbol{\mu}}\dim\sigma_{\boldsymbol{\lambda}+\boldsymbol{\mu}-\rho}\leq\frac{(\lambda_1-\lambda_2+4)(\lambda_2-\lambda_3+4)(\lambda_1-\lambda_3+6)}{2}.$$

Therefore,
$$\left|\frac{\Xi_p(p^{\boldsymbol{\lambda}})}{\Xi_p(\mathbf{1}_p)}\right| \leq \frac{4\delta_B(p^{\boldsymbol{\lambda}})^{\frac12}}{|K_pp^{\boldsymbol{\lambda}}K_p/K_p|}\cdot(\lambda_1-\lambda_2+4)(\lambda_2-\lambda_3+4)(\lambda_1-\lambda_3+6).$$

The absolute value of the right hand side of (5.77) is thus
$$\leq 4\cdot4\cdot10^3\cdot\mu(I_p'(1))\cdot\left(\frac{1}{p^2}+p\sum_{n\geq1}\sum_{m\geq1}\frac{m+n+2}{p^{m+n}}+\sum_{n\geq1}\frac{n+1}{p^n}\right)\leq\frac{10^6\mu(I_p'(1))}{p}.$$
As a consequence, we have
$$\left|\int_{G'(\mathbb{Q}_p)}\frac{\Xi_p(\mathfrak{n}_p^{-1}g_p\mathfrak{n}_p)\Xi_p'(g_p)}{\Xi_p(\mathbf{1}_p)\Xi_p'(\mathbf{1}_p')}dg_p-\mu(I_p'(1))\right|\leq\frac{10^6\mu(I_p'(1))}{p}. \tag{5.78}$$
Now (5.25) follows from (5.78), the identity
$$\mu(I_p'(1))=\frac{1}{(p-1)(p+1)}=\frac{1}{p^2-1} \tag{5.79}$$
and our assumption that if $N'=p>1$ then $p>10^6$.

Remark 5.8. When one works on a spherical vector ξ_p without the translation by $\mathfrak{n}_p$, the periods $\mathcal{P}_p^\natural(\xi_p,\xi_p;\xi_p',\xi_p')$ is vanishing. In fact, applying Cartan-Iwahori decomposition, we obtain by Lemma A.2, Lemma A.3 and Proposition 5.12, that
$$\frac{\mathcal{P}_p(\xi_p,\xi_p;\xi_p',\xi_p')}{\langle\xi_p,\xi_p\rangle_p\cdot\langle\xi_p',\xi_p'\rangle_p}=\sum_{n\geq0}\sum_{w'\in W_n'}\frac{\langle\pi_p(w')\xi_p,\xi_p\rangle_p}{\langle\xi_p,\xi_p\rangle_p}\cdot\frac{(-1)^{\lambda'(w')}}{p+1}, \tag{5.80}$$
which converges absolutely. Therefore, one can switch the sums on the right hand of (5.80), obtaining
$$\frac{\mathcal{P}_p(\xi_p,\xi_p;\xi_p',\xi_p')}{\langle\xi_p,\xi_p\rangle\langle\xi_p',\xi_p'\rangle_p}=\sum_{n\geq0}\frac{\langle\pi_p(A_n)\xi_p,\xi_p\rangle_p}{\langle\xi_p,\xi_p\rangle_p}\cdot\mu(I_p')\sum_{w'\in W_n'}\frac{(-1)^{\lambda'(w')}}{p+1}=0.$$

6. The Geometric Side

In this section and the next three sections, we compute the terms on the right hand side of (3.6) with the choices of $f^{\mathfrak{n}}$ and φ' that have been described in the previous sections. In particular this will show the absolute convergence of the sums/integral appearing in (3.6) and (3.7) so that (3.7) is completely justified.

6.1. Basic Decomposition.

In this subsection we briefly recall some basic decomposition related to $G=U(V)$. These results will be used to calculate representatives of double cosets and estimate regular orbital integrals.

Let P be the parabolic subgroup stabilizing the isotropic line through e_{-1}. Explicitly, $P=MN$, with

$$M=\left\{m(\alpha,\beta):=\begin{pmatrix}\alpha&&\\&\beta&\\&&\overline{\alpha}^{-1}\end{pmatrix}:\ \alpha\in E^\times,\ \beta\in E^1\right\};$$

and the unipotent radical

$$N=\left\{n(b,z):=\begin{pmatrix}1&b&z\\&1&-\overline{b}\\&&1\end{pmatrix}:\ z,b\in E,\ z+\overline{z}=-b\overline{b}\right\}.$$

As algebraic groups defined over $\mathbb{Q}$, M is a 3-dimensional torus with split rank 1, and N is a 3-dimensional unipotent group. The center of G is

$$Z_G=\{m(\beta,\beta)=\mathrm{diag}(\beta,\beta,\beta):\ \beta\in E^1\}\subseteq M$$

so that $Z_G\simeq E^1$; given $\beta\in E^1$, we set

$$\gamma_\beta:=\mathrm{diag}(\beta,\beta,\beta)\in Z_G(\mathbb{Q}). \tag{6.1}$$

Lemma 6.1 (Bruhat decomposition)**.** *Let notation be as before and J given in (2.1). Then*

$$G=P\sqcup PJP, \tag{6.2}$$

and $PJP=NJP=PJN$. The expression of an element from the cell PJP as nJp or pJn with $n\in N$ and $p\in P$ is unique.

Proof. Suppose $g\notin P$. So $ge_{-1}\notin\langle e_{-1}\rangle$. Suppose $ge_{-1}=c_{-1}e_{-1}+c_0e_0$. Then $(ge_{-1},ge_{-1})_V=c_0\overline{c_0}$. On the other hand, $(ge_{-1},ge_{-1})_V=(e_{-1},e_{-1})_V=0$. So we get a contradiction. Thus ge_{-1} must involve the line $\langle e_1\rangle$.

One can write $ge_{-1}=c_{-1}e_{-1}+c_0e_0+c_1e_1$ with $c_1\neq 0$. Then a straightforward computation using linear algebra shows that one can find some $p\in P$ satisfying $e_1=pge_{-1}$. Hence from $(pge_0,pge_{-1})_V=0$ we deduce that $(pge_0,e_1)_V=0$, namely, $pge_0\in\langle e_1\rangle^\perp=\langle e_0,e_1\rangle$. Therefore,

$$pg=\begin{pmatrix}&&*\\&*&*\\ *&*&*\end{pmatrix}\in JP.$$

Thus $g\in PJP$, proving (6.2). The remaining part of this lemma is similar. □

Remark 6.1. Note that (13.14) is not a decomposition as algebraic groups, since there are 6 Bruhat cells required to cover $G(E)=\mathrm{GL}(3,E)$.

6.2. **Representatives of $G'(\mathbb{Q})\backslash G(\mathbb{Q})/G'(\mathbb{Q})$.** To deal with the geometric side of the relative trace formula, we need to describe the double coset $G'(\mathbb{Q})\backslash G(\mathbb{Q})/G'(\mathbb{Q})$. Our main tool is the Bruhat decomposition (13.14).

Lemma 6.2. *Let $c \in E$ be such that $c + \overline{c} = -1$. Then*

$$J\begin{pmatrix} 1 & 1 & c \\ & 1 & -1 \\ & & 1 \end{pmatrix} J = \begin{pmatrix} 1 & -\frac{1}{1+c} & -\frac{1}{1+\overline{c}} \\ & 1 & \frac{1}{1+\overline{c}} \\ & & 1 \end{pmatrix} J \begin{pmatrix} c & 1 & 1 \\ & \frac{\overline{c}}{1+\overline{c}} & -\frac{1}{1+\overline{c}} \\ & & -\frac{1}{1+c} \end{pmatrix}. \tag{6.3}$$

Proof. By making the proof of Lemma 6.1 explicit we find

$$\begin{pmatrix} 1 & \frac{1}{1+c} & -\frac{1}{1+c} \\ & 1 & -\frac{1}{1+\overline{c}} \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ -1 & 1 & \\ c & 1 & 1 \end{pmatrix} = \begin{pmatrix} & & -\frac{1}{1+c} \\ & \frac{\overline{c}}{1+\overline{c}} & -\frac{1}{1+\overline{c}} \\ c & 1 & 1 \end{pmatrix}. \tag{6.4}$$

Note also that the second matrix in (6.4) equals the left hand side of (6.3). Hence

$$\begin{aligned} J\begin{pmatrix} 1 & 1 & c \\ & 1 & -1 \\ & & 1 \end{pmatrix} J &= \begin{pmatrix} 1 & \frac{1}{1+c} & -\frac{1}{1+c} \\ & 1 & -\frac{1}{1+\overline{c}} \\ & & 1 \end{pmatrix}^{-1} \begin{pmatrix} & & -\frac{1}{1+c} \\ & \frac{\overline{c}}{1+\overline{c}} & -\frac{1}{1+\overline{c}} \\ c & 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & \frac{1}{1+c} & -\frac{1}{1+c} \\ & 1 & -\frac{1}{1+\overline{c}} \\ & & 1 \end{pmatrix}^{-1} J \begin{pmatrix} c & 1 & 1 \\ & \frac{\overline{c}}{1+\overline{c}} & -\frac{1}{1+\overline{c}} \\ & & -\frac{1}{1+c} \end{pmatrix}. \end{aligned}$$

Then (6.3) follows from computing the inverse of the first matrix in the last line. ☐

For $x \in E$ we set

$$\gamma(x) = \begin{pmatrix} \frac{x\overline{x}+3\overline{x}-x+1}{4} & \frac{1+x}{2} & -\frac{1}{2} \\ \frac{(x+1)(\overline{x}-1)}{2} & x & -1 \\ -\frac{(1-x)(1-\overline{x})}{2} & 1-x & 1 \end{pmatrix}. \tag{6.5}$$

Lemma 6.3. *Let $x_1, x_2 \in E$, $\alpha \in E^1 - \{1\}$ be such that $x_1 = \alpha x_2$,. Then there exists $g_1, g_2 \in G'(\mathbb{Q})$ such that*

$$\gamma(x_1) = \alpha g_1 \gamma(x_2) g_2. \tag{6.6}$$

Proof. A straightforward calculation shows that

$$\gamma(x_j)J = \begin{pmatrix} 1 & 1 & -\frac{1}{2} \\ & 1 & -1 \\ & & 1 \end{pmatrix} J \begin{pmatrix} 1 & 1-x_j & -\frac{(1-x_j)(1-\overline{x}_j)}{2} \\ & 1 & \overline{x}_j - 1 \\ & & 1 \end{pmatrix}, \quad j = 1, 2. \tag{6.7}$$

By Hilbert 90, there exists $\beta = a' + b'\sqrt{-D} \in E^\times$ such that $\alpha = \beta\overline{\beta}^{-1}$, with $a', b' \in \mathbb{Q}$. Since $\alpha \neq 1$, then $b' \neq 0$. Let $v' = a'/(2b') \in \mathbb{Q}$. Then one has

$$\alpha = \frac{a' + b'\sqrt{-D}}{a' - b'\sqrt{-D}} = -\frac{-1/2 + v'\sqrt{-D}}{-1/2 - v'\sqrt{-D}}.$$

Let $v = v'\sqrt{-D} \in E$. Then $v + \overline{v} = 0$. Let $c = -1/2 + v$. By (6.3) we have

$$J\begin{pmatrix} 1 & 1 & c \\ & 1 & -1 \\ & & 1 \end{pmatrix} J = \begin{pmatrix} \overline{c}^{-1} & -c^{-1} & 1 \\ & -\overline{c}c^{-1} & -1 \\ & & c \end{pmatrix} J \begin{pmatrix} 1 & c^{-1} & c^{-1} \\ & 1 & -\overline{c}^{-1} \\ & & 1 \end{pmatrix}, \tag{6.8}$$

since $c+\overline{c}+1=0$. Let $u,d\in E$ be such that $u+\overline{u}=d+\overline{d}=0$. Let $a,b\in E^\times$. Appealing to the identities (6.7) and (6.8) we then obtain

$$\begin{pmatrix} a & & au\\ & \alpha & \\ & & \overline{a}^{-1}\end{pmatrix} J \begin{pmatrix} 1 & & v\\ & 1 & \\ & & 1\end{pmatrix}\gamma(x_2)J\begin{pmatrix} b & & \\ & 1 & \\ & & \overline{b}^{-1}\end{pmatrix}\begin{pmatrix} 1 & & d\\ & 1 & \\ & & 1\end{pmatrix}$$
$$=\begin{pmatrix} \frac{ab}{\overline{c}} & -\frac{a}{c} & \frac{a(1+uc)}{\overline{b}}\\ & -\frac{\alpha\overline{c}}{c} & -\frac{\alpha}{\overline{b}}\\ & & \frac{c}{\overline{a}\overline{b}}\end{pmatrix} J\begin{pmatrix} 1 & \frac{1-x_2+c^{-1}}{b} & \frac{-\frac{(1-x_2)(1-\overline{x}_2)}{2}+\overline{x}_2c^{-1}}{b\overline{b}}+d\\ & 1 & \frac{\overline{x}_2-1-\overline{c}^{-1}}{\overline{b}}\\ & & 1\end{pmatrix}.$$

To compare the right hand side of this equality with $\gamma(x_1)J$, we consider

$$\begin{cases} ab=\overline{c},\ a=-c,\ \alpha=-c\overline{c}^{-1}\\ a\overline{b}^{-1}+auc\overline{b}^{-1}=-1/2,\ \frac{1-x_2+c^{-1}}{b}=1-x_1\\ \frac{-\frac{(1-x_2)(1-\overline{x}_2)}{2}+\overline{x}_2c^{-1}}{b\overline{b}}+d=-\frac{(1-x_1)(1-\overline{x}_1)}{2}\\ d+\overline{d}=u+\overline{u}=0.\end{cases} \tag{6.9}$$

Solve the system of equations (6.9) we have

$$\begin{cases} a=-c,\ b=-\overline{c}c^{-1},\ \alpha=-c\overline{c}^{-1},\ u=-\frac{1/2+\overline{c}}{c\overline{c}},\ x_1=\alpha_1x_2\\ d=-\frac{(1-x_1)(1-\overline{x}_1)}{2}+\frac{(1-x_2)(1-\overline{x}_2)}{2}-\overline{x}_2c^{-1}=\frac{(\alpha-1)x_2+(\overline{\alpha}-1)\overline{x}_2}{2}-\overline{x}_2c^{-1}.\end{cases} \tag{6.10}$$

Let $a,b,c,d,u,v,\alpha\in E$ be as in (6.10). Then we have

$$\gamma(x_1)J=\begin{pmatrix} a & & au\\ & \alpha & \\ & & \overline{a}^{-1}\end{pmatrix} J\begin{pmatrix} 1 & & v\\ & 1 & \\ & & 1\end{pmatrix}\gamma(x_2)J\begin{pmatrix} b & & bd\\ & 1 & \\ & & \overline{b}^{-1}\end{pmatrix}. \tag{6.11}$$

Then (6.6) follows from (6.11) by setting

$$g_1=\begin{pmatrix} \alpha^{-1}a & & \alpha^{-1}au\\ & 1 & \\ & & \alpha^{-1}\overline{a}^{-1}\end{pmatrix} J\begin{pmatrix} 1 & & v\\ & 1 & \\ & & 1\end{pmatrix},\quad g_2=J\begin{pmatrix} b & & bd\\ & 1 & \\ & & \overline{b}^{-1}\end{pmatrix}J.$$

One verifies that $g_1,g_2\in G'(\mathbb{Q})$. Hence Lemma 6.3 follows. ☐

Proposition 6.4. *For γ_β and $\gamma(x)$ defined in (6.1) and (6.5), the set*

$$\Phi=\{\gamma_\beta,\ \gamma(x),\ \beta\in E^1,\ x\in E\}$$

form a complete set of representatives for the double quotient $G'(\mathbb{Q}\backslash G(\mathbb{Q})/G'(\mathbb{Q})$.

Proof. Let $g\in P(\mathbb{Q})$. We can write $g=\beta m(\alpha,1)n(b,z)$ for some $\alpha\in E^\times$, $\beta\in E^1$ and $b,z\in E$. We then have

$$G'(\mathbb{Q})gG'(\mathbb{Q})=G'(\mathbb{Q})\begin{pmatrix} \alpha & & \\ & \beta & \\ & & \overline{\alpha}^{-1}\end{pmatrix}\begin{pmatrix} 1 & b & z\\ & 1 & -\overline{b}\\ & & 1\end{pmatrix}G'(\mathbb{Q})$$

(i) If $b=0$, then $G'(\mathbb{Q})gG'(\mathbb{Q})=G'(\mathbb{Q})\gamma_\beta G'(\mathbb{Q})$, because the semisimple part $\mathrm{diag}(\alpha,1,\overline{\alpha}^{-1})\in G'(\mathbb{Q})$.

(ii) Suppose $b\neq 0$. Note that for any z satisfying $z+\overline{z}=0$, one has $n(0,z)\in G'(\mathbb{Q})$. Hence

$$G'(\mathbb{Q})gG'(\mathbb{Q})=G'(\mathbb{Q})\begin{pmatrix} 1 & 1 & \frac{z}{b\overline{b}}\\ & 1 & -1\\ & & 1\end{pmatrix}G'(\mathbb{Q})=G'(\mathbb{Q})\gamma(1)G'(\mathbb{Q}).$$

Let $g \in P(\mathbb{Q})JP(\mathbb{Q})$. We can write $g = m(\alpha,\beta)n(b,z)Jn(b',z')$ for some $\alpha \in E^\times$, $\beta \in E^1$ and $b, b', z, z' \in E$. We then have

$$G'(\mathbb{Q})gG'(\mathbb{Q}) = G'(\mathbb{Q}) \begin{pmatrix} \alpha & & \\ & \beta & \\ & & \overline{\alpha}^{-1} \end{pmatrix} \begin{pmatrix} 1 & b & z \\ & 1 & -\overline{b} \\ & & 1 \end{pmatrix} J \begin{pmatrix} 1 & b' & z' \\ & 1 & -\overline{b}' \\ & & 1 \end{pmatrix} G'(\mathbb{Q})$$

(iii) Suppose $b = 0$. Then $z+\overline{z} = -b\overline{b} = 0$, implying that $m(\alpha,1)n(b,z) \in G'(\mathbb{Q})$. Note that $J \in G'(\mathbb{Q})$. Hence this situation will boil down to case (i) or (ii), namely, we obtain, under the hypothesis $b = 0$, that

$$G'(\mathbb{Q})gG'(\mathbb{Q}) \subseteq G'(\mathbb{Q})\gamma_\beta \cup G'(\mathbb{Q})\gamma_\beta\gamma(1)G'(\mathbb{Q}).$$

(iv) Suppose $b' = 0$. Then similarly we obtain that

$$G'(\mathbb{Q})gG'(\mathbb{Q}) \subseteq G'(\mathbb{Q})\gamma_\beta \cup G'(\mathbb{Q})\gamma_\beta\gamma(1)G'(\mathbb{Q}).$$

(v) Suppose $b \neq 0$ and $b' \neq 0$. Let $x = 1 - \overline{b}b' \neq 1$ and $\widetilde{z} = b\overline{b}z'$. Then

$$G'(\mathbb{Q})gG'(\mathbb{Q}) = G'(\mathbb{Q})\gamma_\beta \begin{pmatrix} 1 & 1 & \frac{z}{b\overline{b}} \\ & 1 & -1 \\ & & 1 \end{pmatrix} J \begin{pmatrix} 1 & 1-x & \widetilde{z} \\ & 1 & \overline{x}-1 \\ & & 1 \end{pmatrix} G'(\mathbb{Q}).$$

Using the fact that $n(0,z) \in G'(\mathbb{Q})$ when $z + \overline{z} = 0$, we can further deduce

$$G'(\mathbb{Q})gG'(\mathbb{Q}) = G'(\mathbb{Q})\gamma_\beta \begin{pmatrix} 1 & 1 & -1/2 \\ & 1 & -1 \\ & & 1 \end{pmatrix} J \begin{pmatrix} 1 & 1-x & -\frac{(1-x)(1-\overline{x})}{2} \\ & 1 & \overline{x}-1 \\ & & 1 \end{pmatrix} G'(\mathbb{Q}).$$

Then it follows from Lemma 6.1 and the identity

$$\begin{pmatrix} 1 & 1 & -\frac{1}{2} \\ & 1 & -1 \\ & & 1 \end{pmatrix} J \begin{pmatrix} 1 & 1-x & -\frac{(1-x)(1-\overline{x})}{2} \\ & 1 & \overline{x}-1 \\ & & 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{1+x}{2} & \frac{x\overline{x}+3\overline{x}-x+1}{4} \\ -1 & x & \frac{(x+1)(\overline{x}-1)}{2} \\ 1 & 1-x & -\frac{(1-x)(1-\overline{x})}{2} \end{pmatrix}$$

that

$$G'(\mathbb{Q})\backslash G(\mathbb{Q})/G'(\mathbb{Q}) = \bigcup_{\gamma\in\Phi} Z_G(\mathbb{Q})G'(\mathbb{Q})\gamma G'(\mathbb{Q}), \tag{6.12}$$

where Z_G is the center of G. By Lemma 6.3 we can write (6.12) as

$$G(\mathbb{Q}) = \bigcup_{\gamma\in\Phi} G'(\mathbb{Q})\gamma G'(\mathbb{Q}). \tag{6.13}$$

Now we show the union in (6.13) is actually disjoint.

Let $\gamma(x_1), \gamma(x_2) \in \Phi$. Suppose $\gamma(x_i)$ $(1 \leq i \leq 2)$ are such that

$$G'(\mathbb{Q})\gamma(x_1)G'(\mathbb{Q}) = G'(\mathbb{Q})\gamma(x_2)G'(\mathbb{Q}).$$

Combining the definition of $\gamma(x_1), \gamma(x_2)$ in (6.5) and the identity

$$\begin{pmatrix} * & & * \\ & 1 & \\ * & & * \end{pmatrix} \begin{pmatrix} * & * & * \\ * & x & * \\ * & * & * \end{pmatrix} \begin{pmatrix} * & & * \\ & 1 & \\ * & & * \end{pmatrix} = \begin{pmatrix} * & * & * \\ * & x & * \\ * & * & * \end{pmatrix},$$

we deduce (by comparing the $(2,2)$-th entry) that $x_1 = x_2$.

By the Bruhat decomposition, the orbit $G'(\mathbb{Q})\gamma_\beta G'(\mathbb{Q})$ does not intersect the orbit $G'(\mathbb{Q})\gamma(x)G'(\mathbb{Q})$ for any $\beta \in E^1$ and $x \in E$. In conclusion, (6.13) is a disjoint union. ∎

Given $\gamma\in\Phi$, we denote by $H_\gamma\subset G'\times G'$ the stabilizer of γ, namely,

$$H_\gamma=\big\{(u,v)\in G'\times G':\ u^{-1}\gamma v=\gamma\big\}.$$

Let φ' be an automorphic form on $G'(\mathbb{Q})\backslash G'(\mathbb{A})$. We define (at least formally) the orbital integral

$$\mathcal{O}_\gamma(f^{\mathfrak{n}},\varphi')=\int_{H_\gamma(\mathbb{Q})\backslash(G'\times G')(\mathbb{A})}f^{\mathfrak{n}}(u^{-1}\gamma v)\varphi'(u)\overline{\varphi}'(v)dudv. \tag{6.14}$$

By Proposition 6.4 we can rewrite $J(f)$ (at least formally) as

$$J(f^{\mathfrak{n}},\varphi')=\sum_{\beta\in E^1}\mathcal{O}_{\gamma_\beta}(f^{\mathfrak{n}},\varphi')+\sum_{x\in E}\mathcal{O}_{\gamma(x)}(f^{\mathfrak{n}},\varphi'). \tag{6.15}$$

The analysis of these integrals of course depend heavily on the structure of the stabilizer H_γ and we give here a roadmap of what is to come.

– For $\beta\in E^1$, $\gamma_\beta\in Z_G(\mathbb{Q})$ and the stabilizer H_{γ_β} is the diagonal subgroup
$$H_{\gamma_\beta}=\Delta G'\subset G'\times G'.$$
– We will see in Sec. 8 that the the stabilizer $H_{\gamma(1)}$ (and more generally $H_{\gamma(x)}$ for $x\in E^1$) is isomorphic to the unipotent radical of the Borel subgroup of G'.
– Finally for $x\in E-E^1$ we will see that $H_{\gamma(x)}$ is a torus isomorphic to the unitary group $U(1)$.

Moreover we can use the support and invariance properties of $f^{\mathfrak{n}}$ to infer further restrictions on the γ_β and $\gamma(x)$ whose orbital integral is non-zero.

Regarding the former, notice that $u,v\in G'(\mathbb{A})$, $u^{-1}\gamma_\beta v$ is of the form $\begin{pmatrix}* & & *\\ & \beta & \\ * & & *\end{pmatrix}$. Therefore, by definition of $f^{\mathfrak{n}}$ one has

$$f^{\mathfrak{n}}(u^{-1}\gamma_\beta v)=0$$

unless

$$\beta\in\mathcal{O}_E^1=E^1\cap\mathcal{O}_E.$$

In this case $f^{\mathfrak{n}}$ is invariant under $Z_G(\mathcal{O}_E^1)$, by Lemma 6.3 we have

$$\mathcal{O}_{\gamma_\beta}(f^{\mathfrak{n}},\varphi')=\mathcal{O}_{\gamma_1}(f^{\mathfrak{n}},\varphi'),\quad \beta\in\mathcal{O}_E^1,$$

therefore

$$\sum_{\beta\in E^1}\mathcal{O}_{\gamma_\beta}(f^{\mathfrak{n}},\varphi')=w_E\mathcal{O}_{\gamma_1}(f^{\mathfrak{n}},\varphi')$$

where $w_E=\#\mathcal{O}_E^1$ is finite (as E is an imaginary quadratic field).

Regarding the orbital integrals $\mathcal{O}_{\gamma(x)}(f^{\mathfrak{n}},\varphi')$, $x\in E^1$, we will see in §8 that these converge absolutely. Moreover, using by Lemma 6.3 we will show, in Proposition 8.13 in Sec. 8.6, that for $x\in E^1$, one has

$$\mathcal{O}_{\gamma(x)}(f^{\mathfrak{n}},\varphi')=\begin{cases}\mathcal{O}_{\gamma(1)}(f^{\mathfrak{n}},\varphi'), & \text{if } x\in\mathcal{O}_E^1;\\ 0, & \text{otherwise.}\end{cases} \tag{6.16}$$

This implies that

$$\sum_{x\in E^1}\mathcal{O}_{\gamma(x)}(f^{\mathfrak{n}},\varphi')=w_E\mathcal{O}_{\gamma(1)}(f^{\mathfrak{n}},\varphi')$$

Finally, for the (more complicated) orbital integrals $\mathcal{O}_{\gamma(x)}(f^{\mathfrak{n}},\varphi'),\ x\in E-E^1$, we will see in §10, that these orbital integrals as well as their sum converge absolutely. From (6.16) we then obtain

$$J(f^{\mathfrak{n}},\varphi')=w_E\mathcal{O}_{\gamma_1}(f^{\mathfrak{n}},\varphi')+w_E\mathcal{O}_{\gamma(1)}(f^{\mathfrak{n}},\varphi')+\sum_{x\in E-E^1}\mathcal{O}_{\gamma(x)}(f^{\mathfrak{n}},\varphi'),\tag{6.17}$$

In the sequel, we will call

- $\mathcal{O}_{\gamma_1}(f^{\mathfrak{n}},\varphi')$ the *identity orbital integral*;
- $\mathcal{O}_{\gamma(1)}(f^{\mathfrak{n}},\varphi')$ and more generally the integrals $\mathcal{O}_{\gamma(x)}(f^{\mathfrak{n}},\varphi'),\ x\in E^1$ the *unipotent orbital integrals*,
- $\mathcal{O}_{\gamma(x)}(f^{\mathfrak{n}},\varphi'),\ x\in E-E^1$ *the regular orbital integrals* (as these $\gamma(x)$'s are regular).

7. The Identity Orbital Integral

In this section, we deal with the identity orbital integral (associated to the identity element $\gamma_1=\mathrm{Id}_3$):

$$\mathcal{O}_{\gamma_1}(f^{\mathfrak{n}},\varphi')=\int_{H_{\gamma_1}(\mathbb{Q})\backslash H(\mathbb{A})}f^{\mathfrak{n}}(x^{-1}y)\overline{\varphi}'(x)\varphi'(y)dxdy.$$

Let $\nu(f)$ be the set of primes p such that

$$f_p=1_{G(\mathbb{Z}_p)A_{p^{r_p}}G(\mathbb{Z}_p)}$$

for some integer $r_p\geq 1$. By construction, as $p\in\nu(f)$, p is inert $(p,NN')=1$; in particular π'_p is unramified.

The matrix $A_{p^{r_p}}$ also belongs to $G'(\mathbb{Q}_p)$ and convolution by the function

$$1_{G'(\mathbb{Z}_p)A'_{p^r}G'(\mathbb{Z}_p)}$$

is an Hecke operator; the space of $G'(\mathbb{Z}_p)$-invariant function ${\pi'_p}^{G'(\mathbb{Z}_p)}$ is an eigenspace with eigenvalue

$$p^{r_p}\lambda_{\pi'_p}(p^{r_p}).$$

Moreover since π' is tempered one has $|\lambda_{\pi'_p}(p^{r_p})|\leq 2$ (and for $r_p=0$ $\lambda_{\pi'_p}(1)=1$)

Proposition 7.1. *Let notation be as before. Then $\mathcal{O}_{\gamma_1}(f^{\mathfrak{n}},\varphi')$ is equal to*

$$\frac{\langle\varphi',\varphi'\rangle}{d_k}\frac{N^2}{N'^2}\Psi(N)\mathfrak{S}(N')\prod_{p\in\nu(f)}p^{r_p}\lambda_{\pi'}(p^{r_p}).\tag{7.1}$$

where

$$d_k=k-1,\ \Psi(N)=\prod_{p|N}\left(1-\frac{1}{p}+\frac{1}{p^2}\right),\ \mathfrak{S}(N')=\prod_{p|N'}\frac{1}{1-p^{-2}}.$$

Proof. Note that $H_{\gamma_1}=\Delta G'$, the diagonal embedding of G' into $G'\times G'$. Hence we can change variables to obtain

$$\mathcal{O}_{\gamma_1}(f^{\mathfrak{n}})=\int_{\Delta G'(\mathbb{A})\backslash G'(\mathbb{A})\times G'(\mathbb{A})}f(\widetilde{\mathfrak{n}}^{-1}x^{-1}y\widetilde{\mathfrak{n}})\int_{G'(\mathbb{Q})\backslash G'(\mathbb{A})}\overline{\varphi}'(hx)\varphi'(hy)dhdxdy.$$

We can write it as the Petersson inner product of cusp forms:

$$\mathcal{O}_{\gamma_1}(f^{\mathfrak{n}})=\int_{G'(\mathbb{Q})\backslash G'(\mathbb{A})}\overline{\varphi}'(x)\pi'(f^{\mathfrak{n}})\varphi'(x)dx=\langle\pi'(f^{\mathfrak{n}})\varphi',\varphi'\rangle.\tag{7.2}$$

Since φ' is rapidly decreasing, each integral above is absolutely convergent.

Write

$$(\pi',V)=(\pi'_\infty,V_\infty)\otimes(\pi'_{\mathrm{fin}},V_{\mathrm{fin}});$$

we have $\varphi'\simeq\xi'_\infty\otimes\xi'_{\mathrm{fin}}\in V_\infty\otimes V_{\mathrm{fin}}$ for some local new vector $\xi'_{\mathrm{fin}}\in V_{\mathrm{fin}}$. Let us also recall that through the isomorphism ι (see (4.35)) we have

$$\pi'\simeq\pi_k^+,$$

the cuspidal representation of $\mathrm{SL}(2,\mathbb{A})$ of level 1 and whose archimedean component is the holomorphic discrete series of weight k and set $\xi'_\infty=v_k^\circ\circ\iota$.

Wrinting $f=f_\infty\times f_{\mathrm{fin}}$ we have

$$\pi'(f^{\mathfrak{n}})(\xi'_\infty\otimes\xi'_{\mathrm{fin}})=\int_{G'(\mathbb{R})}\int_{G'(\mathbb{A}_{\mathrm{fin}})}f_\infty(y_\infty)\pi'_\infty(y_\infty)\xi'_\infty\otimes f^{\mathfrak{n}}_{\mathrm{fin}}(y_{\mathrm{fin}})\pi'_f(y_{\mathrm{fin}})\xi'_{\mathrm{fin}}dy,$$

where $dy=dy_\infty dy_{\mathrm{fin}}$ with $y_{\mathrm{fin}}=\otimes_{p<\infty}y_p\in G'(\mathbb{A}_{\mathrm{fin}})$ and we have

$$\pi'(f^{\mathfrak{n}})\varphi'=\pi'(f^{\mathfrak{n}})(\xi'_\infty\otimes\xi'_{\mathrm{fin}})=\pi'_\infty(f_\infty)\xi'_\infty\bigotimes_{p<\infty}\pi'_p(f_p^{\mathfrak{n}})\xi'_p.$$

We recall that that ξ'_{fin} is unique up to scalar. Specifically, at $p\nmid N'$, ξ'_p is spherical; and at $p\mid N'$, ξ'_p is a nonzero Iwahori fixed vector.

Note that by (4.22) we can regard f_∞ as the matrix coefficient F_{-k_1}. Since $SU(1,1;\mathbb{R})$ is unimodular and the central characters in the above representations are trivial, we can apply Schur orthogonality relations to conclude that

$$\pi'_\infty(f_\infty)\xi'_\infty=0$$

if $k\neq-k_1$; and when $k=-k_1$, we have

$$\pi'_\infty(f_\infty)\xi'_\infty=\frac{1}{k-1}\xi'_\infty. \tag{7.3}$$

(1) Suppose $p\mid N$. Then ξ'_p is I'_p-fixed, $f_p^{\mathfrak{n}_p}=f_p$ and $G'(\mathbb{Q}_p)\cap\operatorname{supp}f_p=I'_p$. Therefore, we have

$$\begin{aligned}\pi'_p(f_p^{\mathfrak{n}_p})\xi'_p=\pi'_p(f_p)\xi'_p&=\int_{G'(\mathbb{Q}_p)}f_p(y_p)\pi'_p(y_p)\xi'_p dy_p\\ &=\frac{\mu(I'_p)}{\mu(K_p(N))}\cdot\xi'_p=(p^2-p+1)\xi'_p\end{aligned} \tag{7.4}$$

by Lemma A.4 and Lemma A.1 (see the Appendix).

(2) Suppose $p\mid N'$. In that case $f_p^{\mathfrak{n}_p}\neq f_p$ and we then write $y_p=\begin{pmatrix}a&&b\\&1&\\c&&d\end{pmatrix}$.

By definition the non-vanishing

$$f_p(\widetilde{\mathfrak{n}}_p^{-1}y_p\widetilde{\mathfrak{n}}_p)\neq0$$

amounts to

$$\mathfrak{n}^{-1}y\mathfrak{n}=\mathfrak{n}_p^{-1}z_p\begin{pmatrix}a&b&\\c&d&\\&&1\end{pmatrix}\mathfrak{n}_p=\begin{pmatrix}a&b&(a-1)p^{-1}\\c&d&cp^{-1}\\&&1\end{pmatrix}\in K_p,$$

which is equivalent to

$$y_p=\begin{pmatrix}a&&b\\&1&\\c&&d\end{pmatrix}\in I'_p(1)=\left\{\begin{pmatrix}g_{11}&&g_{12}\\&1&\\g_{21}&&g_{22}\end{pmatrix}\in I'_p:\ g_{11}\in1+p\mathbb{Z}_p\right\}.$$

Therefore, we have by (5.79) that

$$\pi'_p(f_p^{\mathfrak{n}_p})\xi'_p=\frac{1}{\mu(K_p)}\int_{I'_p(1)}\pi'(y_p)\xi'_p dy_p=\frac{\mu(I'_p(1))}{\mu(K_p)}\xi'_p=\frac{1}{p^2-1}\xi'_p. \tag{7.5}$$

(3) Suppose p is inert and $p\nmid NN'$, and
$$f_p^{\mathfrak{n}_{\mathfrak{p}}}=f_p=1_{G(\mathbb{Z}_p)A_{p^r}G(\mathbb{Z}_p)}$$
for some $r\geq 0$. Since ξ_p' is spherical and
$$G(\mathbb{Z}_p)A_{p^{r_p}}G(\mathbb{Z}_p)\cap G'(\mathbb{Q}_p)=G'(\mathbb{Z}_p)A_{p^{r_p}}G'(\mathbb{Z}_p).$$
we have
$$\pi_p'(f_p^{\mathfrak{n}_{\mathfrak{p}}})\xi_p'=p^{r_p}\lambda_{\pi'}(p^{r_p})\xi_p'.$$
Combining (7.3) with (7.4) and (7.5) we obtain

$$\pi'(f)\varphi'=\frac{1}{k-1}\frac{N^2}{N'^2}\Psi(N)\mathfrak{S}(N')\prod_{p\in\nu(f)}p^{r_p}\lambda_{\pi'}(p^{r_p})\cdot\varphi'. \tag{7.6}$$

Substituting (7.6) into (7.2), (7.1) follows. □

8. **The unipotent Orbital Integrals**

In this section, we will deal with the orbital integral with respect to $\gamma(x)$ when $x\in E^1$. We will start with the special case that $x=1$; as we will see from Lemma 8.13, the general case reduces to this special case.

Recall that (see (6.5))
$$\gamma(1)=\begin{pmatrix}1&1&-1/2\\&1&-1\\&&1\end{pmatrix}.$$

Let $H_{\gamma(1)}$ be the stabilizer of $\gamma(1)$ and define (at least formally) the following integral
$$\mathcal{O}_{\gamma(1)}(f,\varphi')=\int_{H_{\gamma(1)}(\mathbb{Q})\backslash H(\mathbb{A})}f(x^{-1}\gamma(1)y)\overline{\varphi}'(x)\varphi'(y)dxdy.$$
where f is the function noted $f^{\mathfrak{n}}$ in (4.31). We will show below that $\mathcal{O}_{\gamma(1)}(f,\varphi')$ converges absolutely so that this integral is well defined.

8.1. **Factorization of the Unipotent Orbital Integral.**

The orbital integral $\mathcal{O}_{\gamma(1)}(f,\varphi')$ is not factorable into a product of local components over $p\leq\infty$ but we will apply the Fourier expansion to it which will provide an infinite sum of factorable integrals over $\mathbb{A}$ from which a sharp upper bound will be deduced.

We start with the following explicit expression:

Lemma 8.1. *Let notations be as before and let*[3] *N' be the unipotent of the standard parabolic subgroup of $G'=U(W)$, i.e., for any $\mathbb{Q}$-algebra R,*
$$N'(R)=\left\{n(0,b):=\begin{pmatrix}1&&b\\&1&\\&&1\end{pmatrix}\in\mathrm{GL}(3,E\otimes_{\mathbb{Q}}R):\ b+\overline{b}=0\right\}.$$
Then
$$H_{\gamma(1)}=\Delta N'\subset G'\times G'$$
and
$$\mathcal{O}_{\gamma(1)}(f,\varphi')=\int_{N'(\mathbb{A})\backslash G'(\mathbb{A})}\int_{G'(\mathbb{A})}f\left(x^{-1}\gamma(1)y\right)\int_{[N']}\overline{\varphi}'(vx)\varphi'(vy)dvdxdy. \tag{8.1}$$

[3]Hopefully this will not create a confusion with the conductor of π'

Proof. To compute the stabilizer $H_{\gamma(1)}$, we consider the equation

$$\begin{pmatrix} a & & b \\ & 1 & \\ c & & d \end{pmatrix}\begin{pmatrix} 1 & 1 & -1/2 \\ & 1 & -1 \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1/2 \\ & 1 & -1 \\ & & 1 \end{pmatrix}\begin{pmatrix} a' & & b' \\ & 1 & \\ c' & & d' \end{pmatrix}.$$

The solution is $a=d=a'=d'=1$, $c=c'=0$ and $b=b'$.

In other terms, $H_{\gamma(1)}=\Delta N'$, the image of the diagonal embedding

$$\Delta : N' \hookrightarrow N'\times N'.$$

We have

$$\mathcal{O}_{\gamma(1)}(f) = \int_{\Delta N'(\mathbb{Q})\backslash H(\mathbb{A})} f(x^{-1}\gamma(1)y)\overline{\varphi}'(x)\varphi'(y)dxdy. \tag{8.2}$$

Let

$$H_1 = N'(\mathbb{A})^2\backslash G'(\mathbb{A})^2,\ H_2=\Delta N'(\mathbb{Q})\backslash N'(\mathbb{A})^2,\ H_3 = N'(\mathbb{A})\backslash G'(\mathbb{A}).$$

Then the right hand side is equal to

$$\begin{aligned}
&\int_{H_1}\int_{H_2} f(x^{-1}n(0,\overline{b})\gamma(1)n(0,b')y)\overline{\varphi}'(n(0,b)x)\varphi'(n(0,b')y)dbdb'dxdy\\
&=\int_{H_1}\int_{H_2} f\left(x^{-1}\begin{pmatrix} 1 & 1 & -\frac{1}{2}+b'-b \\ & 1 & -1 \\ & & 1\end{pmatrix} y\right)\overline{\varphi}'(n(0,b)x)\varphi'(n(0,b')y)dbdb'dxdy\\
&=\int_{H_3}\int_{H_3}\int_{H_2} f\left(x^{-1}\begin{pmatrix} 1 & 1 & -\frac{1}{2}+b' \\ & 1 & -1 \\ & & 1\end{pmatrix} y\right)\overline{\varphi}'(n(0,b)x)\varphi'(n(0,b+b')y)dbdb'dxdy\\
&=\int_{H_3}\int_{[N']}\int_{G'(\mathbb{A})} f\left(x^{-1}\begin{pmatrix} 1 & 1 & -\frac{1}{2} \\ & 1 & -1 \\ & & 1\end{pmatrix} y\right)\overline{\varphi}'(n(0,b)x)\varphi'(n(0,b)y)dbdxdy.
\end{aligned}$$

Therefore, we obtain

$$\mathcal{O}_{\gamma(1)}(f,\varphi') = \int_{N'(\mathbb{A})\backslash G'(\mathbb{A})}\int_{G'(\mathbb{A})} f\left(x^{-1}\gamma(1)y\right)\int_{[N']}\overline{\varphi}'(vx)\varphi'(vy)dvdxdy,$$

which proves Lemma 8.1. □

Let us recall (see § 4.5) that the automorphic forms φ' on $G'(\mathbb{A})$ correspond to a $\mathrm{GL}_2(\mathbb{A})$-cusp form φ_1. The later admits a Fourier expansion which translates to a corresponding expansion for φ'. We spell this out below.

Let

$$\theta=\theta_\infty.\theta_f=\theta_\infty\prod_p\theta_p$$

be the usual unramified additive character of $\mathbb{A}/\mathbb{Q}$: ie. $\theta_\infty(x):=e^{-2\pi i x}$, and for p a prime, $\theta_p(x)=e^{2\pi i r_p(x)}$, where $r_p(x)$ is the principal part of $x\in\mathbb{Q}_p$. For $n\in\mathbb{A}$ and $x\in\mathbb{A}$, we define

$$\theta_n(x):=\theta(nx).$$

The additive character θ_n defines a character on $N'(\mathbb{Q})\backslash N'(\mathbb{A})$ by setting

$$\psi_n(u):=\theta_n(x)$$

for

$$u=u(x)=\begin{pmatrix}1&0&x\Delta\\0&1&0\\0&0&1\end{pmatrix}\in N'(\mathbb{A}), \tag{8.3}$$

We also set for R a commutative $\mathbb{Q}$-algebra and $v, w \in R^\times$

$$a(v,w) := \begin{pmatrix} v & & \\ & 1 & \\ & & w \end{pmatrix},\ a(v) := a(v,1) = \begin{pmatrix} v & & \\ & 1 & \\ & & 1 \end{pmatrix}.$$

We denote by

$$W_n(g;\varphi') := \int_{N'(\mathbb{Q})\backslash N'(\mathbb{A})} \varphi'(ug)\overline{\psi_n(u)}du$$

the n-Whittaker function of φ'. By $G'(\mathbb{Q})$-invariance of φ' and the identity

$$\iota^{-1}(u(x)) = \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix},$$

for $\iota : \mathrm{SL}_2 \simeq SU(W)$ the exceptional isomorphism discussed in §4.5.2, we have

$$W_n(g;\varphi') = W_1(a(n)g;\varphi_1),$$

the Whittaker function associated to φ_1 relative to the character θ and the Fourier expansion

$$\varphi'(ug) = \sum_{n\in\mathbb{Q}^\times} W_n(g;\varphi')\psi_n(u). \tag{8.4}$$

Substituting (8.4) into the expression (8.1) of $\mathcal{O}_{\gamma(1)}(f)$ we then get

$$\begin{aligned}\mathcal{O}_{\gamma(1)}(f,\varphi') &= \int_{N'(\mathbb{A})\backslash G'(\mathbb{A})}\int_{G'(\mathbb{A})} f\left(x^{-1}\gamma(1)y\right)\sum_{n\in\mathbb{Q}} \overline{W_n}(x;\varphi')W_n(y;\varphi')dxdy \\ &= \sum_{n\in\mathbb{Q}}\int_{N'(\mathbb{A})\backslash G'(\mathbb{A})}\int_{G'(\mathbb{A})} f\left(x^{-1}\gamma(1)y\right)\overline{W_n}(x;\varphi')W_n(y;\varphi')dxdy.\end{aligned}$$

Since ϕ' has been chosen to be a primitive cusp form, φ' is decomposable. Let

$$\varphi' \simeq \otimes_v \xi_v' \in \otimes_v' V_{\pi_v'}.$$

Decomposing ψ_n into local characters

$$\psi_n = \prod_v \psi_{n,v}$$

yields a decomposition of the corresponding Whittaker function ($g = (g_v)_v \in G'(\mathbb{A})$)

$$W_n(g;\varphi') = \prod_v W_{n,v}(g_v;\xi_v') = \prod_v W_{1,v}(a(n)g_v;\xi_v'), \tag{8.5}$$

where the $W_{n,v}(g_v;\xi_v') = W_{1,v}(a(n)g_v;\xi_v')$ are the local Whittaker functions which satisfy

$$W_{n,v}(g_v;\xi_v') = W_{n,v}(u_vg_v;\xi_v') = \psi_{n,v}(u_v)W_{n,v}(g_v;\xi_v')$$

for all $u_v \in N'(\mathbb{Q}_v)$, $g_v \in G'(\mathbb{Q}_v)$. Also to simplify notation we write in the rest of this section

$$W_{n,v}(g_v) := W_{1,v}(a(n)g_v;\xi_v') := W_{n,v}(g_v;\xi_v'). \tag{8.6}$$

For p a prime, let $\mathbf{1}_p$ be the identity element in $G'(\mathbb{Q}_p)$ and $\mathbf{1}_f$ for the identity element of $G'(\mathbb{A}_{\mathrm{fin}})$. We choose for all primes p, ξ_p''s to be the local new vector normalized such that

$$W_{1,p}(\mathbf{1}_p;\xi_p') = 1.$$

This normalization will be our choice for ξ_p''s henceforth. Using this decomposition of Whittaker functions we can write $\mathcal{O}_{\gamma(1)}(f)$ into a sum of product of local orbital integrals.

Lemma 8.2. *Let notation be as before. Then*

$$\mathcal{O}_{\gamma(1)}(f,\varphi')=\sum_{n\in\mathbb{Q}^\times}\mathcal{O}(f;n)=\sum_{n\in\mathbb{Q}^\times}\prod_v\mathcal{O}_v(f;n), \tag{8.7}$$

where

$$\mathcal{O}(f;n)=\prod_v\mathcal{O}_v(f;n)$$

and v runs through all the places of $\mathbb{Q}$ and

(8.8)

$$\mathcal{O}_v(f;n):=\int_{N'(\mathbb{Q}_v)\backslash G'(\mathbb{Q}_v)}\int_{G'(\mathbb{Q}_v)}\overline{W_{n,v}}(x_v;\xi_v')W_{n,v}(y_v;\xi_v')f_v\left(x_v^{-1}\gamma(1)y_v\right)dx_v dy_v,$$

Proof. We have

$$\mathcal{O}_{\gamma(1)}(f,\varphi')=\sum_{n\in\mathbb{Q}^\times}\int_{N'(\mathbb{A})\backslash G'(\mathbb{A})}\int_{G'(\mathbb{A})}f\left(x^{-1}\gamma(1)y\right)\overline{W_n}(x;\varphi')W_n(y;\varphi')dxdy.$$

Then (8.7) follows from the factorization of Whittaker functions. □

Remark 8.1. As we will see below n is in fact a non-zero integer.

8.2. **Computation of $\mathcal{O}_\infty(f;n)$.** In this section, we compute compute the local orbital integral $\mathcal{O}_\infty(f;n)$. For this we compute explicitly the archimedean Whittaker functions $W_{n,\infty}(g_\infty)$ and $f_\infty(x_\infty^{-1}\gamma(1)y_\infty)$. Identifying $g_\infty\in G'(\mathbb{R})$ with $g_\infty.\mathbf{1}_f\in G'(\mathbb{A})$ we have

$$W_n(g_\infty;\varphi')=W_{n,\infty}(g_\infty)\prod_p W_{n,p}(\mathbf{1}_p). \tag{8.9}$$

8.2.1. *Whittaker Functions.* Let $g_\infty\in G'(\mathbb{R})$. Write g_∞ into its Iwasawa form:

$$g_\infty=n_\infty a_\infty k_\infty=\begin{pmatrix}1&&-\Delta t\\&1&\\&&1\end{pmatrix}\begin{pmatrix}a&&\\&1&\\&&\overline{a}^{-1}\end{pmatrix}\begin{pmatrix}\cos\alpha&&\Delta\sin\alpha\\&1&\\-\Delta^{-1}\sin\alpha&&\cos\alpha\end{pmatrix},$$

where $\alpha\in[-\pi,\pi)$; $a\in\mathbb{C}^\times$ and $t\in\mathbb{R}$ and

$$\Delta=i\sqrt{|D_E|}\in i\mathbb{R}_{>0}.$$

We also write

$$a_\infty=z_\infty.a_\infty^1=\begin{pmatrix}(a/\overline{a})^{1/2}&&\\&1&\\&&(a/\overline{a})^{1/2}\end{pmatrix}\begin{pmatrix}(a\overline{a})^{1/2}&&\\&1&\\&&(a\overline{a})^{-1/2}\end{pmatrix}$$

where z_∞ is in the center $Z_{G'}(\mathbb{R})$ and a_∞^1 has determinant 1.

Lemma 8.3. *Let notation be as before. Then*

$$W_{n,\infty}(g_\infty)\cdot\prod_{p<\infty}W_{n,p}(\mathbf{1}_p)=e^{-ik\alpha}e^{-2\pi na\overline{a}+2\pi nit}\cdot(a\overline{a})^{k/2}a_n. \tag{8.10}$$

where a_n denote the n-th Fourier coefficient of the classical form ϕ' (cf. (4.39)).

Proof. By (4.36), we have (since φ' is invariant by $Z_{G'}(\mathbb{R})$)

$$\begin{aligned}
W_n(g_\infty;\varphi') =&\psi_{n,\infty}(n_\infty)\int_{N'(\mathbb{Z})\backslash N'(\mathbb{R})}\varphi'(u_\infty a_\infty^1 k_\infty)\overline{\psi_{n,\infty}(u_\infty)}du_\infty\\
=&\psi_{n,\infty}(n_\infty)\int_{N'(\mathbb{Z})\backslash N'(\mathbb{R})}\frac{\phi'(\iota^{-1}(u_\infty a_\infty^1 k_\infty).i)}{j(\iota^{-1}(u_\infty a_\infty^1 k_\infty),i)^k}\overline{\psi_{n,\infty}(u_\infty)}du_\infty\\
=&\frac{\psi_{n,\infty}(n_\infty)}{j(\iota^{-1}(a_\infty^1),i)^k j(\iota^{-1}(k_\infty),i)^k}\int_{N'(\mathbb{Z})\backslash N'(\mathbb{R})}\phi'(\iota^{-1}(u_\infty a_\infty^1).i)\overline{\psi_{n,\infty}(u_\infty)}du_\infty\\
=&\frac{\psi_{n,\infty}(n_\infty)}{j(\iota^{-1}(a_\infty^1),i)^k j(\iota^{-1}(k_\infty),i)^k}\int_0^1\phi'(a\overline{a}i+t)e^{-2\pi nit}dt\\
=&\frac{\psi_{n,\infty}(n_\infty)}{(a\overline{a})^{-k/2}e^{ik\alpha}}\int_0^1\phi'(a\overline{a}i+t)e^{-2\pi nit}dt.
\end{aligned}$$

Then (8.10) follows from (8.9), (4.39) and the equality $\psi_{n,\infty}(n_\infty)=e^{2\pi nit}$. □

8.2.2. *The archimedean test tunction.* Let $x_\infty, y_\infty\in G'(\mathbb{R})$ written in their Iwasawa forms:

$$x_\infty=\begin{pmatrix}e^{i\alpha_1}&&\\&1&\\&&e^{i\alpha_1}\end{pmatrix}\begin{pmatrix}a_1&&\\&1&\\&&a_1^{-1}\end{pmatrix}\begin{pmatrix}\cos\alpha&&\Delta\sin\alpha\\&1&\\-\Delta^{-1}\sin\alpha&&\cos\alpha\end{pmatrix};$$

$$y_\infty=\begin{pmatrix}e^{i\alpha_2}&&\\&1&\\&&e^{i\alpha_2}\end{pmatrix}\begin{pmatrix}a_2&&-a_2^{-1}\Delta t\\&1&\\&&a_2^{-1}\end{pmatrix}\begin{pmatrix}\cos\beta&&\Delta\sin\beta\\&1&\\-\Delta^{-1}\sin\beta&&\cos\beta\end{pmatrix},$$

where $\alpha_1,\alpha_2,\alpha,\beta\in[-\pi,\pi)$; $a_1,a_2\in\mathbb{R}^\times$ and $t\in\mathbb{R}$.

Lemma 8.4. *With the notations above, we have*

$$f_\infty(x_\infty^{-1}\gamma(1)y_\infty)=2^k\left[(a_1^{-1}a_2+a_1a_2^{-1})-a_1^{-1}a_2^{-1}i(t+\Delta^{-1}/2)\right]^{-k}\cdot e^{ik(\alpha-\beta)}.$$

Proof. It follows from the definition that $f_\infty(x_\infty^{-1}\gamma(1)y_\infty)$ does not depend on $\alpha_i,\ i=1,2$ so we may assume that $\alpha_1=\alpha_2=0$.

Let $t'=t-\Delta^{-1}/2$. Computing the matrices one obtains

$$x_\infty=\begin{pmatrix}a_1\cos\alpha&&a_1\Delta\sin\alpha\\&1&\\-a_1^{-1}\Delta^{-1}\sin\alpha&&a_1^{-1}\cos\alpha\end{pmatrix},$$

$$y_\infty=\begin{pmatrix}a_2\cos\beta+a_2^{-1}t\sin\beta&&a_2\Delta\sin\beta-a_2^{-1}\Delta t\cos\beta\\&1&\\-a_2^{-1}\Delta^{-1}\sin\beta&&a_2^{-1}\cos\beta\end{pmatrix}.$$

We set

$$a=a_1\cos\alpha,\ b=a_1\Delta\sin\alpha,\ c=-a_1^{-1}\Delta^{-1}\sin\alpha,\ d=a_1^{-1}\cos\alpha;$$

and

$$a'=a_2\cos\beta+a_2^{-1}t\sin\beta,\ b'=a_2\Delta\sin\beta-a_2^{-1}\Delta t\cos\beta,$$
$$c'=-a_2^{-1}\Delta^{-1}\sin\beta,\ d'=a_2^{-1}\cos\beta.$$

With these notations we have that $x_\infty^{-1}\gamma(1)y_\infty$ is equal to

$$\begin{aligned}
\begin{pmatrix}\overline{d}&&\overline{b}\\&1&\\\overline{c}&&\overline{a}\end{pmatrix}\gamma(1)\begin{pmatrix}a'&&b'\\&1&\\c'&&d'\end{pmatrix}&=\begin{pmatrix}d&&-b\\&1&\\-c&&a\end{pmatrix}\gamma(1)\begin{pmatrix}a'&&b'\\&1&\\c'&&d'\end{pmatrix}\\
&=\begin{pmatrix}d(a'-\frac{c'}{2})-bc'&d&d(b'-\frac{d'}{2})-bd'\\-c'&1&-d'\\-c(a'-\frac{c'}{2})+ac'&-c&-c(b'-\frac{d'}{2})+ad'\end{pmatrix}.
\end{aligned}$$

Then by definition (4.19) we have (recall that $k_1=-k,\ k_2=k/2$)

$$f_\infty(x_\infty^{-1}\gamma(1)y_\infty)=2^k\cdot(\overline{A}-\overline{B})^{-k}, \tag{8.11}$$

where

$$A=d(a'-\frac{c'}{2})-bc'-c(b'-\frac{d'}{2})+ad'$$

and

$$B=\big[d(b'-\frac{d'}{2})-bd'\big]\cdot|D_E|^{-1/2}+\big[-c(a'-\frac{c'}{2})+ac'\big]\cdot|D_E|^{1/2}.$$

Substituting expressions of a,b,c,d and a',b',c',d' we then have

$$A=(a_1^{-1}a_2+a_1a_2^{-1})\cos(\alpha-\beta)-a_1^{-1}a_2^{-1}t'\sin(\alpha-\beta). \tag{8.12}$$

Then

$$B=-i\Big[(a_1^{-1}a_2+a_1a_2^{-1})\sin(\alpha-\beta)+a_1^{-1}a_2^{-1}t'\cos(\alpha-\beta)\Big]. \tag{8.13}$$

so that

$$A-B=\big[(a_1^{-1}a_2+a_1a_2^{-1})+a_1^{-1}a_2^{-1}it'\big]\cdot e^{i(\alpha-\beta)}.$$

Hence the Lemma follows from (8.11), (8.12), (8.13) and this last identity. □

8.2.3. *Orbital Integrals.* In this subsection, we will combine Lemma 8.3 and Lemma 8.4 to compute the archimedean unipotent orbital integral. We set

$$|W_{n,f}(\mathbf{1})|^2:=\prod_{p<\infty}|W_{n,p}(\mathbf{1}_p)|^2 \tag{8.14}$$

Proposition 8.5. *Let notation be as before. If $|W_{n,f}(\mathbf{1})|^2=0$ we have*

$$\mathcal{O}_\infty(f;n)=0.$$

Otherwise we have

$$\begin{aligned}\frac{\mathcal{O}_\infty(f;n)}{|W_{n,f}(\mathbf{1})|^2}&=\frac{2^3\pi^2}{2^{4(k-1)}}\frac{\Gamma(k-1)^2}{\Gamma(k)}\frac{|a_n|^2}{(4\pi n)^{k-1}}e^{-\pi nD_E^{-1/2}}\\&=\frac{2^4\pi^5}{3.2^{4(k-1)}}\frac{1}{(k-1)^2}\prod_{p|N'}(1+\frac{1}{p})\frac{|\lambda_{\pi'}(n)|^2}{L(\pi',\mathrm{Ad},1)}e^{-\pi nD_E^{-1/2}}.\end{aligned} \tag{8.15}$$

In particular we have

$$\mathcal{O}_\infty(f;n)\leq\frac{(kN'n)^{o(1)}}{2^{4k}k^2}e^{-\pi nD_E^{-1/2}}\prod_{p<\infty}|W_{n,p}(\mathbf{1}_p)|^2. \tag{8.16}$$

Proof. Suppose $|W_{n,f}(\mathbf{1})|^2\neq0$. We recall that

$$\frac{\mathcal{O}_\infty(f;n)}{|W_{n,f}(\mathbf{1})|^2}=\int_{N'(\mathbb{R})\backslash G'(\mathbb{R})}\int_{G'(\mathbb{R})}\frac{\overline{W_{n,\infty}(x)}W_{n,\infty}(y)}{|W_{n,f}(\mathbf{1})|^2}f_\infty\left(x^{-1}\gamma(1)y\right)dxdy.$$

By Lemma 8.3 and Lemma 8.4 we have

$$\begin{aligned}\frac{\mathcal{O}_\infty(f;n)}{|W_{n,f}(\mathbf{1})|^2}=&2^k|a_n|^2\int_{\mathbb{R}}\int_{\mathbb{R}_+}\int_{\mathbb{R}_+}\int_0^{2\pi}\int_0^{2\pi}e^{-2\pi n(a_1^2+a_2^2)-2\pi nit}\\&\frac{a_1^ka_2^k}{\left[(a_1^{-1}a_2+a_1a_2^{-1})-a_1^{-1}a_2^{-1}i(t+\Delta^{-1}/2)\right]^k}\frac{d\alpha d\beta}{4\pi^2}\frac{da_1}{a_1^3}\frac{da_2}{a_2^3}dt.\end{aligned}$$

Since $k=|k_1|\geq 8$, the integral $\mathcal{O}_\infty(f;n)$ converges absolutely. After passing to polar coordinates, $a_1^2+a_2^2=r^2(\cos^2\theta+\sin^2\theta)$, we obtain

$$\begin{aligned}\frac{\mathcal{O}_\infty(f;n)}{|W_{n,f}(\mathbf{1})|^2}=&2^k|a_n|^2\cdot\int_{\mathbb{R}_+}\int_{\mathbb{R}_+}\int_{\mathbb{R}}\frac{e^{-2\pi n(a_1^2+a_2^2)-2\pi nit}\cdot(a_1a_2)^{2k}}{\left[a_1^2+a_2^2-i(t+\Delta^{-1}/2)\right]^k}dt\frac{da_1}{a_1^3}\frac{da_2}{a_2^3}\\
=&2^k|a_n|^2e^{\pi in\Delta^{-1}}\iint_{\mathbb{R}_+^2}\int_{\mathbb{R}}\frac{e^{-2\pi n(a_1^2+a_2^2)-2\pi nit}\cdot(a_1a_2)^{2k}}{(a_1^2+a_2^2-it)^k}dt\frac{da_1}{a_1^3}\frac{da_2}{a_2^3}\\
=&2^{k-2k+3}|a_n|^2e^{\pi in\Delta^{-1}}\int_0^\infty\int_0^{\frac{\pi}{2}}\int_{\mathbb{R}}\frac{e^{-2\pi nr^2-2\pi nit}\cdot(r^22\cos\theta\sin\theta)^{2k-3}}{(r^2-it)^k}dtrdrd\theta\\
=&\frac{2}{2^{k-1}}|a_n|^2e^{\pi in\Delta^{-1}}\int_0^\pi(\sin\theta)^{2k-3}d\theta\cdot\int_0^\infty\int_{\mathbb{R}}\frac{e^{-2\pi nr-2\pi nit}\cdot r^{2k-3}}{(r-it)^k}dtdr.\end{aligned}$$

after making the changes of variable $2\theta\leftrightarrow\theta$, $r^2\leftrightarrow r$.

We have

$$\int_0^\pi\sin^{2k-3}\theta d\theta=\pi^{1/2}\frac{\Gamma(k-1)}{\Gamma(k-1/2)}.$$

Appealing to Cauchy integral formula we then obtain

$$\begin{aligned}\frac{1}{e^{\pi in\Delta^{-1}}}\frac{\mathcal{O}_\infty(f;n)}{|W_{n,f}(\mathbf{1})|^2}=&\frac{\pi^{1/2}}{2^{k-2}}|a_n|^2\frac{\Gamma(k-1)}{\Gamma(k-1/2)}\frac{(-1)^k}{i}\int_0^\infty\int_{i\mathbb{R}}\frac{e^{-2\pi nr-2\pi nz}\cdot r^{2k-3}}{(z-r)^k}dzdr\\
=&\frac{2\pi^{1/2}}{2^{k-1}}|a_n|^2\frac{\Gamma(k-1)}{\Gamma(k-1/2)}\cdot\frac{2\pi(-1)^{k-1}}{(k-1)!}\int_0^\infty\frac{r^{2k-3}}{e^{2\pi nr}}\left[\frac{d^{k-1}e^{-2\pi nz}}{dz^{k-1}}\right]_{z=r}dr\\
=&\frac{2\pi^{1/2}}{2^{k-1}}|a_n|^2\frac{\Gamma(k-1)}{\Gamma(k-1/2)}\frac{2\pi(-1)^{k-1}}{\Gamma(k)}(-2\pi n)^{k-1}\cdot\int_0^\infty e^{-4\pi nr}\cdot r^{2k-2}\frac{dr}{r}\\
=&\frac{2\pi^{1/2}}{2^{k-1}}|a_n|^2\frac{\Gamma(k-1)}{\Gamma(k-1/2)}\frac{2\pi}{\Gamma(k)}\frac{(2\pi n)^{k-1}}{(4\pi n)^{2(k-1)}}\cdot\Gamma(2(k-1))\\
=&\frac{2\pi^{1/2}}{2^{k-1}}|a_n|^2\frac{\Gamma(k-1)}{\Gamma(k-1/2)}\frac{2\pi}{\Gamma(k)}\frac{(2\pi n)^{k-1}}{(4\pi n)^{2(k-1)}}\pi^{1/2}2^{1-2(k-1)}\Gamma(k-1)\Gamma(k-1/2)\\
=&\frac{2^3\pi^2}{2^{4(k-1)}}\frac{\Gamma(k-1)^2}{\Gamma(k)}\frac{|a_n|^2}{(4\pi n)^{k-1}}\end{aligned}$$

on using the duplication formula

$$\Gamma(2(k-1))=\pi^{1/2}2^{1-2(k-1)}\Gamma(k-1)\Gamma(k-1/2).$$

Then formula (8.15) follows from (4.40) and the bound (8.16) results from (4.41) and (4.44). ☐

Remark 8.2. The reason for this normalization by the factor

$$|W_{n,f}(\mathbf{1})|^2:=\prod_{p<\infty}|W_{n,p}(\mathbf{1}_p)|^2$$

is that as we will see in the forthcoming lemma, for any n, the local orbital integrals $\mathcal{O}_p(f;n)$ appearing in Lemma 8.2 are equal to $|W_{n,p}(\mathbf{1}_p)|^2$ for almost every p.

8.3. **Computation of $\mathcal{O}_p(f;n)$ when π'_p is unramified.** In this section and the next ones, we compute the local orbital integrals over nonarchimedean places.

Let p be a rational prime we denote by ν the usual p-adic valution.

In this subsection, we will assume $p\nmid N'$. Thus ξ'_p is a spherical vector.

The next three lemmatas consider this situation when

- p splits in E.
- p is inert in E and π_p is unramified (ie. $p\not|N$)
- p is inert in E and π_p is ramified (ie. $p|N$).

- p is ramified in E.

For the sequel we denote by ν the usual valuation on $\mathbb{Q}_p$.

Lemma 8.6. *Let notation be as before. Let p be a prime split in E and coprime to N. Under the isomorphism*

$$G'(\mathbb{Q}_p)\simeq \mathrm{GL}(2,\mathbb{Q}_p),$$

π_p' is isomorphic to a principal series representation

$$\pi_p'\simeq \mathrm{Ind}(\chi\otimes\overline{\chi})$$

for some unramified unitary character χ. We have

(8.17)
$$\mathcal{O}_p(f;n)=|W_{n,p}(\mathbf{1}_p)|^2\frac{e^{2\pi n i r_p(-\frac{1}{2\Delta})}}{p^{\nu(n)}}\cdot\sum_{l=0}^{\nu(n)}(l+1)\cdot\left|\frac{\chi(p)^{\nu(n)-l+1}-\overline{\chi}(p)^{\nu(n)-l+1}}{\chi(p)^{\nu(n)+1}-\overline{\chi}(p)^{\nu(n)+1}}\right|^2.$$

In particular, (8.17) equals zero if $\nu(n)<0$ and equals 1 if $\nu(n)=0$.

Proof. Given $g_p\in G'(\mathbb{Q}_p)$ with the Iwasawa decomposition $g_p=u_pa_p\kappa_p$. Then

$$W_{n,p}(g_p)=|n|_p^{-1/2}\psi_{n,p}(u_p)\overline{\chi}(n)W_{1,p}\left(\begin{pmatrix}n&&\\&1&\\&&1\end{pmatrix}a_p\right). \tag{8.18}$$

Write $a_p=\mathrm{diag}(p^{j_1},1,p^{j_2})$, where $(j_1,j_2)\in\mathbb{Z}^2$. Then it follows from (8.18) that $W_{n,p}(g_p)=0$ unless $j_1\geq j_2-\nu(n)$.

Let $x_p\in N'(\mathbb{Q}_p)\backslash G'(\mathbb{Q}_p)$ and $y_p\in G'(\mathbb{Q}_p)$. We can write them into their Iwasawa coordinates:

$$x_p=\begin{pmatrix}p^{i_1}&&\\&1&\\&&p^{i_2}\end{pmatrix}\kappa_1,\quad y_p=\begin{pmatrix}1&&\Delta t\\&1&\\&&1\end{pmatrix}\begin{pmatrix}p^{j_1}&&\\&1&\\&&p^{j_2}\end{pmatrix}\kappa_2,$$

where $(i_1,i_2)\in\mathbb{Z}^2$, $(j_1,j_2)\in\mathbb{Z}^2$; $t\in\mathbb{Q}_p$ and $\kappa_1,\kappa_2\in G'(\mathbb{Z}_p)$.

Since $p\nmid N$, $\iota(\kappa_1),\iota(\kappa_2)\in K_p=\mathrm{GL}(3,\mathbb{Z}_p)$ via the natural embedding (2.3) and in this case we have

$$f_p=\mathbf{1}_{\mathrm{GL}(3,\mathbb{Z}_p)}.$$

Hence $f_p(x_p^{-1}\gamma(1)y_p)\neq 0$ if and only if

$$\begin{pmatrix}p^{-i_1}&&\\&1&\\&&p^{-i_2}\end{pmatrix}\begin{pmatrix}1&1&-1/2\\&1&-1\\&&1\end{pmatrix}\begin{pmatrix}p^{j_1}&&p^{j_2}\Delta t\\&1&\\&&p^{j_2}\end{pmatrix}\in\mathrm{GL}(3,\mathbb{Z}_p).$$

which, by a direct computation, can be further reduced to

$$\begin{pmatrix}p^{j_1-i_1}&p^{-i_1}&p^{j_2-i_1}(\Delta t-\frac{1}{2})\\&1&-p^{j_2}\\&&p^{j_2-i_2}\end{pmatrix}\in\mathrm{GL}(3,\mathbb{Z}_p).$$

Hence $j_1=i_1\leq 0$ and $j_2=i_2\geq 0$. By the support of Whittaker functions we have

$$\overline{W_{n,p}}(x_p)W_{n,p}(y_p)=0$$

unless $i_1\geq i_2-\nu(n)$ and $j_1\geq j_2-\nu(n)$. Hence $\mathcal{O}_p(f;n)=0$ unless

$$0\geq i_1\geq i_2-\nu(n)\geq-\nu(n).$$

In particular we have $\nu(n)\geq 0$.

Since $\mathrm{vol}(G'(\mathbb{Z}_p))=1$. Therefore, we have

$$\begin{aligned}\mathcal{O}_p(f;n)=&\int_{N'(\mathbb{Q}_p)\backslash G'(\mathbb{Q}_p)}\int_{G'(\mathbb{Q}_p)}f_p(x_p^{-1}\gamma(1)y_p)\overline{W_{n,p}}(x_p)W_{n,p}(y_p)dx_pdy_p\\=&|n|_p^{-1}\sum_{i_1=-\nu(n)}^{0}\sum_{i_2=0}^{i_1+\nu(n)}p^{2i_1-2i_2}I_p(i_1,i_2)\Bigg|W_{1,p}\begin{pmatrix}np^{i_1}&&\\&1&\\&&p^{i_2}\end{pmatrix}\Bigg|^2,\end{aligned}$$

where

$$I_p(i_1,i_2):=\int_{\mathbb{Q}_p}\mathbf{1}_{\mathbb{Z}_p}(p^{i_2-i_1}(\Delta t-1/2))\theta_{n,p}(t)dt.$$

Since $\Delta\in\mathbb{Z}_p{}^\times$ and $i_2-i_1\le\nu(n)$, we have

$$I_p(i_1,i_2)=e^{2\pi nir_p(-\Delta^{-1}/2)}p^{i_2-i_1}.$$

Hence we have

$$\mathcal{O}_p(f;n)=\frac{e^{2\pi nir_p(-\Delta^{-1}/2)}}{|n|_p}\cdot\sum_{l=0}^{\nu(n)}(\nu(n)-l+1)p^{l-\nu(n)}\Bigg|W_{1,p}\left(\iota\begin{pmatrix}p^l&\\&1\end{pmatrix}\right)\Bigg|^2.$$

Substituting Casselman-Shalika formula (cf. [CS80]) into the above computation we then obtain

$$\begin{aligned}\frac{\mathcal{O}_p(f;n)}{e^{2\pi nir_p(-\frac{1}{2\Delta})}}=&\frac{|W_{1,p}(\mathbf{1}_p)|^2}{|n|_p}\cdot\sum_{l=0}^{\nu(n)}(\nu(n)-l+1)p^{l-\nu(n)}\cdot\left|\frac{p^{-\frac{l}{2}}(\chi(p)^{l+1}-\overline{\chi}(p)^{l+1})}{\chi(p)-\overline{\chi}(p)}\right|^2\\=&|W_{n,p}(\mathbf{1}_p)|^2\cdot\sum_{l=0}^{\nu(n)}(l+1)p^{-\nu(n)}\cdot\left|\frac{\chi(p)^{\nu(n)-l+1}-\overline{\chi}(p)^{\nu(n)-l+1}}{\chi(p)^{\nu(n)+1}-\overline{\chi}(p)^{\nu(n)+1}}\right|^2.\end{aligned}$$

Then (8.17) follows. □

Lemma 8.7. *Let notation be as before. Let p be a rational prime inert in E and such that $p\nmid N'$.*

For

$$f_p=1_{G(\mathbb{Z}_p)},$$

we have

$$\mathcal{O}_p(f;n)=|W_{n,p}(\mathbf{1}_p)|^2\sum_{-\nu(n)/2\le i\le 0}\left|\frac{\chi(p)^{\nu(n)+2i+1}-\overline{\chi}(p)^{\nu(n)+2i+1}}{\chi(p)^{\nu(n)+1}-\overline{\chi}(p)^{\nu(n)+1}}\right|^2.\tag{8.19}$$

In particular, (8.19) equal 0 if $\nu(n)<0$ and equals 1 if $\nu(n)=0$.

For

$$f_p=1_{G(\mathbb{Z}_p)A_rG(\mathbb{Z}_p)},$$

for $r\ge1$ (cf. (A.12) in the Appendix), we have

$$\mathcal{O}_p(f;n)\ll(r+\nu(n))^4p^r|W_{n,p}(\mathbf{1}_p)|^2,\tag{8.20}$$

where the implied constant is absolute.

Proof. By our assumptions

$$K'_p(N)=G'(\mathbb{Z}_p)=U(L\otimes_{\mathbb{Q}}\mathbb{Q}_p).$$

Let $g_p\in G'(\mathbb{Q}_p)$. Write $g_p=u_pa_pk_p$ be the Iwasawa decomposition of g_p. Then one has

$$W_{n,p}(g_p)=W_{n,p}(u_pa_pk_p)=\psi_{n,p}(u_p)W_{n,p}(a_p).\tag{8.21}$$

For $a_p=\operatorname{diag}(p^i,1,p^{-i})$, where $i\in\mathbb{Z}$, we have, by Casselman-Shalika formula, that (8.22)

$$W_{n,p}\begin{pmatrix}p^i&&\\&1&\\&&p^{-i}\end{pmatrix}=\frac{\mathbf{1}_{\mathcal{O}_{E_p}}(np^{2i})}{p^i}\cdot\frac{\chi(p)^{\nu(n)+2i+1}-\overline{\chi}(p)^{\nu(n)+2i+1}}{\chi(p)^{\nu(n)+1}-\overline{\chi}(p)^{\nu(n)+1}}\cdot W_{n,p}(\mathbf{1}_p).$$

Let $x_p\in N'(\mathbb{Q}_p)\backslash G'(\mathbb{Q}_p)$ and $y_p\in G'(\mathbb{Q}_p)$. We can write them into their Iwasawa coordinates:

$$x_p=\begin{pmatrix}p^i&&\\&1&\\&&p^{-i}\end{pmatrix}\kappa_1,\quad y_p=\begin{pmatrix}1&&\Delta t\\&1&\\&&1\end{pmatrix}\begin{pmatrix}p^j&&\\&1&\\&&p^{-j}\end{pmatrix}\kappa_2,$$

for $i,j\in\mathbb{Z}^2$; $t\in\mathbb{Q}_p$ and $\kappa_1,\kappa_2\in K_p'(N)$.

By definition, $f_p=\mathbf{1}_{K_p}$. Noting that $p\nmid N$, then $\kappa_1,\kappa_2\in K_p$ via the natural embedding (2.3). Hence $f_p(x_p^{-1}\gamma(1)y_p)\neq 0$ if and only if

$$\begin{pmatrix}p^{-i}&&\\&1&\\&&p^i\end{pmatrix}\begin{pmatrix}1&1&-1/2\\&1&-1\\&&1\end{pmatrix}\begin{pmatrix}1&&\Delta t\\&1&\\&&1\end{pmatrix}\begin{pmatrix}p^j&&\\&1&\\&&p^{-j}\end{pmatrix}\in G(\mathbb{Z}_p),$$

which could be further reduced to

$$\begin{pmatrix}p^{j-i}&p^{-i}&p^{-j-i}(\Delta t-\frac{1}{2})\\&1&-p^{-j}\\&&p^{-j+i}\end{pmatrix}\in G(\mathbb{Z}_p).$$

Hence $j=i\leq 0$. By the support of Whittaker functions we have

$$\overline{W_{n,p}}(x_p)W_{n,p}(y_p)=0$$

unless

$$\nu(n)+2i\geq 0,\ \ \nu(n)+2j\geq 0.$$

In particular $\nu(n)\geq 0$.

It follows that $\mathcal{O}_p(f;n)=0$ unless

$$i=j\geq -\nu(n)/2.$$

Therefore, we have

$$\begin{aligned}\mathcal{O}_p(f;n)&=\int_{N'(\mathbb{Q}_p)\backslash G'(\mathbb{Q}_p)}\int_{G'(\mathbb{Q}_p)}f_p(x_p^{-1}\gamma(1)y_p)\overline{W_{n,p}}(x_p)W_{n,p}(y_p)dx_pdy_p\\&=\sum_{-\nu(n)/2\leq i\leq 0}p^{4i}\int_{\mathbb{Q}_p}\mathbf{1}_{\mathcal{O}_{E_p}}(p^{-2i}(\Delta t-1/2))\theta_{n,p}(t)dt\cdot\left|W_{n,p}\begin{pmatrix}p^i&&\\&1&\\&&p^{-i}\end{pmatrix}\right|^2.\end{aligned}$$

The condition $p^{-2i}(\Delta t-1/2)\in\mathcal{O}_{E_p}$ is equivalent to $t\in p^{2i}\mathbb{Z}_p$ and since $-2i\leq\nu(n)$, we have

$$\int_{\mathbb{Q}_p}\mathbf{1}_{\mathcal{O}_{E_p}}(p^{-2i}(\Delta t-1/2))\theta_{n,p}(t)dt=p^{-2i}.$$

This in conjunction with formula (8.22) implies that

$$\frac{\mathcal{O}_p(f;n)}{|W_{n,p}(\mathbf{1}_p)|^2}=\sum_{-\nu(n)/2\leq i\leq 0}\left|\frac{\chi(p)^{\nu(n)+2i+1}-\overline{\chi}(p)^{\nu(n)+2i+1}}{\chi(p)^{\nu(n)+1}-\overline{\chi}(p)^{\nu(n)+1}}\right|^2,$$

proving (8.19).

Now let's take $f_p = 1_{G(\mathbb{Z}_p)A_rG(\mathbb{Z}_p)}$, for $r \geq 1$. Then $f_p(x_p^{-1}\gamma(1)y_p) \neq 0$ if and only if

$$\begin{pmatrix} p^{-i} & & \\ & 1 & \\ & & p^i \end{pmatrix}\begin{pmatrix} 1 & 1 & -1/2 \\ & 1 & -1 \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \Delta t \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} p^j & & \\ & 1 & \\ & & p^{-j} \end{pmatrix} \in G(\mathbb{Z}_p)A_rG(\mathbb{Z}_p),$$

which could be further reduced to $t \in \mathcal{E}(i,j;r)$. Here $\mathcal{E}(i,j;r)$ is defined by

$$\left\{t \in \mathbb{Q}_p : \begin{pmatrix} p^{j-i} & p^{-i} & p^{-j-i}(\Delta t - \frac{1}{2}) \\ & 1 & -p^{-j} \\ & & p^{-j+i} \end{pmatrix} \in G(\mathbb{Z}_p)\begin{pmatrix} p^r & & \\ & 1 & \\ & & p^{-r} \end{pmatrix} G(\mathbb{Z}_p)\right\}.$$

Note that $\mathcal{E}(i,j;r)$ is empty unless $i \leq r$, $j \leq r$, and $\Delta t - 1/2 \in p^{i+j-r}\mathbb{Z}_p$.

Considering the support of Whittaker functions as above, it follows that

$$\mathcal{O}_p(f;n) = \int_{N'(\mathbb{Q}_p)\backslash G'(\mathbb{Q}_p)}\int_{G'(\mathbb{Q}_p)} f_p(x_p^{-1}\gamma(1)y_p)\overline{W_{n,p}}(x_p)W_{n,p}(y_p)dx_pdy_p$$

is equal to

$$\sum_{\substack{-\frac{\nu(n)}{2}\leq i\leq r \\ -\frac{\nu(n)}{2}\leq j\leq r}} p^{2(i+j)}\int_{\mathbb{Q}_p} \mathbf{1}_{\mathcal{E}(i,j;r)}(t)\theta_{n,p}(t)dtW_{n,p}\begin{pmatrix} p^i & & \\ & 1 & \\ & & p^{-i} \end{pmatrix}\overline{W_{n,p}\begin{pmatrix} p^j & & \\ & 1 & \\ & & p^{-j} \end{pmatrix}}.$$

Appealing to the triangle inequality, $|\mathcal{O}_p(f;n)|$ is

$$\leq \sum_{\substack{-\frac{\nu(n)}{2}\leq i\leq r \\ -\frac{\nu(n)}{2}\leq j\leq r}} p^{2(i+j)}\int_{\mathbb{Q}_p} \mathbf{1}_{\mathcal{E}(i,j;r)}(t)dt\left|W_{n,p}\begin{pmatrix} p^i & & \\ & 1 & \\ & & p^{-i} \end{pmatrix}\overline{W_{n,p}\begin{pmatrix} p^j & & \\ & 1 & \\ & & p^{-j} \end{pmatrix}}\right|$$

$$\leq \sum_{\substack{-\frac{\nu(n)}{2}\leq i\leq r \\ -\frac{\nu(n)}{2}\leq j\leq r}} p^r\left|\frac{\chi(p)^{\nu(n)+2i+1}-\overline{\chi}(p)^{\nu(n)+2i+1}}{\chi(p)^{\nu(n)+1}-\overline{\chi}(p)^{\nu(n)+1}}\cdot\frac{\chi(p)^{\nu(n)+2j+1}-\overline{\chi}(p)^{\nu(n)+2j+1}}{\chi(p)^{\nu(n)+1}-\overline{\chi}(p)^{\nu(n)+1}}\right||W_{n,p}(\mathbf{1}_p)|^2.$$

Hence (8.20) follows. □

If $p \mid N$ then p is inert in E and $\pi_p \simeq \mathrm{St}_p$ is the Steinberg representation. From (4.23) we have

$$G(\mathbb{Z}_p) = U(\Gamma\otimes_{\mathbb{Q}}\mathbb{Q}_p)\text{and } G'(\mathbb{Z}_p) = U(L\otimes_{\mathbb{Q}}\mathbb{Q}_p).$$

Explicitly,

$$G(\mathbb{Z}_p) = \{g \in G(\mathbb{Q}_p) :\ g.\Gamma\otimes_{\mathbb{Q}}\mathbb{Q}_p = \Gamma\otimes_{\mathbb{Q}}\mathbb{Q}_p\} = G(\mathbb{Q}_p)\cap \mathrm{GL}(3,\mathcal{O}_{E_p}),$$
$$G'(\mathbb{Z}_p) = \{g \in G'(\mathbb{Q}_p) :\ g.L\otimes_{\mathbb{Q}}\mathbb{Q}_p = L\otimes_{\mathbb{Q}}\mathbb{Q}_p\} = G'(\mathbb{Q})\cap \mathrm{GL}(2,\mathcal{O}_{E_p}).$$

Lemma 8.8. *Let notation be as before. Suppose that $p \mid N$ (in particular p is inert) and $(p,n)=1$. Then*

$$\mathcal{O}_p(f;n) = |W_{n,p}(\mathbf{1}_p)|^2\frac{\mu(I_p')^2}{\mu(I_p)}. \tag{8.23}$$

Proof. Let $x_p \in N'(\mathbb{Q}_p)\backslash G'(\mathbb{Q}_p)$ and $y_p \in G'(\mathbb{Q}_p)$. By Lemma A.1 and Iwasawa decomposition, we can write

$$x_p = \begin{pmatrix} p^i & & \\ & 1 & \\ & & p^{-i} \end{pmatrix}\mu_1\kappa_1',\ y_p = \begin{pmatrix} 1 & & t \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} p^j & & \\ & 1 & \\ & & p^{-j} \end{pmatrix}\mu_2\kappa_2',$$

where $\kappa_1', \kappa_2' \in I_p'$; $i,j\in\mathbb{Z}$; and μ_1, μ_2 are either trivial or of form $\begin{pmatrix} \delta & & 1\\ & 1 & \\ 1 & & \end{pmatrix}$ as in Lemma A.1. By definition, $f_p(x_p^{-1}\gamma(1)y_p)\neq 0$ is equivalent to

$$x_p^{-1}\gamma(1)y_p=\mu_1^{-1}\begin{pmatrix} p^{-i} & & \\ & 1 & \\ & & p^i\end{pmatrix}\begin{pmatrix} 1 & & \Delta t\\ & 1 & \\ & & 1\end{pmatrix}\begin{pmatrix} p^{j} & & \\ & 1 & \\ & & p^{-j}\end{pmatrix}\mu_2\in I_p,$$

which could be further simplified to

$$\mu_1^{-1}\begin{pmatrix} p^{j-i} & p^{-i} & p^{-j-i}(\Delta t-\frac{1}{2})\\ & 1 & -p^{-j}\\ & & p^{-j+i}\end{pmatrix}\mu_2\in I_p\subseteq G(\mathbb{Z}_p). \tag{8.24}$$

Hence $j=i\leq 0$. However, when $i<0$, from the properties of Whittaker functions (since $(n,p)=1$) we have $W_{n,p}(x_p)=W_{n,p}(\mathrm{diag}(p^i,1,p^{-i}))=0$. Then $\mathcal{O}_p(f;n)=0$ is this case. So $i=j=0$ and (8.24) becomes

$$\mu_1^{-1}\begin{pmatrix} 1 & 1 & \Delta t-\frac{1}{2}\\ & 1 & -1\\ & & 1\end{pmatrix}\mu_2\in I_p. \tag{8.25}$$

If $\mu_1\neq \mathbf{1}_p$ and $\mu_2\neq\mathbf{1}_p$, we may write $\mu_1=\begin{pmatrix} \delta_1 & & 1\\ & 1 & \\ 1 & & \end{pmatrix}$, $\mu_2=\begin{pmatrix} \delta_2 & & 1\\ & 1 & \\ 1 & & \end{pmatrix}$, with $\delta_j+\overline{\delta}_j=0$, $j=1,2$. Since

$$\begin{pmatrix} \delta_1 & & 1\\ & 1 & \\ 1 & & \end{pmatrix}^{-1}\begin{pmatrix} 1 & 1 & \Delta t-\frac{1}{2}\\ & 1 & -1\\ & & 1\end{pmatrix}\begin{pmatrix} \delta_2 & & 1\\ & 1 & \\ 1 & & \end{pmatrix}=\begin{pmatrix} 1 & 0 & 1\\ -1 & 1 & -1\\ \Delta t-\frac{1}{2}-\delta_1+\delta_2 & 1 & 1\end{pmatrix}$$

does not belong to I_p, (8.25) cannot hold. A contradiction! Hence our assumption fails, namely, at least one μ_1 and μ_2 is equal to $\mathbf{1}_p$.

Let $b\in\mathcal{O}_{E_p}^{\times}$. From the following computations

$$\begin{pmatrix} 1 & 1 & \Delta t-\frac{1}{2}\\ & 1 & -1\\ & & 1\end{pmatrix}\begin{pmatrix} \delta_2 & & 1\\ & 1 & \\ 1 & & \end{pmatrix}=\begin{pmatrix} \Delta t-\frac{1}{2}+\delta_2 & 1 & 1\\ -1 & 1 & \\ 1 & & \end{pmatrix}\notin I_p,$$

$$\begin{pmatrix} \delta_1 & & 1\\ & 1 & \\ 1 & & \end{pmatrix}^{-1}\begin{pmatrix} 1 & 1 & \Delta t-\frac{1}{2}\\ & 1 & -1\\ & & 1\end{pmatrix}=\begin{pmatrix} & & 1\\ & 1 & -1\\ 1 & 1 & \Delta t-\frac{1}{2}-\delta_1\end{pmatrix}\notin I_p,$$

we see that in order for (8.25) to hold, one must have $\mu_1=\mu_2=\mathbf{1}_p$.

Therefore,

$$\begin{aligned}\frac{\mathcal{O}_p(f;n)}{|W_{n,p}(\mathbf{1}_p)|^2}=&\int_{N'(\mathbb{Q}_p)\backslash G'(\mathbb{Q}_p)}\int_{G'(\mathbb{Q}_p)}\frac{f_p(x_p^{-1}\gamma(1)y_p)\overline{W_{n,p}}(x_p)W_{n,p}(y_p)}{|W_{n,p}(\mathbf{1}_p)|^2}dx_pdy_p\\ =&\frac{\mu(I_p')^2}{\mu(I_p)}\int_{\mathbb{Q}_p}\mathbf{1}_{\mathcal{O}_{E,p}}(\Delta t-\frac{1}{2})\psi_p(nt)dt=\frac{\mu(I_p')^2}{\mu(I_p)}.\end{aligned}$$

Thus (8.23) follows. □

Lemma 8.9. *Let notation be as before. Suppose that $p\mid(n,N)$. Then we have*

$$\frac{\mathcal{O}_p(f;n)}{|W_{n,p}(\mathbf{1}_p)|^2}=\frac{\mu(I_p')^2}{\mu(I_p)}\cdot\left\{1+2p\sum_{-\nu(n)/2\leq i\leq -1}\left|\frac{\chi(p)^{\nu(n)+2i+1}-\overline{\chi}(p)^{\nu(n)+2i+1}}{\chi(p)^{\nu(n)+1}-\overline{\chi}(p)^{\nu(n)+1}}\right|^2\right\}. \tag{8.26}$$

In particular (8.26) *equals* 0 *if* $\nu(n) \leq 0$.

Proof. By the proof of Lemma 8.8 (we use the same notation here), $f_p(x_p^{-1}\gamma(1)y_p) \neq 0$ is equivalent to (8.24), i.e.,

$$\mu_1^{-1}\begin{pmatrix} p^{j-i} & p^{-i} & p^{-j-i}(\Delta t-\frac{1}{2}) \\ & 1 & -p^{-j} \\ & & p^{-j+i} \end{pmatrix}\mu_2 \in I_p \subseteq G(\mathbb{Z}_p).$$

If $\mu_1 \neq \mathbf{1}_p$ and $\mu_2 \neq \mathbf{1}_p$, we may write $\mu_1 = \begin{pmatrix} \delta_1 & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}$, $\mu_2 = \begin{pmatrix} \delta_2 & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}$, with $\delta_j + \overline{\delta}_j = 0$, $j = 1, 2$. Then the condition (8.24) becomes

$$\begin{pmatrix} 1 & & \\ p^{-i} & 1 & \\ p^{-2i}(\Delta t-\frac{1}{2})+\delta_2-\delta_1 & p^{-i} & 1 \end{pmatrix} \in I_p \subseteq G(\mathbb{Z}_p),$$

which is equivalent to

$$i \leq -1,\ p^{-2i}(\Delta t-\frac{1}{2}) \in \mathcal{O}_{E_p},\ \text{and}\ \delta_1-\delta_2 \equiv p^{-2i}(\Delta t-\frac{1}{2}) \pmod{N_1}.$$

On the other hand, we have

$$\begin{pmatrix} 1 & p^{-i} & p^{-2i}(\Delta t-\frac{1}{2}) \\ & 1 & -p^{-i} \\ & & 1 \end{pmatrix}\begin{pmatrix} \delta_2 & & 1 \\ & 1 & \\ 1 & & \end{pmatrix} = \begin{pmatrix} \frac{\Delta t-1/2}{p^{2i}}+\delta_2 & p^{-i} & 1 \\ -p^{-i} & 1 & \\ 1 & & \end{pmatrix} \notin I_p,$$

$$\begin{pmatrix} \delta_1 & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}^{-1}\begin{pmatrix} 1 & p^{-i} & p^{-2i}(\Delta t-\frac{1}{2}) \\ & 1 & -p^{-i} \\ & & 1 \end{pmatrix} = \begin{pmatrix} & & 1 \\ & 1 & -p^{-i} \\ 1 & p^{-i} & \frac{\Delta t-1/2}{p^{2i}}-\delta_1 \end{pmatrix} \notin I_p,$$

therefore to make (8.25) hold, one must have either $\mu_1 = \mu_2 = \mathbf{1}_p$ or

$$\mu_1 = \begin{pmatrix} \delta_1 & & 1 \\ & 1 & \\ 1 & & \end{pmatrix},\ \mu_2 = \begin{pmatrix} \delta_2 & & 1 \\ & 1 & \\ 1 & & \end{pmatrix},$$

for some $\delta_j \in \mathcal{O}_{E_p}/p\mathcal{O}_{E_p}$, $\delta_j + \overline{\delta}_j = 0$, $j = 1, 2$.

By (8.22) we obtain

$$\begin{aligned}\frac{\mathcal{O}_p(f;n)}{\mu(I_p')^2/\mu(I_p)} &= \iint f_p(x_p^{-1}\gamma(1)y_p)\overline{W_{n,p}}(x_p)W_{n,p}(y_p)dx_pdy_p\\ &= \sum_{\substack{i\geq -\nu(n)/2\\ i\leq 0}}\int_{\mathbb{Q}_p} p^{4i}\mathbf{1}_{\mathcal{O}_{E_p}}(p^{-2i}(\Delta t-1/2))\theta_{n,p}(t)dt\left|W_{n,p}\begin{pmatrix}p^i&&\\&1&\\&&p^{-i}\end{pmatrix}\right|^2\\ &\quad+\sum_{-\nu(n)/2\leq i\leq -1}\int_{\mathbb{Q}_p} p^{4i}\cdot\left|W_{n,p}\begin{pmatrix}p^i&&\\&1&\\&&p^{-i}\end{pmatrix}\right|^2\\ &\qquad\sum_{\substack{\delta_1,\delta_2\in\mathcal{O}_{E_p}/N\mathcal{O}_{E_p}\\ \delta_1+\overline{\delta}_1=0,\ \delta_2+\overline{\delta}_2=0\\ \delta_1-\delta_2\equiv p^{-2i}(\Delta t-\frac{1}{2})\pmod N}}\mathbf{1}_{\mathcal{O}_{E_p}}(p^{-2i}(\Delta t-1/2))\theta_{n,p}(t)dt\\ &=|W_{n,p}(\mathbf{1}_p)|^2\left\{\sum_{-\nu(n)/2\leq i\leq 0}\left|\frac{\chi(p)^{\nu(n)+2i+1}-\overline{\chi}(p)^{\nu(n)+2i+1}}{\chi(p)^{\nu(n)+1}-\overline{\chi}(p)^{\nu(n)+1}}\right|^2\right.\\ &\quad\left.+\sum_{-\nu(n)/2\leq i\leq -1}p\left|\frac{\chi(p)^{\nu(n)+2i+1}-\overline{\chi}(p)^{\nu(n)+2i+1}}{\chi(p)^{\nu(n)+1}-\overline{\chi}(p)^{\nu(n)+1}}\right|^2\right\}.\end{aligned}$$

Then Lemma 8.9 follows. □

Lemma 8.10. *Let notation be as before. Let p be a prime ramified in E. We have*

$$\frac{\mathcal{O}_p(f;n)}{|W_{n,p}(\mathbf{1}_p)|^2}=\sum_{-\nu(n)+1\leq i\leq -2\nu(D_E)}\left|\frac{\chi(p)^{\nu(n)+2i+1}-\overline{\chi}(p)^{\nu(n)+2i+1}}{\chi(p)^{\nu(n)+1}-\overline{\chi}(p)^{\nu(n)+1}}\right|^2. \tag{8.27}$$

In particular, $\mathcal{O}_p(f;n)=0$ unless $\nu(n)\geq 2\nu(D_E)+1$.

Proof. Let $p=\mathfrak{p}^2$ be ramified in E and $\nu_{\mathfrak{p}}$ be the corresponding valuation ($\nu_{\mathfrak{p}}=2\nu$ on $\mathbb{Q}_p$). Then $p\mid D_E$. Let ϖ be a uniformizer in $\mathfrak{p}$. Writing $g_p=u_pa_pk_p$ for the Iwasawa decomposition of $g_p\in G'(\mathbb{Q}_p)$, we have again

$$W_{n,p}(g_p)=W_{n,p}(u_pa_pk_p)=\psi_{n,p}(u_p)W_{n,p}(a_p). \tag{8.28}$$

Let $a_p=\operatorname{diag}(\varpi^j,1,\varpi^{-j})$, where $j\in\mathbb{Z}$. From the support properties of the Whittaker function, we have $W_{n,p}(g_p)=0$ unless $2j+\nu(n)\geq 0$.

Let $x_p\in N'(\mathbb{Q}_p)\backslash G'(\mathbb{Q}_p)$ and $y_p\in G'(\mathbb{Q}_p)$. We can write them into their Iwasawa coordinates:

$$x_p=\begin{pmatrix}\varpi^i&&\\&1&\\&&\overline{\varpi}^{-i}\end{pmatrix}\kappa_1,\quad y_p=\begin{pmatrix}1&&\Delta t\\&1&\\&&1\end{pmatrix}\begin{pmatrix}\varpi^j&&\\&1&\\&&\overline{\varpi}^{-j}\end{pmatrix}\kappa_2,$$

where $i,j\in\mathbb{Z}^2$, $t\in\mathbb{Q}_p$ and $\kappa_1,\kappa_2\in K_p'(N)$.

By definition, $f_p=\mathbf{1}_{K_p}$. Since $p\nmid N$, we have $\kappa_1,\kappa_2\in K_p$ and $f_p(x_p^{-1}\gamma(1)y_p)\neq 0$ if and only if

$$\begin{pmatrix}\varpi^{-i}&&\\&1&\\&&\overline{\varpi}^{i}\end{pmatrix}\begin{pmatrix}1&1&-1/2\\&1&-1\\&&1\end{pmatrix}\begin{pmatrix}1&&\Delta t\\&1&\\&&1\end{pmatrix}\begin{pmatrix}\varpi^j&&\\&1&\\&&\overline{\varpi}^{-j}\end{pmatrix}\in G(\mathbb{Z}_p),$$

which could be further reduced to

$$\begin{pmatrix}\varpi^{j-i}&\varpi^{-i}&\varpi^{-j}\overline{\varpi}^{-i}(\Delta t-\frac{1}{2})\\&1&-\overline{\varpi}^{-j}\\&&\overline{\varpi}^{-j+i}\end{pmatrix}\in G(\mathbb{Z}_p).$$

Hence $j = i \leq -2\nu(D_E)$. By the support of Whittaker functions we have $\overline{W_{n,p}}(x_p)W_{n,p}(y_p) = 0$ unless $2i + \nu_{\mathfrak{p}}(n) \geq 0$. Hence $\mathcal{O}_p(f;n) = 0$ unless

$$-\nu(n) \leq i = j \leq -2\nu(D_E).$$

Therefore, we have

$$\begin{aligned}\mathcal{O}_p(f;n) &= \int_{N'(\mathbb{Q}_p)\backslash G'(\mathbb{Q}_p)} \int_{G'(\mathbb{Q}_p)} f_p(x_p^{-1}\gamma(1)y_p)\overline{W_{n,p}}(x_p)W_{n,p}(y_p)dx_p dy_p \\ &= \sum_{\substack{i \geq -\nu(n) \\ i \leq -2\nu(D_E)}} p^{2i} \int_{E_{\mathfrak{p}}} \mathbf{1}_{\mathcal{O}_{E_p}}(p^{-i}(\Delta t - 1/2))\theta_{n,p}(t)dt \cdot \left| W_{n,p}\begin{pmatrix} \varpi^i & & \\ & 1 & \\ & & \overline{\varpi}^{-i}\end{pmatrix}\right|^2 \\ &= |W_{n,p}(\mathbf{1}_p)|^2 \cdot \sum_{-\nu(n)+\nu(D_E)/2 \leq i \leq -2\nu(D_E)} \left| \frac{\chi(p)^{\nu(n)+2i+1} - \overline{\chi}(p)^{\nu(n)+2i+1}}{\chi(p)^{\nu(n)+1} - \overline{\chi}(p)^{\nu(n)+1}} \right|^2.\end{aligned}$$

We then conclude (8.27). $\square$

8.4. **Computation of $\mathcal{O}_p(f;n)$ when π'_p is ramified.** In this subsection we deal with the case $p = N'$. In particular p is split and $\pi'_p \simeq \mathrm{St}'_p$ is the Steinberg representation. We continue to denote by ν the usual valuation on $\mathbb{Q}_p$.

Lemma 8.11. *Suppose that $p = N'$.*

For $\nu(n) < 0$, we have

$$\mathcal{O}_p(f^{\mathfrak{n}_p};n) = 0.$$

For $\nu(n) = 0$ we have

$$\frac{\mathcal{O}_p(f^{\mathfrak{n}_p};n)}{|W_{n,p}(\mathbf{1}_p)|^2} = \frac{(p-2)\mu(I'_p(1))^2}{\mu(K_p)}. \tag{8.29}$$

For $\nu(n) \geq 1$ we have

$$\frac{\mathcal{O}_p(f^{\mathfrak{n}_p};n)}{|W_{n,p}(\mathbf{1}_p)|^2} \ll \frac{\nu(n)^2}{p\mu(K_p)}, \tag{8.30}$$

where the implied constant is absolute. We recall that $I'_p(1)$ is defined in (5.69) (when we identify it with a subgroup of $\mathrm{GL}_2(\mathbb{Z}_p)$).

Proof. By definition of the test function f_p, the orbital integral

$$\mathcal{O}_p(f;n) = \int_{N'(\mathbb{Q}_p)\backslash G'(\mathbb{Q}_p)} \int_{G'(\mathbb{Q}_p)} f_p(\mathfrak{n}_p^{-1}x_p^{-1}\gamma(1)y_p\mathfrak{n}_p)\overline{W_{n,p}}(x_p)W_{n,p}(y_p)dx_p dy_p,$$

is zero unless

$$\mathfrak{n}_p^{-1}\begin{pmatrix} a & b & \\ c & d & \\ & & 1\end{pmatrix}^{-1}\begin{pmatrix} 1 & -\frac{1}{2} & 1 \\ & 1 & \\ & -1 & 1\end{pmatrix}\begin{pmatrix} a' & b' & \\ c' & d' & \\ & & 1\end{pmatrix}\mathfrak{n}_p \in K_p. \tag{8.31}$$

By the Iwasawa decomposition for $G(\mathbb{Q}_p)$ and the Iwahori decomposition for K'_p, we write

$$\begin{pmatrix} a' & b' & \\ c' & d' & \\ & & 1\end{pmatrix} \in \begin{pmatrix} p^{j'} & & \\ & p^{k'} & \\ & & 1\end{pmatrix}\begin{pmatrix} 1 & b' & \\ & 1 & \\ & & 1\end{pmatrix}\begin{pmatrix} \delta & & \\ & 1 & \\ & & 1\end{pmatrix} I'_p(1) \tag{8.32}$$

or

$$\begin{pmatrix} a' & b' & \\ c' & d' & \\ & & 1\end{pmatrix} \in \begin{pmatrix} p^{j'} & b'p^{k'} & \\ & p^{k'} & \\ & & 1\end{pmatrix}\begin{pmatrix} \mu_2 & 1 & \\ 1 & & \\ & & 1\end{pmatrix}\begin{pmatrix} \delta & & \\ & 1 & \\ & & 1\end{pmatrix} I'_p(1) \tag{8.33}$$

1. Suppose x_p and y_p are both of the form in (8.32), namely, suppose

$$x_p=\begin{pmatrix}p^j&&\\&p^k&\\&&1\end{pmatrix}\begin{pmatrix}\delta&&\\&1&\\&&1\end{pmatrix}\gamma_1,$$
$$y_p=\begin{pmatrix}p^{j'}&&\\&p^{k'}&\\&&1\end{pmatrix}\begin{pmatrix}1&b'&\\&1&\\&&1\end{pmatrix}\begin{pmatrix}\delta&&\\&1&\\&&1\end{pmatrix}\gamma_2,$$

where $\gamma_1,\gamma_2\in I_p'(1)$. Denote by $\mathcal{O}_p^{(1)}(f;n)$ the contribution of x_p,y_p in the above forms. Then (8.31) is equivalent to

$$\mathfrak{n}_p^{-1}\begin{pmatrix}\tau p^j&&\\&p^k&\\&&1\end{pmatrix}^{-1}\begin{pmatrix}1&-1/2&1\\&1&\\&-1&1\end{pmatrix}\begin{pmatrix}p^{j'}&&\\&p^{k'}&\\&&1\end{pmatrix}\begin{pmatrix}\delta&b'&\\&1&\\&&1\end{pmatrix}\mathfrak{n}_p\in K_p$$

A direct calculation shows that

$$\begin{pmatrix}p^{j'-j}&p^{j'-j}b'-2^{-1}p^{k'-j}+\tau p^{k-1}&\delta p^{j'-j-1}+p^{-j}-\tau p^{-1}\\&p^{k'-k}&\\&-p^{k'}&1\end{pmatrix}\in K_p.$$

Similarly, taking the inverse, we have

$$\begin{pmatrix}p^{-j'}&-b'p^{-k'}&-\delta p^{-1}\\&p^{-k'}&\\&&1\end{pmatrix}\begin{pmatrix}1&-1/2&-1\\&1&\\&1&1\end{pmatrix}\begin{pmatrix}p^j&&\tau p^{j-1}\\&p^k&\\&&1\end{pmatrix}\in K_p.$$

Expanding the matrices we then obtain

$$\begin{pmatrix}p^{j-j'}&-2^{-1}p^{k-j'}-b'p^{k-k'}-\delta p^{k-1}&\tau p^{k-j'-1}-p^{-j'}-\delta p^{-1}\\0&p^{k-k'}&0\\0&p^k&1\end{pmatrix}\in K_p.$$

So $j-j'=k-k'$ and $k=k'\geq 0$, therefore $j=j'$. Hence the above constraints reduces to the following:

$$\begin{pmatrix}1&b'-2^{-1}p^{k-j}&(\delta-\tau)p^{-1}+p^{-j}\\0&1&0\\0&-p^k&1\end{pmatrix}\in K_p.$$

$$\begin{pmatrix}1&-2^{-1}p^{k-j}-b'-\delta p^{k-1}&\tau p^{k-j-1}-p^{-j}-\delta p^{-1}\\0&1&0\\0&p^k&1\end{pmatrix}\in K_p.$$

Since $\tau p^{k-j-1}-p^{-j}-\delta^{-1}p^{-1}\in\mathbb{Z}_p$ and $k\geq 0$, then $0\leq j\leq 1$. Also, $p^{k-j}+\delta p^{k-1}\in\mathbb{Z}_p$. So $j\neq 0$, forcing that $j=1$. From

$$\tau p^{k-j-1}-p^{-j}-\delta^{-1}p^{-1}\in\mathbb{Z}_p$$

we then see that $k\geq 1$.

Also, by the support properties of Whittaker functions, we have

$$\nu(n)+1\geq k\geq 1.$$

If $\nu(n)=0$ we have thus $k=1$ and

$$\begin{aligned}\mathcal{O}_p^{(1)}(f;n)=&\frac{\mu(I_p'(1))^2}{\mu(K_p)}\sum_{\substack{\delta,\tau\\(\delta-\tau)p^{-1}+p^{-1}\in\mathbb{Z}_p}}\left|W_{n,p}\left(\mathbf{1}_p\right)\right|^2\\=&\frac{(p-2)\mu(I_p'(1))^2}{\mu(K_p)}|W_{n,p}(\mathbf{1}_p)|^2.\end{aligned}$$

Assume now that $\nu(n)\geq 1$. If $k\geq 2$, then there are only one choice for the pair (δ,τ) and

$$\mathcal{O}_p^{(1)}(f;n)=\frac{\mu(I_p'(1))^2}{\mu(K_p)}\sum_{1\leq k\leq\nu(n)}p^{1-k}\left|W_{n,p}\begin{pmatrix}p&&\\&1&\\&&p^k\end{pmatrix}\right|^2\ll\frac{\mu(I_p'(1))^2\nu(n)}{\mu(K_p)}|W_{n,p}(\mathbf{1}_p)|^2.$$

2. Suppose x_p and y_p are both of the form in (8.33), namely, suppose

$$\begin{aligned}x_p&=\begin{pmatrix}p^j&&\\&p^k&\\&&1\end{pmatrix}\begin{pmatrix}\mu_1&1&\\1&&\\&&1\end{pmatrix}\begin{pmatrix}\tau^{-1}&&\\&1&\\&&1\end{pmatrix}\gamma_1,\\y_p&=\begin{pmatrix}p^{j'}&&\\&p^{k'}&\\&&1\end{pmatrix}\begin{pmatrix}1&b'&\\&1&\\&&1\end{pmatrix}\begin{pmatrix}\mu_2&1&\\1&&\\&&1\end{pmatrix}\begin{pmatrix}\delta^{-1}&&\\&1&\\&&1\end{pmatrix}\gamma_2,\end{aligned}$$

where $\gamma_1,\gamma_2\in I_p'(1)$. Then (8.31) is equivalent to

$$\begin{pmatrix}p^{-j}&&\\&p^{-k}&-\tau p^{-1}\\&&1\end{pmatrix}\begin{pmatrix}1&-1/2&1\\&1&\\&-1&1\end{pmatrix}\begin{pmatrix}p^{j'}&bp^{j'}&\delta^{-1}bp^{j'-1}\\&p^{k'}&\delta^{-1}p^{k'-1}\\&&1\end{pmatrix}\in K_p,$$

where $b=b'+\mu_2-\mu_1p^{j-k+k'-j'}$. Expanding the left hand side we obtain

$$\begin{pmatrix}p^{j'-j}&bp^{j'-j}-2^{-1}p^{k'-j}&z\\0&p^{k'-k}+\tau p^{k'-1}&\delta^{-1}p^{k'-1-k}-\tau p^{-1}+\tau\delta^{-1}p^{k'-2}\\0&-p^{k'}&1-\delta^{-1}p^{k'-1}\end{pmatrix}\in K_p,$$

where $z=\delta^{-1}bp^{j'-j-1}-2^{-1}\delta^{-1}p^{k'-1-j}+p^{-j}$.

Taking the inverse of the above matrix we then obtain

$$\begin{pmatrix}p^{j-j'}&-2^{-1}p^{k-j'}-bp^{k-k'}&z'\\0&p^{k-k'}-\delta p^{k-1}&\tau^{-1}p^{k-k'-1}-\delta p^{-1}-\delta\tau^{-1}p^{k-2}\\0&p^k&1+\tau^{-1}p^{k-1}\end{pmatrix}\in K_p,$$

where

$$z'=-2^{-1}\tau^{-1}p^{k-j'-1}-p^{-j'}-b\tau^{-1}p^{k-k'-1}.$$

In particular we have

$$k\geq 1,\ k'\geq 1,\ j-j'=k-k'.$$

If $k>k'$ then $k\geq 2$, which contradicts the condition

$$\tau^{-1}p^{k-k'-1}-\delta p^{-1}-\delta\tau^{-1}p^{k-2}\in\mathbb{Z}_p.$$

Hence $k\leq k'$. Likewise, $k'\leq k$. So we must have $k=k'$ and therefore $j=j'$. Eventually, the above constraints reduces to

$$\begin{pmatrix}1&b-2^{-1}p^{k-j}+\mu_1p^{k-1}&z_1\\0&1+\tau p^{k-1}&\delta^{-1}p^{-1}-\tau p^{-1}+\tau\delta^{-1}p^{k-2}\\0&-p^k&1-\delta^{-1}p^{k-1}\end{pmatrix}\in K_p,$$

where $z_1=\delta^{-1}bp^{-1}-2^{-1}\delta^{-1}p^{-1-j}+p^{-j}$. From

$$bp^{j'-j}-2^{-1}p^{k'-j}\in\mathbb{Z}_p,\quad -2^{-1}p^{k-j'}-bp^{k-k'}\in\mathbb{Z}_p$$

one has $k\geq j$ and $b\in\mathbb{Z}_p$. On the other hand, from the support of Whittaker functions we have necessarily that $\nu(n)+j\geq k\geq 1$.

We have the following cases

(a) Suppose $\nu(n)=0$. Then $k=j\geq 1$. Since $b\in\mathbb{Z}_p$, then from

$$z'=-2^{-1}\tau^{-1}p^{k-j'-1}-p^{-j'}-b\tau^{-1}p^{k-k'-1}\in\mathbb{Z}_p$$

one concludes that $-j\geq -1$, i.e., $j\leq 1$. Hence $j=k=1$. Then it follows from

$$\delta^{-1}p^{k'-1-k}-\tau p^{-1}+\tau\delta^{-1}p^{k'-2},\ \tau^{-1}p^{k-k'-1}-\delta p^{-1}-\delta\tau^{-1}p^{k-2}\in\mathbb{Z}_p \tag{8.34}$$

that $\tau\equiv-\delta\pmod p$ and $1+\tau+\tau^2\equiv 0\pmod p$. From $\tau z'+\delta z\in\mathbb{Z}_p$ we have $1+2\tau\equiv 0\pmod p$, which in conjunction with $1+\tau+\tau^2\equiv 0\pmod p$ implies $p\mid 3$. However, since we have assumed that $p\geq 5$ we encounter a contradiction if $\nu(n)=0$.

(b) Therefore $\nu(n)\geq 1$. Note that from (8.34) δ and τ should satisfy either $\delta\tau\equiv 1\pmod p$ or $\tau^2+\tau+1\equiv 0\pmod p$ and $\tau+\delta\equiv 0\pmod p$. Hence, in conjunction with

$$W_{n,p}(x_p)=\theta_{n,p}(p^{j-k}\mu_1)W_{n,p}\begin{pmatrix}p^{j-k}&&\\&1&\\&&1\end{pmatrix}$$

$$W_{n,p}(y_p)=\theta_{n,p}(p^{j-k}\mu_2)\theta_{n,p}(p^{j-k}b')W_{n,p}\begin{pmatrix}p^{j-k}&&\\&1&\\&&1\end{pmatrix},$$

we conclude that

$$\mathcal{O}_p^{(2)}(f;n)\leq p^3\mu(I_p'(1))^2\sum_{k=1}^{\nu(n)+1}\sum_{j=k-\nu(n)}^{1}p^{j-k}\left|W_{n,p}\begin{pmatrix}p^{j-k}&&\\&1&\\&&1\end{pmatrix}\right|^2\ll\frac{\nu(n)^2|W_{n,p}(\mathbf{1}_p)|^2}{p\mu(K_p)}.$$

3. Suppose x_p is of the form in (8.32) and y_p is of the form in (8.33), namely, suppose

$$x_p=\begin{pmatrix}p^j&&\\&p^k&\\&&1\end{pmatrix}\begin{pmatrix}\tau&&\\&1&\\&&1\end{pmatrix}\gamma_1,$$

$$y_p=\begin{pmatrix}p^{j'}&&\\&p^{k'}&\\&&1\end{pmatrix}\begin{pmatrix}1&b'&\\&1&\\&&1\end{pmatrix}\begin{pmatrix}\mu_2&1&\\1&&\\&&1\end{pmatrix}\begin{pmatrix}\delta&&\\&1&\\&&1\end{pmatrix}\gamma_2,$$

where $\gamma_1,\gamma_2\in I_p'(1)$. Denote by $\mathcal{O}_p^{(3)}(f;n)$ the contribution of x_p,y_p in the above forms. Then (8.31) is equivalent to

$$\begin{pmatrix}\tau^{-1}p^{-j}&&-p^{-1}\\&p^{-k}&\\&&1\end{pmatrix}\begin{pmatrix}1&-1/2&1\\&1&\\&-1&1\end{pmatrix}\begin{pmatrix}p^{j'}&bp^{j'}&\delta^{-1}bp^{j'-1}\\&p^{k'}&\delta^{-1}p^{k'-1}\\&&1\end{pmatrix}\in K_p$$

where $b=b'+\mu_2$. Expanding the left hand side, the above constraint becomes

$$\begin{pmatrix}p^{j'-j}&bp^{j'-j}-2^{-1}p^{k'-j}+\tau p^{k'-1}&z_2\\0&p^{k'-k}&\delta^{-1}p^{k'-k-1}\\0&-p^{k'}&1-\delta^{-1}p^{k'-1}\end{pmatrix}\in K_p,$$

where

$$z_2 := \delta^{-1}bp^{j'-j-1} - 2^{-1}\delta^{-1}p^{k'-j-1} + p^{-j} - \tau p^{-1} + \tau\delta^{-1}p^{k'-2}.$$

Considering the inverse as before, we obtain

$$\begin{pmatrix} p^{-j'} & -bp^{-k'} & \\ & p^{-k'} & -\delta p^{-1} \\ & & 1 \end{pmatrix} \begin{pmatrix} p^{j} & -2^{-1}p^{k} & \tau^{-1}p^{j-1} - 1 \\ & p^{k} & \\ & p^{k} & 1 \end{pmatrix} \in K_p,$$

which amounts to the following condition:

$$\begin{pmatrix} p^{j-j'} & -2^{-1}p^{k-j'} - bp^{k-k'} & \tau^{-1}p^{j-j'-1} - p^{-j'} \\ 0 & p^{k-k'} - \delta p^{k-1} & -\delta p^{-1} \\ 0 & p^{k} & 1 \end{pmatrix} \in K_p.$$

Then we get a contradiction. So in this case, one has $\mathcal{O}_p^{(3)}(f;n) = 0$.

4. Suppose x_p is of the form in (8.33) and y_p is of the form in (8.32), namely, suppose

$$x_p = \begin{pmatrix} p^{j} & & \\ & p^{k} & \\ & & 1 \end{pmatrix} \begin{pmatrix} \mu_1 & 1 & \\ 1 & & \\ & & 1 \end{pmatrix} \begin{pmatrix} \tau & & \\ & 1 & \\ & & 1 \end{pmatrix} \gamma_1,$$

$$y_p = \begin{pmatrix} p^{j'} & & \\ & p^{k'} & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & b' & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} \delta & & \\ & 1 & \\ & & 1 \end{pmatrix} \gamma_2,$$

where $\gamma_1, \gamma_2 \in I_p'(1)$. Denote by $\mathcal{O}_p^{(4)}(f;n)$ the contribution of x_p, y_p in the above forms. Then (8.31) is equivalent to

$$\begin{pmatrix} p^{-j} & & -\mu_1 p^{-1} \\ & p^{-k} & -\tau p^{-1} \\ & & 1 \end{pmatrix} \begin{pmatrix} p^{j'} & -2^{-1}p^{k'} & 1 \\ & p^{k'} & \\ & -p^{k'} & 1 \end{pmatrix} \begin{pmatrix} 1 & b' & \delta^{-1}p^{-1} \\ & 1 & \\ & & 1 \end{pmatrix} \in K_p.$$

Expanding the left hand side this becomes

$$\begin{pmatrix} p^{j'-j} & p^{j'-j}b' - 2^{-1}p^{k'-j} + \mu_1 p^{k'-1} & \delta^{-1}p^{j'-j-1} + p^{-j} - \mu_1 p^{-1} \\ 0 & p^{k'-k} + \tau p^{k'-1} & -\tau p^{-1} \\ 0 & -p^{k'} & 1 \end{pmatrix} \in K_p.$$

Taking the inverse we then obtain

$$\begin{pmatrix} p^{-j'} & -bp^{-k'} & -\delta p^{-1} \\ & p^{-k'} & \\ & & 1 \end{pmatrix} \begin{pmatrix} p^{j} & -2^{-1}p^{k} & -1 \\ & p^{k} & \\ & p^{k} & 1 \end{pmatrix} \begin{pmatrix} 1 & \mu_1 & \mu_1\tau^{-1}p^{-1} \\ & 1 & \tau^{-1}p^{-1} \\ & & 1 \end{pmatrix} \in K_p$$

By a calculation, this is equivalent to

$$\begin{pmatrix} p^{j-j'} & -2^{-1}p^{k-j'} - bp^{k-k'} - \delta p^{k-1} & z_2' \\ 0 & p^{k-k'} & \tau^{-1}p^{k-k'-1} \\ 0 & p^{k} & 1 + \tau^{-1}p^{k-1} \end{pmatrix} \in K_p,$$

where

$$z_2' = \mu_1\tau^{-1}p^{j-j'-1} - 2^{-1}\tau^{-1}p^{k-j'-1} - p^{-j'} - \tau^{-1}bp^{k-k'-1} - \delta p^{-1} - \tau^{-1}\delta p^{k-2}.$$

So $j=j'$ and $k=k'\geq 1$. Thus,

$$\begin{pmatrix} 1 & b'-2^{-1}p^{k-j}+\mu_1p^{k-1} & \delta^{-1}p^{-1}+p^{-j}-\mu_1p^{-1}\\ 0 & 1+\tau p^{k-1} & -\tau p^{-1}\\ 0 & -p^k & 1\end{pmatrix}\in K_p.$$

Then we get a contradiction and conclude that $\mathcal{O}_p^{(4)}(f;n)=0$.

In conclusion, (8.29) and (8.30) hold, giving Lemma 8.11. $\square$

8.5. **An upper bound for the global orbital integral $\mathcal{O}_{\gamma(1)}(f;\varphi')$.** In this section, we combine results from previous sections to deduce an upper bound for $\mathcal{O}_{\gamma(1)}(f,\varphi')$.

Let $\nu(f)$ be the set of (inert) primes p (coprime with $NN'D$) such that

$$f_p=1_{G(\mathbb{Z}_p)A_{r_p}G(\mathbb{Z}_p)}$$

for some $r_p\geq 1$.

Proposition 8.12. *Let notation be as before. We have*

$$\mathcal{O}_{\gamma(1)}(f,\varphi')\leq\frac{(\ell N')^{o(1)}}{2^{4k}k^2}\frac{N}{N'^3}\prod_{p\in\nu(f)}p^{r_p}. \tag{8.35}$$

where the implicit constants depend on E and π' (via $L(\pi',\mathrm{Ad},1)$).

Proof. It follows from (8.7), (8.15), and Lemmas 8.6, 8.7, 8.8, 8.9, 8.10, 8.11 that

$$\mathcal{O}_{\gamma(1)}(f,\varphi')=\eta(\Delta)\frac{C_kC_{N,N'}}{L(\pi',\mathrm{Ad},1)}\sum_{n\geq 1}|\lambda_{\pi'}(n)|^2e^{-\frac{\pi n}{|D_E|^{1/2}}}\prod_{p<\infty}I_p(n,\varphi'), \tag{8.36}$$

where $\eta(\Delta)$ is a complex number of modulus 1,

$$C_k=\frac{2^4\pi^5}{3.2^{4(k-1)}}\frac{1}{(k-1)^2},$$

$$\begin{aligned} C_{N,N'}&=\prod_{p|N}\frac{\mu(I_p')^2}{\mu(K_p)}\prod_{p\in N'}(1+\frac{1}{p})(p-2)\mu(I_p'(1))^2\\ &=\prod_{p|N}\frac{p^2-p+1}{p+1}\prod_{p|N'}\frac{(p-2)(p+1)}{p(p^2-1)^2}\ll\frac{N}{N'^3}. \end{aligned} \tag{8.37}$$

and where $I_p(n,\varphi')=1$ if $(n,p\ell)=1$ and in general satisfies,

$$\begin{cases} I_p(n,\varphi')\leq (n,p)^{1+o(1)}, & \text{if } p\notin\nu(f)\\ I_p(n,\varphi')\ll (r_p+1)^4(n,p)^{1+o(1)}p^{r_p}, & \text{if } p\in\nu(f). \end{cases} \tag{8.38}$$

Hence, (8.36), Deligne's bound (4.41) and (4.44) yields

$$\mathcal{O}_{\gamma(1)}(f,\varphi')\ll\frac{(\ell N')^{o(1)}}{2^{4k}k^2}\frac{N}{N'^3}\cdot\prod_{p\in\nu(f)}p^{r_p}. \tag{8.39}$$

$\square$

8.6. **Computation of the remaining unipotent orbital integrals.** In this subsection, we prove the claim (6.16) and reduce the computation of $\mathcal{O}_{\gamma(x)}(f^{\mathfrak{n}},\varphi')$ $x\in E^1$ to $\mathcal{O}_{\gamma(1)}(f^{\mathfrak{n}},\varphi')$. Our result is the following:

Proposition 8.13. *Let notation be as before. Let $x\in E^1$. Then*
$$\mathcal{O}_{\gamma(x)}(f^{\mathfrak{n}},\varphi')=\begin{cases}\mathcal{O}_{\gamma(1)}(f^{\mathfrak{n}},\varphi'), & \textit{if } x\in\mathcal{O}_E^1;\\ 0, & \textit{otherwise.}\end{cases}$$

Proof. Let $x\neq 1$. By Lemma 6.3 we have
$$\gamma(x)=g_1.x\gamma(1).g_2,\ \ g_1,g_2\in G'(\mathbb{Q})$$
and by $G'(\mathbb{Q})$ invariance of φ', we have

$$\begin{aligned}\mathcal{O}_{\gamma(x)}(f^{\mathfrak{n}},\varphi')&=\int_{H_{\gamma(x)}(\mathbb{Q})\backslash G'(\mathbb{A})^2}f^{\mathfrak{n}}(u^{-1}g_1.x\gamma(1).g_2v)\overline{\varphi}'(u)\varphi'(v)dudv\\&=\mathcal{O}_{x\gamma(1)}(f^{\mathfrak{n}},\varphi').\end{aligned}\tag{8.40}$$

Similar to the proof of Lemma 8.1 the orbital integral $\mathcal{O}_{x\gamma(1)}(f^{\mathfrak{n}},\varphi')$ is equal to

$$\int_{N'(\mathbb{A})\backslash G'(\mathbb{A})}\int_{G'(\mathbb{A})}f^{\mathfrak{n}}\left(y_1^{-1}x\gamma(1)y_2\right)\int_{[N']}\overline{\varphi}'(vy_1)\varphi'(vy_2)dvdy_1dy_2,\tag{8.41}$$

which is absolutely converging by Proposition 8.12.

Note that $f^{\mathfrak{n}}\left(y_1^{-1}x\gamma(1)y_2\right)=0$ unless
$$\widetilde{\mathfrak{n}}^{-1}y_1^{-1}x\gamma(1)y_2\widetilde{\mathfrak{n}}\in\prod_{\substack{p<\infty\\ p\notin\nu(f)}}K_p(N)\prod_{p\in\nu(f)}G(\mathbb{Z}_p)A_{p^{r_p}}G(\mathbb{Z}_p).$$

Also,
$$\begin{pmatrix}a&&b\\&1&\\c&&d\end{pmatrix}\begin{pmatrix}1&1&-1/2\\&1&-1\\&&1\end{pmatrix}\begin{pmatrix}a'&&b'\\&1&\\c'&&d'\end{pmatrix}=\begin{pmatrix}*&&*\\&1&\\ *&&*\end{pmatrix}.$$

Hence $f^{\mathfrak{n}}\left(y_1^{-1}x\gamma(1)y_2\right)=0$ unless $\nu_v(x)\geq 0$ (i.e., $x\in\mathcal{O}_{E_v}$) for all places v in E such that v is not above N' and $v\notin\nu(f)$.

Let $p\in\nu(f)$. Then $f_p^{\mathfrak{n}_p}\left(y_1^{-1}x\gamma(1)y_2\right)=0$ unless

$$\widetilde{\mathfrak{n}}_p^{-1}y_{1,p}^{-1}\gamma(1)y_{2,p}\widetilde{\mathfrak{n}}_p\in x^{-1}G(\mathbb{Z}_p)A_{p^{r_p}}G(\mathbb{Z}_p).\tag{8.42}$$

Taking the determinant on both sides of (8.42) we then derive that $|x|_p=1$, implying that $x\in\mathcal{O}_{E_p}^\times$.

Let $p\mid N'$. Then $f_p^{\mathfrak{n}_p}\left(y_1^{-1}x\gamma(1)y_2\right)=0$ unless

$$\widetilde{\mathfrak{n}}_p^{-1}y_{1,p}^{-1}\gamma(1)y_{2,p}\widetilde{\mathfrak{n}}_p\in x^{-1}K_p.\tag{8.43}$$

We then apply the manipulation as in the proof of Lemma 8.11 to show that, if (8.43) holds for some $x\in E^1$, then necessarily $x\in\mathcal{O}_{E_p}^\times$.

Since $p|N'$ we may identify $G(\mathbb{Q}_p)$ (resp. $G'(\mathbb{Q}_p)$) with $\mathrm{GL}(3,\mathbb{Q}_p)$ (resp. $\mathrm{GL}(2,\mathbb{Q}_p)$); suppose for the contrary that $\nu(x)\neq 0$. In the notations of the proof of Lemma 8.11 we have four cases to consider.

In Case 1, (8.43) implies that
$$\begin{pmatrix}p^{j-j'}&-2^{-1}p^{k-j'}-b'p^{k-k'}-\delta p^{k-1}&\tau p^{k-j'-1}-p^{-j'}-\delta p^{-1}\\0&p^{k-k'}&0\\0&p^k&1\end{pmatrix}\in x^{-1}K_p,$$
which contradicts the assumption that $\nu(x)<0$. Similarly, taking the inverse, we see that $\nu(x)>0$ cannot happen as well.

In the Case 3, (8.43) implies that

$$x\begin{pmatrix} p^{j'-j} & bp^{j'-j}-2^{-1}p^{k'-j}+\tau p^{k'-1} & z_2\\ & p^{k'-k} & \delta^{-1}p^{k'-k-1}\\ & -p^{k'} & 1-\delta^{-1}p^{k'-1}\end{pmatrix}\in K_p, \tag{8.44}$$

where

$$z_2:=\delta^{-1}bp^{j'-j-1}-2^{-1}\delta^{-1}p^{k'-j-1}+p^{-j}-\tau p^{-1}+\tau\delta^{-1}p^{k'-2}.$$

Considering the inverse as before, we then obtain

$$x^{-1}\begin{pmatrix} p^{j-j'} & -2^{-1}p^{k-j'}-bp^{k-k'} & \tau^{-1}p^{j-j'-1}-p^{-j'}\\ 0 & p^{k-k'}-\delta p^{k-1} & -\delta p^{-1}\\ 0 & p^{k} & 1\end{pmatrix}\in K_p. \tag{8.45}$$

By (8.45) one has $\nu(x)\leq -1$ and $j-j'=\nu(x)$. Computing the determinant of (8.44) we then have

$$j-j'+k-k'=3\nu(x),\ \textit{i.e.},\ k-k'=2\nu(x).$$

Therefore

$$j=1, j'=2, k=-1, k'=1,\ \text{and}\ \nu(x)=-1.$$

Applying inversion in (8.45) we then get $\nu(x)-1\geq 0$ by considering its $(2,3)$-th entry and therefore $\nu(x)\geq 1$, a contradiction! In conclusion, the Case 3 does not contribute to the orbital integral, just as the situation in Lemma 8.11. Similarly the contribution of Case 4 is also zero.

Finally we consider the Case 2; where (8.43) implies that

$$\begin{pmatrix} p^{j'-j} & bp^{j'-j}-2^{-1}p^{k'-j}+\mu_1p^{k'-1} & z\\ & p^{k'-k}+\tau p^{k'-1} & \delta^{-1}p^{k'-1-k}-\tau p^{-1}+\tau\delta^{-1}p^{k'-2}\\ & -p^{k'} & 1-\delta^{-1}p^{k'-1}\end{pmatrix}\in x^{-1}K_p,$$

where

$$z=\delta^{-1}bp^{j'-j-1}-2^{-1}\delta^{-1}p^{-1-j}+p^{-j}-\mu_1p^{-1}+\mu_1\delta^{-1}p^{k'-2}.$$

Suppose $\nu(x)\leq -1$. Then $k'=1=k$ and $j'-j=-\nu(x)$ by analyzing the diagonals. However, taking determinant we then obtain

$$j'-j+k'-k=-3\nu(x),$$

a contradiction! Suppose $\nu(x)\geq 1$. Then taking the inverse of the above matrix we then obtain

$$\begin{pmatrix} p^{j-j'} & \mu_1p^{j-j'}-2^{-1}p^{k-j'}-bp^{k-k'} & z'\\ & p^{k-k'}-\delta p^{k-1} & \tau^{-1}p^{k-k'-1}-\delta p^{-1}-\delta\tau^{-1}p^{k-2}\\ & p^{k} & 1+\tau^{-1}p^{k-1}\end{pmatrix}\in xK_p,$$

where

$$z'=\mu_1\tau^{-1}p^{j-j'-1}-2^{-1}\tau^{-1}p^{k-j'-1}-p^{-j'}-b\tau^{-1}p^{k-k'-1}.$$

So $k=k'=1$ and $j-j'=1$. Again, we encounter an contradiction by taking determinant. Hence, $\nu(x)=0$, i.e., $x\in\mathcal{O}_{E_p}^{\times}$.

In summary, we have shown that $\nu(x)\geq 0$ in all finite places. So $x\in\mathcal{O}_E$. Since $x\overline{x}=1$ we have $x\in\mathcal{O}_E^1$.

Since the test function f is $Z_G(\mathcal{O}_E^1)$-invariant, then Proposition 8.13 follows. $\square$

As a consequence of Proposition 8.12 and 8.13 we have

Corollary 8.14. *The sum of the unipotent orbital integrals satisfies the bound*

$$\sum_{x\in E^1}\mathcal{O}_{\gamma(x)}(f,\varphi')\ll\frac{(\ell N')^{o(1)}}{2^{4k}k^2}\frac{N}{N'^3}\prod_{p\in\nu(f)}p^{r_p}$$

where the implicit constant depend on E and π'.

9. The Regular Orbital Integrals

In this section and the next we handle the contribution to (6.17) of the regular orbital integrals, ie. the $\mathcal{O}_{\gamma(x)}(f,\varphi')$ when $x\in E^\times-E^1$.

In the case of $\mathrm{GL}(2)\times\mathrm{GL}(1)$ this was handled in [RR05, §2.6]; here the geometric structure of $U(2,1)\times U(1,1)$ is much more complicated.

In this section, we provide bounds for some local integrals which we combine together in Section 10 to prove Theorem 10.1.

9.1. **The Stabilizer $H_{\gamma(x)}$.** We start by computing the stabilizer $H_{\gamma(x)}$ associated to $\gamma(x)$ when $x\in E^\times-E^1$. Recall that for any $\mathbb{Q}$-algebra R

$$H_{\gamma(x)}(R):=\big\{(g_1,g_2)\in G'(R):\ g_1^{-1}\gamma(x)g_2=\gamma(x)\big\}.$$

To save space we will represent matrices in G' either as 2×2 or a 3×3 matrices (in the standard bases $\{e_{-1},e_1\}$ or $\{e_{-1},e_0,e_1\}$), that is either $\begin{pmatrix}a&b\\c&d\end{pmatrix}$ or $\begin{pmatrix}a&&b\\&1&\\c&&d\end{pmatrix}$. Of course whenever matrices from G are involved we will use the 3×3 notation.

Lemma 9.1. *Let notation be as above, for any $\mathbb{Q}$-algebra R, $H_{\gamma(x)}(R)$ is equal to*

$$\left\{\begin{pmatrix}1+\frac{y(x-1)(\overline{x}+1)}{2}&\frac{y(1+x)(1+\overline{x})}{4}\\ y(1-x)(1-\overline{x})&1+\frac{y(x+1)(\overline{x}-1)}{2}\end{pmatrix}\times\begin{pmatrix}1+\frac{y(x-1)(\overline{x}+1)}{2}&y\\ \frac{y(1+x)(1+\overline{x})(1-x)(1-\overline{x})}{4}&1+\frac{y(x+1)(\overline{x}-1)}{2}\end{pmatrix}\right.$$
$$\left.\in G'(R)\times G'(R):y\in E^\times(R)\ \textit{and}\ y+\overline{y}+y\overline{y}(x\overline{x}-1)=0\right\}.$$

Proof. Let

$$g=\begin{pmatrix}a&b\\c&d\end{pmatrix},g'=\begin{pmatrix}a'&b'\\c'&d'\end{pmatrix}\in G'(R).$$

Assume

$$g\begin{pmatrix}\frac{x\overline{x}+3\overline{x}-x+1}{4}&\frac{1+x}{2}&-\frac{1}{2}\\ \frac{(x+1)(\overline{x}-1)}{2}&x&-1\\ -\frac{(1-x)(1-\overline{x})}{2}&1-x&1\end{pmatrix}=\begin{pmatrix}\frac{x\overline{x}+3\overline{x}-x+1}{4}&\frac{1+x}{2}&-\frac{1}{2}\\ \frac{(x+1)(\overline{x}-1)}{2}&x&-1\\ -\frac{(1-x)(1-\overline{x})}{2}&1-x&1\end{pmatrix}g'.\tag{9.1}$$

Expanding both sides of equation (9.1) we then obtain an equality between the matrix

$$\begin{pmatrix}a'(\overline{x}+\frac{(1-x)(1-\overline{x})}{4})-\frac{c'}{2}&\frac{1+x}{2}&b'(\overline{x}+\frac{(1-x)(1-\overline{x})}{4})-\frac{d'}{2}\\ \frac{a'(x+1)(\overline{x}-1)}{2}-c'&x&\frac{b'(x+1)(\overline{x}-1)}{2}-d'\\ -\frac{a'(1-x)(1-\overline{x})}{2}+c'&1-x&-\frac{b'(1-x)(1-\overline{x})}{2}+d'\end{pmatrix}$$

and the matrix

$$\begin{pmatrix}a(\overline{x}+\frac{(1-x)(1-\overline{x})}{4})-\frac{b(1-x)(1-\overline{x})}{2}&\frac{a(1+x)}{2})+b(1-x)&-\frac{a}{2}+b\\ \frac{(x+1)(\overline{x}-1)}{2}&x&-1\\ c(\overline{x}+\frac{(1-x)(1-\overline{x})}{4})-\frac{d(1-x)(1-\overline{x})}{2}&\frac{c(1+x)}{2}+d(1-x)&-\frac{c}{2}+d\end{pmatrix},$$

which is equivalent to the system of equations

$$
\begin{cases}
-\frac{a}{2}+b=b'(\overline{x}+\frac{(1-x)(1-\overline{x})}{4})-\frac{d'}{2}\\
\frac{a(1+x)}{2})+b(1-x)=\frac{1+x}{2}\\
a(\overline{x}+\frac{(1-x)(1-\overline{x})}{4})-\frac{b(1-x)(1-\overline{x})}{2}=a'(\overline{x}+\frac{(1-x)(1-\overline{x})}{4})-\frac{c'}{2}\\
-\frac{c}{2}+d=-\frac{b'(1-x)(1-\overline{x})}{2}+d'\\
\frac{c(1+x)}{2}+d(1-x)=1-x\\
c(\overline{x}+\frac{(1-x)(1-\overline{x})}{4})-\frac{d(1-x)(1-\overline{x})}{2}=-\frac{a'(1-x)(1-\overline{x})}{2}+c'\\
-1=\frac{b'(x+1)(\overline{x}-1)}{2}-d'\\
\frac{(x+1)(\overline{x}-1)}{2}=\frac{a'(x+1)(\overline{x}-1)}{2}-c'
\end{cases}
\tag{9.2}
$$

Now we need to solve (9.2) to find the group $H_{\gamma(x)}(R)$. Note that

$$
(-\frac{1}{2},1)\frac{(1-x)(1+\overline{x})}{2}+(\frac{1+x}{2},1-x)\overline{x}=(\frac{x\overline{x}+3\overline{x}-x-1}{4},-\frac{(1-x)(1-\overline{x})}{2}).
$$

Hence we obtain from (9.2) the following equations on (a',b',c',d') :

$$
\begin{cases}
(\frac{b'(x\overline{x}+3\overline{x}-x-1)}{4}-\frac{d'(x-1)(1+\overline{x})}{4}+\frac{(1+x)\overline{x}}{2}=\frac{a'(x\overline{x}+3\overline{x}-x-1)}{4}-\frac{c'}{2}\\
(-\frac{b'(1-x)(1-\overline{x})}{2}+d')\frac{(x-1)(1+\overline{x})}{2}+\overline{x}(1-x)=-\frac{a'(1-x)(1-\overline{x})}{2}+c'\\
-1=\frac{b'(x+1)(\overline{x}-1)}{2}-d'\\
\frac{(x+1)(\overline{x}-1)}{2}=\frac{a'(x+1)(\overline{x}-1)}{2}-c'.
\end{cases}
\tag{9.3}
$$

A computation shows that the companion matrix is singular. By Gaussian elimination, we can further simplify the system of equations (9.3) to get

$$
\begin{cases}
c'=\frac{b'(x+1)(\overline{x}-1)(x-1)(\overline{x}+1)}{4}\\
d'=\frac{b'(x+1)(\overline{x}-1)}{2}+1\\
c'=\frac{(a'-1)(x+1)(\overline{x}-1)}{2}.
\end{cases}
\tag{9.4}
$$

Denote by $y=\frac{(x-1)(\overline{x}+1)}{2}$. Since $\begin{pmatrix}a'&b'\\c'&d'\end{pmatrix}\in G'(R)$, we have

$$
\begin{pmatrix}\overline{b}'y+1&\overline{b}'\\\overline{b}'y\overline{y}&\overline{b}'\overline{y}+1\end{pmatrix}\begin{pmatrix}b'y+1&b'\\b'y\overline{y}&b'\overline{y}+1\end{pmatrix}=\mathrm{Id},
$$

which turns out to be equivalent to $b'\overline{b}'(y+\overline{y})+b'+\overline{b}'=0$, i.e.,

$$
b'\overline{b}'(x\overline{x}-1)+b'+\overline{b}'=0.
\tag{9.5}
$$

Substituting (9.4) into (9.2) we then obtain

$$
\begin{cases}
-\frac{a}{2}+b=b'(\overline{x}+\frac{(1-x)(1-\overline{x})}{4})-b'(\frac{(1-x)(1-\overline{x})}{4}-\frac{1-\overline{x}}{2})-\frac{1}{2}=\frac{b'(1+\overline{x})}{2}-\frac{1}{2}\\
\frac{a(1+x)}{2}+b(1-x)=\frac{1+x}{2}\\
a(\overline{x}+\frac{(1-x)(1-\overline{x})}{4})-\frac{b(1-x)(1-\overline{x})}{2}=a'(\overline{x}+\frac{(1-x)(1-\overline{x})}{4})-(a'-1)(\frac{(1-x)(1-\overline{x})}{4}-\frac{1-\overline{x}}{2})\\
-\frac{c}{2}+d=-\frac{b'(1-x)(1-\overline{x})}{2}+d'=b'(\frac{(1-x)(1-\overline{x})}{2}+\overline{x}-1)+1-\frac{b'(1-x)(1-\overline{x})}{2}\\
\frac{c(1+x)}{2}+d(1-x)=1-x\\
c(\overline{x}+\frac{(1-x)(1-\overline{x})}{4})-\frac{d(1-x)(1-\overline{x})}{2}=-\frac{a'(1-x)(1-\overline{x})}{2}+(a'-1)(\frac{(1-x)(1-\overline{x})}{2}-1+\overline{x}).
\end{cases}
$$

Using Gaussian elimination method we then see the above system becomes

$$\begin{cases} a = 1 - b'(1-x)\frac{1+\overline{x}}{2} \\ b = \frac{b'(1+\overline{x})}{2} - \frac{b'(1-x)}{2}(1-\frac{\overline{x}}{2}) = \frac{b'(1+x)(1+\overline{x})}{4} \\ c = b'(1-x)(1-\overline{x}) \\ d = 1 - b'(1-\overline{x}) + \frac{b'(1-x)(1-\overline{x})}{2} \\ a(\overline{x} + \frac{(1-x)(1-\overline{x})}{4}) - \frac{b(1-x)(1-\overline{x})}{2} = \frac{a'(1+\overline{x})}{2}) + (\frac{(1-x)(1-\overline{x})}{4} - \frac{(1-\overline{x})}{2}) \\ c(\overline{x} + \frac{(1-x)(1-\overline{x})}{4}) - \frac{d(1-x)(1-\overline{x})}{2} = -a'(1-\overline{x}) - (\frac{(1-x)(1-\overline{x})}{2} - 1 + \overline{x}). \end{cases} \tag{9.6}$$

A calculation shows that the last two equations in (9.6) are redundant. That is, (9.6) is equivalent to

$$\begin{cases} a = 1 - \frac{b'(1-x)(1+\overline{x})}{2} \\ b = \frac{b'(1+\overline{x})}{2} - \frac{b'(1-x)(1+\overline{x})}{4} = \frac{b'(1+x)(1+\overline{x})}{4} \\ c = b'(1-x)(1-\overline{x}) \\ d = 1 - \frac{b'(1+x)(1-\overline{x})}{2}. \end{cases} \tag{9.7}$$

Since $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G'(\mathbb{Q})$, we then have

$$\begin{pmatrix} \overline{d} & \overline{b} \\ \overline{c} & \overline{a} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \mathrm{Id}\,. \tag{9.8}$$

Substituting (9.7) into (9.8) we then conclude together with (9.5) that

$$\begin{cases} \left[b' + \overline{b}' + b'\overline{b}'(x\overline{x} - 1)\right]\frac{(1-x)(1+\overline{x})}{2} = 0 \\ b' + \overline{b}' + b'\overline{b}'(x\overline{x} - 1) = 0. \end{cases} \tag{9.9}$$

Then it follows from (9.9) that $b' + \overline{b}' + b'\overline{b}'(x\overline{x} - x - \overline{x}) = 0$. Hence Lemma 9.1 follows. ☐

We will now use that $x \notin E^1$, that is $x.\overline{x} - 1 \neq 0$. It turns out that the conjugate stabilizer of $\gamma(x)J$ (which is a conjugate to that of $\gamma(x)$ since $J \in G'(\mathbb{Q})$) has a nice parametrization and we will use it instead:

Corollary 9.2. *Notation be as before, for $x \in E^\times - E^1$ set*

$$s(x) := \frac{(x-1)(\overline{x}+1)}{2} \in E^\times.$$

For R any $\mathbb{Q}$-algebra, the stabilizer of $\gamma(x)J$ is given by

$$H_{\gamma(x)J}(R) = \left\{(h_1(w), h_2(w)) \in G'(R) \times G'(R) :\ w \in E^\times(R),\ w\overline{w} = 1\right\},$$

where

$$\begin{cases} h_1(w) := \begin{pmatrix} 1 - \frac{(1-w)s(x)}{s(x)+s(\overline{x})} & -\frac{(1-w)s(x)s(\overline{x})}{(s(x)+s(\overline{x}))x\overline{x}} \\ -\frac{(1-w)x\overline{x}}{s(x)+s(\overline{x})} & 1 - \frac{(1-w)s(\overline{x})}{s(x)+s(\overline{x})} \end{pmatrix} \\ \\ h_2(w) := \begin{pmatrix} 1 - \frac{(1-w)s(\overline{x})}{s(x)+s(\overline{x})} & -\frac{(1-w)s(x)s(\overline{x})}{s(x)+s(\overline{x})} \\ -\frac{(1-w)}{s(x)+s(\overline{x})} & 1 - \frac{(1-w)s(x)}{s(x)+s(\overline{x})} \end{pmatrix}. \end{cases} \tag{9.10}$$

In particular, $h_1(w)$ and $h_2(w)$ are determined uniquely by their determinant:

$$\det h_1(w) = \det h_2(w) = w.$$

Proof. Since $x \notin E^1$, $x\overline{x} - 1 \neq 0$. Now we consider the equation

$$y + \overline{y} + y\overline{y}(x\overline{x} - 1) = 0,$$

whose locus is a conic (since $x\overline{x} - 1 \neq 0$) and by the intersection method, its R-rational points are parametrized by $\mathbb{P}^1(R)$. Explicitely, write $y = a + at\sqrt{-D}$, where $a, t \in R$. Then

$$a(1 + t^2)(x\overline{x} - 1) + 2 = 0.$$

and we obtain

$$y = -\frac{2(1 + t\sqrt{-D})}{(1 + t^2 D)(x\overline{x} - 1)} = -\frac{2}{(1 - t\sqrt{-D})(x\overline{x} - 1)}, \quad t \in \mathbb{P}^1(R).$$

Then the parametrization (9.10) follows from Lemma 9.1 and the fractional linear transform replacing t with w such that $(1 - w)/2 = (1 - t\sqrt{-D})^{-1}$. We have again used the fact that $J \in G'(\mathbb{Q})$. □

Remark 9.1. For $x_0 \in E^1$, the above computation of $H_{\gamma(x)J}$ is consistent with what we obtained before for $H_{\gamma(x_0)}$ after taking limit $x \mapsto x_0$ in (9.10).

We will henceforth write H_x for the image of $H_{\gamma(x)J} \subset G' \times G'$ under the projection to the first component, i.e.,

$$H_x(R) = \{h_1(u), \ u \in E^\times(R), \ u\overline{u} = 1\}$$

for $h_1(u)$ defined in (9.10).

As a consequence of Corollary 9.2, we obtain

Corollary 9.3. *Let $x \in E^\times - E^1$. Then H_x is isomorphic to $U(1)$, the unitary group of rank 1. In particular, $H_x(\mathbb{Q}) \backslash H_x(\mathbb{A})$ has volume 2 for the Tamagawa measure.*

Proof. A direct computation shows that

$$h_1(u_1)h_1(u_2) = h_1(u_1 u_2).$$

This gives the group law of $H_x(R)$, where R is a $\mathbb{Q}$-algebra. In particular, we have a natural isomorphism with the unitary group

$$H_x \simeq U(1) = E^1.$$

□

9.2. **Reduction to factorable integrals.** To simplify notations we will write

$$\mathcal{O}_x(f^{\mathfrak{n}}, \varphi') := \mathcal{O}_{\gamma(x)}(f^{\mathfrak{n}}, \varphi')$$

Observe that since $J \in G'(\mathbb{Q})$ and φ' is $G'(\mathbb{Q})$-invariant we have by suitable changes of variables that for $x \in E^\times - E^1$

$$\mathcal{O}_x(f^{\mathfrak{n}}, \varphi') = \mathcal{O}_{\gamma(x)}(f^{\mathfrak{n}}, \varphi') = \mathcal{O}_{\gamma(x)J}(f^{\mathfrak{n}}, \varphi')$$

$$= \int_{H_{\gamma(x)J}(\mathbb{Q}) \backslash G'(\mathbb{A}) \times G'(\mathbb{A})} f^{\mathfrak{n}}(y_1^{-1}\gamma(x)Jy_2)\overline{\varphi}'(y_1)\varphi'(y_2)dy_1dy_2$$

$$= \int_{H_x(\mathbb{A}) \backslash G'(\mathbb{A}) \times G'(\mathbb{A})} f^{\mathfrak{n}}(y_1^{-1}\gamma(x)Jy_2) \int_{[H_{\gamma(x)J}]} \overline{\varphi}'(h_1y_1)\varphi'(h_2y_2)dudy_1dy_2$$

where

$$[H_{\gamma(x)J}] := H_{\gamma(x)J}(\mathbb{Q}) \backslash H_{\gamma(x)J}(\mathbb{A}),$$

and

$$(h_1, h_2) = (h_1(u), h_2(u)) \in [H_{\gamma(x)J}].$$

For any (y_1, y_2), the u-integral can be evaluated as an (infinite) sum of Waldspurger's period integrals (see [Wal85]). However, we will be much more coarse and will content ourselves with the upperbound

$$\int_{[H_{\gamma(x)J}]} \varphi'(h_1y_1)\overline{\varphi}'(h_2y_2)du \ll \|\varphi'\|_\infty^2. \tag{9.11}$$

Therefore we obtain

$$|\mathcal{O}_x(f^{\mathfrak{n}}, \varphi')| \ll \|\varphi'\|_\infty^2 \mathcal{I}(f^{\mathfrak{n}}, x) \tag{9.12}$$

where

$$\mathcal{I}(f^{\mathfrak{n}}, x) = \int_{H_{\gamma(x)J}(\mathbb{A})\backslash G'(\mathbb{A})^2} |f^{\mathfrak{n}}(u^{-1}\gamma(x)Jv)|dudv,$$

say.

The main benefit of this rather crude treatment is that the resulting integral is factorable: we have

$$\mathcal{I}(f^{\mathfrak{n}}, x) = \prod_v \mathcal{I}_v(f_v^{\mathfrak{n}}, x)$$

where

$$\mathcal{I}_v(f_v^{\mathfrak{n}}, x) := \int_{H_{\gamma(x)J}(\mathbb{Q}_v)\backslash(G'\times G')(\mathbb{Q}_v)} |f_v^{\mathfrak{n}}(u^{-1}\gamma(x)Jv)|dudv; \tag{9.13}$$

indeed as we will verify in the next subsections below, for any given $x \in E - E_1$, we have

$$\mathcal{I}_p(f_p^{\mathfrak{n}}, x) = 1$$

for all but finitely many p.

In the subsequent subsections we analyse these local integrals $\mathcal{I}_v(f_v^{\mathfrak{n}}, x)$ give critera for vanishing and provide upper bounds.

In the sequel to simplify notations we will write $\mathcal{I}(x), \mathcal{I}_v(x)$ in place of $\mathcal{I}(f^{\mathfrak{n}}, x)$ and $\mathcal{I}_v(f_v^{\mathfrak{n}}, x)$.

9.3. **A vanishing criterion for the non-archimedan local integrals.** Let p be a prime, $\mathfrak{p}$ the place above p we have fixed (we take $\varpi_p = p$ if p is inert) and let ν be the associated valuation in the local field $E_{\mathfrak{p}}$.

Proposition 9.4. *Let $x \in E^\times - E^1$ a non-zero regular element. The following holds*

- *If p is ramified, then $\mathcal{I}_p(x) = 0$ unless*
$$x \in \mathcal{O}_{E,p}.$$
- *If p is inert, then $\mathcal{I}_p(x) = 0$ unless*
$$x \in p^{-\nu(\ell)}\mathcal{O}_{E,p} \quad \text{and } x\overline{x} \equiv 1 \,(\mathrm{mod}\,(N, p)).$$
- *If p is split then $\mathcal{I}_p(x) = 0$ unless*
$$\nu(N'x), \nu(N'\overline{x}) \geq 0.$$

From this we deduce immediately the a global vanishing criterion:

Theorem 9.5. *Notations being as above, let*

$$\mathfrak{X}(N, N', \ell) = \big\{x \in E^\times - E^1,\ x \in (\ell N')^{-1}\mathcal{O}_E :\ x\overline{x} \equiv 1 \pmod N\big\}. \tag{9.14}$$

For $x \in E^\times - E^1$ and not contained in $\mathfrak{X}(N, N', \ell)$, we have

$$\mathcal{I}(f^{\mathfrak{n}}, x) = 0$$

and by (9.12) we have

$$\mathcal{O}_{\gamma(x)}(f^{\mathfrak{n}}, \varphi') = 0 \tag{9.15}$$

Remark 9.2. This criterion is similar to the phenomenon encountered in the case of the forms of $\mathrm{GL}(2)\times\mathrm{GL}(1)$ in [RR05, §2.6] and [FW09].

We will prove this through case by case analysis: see Propositions 9.6, 9.7, 9.8, In fact we will also provide upper bounds for $\mathcal{I}_p(x)$. The non-split case is discussed in §9.4 and the following subsection and the split case in §9.5.

For any $\mathbb{Q}$-algebra R let

$$SG'(R):=SU(W)=\{g\in G'(R),\ \det g=1\}.$$

We start with the observation that a fundamental domain for the quotient $H_{\gamma(x)J}(\mathbb{Q}_p)\backslash(G'\times G')(\mathbb{Q}_p)$ is the subgroup

$$SG'(\mathbb{Q}_p)\times G'(\mathbb{Q}_p).$$

Indeed any $(u,v)\in G'\times G'(\mathbb{Q}_p)$ can be written

$$(u,v)=(g_1(w)u',g_2(w)v')$$

with

$$w=\det(u)=\det(g_1(w)),\ u'=g_1(w)^{-1}u\in SG'(\mathbb{Q}_p),\ v'=g_2(w)^{-1}v.$$

Moreover for $u_1,u_1'\in SG'(\mathbb{Q}_p)$ and $v,v'\in G'(\mathbb{Q}_p)$ we have

$$(u_1',v')=(g_1(w)u_1,g_2(w)v)\Longrightarrow w=1,\ g_1(w)=g_2(w)=\mathrm{Id}_3,\ u_1=u_1',\ v'=v.$$

We have therefore

$$\mathcal{I}_p(x):=\int_{SG'(\mathbb{Q}_p)\times G'(\mathbb{Q}_p)}|f_v^{\mathfrak{n}}(u^{-1}\gamma(x)Jv)|dudv.$$

9.4. **The non-split case.** In this section we evaluate the local integral $\mathcal{I}_p(x)$ when p is non-split: this implies that $\overline{\varpi}_p=\pm\varpi_p$, that $\mathfrak{n}_p=\mathrm{Id}_3$ and that $f_p^{\mathfrak{n}}=f_p$ is a scalar times the characteristic function of either $K_p(N)$ or of

$$G(\mathbb{Z}_p)A_rG(\mathbb{Z}_p)$$

for $A_r,\ r\geq 0$ the diagonal matrix defined in the Appendix (A.12) .

We use the Iwasawa decomposition of for $u,v\in SG'(\mathbb{Q}_p)\times G'(\mathbb{Q}_p)$.

We have $\det u=1$ and from the description of $G'(\mathbb{Q}_p)$ $\det v$ has valuation 0 so that (here we represent u and v as 2×2 matrices)

(9.16)
$$u=\begin{pmatrix}\varpi_p^i & \\ & \overline{\varpi}_p^{-i}\end{pmatrix}\begin{pmatrix}1 & b\\ & 1\end{pmatrix}k_1=u'k_1\kappa_1^i,\quad v=\begin{pmatrix}\varpi_p^j & \\ & \overline{\varpi}_p^{-j}\end{pmatrix}\begin{pmatrix}1 & b'\\ & 1\end{pmatrix}k_2=v'k_2,$$

with $i,j\in\mathbb{Z}$, $b,b'\in E_p^0$, $k_1,k_2\in G'(\mathbb{Z}_p)$, $\det k_1=1$ and $\kappa_1=\begin{pmatrix}\overline{\varpi}_p & \\ & \varpi_p\end{pmatrix}$.

By definition, the function f_p is bi-$K_p(N)$-invariant so that

$$f_p(u^{-1}\gamma(x)Jv)=f_p(k_1^{-1}u'^{-1}\gamma(x)Jv'k_2).$$

We have

$$\gamma(x)J=\begin{pmatrix}-\frac{1}{2} & \frac{1+x}{2} & \frac{x\overline{x}+3\overline{x}-x+1}{4}\\ -1 & x & \frac{(x+1)(\overline{x}-1)}{2}\\ 1 & 1-x & -\frac{(1-x)(1-\overline{x})}{2}\end{pmatrix}$$

and

$$u'^{-1}\gamma(x)Jv'=\begin{pmatrix}-\frac{\varpi_p^{j-i}}{2}-\overline{\varpi}_p^i\varpi_p^jb & e & z\\ -\varpi_p^j & x & f\\ \varpi_p^{i+j} & \overline{\varpi}_p^i(1-x) & g\end{pmatrix},\tag{9.17}$$

where

$$\begin{aligned} e &= \frac{\varpi_p^{-i}(1+x)}{2} - \overline{\varpi}_p^i b(1-x), \\ f &= -\varpi_p^j b' + \overline{\varpi}_p^{-j}\frac{(x+1)(\overline{x}-1)}{2}, \\ g &= \overline{\varpi}_p^i \varpi_p^j b' - \overline{\varpi}_p^{i-j}\frac{(x-1)(\overline{x}-1)}{2}, \end{aligned} \tag{9.18}$$

and

$$z = -\frac{1}{2}\varpi_p^{j-i}b' + \varpi_p^{-i}\overline{\varpi}_p^{-j}\frac{x\overline{x}+3\overline{x}-x+1}{4} - \overline{\varpi}_p^i\varpi_p^j bb' + \overline{\varpi}_p^{i-j}b\frac{(x-1)(\overline{x}-1)}{2}.$$

$$= \frac{1-x\overline{x}}{1-x}\varpi_p^{-i}\overline{\varpi}_p^{-j} + \frac{f}{1-x}\varpi_p^{-i} - \frac{1-\overline{x}}{1-x}e\overline{\varpi}_p^{-j} + \frac{ef}{1-x} \tag{9.19}$$

9.4.1. *The non-split case $f_p = 1_{K_p(N)}$.*

Proposition 9.6. *We assume here that p is non-split and that $f_p = 1_{K_p(N)}$. The following hold:*

- *If $\nu(x) < 0$ we have $\mathcal{I}_p(x) = 0$.*

Assume that $\nu(x) \geq 0$ (ie. $x \in \mathcal{O}_{E,p}$)

- *if $p \nmid 2D_E N$ and $\nu(x\overline{x}-1) = 0$, we have*
$$\mathcal{I}_p(x) = 1.$$
- *if $p \mid N$, and $\nu(x\overline{x}-1) \leq 0$ we have $f_p(u^{-1}\gamma(x)Jv) = 0$.*
- *In general, we have the bound*
$$\mathcal{I}_p(x) \ll e_p(x)(N,p)N(\varpi_p)^{3\nu(x\overline{x}-1)}; \tag{9.20}$$
here we have set
$$e_p(x) = (1+\nu(x-1))^2(1+\nu(x\overline{x}-1)),$$
$$N(\varpi_p) = |\mathcal{O}_{E_\mathfrak{p}}/\varpi_p\mathcal{O}_{E_\mathfrak{p}}| = p^{f_p},\ f_p = \begin{cases} 2 & \textit{if } p \textit{ is inert} \\ 1 & \textit{if } p \textit{ is ramified} \end{cases}$$
and the implicit constant is absolute.

Proof. As noted previously $f_p(u^{-1}\gamma(x)Jv)$ is non zero if and only if

$$u'^{-1}\gamma(x)Jv' \in k_1K_p(N)k_2^{-1} \subset G(\mathbb{Z}_p). \tag{9.21}$$

This implies that

$$\begin{cases} \nu(x) \geq 0,\ \nu(z) \geq 0 \\ j \geq 0 \\ i + \nu(1-x) \geq 0 \\ e, f, g, z, -\frac{\varpi_p^{j-i}}{2} - \overline{\varpi}_p^i\varpi_p^j b \in \mathcal{O}_{E_p} \end{cases} \tag{9.22}$$

In particular we have $f_p(u^{-1}\gamma(x)Jv) = 0$ unless $\nu(x) \geq 0$. The proves the first part of Proposition 9.6.

Since $j \geq 0$, then it follows from the equality

$$\varpi_p^j e - [-\frac{1}{2}\varpi_p^{j-i} - \overline{\varpi}_p^i\varpi_p^j b](1-x) = \varpi_p^{j-i}$$

that $j \geq i$.

Note that

$$\overline{\varpi}_p^i f + g = \overline{\varpi}_p^{i-j}(\overline{x}-1).$$

Hence

$$i - j + \nu(x-1) \geq \min\{0, i\}. \tag{9.23}$$

From the condition $e,f,z\in\mathcal{O}_{E,p}$, the equality (9.19) and the fact that $\nu(1-x)=\nu(1-\overline{x})$, we have

$$-i-j-\nu(x-1)+\nu(x\overline{x}-1)\geq\min\{-i-\nu(x-1),-j\}. \tag{9.24}$$

Suppose now that $\nu(x\overline{x}-1)=0$; this implies that $\nu(x-1)=0$ and therefore by (9.22) $i\geq 0$ and eventually $i\geq j$ by (9.23), therefore $i=j\geq 0$. By (9.24) we have $-2i\geq -i$ and therefore

$$i=j=0$$

and from (9.17) we conclude that

$$b,b'\in\mathcal{O}_{E_p}.$$

If p does not divide $2N$ we have $K_p(N)=G(\mathbb{Z}_p)$ so that if $\nu(x\overline{x}-1)=0$ we have

$$|f_p(u^{-1}\gamma(x)Jv)|=1$$

precisely, if

$$i=j=0,\ k_1\in SK_p,\ k_2\in K_p,\ b,b'\in\mathcal{O}_{E_p}$$

and otherwise it is zero. It follows that

$$\int_{H_{\gamma(x)J}(\mathbb{Q}_p)\backslash(G'\times G')(\mathbb{Q}_p)}\big|f_p(u^{-1}\gamma(x)Jv)\big|dudv=\mu(\mathcal{O}_{E_p})^2=1.$$

This proves the generic part of Proposition 9.6.

We return to the general case: given two integers i,j such that u,v have Iwasawa decomposition given by (9.16) and such that $f_p(u^{-1}\gamma(x)Jv)\neq 0$. From the previous discussion we see that

$$b=\frac{\varpi_p^{-i}\overline{\varpi}_p^{-i}(1+x)}{2(1-x)}-\frac{e}{\overline{\varpi}_p^{i}(1-x)}\in B_i,$$

$$b'=\varpi_p^{-j}\overline{\varpi}_p^{-j}\frac{(x+1)(\overline{x}-1)}{2}-\varpi_p^{-j}f\in B'_j$$

where

$$B_i:=\frac{\varpi_p^{-i}\overline{\varpi}_p^{-i}(1+x)}{2(1-x)}-\frac{1}{\overline{\varpi}_p^{i}(1-x)}\mathcal{O}_{E_p},$$

$$B'_j:=\varpi_p^{-j}\overline{\varpi}-p^{-j}\frac{(x+1)(\overline{x}-1)}{2}-\varpi_p^{-j}\mathcal{O}_{E_p}$$

We have

$$\int_{B_i}db\int_{B'_j}db'\ll\mathrm{Nr}(\varpi_p)^{i+\nu(1-x)+j}. \tag{9.25}$$

Suppose $i\leq 0$. Then by (9.23) we have $j\leq\nu(x-1)$ and

$$-\nu(x-1)\leq i\leq 0\leq j\leq\nu(x-1).$$

Then by (9.25)

$$\sum\sum_{-\nu(x-1)\leq i\leq 0\leq j\leq\nu(x-1)}\int_{B_i}db\int_{B'_j}db'\leq(1+\nu(x-1))^2N(\varpi_p)^{2\nu(x-1)}. \tag{9.26}$$

Suppose $i\geq 1$. Then similar arguments as before show that $j-\nu(x-1)\leq i\leq j$ and $1\leq j\leq\nu(x\overline{x}-1)$. So in this range we have

$$\sum\sum_{i,j\cdots}\int_{B_i}db\int_{B'_j}db'\leq e_p(x)N(\varpi_p)^{2\nu(x\overline{x}-1)+\nu(x-1)}. \tag{9.27}$$

Then (9.20) follows from (9.26) and (9.27). When $p=2$, a similar argument also applies with some worse (but absolute) implied constant. The proves the last part of Proposition 9.6.

Suppose now that $p \mid N$. Then $K_p(N) = K_p(p) \subset K_p$ is the Iwahori subgroup and its intersection with $G'(\mathbb{Z}_p)$ is $K'_p(N) = K'_p(p)$ the Iwahori subgroup of $G'(\mathbb{Z}_p)$. It follows that the function on $G'(\mathbb{Z}_p) \times G'(\mathbb{Z}_p)$

$$(u, v) \mapsto f_p(u^{-1}\gamma(x)Jv)$$

is bi-$K'_p(p)$ invariant.

Because of this we will evaluate the integral $\mathcal{I}_p(x)$ using the Iwasawa decompositions of u and v (9.16) and in particular decompose the two integrals over the $k_1 \in SG'(\mathbb{Z}_p)$ and $k_2 \in G'(\mathbb{Z}_p)$ variable into a sum of $(p+1)^2$-integrals supported along $SK'_p(p) \times K'_p(p)$-cosets using Lemma A.1.

Given u, v such that $f_p(u^{-1}\gamma(x)v) \neq 0$ and whose Iwasawa decompositions are given by (9.16) and let

$$g := \begin{pmatrix} 1 & & -b \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} \varpi_p^{-i} & & \\ & 1 & \\ & & \overline{\varpi}_p^{i} \end{pmatrix} \gamma(x) J \begin{pmatrix} \varpi_p^{j} & & \\ & 1 & \\ & & \overline{\varpi}_p^{-j} \end{pmatrix} \begin{pmatrix} 1 & & b' \\ & 1 & \\ & & 1 \end{pmatrix},$$

Since f_p is supported on $K_p(p)$, for the integrand in one of these to be non-zero one of the following hold :

$$g \in K_p(p), \text{ or } J \begin{pmatrix} 1 & & -\delta \\ & 1 & \\ & & 1 \end{pmatrix} g \in K_p(p) \text{ or } g \begin{pmatrix} 1 & & \delta' \\ & 1 & \\ & & 1 \end{pmatrix} J \in K_p(p)$$

$$\text{or } J \begin{pmatrix} 1 & & -\delta \\ & 1 & \\ & & 1 \end{pmatrix} g \begin{pmatrix} 1 & & \delta' \\ & 1 & \\ & & 1 \end{pmatrix} J \in K_p(p) \text{ for some } \delta, \delta' \in \mathbb{Z}_p/p\mathbb{Z}_p.$$

We will show in each case that $\nu(x\overline{x} - 1) \geq 1$.

If we assume instead that $\nu(x\overline{x} - 1) = 0$, we have

$$\nu(x - 1) = \nu(\overline{x} + 1) = \nu(1 - \overline{x}) = 0.$$

We will obtain a contradiction for each of the $1 + 2p + p^2$ possible locations of g:

(i) Assume $g \in K_p(p)$. Then by (9.17) we have $i \geq 1$, $j \geq 1$ and $e, f, z \in \mathcal{O}_{E_p}$. But this contradicts the algebraic relation (9.19), which forces that $z \notin \mathcal{O}_{E_p}$ a contradiction and therefore $\nu(x\overline{x} - 1) \geq 1$.

(ii) Suppose $J \begin{pmatrix} 1 & & -\delta \\ & 1 & \\ & & 1 \end{pmatrix} g \in K_p(p)$. This implies that

$$\begin{pmatrix} -\frac{\varpi_p^{j-i}}{2} - \overline{\varpi}_p^{i}\varpi_p^{j} b & e & z \\ -\varpi_p^{j} & x & f \\ \varpi_p^{i+j} & \overline{\varpi}_p^{i}(1 - x) & \overline{\varpi}_p^{i}\varpi_p^{j} b' - \overline{\varpi}_p^{i-j}\frac{(x-1)(\overline{x}-1)}{2} \end{pmatrix} \in JK_p(p). \tag{9.28}$$

here we have made the change of variable

$$b \mapsto b + \delta.$$

Hence $i \geq 0$, $j \geq 1$, $\nu(e) \geq 1$, $\nu(f) \geq 0$ and $\nu(z) \geq 0$. However, this contradicts (9.19) as well.

Notice then, that for $\nu(x\overline{x} - 1) \geq 1$, (9.28) leads to

$$-\frac{\varpi_p^{j-i}}{2} - \overline{\varpi}_p^{i}\varpi_p^{j} b \in \varpi_p \mathcal{O}_{E_p}$$

and $\varpi_p^{i+j} \in \mathcal{O}_{E_p}^{\times}$. So there is only one possible choice for $b \mod \varpi_p$.

(iii) Suppose that $g\begin{pmatrix}1&&\delta'\\&1&\\&&1\end{pmatrix}J\in K_p(p)$. This is similar to the preceding case (ii). Again we must have $\nu(x\overline{x}-1)\geq 1$ and there is only one choice for $b' \mod \varpi_p$.

(iv) Suppose finally that $J\begin{pmatrix}1&&-\delta\\&1&\\&&1\end{pmatrix}g\begin{pmatrix}1&\delta'&\\&1&\\&&1\end{pmatrix}J\in K_p(p)$. We then have again

$$\begin{pmatrix}-\frac{\varpi_p^{j-i}}{2}-\overline{\varpi}_p^i\varpi_p^jb & e & z\\ -\varpi_p^j & x & f\\ \varpi_p^{i+j} & \overline{\varpi}_p^i(1-x) & \overline{\varpi}_p^i\varpi_p^jb'-\overline{\varpi}_p^{i-j}\frac{(x-1)(\overline{x}-1)}{2}\end{pmatrix}\in JK_p(p);$$

where this time we have made the change of variables

$$b\mapsto b+\delta,\ b'\mapsto b'+\delta'.$$

We have therefore $i\geq 0$, $j\geq 0$, $\nu(e)\geq 1$, $\nu(f)\geq 1$ and $\nu(z)\geq 1$, which contradicts (9.19) and necessarily $\nu(x\overline{x}-1)\geq 1$. Since $\nu(e)\geq 1$ and $\nu(f)\geq 1$, this implies that

$$b\in C_i:=\frac{\varpi_p^{-i}\overline{\varpi}_p^{-i}(1+x)}{2(1-x)}-\frac{1}{\overline{\varpi}_p^i(1-x)}\varpi_p\mathcal{O}_{E_p},$$
$$b'\in C_j':=\varpi_p^{-j}\overline{\varpi_p}-p^{-j}\frac{(x+1)(\overline{x}-1)}{2}-\varpi_p^{1-j}\mathcal{O}_{E_p}.$$

Therefore,

$$\int_{C_i}db\int_{C_j'}db'\ll N(\varpi_p)^{i+\nu(x-1)+j-2}.$$

Notice the extra saving p^{-2} (compared with (9.25)): it comes to compensate the contribution of the p^2 choices of δ and δ' in $\mathbb{Z}_p/p\mathbb{Z}_p$.

In all case we have $\nu(x\overline{x}-1)\geq 1$. Integrating over these $SK_p'(p)\times K_p'(p)$ cosets, we conclude that when $p\mid N$, one has

$$\mathcal{I}_p(x)\ll\frac{e_p(x)N(\varpi_p)^{3\nu(x\overline{x}-1)}\mu(SK_p'(p))\mu(K_p'(p))}{\mu(K_p(N))}\ll\frac{(1/p)^2}{1/p^3}e_p(x)N(\varpi_p)^{3\nu(x\overline{x}-1)}.$$

This completes the proof of the Proposition when $p|N$. □

9.4.2. *The inert Hecke case.* We now evaluate $\mathcal{I}_p(x)$ for p inert in the last remaining case.

Proposition 9.7. *Let p be an inert prime, $r\geq 0$ and suppose that*

$$f_p=1_{G(\mathbb{Z}_p)A_{p^r}G(\mathbb{Z}_p)}.$$

We have

$$\mathcal{I}_p(x)=0$$

unless $\nu(x)\geq -r$ in which case we have

$$\mathcal{I}_p(x)\ll(r+|\nu(1-x\overline{x})|+|\nu(1-x)|)(p^{3r+2\nu(1-x)}+p^{7r+2\nu(1-x\overline{x})}).\tag{9.29}$$

Here the implied constant is absolute.

Proof. We use again the notations on §9.4. Let u,v such that

$$f_p(u^{-1}\gamma(x)Jv)=f_p(u'^{-1}\gamma(x)Jv')\neq 0,$$

which, by (9.17), amounts to

$$u'^{-1}\gamma(x)Jv' = \begin{pmatrix} -\frac{p^{j-i}}{2} - p^{j+i}b & e & z \\ -p^j & x & f \\ p^{i+j} & p^i(1-x) & g \end{pmatrix} \in G(\mathbb{Z}_p)A_rG(\mathbb{Z}_p). \tag{9.30}$$

where

$$\begin{aligned} e &= \frac{p^{-i}(1+x)}{2} - p^i b(1-x), \\ f &= -p^j b' + p^{-j}\frac{(x+1)(\overline{x}-1)}{2}, \\ g &= p^{i+j}b' - p^{i-j}\frac{(x-1)(\overline{x}-1)}{2}. \end{aligned} \tag{9.31}$$

and

$$z = \frac{1-x\overline{x}}{1-x}p^{-i-j} + \frac{f}{1-x}p^{-i} - \frac{1-\overline{x}}{1-x}ep^{-j} + \frac{ef}{1-x}.$$

Analyzing the entries in (9.30) we then derive that $z, e, f \in p^{-r}\mathbb{Z}_p$. Hence,

$$\begin{cases} \nu(x) \geq -r, \ \nu(g) \geq -r, \\ j \geq -r, \ i \geq -r - \nu(1-x), \\ f = -p^j b' + p^{-j}\frac{(x+1)(\overline{x}-1)}{2} \in p^{-r}\mathbb{Z}_p, \\ e = \frac{p^{-i}(1+x)}{2} - p^i b(1-x) \in p^{-r}\mathbb{Z}_p, \\ z = \frac{1-x\overline{x}}{1-x}p^{-i-j} + \frac{f}{1-x}p^{-i} - \frac{1-\overline{x}}{1-x}ep^{-j} + \frac{ef}{1-x} \in p^{-r}\mathbb{Z}_p, \end{cases} \tag{9.32}$$

where the last constraint yields that

$$-i - j + \nu(1 - x\overline{x}) \geq \min\big\{ -r - i, -r - j + \nu(1-x), -2r, -r + \nu(1-x) \big\}. \tag{9.33}$$

Since $p^i f + g = p^{i-j}(\overline{x} - 1)$, then

$$i - j + \nu(1-x) = \min\{i + \nu(f), \nu(g)\} \geq \min\{i - r, -r\}. \tag{9.34}$$

Since $p^j e - (1-x)a = p^{j-i}$, then

$$j - i = \min\{j + \nu(e), \nu(1-x) + \nu(a)\} \geq \min\{j - r, \nu(1-x) - r\}. \tag{9.35}$$

We now separate the cases to derive the ranges of i and j as follows.

(1) Suppose $i \leq 0$. Then by (9.34) we obtain that $i - j + \nu(1-x) \geq i - r$, implying that $j \leq r + \nu(1-x)$. Thus in this case we have $-r \leq i \leq 0$ and $-r \leq j \leq r + \nu(1-x)$.

(2) Suppose that $i \geq 1$. Then (9.34) gives

$$i - j + \nu(1-x) \geq -r. \tag{9.36}$$

- Suppose further that $j \leq \nu(1-x)$. Then it follows from (9.35) that $j - i \geq j - r$, i.e., $i \leq r$. Hence, in this case we have $1 \leq i \leq r$ and $-r \leq j \leq \nu(1-x)$.
- Suppose that $j > \nu(1-x)$. Then by (9.35) we have $j - i \geq \nu(1-x) - r$, i.e., $-r - i \geq -2r - j + \nu(1-x)$. Substituting this into (9.33) to obtain

$$\begin{aligned} -i - j + \nu(1 - x\overline{x}) &\geq \min\big\{ -2r - j + \nu(1-x), -2r, -r + \nu(1-x) \big\} \\ &= \min\big\{ -2r - j + \nu(1-x), -2r \big\} = -2r - j + \nu(1-x). \end{aligned}$$

Here we note the fact that $-2r \leq -r + \nu(1-x)$ and $j > \nu(1-x)$. Therefore, we have $1 \leq i \leq 2r + \nu(1 - x\overline{x}) - \nu(1-x)$. By (9.36),

$$\nu(1-x) < j \leq i + r + \nu(1-x) \leq 3r + \nu(1 - x\overline{x}).$$

Denote by $\mathcal{S}$ the support of $(i,j)\in\mathbb{Z}^2$ determined by (9.32). From the above discussion we see that $\#\mathcal{S}\ll r+|\nu(1-x\overline{x})|+|\nu(1-x)|$ and for $(i,j)\in\mathcal{S}$,

$$i+j\leq\max\{r+\nu(1-x),5r+2\nu(1-x\overline{x})-\nu(1-x)\}.$$

Note that (9.32) also implies that b (resp. b') ranges over a translate of $p^{-i-r-\nu(1-x)}\mathbb{Z}_p$ (resp. $p^{-j-r}\mathbb{Z}_p$). Therefore,

$$\mathcal{I}_p(x)\ll\sum_{(i,j)\in\mathcal{S}}\int_{p^{-i-r-\nu(1-x)}\mathbb{Z}_p}db\int_{p^{-j-r}\mathbb{Z}_p}db'\ll\sum_{(i,j)\in\mathcal{S}}p^{i+j}\cdot p^{\nu(1-x)+2r}.$$

Hence, (9.29) follows. $\square$

9.5. **The split case.** Let p be a prime split in E, the corresponding ideal of K decomposes as $\mathfrak{p}\overline{\mathfrak{p}}=(p)$ and we have $E\otimes_{\mathbb{Q}}\mathbb{Q}_p\simeq E_{\mathfrak{p}}\times E_{\overline{\mathfrak{p}}}\simeq\mathbb{Q}_p\times\mathbb{Q}_p$. Let ϖ in $E_{\mathfrak{p}}\simeq\mathbb{Q}_p$ be an uniformizer.

In the split case, we have $G'(\mathbb{Q}_p)\simeq\mathrm{GL}(2,\mathbb{Q}_p)$ and by Corollary 9.2 one has

$$H_x(\mathbb{Q}_p)\simeq\mathrm{GL}(1,\mathbb{Q}_p).$$

Given $u,v\in G'(\mathbb{Q}_p)$, we write them in Iwasawa coordinates:

$$u=\begin{pmatrix}p^{i_1}&\\&p^{i_2}\end{pmatrix}\begin{pmatrix}1&b\\&1\end{pmatrix}k_1,\quad v=\begin{pmatrix}p^{j_1}&\\&p^{j_2}\end{pmatrix}\begin{pmatrix}1&b'\\&1\end{pmatrix}k_2,\tag{9.37}$$

where $i_1,i_2,j_1,j_2\in\mathbb{Z}$, $b,b'\in\mathbb{Q}_p$ and $k_1,k_2\in G'(\mathbb{Z}_p)$. In particular if $\nu(\det u)=0$ we have $i_2=-i_1$.

Applying this decomposition to the variable u in the integral $\mathcal{I}_p(x)$ we have

$$\mathcal{I}_p(x)=\sum_{i\in\mathbb{Z}}\int_{SG'(\mathbb{Z}_p)}\int_{\mathbb{Q}_p}\int_{G'(\mathbb{Q}_p)}\left|f_p^{\mathfrak{n}}\left(k_1^{-1}\begin{pmatrix}p^{-i}&&-p^ib\\&1&\\&&p^i\end{pmatrix}\gamma(x)Jv\right)\right|dbdk_1dv.$$

Proposition 9.8. *Let p be a prime, split in E and $x\in E^\times-E^1$.*

(1) *If $p\nmid N'$ we have*

$$\mathcal{I}_p(x)=0$$

unless $\nu(x)$, $\nu(\overline{x})\geq0$; in that case we have the bound

$$\mathcal{I}_p(x)\ll(1+\nu(P(x,\overline{x}))p^{3\nu(x\overline{x}-1)},\tag{9.38}$$

where $P(X,Y)\in\mathbb{Z}[X,Y]$ is a polynomial independent of p whose degree and coefficients are absolutely bounded; moreover the implicit constant is absolute.

– *In addition, if $p\neq2$ and*

$$\nu(x\overline{x}-1)=\nu(1-x)=\nu(1-\overline{x})=0$$

we have $\mathcal{I}_p(x)=1$.

(2) *If $p|N'$, we have*

$$\mathcal{I}_p(x)=0\tag{9.39}$$

unless $x\in p^{-1}\mathcal{O}_{E_p}$, i.e., $\nu(x)\geq-1$ and $\nu(\overline{x})\geq-1$.

– *Moreover if*

$$\nu(x),\nu(\overline{x})\geq0,$$

one has

$$\mathcal{I}_p(x)\ll(1+\nu(P(x,\overline{x})))^2\left(p^{\nu(x\overline{x}(1-x)^2(1-\overline{x})^2)}+p^{\nu(1-x\overline{x})-1}+p^{2\nu(1-x\overline{x})-2}+p^{2\nu(1-x\overline{x})-1}\right)\tag{9.40}$$

where $P(X,Y)\in\mathbb{Z}[X,Y]$ is a polynomial independent of p whose degree and coefficients are absolutely bounded; moreover the implicit constant is absolute.

Proof. We start with the case $p \nmid N'$. We have $f_p^{\mathfrak{n}} = f_p$ and

$$f_p(u^{-1}\gamma(x)Jv) = 0$$

if and only if $u^{-1}\gamma(x)Jv \in G(\mathbb{Z}_p)$. Since the determinant of v has valuation 0, its Iwasawa coordinates (9.37) satisfy $j_1 + j_2 = 0$. Hence $\mathcal{I}_p(x)$ is bounded from above by

$$\sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}\int_{\mathbb{Q}_p}\int_{\mathbb{Q}_p}|f_p\left(\begin{pmatrix}p^i & & p^ib\\ & 1 & \\ & & p^{-i}\end{pmatrix}^{-1}\gamma(x)J\begin{pmatrix}p^j & & p^jb'\\ & 1 & \\ & & p^{-j}\end{pmatrix}\right)|dbdb'.$$

The proof then follow the same lines as for the nonsplit case considered in Proposition 9.6. We also note that at a split place p, we always have $G'(\mathbb{Z}_p) \subset G(\mathbb{Z}_p)$ if $p \nmid N'$.

We now consider the case of a split prime $p \mid N'$. Let $b, b' \in \mathbb{Q}_p$ and $i, j \in \mathbb{Z}$. We have

$$\begin{pmatrix}p^i & & p^ib\\ & 1 & \\ & & p^{-i}\end{pmatrix}^{-1}\gamma(x)J\begin{pmatrix}p^j & & p^jb'\\ & 1 & \\ & & p^{-j}\end{pmatrix} = \begin{pmatrix}a & e & z\\ -p^j & x & f\\ p^{i+j} & p^i(1-x) & g\end{pmatrix}, \tag{9.41}$$

where

$$\begin{cases} a = -\frac{1}{2}p^{j-i} - p^{i+j}b\\ e = \frac{p^{-i}(1+x)}{2} - p^ib(1-x)\\ f = -p^jb' + p^{-j}.\frac{(x+1)(\overline{x}-1)}{2}\\ g = p^{i+j}b' - \frac{(x-1)(\overline{x}-1)}{2}p^{i-j}\\ z = -\frac{1}{2}p^{j-i}b' + p^{-i-j}y - p^{i+j}bb' + p^{i-j}b\frac{(x-1)(\overline{x}-1)}{2}. \end{cases} \tag{9.42}$$

where

$$y = \frac{x\overline{x} + 3\overline{x} - x + 1}{4}.$$

Then one has an explicit algebraic relation

$$z = \frac{1-x\overline{x}}{1-x}p^{-i-j} + \frac{f}{1-x}p^{-i} - \frac{1-\overline{x}}{1-x}ep^{-j} - \frac{ef}{1-x}. \tag{9.43}$$

Taking inverse of (9.41) to obtain

$$\begin{pmatrix}p^j & & p^jb'\\ & 1 & \\ & & p^{-j}\end{pmatrix}^{-1}J\gamma(x)^{-1}\begin{pmatrix}p^i & & p^ib\\ & 1 & \\ & & p^{-i}\end{pmatrix} = \begin{pmatrix}a' & e' & z'\\ p^i(1-\overline{x}) & \overline{x} & f'\\ p^{i+j} & -p^j & g'\end{pmatrix}, \tag{9.44}$$

where

$$\begin{cases} a' = -\frac{(1-x)(1-\overline{x})}{2}p^{i-j} - p^{i+j}b'\\ e' = \frac{(x-1)(\overline{x}+1)}{2}p^{-j} + p^jb'\\ f' = (1-\overline{x})p^ib + \frac{1+\overline{x}}{2}p^{-i}\\ g' = p^{i+j}b - \frac{1}{2}p^{j-i}\\ z' = -p^{i-j}b\frac{(x-1)(\overline{x}-1)}{2} + p^{-i-j}\overline{y} - p^{i+j}bb' + \frac{1}{2}p^{j-i}b' \end{cases} \tag{9.45}$$

and one notes the algebraic relation

$$z' = \frac{1-x\overline{x}}{1-\overline{x}}p^{-i-j} + \frac{e'}{1-\overline{x}}p^{-i} - \frac{1-x}{1-\overline{x}}f'p^{-j} - \frac{e'f'}{1-\overline{x}}. \tag{9.46}$$

We also note that

$$\begin{aligned} &a + g' = -p^{j-i},\ a' + g = -(1-x)(1-\overline{x})p^{i-j},\\ &p^i.f + g = p^{i-j}(\overline{x}-1),\ p^jf' - (1-\overline{x})g' = p^{j-i} \end{aligned} \tag{9.47}$$

By definition (9.13), we have

$$\mathcal{I}_p(x)=\int_{SG'(\mathbb{Q}_p)\times G'(\mathbb{Q}_p)}\left|f_p(\widetilde{\mathfrak{n}}_p^{-1}u^{-1}\gamma(x)Jv\widetilde{\mathfrak{n}}_p)\right|dudv.$$

where we recall that $f_p=\mathbf{1}_{K_p}/\mu(K_p),\ K_p=G(\mathbb{Z}_p)$ and

$$\widetilde{\mathfrak{n}}_p=w'\mathfrak{n}_pw'=\begin{pmatrix}1&p^{-1}&\\&1&\\&&1\end{pmatrix}$$

We will apply the Iwasawa decomposition as in the beginning of the proof of Proposition 9.8.

Due to the conjugation by $\widetilde{\mathfrak{n}}_p$, the function

$$(u,v)\mapsto f_p(\widetilde{\mathfrak{n}}_p^{-1}u^{-1}\gamma(x)Jv\widetilde{\mathfrak{n}}_p)$$

is bi $I_p'(1)$-invariant (since $\widetilde{\mathfrak{n}}_pI_p'(1)\widetilde{\mathfrak{n}}_p^{-1}\subset K_p$) so we shall further decompose $G'(\mathbb{Z}_p)$ into a union of disjoint cosets of $I_p'(1)$: we start with the Iwahori decomposition for G' (see (5.66))

$$G'(\mathbb{Z}_p)=I_p'\sqcup I_p'JI_p'=I_p'\sqcup\bigsqcup_{\delta\in(\mathbb{Z}/p\mathbb{Z})^\times}I_p'J\begin{pmatrix}\delta&&\\&1&\\&&1\end{pmatrix}I_p'(1)$$

We then check four cases as in Lemma 9.6.

Case I. We start with the most complicated case:

$$u=\begin{pmatrix}p^i&&p^ib\\&1&\\&&p^{-i}\end{pmatrix}\begin{pmatrix}1&&\mu\\&1&\\&&1\end{pmatrix}J\begin{pmatrix}\tau^{-1}&&\\&1&\\&&1\end{pmatrix}k_1,$$
$$v=\begin{pmatrix}p^j&&p^jb'\\&1&\\&&p^{-j}\end{pmatrix}\begin{pmatrix}1&&\mu'\\&1&\\&&1\end{pmatrix}J\begin{pmatrix}\delta&&\\&1&\\&&1\end{pmatrix}k_2,$$

with $\mu,\mu'\in\mathbb{Z}_p$ run over representatives of $\mathbb{Z}_p/p\mathbb{Z}_p$ and $\tau,\delta\in\mathbb{Z}_p^\times$ run over representatives of $(\mathbb{Z}_p/p\mathbb{Z}_p)^\times$, and $k_1,k_2\in I_p'(1)$.

Let $\mathcal{I}_p(x;1)$ be the contribution to $\mathcal{I}_p(x)$ of all the u,v whose Iwasawa decomposition is of the above form. We first look for some necessary condition for $\mathcal{I}_p(x;1)$ to be non zero.

For $\delta\in\mathbb{Z}_p^\times$, we denote by

$$\mathfrak{u}_\delta=\begin{pmatrix}1&&\\&1&\\&\delta p^{-1}&1\end{pmatrix}=J\begin{pmatrix}\delta&&\\&1&\\&&1\end{pmatrix}\widetilde{\mathfrak{n}}_pJ.$$

Then, since $J\in K_p$, $f_p(\widetilde{\mathfrak{n}}_p^{-1}u^{-1}\gamma(x)Jv\widetilde{\mathfrak{n}}_p)\neq0$ if and only if

$$\mathfrak{u}_\tau^{-1}\begin{pmatrix}p^i&&p^i(b+\mu)\\&1&\\&&p^{-i}\end{pmatrix}^{-1}\gamma(x)J\begin{pmatrix}p^j&&p^j(b'+\mu')\\&1&\\&&p^{-j}\end{pmatrix}\mathfrak{u}_\delta\in K_p\tag{9.48}$$

or equivalently

$$\begin{pmatrix}a&e+\frac{\delta}{p}z&z\\-p^j&x+\frac{\delta}{p}f&f\\p^{i+j}+\tau p^{j-1}&p^i(1-x)+\frac{\delta}{p}g-\frac{\tau}{p}(x+\delta p^{-1}f)&g-\tau p^{-1}f\end{pmatrix}\in K_p,\tag{9.49}$$

and taking the inverse of (9.49) we also obtain the condition

$$\begin{pmatrix} a' & e'+\frac{\tau}{p}z' & z' \\ p^i(1-\overline{x}) & \overline{x}+\frac{\tau}{p}f' & f' \\ p^{i+j}-\delta p^{i-1}(1-\overline{x}) & -p^j+\frac{\tau}{p}g'-\frac{\delta}{p}(\overline{x}+\frac{\tau}{p}f') & g'-\frac{\delta}{p}f' \end{pmatrix}\in K_p. \tag{9.50}$$

Here a,e,f,g,z are defined in (9.42) and a',e',f',g',z' as defined in (9.45).

These conditions imply that

$$\nu(f),\ \nu(f'),\ \nu(x+\delta p^{-1}f),\ \nu(\overline{x}+\tau p^{-1}f')\geq 0$$

which in turn imply that

$$\nu(x),\ \nu(\overline{x})\geq -1.$$

This proves Proposition 9.8 in Case I.

Note also that (9.49) and (9.50) imply that

$$i':=i+\nu(1-\overline{x})\geq 0,\ j\geq 0, \tag{9.51}$$

$$\nu(z),\ \nu(z')\geq 0,\ \nu(e),\ \nu(e')\geq -1.$$

We have

$$\mathcal{I}_p(x;1)\leq \mu(I_p'(1))^2p^2\sum_{i\geq -\nu(1-\overline{x})}\sum_{j\geq 0}\int_{\mathbb{Q}_p}\int_{\mathbb{Q}_p}\sum_{\tau}\sum_{\delta}\mathbf{1}_{K_p}(\cdots)dbdb', \tag{9.52}$$

where the "$\cdots$" in the parenthesis indicates the left hand side of (9.49) with

$$\mu=\mu'=0.$$

Indeed, up to changing variables in the b and b' integrals, we may assume this is the case.

Define

$$\mathcal{I}_p^{j=0}(x;1):=\mu(I_p'(1))^2p^2\sum_{i\geq -\nu(1-\overline{x})}\sum_{j=0}\int_{\mathbb{Q}_p}\int_{\mathbb{Q}_p}\sum_{\tau}\sum_{\delta}\mathbf{1}_{K_p}(\cdots)dbdb',$$

$$\mathcal{I}_p^{j>0}(x;1):=\mu(I_p'(1))^2p^2\sum_{i\geq -\nu(1-\overline{x})}\sum_{j>0}\int_{\mathbb{Q}_p}\int_{\mathbb{Q}_p}\sum_{\tau}\sum_{\delta}\mathbf{1}_{K_p}(\cdots)dbdb'.$$

We now prove Proposition 9.8 in case I and when assuming that

$$\nu(x),\ \nu(\overline{x})\geq 0$$

and therefore

$$\nu(1\pm x),\ \nu(1\pm\overline{x})\geq 0.$$

By (9.49) and (9.50), we have

$$\nu(f)\geq 1,\ \nu(f')\geq 1 \tag{9.53}$$

$$g=p^{i+j}b'-\frac{(x-1)(\overline{x}-1)}{2}p^{i-j}\in\mathbb{Z}_p,\ g'=-\frac{p^{j-i}}{2}-p^{i+j}b\in\mathbb{Z}_p.$$

Also since (see (9.47))

$$p^{j-i}=a+g'\in\mathbb{Z}_p \text{ and } -(x-1)(\overline{x}-1)p^{i-j}=a'+g\in\mathbb{Z}_p$$

we have

$$0\leq j-i\leq\ \nu(1-x)+\nu(1-\overline{x}). \tag{9.54}$$

From $\nu(e')\geq -1$ and $\nu(f)\geq 1$ we then have

$$j\leq 1+\nu(x(\overline{x}-1)). \tag{9.55}$$

and likewise

$$j\leq 1+\nu(\overline{x}(x-1))$$

so that

$$j\leq 1+\frac{\nu(x\overline{x}(x-1)(\overline{x}-1))}{2}. \tag{9.56}$$

Localization of b, b'. Another general observation in Case I is that, since

$$f=-p^jb'+p^{-j}.\frac{(x+1)(\overline{x}-1)}{2}\in p\mathbb{Z}_p,\ f'=(1-\overline{x})p^ib+\frac{1+\overline{x}}{2}p^{-i}\in p\mathbb{Z}_p, \tag{9.57}$$

b and b' are contained, respectively, in translates of $p^{-i-\nu(1-\overline{x})+1}\mathbb{Z}_p$ and $p^{-j+1}\mathbb{Z}_p$ (depending only on x) whose volumes are $p^{i+\nu(1-\overline{x})-1}$ and p^{j-1}.

Similarly since

$$g=p^{i+j}b'-\frac{(x-1)(\overline{x}-1)}{2}p^{i-j}\in\mathbb{Z}_p,\ g'=-\frac{p^{j-i}}{2}-p^{i+j}b\in\mathbb{Z}_p, \tag{9.58}$$

b and b' are contained in translates of $p^{-(i+j)}\mathbb{Z}_p$ (depending only on x) whose volumes are p^{i+j}.

We will use either of these informations depending on the values of

$$\min(i+\nu(1-\overline{x})-1,i+j)\text{ and }\min(j-1,i+j).$$

For this we need to split the discussion into further cases:

The case $j=0$. Since $p^i+\tau p^{-1}\in\mathbb{Z}_p$ we have $i=-1$ and $\tau\equiv-1 \pmod p$. Since $i+j=-1$ we have that b and b' each belong to some translates of $p\mathbb{Z}_p$. Hence we have

$$\mathcal{I}_p^0(x;1)\leq\frac{\mu(I_p'(1))^2p^2(p-1)}{\mu(K_p)}\int_{p\mathbb{Z}_p}\int_{p\mathbb{Z}_p}dbdb'\leq\frac{1}{p^2(p-1)}. \tag{9.59}$$

The case $j\geq 1$: $\nu(x)=\nu(\overline{x}-1)=0$. Suppose first that $\nu(x)=\nu(\overline{x}-1)=0$. From (9.51), (9.55) and (9.54), we have

$$0\leq i\leq j=1.$$

Finally, since

$$p^{i+1}-\delta p^{i-1}(1-\overline{x})\in\mathbb{Z}_p$$

we conclude that

$$i=j=1.$$

In particular by (9.57), b and b' are contained in single translates of $\mathbb{Z}_p$.

We need to split the discussion into two further cases:

– Suppose that, given x,e,f,z, there exists at most one $\delta \pmod p$ satisfying (9.49) and (9.50) we then have (under the assumption that $\nu(x)=\nu(\overline{x}-1)=0$)

$$\mathcal{I}_p^{>0}(x;1)\leq\frac{\mu(I_p'(1))^2(p-1)}{\mu(K_p)}\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}dbdb'\leq\frac{1}{p^2(p-1)}. \tag{9.60}$$

– Suppose instead that, given x,e,f,z, there exists $\delta_1\not\equiv\delta_2 \pmod p$ satisfying (9.49) and (9.50); we then have $(\delta_1-\delta_2)p^{-1}z\in\mathbb{Z}_p$; this implies that $\nu(e)\geq 0$ and

$$(1-x\overline{x})p^{-2}+fp^{-1}-(1-\overline{x})ep^{-1}-ef\in(1-x)p\mathbb{Z}_p.$$

Since the last three terms belong to $p^{-1}\mathbb{Z}_p$ we must have

$$\nu(x\overline{x}-1)\geq 1.$$

In all cases, we conclude that

$$\mathcal{I}_p^{>0}(x;1)\leq\frac{\mu(I_p'(1))^2(p-1)^2}{\mu(K_p)}\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}dbdb'\leq\frac{1}{p^2}\leq\frac{p^{\nu(x\overline{x}-1)}}{p^2(p-1)}. \tag{9.61}$$

The case $j \geq 1$: $\nu(x)+\nu(\overline{x}-1) \geq 1$. Suppose now that $\nu(\overline{x}-1)+\nu(x) \geq 1$. From (9.51), (9.54) and (9.56) we find that $\mathcal{I}_p^{>0}(x;1)$ is bounded by

$$\begin{aligned}\frac{\mu(I_p'(1))^2p^2(p-1)^2}{\mu(K_p)}&\sum_{j=1}^{\frac{\nu(x\overline{x}(x-1)(\overline{x}-1))}{2}+1}\sum_{i=-\nu(x-1)}^{j}\int_{p^{-(i+\nu(1-\overline{x})-1)}\mathbb{Z}_p}\int_{p^{-(j-1)}\mathbb{Z}_p}dbdb'\\ &\ll \sum_{j=1}^{\frac{\nu(x\overline{x}(x-1)(\overline{x}-1))}{2}+1}\sum_{i=-\nu(x-1)}^{j}p^{i+j+\nu(1-\overline{x})-2}\\ &\ll (1+\nu(x\overline{x}(x-1)(\overline{x}-1)))^2p^{\nu(x\overline{x}(x-1)(\overline{x}-1))+\nu(1-\overline{x})}\end{aligned}$$

Since $\nu(1-x) \geq 0$ we can replace the exponent in p above by the more symmetric expression

$$\nu(x\overline{x})+2\nu((x-1)(\overline{x}-1)),$$

so that combining this with (9.59), (9.60), (9.61)

$$\mathcal{I}_p(x;1) \ll (1+\nu(x\overline{x}(x-1)(\overline{x}-1)))^2p^{\nu(x\overline{x})+2\nu((x-1)(\overline{x}-1))}+\frac{p^{\nu(1-x\overline{x})}}{p^3}. \tag{9.62}$$

Case II. We assume now that the Iwasawa decomposition of u and v are of the form

$$u=\begin{pmatrix}p^i & & p^ib\\ & 1 & \\ & & p^{-i}\end{pmatrix}\begin{pmatrix}\tau^{-1} & & \\ & 1 & \\ & & 1\end{pmatrix}k_1,\ v=\begin{pmatrix}p^j & & p^jb'\\ & 1 & \\ & & p^{-j}\end{pmatrix}\begin{pmatrix}\delta & & \\ & 1 & \\ & & 1\end{pmatrix}k_2,$$

where $\tau,\delta\in(\mathbb{Z}/p\mathbb{Z})^\times$, and $k_1,k_2\in I_p'(1)$.

Let $\mathcal{I}_p(x;2)$ be the contribution of u,v of the above forms to $\mathcal{I}_p(x)$. We first look for some necessary condition for $\mathcal{I}_p(x;2)$ to be non zero.

Computing $u^{-1}\gamma(x)Jv$ we see that $f_p^{\mathfrak{n}_p}(u^{-1}\gamma(x)Jv)\neq 0$ if and only if

$$\begin{pmatrix}a+\tau p^{j-1} & e+\delta p^{-1}a-\tau p^{-1}(x-\delta p^{j-1}) & z-\tau p^{-1}f\\ -p^j & x-\delta p^{j-1} & f\\ p^{i+j} & p^i(1-x)+\delta p^{i+j-1} & g\end{pmatrix}\in K_p. \tag{9.63}$$

Taking inverse, we obtain

$$\begin{pmatrix}a'-\delta p^{i-1}(1-\overline{x}) & e'+\frac{\tau}{p}a'-\frac{\delta}{p}(\overline{x}+\tau p^{i-1}(1-\overline{x})) & z'-\frac{\delta}{p}f'\\ p^i(1-\overline{x}) & \overline{x}+\tau p^{i-1}(1-\overline{x}) & f'\\ p^{i+j} & -p^j+\tau p^{i+j-1} & g'\end{pmatrix}\in K_p. \tag{9.64}$$

Here, as in Case I, a,e,f,g,z are defined in (9.42) and a',e',f',g',z' as defined in (9.45).

These conditions imply in particular that

$$i+\nu(1-\overline{x})\geq 0,\ j\geq 0,\ i+j\geq 0 \tag{9.65}$$

which together with $\nu(x-\delta p^{j-1}),\ \nu(\overline{x}-\tau p^i(1-\overline{x})p^{-1})\geq 0$ implies that

$$\nu(x),\ \nu(\overline{x})\geq -1.$$

This proves Proposition 9.8 in Case II.

We also note that since

$$f,f',g,g',\ z-\tau p^{-1}f,\ z'-\delta p^{-1}f'\in\mathbb{Z}_p$$

we have

$$\nu(z),\nu(z')\geq -1$$

and

$$\frac{1-x\overline{x}}{1-x}p^{-i-j}+\frac{f}{1-x}p^{-i}-\frac{1-\overline{x}}{1-x}ep^{-j}-\frac{ef}{1-x}=z\in p^{-1}\mathbb{Z}_p. \tag{9.66}$$

We considered Case II when $p|N'$ and assume that

$$\nu(x),\ \nu(\overline{x})\geq 0.$$

In particular $\nu(1\pm x),\ \nu(1\pm\overline{x})\geq 0$.

A few general remarks:

- Since $\nu(x)\geq 0$ and $\nu(x-\delta p^{j-1})\geq 0$ we have $j\geq 1$ and therefore since $a+\tau p^{j-1}\in\mathbb{Z}_p$ we have $a\in\mathbb{Z}_p$.
- This together with $e+\delta p^{-1}a-\tau p^{-1}(x-\delta p^{j-1})\in\mathbb{Z}_p$ implies that
$$\nu(e)\geq -1.$$
- Since $f,f',g,g'\in\mathbb{Z}_p$, we see that b,b' are contained in translates of respectively
$$p^{-\min(i+\nu(1-\overline{x}),i+j)}\mathbb{Z}_p \text{ and } p^{-\min(j,i+j)}\mathbb{Z}_p$$
which have volumes
$$p^{\min(i+\nu(1-\overline{x}),i+j)}\mathbb{Z}_p \text{ and } p^{\min(j,i+j)}\mathbb{Z}_p.$$

The case $\nu(\overline{x}-1)=0$. Since $\overline{x}+\tau p^{i-1}(1-\overline{x})\in\mathbb{Z}_p$ we have $i\geq 1$ and since

$$p^i.f+p^{i-j}(1-\overline{x})=-g\in\mathbb{Z}_p,\ p^jf'-(1-\overline{x})g'=p^{j-i}\in\mathbb{Z}_p \tag{9.67}$$

we have

$$j=i\geq 1.$$

Since

$$x-\delta p^{j-1},a,e+\delta p^{-1}a-\tau p^{-1}(x-\delta p^{j-1})\in\mathbb{Z}_p$$

we have $\nu(e)\geq -1$

We now look at the variable z: under our current assumptions (9.66) becomes

$$\frac{1-x\overline{x}}{1-x}p^{-2i}+\frac{f}{1-x}p^{-i}-\frac{1-\overline{x}}{1-x}ep^{-i}-\frac{ef}{1-x}=z\in p^{-1}\mathbb{Z}_p.$$

The valuation of the first term is $\geq$ of the minimum of the valuations of the three other terms and of z:

$$\nu(1-x\overline{x})-2i\geq\min(-i+\nu(f),\nu(1-\overline{x})+\nu(e)-i,\nu(e)+\nu(f),\nu(z)+\nu(1-x))$$

which yields (since $\nu(1-\overline{x}),\nu(f)\geq 0$ and $\nu(e),\ \nu(z)\geq -1$)

$$\nu(1-x\overline{x})-2i\geq -i-1$$

so that

$$1\leq i=j\leq\nu(1-x\overline{x})+1.$$

Notice also that if $\nu(f)=0$ or $\nu(f')=0$ (which is the generic case) the relations

$$z-\tau p^{-1}f\in\mathbb{Z}_p,\ z'-\delta p^{-1}f'$$

uniquely determine $\delta\,(\mathrm{mod}\,p)$ and $\tau\,(\mathrm{mod}\,p)$. On the other hand, if either $f\in p\mathbb{Z}_p$ or $f'\in p\mathbb{Z}_p$, then b' or b belong to fixed translates of $p^{-i+1}\mathbb{Z}_p$ (whose volume is smaller by a factor p): Therefore, we have

$$\begin{aligned}\mathcal{I}_p(x;2) \ll &\frac{\mu(I_p'(1))^2}{\mu(K_p)}\sum_{1\leq i\leq 1+\nu(1-x\overline{x})}\iint_{b\in p^{-i+1}\mathbb{Z}_p, b'\in p^{-i+1}\mathbb{Z}_p}\sum\sum_{\delta,\tau\,(\mathrm{mod}\,p)} dbdb'\\ &+2\frac{\mu(I_p'(1))^2}{\mu(K_p)}\sum_{1\leq i\leq 1+\nu(1-x\overline{x})}\iint_{b\in p^{-i}\mathbb{Z}_p, b'\in p^{-i+1}\mathbb{Z}_p}\sum_{\tau\,(\mathrm{mod}\,p)} dbdb'\\ &+\frac{\mu(I_p'(1))^2}{\mu(K_p)}\sum_{1\leq i\leq 1+\nu(1-x\overline{x})}\iint_{b\in p^{-i}\mathbb{Z}_p, b'\in p^{-i}\mathbb{Z}_p} dbdb'\end{aligned}$$

$$\ll \frac{1}{p^4}(1+\nu(1-x\overline{x}))p^{2\nu(1-x\overline{x})+2} \ll (1+\nu(1-x\overline{x}))\frac{p^{2\nu(1-x\overline{x})}}{p^2} \tag{9.68}$$

if $\nu(1-\overline{x})=0$.

The case $\nu(1-\overline{x})\geq 1$. Since $\overline{x}+\tau p^{i-1}(1-\overline{x})\in\mathbb{Z}_p$, $j\geq 1$, $f',g'\in\mathbb{Z}_p$ and $p^{j-i}=p^jf'-(1-\overline{x})g'$ we have

$$i\geq 1-\nu(1-\overline{x}),\ j\geq i+1.$$

By (9.66) we have (since $\nu(f)\geq 0$, $\nu(e),\nu(z),\nu(1-x)\geq 0$)

$$\nu(1-x\overline{x})-i-j\geq \min(-i,-j+\nu(1-\overline{x})-1,-1)$$

or equivalently

$$i+j\leq \nu(1-x\overline{x})+\max(i,j+1-\nu(1-\overline{x}),1). \tag{9.69}$$

Also since $p^i.f+p^{i-j}(1-\overline{x})=-g\in\mathbb{Z}_p$ we have $i-j+\nu(1-\overline{x})\geq\min(i,0)$ or equivalently

$$j\leq\max(i,0)+\nu(1-\overline{x}). \tag{9.70}$$

If $i\leq 0$, this gives $j\leq\nu(1-\overline{x})$ and by (9.69)

$$i+j\leq\nu(1-x\overline{x})+1. \tag{9.71}$$

We have therefore

$$1-\nu(1-\overline{x})\leq i\leq 0,\ 1\leq j\leq \nu(1-x\overline{x})+1-i.$$

The contribution of this configuration is bounded by

$$\begin{aligned}\ll &\frac{\mu(I_p'(1))^2}{\mu(K_p)}\sum\sum_{\substack{1-\nu(1-\overline{x})\leq i\leq 0\\ 1\leq j\leq\nu(1-x\overline{x})+1-i}}\iint_{\substack{b\in p^{-i-\nu(1-\overline{x})+1}\mathbb{Z}_p\\ b'\in p^{-j+1}\mathbb{Z}_p}}\sum\sum_{\delta,\tau\,(\mathrm{mod}\,p)} 1dbdb'\\ &+\frac{\mu(I_p'(1))^2}{\mu(K_p)}\sum\sum_{\substack{1-\nu(1-\overline{x})\leq i\leq 0\\ 1\leq j\leq\nu(1-x\overline{x})+1-i}}\iint_{\substack{b\in p^{-i-\nu(1-\overline{x})+1}\mathbb{Z}_p\\ b'\in p^{-j}\mathbb{Z}_p}}\sum_{\tau\,(\mathrm{mod}\,p)} 1dbdb'\\ &+\frac{\mu(I_p'(1))^2}{\mu(K_p)}\sum\sum_{\substack{1-\nu(1-\overline{x})\leq i\leq 0\\ 1\leq j\leq\nu(1-x\overline{x})+1-i}}\iint_{\substack{b\in p^{-i-\nu(1-\overline{x})}\mathbb{Z}_p\\ b'\in p^{-j+1}\mathbb{Z}_p}}\sum_{\delta\,(\mathrm{mod}\,p)} 1dbdb'\\ &+\frac{\mu(I_p'(1))^2}{\mu(K_p)}\sum\sum_{\substack{1-\nu(1-\overline{x})\leq i\leq 0\\ 1\leq j\leq\nu(1-x\overline{x})+1-i}}\iint_{\substack{b\in p^{-i-\nu(1-\overline{x})}\mathbb{Z}_p\\ b'\in p^{-j}\mathbb{Z}_p}} dbdb'\end{aligned}$$

and using (9.71) this is bounded by

$$\ll (1+\nu(1-x\overline{x}))(1+\nu(1-\overline{x}))\frac{p^{\nu(1-x\overline{x})+\nu(1-\overline{x})}}{p^3}. \tag{9.72}$$

If $i\geq 1$ then by (9.70) we have $j\leq \nu(1-\overline{x})+i$ and by (9.69)

$$j\leq \nu(1-x\overline{x})+1 \text{ and } i\leq j-1\leq \nu(1-x\overline{x}).$$

$$\begin{aligned}
&\ll \frac{\mu(I_p'(1))^2}{\mu(K_p)}\sum_{\substack{1\leq i\leq \nu(1-x\overline{x})\\ i+1\leq j\leq \nu(1-x\overline{x})+1}}\sum \iint_{\substack{b\in p^{-i-\nu(1-\overline{x})+1}\mathbb{Z}_p\\ b'\in p^{-j+1}\mathbb{Z}_p}}\sum_{\delta,\tau \,(\mathrm{mod}\, p)}\sum 1dbdb'\\
&+\frac{\mu(I_p'(1))^2}{\mu(K_p)}\sum_{\substack{1\leq i\leq \nu(1-x\overline{x})\\ i+1\leq j\leq \nu(1-x\overline{x})+1}}\sum \iint_{\substack{b\in p^{-i-\nu(1-\overline{x})+1}\mathbb{Z}_p\\ b'\in p^{-j}\mathbb{Z}_p}}\sum_{\tau \,(\mathrm{mod}\, p)} 1dbdb'\\
&+\frac{\mu(I_p'(1))^2}{\mu(K_p)}\sum_{\substack{1\leq i\leq \nu(1-x\overline{x})\\ i+1\leq j\leq \nu(1-x\overline{x})+1}}\sum \iint_{\substack{b\in p^{-i-\nu(1-\overline{x})}\mathbb{Z}_p\\ b'\in p^{-j+1}\mathbb{Z}_p}}\sum_{\delta \,(\mathrm{mod}\, p)} 1dbdb'\\
&+\frac{\mu(I_p'(1))^2}{\mu(K_p)}\sum_{\substack{1\leq i\leq \nu(1-x\overline{x})\\ i+1\leq j\leq \nu(1-x\overline{x})+1}}\sum \iint_{\substack{b\in p^{-i-\nu(1-\overline{x})}\mathbb{Z}_p\\ b'\in p^{-j}\mathbb{Z}_p}} dbdb'
\end{aligned}$$

which is bounded by

$$\ll (1+\nu(1-x\overline{x}))^2\frac{p^{2\nu(1-x\overline{x})+\nu(1-\overline{x})}}{p^3}. \tag{9.73}$$

Combining (9.68) with the bounds (9.72) and (9.73) for $\nu(1-\overline{x})\geq 1$ we obtain

$$\mathcal{I}_p(x;2)\ll (1+\nu(1-x\overline{x}))(1+\nu(1-x\overline{x})+\nu(1-\overline{x}))\frac{p^{2\nu(1-x\overline{x})+\max(1,\nu(1-\overline{x}))}}{p^3}. \tag{9.74}$$

Case III. We assume now that the Iwasawa decomposition of u and v are of the form

$$u=\begin{pmatrix}p^i & & p^ib\\ & 1 & \\ & & p^{-i}\end{pmatrix}\begin{pmatrix}1 & & \mu\\ & 1 & \\ & & 1\end{pmatrix}J\begin{pmatrix}\tau & & \\ & 1 & \\ & & 1\end{pmatrix}\gamma_1,$$

$$v=\begin{pmatrix}p^j & & p^jb'\\ & 1 & \\ & & p^{-j}\end{pmatrix}\begin{pmatrix}\delta & & \\ & 1 & \\ & & 1\end{pmatrix}\gamma_2,$$

where $\tau,\delta\in(\mathbb{Z}/p\mathbb{Z})^\times$, $\mu\in\mathbb{Z}/p\mathbb{Z}$, and $\gamma_1,\gamma_2\in I_p'(1)$.

Let $\mathcal{I}_p(x;3)$ be the contribution of u,v of the above forms to $\mathcal{I}_p(x)$. We first look for some necessary condition for $\mathcal{I}_p(x;3)$ to be non zero.

We have $f_p^{\mathfrak{n}_p}(u^{-1}\gamma(x)Jv)\neq 0$ if and only if

$$\begin{pmatrix} a & e+\delta p^{-1}a & z\\ -p^j & x-\delta p^{j-1} & f\\ p^{i+j}-\tau p^{j-1} & p^i(1-x)+\delta p^{i+j-1}-\frac{\tau}{p}(x-\delta p^{j-1}) & g-\tau p^{-1}f\end{pmatrix}\in K_p. \tag{9.75}$$

Taking inverse of (9.75) we then obtain that

$$\begin{pmatrix} a'-\delta p^{i-1}(1-\overline{x}) & e'-\frac{\delta}{p}\overline{x}+\frac{\tau}{p}(z'-\delta p^{-1}f') & z'-\delta p^{-1}f'\\ p^i(1-\overline{x}) & \overline{x}+\tau p^{-1}f' & f'\\ p^{i+j} & -p^j+\tau p^{-1}g' & g'\end{pmatrix}\in K_p. \tag{9.76}$$

These conditions imply in particular that

$$i+j\geq 0,\ \ j\geq 1, i+\nu(1-\overline{x})\geq 0 \tag{9.77}$$

and therefore $\nu(x) \geq 0$. Also $f', \overline{x} + \tau p^{-1} f' \in \mathbb{Z}_p$ implies that $\nu(\overline{x}) \geq -1$. This proves (9.8) (in a stronger form) in Case III.

We assume now that $x, \overline{x} \in \mathbb{Z}_p$. We have

$$f', g' \in p\mathbb{Z}_p$$

So $j \geq 1$, $\nu(f') \geq 1$ and $\nu(g') \geq 1$. Consequently, since

$$p^{j-i} = p^j f' - (1-\overline{x})g' \in p\mathbb{Z}_p$$

(see (9.47)) we have

$$i \leq j-1 \tag{9.78}$$

and $a \in p\mathbb{Z}_p$ which implies that $e \in \mathbb{Z}_p$.

Since $z \in \mathbb{Z}_p$ by (9.43) we have

$$\frac{1-x\overline{x}}{1-x}p^{-i-j} + \frac{f}{1-x}p^{-i} - \frac{1-\overline{x}}{1-x}ep^{-j} - \frac{ef}{1-x} = z \in \mathbb{Z}_p$$

and since $f, e \in \mathbb{Z}_p$ we have

$$\nu(1-x\overline{x}) - i - j \geq \min\{\nu(1-x), -i, -j+\nu(1-\overline{x}), 0\} = \min\{-i, -j+\nu(1-\overline{x}), 0\}.$$

or

$$i + j \leq \nu(1-x\overline{x}) + \max\{i, j-\nu(1-\overline{x}), 0\}. \tag{9.79}$$

Also, we have $\nu(g) \geq -1$ so from the relation $p^i f + g = p^{i-j}(1-\overline{x})$ we conclude that

$$j - i - \nu(1-\overline{x}) \leq \max\{1, -i\} \tag{9.80}$$

and substracting i from (9.79) we obtain

$$1 \leq j \leq \nu(1-x\overline{x}) + \max\{1, -i\}. \tag{9.81}$$

Localisation of b and b'. Finally we observe that since $f \in \mathbb{Z}_p$, $f' \in p\mathbb{Z}_p$, we see from the expression of f and f' in (9.42) and (9.45) that b and b' are contained in translates of

$$p^{1-i-\nu(1-\overline{x})}\mathbb{Z}_p \text{ and } p^{-j}\mathbb{Z}_p \text{ respectively}$$

which have volumes

$$p^{i+\nu(1-\overline{x})-1} \text{ and } p^j.$$

Moreover we notice that if $\nu(f) = 0$ then since $g - \tau p^{-1} f \in \mathbb{Z}_p$ the congruence class τ is uniquely determined by g and f (which depend on x and b') while for $\nu(f) \geq 1$ there is no constraint on τ but b' varies over a translate of $p^{-j+1}\mathbb{Z}_p$.

The case $\nu(1-\overline{x}) = 0$. We have $1 \leq j \leq \max\{i+1, 0\} \leq j$ so $j \leq i+1$ and

$$j = i + 1.$$

Therefore, the contribution from this case to $\mathcal{I}_p(x;3)$ is bounded by

$$\frac{\mu(I_p'(1))^2}{\mu(K_p)} \sum_{\mu \in \mathbb{Z}/p\mathbb{Z}} \sum_{0 \leq i \leq \nu(1-x\overline{x})} \sum_{\tau} \sum_{\delta} \int_{p^{1-i}\mathbb{Z}_p} db \int_{p^{-i-1}\mathbb{Z}_p} db'$$
$$\ll p\mu(I_p'(1))^2(1+\nu(1-x\overline{x}))p^{2\nu(1-x\overline{x})} \ll (1+\nu(1-x\overline{x}))\frac{p^{2\nu(1-x\overline{x})}}{p^3}.$$

The case $\nu(1-\overline{x})\geq 1$. We have $j-\nu(1-\overline{x})\leq j-1$ and (since $i\leq j-1$) we have

$$i+j\leq \nu(1-x\overline{x})+j-1 \Longleftrightarrow i\leq \nu(1-x\overline{x})-1.$$

– If $i\geq 0$ we have by (9.81)

$$j\leq \nu(1-x\overline{x})+1$$

and by (9.78)

$$i+j\leq 2\nu(1-x\overline{x})+1.$$

– If $i\leq -1$ we have by (9.80)

$$-i,j\leq \nu(1-\overline{x})$$

and

$$i+j\leq \nu(1-x\overline{x})$$

We conclude that the contribution from the case $\nu(1-\overline{x})\geq 1$ to $\mathcal{I}_p(x;3)$ is bounded by (see the paragraph on the localisation of b and b')

$$\frac{\mu(I_p'(1))^2}{\mu(K_p)}\sum_{\mu\in\mathbb{Z}/p\mathbb{Z}}\sum\sum_{\substack{-\nu(1-\overline{x})\leq i\leq j-1\leq \nu(1-x\overline{x})\\ i+j\leq 2\nu(1-x\overline{x})+1}}\int_{p^{1-i-\nu(1-\overline{x})}\mathbb{Z}_p}\int_{p^{-j+1}\mathbb{Z}_p}\sum_{\tau}\sum_{\delta}1dbdb'$$

$$+\frac{\mu(I_p'(1))^2}{\mu(K_p)}\sum_{\mu\in\mathbb{Z}/p\mathbb{Z}}\sum\sum_{\substack{-\nu(1-\overline{x})\leq i\leq j-1\leq \nu(1-x\overline{x})\\ i+j\leq 2\nu(1-x\overline{x})+1}}\int_{p^{1-i-\nu(1-\overline{x})}\mathbb{Z}_p}\int_{p^{-j}\mathbb{Z}_p}\sum_{\delta}1dbdb'$$

$$\ll (1+\nu(1-\overline{x})+\nu(1-x\overline{x}))^2\frac{p^{2\nu(1-x\overline{x})+\nu(1-\overline{x})}}{p^2}.$$

Putting the above discussion together we then obtain

$$\mathcal{I}_p(x;3)\ll (1+\nu(1-\overline{x})+\nu(1-x\overline{x}))^2\frac{p^{2\nu(1-x\overline{x})+\nu(1-\overline{x})}}{p^2}. \tag{9.82}$$

Case IV. Consider u and v in their Iwasawa forms:

$$u=\begin{pmatrix}p^i&&p^ib\\&1&\\&&p^{-i}\end{pmatrix}\begin{pmatrix}\tau&&\\&1&\\&&1\end{pmatrix}\gamma_1,$$

$$v=\begin{pmatrix}p^j&&p^jb'\\&1&\\&&p^{-j}\end{pmatrix}\begin{pmatrix}1&&\mu\\&1&\\&&1\end{pmatrix}J\begin{pmatrix}\delta&&\\&1&\\&&1\end{pmatrix}\gamma_2,$$

where $\tau,\delta\in(\mathbb{Z}/p\mathbb{Z})^\times$, $\mu\in\mathbb{Z}/p\mathbb{Z}$, and $\gamma_1,\gamma_2\in I_p'(1)$.

Let $\mathcal{I}_p(x;4)$ be the contribution of u,v of the above forms to $\mathcal{I}_p(x)$. We first look for some necessary condition for $\mathcal{I}_p(x;4)$ to be non zero.

Then $f_p^{\mathfrak{n}_p}(u^{-1}\gamma(x)Jv)\neq 0$ if and only if

$$\begin{pmatrix}a+\tau p^{j-1}&e+\frac{\delta}{p}z-\frac{\tau}{p}(x+\frac{\delta}{p}f)&z-\frac{\tau}{p}f\\-p^j&x+\frac{\delta}{p}f&f\\p^{i+j}&p^i(1-x)+\frac{\delta}{p}g&g\end{pmatrix}\in K_p. \tag{9.83}$$

Taking inverse of (9.83) we then obtain that

$$\begin{pmatrix}a'&e'+\frac{\tau}{p}a'&z'\\p^i(1-\overline{x})&\overline{x}+\tau p^{i-1}(1-\overline{x})&f'\\p^{i+j}-\frac{\delta}{p}p^i(1-\overline{x})&-p^j+p^{i+j}\frac{\tau}{p}-\frac{\delta}{p}(\overline{x}+\frac{\tau}{p}p^i(1-\overline{x}))&g'-\frac{\delta}{p}f'\end{pmatrix}\in K_p. \tag{9.84}$$

These conditions imply in particular that

$$j\geq 0, i+j\geq 0,\ i+\nu(1-\overline{x})\geq 1.$$

Moreover since $f, x+\delta f/p \in \mathbb{Z}_p$ we have $\nu(x) \geq -1$. Since

$$p^i(1-\overline{x}) \in p\mathbb{Z}_p, \overline{x}+\tau p^i(1-\overline{x})/p \in \mathbb{Z}_p$$

we have $\nu(\overline{x}) \geq 0$. This proves Proposition 9.8 in case IV (in a stronger form).

We assume now that $x, \overline{x} \in \mathbb{Z}_p$. We have

$$f \in p\mathbb{Z}_p, z \in \mathbb{Z}_p.$$

Since $z \in \mathbb{Z}_p$ by (9.43) we have

$$\frac{1-x\overline{x}}{1-x}p^{-i-j}+\frac{f}{1-x}p^{-i}-\frac{1-\overline{x}}{1-x}ep^{-j}-\frac{ef}{1-x}=z \in \mathbb{Z}_p$$

and since $f \in p\mathbb{Z}_p,\ e \in p^{-1}\mathbb{Z}_p$ we have

$$\nu(1-x\overline{x})-i-j \geq \min\{\nu(1-x),-i+1,-j+\nu(1-\overline{x})-1,0\}=\min\{-i+1,-j+\nu(1-\overline{x})-1,0\}.$$

or

$$i+j \leq \nu(1-x\overline{x})+\max\{i-1,j-\nu(1-\overline{x})+1,0\}.$$

Also from $g \in \mathbb{Z}_p$ and $p^if+g=p^{i-j}(1-\overline{x})$ we have

$$j-\nu(1-\overline{x})-i \leq \max(-i-1,0)$$

which implies that

$$j-\nu(1-\overline{x}) \leq \max(-1,i)$$

and

$$i+j \leq \nu(1-x\overline{x})+\max\{i+1,0\}.$$

We also have $f' \in \mathbb{Z}_p, g' \in p^{-1}\mathbb{Z}_p$ and since

$$p^{j-i}=p^jf'-(1-\overline{x})g'$$

we have

$$j-i \geq \min(j,\nu(1-\overline{x})-1)$$

and therefore

$$1-\nu(1-\overline{x}) \leq i \leq \max(0,j+1-\nu(1-\overline{x})).$$

The case $i \geq 0$. In that case we have

$$j-\nu(1-\overline{x}) \leq i,\ j \leq \nu(1-x\overline{x})+1$$

and therefore

$$0 \leq i \leq \nu(1-x\overline{x})+2$$

so that

$$i+j \leq 2\nu(1-x\overline{x})+3$$

The case $i<0$. We have

$$1-\nu(1-\overline{x}) \leq i \leq 0 \leq j \leq \nu(1-\overline{x})-1$$

and

$$i+j \leq \nu(1-x\overline{x}) \leq 2\nu(1-x\overline{x})+3.$$

Localisation of b and b'. we observe that since $f\in p\mathbb{Z}_p$, $f'\in\mathbb{Z}_p$, we see from the expression of f and f' in (9.42) and (9.45) that b and b' are contained in translates of

$$p^{-i-\nu(1-\overline{x})}\mathbb{Z}_p \text{ and } p^{-j+1}\mathbb{Z}_p \text{ respectively}$$

which have volumes

$$p^{i+\nu(1-\overline{x})} \text{ and } p^{j-1}.$$

Moreover we notice that if $\nu(f')=0$, since $g'-\frac{\delta}{p}f'\in\mathbb{Z}_p$ the congruence class δ is uniquely determined by g' and f' (which depend on x and b) while for $\nu(f')\geq 1$ there is no constraint on δ but b varies over a translate of $p^{i+\nu(1-\overline{x})+1}\mathbb{Z}_p$ (whose volume is smaller by a factor p).

Arguing as before, we deduce that

$$\begin{aligned}\mathcal{I}_p(x;4)&\ll(1+\nu(1-\overline{x})+\nu(1-x\overline{x}))^2\frac{p}{p^4}p^{2\nu(1-x\overline{x})+3+\nu(1-\overline{x})-1+1}\\&\ll(1+\nu(1-\overline{x})+\nu(1-x\overline{x}))^2p^{2\nu(1-x\overline{x})+\nu(1-\overline{x})}\end{aligned}$$

Putting the above discussion together we then obtain

$$\mathcal{I}_p(x;4)\leq\mu(I'_p(1))\big[4\nu(1-\overline{x})+\nu(1-x\overline{x})^2\big]p^{2\nu(1-x\overline{x})}. \tag{9.85}$$

Combining (9.62), (9.74), (9.82) with (9.85) we then obtain

$$\begin{aligned}\mathcal{I}_p(x)\leq&\frac{2+2p^{2\nu(1-x\overline{x})}}{p}+\nu(x\overline{x}(x-1)(\overline{x}-1))^2p^{\nu(x\overline{x})+2\nu((\overline{x}-1)(x-1))}\\&+\frac{8\nu((1-x\overline{x})(1-x\overline{x}))^2p^{2\nu(1-x\overline{x})}}{p^2}+\frac{4\nu(1-\overline{x})\nu(1-x\overline{x})^2p^{2\nu(1-x\overline{x})}}{p}.\end{aligned}$$

Consequently, (9.40) follows readily. □

9.6. **Analysis of the non-integral cases.** We now deal with the remaining cases when x or $\overline{x}$ have negative valuation.

Proposition 9.9. *Let notation be as before. Let p be a prime divisor of N'. Let $x\in E^\times-E^1$ be such that*

$$\nu(x),\nu(\overline{x})\geq-1,\ \textit{and}\ \nu(x)\ \textit{or}\ \nu(\overline{x})=-1.$$

Then

(I) we have $\mathcal{I}_p(x;1)=0$ unless $\nu(x)=-1$ and $\nu(1-\overline{x})\geq1$, in which case

$$\mathcal{I}_p(x;1)\ll\begin{cases}p^{-4}, & \textit{if } \nu(x)=-1 \textit{ and } \nu(1-\overline{x})=1,\\ p^{-2}, & \textit{if } \nu(x)=-1 \textit{ and } \nu(1-\overline{x})\geq2.\end{cases}$$

(II): we have $\mathcal{I}_p(x;2)=0$ unless wither $\nu(x)=-1$, $\nu(\overline{x})=-1$, or $\nu(x)=-1$, $\nu(\overline{x})=\nu(1-\overline{x})=0$, in which case

$$\mathcal{I}_p(x;2)\ll\begin{cases}p^{-4}, & \textit{if } \nu(x)=-1 \textit{ and } \nu(\overline{x})=-1,\\ p^{-3}, & \textit{if } \nu(x)=-1 \textit{ and } \nu(\overline{x})=\nu(1-\overline{x})=0.\end{cases}$$

(III): we have $\mathcal{I}_p(x;3)=0$ unless $\nu(x)\geq0$ and $\nu(\overline{x})=-1$, in which case

$$\mathcal{I}_p(x;3)\ll\begin{cases}p^{-3}, & \textit{if } \nu(x)=0 \textit{ and } \nu(\overline{x})=-1,\\ p^{2\nu(1-x\overline{x})-1}, & \textit{if } \nu(x)\geq1 \textit{ and } \nu(\overline{x})=-1.\end{cases}$$

(IV): we have $\mathcal{I}_p(x;4)=0$ unless $\nu(x)=-1$ and $\nu(\overline{x})\geq 0$, in which case

$$\mathcal{I}_p(x;4)\ll\begin{cases}p^{-3}, & \text{if } \nu(x)=-1 \text{ and } \nu(\overline{x})=\nu(1-\overline{x})=0;\\ p^{2\nu(1-x\overline{x})-1}, & \text{if } \nu(x)=-1 \text{ and } \nu(\overline{x})\geq 1;\\ p^{-3}, & \text{if } \nu(x)=-1 \text{ and } \nu(1-\overline{x})=1;\\ \nu(1-\overline{x})p^{2\nu(1-\overline{x})-5}, & \text{if } \nu(x)=-1 \text{ and } \nu(1-\overline{x})\geq 2.\end{cases}$$

Proof. We shall keep the notation in the proof of Proposition 9.8 and investigate the four cases therein.

Case I. We start by recalling the integrality conditions in that case:

$$\begin{pmatrix} a & e+\frac{\delta}{p}z & z\\ -p^j & x+\frac{\delta}{p}f & f\\ p^{i+j}+\tau p^{j-1} & p^i(1-x)+\frac{\delta}{p}g-\frac{\tau}{p}(x+\delta p^{-1}f) & g-\tau p^{-1}f\end{pmatrix}\in K_p, \tag{9.86}$$

$$\begin{pmatrix} a' & e'+\frac{\tau}{p}z' & z'\\ p^i(1-\overline{x}) & \overline{x}+\frac{\tau}{p}f' & f'\\ p^{i+j}-\delta p^{i-1}(1-\overline{x}) & -p^j+\frac{\tau}{p}g'-\frac{\delta}{p}(\overline{x}+\frac{\tau}{p}f') & g'-\frac{\delta}{p}f'\end{pmatrix}\in K_p. \tag{9.87}$$

Suppose that $\nu(\overline{x})=-1$. The inclusions

$$f',\overline{x}+\tau p^{-1}f'\in\mathbb{Z}_p$$

imply that $\nu(f')=0$ and since $\nu(g'-\delta p^{-1}f')\in\mathbb{Z}_p$, we have $\nu(g')=-1$ which in turn would imply (since $j\geq 0$)

$$\nu(-p^j+\tau p^{-1}g'-\delta p^{-1}(\overline{x}+\tau p^{-1}f'))=\nu(\tau p^{-1}g')=-2$$

a contradiction.

We have therefore $\nu(\overline{x})\geq 0$ and $\nu(x)=-1$. It follows from

$$\nu(x+\delta p^{-1}f),\ \nu(f)\geq 0$$

that $\nu(f)=0$ and since $\nu(g-\tau p^{-1}f)\geq 0$ we then have $\nu(g)=-1$.

We have also

$$p^i(1-x)+\delta p^{-1}g-\tau p^{-1}(x+\delta p^{-1}f)\in\mathbb{Z}_p.$$

In the above expression, the three terms on the lefthand side have respective valuations $i-1,\ -2,\geq -1$; this forces $i=-1$ and since $p^i(1-\overline{x})\in\mathbb{Z}_p$ we have the congruence

$$\nu(1-\overline{x})\geq 1;$$

this implies that $\nu(\overline{x})=0$ and $\nu(x\overline{x}-1)=-1$. This implies also that $\nu(f')\geq 1$ and $\nu(g')\geq 0$.

If $\nu(1-\overline{x})=1$, then since

$$p^{-1+j}-\delta p^{-2}(1-\overline{x})\in\mathbb{Z}_p$$

and the second term has valuation -1 we must have $j=0$.

If $\nu(1-\overline{x})\geq 2$ then since

$$(1-x\overline{x})p^{1-j}+fp-(1-\overline{x})ep^{-j}-ef=(1-x)z\in p^{-1}\mathbb{Z}_p$$

the terms above have respective valuations

$$=-j,\ \geq 1,\ \geq 1-j,\ \geq -1$$

we conclude that $0\leq j\leq 1$. Moreover, looking at the $(3,1)$-th entry of (9.87), we derive that $p^{-1+j}\in\mathbb{Z}_p$, implying that $j\geq 1$. So the assumption that $\nu(1-\overline{x})\geq 2$ forces that $j=1$.

Localisation of b and b'. Since $g'\in\mathbb{Z}_p$, we see that b belong to a translate of $p^{1-j}\mathbb{Z}_p$ and since $a'\in\mathbb{Z}_p$ we see that b' belong to a translate of $p^{-i-j}\mathbb{Z}_p=p^{1-j}\mathbb{Z}_p$. In particular, the translations depends *only* on x.

Localisation of δ and τ. For fixed b and b', we show that δ and τ are determined uniquely. Since $\nu(x)=-1$, $\nu(f)=0$ and $\delta\,(\mathrm{mod}\,p)$ is determined by f.

If $j=0$ then since $p^{-1}+\tau p^{-1}$ we have $\tau\equiv-1\,(\mathrm{mod}\,p)$. Suppose now that $j=1$; we have, by considering the $(3,2)$-th entry of (9.87), that

$$\tau(g'-\delta p^{-1}f')-\delta\overline{x}\in p\mathbb{Z}_p$$

so if $\nu(g'-\delta p^{-1}f')=0$, $\tau\,(\mathrm{mod}\,p)$ is determined.

Otherwise $\nu(z')=0$ because the last column of (9.87) cannot be divisible by p and the last two entries are. In that case, the condition $e'+\frac{\tau}{p}z'\in\mathbb{Z}_p$ determines $\tau\,(\mathrm{mod}\,p)$ in terms of e' and z'.

Hence the corresponding contribution in the case $\nu(1-\overline{x})=1$ to $\mathcal{I}_p(x;1)$ is

$$\ll\frac{1}{p^4}\sum_{\mu\in\mathbb{Z}_p/p\mathbb{Z}_p}\sum_{\mu'\in\mathbb{Z}_p/p\mathbb{Z}_p}\int_{p\mathbb{Z}_p}\int_{p\mathbb{Z}_p}db'db\ll p^{-4}.$$

and the corresponding contribution in the case$\nu(1-\overline{x})\geq 2$ to $\mathcal{I}_p(x;1)$ is

$$\ll\frac{1}{p^4}\sum_{\mu\in\mathbb{Z}_p/p\mathbb{Z}_p}\sum_{\mu'\in\mathbb{Z}_p/p\mathbb{Z}_p}\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}db'db\ll p^{-2}.$$

In conclusion we obtain that

$$\mathcal{I}_p(x;1)\ll\delta_{\nu(1-\overline{x})=1}p^{-4}+\delta_{\nu(1-\overline{x})\geq 2}p^{-2}.$$

Case II. We suppose we are in case II and assume that $\nu(x)\geq 0$ and $\nu(\overline{x})=-1$. Considering (9.63) and (9.64) we see that

$$i=1,\ j\geq 1.$$

Also from the above equations, we have

$$p^if+g=(\overline{x}-1)p^{i-j},\ \ f,g\in\mathbb{Z}_p.$$

So $i-j\geq 1$, and $j\leq 0$. A contradiction!

Hence, we only have the following two possible cases:

(i) $\nu(x)=\nu(\overline{x})=-1$. We then have $j=0$, $i=1$, $\tau=1\,(\mathrm{mod}\,p)$ and $\delta=px\,(\mathrm{mod}\,p)$. Moreover, since $f,f'\in\mathbb{Z}_p$ we see that b,b' belong to translates of $\mathbb{Z}_p$ determined by x and $\overline{x}$. We then have

$$\mathcal{I}_p(x;2)\ll p^{-4}.$$

(ii) $\nu(x)=-1$ and $\nu(\overline{x})\geq 0$. We have $j=0$ and $\delta\equiv px\,(\mathrm{mod}\,p)$. Also, since $f\in\mathbb{Z}_p$ we see that b' belong to a translate of $\mathbb{Z}_p$ determined by x and $\overline{x}$. Since

$$-p^j+\tau p^{i+j-1}\in\mathbb{Z}_p$$

we obtain $i\geq 1$, and since

$$a\in p^{-1}\mathbb{Z}_p,\ g'\in\mathbb{Z}_p\text{ and }a+g'=-p^{j-i}=-p^{-i}\in p^{-1}\mathbb{Z}_p,$$

we have $i\leq 1$ and therefore $i=1$.

Since $a(1-\overline{x})+f'=\overline{x}p^{-1}$ and $a+\tau p^{-1}\in\mathbb{Z}_p$ we have

$$f'\in\overline{x}p^{-1}+\tau(1-\overline{x})p^{-1}+\mathbb{Z}_p;$$

but $f'\in\mathbb{Z}_p$, therefore

$$\overline{x}p^{-1}+\tau(1-\overline{x})p^{-1}\in\mathbb{Z}_p,$$

which implies that $\nu(\overline{x})=\nu(1-\overline{x})=0$ and $\tau \pmod p$ is uniquely determined by $\overline{x}$. Finally, since $g'\in\mathbb{Z}_p$, one has $b\in\frac{1}{2}p^{-2}+p^{-1}\mathbb{Z}_p$.

It follows that in this case we have

$$\mathcal{I}_p(x)\ll\frac{1}{p^4}\int_{\mathbb{Z}_p}db'\int_{p^{-1}\mathbb{Z}_p}db=\frac{1}{p^3}.$$

Case III. Consider now Case III; we recall the two integrality conditions (9.75) and (9.76):

$$\begin{pmatrix} a & e+\delta p^{-1}a & z\\ -p^j & x-\delta p^{j-1} & f\\ p^{i+j}-\tau p^{j-1} & p^i(1-x)+\delta p^{i+j-1}-\frac{\tau}{p}(x-\delta p^{j-1}) & g-\tau p^{-1}f\end{pmatrix}\in K_p. \tag{9.88}$$

$$\begin{pmatrix} a'-\delta p^{i-1}(1-\overline{x}) & e'-\frac{\delta}{p}\overline{x}+\frac{\tau}{p}(z'-\delta p^{-1}f') & z'-\delta p^{-1}f'\\ p^i(1-\overline{x}) & \overline{x}+\tau p^{-1}f' & f'\\ p^{i+j} & -p^j+\tau p^{-1}g' & g'\end{pmatrix}\in K_p. \tag{9.89}$$

$$\begin{cases} a=-\frac{1}{2}p^{j-i}-p^{i+j}b\\ e=\frac{p^{-i}(1+x)}{2}-p^ib(1-x)\\ f=-p^jb'+p^{-j}.\frac{(x+1)(\overline{x}-1)}{2}\\ g=p^{i+j}b'-\frac{(x-1)(\overline{x}-1)}{2}p^{i-j}\\ z=-\frac{1}{2}p^{j-i}b'+p^{-i-j}y-p^{i+j}bb'+p^{i-j}b\frac{(x-1)(\overline{x}-1)}{2}.\end{cases}$$

where

$$y=\frac{x\overline{x}+3\overline{x}-x+1}{4}.$$

Then one has an explicit algebraic relation

$$z=\frac{1-x\overline{x}}{1-x}p^{-i-j}+\frac{f}{1-x}p^{-i}-\frac{1-\overline{x}}{1-x}ep^{-j}-\frac{ef}{1-x}. \tag{9.90}$$

$$\begin{cases} a'=-\frac{(1-x)(1-\overline{x})}{2}p^{i-j}-p^{i+j}b'\\ e'=\frac{(x-1)(\overline{x}+1)}{2}p^{-j}+p^jb'\\ f'=(1-\overline{x})p^ib+\frac{1+\overline{x}}{2}p^{-i}\\ g'=p^{i+j}b-\frac{1}{2}p^{j-i}\\ z'=-p^{i-j}b\frac{(x-1)(\overline{x}-1)}{2}+p^{-i-j}\overline{y}-p^{i+j}bb'+\frac{1}{2}p^{j-i}b'\end{cases} \tag{9.91}$$

and one notes the algebraic relation

$$z'=\frac{1-x\overline{x}}{1-\overline{x}}p^{-i-j}+\frac{e'}{1-\overline{x}}p^{-i}-\frac{1-x}{1-\overline{x}}f'p^{-j}-\frac{e'f'}{1-\overline{x}}.$$

We also recall that

$$a+g'=-p^{j-i},\ a'+g=-(1-x)(1-\overline{x})p^{i-j},$$
$$p^if+g=p^{i-j}(\overline{x}-1),\ p^jf'-(1-\overline{x})g'=p^{j-i}$$

Suppose that $\nu(x)=-1$. We have $j=0$ and since $p^{i+j}-\tau p^{j-1}\in\mathbb{Z}_p$ we have $i=-1$, which contradicts the condition $p^{i+j}\in\mathbb{Z}_p$.

So we must have $\nu(x)\geq 0$ and therefore $\nu(\overline{x})=-1$.

This implies that $i,j\geq 1$ and since

$$p^{i-j}(\overline{x}-1)=p^if+g\in p^{-1}\mathbb{Z}_p$$

we have $i\geq j\geq 1$. We also have $f',p^{-1}g'\in\mathbb{Z}_p$ and since

$$p^{j-i}=p^jf'-(1-\overline{x})g'\in\mathbb{Z}_p$$

we have $j\geq i$ and
$$i=j\geq 1.$$

Localization of b and b'. We observe that the conditions $\nu(\overline{x})=-1$ and $\overline{x}+\tau f'/p\in\mathbb{Z}_p$ imply that $f'\in\mathbb{Z}_p^\times$ and that
$$f'\in\tau^{-1}p\overline{x}+p\mathbb{Z}_p.$$
We also note that since
$$f'=(1-\overline{x})p^ib+\frac{1+\overline{x}}{2}p^{-i}$$
we have
$$(1-\overline{x})p^ib+\frac{1+\overline{x}}{2}p^{-i}-\tau^{-1}p\overline{x}\in p\mathbb{Z}_p$$
which implies that b belongs to a translate (depending on x and τ) of
$$(1-\overline{x})^{-1}p^{1-i}\mathbb{Z}_p=p^{2-i}\mathbb{Z}_p.$$
We also have
$$p^if+g=\overline{x}-1,\ g-\tau p^{-1}f\in\mathbb{Z}_p$$
so that
$$(\tau p^{-1}+p^i)f+\overline{x}\in\mathbb{Z}_p.$$
Since
$$f=-p^ib'+p^{-i}.\frac{(x+1)(\overline{x}-1)}{2}$$
we conclude that b' belong to a translate (depending on x and τ) of
$$(\tau p^{-1}+p^i)^{-1}p^{-i}\mathbb{Z}_p=p^{1-i}\mathbb{Z}_p.$$
We now consider the possible values of $i=j\geq 1$.

9.6.1. *The case $i=1$.* Suppose that $i=j=1$. The inclusion
$$p(1-x)+\delta p-\frac{\tau}{p}(x-\delta)\in\mathbb{Z}_p$$
implies that $x-\delta\in p\mathbb{Z}_p$. This implies that $\nu(x)=0$ and the congruence class $\delta\,(\mathrm{mod}\,p)$ is determined by x.

Remembering that for $i=j=1$, b and b' belong respectively to additive translates of $p\mathbb{Z}_p$ and $\mathbb{Z}_p$, we conclude that for $\nu(\overline{x})=-1$ and $\nu(x)=0$, the contribution to $\mathcal{I}_p(x;3)$ of the case $i=1$ is bounded by
$$\mathcal{I}_p^{i=1}(x;3)\ll\frac{1}{p^4}p^{1+1-1+0}\leq\frac{1}{p^3}.$$

9.6.2. *The case $i\geq 2$.* Suppose that $i=j\geq 2$. We observe that a is a unit because the two other terms in the first column of (9.88) are divisible by p; this implies that $\nu(e)=-1$ and that the congruence congruence $\delta\,(\mathrm{mod}\,\mathbb{Z}_p)$ is determined by e and a (so by x and b).

By (9.90) we have
$$(1-x\overline{x})+fp^i-(1-\overline{x})ep^i-efp^{2i}=(1-x)zp^{2i}\in p^{2i}\mathbb{Z}_p,\tag{9.92}$$
and since $\nu(f)\geq 0$, $\nu(e)=-1$ we obtain that
$$\nu((1-\overline{x})ep^i)=i-2.$$
Since the second and last term of (9.92) have valuation $>i-2$ we have
$$\nu(1-x\overline{x})=i-2\geq 0.$$
This is only possible if $\nu(x)\geq 1$.

Therefore, the contribution of this case to $\mathcal{I}_p(x;3)$ is bounded by

$$\mathcal{I}_p^{i\geq 2}(x;3) \ll \frac{1}{p^4}\sum_{\mu\in\mathbb{Z}_p/p\mathbb{Z}_p}\sum_{\tau}\int_{p^{2-i}\mathbb{Z}_p}\int_{p^{1-i}\mathbb{Z}_p} db'db \ll p^{-4+2+2i-3} = p^{2\nu(1-x\overline{x})-1}.$$

and when $\nu(\overline{x})=-1,\ \nu(x)\geq 0$ we have

$$\mathcal{I}_p(x;3) \ll p^{2\nu(1-x\overline{x})}.$$

Case IV. Consider Case IV, and recall (9.83) and (9.84):

$$\begin{pmatrix} a+\tau p^{j-1} & e+\frac{\delta}{p}z-\frac{\tau}{p}(x+\frac{\delta}{p}f) & z-\frac{\tau}{p}f \\ -p^j & x+\frac{\delta}{p}f & f \\ p^{i+j} & p^i(1-x)+\frac{\delta}{p}g & g \end{pmatrix} \in K_p. \tag{9.93}$$

$$\begin{pmatrix} a' & e'+\frac{\tau}{p}a' & z' \\ p^i(1-\overline{x}) & \overline{x}+\tau p^{i-1}(1-\overline{x}) & f' \\ p^{i+j}-\frac{\delta}{p}p^i(1-\overline{x}) & -p^j+p^{i+j}\frac{\tau}{p}-\frac{\delta}{p}(\overline{x}+\frac{\tau}{p}p^i(1-\overline{x})) & g'-\frac{\delta}{p}f' \end{pmatrix} \in K_p. \tag{9.94}$$

$$\begin{cases} a=-\frac{1}{2}p^{j-i}-p^{i+j}b \\ e=\frac{p^{-i}(1+x)}{2}-p^ib(1-x) \\ f=-p^jb'+p^{-j}.\frac{(x+1)(\overline{x}-1)}{2} \\ g=p^{i+j}b'-\frac{(x-1)(\overline{x}-1)}{2}p^{i-j} \\ z=-\frac{1}{2}p^{j-i}b'+p^{-i-j}y-p^{i+j}bb'+p^{i-j}b\frac{(x-1)(\overline{x}-1)}{2}. \end{cases}$$

where

$$y=\frac{x\overline{x}+3\overline{x}-x+1}{4}.$$

Then one has an explicit algebraic relation

$$z=\frac{1-x\overline{x}}{1-x}p^{-i-j}+\frac{f}{1-x}p^{-i}-\frac{1-\overline{x}}{1-x}ep^{-j}-\frac{ef}{1-x}. \tag{9.95}$$

$$\begin{cases} a'=-\frac{(1-x)(1-\overline{x})}{2}p^{i-j}-p^{i+j}b' \\ e'=\frac{(x-1)(\overline{x}+1)}{2}p^{-j}+p^jb' \\ f'=(1-\overline{x})p^ib+\frac{1+\overline{x}}{2}p^{-i} \\ g'=p^{i+j}b-\frac{1}{2}p^{j-i} \\ z'=-p^{i-j}b\frac{(x-1)(\overline{x}-1)}{2}+p^{-i-j}\overline{y}-p^{i+j}bb'+\frac{1}{2}p^{j-i}b' \end{cases} \tag{9.96}$$

and one notes the algebraic relation

$$z'=\frac{1-x\overline{x}}{1-\overline{x}}p^{-i-j}+\frac{e'}{1-\overline{x}}p^{-i}-\frac{1-x}{1-\overline{x}}f'p^{-j}-\frac{e'f'}{1-\overline{x}}. \tag{9.97}$$

We also note that

$$a+g'=-p^{j-i},\ a'+g=-(1-x)(1-\overline{x})p^{i-j},$$
$$p^i.f+g=p^{i-j}(\overline{x}-1),\ p^jf'-(1-\overline{x})g'=p^{j-i}$$

We assume now that $\nu(x),\nu(\overline{x})\geq -1$ and that one of the two equals -1. These conditions imply first that

$$j\geq 0, i+j\geq 0,\ i+\nu(1-\overline{x})\geq 0.$$

Suppose $\nu(\overline{x})=-1$. Then $i=1$ (since $\overline{x}+\tau p^{i-1}(1-\overline{x})\in\mathbb{Z}_p$) and we get a contradiction from $p^{i+j}-\delta p^{i-1}(1-\overline{x})\in\mathbb{Z}_p$.

So we must have $\nu(\overline{x})\geq 0$ and $\nu(x)=-1$. Since $p^i(1-x)+\delta p^{-1}g\in\mathbb{Z}_p$ we have $i\geq 0$. We now distinguish two subcases.

The case $\nu(1-\overline{x})=0$. Note that

$$-(1-x)(1-\overline{x})p^{i-j}=a'+g\in\mathbb{Z}_p,$$

which imply that $i-j\geq 1$. Since $j\geq 0$, then $i\geq 1$.

The case $i=1$. Suppose $i=1$. Then $j=0$. Since the $(3,2)$-th entry of (9.94) is in $\mathbb{Z}_p$ we have the $(2,2)$-th entry of (9.94) lies in $p\mathbb{Z}_p$, which implies that τ is determined by $\overline{x}$. From the $(2,2)$-th entry of (9.93) we see that $f+\delta^{-1}px\in p\mathbb{Z}_p$. So b' lies in a translate of $p\mathbb{Z}_p$.

The identity $g'=(1-\overline{x})^{-1}f'+(1-\overline{x})^{-1}p^{-1}$ together with $g'-\delta p^{-1}f'\in\mathbb{Z}_p$ implies that f' belong to a translate of $p\mathbb{Z}_p$ and that b belongs to a translate of $\mathbb{Z}_p$ (depending on x and δ). Therefore, the contribution in this case to $\mathcal{I}_p(x;4)$ is

$$\ll\frac{1}{p^4}\sum_{\mu\in\mathbb{Z}_p/p\mathbb{Z}_p}\sum_{\delta\in(\mathbb{Z}_p/p\mathbb{Z}_p)^\times}\int_{\mathbb{Z}_p}db\int_{p\mathbb{Z}_p}db'\ll p^{-3}.$$

The case $i\geq 2$. Now we suppose $i\geq 2$. Looking at the first column of (9.94) we see that $\nu(a')=0$ (because the whole column cannot be 0 modulo p). This also implies that $\nu(e')=-1$.

Since the $(3,2)$-th entry of (9.93) is integral, we conclude that $\nu(g)\geq 1$ and that $a'+g=-(1-x)(1-\overline{x})p^{i-j}$ is a unit so that

$$i-j=1$$

and $j\geq 1$.

Next we recall that

$$(1-x\overline{x})+e'p^j-(1-x)f'p^i-e'f'p^{i+j}=(1-\overline{x})z'p^{i+j}\in p^{i+j}\mathbb{Z}_p$$

Observe that

$$\nu(e'p^j)=i-2,\ \nu((1-x)f'p^i)\geq i-1,\ \nu(e'f'p^{i+j})\geq 2i-2$$

the first valuation is therefore the smallest as $i\geq 2$; from this we conclude that

$$\nu(1-x\overline{x})=i-2\geq 0.$$

In particular $\nu(\overline{x})\geq 1$.

Localization of b,b'. Finally since $x+\delta p^{-1}f\in\mathbb{Z}_p$ we see that b' belong to a translate of $p^{1-j}\mathbb{Z}_p=p^{2-i}\mathbb{Z}_p$.

The identity $g'=(1-\overline{x})^{-1}p^{i-1}f'+(1-\overline{x})^{-1}p^{-1}$ together with $g'-\delta p^{-1}f'\in\mathbb{Z}_p$ implies that f' belong to a translate of $p\mathbb{Z}_p$ and that b belongs to a translate of $p^{1-i}\mathbb{Z}_p$ (depending on x and δ).

Localization of τ. Comparing evaluations on both sides of (9.95) we derive that

$$\frac{1-x\overline{x}}{1-x}p^{-i-j}-\frac{1-\overline{x}}{1-x}ep^{-j}\in p^{-i+1}\mathbb{Z}_p.$$

As a consequence, we have

$$(1-x\overline{x})p^{-i}-(1-\overline{x})e\in p^{-1}\mathbb{Z}_p. \tag{9.98}$$

On the other hand, considering the $(1,2)$, $(1,3)$ and $(2,2)$-th entries of (9.93), one has

$$e+\delta\tau p^{-2}f\in p^{-1}\mathbb{Z}_p,\quad z-\tau p^{-1}f\in\mathbb{Z}_p,\quad x+\delta p^{-1}f\in\mathbb{Z}_p.$$

Hence $e-\tau p^{-1}x\in p^{-1}\mathbb{Z}_p$. In conjunction with (9.98) we deduce that

$$(1-x\overline{x})p^{-i}-\tau p^{-1}x(1-\overline{x})\in p^{-1}\mathbb{Z}_p.$$

So τ is determined by x.

Hence, in this case, i.e., $i \geq 2$, the corresponding contribution to $\mathcal{I}_p(x;4)$ is

$$\ll \frac{1}{p^4} \sum_{\mu \in \mathbb{Z}_p/p\mathbb{Z}_p} \sum_{\delta \in (\mathbb{Z}_p/p\mathbb{Z}_p)^\times} \int_{p^{1-i}\mathbb{Z}_p} db \int_{p^{2-i}\mathbb{Z}_p} db' \ll p^{2\nu(1-x\overline{x})-1}.$$

In conclusion, for $\nu(x) = -1$ and $\nu(1-\overline{x}) = 0$ we have

$$\mathcal{I}_p(x;4) \ll \delta_{\nu(\overline{x})=\nu(1-\overline{x})=0} p^{-3} + \delta_{\nu(\overline{x})\geq 1} p^{2\nu(1-x\overline{x})}.$$

Case $\nu(1-\overline{x}) \geq 1$**.** We then have $\nu(\overline{x}) = 0$ and $\nu(1-x\overline{x}) = -1$.

Suppose that $i \geq 1$ then since $j \geq 0, i+j-1 \geq 0$ and $i + \nu(1-\overline{x}) - 1 \geq 1$ and $\overline{x}$ is a unit, the condition

$$-p^j + p^{i+j}\frac{\tau}{p} - \frac{\delta}{p}(\overline{x} + \frac{\tau}{p}p^i(1-\overline{x})) \in \mathbb{Z}_p$$

leads to $\delta\overline{x}p^{-1} \in \mathbb{Z}_p$ a contradiction.

Therefore $i \leq 0$. Moreover since $\nu(g) \geq 0$ and $\nu(1-x) = -1$ the condition

$$p^i(1-x) + \frac{\delta}{p}g \in \mathbb{Z}_p$$

forces $i \geq 0$ so we have

$$i = 0.$$

The condition

$$-(1-x)(1-\overline{x})p^{-j} = a' + g \in \mathbb{Z}_p$$

gives the bound

$$0 \leq j \leq \nu(1-\overline{x}) - 1.$$

The case $\nu(\overline{x}-1) = 1$*.* Suppose $\nu(\overline{x}-1) = 1$. Then $j = 0$. From $x + \delta p^{-1}f \in \mathbb{Z}_p$ we derive that b' is in a translate of $p\mathbb{Z}_p$. Looking at the $(1,1)$-th entry of (9.93), we obtain that $a + \tau p^{-1} = -b - 1/2 + \tau p^{-1} \in \mathbb{Z}_p$. So b belongs to a translate of $\mathbb{Z}_p$.

Considering the $(3,2)$-th entry of (9.94), we obtain

$$\tau - \delta(\overline{x} + \tau p^{-1}(1-\overline{x})) \in p\mathbb{Z}_p.$$

Hence δ is determined by τ. Therefore, the contribution in this case to $\mathcal{I}_p(x;4)$ is

$$\ll \frac{1}{p^4} \sum_{\mu \in \mathbb{Z}_p/p\mathbb{Z}_p} \sum_{\tau \in (\mathbb{Z}_p/p\mathbb{Z}_p)^\times} \int_{\mathbb{Z}_p} db \int_{p\mathbb{Z}_p} db' \ll p^{-3}.$$

The case $\nu(\overline{x}-1) \geq 2$*.* Now we consider the situation where $\nu(\overline{x}-1) \geq 2$.

Localization of δ, τ*.* Since $\nu(x) = -1$, we see that f is a unit and that $\delta \,(\mathrm{mod}\, p)$ is determined by x and f (which depend on b'). In addition we see that $\nu(z) = -1$ and $\nu(\delta z/p) = -2$ ans since $\nu(\frac{\tau}{p}(x + \frac{\delta}{p}f)) \geq -1$ we must have $\nu(e) = -2$. Since

$$e + \frac{\delta}{p}z - \frac{\tau}{p}(x + \frac{\delta}{p}f) \in \mathbb{Z}_p$$

we see that $e + \frac{\delta}{p}z$ has valuation -1 which implies that $\delta \,(\mathrm{mod}\, p)$ is determined by e and z which depends on b and b'.

Localization of b,b'. Since $x+\delta p^{-1}f\in\mathbb{Z}_p$ we derive that b' is in a translate of $p^{1-j}\mathbb{Z}_p$, and since $a+\tau p^{j-1}\in\mathbb{Z}_p$, b belongs to a translate of $p^{-j}\mathbb{Z}_p$. Therefore, the contribution in this case to $\mathcal{I}_p(x;4)$ is

$$\ll \frac{1}{p^4}\sum_{0\leq j\leq \nu(1-\overline{x})-1}\sum_{\mu\in\mathbb{Z}_p/p\mathbb{Z}_p}\sum_{\tau\in(\mathbb{Z}_p/p\mathbb{Z}_p)^\times}\int_{p^{-j}\mathbb{Z}_p}db\int_{p^{1-j}\mathbb{Z}_p}db'\ll \nu(1-\overline{x})p^{2\nu(1-\overline{x})-5}.$$

In conclusion, for $\nu(x)=-1,\ \nu(\overline{x})\geq 0$ and $\nu(1-\overline{x})\geq 1$ we have

$$\mathcal{I}_p(x;4)\ll \delta_{\nu(1-\overline{x})=1}p^{-3}+\delta_{\nu(1-\overline{x})\geq 2}\nu(1-\overline{x})p^{2\nu(1-\overline{x})-5}.$$

This concludes Proposition 9.9.

□

9.6.3. *The archimedean place.* In this subsection we study the archimedean local orbital integral $\mathcal{I}_\infty(x)$. Recall the definition (given in (9.13)):

$$\mathcal{I}_\infty(x)=\int_{H_x(\mathbb{R})\backslash G'(\mathbb{R})\times G'(\mathbb{R})}\left|f_\infty(y_1^{-1}\gamma(x)Jy_2)\right|dy_1dy_2,\ x\in E^\times-E^1.$$

Proposition 9.10. *Let notation be as before. Let* $x\in E^\times-E^1$. *Define*

$$\langle x\rangle:=\begin{cases}|x|^2+1 & \text{if } |x|<1,\\ |x|^2 & \text{if } |x|>1.\end{cases}\tag{9.99}$$

Then for $k\geq 32$

$$\mathcal{I}_\infty(x)\ll\frac{1}{k\langle x\rangle^{\frac{k}{4}-2}(|x|^2-1)^2},\tag{9.100}$$

where the implied constant is absolute, and the absolutely value $|\cdot|$ *is the usual norm in* $\mathbb{C}$.

Before engaging the proof we will need two elementary lemmatas

Lemma 9.11. *Let* $A,B,C>0$. *Let* $m\geq 2$. *Then*

$$\int_0^\infty\frac{1}{\left[A+(Ba-Ca^{-1})^2\right]^m}\frac{da}{a^2}\ll\frac{1}{A^{m-\frac12}C},\tag{9.101}$$

where the implied constant is absolute.

Proof. Denote by LHS the left hand side of (9.101). Then

$$\begin{aligned}\mathrm{LHS}&=\int_{\sqrt{\frac{C}{B}}}^\infty\frac{1}{\left[A+(Ba-Ca^{-1})^2\right]^m}\frac{da}{a^2}+\int_0^{\sqrt{\frac{C}{B}}}\frac{1}{\left[A+(Ba-Ca^{-1})^2\right]^m}\frac{da}{a^2}\\&\leq\int_{\sqrt{\frac{C}{B}}}^\infty\frac{1}{\left[A+(Ba-\sqrt{BC})^2\right]^m}\frac{da}{a^2}+\int_0^{\sqrt{\frac{C}{B}}}\frac{1}{\left[A+(Ca^{-1}-\sqrt{BC})^2\right]^m}\frac{da}{a^2}\\&=\int_0^\infty\frac{da}{(a+\sqrt{CB^{-1}})^2(A+B^2a^2)^m}+\int_0^\infty\frac{1}{(A+C^2a^2)^m}da\\&\leq\int_0^\infty\frac{da}{(\sqrt{CB^{-1}})^2(A+B^2a^2)^m}+\frac{1}{A^{m-\frac12}C}\ll\frac{1}{A^{m-\frac12}C},\end{aligned}$$

where the implied constant is absolute. Then (9.101) holds. □

Remark 9.3. One has

$$\int_0^\infty\frac{da}{(1+a^2)^m}=\frac{2\pi}{2^{2m}}\frac{(2m-2)!}{(m-1)!^2}=\frac{2\pi}{2^{2m}}\frac{2^{2(m-1)}}{(\pi m)^{1/2}}(1+o(1))\ll\frac{1}{m^{1/2}}\tag{9.102}$$

from which one can extract slightly better bounds.

Similarly we have

Lemma 9.12. *Let $A, B, C > 0$. Let $m \geq 2$. Then*

$$\int_0^\infty \frac{1}{\left[A+(Ba+Ca^{-1})^2\right]^m}\frac{da}{a^2} \ll \frac{1}{(A+2BC)^{m-\frac{1}{2}}C}, \tag{9.103}$$

where the implied constant is absolute (independent of m).

Proof. Denote by LHS the left hand side of (9.103). Then

$$\begin{aligned}\mathrm{LHS} &= \int_{\sqrt{\frac{C}{B}}}^\infty \frac{1}{\left[A+(Ba+Ca^{-1})^2\right]^m}\frac{da}{a^2} + \int_0^{\sqrt{\frac{C}{B}}} \frac{1}{\left[A+(Ba+Ca^{-1})^2\right]^m}\frac{da}{a^2}\\ &\leq \int_{\sqrt{\frac{C}{B}}}^\infty \frac{1}{\left[A+2BC+(Ba)^2\right]^m}\frac{da}{a^2} + \int_{\sqrt{\frac{B}{C}}}^\infty \frac{1}{\left[A+2BC+(Ca)^2\right]^m}da.\end{aligned}$$

For the first term, we note that in the range of integration we have

$$(Ba)^2 \geq (Ba - C/a)^2$$

so that the first integral is bounded using Lemma 9.11 and the second is bounded using a linear change of variable. This yields to (9.103). ☐

Proof. (of Proposition 9.100) Recall in §4.1.5 the notation

$$g_E = \mathrm{diag}(|D_E|^{1/4}, 1, |D_E|^{-1/4}).$$

Write $y_{1,\infty}$ and $y_{2,\infty}$ into their Iwasawa coordinates:

$$y_{1,\infty} = g_E\begin{pmatrix} a & \\ & a^{-1}\end{pmatrix}\begin{pmatrix} 1 & -ib \\ & 1\end{pmatrix}k_1 g_E^{-1},\ y_{2,\infty} = g_E\begin{pmatrix} a' & \\ & a'^{-1}\end{pmatrix}\begin{pmatrix} 1 & -ib' \\ & 1\end{pmatrix}k_2 g_E^{-1},$$

where $a, a' \in \mathbb{R}_+^\times$ and k_1, k_2 lie in the maximal compact subgroup.

Then $g_E^{-1}y_{1,\infty}^{-1}\gamma(x)y_{2,\infty}g_E$ is equal to

$$k_1^{-1}\begin{pmatrix} a^{-1} & & iab\\ & 1 & \\ & & a\end{pmatrix} g_E^{-1}\gamma(x)Jg_E\begin{pmatrix} a' & & -ia'b'\\ & 1 & \\ & & a'^{-1}\end{pmatrix}k_2.$$

Noting the K-type, we then obtain

$$|f_\infty(y_{1,\infty}^{-1}\gamma(x)y_{2,\infty})| = \left|M\left(\begin{pmatrix} a^{-1} & & iab\\ & 1 & \\ & & a\end{pmatrix} g_E^{-1}\gamma(x)Jg_E\begin{pmatrix} a' & & ia'b'\\ & 1 & \\ & & a'^{-1}\end{pmatrix}\right)\right|,$$

where $M(g) := \langle D^\Lambda(g)\phi_\circ, \phi_\circ\rangle_\Lambda$ is defined in (10.7) (cf. Lemma 4.7). A direct computation shows that $\begin{pmatrix} a^{-1} & & iab\\ & 1 & \\ & & a\end{pmatrix} g_E^{-1}\gamma(x)Jg_E\begin{pmatrix} a' & & ia'b'\\ & 1 & \\ & & a'^{-1}\end{pmatrix}$ is equal to

$$\begin{pmatrix} * & \frac{1+x}{2a|D_E|^{\frac14}} + iab(1-x)|D_E|^{\frac14} & * \\ -a'|D_E|^{\frac14} & x & ia'b'|D_E|^{\frac14} + \frac{(x+1)(\overline{x}-1)}{2a'|D_E|^{\frac14}} \\ aa' & (1-x)a|D_E|^{\frac14} & * \end{pmatrix}. \tag{9.104}$$

Denote by (g_{ij}) the matrix given in (9.104). Then by Lemma 4.7

$$|f_\infty(y_{1,\infty}^{-1}\gamma(x)y_{2,\infty})| = |M((g_{ij}))| = \frac{2^k|g_{22}|^{k/2}}{|g_{11}-g_{13}-g_{31}+g_{33}|^k}. \tag{9.105}$$

Since $g = (g_{ij})$ is unitary, i.e., ${}^t\overline{g}Jg = J$, its conjugate by $\mathbf{B}$ (defined in (4.2)) satisfies

$$g' = (g'_{ij}) = \mathbf{B}^{-1}g\mathbf{B} \in G_{J'}(\mathbb{R}),$$

where $J' = \mathrm{diag}(1,1,-1)$. Since ${}^t\overline{g'}Jg' = J'$ we have

$$|g'_{33}|^2 = |g'_{22}|^2 + |g'_{31}|^2 + |g'_{12}|^2 = |g'_{22}|^2 + |g'_{13}|^2 + |g'_{21}|^2$$

and

$$|g'_{33}|^2=|g'_{22}|^2+\frac{|g'_{13}|^2+|g'_{31}|^2+|g'_{12}|^2+|g'_{21}|^2}{2}\geq |g'_{22}|^2+\frac{|g'_{12}|^2+|g'_{21}|^2}{2}. \tag{9.106}$$

By

$$\mathbf{B}\begin{pmatrix} g_{11} & g_{12} & g_{13}\\ g_{21} & g_{22} & g_{23}\\ g_{31} & g_{32} & g_{33}\end{pmatrix}\mathbf{B}=\begin{pmatrix} \frac{g_{11}+g_{13}+g_{31}+g_{33}}{2} & \frac{g_{12}+g_{32}}{\sqrt{2}} & \frac{g_{11}-g_{13}+g_{31}-g_{33}}{2}\\ \frac{g_{21}+g_{23}}{\sqrt{2}} & g_{22} & \frac{g_{21}-g_{23}}{\sqrt{2}}\\ \frac{g_{11}+g_{13}-g_{31}-g_{33}}{2} & \frac{g_{12}-g_{32}}{\sqrt{2}} & \frac{g_{11}-g_{13}-g_{31}+g_{33}}{2}\end{pmatrix}.$$

we then have from (9.106) that

$$|g_{11}-g_{13}-g_{31}+g_{33}|^2\geq 4|g_{22}|^2+2|g_{12}+g_{32}|^2+2|g_{21}+g_{23}|^2. \tag{9.107}$$

Substituting (9.107) into (9.105) we then get

$$|f_\infty(y_{1,\infty}^{-1}\gamma(x)y_{2,\infty})|\leq\left(\frac{2|x|}{2|g_{22}|^2+|g_{12}+g_{32}|^2+|g_{21}+g_{23}|^2}\right)^{k/2}. \tag{9.108}$$

We can write $x=m+ni\sqrt{|D_E|}$ with $m,n\in\mathbb{Q}$. Plugging (9.104) into the right hand side of (9.108) we then see that $|f_\infty(y_{1,\infty}^{-1}\gamma(x)y_{2,\infty})|$ is bounded by We have

$$\begin{aligned} &2|g_{22}|^2+|g_{12}+g_{32}|^2+|g_{21}+g_{23}|^2\\ =&\ 2|x|^2+h_1(a|D_E|^{1/4},b,x)^2+h_2(|D_E|^{1/4}a,b,x)^2\\ &+h_1'(|D_E|^{1/4}a',b',x)^2+h_2'(|D_E|^{1/4}a',b',x)^2\end{aligned}$$

where

$$\begin{cases} h_1(a,b,x)=\frac{m+1}{2a}+abn|D_E|^{\frac12}-(m-1)a\\ h_2(a,b,x)=ab(m-1)+an|D_E|^{\frac12}-\frac{n|D_E|^{\frac12}}{2a}\\ h_1'(a',b',x)=a'-\frac{m^2+n^2|D_E|-1}{2a'}\\ h_2'(a',b',x)=a'b'-\frac{n|D_E|^{\frac12}}{a'}.\end{cases} \tag{9.109}$$

Then after the change of variables

$$a|D_E|^{1/4}\longleftrightarrow a,\ \ a'|D_E|^{1/4}\longleftrightarrow a',$$

$\mathcal{I}_\infty(x)$ is bounded by

$$\int_{\mathbb{R}_{>0}}\int_{\mathbb{R}_{>0}}\int_{\mathbb{R}}\int_{\mathbb{R}}\left[\frac{2|x|}{2|x|^2+\sum_{j=1}^2\left[h_j(a,b,x)^2+h_j'(a',b',x)^2\right]}\right]^{\frac{k}{2}}\frac{dbdb'dada'}{aa'}. \tag{9.110}$$

Note that $x\notin E^1$, i.e., $|x|^2=m^2+n^2|D_E|\neq 1$. So $(m,n)\neq(1,0)$ and if furthermore $|D_E|=1$, then $(m,n)\neq(\pm1,0)$ or $(0,\pm1.)$. Suppose first that $|x|^2>1$.

1. Suppose $n\neq 0$. Then we make the linear change of variables

$$h_1(a,b,x)\longleftrightarrow b,\ \ h_2'(a',b',x)\longleftrightarrow b'$$

and find that(9.110) is bounded by

$$\int_{\mathbb{R}_{>0}}\int_{\mathbb{R}_{>0}}\int_{\mathbb{R}}\int_{\mathbb{R}}\left[\frac{2|x|}{2|x|^2+b^2+h(a,b,x)^2+b'^2+h'(a',x)^2}\right]^{\frac{k}{2}}\frac{dbdb'dada'}{|n||D_E|^{1/2}a^2a'^2}, \tag{9.111}$$

where

$$h'(a',x)=h_1'(a',b',x)=a'-\frac{|x|^2-1}{2}\frac{1}{a'}$$

defined in (9.109) and

$$h(a,b,x)=\frac{m-1}{n|D_E|^{\frac{1}{2}}}b+\frac{|x-1|^2}{n|D_E|^{\frac{1}{2}}}a-\frac{|x|^2-1}{2n|D_E|^{\frac{1}{2}}}\frac{1}{a}=\alpha.b+\beta,$$

say. Making a linear change of variable

$$(1+\alpha^2)^{1/2}.b+\beta\longleftrightarrow b,$$

and noting that

$$1+\alpha^2=\frac{|x-1|^2}{n^2|D_E|},$$

(9.111) becomes

$$\int_{\mathbb{R}_{>0}}\int_{\mathbb{R}_{>0}}\int_{\mathbb{R}}\int_{\mathbb{R}}\left[\frac{2|x|}{2|x|^2+b^2+h(a,x)^2+b'^2+h'(a',x)^2}\right]^{\frac{k}{2}}\frac{dbdb'dada'}{|x-1|a^2a'^2},\tag{9.112}$$

where

$$h(a,x)=\frac{\beta}{(1+\alpha^2)^{1/2}}=|x-1|a-\frac{|x|^2-1}{2|x-1|}\frac{1}{a}.$$

By two changes of variable (9.112) is equal to

$$T_k.T_{k-1}\frac{(2|x|)^{k/2}}{|x-1|}\int_{\mathbb{R}_{>0}}\int_{\mathbb{R}_{>0}}\left[\frac{1}{2|x|^2+h(a,x)^2+h'(a',x)^2}\right]^{\frac{k}{2}-1}\frac{dada'}{a^2a'^2}.\tag{9.113}$$

where

$$T_k=\int_{-\infty}^{\infty}\frac{db}{(1+b^2)^{k/2}}\ll\frac{1}{k^{1/2}}$$

by (9.102). Applying twice the computational Lemma 9.11 above, with

$$A=2|x|^2+h'(a',x)^2,\ C=\frac{|x|^2-1}{2|x-1|}>0,\ A'=2|x|^2,C'=\frac{|x|^2-1}{2},$$

we have

$$\int_{\mathbb{R}_{>0}}\int_{\mathbb{R}_{>0}}\left[\frac{1}{2|x|^2+h(a,x)^2+h'(a',x)^2}\right]^{\frac{k}{2}-1}\frac{dada'}{a^2a'^2}\ll\frac{1}{k}\frac{|x-1|}{(2|x|^2)^{k/2-2}(|x|^2-1)^2}$$

where the implicit constant is absolute. Therefore,

$$\mathcal{I}_\infty(x)\ll\frac{1}{k}\frac{1}{|x|^{\frac{k}{2}-4}(|x|^2-1)^2},\tag{9.114}$$

where the implicit constant is absolute.

2. Suppose $n=0$. We then have $x=m\neq 1$ and $\mathcal{I}_\infty(x)$ is bounded by (9.110) with

$$\begin{cases}h_1(a,b,x)=(m-1)a-\frac{m+1}{2}\frac{1}{a}\\ h_2(a,b,x)=a(m-1)b\\ h_1'(a',b',x)=a'-\frac{m^2-1}{2}\frac{1}{a'}\\ h_2'(a',b',x)=a'b'.\end{cases}\tag{9.115}$$

In particular both $h_1(a,b,x)$ and $h_1'(a,b,x)$ do not depends on b and b' and we note them $h(a,x)$ and $h'(a,x)$ respectively. Changing variables this expression equals

(9.116)

$$\frac{1}{|x-1|}\int_{\mathbb{R}_{>0}}\int_{\mathbb{R}_{>0}}\int_{\mathbb{R}}\int_{\mathbb{R}}\left[\frac{2|x|}{2|x|^2+b^2+h(a,x)+b'^2+h'(a',x)}\right]^{\frac{k}{2}}\frac{dbdb'dada'}{a^2a'^2}.$$

By Lemma 9.11, (9.116) is majorized by

$$T_k.T_{k-1}.\frac{(2|x|)^{k/2}}{|x-1|}\int_0^\infty\int_0^\infty\left[\frac{1}{2|x|^2+h(a,x)+h'(a',x)}\right]^{\frac{k}{2}-1}\frac{dada'}{a^2a'^2}\ll\frac{1}{k}\frac{1}{|x|^{\frac{k}{2}-4}(|x|^2-1)^2},$$

where the implied constant is absolute. We have also used that

$$m^2-1=|x|^2-1.$$

Therefore, assuming that $|x|>1$, it follows from (9.114) and the above estimates that

$$\mathcal{I}_\infty(x)\ll\frac{1}{k}\frac{1}{|x|^{\frac{k}{2}-4}(|x|^2-1)^2}.$$

If we assume that $|x|<1$, i.e., $1-|x|^2>0$. Then similarly as above we find (using Lemma 9.12 instead of Lemma 9.11)

$$\mathcal{I}_\infty(x)\ll\frac{1}{k}\frac{|x|^{\frac{k}{2}}}{(|x|^2+1)^{\frac{k}{2}-2}(|x|^2-1)^2}\ll\frac{1}{k}\frac{1}{(|x|^2+1)^{\frac{k}{4}-2}(|x|^2-1)^2}.$$

Thus (9.100) holds. ☐

Remark 9.4. Recall that in the regular orbital integral case we have $x\notin E^1$, so it does not lead to the divergence of the integral (9.110). That is why bounds (9.11) was too coarse to deal with the unipotent orbital integrals and we have had to make explicit computations instead.

10. Bounds for the sum of global regular orbital integrals

In this section we collect the local bounds from the previous section to bound the sum of global regular orbital integrals in (6.17)

$$\sum_{x\in E^\times-E^1}\mathcal{O}_{\gamma(x)}(f^{\mathfrak{n}},\varphi') \tag{10.1}$$

(and in particular establish absolute convergence when k is large enough).

We recall that to the test function $f^{\mathfrak{n}}$ is associated an integer $\ell\geq 1$ given by (4.25) and that in Theorem 9.5 we have introduced the set

$$\mathfrak{X}(N,N',\ell)=\left\{x\in E^\times-E^1,\ x\in(\ell N')^{-1}\mathcal{O}_E,\ x\overline{x}\equiv 1\ (\mathrm{mod}\ N)\right\}.$$

and have proved that for $x\in E^\times-E^1$ not contained in $\mathfrak{X}(N,N',\ell)$ we have

$$\mathcal{I}(f^{\mathfrak{n}};x)=\mathcal{O}_{\gamma(x)}(f^{\mathfrak{n}},\varphi')=0.$$

The main result of this section is the following upper bound for the sum (10.1):

Theorem 10.1. *Let notations and assumption be as before. Let φ' be the new form of weight $k\geq 32$, level N' described in §4.5 subject to the normalization (4.38) and set*

$$\kappa=\frac{k}{4}-2. \tag{10.2}$$

We have

$$\sum_{x\in E^\times-E^1}\frac{\mathcal{O}_{\gamma(x)}(f^{\mathfrak{n}},\varphi')}{\langle\varphi',\varphi'\rangle}\ll_E(k\ell NN')^{o(1)}k^{-\frac{1}{2}}\ell^{17}N'^{\frac{14}{3}}N^2(1+\frac{\ell^2N'^2}{N})^2\mathcal{E} \tag{10.3}$$

where

$$\mathcal{E}:=e^{-\frac{\kappa}{(\ell N')^2+1}}+2^{-\kappa}$$

Moreover if we assume that

$$\ell^2N'^2<N \tag{10.4}$$

we have

$$\sum_{x\in E^\times-E^1}\mathcal{O}_{\gamma(x)}(f^{\mathfrak{n}},\varphi')\ll(k\ell NN')^{o(1)}k^{-\frac{1}{2}}\ell^7N^4N'^{2+2/3}(\frac{\ell^2N'^2}{N})^{\kappa}. \tag{10.5}$$

Here the implicit constants depends only on E.

Remark 10.1. The above estimate could be improved by carefully analyzing each situation determined by Proposition 9.9.

10.1. **Decomposition of Regular Orbital Integrals.** Let us recall that we have made the following reductions in §9.2:

$$\begin{aligned}
\Big|\sum_{x\in E^\times-E^1}\mathcal{O}_{\gamma(x)}(f^{\mathfrak{n}},\varphi')\Big| &\leq \sum_{x\in E^\times-E^1}|\mathcal{O}_{\gamma(x)}(f^{\mathfrak{n}},\varphi')|\\
&\ll \|\varphi'\|_\infty^2\sum_{x\in\mathfrak{X}(N,N',\ell)}\mathcal{I}(x)\\
&\ll \|\varphi'\|_\infty^2\sum_{x\in\mathfrak{X}(N,N',\ell)}\mathcal{I}_\infty(x)\prod_p\mathcal{I}_p(x)\\
&\leq (kN')^{o(1)}k^{1/2}N'^{2/3}\sum_{x\in\mathfrak{X}(N,N',\ell)}\mathcal{I}_\infty(x)\prod_p\mathcal{I}_p(x) \qquad (10.6)
\end{aligned}$$

here for the last step we have used the following bound from [HT13, Thm. 1] for φ' satisfying (4.38)

$$\|\varphi'\|_\infty^2\leq(kN')^{o(1)}k^{1/2}N'^{\frac{2}{3}}\langle\varphi',\varphi'\rangle.$$

Set

$$S(N,N',\ell):=\sum_{x\in\mathfrak{X}(N,N',\ell)}\mathcal{I}_\infty(x)\prod_p\mathcal{I}_p(x).$$

By Proposition 9.10 we have

$$\mathcal{I}_\infty(x)\ll\frac{1}{k}\frac{1}{\langle x\rangle^\kappa(|x|^2-1)^2}, \tag{10.7}$$

where

$$\langle x\rangle=\begin{cases}|x|^2+1 & \text{if } |x|<1,\\ |x|^2 & \text{if } |x|>1.\end{cases}$$

We consider the decomposition

$$\prod_p\mathcal{I}_p(x)=\mathcal{I}_{n-sp}(x)\mathcal{I}_{sp}(x)\mathcal{I}_{N'}(x)\mathcal{I}_\ell(x)$$

where $\mathcal{I}_{n-sp}(x)$, $\mathcal{I}_{sp}(x)$, $\mathcal{I}_{N'}(x)$ and $\mathcal{I}_\ell(x)$ denote respectively the product of the $\mathcal{I}_p(x)$ over all the non-split primes (coprime to ℓ), over the split primes (coprime to N') and over the primes dividing N' (if any) and ℓ. These integrals have been bounded in §9.4 and in §9.5.

To implement these bounds, we need some extra notation: for any $z\in E^\times$ and p a prime we set

$$\mathrm{Nr}(z)_p=\prod_{\mathfrak{p}|p}p^{e_pf_p\nu_{\mathfrak{p}}(z)}$$

where e_p, f_p and $\nu_{\mathfrak{p}}$ are respectively the ramification index, residual degree and valuation at $\mathfrak{p}$; for S a subset of prime numbers we set

$$\mathrm{Nr}(z)_S:=\prod_{p\in S}\mathrm{Nr}(z)_p.$$

We have

$$\mathrm{Nr}(z)=z\overline{z}=|z|^2=\prod_p \mathrm{Nr}(z)_p=\mathrm{Nr}(z)_{\ell N'}\mathrm{Nr}(z)_{sp}\mathrm{Nr}(z)_{n-sp} \tag{10.8}$$

where sp (resp. $n-sp$) denote the product over the split (resp. non-split) primes.

By Propositions 9.6 (for the non-split case) and 9.8 (for the split case) we have for any $\varepsilon>0$ and $x\in\mathfrak{X}(N,N',\ell)$ (in particular $x\notin E^1$)

$$\mathcal{I}_{n-sp}(x)\ll_\varepsilon \delta_{x\overline{x}\equiv 1\,(\mathrm{mod}\,N)}N\mathrm{Nr}(x\overline{x}-1)^{3+\varepsilon}_{n-sp}, \tag{10.9}$$

$$\mathcal{I}_{sp}(x)\ll_\varepsilon \mathrm{Nr}(x\overline{x}-1)^{3/2+\varepsilon}_{sp}\leq \mathrm{Nr}(x\overline{x}-1)^{3+\varepsilon}_{sp} \tag{10.10}$$

(the last inequality because $x\overline{x}-1$ has non-negative valuation at every prime $p\nmid N'\ell$).

To bound $\mathcal{I}_\ell(x)=\prod_{p|\ell}\mathcal{I}_p(x)$ we apply Proposition 9.7: for $p\mid\ell$ and $r=\nu_p(\ell)$, we have

$$\mathcal{I}_p(x)\ll \delta_{\nu(x)\geq -r}(1+|\nu(1-x))|+|\nu(1-x\overline{x})|)p^{7r}\left(p^{2\nu(1-x)}+p^{2\nu(1-x\overline{x})}\right). \tag{10.11}$$

To bound $\mathcal{I}_{N'}(x)$ (granted $N'>1$) we use Propositions 9.8 and 9.9, which depend on the valuations $\nu(x)$, $\nu(\overline{x})$, $\nu(1-\overline{x})$ at the prime N'. Define

$$\begin{aligned}\mathfrak{X}(N,N',\ell)_0:=&\left\{x\in\mathfrak{X}(N,N',\ell):\ \nu(x)\geq 0,\ \nu(\overline{x})\geq 0\right\},\\ \mathfrak{X}(N,N',\ell)_1:=&\left\{x\in\mathfrak{X}(N,N',\ell):\ \nu(x)=-1,\ \nu(1-\overline{x})\geq 1\right\},\\ \mathfrak{X}(N,N',\ell)_2:=&\left\{x\in\mathfrak{X}(N,N',\ell):\ \nu(x)=-1,\ \nu(\overline{x})\geq -1\right\},\\ \mathfrak{X}(N,N',\ell)_3:=&\left\{x\in\mathfrak{X}(N,N',\ell):\ \nu(x)\geq 0,\ \nu(\overline{x})=-1\right\},\\ \mathfrak{X}(N,N',\ell)_4:=&\left\{x\in\mathfrak{X}(N,N',\ell):\ \nu(x)=-1,\ \nu(\overline{x})\geq 0\right\}.\end{aligned} \tag{10.12}$$

Then (10.7), together with Propositions 9.8 and 9.9, implies that

$$\sum_{x\in E^\times-E^1}\mathcal{O}_{\gamma(x)}(f^{\mathfrak{n}},\varphi')\ll (kN')^{o(1)}k^{-1/2}N'^{2/3}\sum_{i=0}^4 S_i(N,N',\ell), \tag{10.13}$$

where

$$S_i(N,N',\ell):=\sum_{x\in\mathfrak{X}(N,N',\ell)_i}\frac{1}{\langle x\rangle^\kappa(|x|^2-1)^2}\prod_p\mathcal{I}_p(x)$$

where we recall that $\kappa=k/4-2$. By (10.9), (10.10) and (10.11), for $0\leq i\leq 4$, $S_i(N,N',\ell)$ is majorized by

$$\ell^7\sum_{x\in\mathfrak{X}(N,N',\ell)_i}\frac{N\mathrm{Nr}(x\overline{x}-1)^{3+\varepsilon}_{n-sp}\mathrm{Nr}(x\overline{x}-1)^{3+\varepsilon}_{sp}\mathcal{I}_{N'}(x)}{\langle x\rangle^\kappa(|x|^2-1)^2}\prod_{p|\ell}(\mathrm{Nr}(1-x)_p+\mathrm{Nr}(1-x\overline{x})_p)^{1+\varepsilon}.$$

10.2. **Bounding $S_0(N,N',\ell)$.** We first bound the simplest term $S_0(N,N',\ell)$, which is the "generic" case if N' is not too large. In fact, since we are in the stable range, one can conceptually think $N'=1$, then $S_i(N,N',\ell)=0$ for $1\leq i\leq 4$.

Lemma 10.2. *Let notations be as before. Assuming that $\kappa>6$, we have for all $\varepsilon>0$ and $\ell\geq 1$*

$$S_0(N,N',\ell)\ll(\ell NN')^\varepsilon\ell^{11}N^2(1+\frac{\ell^2N'^2}{N})^2\left(e^{-\frac{\kappa}{\ell^2+1}}+2^{-\kappa}\right).$$

If in addition we have

$$\ell^2<N \tag{10.14}$$

we have

$$S_0(N,N',\ell)\ll(\ell N)^{\varepsilon}(\frac{\ell^2}{N})^{\kappa-4}.$$

Proof. By Proposition 9.8, for $x\in\mathfrak{X}(N,N',\ell)_0$, we have

$$\mathcal{I}_{N'}(x)\ll_{\varepsilon}\mathrm{Nr}(x\overline{x})^{\varepsilon}\big(\mathrm{Nr}(X)_{N'}+\frac{\mathrm{Nr}(x\overline{x}-1)_{N'}}{N'}\big) \tag{10.15}$$

where $X=X(x)=x\overline{x}(1-x)(1-\overline{x})$.

For $x\in\mathfrak{X}(N,N',\ell)_0$, we may write

$$x=z\ell^{-1},\ z\in\mathcal{O}_E-\{0\},\ \mathrm{Nr}(z)\neq\ell^2,\ z\overline{z}\equiv\ell^2\ (\mathrm{mod}\ N).$$

We have therefore

$$|z|^2=qN+\ell^2>0,\ q\in\mathbb{Z}-\{0\}. \tag{10.16}$$

Then $S_0(N,N',\ell)$ is majorized by

$$\sum_{q>-\ell^2N^{-1}}\frac{r(q)\ell^2N(|q|N)^{3+\varepsilon}\ell^7}{\langle x\rangle^{\kappa}q^2N^2}$$
$$\times\left[\left((qN+\ell^2)\big|1-z/\ell\big|^4+\frac{qN+\ell^2}{N'}\right)\left(\big|1-z/\ell\big|^2+\frac{qN}{\ell^2}\right)\right]^{1+\varepsilon},$$

where

$$r(q)=|\{z\in\mathcal{O}_E-\{0\},\ z\overline{z}=Nq+\ell^2\}|\ll_{\varepsilon}(N\ell)^{\varepsilon}$$

for any $\varepsilon>0$.

By triangle inequality,

$$\big|1-z/\ell\big|^2\leq2(1+z\overline{z}/\ell^2)=2(2+qN\ell^{-2}).$$

Hence, $S_0(N,N',\ell)$ is bounded by

$$\ll(\ell N)^{\varepsilon}\sum_{q>-\ell^2/N}\frac{r(q)\ell^2|q|^{1+\varepsilon}N^2\ell^7}{\langle x\rangle^{\kappa}}$$
$$\times\left(1+\frac{qN}{\ell^2}\right)\left((qN+\ell^2)(1+\frac{qN}{\ell^2})^2+\frac{qN+\ell^2}{N'}\right)$$
$$\ll(\ell N)^{\varepsilon}\ell^{11}N^2\sum_{q>-\ell^2/N}\frac{|q|^{1+\varepsilon}}{\langle x\rangle^{\kappa}}(1+\frac{qN}{\ell^2})^{4+\varepsilon}\ll(\ell N)^{\varepsilon}\ell^{11}N^2(S_{01}+S_{02}),$$

say, where

$$S_{01}:=\sum_{-\ell^2/N<q<0}\frac{|q|^{1+\varepsilon}}{\left(\frac{qN+\ell^2}{\ell^2}+1\right)^{\kappa}}(1+\frac{qN}{\ell^2})^{4+\varepsilon}$$

and

$$S_{02}:=\sum_{q\geq1}\frac{|q|^{1+\varepsilon}}{\left(\frac{qN+\ell^2}{\ell^2}\right)^{\kappa}}(1+\frac{qN}{\ell^2})^{4+\varepsilon}.$$

Note that

$$(A+1)^{-\kappa}=\exp\left(-\kappa\log\left(1+A\right)\right)\leq\exp\left(-\frac{\kappa}{A^{-1}+1}\right); \tag{10.17}$$

this implies that for all $q>-\ell^2/N$ one has (since $qN+\ell^2\geq1$)

$$\exp\left(-\frac{\kappa}{\left(\frac{qN+\ell^2}{\ell^2}\right)^{-1}+1}\right)\leq e^{-\frac{\kappa}{\ell^2+1}}.$$

This implies that

$$S_{01} \ll \sum_{-\ell^2N^{-1}<q<0} |q|^{1+\varepsilon} e^{-\frac{\kappa}{\ell^2+1}} \ll (1+\ell^2/N)^{2+\varepsilon} e^{-\frac{\kappa}{\ell^2+1}}.$$

To estimate S_{02} we break it into two further pieces:

$$S_{02} = \sum_{1\leq q\leq \ell^2/N} \cdots + \sum_{q>\ell^2/N} \cdots .$$

The first piece is bounded by

$$\sum_{1\leq q\leq \ell^2/N} \cdots \ll \sum_{1\leq q\leq \ell^2/N} \frac{|q|^{1+\varepsilon}}{(\frac{qN}{\ell^2}+1)^{\kappa}}$$
$$\ll (\ell N)^{\varepsilon} \frac{\ell^4}{N^2} e^{-\frac{\kappa}{\ell^2/N+1}} \ll (\ell N)^{\varepsilon}(1+\frac{\ell^2}{N})^2 e^{-\frac{\kappa}{\ell^2+1}}.$$

The second piece is bounded by

$$\sum_{q>\ell^2/N} \cdots \ll \frac{\ell^2}{N} \sum_{q>\ell^2/N} \frac{(qN\ell^{-2})^{5+\varepsilon}}{\left(\frac{qN}{\ell^2}+1\right)^{\kappa}}$$
$$\ll (1+\ell^2/N)^2 \int_1^{\infty} \frac{t^{5+\varepsilon}}{(t+1)^{\kappa}} dt \ll \frac{(1+\ell^2/N)^2}{2^{\kappa}}.$$

Consequently

$$S_{02} \ll (\ell N)^{\varepsilon}(1+\ell^2/N)^2 (e^{-\frac{\kappa}{\ell^2+1}} + 2^{-\kappa}).$$

Let us now assume that (10.14) holds, then (10.16) implies that $q\geq 1$ and in the discussion above, the terms S_{01} and the first piece of S_{02} are empty and we have

$$S_0(N,N',\ell) = \sum_{q\geq 1} \cdots \ll \frac{\ell^2}{N} \sum_{q\geq 1} \frac{(qN\ell^{-2})^{5+\varepsilon}}{\left(\frac{qN}{\ell^2}+1\right)^{\kappa}} \ll (\ell N)^{\varepsilon} (\frac{\ell^2}{N})^{\kappa-4}.$$

This concludes the proof of Lemma 10.2. ☐

Remark 10.2. In the above, the series are absolutely converging since $\kappa-5>1$.This is indeed our treatment of $S_0(N,N',\ell)$ which is responsible for the constraint $k\geq 32$. The following sums will be absolutely converging for smaller values of k.

10.3. **Bounding $S_2(N,N',\ell)$.** The worse case scenario is achieved when $x\in\mathfrak{X}(N,N',\ell)_2$ (see (10.12)). In this section we bound $S_2(N,N',\ell)$. The approach is similar to that in §10.2, with a mild modification.

Lemma 10.3. *Let notation be as before. We have for any $\varepsilon>0$*

$$S_2(N,N',\ell) \ll (\ell N' N)^{\varepsilon} \ell^9 N'^3 N^2 (1+\ell^2 {N'}^2/N)^2 \left(e^{-\frac{\kappa}{(\ell N')^2+1}} + 2^{-\kappa}\right). \tag{10.18}$$

If we assume in addition that

$$\ell^2 {N'}^2 < N \tag{10.19}$$

we have

$$S_2(N,N',\ell) \ll (\ell N N')^{\varepsilon} \frac{\ell^7 N^3}{N'^3} (\frac{\ell^2 {N'}^2}{N})^{\kappa}. \tag{10.20}$$

Proof. For $x\in\mathfrak{X}(N,N',\ell)_2$, we may write

$$x = z/(\ell N'),\ z\in\mathcal{O}_E-\{0\},\ \mathrm{Nr}(x)\neq 1,\ z\overline{z}\equiv(\ell N')^2 \,(\mathrm{mod}\, N)$$

and we can write

$$0 < z\overline{z} = qN + (\ell N')^2,\ q\in\mathbb{Z}-\{0\}. \tag{10.21}$$

We have

$$\prod_{p|\ell} \mathrm{Nr}(1-x)_p \leq \left|(1-z/\ell)N'\right|^2 \ll N'^2(1+\frac{\mathrm{Nr}(z)}{\ell^2}).$$

We have

$$S_2(N,N',\ell) \ll S_2^{\heartsuit}(N,N',\ell),$$

where $S_2^{\heartsuit}(N,N',\ell)$ is defined by

$$\sum_{q>-(\ell N')^2N^{-1}} \frac{r(q)(\ell N')^2 N(|q|N)^{3+\varepsilon}\ell^7}{\langle x\rangle^{\kappa} q^2N^2}\left(N'^2(1+\frac{\mathrm{Nr}(z)}{\ell^2})+\frac{qN}{\ell^2}\right)^{1+\varepsilon} N'^{-3}.$$

We break the sum into two pieces as above:

$$S_2^{\heartsuit}(N,N',\ell) = \sum_{-\frac{(\ell N')^2}{N}<q<0} \cdots + \sum_{q\geq 1}\cdots$$

For the first sum we use the trivial bound

$$\left(N'^2(1+\frac{\mathrm{Nr}(z)}{\ell^2})+\frac{qN}{\ell^2}\right)^{1+\varepsilon} \ll N'^{4+\varepsilon},$$

and use (10.17) to bound the remaining terms as in the treatment of S_{01} in the proof of Lemma 10.2.

In the range $q \geq 1$ the sum decays exponentially as in the treatment of S_{02} in the proof of Lemma 10.2. Here we provide an explicit calculation (the value of ε may change from line to line)

$$\begin{aligned}
\sum_{q\geq 1}\cdots &\ll (\ell NN')^{\varepsilon}\ell^9 N'N^2 \sum_{q\geq 1}\frac{|q|^{1+\varepsilon}}{\left(\frac{qN}{(\ell N')^2}+1\right)^{\kappa}}\left(N'^2+\frac{qN}{\ell^2}\right)\\
&\ll (\ell NN')^{\varepsilon}\ell^9 N'^3N^2 \sum_{1\leq q\leq \frac{(\ell N')^2}{N}} q^{1+\varepsilon}\exp\left(-\frac{\kappa}{\frac{(\ell N')^2}{qN}+1}\right)\\
&\quad + (\ell NN')^{\varepsilon}\ell^9 N'N^2 \sum_{q>\frac{(\ell N')^2}{N}} \frac{q^{1+\varepsilon}}{\left(\frac{qN}{(\ell N')^2}+1\right)^{\kappa}}\left(\frac{qN}{\ell^2}\right)\\
&\ll (\ell NN')^{\varepsilon}\ell^9 N'^3N^2(1+\frac{\ell^2N'^2}{N})^2 e^{-\frac{\kappa}{(\ell N')^2/N+1}}\\
&\quad + (\ell NN')^{\varepsilon}\ell^9 N'N^2 \sum_{q>\frac{\ell^2N'^2}{N}} \frac{|q|^{2+\varepsilon}}{\left(\frac{qN}{(\ell N')^2}+1\right)^{\kappa}}\\
&\ll (\ell NN')^{\varepsilon}\ell^9 N'^3N^2(1+\frac{(\ell N')^2}{N})^2\left(e^{-\frac{\kappa}{(\ell N')^2/N+1}}+\int_1^{\infty}\frac{t^{2+\varepsilon}}{(t+1)^{\kappa}}dt\right)\\
&\ll (\ell NN')^{\varepsilon}\ell^9 N'^3N^2(1+\frac{\ell^2N'^2}{N})^2\left(e^{-\frac{\kappa}{(\ell N')^2+1}}+2^{-\kappa}\right).
\end{aligned}$$

Then (10.18) follows.

If we moreover assume (10.19) then (10.21) implies that $q \geq 1$ and we have

$$S_2(N,N',\ell) \ll (\ell NN')^{\varepsilon}\frac{\ell^7N^3}{N'^3}(\frac{\ell^2N'^2}{N})^{\kappa}$$

□

10.4. **Bounding $S_i(N,N',\ell): i=1,3,4$.**

Lemma 10.4. *Let notations be as before. Then for $i=1,3,4$, we have for any $\varepsilon>0$*

$$S_i(N,N',\ell)\ll(\ell NN')^{\varepsilon}\ell^{17}N'^4N^2(1+\ell^2/N)^2\left(e^{-\frac{\kappa}{(\ell N')^2+1}}+2^{-\kappa}\right). \tag{10.22}$$

If we assume in addition that

$$\ell^2N'^2<N \tag{10.23}$$

we have

$$S_i(N,N',\ell)\ll(\ell NN')^{\epsilon}\ell^7N^3N'^2(\frac{\ell^2N'^2}{N})^{\kappa}. \tag{10.24}$$

Proof. Investigating the situations in Proposition 9.9 we see that $\mathcal{I}_{N'}(x)$ is majorized by N'^{-2} or $N'^{2\nu(1-x\overline{x})-1}$ or $\nu(1-x)N'^{2\nu(1-\overline{x})-5}$, depending on $x\in\mathfrak{X}(N,N',\ell)_i$, $1\leq i\leq 4$. In these cases we may still write

$$x=z/(\ell N'),\ z\in\mathcal{O}_E,\ \mathrm{Nr}(x)\neq 1,\ z\overline{z}\equiv(\ell N')^2\,(\mathrm{mod}\,N)$$

and we can write

$$0<z\overline{z}=qN+(\ell N')^2,\ q\in\mathbb{Z}-\{0\}. \tag{10.25}$$

We call x is *good* if

$$\mathcal{I}_{N'}\ll N'^{-2},$$

i.e., $x\in\mathfrak{X}(N,N',\ell)_1$ and some subsets of $\mathfrak{X}(N,N',\ell)_3$ and $\mathfrak{X}(N,N',\ell)_4$. We don't need to cover $\mathfrak{X}(N,N',\ell)_2$ here since it has been handled in Lemma 10.3 already.

The same arguments as in the proof of Lemma 10.3 yields the following upper bound for x good:

$$S_i(N,N',\ell)\ll\sum_{q>-(\ell N')^2/N}\frac{r(q)(\ell N')^2N(|q|N)^{3+\varepsilon}\ell^7}{\langle x\rangle^{\kappa}q^2N^2}\left(N'^2(1+\frac{\mathrm{Nr}(z)}{\ell^2})+\frac{qN}{\ell^2}\right)^{1+\varepsilon}\frac{1}{N'^2}.$$

Note that the above bound is obtained by replacing $\mathcal{I}_{N'}(x)\ll N'^{-3}$ for $x\in\mathfrak{X}(N,N',\ell)_2$ with $\mathcal{I}_{N'}(x)\ll N'^{-2}$ when x is good. By Lemma 10.3 the corresponding contribution from good x is

$$\ll(\ell NN')^{\varepsilon}\ell^9N'^4N^2(1+\ell^2N'^2/N)^2\left(e^{-\frac{\kappa}{(\ell N')^2+1}}+2^{-\kappa}\right).$$

Now we consider the remaining cases where one has only

$$\mathcal{I}_{N'}\ll N'^{2\nu(1-x\overline{x})-1}\text{ or }\mathcal{I}_{N'}\ll\nu(1-x)N'^{2\nu(1-\overline{x})-5}.$$

Write the prime decomposition of $N'\mathcal{O}_E$

$$\mathfrak{p}\overline{\mathfrak{p}}=N'\mathcal{O}_E$$

10.4.1. *First case.* If

$$\mathcal{I}_{N'}\ll N'^{2\nu(1-x\overline{x})-1},$$

by Proposition 9.9 we see that $z\in\mathfrak{p}^2\mathcal{O}_E$ or $z\in\overline{\mathfrak{p}}^2\mathcal{O}_E$. So

$$\mathcal{I}_{N'}(x)\ll N'^{2\nu(1-x\overline{x})-1}\ll(|x|^2-1)^2\ell^4/N'.$$

We can write

$$0<x\overline{x}=qN/\ell^2+1,\ q\in\mathbb{Z}-\{0\}.$$

Therefore, the contribution from these x's is bounded by

$$\ll \ell^7 \sum_x \frac{N\mathrm{Nr}(x\overline{x}-1)_{n-sp}^{3+\varepsilon}\mathrm{Nr}(x\overline{x}-1)_{sp}^{3+\varepsilon}\mathcal{I}_{N'}(x)}{\langle x\rangle^{\kappa}(|x|^2-1)^2} \times \prod_{p|\ell}(\mathrm{Nr}(1-x)_p+\mathrm{Nr}(1-x\overline{x})_p)^{1+\varepsilon}$$

$$\ll (\ell NN')^{\varepsilon}\ell^7 \sum_{q>-\ell^2/N} \frac{r(q)(|q|N)^{3+\varepsilon}}{\langle x\rangle^{\kappa}}\frac{\ell^4}{N'}(1+\frac{qN}{\ell^2})$$

$$\ll (\ell NN')^{\varepsilon}\ell^{17}\frac{N}{N'}\sum_{q>-\ell^2/N}\frac{|q|^{1+\varepsilon}}{\langle x\rangle^{\kappa}}(1+\frac{qN}{\ell^2})^3.$$

This sum is bounded by

$$\ll (\ell NN')^{\varepsilon}\ell^{17}\frac{N}{N'}(S_{01}+S_{02})$$

where S_{01} and S_{02} were introduced in the proof of Lemma 10.2. Therefore, the contribution from the x's under the consideration is bounded by

$$\ll (\ell NN')^{\varepsilon}\ell^{17}\frac{N}{N'}(1+\frac{\ell^2}{N})^2\left(e^{-\frac{\kappa}{\ell^2+1}}+2^{-\kappa}\right).$$

10.4.2. *Second case.* In the case

$$\mathcal{I}_{N'}(x)\ll \nu(1-x)N'^{2\nu(1-x)-5},$$

by Proposition 9.9 we see that $\nu(x)=-1$ and $\nu(1-\overline{x})\geq 2$. So $z\in N'\ell+\overline{\mathfrak{p}}^3\mathcal{O}_E$. Write

$$x=1+\frac{\overline{\varpi}^3u}{N'\ell},\ u\in\mathcal{O}_E,\ \overline{\varpi}\in\overline{\mathfrak{p}}\mathcal{O}_E,\ \mathrm{Nr}(\overline{\varpi})=N'.$$

So

$$\mathcal{I}_{N'}(x)\ll \nu(1-x)N'^{2\nu(1-x)-5}\ll |u|^{\varepsilon}N'^{-5}\ell^2\frac{N'^3|u|^2}{N'^2\ell^2}=\frac{|u|^{2+\varepsilon}}{N'^4}.$$

The congruence condition $x\overline{x}\equiv 1 \pmod N$ becomes

$$\mathrm{Nr}(N'\ell+\overline{\varpi}^3u)\equiv (N'\ell)^2 \pmod N. \tag{10.26}$$

Write

$$x\overline{x}=qN+(\ell N')^2,\ q\in\mathbb{Z}-\{0\}. \tag{10.27}$$

The contribution from these x's is bounded by

$$\ll \ell^7\sum_x \frac{N\mathrm{Nr}(x\overline{x}-1)_{n-sp}^{3+\varepsilon}\mathrm{Nr}(x\overline{x}-1)_{sp}^{3+\varepsilon}\mathcal{I}_{N'}(x)}{\langle x\rangle^{\kappa}(|x|^2-1)^2}\prod_{p|l}(\mathrm{Nr}(1-x)_p+\mathrm{Nr}(1-x\overline{x})_p)^{1+\varepsilon}$$

$$\ll (\ell NN')^{\varepsilon}\sum_{q>-(\ell N')^2N^{-1}}\frac{(\ell N')^2N(|q|N)^{3+\varepsilon}\ell^7}{\langle x\rangle^{\kappa}q^2N^2}\left(N'^2(1+\frac{\mathrm{Nr}(z)}{\ell^2})+\frac{qN}{\ell^2}\right)\frac{|u|^2}{N'^4}$$

$$\ll (\ell NN')^{\varepsilon}(S_1+S_2),$$

where

$$S_1=\sum_{\substack{q>-(\ell N')^2N^{-1}\\ |u|\leq 2\ell N'^{-2}}}\cdots,\ S_2=\sum_{\substack{q>-(\ell N')^2N^{-1}\\ |u|>2\ell N'^{-2}}}\cdots.$$

By definition, we have

$$S_1\ll (\ell NN')^{\varepsilon}\frac{1}{N'}(\frac{\ell}{N'^2})^2S_2^{\heartsuit}(N,N',\ell)=(\ell NN')^{\varepsilon}\frac{\ell^2}{N'^5}S_2^{\heartsuit}(N,N',\ell),$$

where $S_2^{\heartsuit}(N,N',\ell)$ was defined in the proof of Lemma 10.3.

To handle S_2, we observe that (10.26) in the range $|u|>2\ell N'^{-2}$ implies that

$$1+\frac{qN}{(\ell N')^2}=\left|1+\frac{\overline{\varpi}^3u}{\ell N'}\right|^2\gg\frac{N'^4|u|^2}{\ell^2},$$

i.e.,

$$|u|^2\ll\frac{\ell^2}{N'^4}\left(1+\frac{qN}{(\ell N')^2}\right).$$

Therefore, S_2 is bounded by

$$\ll(\ell NN')^{\varepsilon}\sum_{q>-(\ell N')^2N^{-1}}\frac{(\ell N')^2N(|q|N)^{3+\varepsilon}\ell^7}{\langle x\rangle^{\kappa}q^2N^2}\left(N'^2(1+\frac{\mathrm{Nr}(z)}{\ell^2})+\frac{qN}{\ell^2}\right)^1\frac{\ell^2}{N'^8}.$$

and we obtain

$$S_2\ll(\ell NN')^{\varepsilon}\frac{\ell^2}{N'^5}S_2^{\heartsuit}(N,N',\ell).$$

As a consequence, by Lemma 10.3, the contribution from x's in the second case is bounded by

$$\ll(\ell NN')^{\varepsilon}\ell^{11}\frac{N^2}{N'^2}(1+\frac{\ell^2N'^2}{N})^2\left(e^{-\frac{\kappa}{(\ell N')^2+1}}+2^{-\kappa}\right).$$

This proves the first part of Lemma 10.4.

Suppose now that in addition (10.23) holds. For the good x's we have $q\geq1$ in (10.25) and that contribution is bounded by

$$\ll(\ell NN')^{\varepsilon}\sum_{q\geq1}\frac{(\ell N')^2N(|q|N)^{3+\varepsilon}\ell^7}{(\frac{qN}{(\ell N')^2})^{\kappa}q^2N^2}\left(N'^2\frac{qN}{\ell^2}\right)\frac{1}{N'^2}$$
$$\ll(\ell NN')^{\epsilon}\ell^7N^3N'^2(\frac{\ell^2N'^2}{N})^{\kappa}$$

Next the contribution of the non good x's in the first case is bounded by

$$\ll(\ell NN')^{\varepsilon}\ell^{17}\frac{N}{N'}\sum_{q\geq1}\frac{|q|^{1+\varepsilon}}{(qN/\ell^2)^{\kappa}}(\frac{qN}{\ell^2})^3\ll(\ell NN')^{\varepsilon}\frac{\ell^{11}N^4}{N'}(\frac{\ell^2}{N})^{\kappa}.$$

and in the second case, their contribution is bounded by

$$\ll(\ell NN')^{\varepsilon}\sum_{q\geq1}\frac{(\ell N')^2N(|q|N)^{3+\varepsilon}\ell^7}{(qN)^{\kappa}q^2N^2}\left(N'^2\frac{qN}{\ell^2}\right)\frac{qN\ell^2/N'}{N'^4}\ll(\ell NN')^{\varepsilon}\frac{\ell^9N}{N'}N^{-\kappa}$$

$\square$

10.5. **Proof of Theorem 10.1.** Combining Lemma 10.2, 10.3 and 10.4 we obtain

$$\sum_{i=0}^{4}S_i(N,N',\ell)\ll(\ell NN')^{\varepsilon}\ell^{17}N^2N'^4(1+\frac{\ell^2N'^2}{N})^2\left(e^{-\frac{\kappa}{(\ell N')^2+1}}+2^{-\kappa}\right).$$

and if in addition we have

$$\ell^2N'^2<N$$

we have

$$\sum_{i=0}^{4}S_i(N,N',\ell)\ll(\ell NN')^{\varepsilon}\ell^7N^4N'^2(\frac{\ell^2N'^2}{N})^{\kappa}.$$

Substituting the above estimates into (10.13) yields

$$\sum_{x\in E^{\times}-E^1}\mathcal{O}_{\gamma(x)}(f^{\mathfrak{n}},\varphi')\ll(k\ell NN')^{o(1)}\frac{\ell^{17}N'^{\frac{14}{3}}N^2}{k^{1/2}}(1+\frac{\ell^2N'^2}{N})^2\left(e^{-\frac{\kappa}{(\ell N')^2+1}}+2^{-\kappa}\right).$$

and, assuming that $\ell^2 N'^2 < N$,

$$\sum_{x \in E^\times - E^1} \mathcal{O}_{\gamma(x)}(f^{\mathfrak{n}}, \varphi') \ll (k\ell N N')^{o(1)} \frac{\ell^7 N^4 N'^{8/3}}{k^{1/2}} (\frac{\ell^2 N'^2}{N})^\kappa.$$

So Theorem 10.1 follows.

11. **Twisted moments of Bessel periods**

In this section, we establish an asymptotic formula for the average of the Bessel periods $|\mathcal{P}(\varphi, \varphi')|^2$ twisted by eigenvalues of Hecke operators supported at inert primes. Theorem 1.2 will follow as a consequence.

Theorem 11.1. *Let notations be as in Theorem 1.2; in particular we recall that*

$$k > 32, \ \kappa = \frac{k}{4} - 2 > 6,$$

$$d_\Lambda = \frac{(2k-2)(k+2)(k-6)}{3}, d_k = k - 1$$

and

$$\Psi(N) = \prod_{p|N} \left(1 - \frac{1}{p} + \frac{1}{p^2}\right), \ \mathfrak{S}(N') = \prod_{p|N'} (1 - \frac{1}{p^2})^{-1}.$$

For $\ell \geq 1$ be an integer coprime to N and divisible only by primes which are inert in E let $\lambda_\pi(\ell)$ be the eigenvalue of the corresponding Hecke operator at π (see (11.6) below). We have

$$(11.1) \quad \frac{1}{d_\Lambda} \sum_{\pi \in \mathcal{A}_k(N)} \lambda_\pi(\ell) \sum_{\varphi \in \mathcal{B}_k^{\mathfrak{n}}(\pi)} \frac{\left|\mathcal{P}(\varphi, \varphi')\right|^2}{\langle \varphi, \varphi \rangle \langle \varphi', \varphi' \rangle} = \frac{w_E}{d_k} (\frac{N}{N'})^2 \Psi(N) \mathfrak{S}(N') \frac{\lambda_{\pi'}(\ell)}{\ell}$$
$$+ O((\ell N N')^{o(1)} \frac{1}{2^{4k} k^2} \frac{N}{N'^3} \frac{1}{\ell} + (k\ell N N')^{o(1)} \frac{\ell^{15} N'^{\frac{14}{3}} N^2}{k^{1/2}} (1 + \frac{\ell^2 N'^2}{N})^2 \mathcal{E})$$

where

$$\mathcal{E} = e^{-\frac{\kappa}{(\ell N')^2 + 1}} + 2^{-\kappa}.$$

Moreover, if we assume that

$$\ell^2 N'^2 < N,$$

then the third term on the right-hand side of (11.1) can be replaced by

$$(k\ell N N')^{o(1)} \frac{\ell^5 N^4 N'^{8/3}}{k^{1/2}} (\frac{\ell^2 N'^2}{N})^\kappa.$$

11.1. **The Hecke algebra at inert primes.** We refer to [Now18, §2.1] for proofs of the well known facts listed below.

Given p a prime inert in E and coprime with N and $r \geq 0$ an integer, the (normalized) p^r-th Hecke operator is the convolution operator by the function

$$(11.2) \qquad T(p^r) = \frac{1}{p^{2r}} 1_{G(\mathbb{Z}_p) A_r G(\mathbb{Z}_p)}.$$

These satisfy for the recurrence relation

$$(11.3) \qquad T(p^r) T(p) = T(p^{r+1}) + \frac{1}{p} T(p^r) + T(p^{r-1}), \ r \geq 1.$$

Given $\pi \in \mathcal{A}_k(N)$, the $G(\mathbb{Z}_p)$-invariant vectors of π are eigenvectors of the $T(p^r)$ are share the same eigenvalue which we denote by $\lambda_\pi(p^r)$. From (11.3) we have therefore

$$(11.4) \qquad \lambda_\pi(p^r) \lambda_\pi(p) = \lambda_\pi(p^{r+1}) + \frac{1}{p} \lambda_\pi(p^r) + \lambda_\pi(p^{r-1}), \ r \geq 1$$

or in other terms

$$\sum_{r\geq 0}\frac{\lambda_\pi(p^r)}{p^{rs}}=(1+\frac{1}{p^{1+s}})(1-\frac{\alpha_\pi(p)}{p^s})^{-1}(1-\frac{\alpha_\pi^{-1}(p)}{p^s})^{-1} \tag{11.5}$$

where

$$\lambda_\pi(p)=\alpha_\pi(p)+1/p+\alpha_\pi^{-1}(p)$$

for some $\alpha_\pi(p)\in\mathbb{C}^\times$.

For any integer $\ell=\prod_p p^{r_p}$ coprime to N and divisible only by primes inert in E we set

$$\lambda_\pi(\ell):=\prod_p \lambda_\pi(p^r) \tag{11.6}$$

Finally, since the representation π is cohomological, it satisfies the Ramanujan-Petersson conjectures [LR92]) and one has

$$|\alpha_\pi(p)|=1;$$

therefore for $r\geq 1$, one has

$$|\lambda_\pi(p^r)|\leq r+2 \tag{11.7}$$

and for ℓ as above one has

$$\lambda_\pi(\ell)\ll \ell^{o(1)}. \tag{11.8}$$

Remark 11.1. The Satake parameters at the prime p of the base change π_E of π are given by $\{\alpha_\pi(p),1,\alpha_\pi^{-1}(p)\}$.

11.2. **Proof of Theorem 11.1.** Let $\ell=\prod_p p^{r_p}$ be as above and let

$$f^{\mathfrak{n}}=f^{\mathfrak{n}}_\infty\prod_p f^{\mathfrak{n}}_p$$

be the smooth function (which depends on ℓ) that was constructed in §4.4; in particular for $p|\ell$, one has

$$f^{\mathfrak{n}}_p=1_{G(\mathbb{Z}_p)A_{r_p}G(\mathbb{Z}_p)}.$$

By Lemma 5.2 and our normalization for the Hecke operators (11.2), we have

$$\begin{aligned}\frac{1}{d_\Lambda}\sum_{\varphi\in\mathcal{B}^{\tilde{\mathfrak{n}}}_k(N)}\frac{\left|\mathcal{P}(\varphi,\varphi')\right|^2}{\langle\varphi,\varphi\rangle\langle\varphi',\varphi'\rangle}\lambda_\pi(\ell)\ell^2 &= w_E\frac{\mathcal{O}_{\gamma_1}(f^{\mathfrak{n}},\varphi')}{\langle\varphi',\varphi'\rangle}\\ &\quad+\sum_{x\in E^1}\frac{\mathcal{O}_{\gamma(x)}(f^{\mathfrak{n}},\varphi')}{\langle\varphi',\varphi'\rangle}+\sum_{x\in E^\times-E^1}\frac{\mathcal{O}_{\gamma(x)}(f^{\mathfrak{n}},\varphi')}{\langle\varphi',\varphi'\rangle}.\end{aligned} \tag{11.9}$$

The proof then follows immediately from Proposition 7.1, Corollary 8.14 and Theorem 10.1 after dividing by ℓ^2. □

11.3. **Proof of Theorem 1.2.** This is a direct consequence of Theorem 11.1 for $\ell=1$ after observing that, there exists a suitable absolute constant $C\geq 32$, such that given any $\delta>0$, if either of the two following conditions is satisfied

$$N'^2\leq N^{1-\delta},\ N>16, k\geq C(1+1/\delta)$$

or

$$N'^2\leq k^{1-\delta},\ N\leq 2^{4k},$$

then the second and third terms on the right-hand side of (11.1) are negligible compared to the first term. □

12. Weighted Vertical Sato-Tate Distribution

In this section, we interpret Theorem 11.1 as an "vertical" Sato-Tate type joint equidistribution result for products of Hecke eigenvalues $\lambda_\pi(p_i)$ at a finite set of inert prime p_i's, for π varying over $\mathcal{A}_k(N)$ and with the Hecke eigenvalues weighted by the Bessel periods $\left|\mathcal{P}(\varphi,\varphi')\right|^2$. For GL(2) a result of that kind goes back to Royer [Roy00].

12.1. **The Measure.** The Sato-Tate measure is the measure on $[-2,2]$ with density

$$d\mu_{\mathrm{ST}}(x) := \begin{cases} \frac{1}{\pi}\sqrt{1-\frac{x^2}{4}}dx, & \text{if } -2\leq x\leq 2,\\ 0, & \text{otherwise.}\end{cases}$$

We recall that an orthonormal basis for μ_{ST} is provided by the the Chebyshev polynomials $C_r(X),\ r\geq 1$ where $C_r(X)$ (of degree r) is defined by

$$C_r(2\cos\theta)=\frac{\sin(r+1)\theta}{\sin\theta}.$$

Let $x\in[-2,2]$ and p be a prime inert in E and such that $p\nmid NN'$. Let $\sigma_{p,x}$ be the unramified unitary representations of $G'(\mathbb{Q}_p)$ with Satake parameters $(\alpha_x(p),\alpha_x(p)^{-1})$ satisfying

$$\alpha_x(p)+\alpha_x(p)^{-1}=x$$

and $\sigma_{E_p,x}$ its base change. Let $L(1/2,\sigma_{E_p,x}\times\pi'_{E_p})$ be the local Rankin-Selberg L-factor of the base change representations. We define the measure on $\mathbb{R}$ supported on $[-2,2]$

$$d\mu_p(x):=L(1/2,\sigma_{E_{p_i},x}\times\pi'_{E_{p_i}})d\mu_{\mathrm{ST}}(x).$$

Given $\mathbf{p}=(p_1,\cdots,p_m)$ a tuple of inert primes coprime with NN', we define a measure $\mu_{\mathbf{p}}$ on $\mathbb{R}^m$ by

$$d\mu_{\mathbf{p}}(x_1,\cdots,x_m):=d\mu_{p_1}(x_1)\otimes\cdots\otimes d\mu_{p_m}(x_m). \tag{12.1}$$

Remark 12.1. The measure $\mu_{\mathbf{p}}$ is a positive measure since, by temperedness, the local factors satisfy

$$(1-1/p)^6\leq L(1/2,\sigma_{E_{p_i},x}\times\pi'_{E_{p_i}})\leq(1+1/p)^6$$

12.2. **Weighted Equidistribution of Joint Hecke Eigenvalues.**

Theorem 12.1. *Let notation be as in Theorem 11.1. Let $\mathbf{p}=(p_1,\cdots,p_m)$ be a tuple of inert primes coprime with NN' and for any $\pi\in\mathcal{A}_k(N)$ set*

$$\tilde{\lambda}_\pi(\mathbf{p}):=(\lambda_\pi(p_1)-p_1^{-1},\cdots,\lambda_\pi(p_m)-p_m^{-1})\in\mathbb{R}^m$$

where $\lambda_\pi(p)$ denote the p-th Hecke eigenvalue. For any continuous function ϕ on $\mathbb{R}^m$, we have, as $k+N\to\infty$

$$\frac{N'^2}{w_E\mathfrak{S}(N')}\frac{d_k}{d_\Lambda}\frac{1}{N^2\Psi(N)}\sum_{\substack{\pi\in\mathcal{A}_k(N)\\ \varphi\in\mathcal{B}_k^{\tilde{n}}(\pi)}}\frac{\left|\mathcal{P}(\varphi,\varphi')\right|^2}{\langle\varphi,\varphi\rangle\langle\varphi',\varphi'\rangle}\phi(\tilde{\lambda}_\pi(\mathbf{p}))=\mu_{\mathbf{p}}(\phi)+o(1),$$

where $\mu_{\mathbf{p}}$ is defined by (12.1) and the error term depends on $E,N',\mathbf{p}$ and ϕ.

Remark 12.2. Regarding uniformity, it will be clear from the proof that this asymptotic formula is valid as long as $N'\prod_{i=1}^m p_i$ is bounded by some absolute positive power of kN but we will ignore this aspect here.

Proof. As a reminder (cf. §11.1), we recall that $\lambda_\pi(p)$ can be expressed as

$$\lambda_\pi(p)=\alpha_\pi(p)+1/p+\alpha_\pi^{-1}(p),$$

where $\alpha_\pi(p)\in\mathbb{C}^\times$. Moreover, $|\alpha_\pi(p)|=1$ since π_p is tempered. In particular, $\lambda_\pi(p)-p^{-1}\in[-2,2]$. For $r\geq 0$ define $\tilde\lambda_\pi(p^r)$ via the formula

$$\sum_{r\geq 0}\frac{\tilde\lambda_\pi(p^r)}{p^{rs}}=(1-\frac{\alpha_\pi(p)}{p^s})^{-1}(1-\frac{\alpha_\pi^{-1}(p)}{p^s})^{-1}. \tag{12.2}$$

In particular we have

$$\tilde\lambda_\pi(p)=\alpha_\pi(p)+\alpha_\pi^{-1}(p)=\lambda_\pi(p)-1/p$$

and more generally

$$\tilde\lambda_\pi(p^r)=C_r(\tilde\lambda_\pi(p)). \tag{12.3}$$

In view of (11.5) we also have

$$\tilde\lambda_\pi(p^r)=\frac{1}{p^r}\sum_{l=0}^{r}(-1)^{r-l}p^l\lambda_\pi(p^l).$$

The identity above can be rewritten

$$\tilde\lambda_\pi(p^r)=\big(\frac{(-1)^{\Omega(\bullet)}}{\mathrm{Id}}\star\lambda_\pi\big)(p^r)$$

where $\star$ is the Dirichlet convolution and $\frac{(-1)^{\Omega(\bullet)}}{\mathrm{Id}}$ is the multiplicative function

$$n\mapsto\frac{(-1)^{\Omega(n)}}{n}.$$

In particular, for $\mathbf{p}=(p_1,\cdots,p_m)$ a tuple of inert primes and coprime with NN' we have

$$\tilde\lambda_\pi(\mathbf{p})=(\lambda_\pi(p_1),\cdots,\lambda_\pi(p_m)).$$

Moreover if, for a tuple of integers $(r_1,\cdots,r_m)\in\mathbb{N}^m$ and $\ell=p_1^{r_1}\cdots p_m^{r_m}$, we define

$$\tilde\lambda_\pi(\ell):=\prod_{i=1}^{m}\tilde\lambda_\pi(p_i^{r_i}),\ \tilde\lambda_\pi(1)=1,$$

we obtain a multiplicative function which can be expressed as a Dirichlet convolution:

$$\tilde\lambda_\pi(\ell)=\sum_{\ell_1\ell_2=\ell}\frac{(-1)^{\Omega(\ell_1)}}{\ell_1}\lambda_\pi(\ell_2). \tag{12.4}$$

We now turn to the combinatorics of the Hecke eigenvalues $\lambda_{\pi'}(p^r),\ r\geq 0$: the product of Cartan cells

$$K_p'\begin{pmatrix}p&\\&p^{-1}\end{pmatrix}K_p'\cdot K_p'\begin{pmatrix}p^r&\\&p^{-r}\end{pmatrix}K_p'$$

decomposes as the disjoint union

$$K_p'\begin{pmatrix}p^{r+1}&\\&p^{-r-1}\end{pmatrix}K_p'\bigsqcup K_p'\begin{pmatrix}p^r&\\&p^{-r}\end{pmatrix}K_p'\bigsqcup K_p'\begin{pmatrix}p^{r-1}&\\&p^{-r+1}\end{pmatrix}K_p',$$

implies the Hecke relation

$$\lambda_{\pi'}(p)\lambda_{\pi'}(p^r)=\lambda_{\pi'}(p^{r+1})+\lambda_{\pi'}(p^r)+\lambda_{\pi'}(p^{r-1}).$$

So if we set

$$\tilde\lambda_{\pi'}(p^r):=\frac{1}{p^r}\sum_{l=0}^{r}(-1)^{r-l}\lambda_{\pi'}(p^l),$$

we see, by substituting this definition into the above relation, that

$$p\tilde{\lambda}_{\pi'}(p).p^r\tilde{\lambda}_{\pi'}(p^r) = p^{r+1}\tilde{\lambda}_{\pi'}(p^{r+1}) + p^{r-1}\tilde{\lambda}_{\pi'}(p^{r-1}).$$

This in turn implies that

$$p^r\tilde{\lambda}_{\pi'}(p^r) = C_r(\lambda_{\pi'}(p)). \tag{12.5}$$

Remark 12.3. Unlike the case of $\tilde{\lambda}_\pi(p^r)$, there is no factor p^l included in the definition of $\tilde{\lambda}_{\pi'}(p^r)$.

If we set for $\ell = p_1^{r_1}\cdots p_m^{r_m}$

$$\tilde{\lambda}_{\pi'}(\ell) := \prod_{i=1}^m \tilde{\lambda}_{\pi'}(p_i^{r_i}),\ \tilde{\lambda}_{\pi'}(1) = 1$$

we obtain a multiplicative function which is the Dirichlet convolution

$$\tilde{\lambda}_{\pi'}(\ell) = \frac{1}{\ell}\big((-1)^{\Omega(\bullet)}\star\lambda_{\pi'}\big)(\ell) = \frac{1}{\ell}\sum_{\ell_1\ell_2=\ell}(-1)^{\Omega(\ell_1)}\lambda_{\pi'}(\ell_2) \tag{12.6}$$

Suppose $N > \ell^2 N'^2$. By Theorem 11.1, and using (12.4) and (12.6) we have

$$\frac{N'^2}{w_E\mathfrak{S}(N')}\frac{d_k}{d_\Lambda}\frac{1}{N^2\Psi(N)}\sum_{\substack{\pi\in\mathcal{A}_k(N)\\ \varphi\in\mathcal{B}_k^{\tilde{n}}(\pi)}}\tilde{\lambda}_\pi(\ell)\frac{\left|\mathcal{P}(\varphi,\varphi')\right|^2}{\langle\varphi,\varphi\rangle\langle\varphi',\varphi'\rangle} = \tilde{\lambda}_{\pi'}(\ell) + \mathcal{R}(k,\ell,N,N'). \tag{12.7}$$

where

$$\mathcal{R}(k,\ell,N,N') = \frac{(\ell NN')^{o(1)}}{2^{4k}kNN'\ell} + (k\ell NN')^{o(1)}\ell^{15}N'^{\frac{20}{3}}k^{1/2}(1+\frac{\ell^2N'^2}{N})^2\mathcal{E}$$

with

$$\mathcal{E} = e^{-\frac{\kappa}{(\ell N')^2+1}} + 2^{-\kappa}$$

and if we assume that

$$\ell^2 N'^2 < N, \tag{12.8}$$

the second term can be replaced by

$$(k\ell NN')^{o(1)}\ell^5N^2N'^{14/3}k^{1/2}(\frac{\ell^2N'^2}{N})^\kappa.$$

The now interpret this formula in terms of the measure discussed above.
For $x = (x_1,\cdots,x_m)\in[-2,2]^m$ let

$$\phi(x) = \prod_{i=1}^m C_{r_i}(x_i).$$

From (12.3) we have

$$\tilde{\lambda}_\pi(\ell) := \prod_{i=1}^m\tilde{\lambda}_\pi(p_i^{r_i}) = \prod_{i=0}^m C_{r_i}(\lambda_\pi(p_i)-p_i^{-1}) = \phi(\lambda_\pi(\mathbf{p})),$$

Also for $i = 1,\cdots,m$ we have

$$L(1/2,\sigma_{E_{p_i},x_i}\times\pi'_{E_{p_i}}) = \sum_{r=0}^\infty\frac{C_r(x_i)\tilde{\lambda}_{\pi'}(p_i^{r_i})}{p_i^r}$$

and by (12.5) this is equal to

$$\sum_{r=0}^\infty C_r(x_i)C_r(\tilde{\lambda}_{\pi'}(p_i)).$$

Since Chebyshev polynomials are orthonormal relative to $d\mu_{\mathrm{ST}}$, we have

$$\tilde\lambda_{\pi'}(p_i^{r_i})=\int_{\mathbb{R}}\sum_{r=0}^{\infty}\tilde\lambda_{\pi'}(p_i^{r})C_r(x_i)C_{r_i}(x_i)d\mu_{\mathrm{ST}}(x_i)=\mu_{p_i}(C_{r_i}).$$

We have therefore

$$\tilde\lambda_{\pi'}(\ell)=\prod_{i=1}^{m}\tilde\lambda_{\pi'}(p_i^{r_i})=\mu_{\mathbf{p}}(\phi)$$

So (12.7) becomes

$$\frac{N'^2}{w_E\mathfrak{S}(N')}\frac{d_k}{d_\Lambda}\frac{1}{N^2\Psi(N)}\sum_{\substack{\pi\in\mathcal{A}_k(N)\\ \varphi\in\mathcal{B}_k^{\tilde{\mathfrak{n}}}(\pi)}}\frac{\left|\mathcal{P}(\varphi,\varphi')\right|^2}{\langle\varphi,\varphi\rangle\langle\varphi',\varphi'\rangle}\phi(\lambda_\pi(\mathbf{p}))=\mu_{\mathbf{p}}(\phi)+\mathcal{R}(k,\ell,N,N'). \tag{12.9}$$

Suppose that $k+N\to\infty$. If $k\geq N$ we see that

$$\mathcal{R}(k,\ell,N,N')=o_{\phi,N'}(1)$$

since $\mathcal{E}$ converges exponentially fast to 0 while the dependency in N as at most polynomial. If $N\geq k$ then for N large enough (12.8) is satisfied and

$$\mathcal{R}(k,\ell,N,N')=\frac{(\ell NN')^{o(1)}}{2^{4k}kNN'\ell}+(k\ell NN')^{o(1)}\ell^5N^{2+1/2}N'^{14/3}(\frac{\ell^2N'^2}{N})^{\kappa}.$$

The first term in the expression above is always $o_{\phi,N'}(1)$ while the second term is because $\kappa>6$.

Theorem 12.1 for general ϕ follows from the Stone-Weierstrass theorem. ☐

13. Averaging over forms of exact level N

Suppose $N>1$ (and an inert prime). With the choice of the test function made in §4.4 the spectral side of the relative trace formula picks up both newforms and oldforms of level N. In this section, we show that when N is large enough the contribution from the oldforms is smaller than from the new forms; from this, we will eventually deduce (1.6).

We use the notations of §5.1. The set $\mathcal{A}_k(N)$ is the disjoint union of the two subsets $\mathcal{A}_k^{\mathfrak{n}}(N)$ and $\mathcal{A}_k(1)$ where

$$\mathcal{A}_k(1)=\{\pi=\pi_\infty\otimes\pi_f\in\mathcal{A}(G),\omega_\pi=\mathbf{1},\ \pi_\infty\simeq D^{\Lambda},\ \pi_f^{K_f(1)}\neq\{0\}\}$$

is the space automorphic representations "of level 1" and $\mathcal{A}_k^{\mathfrak{n}}(N)$ the space automorphic representations of "new" at N.

Consequently the space of automorphic forms $\mathcal{V}_k(N)$ admits an orthogonal decomposition

$$\mathcal{V}_k(N)=\mathcal{V}_k^{new}(N)\oplus\mathcal{V}_k^{old}(N)$$

(here $\mathcal{V}_k^{old}(N)$ is the subspace generated the forms that belong to the elements of $\mathcal{A}_k(1)$). We choose a corresponding orthogonal basis

$$\mathcal{B}_k(N)=\mathcal{B}_k^{new}(N)\sqcup\mathcal{B}_k^{old}(N)$$

whose elements belong to the π contained in either $\mathcal{A}_k^{\mathfrak{n}}(N)$ or $\mathcal{A}_k(1)$ and are factorable vectors. Accordingly we have a corresponding decomposition

$$\mathcal{B}_k^{\tilde{\mathfrak{n}}}(N)=\mathcal{B}_k^{\tilde{\mathfrak{n}},new}(N)\sqcup\mathcal{B}_k^{\tilde{\mathfrak{n}},old}(N)$$

and the spectral side of the relative trace formula decomposes as

$$J(f^{\mathfrak{n}})=J^{new}(f^{\mathfrak{n}})+J^{old}(f^{\mathfrak{n}}),$$

where

$$J^{new}(f^{\mathfrak{n}}) = \frac{1}{d_\Lambda}\sum_{\varphi\in\mathcal{B}_k^{\tilde{\mathfrak{n}},new}(N)}\frac{\left|\mathcal{P}(\varphi,\varphi')\right|^2}{\langle\varphi,\varphi\rangle\langle\varphi',\varphi'\rangle},$$

$$J^{old}(f^{\mathfrak{n}}) = \frac{1}{d_\Lambda}\sum_{\varphi\in\mathcal{B}_k^{\tilde{\mathfrak{n}},old}(N)}\frac{\left|\mathcal{P}(\varphi,\varphi')\right|^2}{\langle\varphi,\varphi\rangle\langle\varphi',\varphi'\rangle}.$$

We show the contribution from oldforms are negligible. The main result of this section is the following.

Proposition 13.1. *With notations and assumptions as in Theorem 1.1, we have*

$$J^{old}(f^{\mathfrak{n}}) \ll_{N'} \frac{1}{k} \tag{13.1}$$

13.1. **Proof of Theorem 1.1.** Assuming this Proposition let us prove (1.6).

We have by Theorem 11.1

$$\begin{aligned}\sum_{\varphi\in\mathcal{B}_k^{\tilde{\mathfrak{n}}}(N)}\frac{\left|\mathcal{P}(\varphi,\varphi')\right|^2}{\langle\varphi,\varphi\rangle\langle\varphi',\varphi'\rangle} &= \sum_{\varphi\in\mathcal{B}_k^{\tilde{\mathfrak{n}},new}(N)}\frac{\left|\mathcal{P}(\varphi,\varphi')\right|^2}{\langle\varphi,\varphi\rangle\langle\varphi',\varphi'\rangle} + O_{N'}(\frac{d_\Lambda}{d_k}) \\ &= w_E\frac{d_\Lambda}{d_k}(\frac{N}{N'})^2\Psi(N)\mathfrak{S}(N') \\ &\quad + (kN)^{o(1)}\frac{Nk}{2^{4k}} + (kN)^{o(1)}\frac{k^{5/2}}{N^2}(e^{-\frac{\kappa}{(\ell N')^2+1}} + 2^{-\kappa})\end{aligned} \tag{13.2}$$

For N sufficiently large (depending on N') the main term

$$w_E\frac{d_\Lambda}{d_k}(\frac{N}{N'})^2\Psi(N)\mathfrak{S}(N') \asymp \frac{d_\Lambda}{d_k}(\frac{N}{N'})^2 \tag{13.3}$$

will be twice bigger that the term $O_{N'}(d_\Lambda/d_k)$ above; moreover as $k+N\to\infty$ the second and third terms on the righthand side of (13.2) are negligible compared to (13.3). Therefore under the assumptions of Theorem 1.1 we have

$$\sum_{\varphi\in\mathcal{B}_k^{\tilde{\mathfrak{n}},new}(N)}\frac{\left|\mathcal{P}(\varphi,\varphi')\right|^2}{\langle\varphi,\varphi\rangle\langle\varphi',\varphi'\rangle} \asymp_{N'} \frac{d_\Lambda}{d_k}N^2. \tag{13.4}$$

By Proposition 5.4 we have for any $\varphi\in\mathcal{B}_\pi^{\tilde{\mathfrak{n}}}(N),\ \pi\in\mathcal{A}_k^{\mathfrak{n}}(N)$

$$\frac{\left|\mathcal{P}(\varphi,\varphi')\right|^2}{\langle\varphi,\varphi\rangle\langle\varphi',\varphi'\rangle} \asymp \frac{L(1/2,\pi_E\times\pi_E')}{L(1,\pi,\mathrm{Ad})L(1,\pi',\mathrm{Ad})}\cdot\frac{1}{d_k}\frac{1}{NN'^2}, \tag{13.5}$$

which implies that

$$\sum_{\pi\in\mathcal{A}_k^{\mathfrak{n}}(N)}\frac{L(1/2,\pi_E\otimes\pi_E')}{L(1,\pi,\mathrm{Ad})L(1,\pi',\mathrm{Ad})} \asymp_{N'} d_\Lambda N^3 \asymp_{N'} |\mathcal{A}_k^{\mathfrak{n}}(N)|$$

by Weyl's law (1.5). ☐

13.2. **Local Analysis: an elucidation of oldforms.** Let p be a prime inert in E. Given $\pi\in\mathcal{A}_k(1)$, in this section we shall describe explicitly the space, $\pi_p^{I_p}$, of Iwahori-fixed vectors at p. This is certainly well known but we could not find a reference for it.

As this subsection is purely local (at the place p), we will often, to simplify notations, omit the index p: we will write, E for the local field E_p ν for ν_p its valuation, G for $G(\mathbb{Q}_p)$, π for the local component π_p, I for I_p, π^I for $\pi_p^{I_p}$ etc...

Let $\phi^\circ\in\pi$ be the spherical vector normalized such that $\phi^\circ(e)=1$, where e is the identity matrix in G.

It is well known that the subspace π^I has dimension two : π is induced from unramified character, hence trivial on $G(\mathbb{Z}_p)$ and

$$B(\mathbb{Z}_p)\backslash G(\mathbb{Z}_p)/I \simeq B(\mathbb{F}_p)\backslash G(\mathbb{F}_p)/B(\mathbb{F}_p)$$

has two elements. Obviously π^I contains ϕ°. Our first goal is to construct an explicit vector $\phi^* \in \pi^I$ which is not multiple of ϕ°. By the Gram-Schmidt process, we will obtain an orthonormal basis of π^I.

13.3. **Construction of ϕ^*.** Let

$$t = A_1^{-1} = \mathrm{diag}(p^{-1}, 1, p);$$

we set

$$\phi^*(g) := \frac{1}{\mathrm{vol}(I)} \int_I \phi^\circ(gkt)dk.$$

Lemma 13.2. *We have*

$$\phi^* = p^{-1}\pi(t)\phi^\circ + \sum_{\alpha \in \mathbb{F}_p^\times} \pi\left(\begin{pmatrix} 1 & & i\alpha p^{-1} \\ & 1 & \\ & & 1 \end{pmatrix}\right)\phi^\circ \in \pi^I - \{0\}. \tag{13.6}$$

Proof. Given

$$k = \begin{pmatrix} k_{11} & k_{12} & k_{13} \\ pk_{21} & k_{22} & k_{23} \\ pk_{31} & pk_{32} & k_{33} \end{pmatrix} \in I$$

we have $\nu(k_{11}) = \nu(k_{22}) = \nu(k_{33}) = 0$.

Let $z \in i\mathbb{Z}_p$ be such that

$$zk_{11} \equiv -k_{31} \,(\mathrm{mod}\, p),$$

in particular $k_{33} + pzk_{11} \in \mathcal{O}_E^\times$. Let

$$\begin{pmatrix} 1 & & \\ & 1 & \\ pz & & 1 \end{pmatrix} \in I' := I \cap K'$$

and

$$k^* := \begin{pmatrix} 1 & & \\ & 1 & \\ pz & & 1 \end{pmatrix} k = \begin{pmatrix} k_{11} & k_{12} & k_{13} \\ pk_{21} & k_{22} & k_{23} \\ pk_{31} + pzk_{11} & pk_{32} + pzk_{11} & k_{33} + pzk_{11} \end{pmatrix} \in I.$$

Since $\begin{pmatrix} 1 & & \\ & 1 & \\ -pz & & 1 \end{pmatrix} \in I$ we have, by a change of variable,

$$\phi^*(g) := \frac{1}{\mathrm{vol}(I)} \int_I \phi^\circ(gkt)dk = \frac{1}{\mathrm{vol}(I)} \int_{i\mathbb{Z}_p} \int_I \phi^\circ\left(g \begin{pmatrix} 1 & & \\ & 1 & \\ -pz & & 1 \end{pmatrix} kt\right) dkdz.$$

Since $t^{-1}k^*t \in K$ and ϕ° is spherical we see that

$$\begin{aligned} \phi^*(g) &= \int_{i\mathbb{Z}_p} \phi^\circ\left(g \begin{pmatrix} 1 & & \\ & 1 & \\ -pz & & 1 \end{pmatrix} t\right) dz \\ &= \int_{\mathbb{Z}_p - p\mathbb{Z}_p} \phi^\circ\left(g \begin{pmatrix} 1 & & \\ & 1 & \\ ipz & & 1 \end{pmatrix} t\right) dz + \int_{p\mathbb{Z}_p} \phi^\circ\left(g \begin{pmatrix} 1 & & \\ & 1 & \\ ipz & & 1 \end{pmatrix} t\right) dz. \end{aligned}$$

Note that for $z \in p\mathbb{Z}_p$, $\begin{pmatrix} 1 & & \\ & 1 & \\ ipz & & 1 \end{pmatrix} t \in tK$. Hence,

$$\phi^*(g) = \sum_{\alpha \in \mathbb{F}_p^\times} \phi^\circ \left(g \begin{pmatrix} p^{-1} & & \\ & 1 & \\ i\alpha & & p \end{pmatrix} \right) + p^{-1}\phi^\circ(gt).$$

Taking advantage of the identity

$$\begin{pmatrix} 1 & & i\alpha^{-1}p^{-1} \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} p^{-1} & & \\ & 1 & \\ i\alpha & & p \end{pmatrix} = \begin{pmatrix} & & i\alpha^{-1} \\ & 1 & \\ i\alpha & & p \end{pmatrix} \in K \tag{13.7}$$

we obtain the equality (13.6) by the change of variable $\alpha \mapsto \alpha^{-1}$.

Since π is unramified and given our choice for ϕ°, we have

$$\phi^\circ(t) = \delta(t)^{\frac{1}{2}} \overline{\chi}^2(p) = p^2 \overline{\chi}^2(p)$$

where δ is the modulus character, and χ is an unitary unramified character; it follows that, for $J = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}$

$$\phi^*(J) = p^{-1}\phi^\circ(t^{-1}) + \sum_{\alpha \in \mathbb{F}_p^\times} \phi^\circ \begin{pmatrix} 1 & & \\ & 1 & \\ i\alpha p^{-1} & & 1 \end{pmatrix}. \tag{13.8}$$

By (13.7) we have

$$\phi^\circ \begin{pmatrix} 1 & & \\ & 1 & \\ i\alpha p^{-1} & & 1 \end{pmatrix} = \phi^\circ \left(t^{-1} \begin{pmatrix} 1 & & i\alpha^{-1}p^{-1} \\ & 1 & \\ & & 1 \end{pmatrix} \right) = \phi^\circ(t^{-1}) = p^{-2}\chi^2(p).$$

Substituting this into (13.8) we then obtain

$$\phi^*(J) = (p^{-1} + p - 1)p^{-2}\chi^2(p) \neq 0. \tag{13.9}$$

Hence $\phi^* \not\equiv 0$. □

Lemma 13.3. *The vector ϕ^* is not a scalar multiple of ϕ°.*

Proof. Since ϕ° is spherical, we have

$$\phi^\circ(e) = \phi^\circ(J).$$

On the other hand, by Lemma 13.2 we have

$$\begin{aligned} \phi^*(e) &= p^{-1}\phi^\circ(t) + \sum_{\alpha \in \mathbb{F}_p^\times} \phi^\circ \begin{pmatrix} 1 & & i\alpha p^{-1} \\ & 1 & \\ & & 1 \end{pmatrix} \\ &= p^{-1}\phi^\circ(t) + \sum_{\alpha \in \mathbb{F}_p^\times} \phi^\circ(e), \\ &= p(\overline{\chi}^2(p) + 1) - 1. \end{aligned}$$

Since $|\chi(p)| = 1$, by the triangle inequality, we have $|\phi^*(e)| \geq 1$. On the other hand, (13.9) yields that

$$|\phi^*(J)| = p^{-1} - p^{-2} + p^{-3} < 1.$$

Hence, $\phi^*(e) \neq \phi^*(J)$.. However, as ϕ^o is K-invariant and $J \in K$, $\phi^o(J) = \phi^o(e)$ hence ϕ^* cannot be a scalar multiple of ϕ°. □

13.3.1. *The Gram-Schmidt Process.* Let

$$\phi^\dagger=\frac{\phi^*-\frac{\langle\phi^*,\phi^\circ\rangle}{\langle\phi^\circ,\phi^\circ\rangle}\phi^\circ}{\sqrt{\langle\phi^*,\phi^*\rangle-\frac{\langle\phi^*,\phi^\circ\rangle^2}{\langle\phi^\circ,\phi^\circ\rangle}}}.$$

Then by construction

$$\{\frac{\phi^\circ}{\sqrt{\langle\phi^\circ,\phi^\circ\rangle}},\ \phi^\dagger\}$$

is an orthonormal basis of π^I.

Norm computations. Given $\alpha\in\mathbb{Z}_p^\times$ we set

$$n_\alpha=\begin{pmatrix}1&&i\alpha p^{-1}\\&1&\\&&1\end{pmatrix}. \tag{13.10}$$

We have

$$\langle\pi(n_\alpha)\phi^\circ,\phi^\circ\rangle=\langle\pi(t^{-1})\phi^\circ,\phi^\circ\rangle \tag{13.11}$$

as (13.7) implies that $\begin{pmatrix}1&i\alpha p^{-1}\\&1\end{pmatrix}\in K't^{-1}K'$. Thus by (13.6) we have

$$\begin{aligned}\langle\phi^*,\phi^\circ\rangle=&\langle p^{-1}\pi(t)\phi^\circ+\sum_{\alpha\in\mathbb{F}_p^\times}\pi(n_\alpha)\phi^\circ,\phi^\circ\rangle\\=&p^{-1}\langle\pi(t)\phi^\circ,\phi^\circ\rangle+(p-1)\langle\pi(t^{-1})\phi^\circ,\phi^\circ\rangle,\end{aligned}$$

and

$$\begin{aligned}\langle\phi^*,\phi^*\rangle=&\langle p^{-1}\pi(t)\phi^\circ+\sum_{\alpha\in\mathbb{F}_p^\times}\pi(n_\alpha)\phi^\circ,p^{-1}\pi(t)\phi^\circ+\sum_{\alpha\in\mathbb{F}_p^\times}\pi(n_\alpha)\phi^\circ\rangle\\=&p^{-2}\langle\phi^\circ,\phi^\circ\rangle+p^{-1}\sum_{\alpha\in\mathbb{F}_p^\times}\langle\pi(n_\alpha)\phi^\circ,\pi(t)\phi^\circ\rangle\\&+p^{-1}\sum_{\alpha\in\mathbb{F}_p^\times}\langle\pi(t)\phi^\circ,\pi(n_\alpha)\phi^\circ\rangle+\sum_{\alpha\in\mathbb{F}_p^\times}\sum_{\beta\in\mathbb{F}_p^\times}\langle\pi(n_\beta^{-1}n_\alpha)\phi^\circ,\phi^\circ\rangle.\end{aligned}$$

Note that

$$t^{-1}n_\alpha=\begin{pmatrix}p&i\alpha\\&p^{-1}\end{pmatrix}=\begin{pmatrix}1&i\alpha p\\&1\end{pmatrix}\begin{pmatrix}p&\\&p^{-1}\end{pmatrix}.$$

Therefore, we have

$$\langle\pi(n_\alpha)\phi^\circ,\pi(t)\phi^\circ\rangle=\langle\pi(t^{-1}n_\alpha)\phi^\circ,\phi^\circ\rangle=\langle\pi(t^{-1})\phi^\circ,\phi^\circ\rangle=\langle\pi(t)\phi^\circ,\phi^\circ\rangle,$$

where we use the fact that $t^{-1}=JtJ$ and ϕ° is right J-invariant. By (13.7), we have

$$\langle\pi(n_\beta^{-1}n_\alpha)\phi^\circ,\phi^\circ\rangle=\begin{cases}\langle\phi^\circ,\phi^\circ\rangle, & \text{if } \alpha=\beta \text{ in } \mathbb{F}_p^\times\\ \langle\pi(t)\phi^\circ,\phi^\circ\rangle, & \text{otherwise.}\end{cases}$$

By MacDonald's formula for spherical vectors and the temperedness of π we have

$$\frac{\langle t.\phi^\circ,\phi^\circ\rangle}{\langle\phi^\circ,\phi^\circ\rangle}=O(\frac{1}{p^2})$$

and

$$\begin{aligned}\frac{\langle\phi^*,\phi^*\rangle}{\langle\phi^\circ,\phi^\circ\rangle}=&(p+p^{-2}-1)+(p^2-3p+4-2p^{-1})\frac{\langle\pi(t)\phi^\circ,\phi^\circ\rangle}{\langle\phi^\circ,\phi^\circ\rangle}=p+O(1),\\ \frac{\langle\phi^*,\phi^\circ\rangle}{\langle\phi^\circ,\phi^\circ\rangle}=&(p+p^{-1}-1)\frac{\langle\pi(t)\phi^\circ,\phi^\circ\rangle}{\langle\phi^\circ,\phi^\circ\rangle}=O(\frac{1}{p}),\end{aligned} \tag{13.12}$$

where the implied constants are absolute. Consequently we have

$$\frac{1}{\langle\phi^\circ,\phi^\circ\rangle}\big(\langle\phi^*,\phi^*\rangle-\frac{\langle\phi^*,\phi^\circ\rangle^2}{\langle\phi^\circ,\phi^\circ\rangle}\big)=p+O(1) \tag{13.13}$$

13.4. **Global Analysis: Proof of Proposition 13.1.** In this section we are back to the global setting and return to the notation in force at the beginning of section 13.

Given $\pi\simeq\otimes_{p\leq\infty}\pi_p\in\mathcal{A}_k(1)$, by the previous section we may assume that

$$\mathcal{B}_{\pi,k}(N)=\pi\cap\mathcal{B}_k(N)=\{\varphi_\pi^\circ,\varphi_\pi^\dagger\}$$

is made of two factorable vectors such that

$$\varphi_\pi^\circ\simeq\otimes_v\phi_v^\circ,\ \varphi_\pi^\dagger\simeq\phi_N^\dagger\otimes(\otimes_{v\neq N}\phi_v^\circ)$$

where

- $\phi_v^\circ\in\pi_v$ is either spherical for $v<\infty$ or a highest weight vector of the minimal K-type of D^Λ for $v=\infty$ and
- $\phi_N^\dagger\in\pi_N^{I_N}$ is the vector denoted $\phi^\dagger$ in §13.3.1.

We have

$$J^{old}(f^{\mathfrak{n}})=\frac{1}{d_\Lambda}\sum_{\pi\in\mathcal{A}_k(1)}\frac{\left|\mathcal{P}(\varphi_\pi^\circ,\varphi')\right|^2}{\langle\varphi_\pi^\circ,\varphi_\pi^\circ\rangle}+\frac{1}{d_\Lambda}\sum_{\pi\in\mathcal{A}_k(1)}\frac{\left|\mathcal{P}(\varphi_\pi^\dagger,\varphi')\right|^2}{\langle\varphi_\pi^\dagger,\varphi_\pi^\dagger\rangle}. \tag{13.14}$$

We handle the second sum on the RHS of (13.14). Set

$$p=N,\ t=\mathrm{diag}(p^{-1},1,p)\in G(\mathbb{Q}_p)\hookrightarrow G(\mathbb{A}).$$

We have for $\pi\in\mathcal{A}_k(1)$ (in the sequel we drop the index π to ease notations)

$$\frac{|\mathcal{P}(\varphi_\pi^\dagger,\varphi')|^2}{\langle\varphi_\pi^\dagger,\varphi_\pi^\dagger\rangle}\leq 2\frac{|\mathcal{P}(\phi^*,\varphi')|^2+|\frac{\langle\phi^*,\phi^\circ\rangle}{\langle\phi^\circ,\phi^\circ\rangle}|^2|\mathcal{P}(\phi^\circ,\varphi')|^2}{\langle\phi^*,\phi^*\rangle-\frac{\langle\phi^*,\phi^\circ\rangle^2}{\langle\phi^\circ,\phi^\circ\rangle}}. \tag{13.15}$$

We recall that

$$\phi^*=p^{-1}\pi_p(t)\phi^\circ+\sum_{\alpha\in\mathbb{F}_p^\times}\pi_p(n_\alpha)\phi^\circ\in\pi^I,$$

and n_α as in (13.10). By the change of variables

$$x\mapsto xt^{-1},\ xn_\alpha\mapsto xn_\alpha^{-1},\ \alpha\mapsto-\alpha,$$

we derive that

$$\mathcal{P}(\phi^*,\varphi')=p^{-1}\int_{[G']}\phi^\circ(x)\varphi'(xt^{-1})dx+\sum_{\alpha\in\mathbb{F}_p^\times}\int_{[G']}\phi^\circ(x)\varphi'(xn_\alpha)dx.$$

Similar to (13.11), or making use of (13.7), we have for any $\alpha\in\mathbb{F}_p{}^\times$

$$\int_{[G']}\phi^\circ(x)\varphi'(xn_\alpha)dx=\int_{[G']}\phi^\circ(x)\varphi'(xt^{-1})dx$$

so that

$$\mathcal{P}(\phi^*,\varphi')=(p-1+\frac{1}{p})\int_{[G']}\phi^\circ(x)\varphi'(xt^{-1})dx. \tag{13.16}$$

Let K' be the maximal compact subgroup of $G'(\mathbb{A})$. Since ϕ° is K'-invariant, we have by a change of variable,

$$\int_{[G']}\phi^\circ(x)\varphi'(xt^{-1})dx=\frac{1}{\mathrm{vol}(K')}\int_{[G']}\phi^\circ(x)\int_{K'}\varphi'(xk't^{-1})dk'dx,$$

where the inner integral defines a spherical function.

By multiplicity one and MacDonald's formula, we have

$$\int_{[G']} \phi^\circ(x)\varphi'(xt^{-1})dx = c_{\pi'_N}(t^{-1}) \int_{[G']} \phi^\circ(x)\varphi'(x)dx = c_{\pi'_N}(t^{-1})\mathcal{P}(\phi^\circ, \varphi')$$

for some scalar function

$$c_{\pi'_N}(t) \ll \delta'(t^{-1}) \ll p^{-1}.$$

Here the implied constant is absolute since π'_p is tempered. Therefore, by (13.16) we have

$$\mathcal{P}(\phi^*, \varphi') \ll \mathcal{P}(\phi^\circ, \varphi').$$

By (13.15), (13.12) and (13.13), we obtain

$$\frac{\left|\mathcal{P}(\varphi^\dagger, \varphi')\right|^2}{\langle \varphi^\dagger, \varphi^\dagger \rangle} \ll \frac{1}{p} \frac{|\mathcal{P}(\phi^\circ, \varphi')|^2}{\langle \phi^\circ, \phi^\circ \rangle}.$$

so that

$$\frac{1}{d_\Lambda} \sum_{\pi \in \mathcal{A}_k(1)} \frac{\left|\mathcal{P}(\varphi^\dagger_\pi, \varphi')\right|^2}{\langle \varphi^\dagger_\pi, \varphi^\dagger_\pi \rangle} \ll \frac{1}{N} \frac{1}{d_\Lambda} \sum_{\pi \in \mathcal{A}_k(1)} \frac{\left|\mathcal{P}(\varphi^\circ_\pi, \varphi')\right|^2}{\langle \varphi^\circ_\pi, \varphi^\circ_\pi \rangle}; \tag{13.17}$$

consequently we have

$$J^{old}(f^{\mathfrak{n}}) \ll \frac{1}{d_\Lambda} \sum_{\pi \in \mathcal{A}_k(1)} \frac{\left|\mathcal{P}(\varphi^\circ_\pi, \varphi')\right|^2}{\langle \varphi^\circ_\pi, \varphi^\circ_\pi \rangle}$$

and the argument of §11.2 for $N = \ell = 1$ yield

$$J^{old}(f^{\mathfrak{n}}) \ll_{N'} \frac{1}{k} + \frac{k^{o(1)}}{2^{4k}k^2} + k^{1/2+o(1)}(e^{-\frac{\kappa}{N'^2+1}} + 2^{-\kappa}) \ll_{N'} \frac{1}{k} \tag{13.18}$$

Then (13.1) follows from substituting (13.18) and (13.17) into (13.14).

14. Amplification and Non-vanishing

In this section, we prove Theorem 1.4 and Theorem 1.3 . We assume that

$$k, N \geq C(E, N')$$

for a suitable constant depending on E and N'.

14.1. **The Amplifier.** Let $\sigma \in \mathcal{A}_k(N)$. Let $L > 1$. Denote by

$$\mathcal{L} := \{L/2 < \ell < L : \ \ell \text{ is an inert prime in } E, \text{ and } (\ell, NN') = 1\}.$$

By the prime theorem in arithmetic progression, one has $|\mathcal{L}| \asymp_E L/\log L$, where the implied constant depends on E.

Recall that, for $r \geq 1$, one has $\lambda_\sigma(\ell^r) \in \mathbb{R}$, and that by (11.4), one has

$$\lambda_\sigma(\ell)^2 = \lambda_\sigma(\ell^2) + \ell^{-1}\lambda_\sigma(\ell) + 1.$$

Suppose $|\lambda_\sigma(\ell)| < 1/2$ and $|\lambda_\sigma(\ell^2)| < 1/2$. By triangle inequality we obtain

$$1 \leq \lambda_\sigma(\ell)^2 + |\lambda_\sigma(p^2)| + p^{-1}|\lambda_\sigma(\ell)| < \frac{1}{4} + \frac{1}{2} + p^{-1} \cdot \frac{1}{2} < 1,$$

a contradiction! Hence, there exists $r_p \in \{1, 2\}$ such that $|\lambda_\sigma(p^{r_p})| \geq 1/2$.

Let

$$J_{\mathrm{Spec}}(\sigma, L) := \frac{1}{d_\Lambda} \sum_{\pi \in \mathcal{A}_k(N)} \left| \sum_{p \in \mathcal{L}} \lambda_\sigma(p^{r_p})\lambda_\pi(p^{r_p}) \right|^2 \sum_{\varphi \in \mathcal{B}^{\tilde{\mathfrak{n}}}_{\pi,k}(N)} \frac{\left|\mathcal{P}(\varphi, \varphi')\right|^2}{\langle \varphi, \varphi \rangle \langle \varphi', \varphi' \rangle}.$$

14.2. **Spectral Side: a lower bound.** By dropping all π's that are not equal to σ we have

$$\begin{aligned} J_{\mathrm{Spec}}(\sigma,L) &\geq \frac{1}{d_\Lambda}\Big|\sum_{p\in\mathcal{L}}\lambda_\sigma(p^{r_p})^2\Big|^2 \sum_{\varphi\in\mathcal{B}^{\tilde{\mathfrak{n}}}_{\sigma,k}(N)} \frac{\left|\mathcal{P}(\varphi,\varphi')\right|^2}{\langle\varphi,\varphi\rangle\langle\varphi',\varphi'\rangle} \\ &\gg \frac{L^2}{d_\Lambda\log^2 L}\sum_{\varphi\in\mathcal{B}^{\tilde{\mathfrak{n}}}_{\sigma,k}(N)} \frac{\left|\mathcal{P}(\varphi,\varphi')\right|^2}{\langle\varphi,\varphi\rangle\langle\varphi',\varphi'\rangle}. \end{aligned} \tag{14.1}$$

14.3. **Spectral Side: decompostion.** Squaring out of the sum over the primes $\ell\in\mathcal{L}$, we obtain

$$J_{\mathrm{Spec}}(\sigma,L) = J^{=}_{\mathrm{Spec}}(\sigma,L) + J^{\neq}_{\mathrm{Spec}}(\sigma,L)$$

where we have set

$$J^{=}_{\mathrm{Spec}}(\sigma,L) = \sum_{\ell\in\mathcal{L}}\lambda_\sigma(\ell^{r_\ell})^2 J_{\mathrm{Spec}}(\ell),$$

$$J^{\neq}_{\mathrm{Spec}}(\sigma,L) = \sum_{\substack{\ell_1,\ell_2\in\mathcal{L}\\ \ell_1\neq\ell_2}}\lambda_\sigma(\ell_1^{r_{\ell_1}})\lambda_\sigma(\ell_2^{r_{\ell_2}})J_{\mathrm{Spec}}(\ell_1,\ell_2)$$

with

$$J_{\mathrm{Spec}}(\ell) = \frac{1}{d_\Lambda}\sum_{\pi\in\mathcal{A}_k(N)}\lambda_\pi(\ell^{r_\ell})^2 \sum_{\varphi\in\mathcal{B}^{\tilde{\mathfrak{n}}}_{\pi,k}(N)} \frac{\left|\mathcal{P}(\varphi,\varphi')\right|^2}{\langle\varphi,\varphi\rangle\langle\varphi',\varphi'\rangle},$$

$$J_{\mathrm{Spec}}(\ell_1,\ell_2) = \frac{1}{d_\Lambda}\sum_{\pi\in\mathcal{A}_k(N)}\lambda_\pi(\ell_1^{r_{\ell_1}}\ell_2^{r_{\ell_2}}) \sum_{\varphi\in\mathcal{B}^{\tilde{\mathfrak{n}}}_{\pi,k}(N)} \frac{\left|\mathcal{P}(\varphi,\varphi')\right|^2}{\langle\varphi,\varphi\rangle\langle\varphi',\varphi'\rangle}.$$

We will now bound $J_{\mathrm{Spec}}(\ell)$ and $J_{\mathrm{Spec}}(\ell_1,\ell_2)$ using Theorem 11.1.
We notice first that by the Hecke relations (11.4) we have

$$\begin{cases}\lambda_\pi(\ell)^2 = \lambda_\pi(\ell^2)+p^{-1}\lambda_\pi(\ell)+1,\\ \lambda_\pi(\ell^2)^2 = \lambda_\pi(\ell^4)+\ell^{-1}\lambda_\pi(\ell^3)+\lambda_\pi(\ell^2)+\ell^{-1}\lambda_\pi(\ell)+1.\end{cases} \tag{14.2}$$

Il follows that

$$|J_{\mathrm{Spec}}(\ell)| \leq 4\max_{0\leq\alpha\leq4}\Big|\frac{1}{d_\Lambda}\sum_{\pi\in\mathcal{A}_k(N)}\lambda_\pi(\ell^\alpha)\sum_{\varphi\in\mathcal{B}^{\tilde{\mathfrak{n}}}_{\pi,k}(N)} \frac{\left|\mathcal{P}(\varphi,\varphi')\right|^2}{\langle\varphi,\varphi\rangle\langle\varphi',\varphi'\rangle}\Big|.$$

Since $\ell^\alpha\in[1,L^4]$, by Theorem 11.1 and Deligne's bound, $|\lambda_{\pi'}(\ell^\alpha)|\leq\alpha+1$, we obtain

$$J_{\mathrm{Spec}}(\ell) \ll_{N'} (kLN)^{o(1)}\Big(\frac{N^2}{k}+\frac{L^{60}N^2}{k^{1/2}}(1+\frac{L^8}{N})(e^{-\kappa/N'^2L^8}+2^{-\kappa})\Big) \tag{14.3}$$

and if

$$N'^2L^8 < N \tag{14.4}$$

the second term on the righthand side above can be replaced by

$$\frac{L^{40}N^4}{k^{1/2}}(\frac{L^8N'^2}{N})^\kappa \tag{14.5}$$

Averaging over $\ell\in\mathcal{L}$ and using the bound $\lambda_\sigma(\ell^{r_\ell})^2\ll1$ we obtain that

$$J^{=}_{\mathrm{Spec}}(\sigma,L) \ll_{N'} (kLN)^{o(1)}\Big(\frac{LN^2}{k}+\frac{L^{61}N^2}{k^{1/2}}(1+\frac{L^8}{N})(e^{-\kappa/N'^2L^8}+2^{-\kappa})\Big) \tag{14.6}$$

and if (14.4) holds, the second term on the righthand side of the above bound can be replaced by

$$\frac{L^{41}N^4}{k^{1/2}}(\frac{L^8N'^2}{N})^{\kappa}.$$

We treat $J^{\neq}_{\mathrm{Spec}}(\sigma,L)$ is the same way. Since $\ell_1^{r_{\ell_1}}\ell_2^{r_{\ell_2}}\in[L^2/4,L^4]$, using again Theorem 11.1, we obtain the bound

$$J_{\mathrm{Spec}}(\ell_1,\ell_2)\ll_{N'}(kLN)^{o(1)}\Big(\frac{N^2}{kL^2}+\frac{L^{60}N^2}{k^{1/2}}(1+\frac{L^8}{N})(e^{-\kappa/N'^2L^8}+2^{-\kappa})\Big)$$

and if (14.4) holds, the second term on the righthand side of (14.6) can be replaced by (14.5). Averaging over $\ell_1\neq\ell_2\in\mathcal{L}$ we obtain

$$J^{\neq}_{\mathrm{Spec}}(\sigma,L)\ll_{N'}(kLN)^{o(1)}\Big(\frac{N^2}{k}+\frac{L^{62}N^2}{k^{1/2}}(1+\frac{L^8}{N})(e^{-\kappa/N'^2L^8}+2^{-\kappa})\Big)\tag{14.7}$$

and if (14.4) holds, the second term on the righthand side of (14.3) can be replaced by

$$\frac{L^{42}N^4}{k^{1/2}}(\frac{L^8N'^2}{N})^{\kappa}.$$

In conclusion we obtain that

$$J_{\mathrm{Spec}}(\sigma,L)\ll_{N'}(kLN)^{o(1)}\Big(\frac{LN^2}{k}+\frac{L^{62}N^2}{k^{1/2}}(1+\frac{L^8}{N})(e^{-\kappa/N'^2L^8}+2^{-\kappa})\Big)$$

and if in addition (14.4) holds we have

$$J_{\mathrm{Spec}}(\sigma,L)\ll_{N'}(kLN)^{o(1)}\Big(\frac{LN^2}{k}+\frac{L^{42}N^4}{k^{1/2}}(\frac{L^8N'^2}{N})^{\kappa}\Big)$$

combining this with (14.1) we obtain that

$$\begin{aligned}\sum_{\varphi\in\mathcal{B}^{\tilde{\mathfrak{n}}}_{\sigma,k}(N)}&\frac{\big|\mathcal{P}(\varphi,\varphi')\big|^2}{\langle\varphi,\varphi\rangle\langle\varphi',\varphi'\rangle}\\&\ll_{N'}(kNL)^{o(1)}(\frac{k^2N^2}{L}+L^{60}k^{3/2}N^2(1+\frac{L^8}{N})(e^{-\kappa/N'^2L^8}+2^{-\kappa}))\end{aligned}\tag{14.8}$$

and if (14.4) holds, we have

$$\sum_{\varphi\in\mathcal{B}^{\tilde{\mathfrak{n}}}_{\sigma,k}(N)}\frac{\big|\mathcal{P}(\varphi,\varphi')\big|^2}{\langle\varphi,\varphi\rangle\langle\varphi',\varphi'\rangle}\ll_{N'}(kLN)^{o(1)}\Big(\frac{k^2N^2}{L}+L^{40}k^{3/2}N^4(\frac{L^8N'^2}{N})^{\kappa}\Big).\tag{14.9}$$

14.3.1. *The case $k\geq N$.* We choose $L\gg_E 1$ such that $\mathcal{L}$ is not empty and

$$L^8=\frac{(kN)^{1/2}}{N'^2\log^2(kN)}$$

(which requires that $N'\ll_E\frac{(kN)^{1/4}}{\log(kN)}$). Since $k\geq(kN)^{1/2}$ we conclude that the second term on the left-hand side of (14.8) is negligible and that

$$\sum_{\varphi\in\mathcal{B}^{\tilde{\mathfrak{n}}}_{\sigma,k}(N)}\frac{\big|\mathcal{P}(\varphi,\varphi')\big|^2}{\langle\varphi,\varphi\rangle\langle\varphi',\varphi'\rangle}\ll_{N'}(kN)^{o(1)}(kN)^{2-1/16}.$$

14.3.2. *The case $k \leq N$.* We choose $L \gg_E 1$ such that $\mathcal{L}$ is not empty and

$$L^8 \leq \frac{1}{2}\frac{N^{1/2}}{N'^2}; \tag{14.10}$$

this implies that (14.4) is satisfied and requires $N' \ll_E N^{1/4}$.

The second term on the left-hand side of (14.9) is bounded by

$$(kN)^{o(1)}(kN)^2L^{40}k^{-1/2}N^{2-\kappa/2} \leq (kN)^{o(1)}(kN)^{3/2}L^{40}$$

since $\kappa/2 - 2 \geq 1/2$. Choosing

$$L = (kN)^{1/82}$$

(this is compatible with (14.10)) so that that $(kN)^2/L = (kN)^{3/2}L^{40}$ we obtain for $\sigma \in \mathcal{A}_k(N)$ the bound

$$\sum_{\varphi \in \mathcal{B}^{\tilde{\mathrm{n}}}_{\sigma,k}(N)} \frac{\left|\mathcal{P}(\varphi,\varphi')\right|^2}{\langle\varphi,\varphi\rangle\langle\varphi',\varphi'\rangle} \ll_{N'} (kN)^{o(1)}(kN)^{2-1/82}. \tag{14.11}$$

$\square$

14.4. **Proof of Theorem 1.3.** If $N > 1$, by (13.4) we have

$$\sum_{\pi \in \mathcal{A}^{\mathrm{n}}_k(N)} \sum_{\varphi \in \mathcal{B}^{\tilde{\mathrm{n}}}_{\pi,k}(N)} \frac{\left|\mathcal{P}(\varphi,\varphi')\right|^2}{\langle\varphi,\varphi\rangle\langle\varphi',\varphi'\rangle} \asymp_{N'} (kN)^2$$

(note that since $\pi \in \mathcal{A}^{\mathrm{n}}_k(N)$, $\mathcal{B}^{\tilde{\mathrm{n}}}_{\pi,k}(N)$ is a singleton) and from (14.11) (for $k > 32$ and $N' \ll_E (Nk)^{1/8}$) we have

$$\sum_{\pi \in \mathcal{A}^{\mathrm{n}}_k(N)} \sum_{\varphi \in \mathcal{B}^{\tilde{\mathrm{n}}}_{\pi,k}(N)} \delta_{\mathcal{P}(\varphi,\varphi')\neq 0} \gg_{N'} (kN)^{-1/82+o(1)}$$

since $|\mathcal{B}^{\tilde{\mathrm{n}}}_{\pi,k}(N)| \leq 1$ and $L(1/2, \pi_E \times \pi'_E)$ is proportional to $\left|\mathcal{P}(\varphi,\varphi')\right|^2$ we obtain Theorem 1.3 for $N > 1$.

The case $N = 1$ follows the same principle by using (1.13) for $N = 1$.

Appendix A. Explicit double coset decompositions

In this appendix we record several consequences of the Bruhat-Iwahori-Cartan decompositions for the open compact groups $G(\mathbb{Z}_p)$ and $G'(\mathbb{Z}_p)$ which are used in the evaluation of the local period integrals in §5.

A.1. **Decompositions for $U(W)$.** In this section we discuss the case of $G'(\mathbb{Z}_p)$. For this it will be useful to represent the elements of G' by their 2×2 matrices in the basis $\{e_{-1}, e_1\}$; moreover if p is split we will identify $G'(\mathbb{Q}_p)$ with $\mathrm{GL}_2(\mathbb{Q}_p)$.

We denote by

$$I'_p \subset G'(\mathbb{Z}_p)$$

the Iwahori subgroup corresponding to matrices which are upper-triangular modulo p.

The following lemma is a consequence of the Bruhat decomposition for $G'(\mathbb{F}_p)$.

Lemma A.1. *We have the disjoint union decompositions*

$$G'(\mathbb{Z}_p) = I'_p \sqcup \bigsqcup_{\delta \in \mathbb{Z}_p/p\mathbb{Z}_p} \begin{pmatrix} \delta & 1 \\ 1 & \end{pmatrix} I'_p \quad \textit{if } p \textit{ is split;} \tag{A.1}$$

$$G'(\mathbb{Z}_p) = I'_p \sqcup \bigsqcup_{\substack{\delta \in \mathcal{O}_{E_p}/p\mathcal{O}_{E_p} \\ \delta+\overline{\delta}=0}} \begin{pmatrix} \delta & 1 \\ 1 & \end{pmatrix} I'_p \quad \textit{if } p \textit{ is inert.} \tag{A.2}$$

In particular

$$|G'(\mathbb{Z}_p)/I'_p| = p+1 = \mu(I'_p)^{-1}.$$

A.1.1. *Bruhat-Iwahori-Cartan Decomposition on $U(W)$.* We set

$$J' = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \quad A_n = \begin{pmatrix} p^n & \\ & p^{-n} \end{pmatrix}, \quad n \geq 1.$$

We have the following double cosets decomposition:

Lemma A.2. *For p inert in E, we have the disjoint unions*

$$I'_p A_n I'_p = \bigsqcup_{\substack{\tau \in \mathcal{O}_p/p^{2n}\mathcal{O}_p \\ \tau+\overline{\tau}=0}} \begin{pmatrix} 1 & \tau \\ & 1 \end{pmatrix} A_n I'_p,$$

$$I'_p J' A_n I'_p = \bigsqcup_{\substack{\tau \in p\mathcal{O}_p/p^{2n}\mathcal{O}_p \\ \tau+\overline{\tau}=0}} \begin{pmatrix} 1 & \\ \tau & 1 \end{pmatrix} J' A_n I'_p,$$

$$I'_p A_n J' I'_p = \bigsqcup_{\substack{\tau \in \mathcal{O}_p/p^{2n+1}\mathcal{O}_p \\ \tau+\overline{\tau}=0}} \begin{pmatrix} 1 & \tau \\ & 1 \end{pmatrix} A_n J' I'_p,$$

$$I'_p J' A_n J' I'_p = \bigsqcup_{\substack{\tau \in p\mathcal{O}_p/p^{2n+1}\mathcal{O}_p \\ \tau+\overline{\tau}=0}} \begin{pmatrix} 1 & \\ \tau & 1 \end{pmatrix} J' A_n J' I'_p.$$

For p split in E these decompositions holds upon replacing $\mathcal{O}_p$ by $\mathbb{Z}_p$ and by removing the condition $\tau + \overline{\tau} = 0$.

For the proof we refer to §A.2 where we discuss the more complicated case of $G(\mathbb{Z}_p)$.

Lemma A.3. *Notations as in the previous lemma, we have we have*

$$G'(\mathbb{Z}_p) A_n G'(\mathbb{Z}_p) = I'_p A_n I'_p \bigsqcup I'_p A_n J' I'_p \bigsqcup I'_p J' A_n I'_p \bigsqcup I'_p J' A_n J' I'_p. \tag{A.3}$$

Proof. We discuss again only the case p inert.

Taking inverse in the identity (A.2) we have

$$G'(\mathbb{Z}_p) = I'_p \sqcup \bigsqcup_{\substack{\delta \in \mathcal{O}_p/N'\mathcal{O}_p \\ \delta+\overline{\delta}=0}} I'_p \begin{pmatrix} & 1 \\ 1 & \delta \end{pmatrix}. \tag{A.4}$$

We thus have, by (A.2) and (A.4), that $G'(\mathbb{Z}_p) A_n G'(\mathbb{Z}_p) = U'_1 \bigcup U'_2$, where

$$U'_1 := \bigcup_{\substack{\delta \in \mathcal{O}_p/N'\mathcal{O}_p \\ \delta+\overline{\delta}=0}} I'_p A_n \begin{pmatrix} \delta & 1 \\ 1 & \end{pmatrix} I_p \cup \bigcup_{\substack{\delta \in \mathcal{O}_p/N'\mathcal{O}_p \\ \delta+\overline{\delta}=0}} I'_p \begin{pmatrix} & 1 \\ 1 & \delta \end{pmatrix} A_n I'_p$$

$$U'_2 := I'_p A_n I'_p \cup \bigcup_{\substack{\delta_1, \delta_2 \in \mathcal{O}_p/N'\mathcal{O}_p \\ \delta_1+\overline{\delta}_1=0 \\ \delta_2+\overline{\delta}_2=0}} I'_p \begin{pmatrix} & 1 \\ 1 & \delta_1 \end{pmatrix} A_n \begin{pmatrix} \delta_2 & 1 \\ & 1 \end{pmatrix} I'_p.$$

Suppose $n \geq 1$. Then $G'(\mathbb{Z}_p) A_n G'(\mathbb{Z}_p) = I'_p A_n I'_p \bigcup I'_p A_n J' I'_p \bigcup U'_3$, where

$$U'_3 := \bigcup_{\substack{\delta \in \mathcal{O}_p/N'\mathcal{O}_p \\ \delta+\overline{\delta}=0}} I'_p \begin{pmatrix} & 1 \\ 1 & \delta \end{pmatrix} A_n I'_p \cup \bigcup_{\substack{\delta \in \mathcal{O}_p/N'\mathcal{O}_p \\ \delta+\overline{\delta}=0}} I'_p \begin{pmatrix} & 1 \\ 1 & \delta \end{pmatrix} A_n J' I'_p.$$

Note that under the assumption $\delta \in \mathcal{O}_p^\times$, we have

$$\begin{pmatrix} 1 & \frac{1}{\delta} \\ & 1 \end{pmatrix} A_n = \begin{pmatrix} & p^{-n} \\ p^n & \delta p^{-n} \end{pmatrix} = \begin{pmatrix} 1 & \overline{\delta}^{-1} \\ & 1 \end{pmatrix} A_n \begin{pmatrix} \delta^{-1} & \\ p^{2n} & \overline{\delta} \end{pmatrix} \in I_p' A_n I_p';$$

and $J' A_n = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} A_n \in I_p' J' A_n I_p'$. Hence, we obtain

$$\bigcup_{\substack{\delta \in \mathcal{O}_p/N'\mathcal{O}_p \\ \delta+\overline{\delta}=0}} I_p' \begin{pmatrix} & \frac{1}{\delta} \\ 1 & \end{pmatrix} A_n I_p' \subseteq I_p' A_n I_p' \bigcup I_p' J' A_n I_p'. \tag{A.5}$$

Note that, for $\delta \in \mathcal{O}_p^\times$, a straightforward computation shows

$$\begin{pmatrix} & \frac{1}{\delta} \\ 1 & \end{pmatrix} A_n J' = \begin{pmatrix} p^{-n} & \\ \delta p^{-n} & p^n \end{pmatrix} = \begin{pmatrix} \delta^{-1} & \frac{1}{\delta} \\ & \end{pmatrix} A_n J' \begin{pmatrix} 1 & p^{2n}\overline{\delta}^{-1} \\ & 1 \end{pmatrix} \in I_p' A_n J' I_p'.$$

Also, $J' A_n J' = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} A_n J' \in I_p' J' A_n J' I_p'$. Hence, similar to (A.5) we have,

$$\bigcup_{\substack{\delta \in \mathcal{O}_p/N'\mathcal{O}_p \\ \delta+\overline{\delta}=0}} I_p' \begin{pmatrix} & \frac{1}{\delta} \\ 1 & \end{pmatrix} A_n J' I_p' \subseteq I_p' A_n I_p' \bigcup I_p' J' A_n J' I_p'. \tag{A.6}$$

Substituting the relations (A.5) and (A.6) into the definition of U_3' and the decomposition $G'(\mathbb{Z}_p) A_n G'(\mathbb{Z}_p) = I_p' A_n I_p' \bigcup I_p' A_n J' I_p' \bigcup U_3'$ we then conclude

$$G'(\mathbb{Z}_p) A_n G'(\mathbb{Z}_p) = I_p' A_n I_p' \bigcup I_p' A_n J' I_p' \bigcup I_p' J' A_n I_p' \bigcup I_p' J' A_n J' I_p'. \tag{A.7}$$

Then (A.3) follows from the fact that the union in (A.7) is actually disjoint. □

A.2. **Decompositions for $U(V)$.** Let p be a prime which is inert in E (for instance $p = N$); let $E_p = E \otimes_{\mathbb{Q}} \mathbb{Q}_p$ be the corresponding local field, $\overline{\bullet} : z \mapsto \overline{z}$ the complex conjugation on E_p and $\mathcal{O}_p$ be its ring of integers and p is an uniformizer. We denote by $\nu : E_p \mapsto \mathbb{Z}$ the normalized valuation.

We recall that, by definition of the unitary group $G(\mathbb{Q}_p)$, we have for $g = \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix} \in G(\mathbb{Q}_p)$, the relations

$$\begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix} \begin{pmatrix} \overline{g}_{33} & \overline{g}_{23} & \overline{g}_{13} \\ \overline{g}_{32} & \overline{g}_{22} & \overline{g}_{12} \\ \overline{g}_{31} & \overline{g}_{21} & \overline{g}_{11} \end{pmatrix} = I_3 \tag{A.8}$$

and

$$\begin{pmatrix} \overline{g}_{33} & \overline{g}_{23} & \overline{g}_{13} \\ \overline{g}_{32} & \overline{g}_{22} & \overline{g}_{12} \\ \overline{g}_{31} & \overline{g}_{21} & \overline{g}_{11} \end{pmatrix} \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix} = I_3. \tag{A.9}$$

We denote by I_p the Iwahori subgroup

$$I_p := G(\mathbb{Z}_p) \cap \begin{pmatrix} \mathcal{O}_p & \mathcal{O}_p & \mathcal{O}_p \\ p\mathcal{O}_p & \mathcal{O}_p & \mathcal{O}_p \\ p\mathcal{O}_p & p\mathcal{O}_p & \mathcal{O}_p \end{pmatrix}. \tag{A.10}$$

In particular if $p = N$, $K_p(N) = I_p$.

Like in Lemma A.1 the following is a consequence of the Bruhat decomposition for $G(\mathbb{F}_p)$:

$$G(\mathbb{F}_p) = P(\mathbb{F}_p) \sqcup P(\mathbb{F}_p) J N(\mathbb{F}_p)$$

where $P\subset G$ is the Borel subgroup with unipotent radical N, so

$$N(\mathbb{F}_p)=\{\begin{pmatrix}1&\delta&\tau\\&1&-\overline{\delta}\\&&1\end{pmatrix},\ \delta,\tau\in\mathbb{F}_{p^2}\}.$$

Lemma A.4. *Let p be a prime inert in E. We have a disjoint coset decomposition,*

$$G(\mathbb{Z}_p)=I_p\bigsqcup_{\substack{\tau\in\mathcal{O}_p/p\mathcal{O}_p\\ \delta\in\mathcal{O}_p/p\mathcal{O}_p\\ \tau+\overline{\tau}+\delta\overline{\delta}=0}}\bigsqcup\begin{pmatrix}\tau&\delta&1\\-\overline{\delta}&1&\\1&&\end{pmatrix}I_p. \tag{A.11}$$

In particular

$$|G(\mathbb{Z}_p)/I_p|=p^3+1=(p+1)(p^2-p+1)=\mu(I_p)^{-1}$$

For $n\in\mathbb{Z}$ we set

$$A_n=\begin{pmatrix}p^n&&\\&1&\\&&p^{-n}\end{pmatrix}. \tag{A.12}$$

Lemma A.5. *Assume that p is inert in E. For $n\geq 1$, we have the disjoint decompositions*

$$I_pA_nI_p=\bigsqcup_{\substack{\delta\in\mathcal{O}_p/p^n\mathcal{O}_p\\ \tau\in\mathcal{O}_p/p^{2n}\mathcal{O}_p\\ \tau+\overline{\tau}+\delta\overline{\delta}=0}}\bigsqcup\begin{pmatrix}1&\delta&\tau\\&1&-\overline{\delta}\\&&1\end{pmatrix}A_nI_p,$$

$$I_pJA_nI_p=\bigsqcup_{\substack{\delta\in p\mathcal{O}_p/p^n\mathcal{O}_p\\ \tau\in p\mathcal{O}_p/p^{2n}\mathcal{O}_p\\ \tau+\overline{\tau}+\delta\overline{\delta}=0}}\bigsqcup\begin{pmatrix}1&&\\-\overline{\delta}&1&\\\tau&\delta&1\end{pmatrix}JA_nI_p,$$

$$I_pA_nJI_p=\bigsqcup_{\substack{\delta\in\mathcal{O}_p/p^{n+1}\mathcal{O}_p\\ \tau\in\mathcal{O}_p/p^{2n+1}\mathcal{O}_p\\ \tau+\overline{\tau}+\delta\overline{\delta}=0}}\bigsqcup\begin{pmatrix}1&\delta&\tau\\&1&-\overline{\delta}\\&&1\end{pmatrix}A_nJI_p,$$

$$I_pJA_nJI_p=\bigsqcup_{\substack{\delta\in p\mathcal{O}_p/p^{n+1}\mathcal{O}_p\\ \tau\in p\mathcal{O}_p/p^{2n+1}\mathcal{O}_p\\ \tau+\overline{\tau}+\delta\overline{\delta}=0}}\bigsqcup\begin{pmatrix}1&&\\-\overline{\delta}&1&\\\tau&\delta&1\end{pmatrix}JA_nJI_p.$$

Proof. Let

$$X=\begin{pmatrix}g_{11}&g_{12}&g_{13}\\g_{21}&g_{22}&g_{23}\\g_{31}&g_{32}&g_{33}\end{pmatrix}\in I_p$$

and let

$$Y=\begin{pmatrix}1&&\\p^n\overline{g}_{32}&1&\\p^{2n}\overline{g}_{33}&-p^ng_{32}&1\end{pmatrix}\in I_p.$$

A priori we have $g_{33}\in\mathcal{O}_p^\times$ but we may assume as well that $g_{33}=1$.

Using (A.8) one has

$$g_{12}-g_{13}g_{32}=-\overline{g}_{23}(g_{22}-g_{23}g_{32})\text{ and }(g_{22}-g_{23}g_{32})(\overline{g}_{22}-\overline{g}_{23}\overline{g}_{32})=1.$$

We have

$$XA_nY = \begin{pmatrix} 1 & g_{12}-g_{13}g_{32} & g_{13} \\ & g_{22}-g_{23}g_{32} & g_{23} \\ & & 1 \end{pmatrix} A_n$$
$$= \begin{pmatrix} 1 & -\overline{g}_{23} & g_{13} \\ & 1 & g_{23} \\ & & 1 \end{pmatrix} A_n \begin{pmatrix} 1 & & \\ & g_{22}-g_{23}g_{32} & \\ & & 1 \end{pmatrix}.$$

Since $\mathrm{diag}(1, g_{22}-g_{23}g_{32}, 1) \in I_p$, we then obtain

$$I_pA_nI_p = N(\mathbb{Z}_p)A_nI_p = \bigcup_{\substack{\delta,\tau\in\mathcal{O}_p \\ \tau+\overline{\tau}+\delta\overline{\delta}=0}} n(\delta,\tau)A_nI_p, \quad n(\delta,\tau) = \begin{pmatrix} 1 & \delta & \tau \\ & 1 & -\overline{\delta} \\ & & 1 \end{pmatrix}.$$

Since $A_n^{-1}n(\delta,\tau)A_n = n(p^{-n}\delta, p^{2n}\tau)$, we then have

$$I_pA_nI_p = \bigcup_{\substack{\delta,\tau\in\mathcal{O}_p \\ \tau+\overline{\tau}+\delta\overline{\delta}=0}} n(\delta,\tau)A_nI_p = \bigsqcup_{\substack{\delta\in p\mathcal{O}_p/p^n\mathcal{O}_p \\ \tau\in p\mathcal{O}_p/p^{2n}\mathcal{O}_p \\ \tau+\overline{\tau}+\delta\overline{\delta}=0}} \begin{pmatrix} 1 & \delta & \tau \\ & 1 & -\overline{\delta} \\ & & 1 \end{pmatrix} A_nI_p. \tag{A.13}$$

Similarly, there are some $\delta_1, \tau_1 \in p\mathcal{O}_p$ such that

$$\begin{pmatrix} 1 & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix} JA_nJ \begin{pmatrix} 1 & -p^n g_{12} & p^{2n}\overline{g}_{13} \\ & 1 & p^n\overline{g}_{12} \\ & & 1 \end{pmatrix} = Jn(\delta_1,\tau_1)JA_n^{-1}.$$

Note $A_n^{-1} = JA_nJ$. Similar to (A.13), one has

$$I_pJA_nJI_p = \bigcup_{\substack{\delta,\tau\in p\mathcal{O}_p \\ \tau+\overline{\tau}+\delta\overline{\delta}=0}} Jn(\delta,\tau)JA_n^{-1}I_p = \bigsqcup_{\substack{\delta\in p\mathcal{O}_p/p^{n+1}\mathcal{O}_p \\ \tau\in p\mathcal{O}_p/p^{2n+1}\mathcal{O}_p \\ \tau+\overline{\tau}+\delta\overline{\delta}=0}} \begin{pmatrix} 1 & & \\ -\overline{\delta} & 1 & \\ \tau & \delta & 1 \end{pmatrix} JA_nJI_p.$$

By a straightforward computation there are some $\delta_2, \tau_3 \in \mathcal{O}_p$ such that

$$\begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & 1 \end{pmatrix} A_nJ \begin{pmatrix} 1 & -p^n g_{32} & p^{2n}\overline{g}_{31} \\ & 1 & p^n\overline{g}_{32} \\ & & 1 \end{pmatrix} = n(\delta_2,\tau_2)A_nJ.$$

Note $A_n^{-1}n(\delta_2,\tau_2)A_n = n(p^{-n}\delta_2, p^{-2n}\tau_2)$. Therefore, we have

$$I_pA_nJI_p = \bigcup_{\substack{\delta,\tau\in\mathcal{O}_p \\ \tau+\overline{\tau}+\delta\overline{\delta}=0}} n(\delta,\tau)A_nJI_p = \bigsqcup_{\substack{\delta\in\mathcal{O}_p/p^{n+1}\mathcal{O}_p \\ \tau\in\mathcal{O}_p/p^{2n+1}\mathcal{O}_p \\ \tau+\overline{\tau}+\delta\overline{\delta}=0}} \begin{pmatrix} 1 & \delta & \tau \\ & 1 & -\overline{\delta} \\ & & 1 \end{pmatrix} A_nJI_p.$$

Likewise, there are some $\delta_3, \tau_3 \in p\mathcal{O}_p$ such that

$$\begin{pmatrix} 1 & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix} JA_n \begin{pmatrix} 1 & & \\ p^n\overline{g}_{12} & 1 & \\ p^{2n}\overline{g}_{13} & -p^n g_{12} & 1 \end{pmatrix} = Jn(\delta_3,\tau_3)A_n.$$

Again, by $A_n^{-1}n(\delta_3,\tau_3)A_n = n(p^{-n}\delta_3, p^{-2n}\tau_3)$, one has

$$I_pJA_nI_p = \bigcup_{\substack{\delta,\tau\in p\mathcal{O}_p \\ \tau+\overline{\tau}+\delta\overline{\delta}=0}} Jn(\delta,\tau)A_nI_p = \bigsqcup_{\substack{\delta\in p\mathcal{O}_p/p^n\mathcal{O}_p \\ \tau\in p\mathcal{O}_p/p^{2n}\mathcal{O}_p \\ \tau+\overline{\tau}+\delta\overline{\delta}=0}} \begin{pmatrix} 1 & & \\ -\overline{\delta} & 1 & \\ \tau & \delta & 1 \end{pmatrix} JA_nI_p.$$

Lemma A.5 follows. $\square$

Lemma A.6. *Let p be inert in E. We have*

$$G(\mathbb{Z}_p)A_nG(\mathbb{Z}_p)=I_pA_nI_p\bigsqcup I_pA_nJI_p\bigsqcup I_pJA_nI_p\bigsqcup I_pJA_nJI_p. \tag{A.14}$$

Moreover,

$$G(\mathbb{Z}_p)A_nG(\mathbb{Z}_p)=G(\mathbb{Z}_p)A_{-n}G(\mathbb{Z}_p).$$

Proof. Appealing to Lemma A.4 one has the decomposition

$$G(\mathbb{Z}_p)=I_p\bigsqcup_{\substack{\tau\in\mathcal{O}_p/p\mathcal{O}_p\\ \delta\in\mathcal{O}_p/p\mathcal{O}_p\\ \tau+\overline{\tau}+\delta\overline{\delta}=0}}\bigsqcup\begin{pmatrix}\tau&\delta&1\\-\overline{\delta}&1&\\1&&\end{pmatrix}I_p. \tag{A.15}$$

Taking the inverse of the above identity we then obtain

$$G(\mathbb{Z}_p)=I_p\bigsqcup_{\substack{\tau\in\mathcal{O}_p/p\mathcal{O}_p\\ \delta\in\mathcal{O}_p/p\mathcal{O}_p\\ \tau+\overline{\tau}+\delta\overline{\delta}=0}}\bigsqcup I_p\begin{pmatrix}&&1\\&1&\overline{\delta}\\1&-\delta&\overline{\tau}\end{pmatrix}. \tag{A.16}$$

We thus have, by (A.15) and (A.16), that $G(\mathbb{Z}_p)A_nG(\mathbb{Z}_p)=U_1\bigcup U_2$, where

$$U_1:=\bigcup_{\substack{\tau\in\mathcal{O}_p/p\mathcal{O}_p\\ \delta\in\mathcal{O}_p/p\mathcal{O}_p\\ \tau+\overline{\tau}+\delta\overline{\delta}=0}}I_pA_n\begin{pmatrix}\tau&\delta&1\\-\overline{\delta}&1&\\1&&\end{pmatrix}I_p\bigcup_{\substack{\tau\in\mathcal{O}_p/p\mathcal{O}_p\\ \delta\in\mathcal{O}_p/p\mathcal{O}_p\\ \tau+\overline{\tau}+\delta\overline{\delta}=0}}\bigcup I_p\begin{pmatrix}&&1\\&1&\overline{\delta}\\1&-\delta&\overline{\tau}\end{pmatrix}A_nI_p$$

$$U_2:=I_pA_nI_p\bigcup_{\substack{\tau_1,\tau_2\in\mathcal{O}_p/p\mathcal{O}_p\\ \delta_1,\delta_2\in\mathcal{O}_p/p\mathcal{O}_p\\ \tau_1+\overline{\tau}_1+\delta_1\overline{\delta}_1=0\\ \tau_2+\overline{\tau}_2+\delta\overline{\delta}_2=0}}\bigcup I_p\begin{pmatrix}&&1\\&1&\overline{\delta}_1\\1&-\delta_1&\overline{\tau}_1\end{pmatrix}A_n\begin{pmatrix}\tau_2&\delta_2&1\\-\overline{\delta}_2&1&\\1&&\end{pmatrix}I_p.$$

Since

$$G(\mathbb{Z}_p)A_nG(\mathbb{Z}_p)=G(\mathbb{Z}_p)JA_nJG(\mathbb{Z}_p)=G(\mathbb{Z}_p)A_{-n}G(\mathbb{Z}_p),$$

we may suppose $n\geq 1$ without loss of generality. Therefore, with a straightforward computation we have

$$G(\mathbb{Z}_p)A_nG(\mathbb{Z}_p)=I_pA_nI_p\bigcup I_pA_nJI_p\bigcup U_3,$$

where

$$U_3:=\bigcup_{\substack{\tau\in\mathcal{O}_p/p\mathcal{O}_p\\ \delta\in\mathcal{O}_p/p\mathcal{O}_p\\ \tau+\overline{\tau}+\delta\overline{\delta}=0}}I_p\begin{pmatrix}&&1\\&1&\overline{\delta}\\1&-\delta&\tau\end{pmatrix}A_nI_p\bigcup_{\substack{\tau\in\mathcal{O}_p/p\mathcal{O}_p\\ \delta\in\mathcal{O}_p/p\mathcal{O}_p\\ \tau+\overline{\tau}+\delta\overline{\delta}=0}}\bigcup I_p\begin{pmatrix}&&1\\&1&\overline{\delta}\\1&-\delta&\tau\end{pmatrix}A_nJI_p.$$

Let $\delta\in\mathcal{O}_p^\times$. Then for $\tau\in\mathcal{O}_p$ such that $\tau+\overline{\tau}+\delta\overline{\delta}=0$, one has $\tau\in\mathcal{O}_p^\times$. Then

$$\begin{pmatrix}&&1\\&1&\overline{\delta}\\1&-\delta&\tau\end{pmatrix}A_n=\begin{pmatrix}-1&-\delta\overline{\tau}^{-1}&-\tau^{-1}\\&1&-\overline{\delta}\tau^{-1}\\&&-1\end{pmatrix}A_n\begin{pmatrix}-\overline{\tau}^{-1}&&\\-p^n\overline{\delta}\tau^{-1}&-\overline{\tau}\tau^{-1}&\\-p^{2n}&p^n\delta&-\tau\end{pmatrix}.$$

Denote by $\mathrm{LHS}_{\delta,\tau}^{(1)}$ the left hand side of the above identity. Note that

$$\begin{pmatrix}-1&-\delta\overline{\tau}^{-1}&-\tau^{-1}\\&1&-\overline{\delta}\tau^{-1}\\&&-1\end{pmatrix}\in I_p,\quad\begin{pmatrix}-\overline{\tau}^{-1}&&\\-p^n\overline{\delta}\tau^{-1}&-\overline{\tau}\tau^{-1}&\\-p^{2n}&p^n\delta&-\tau\end{pmatrix}\in I_p.$$

Then we have $\mathrm{LHS}_{\delta,\tau}^{(1)} \in I_pA_nI_p$. Suppose, on the other hand, $\delta = 0$. Then $\tau+\overline{\tau} = 0$. When $\tau \in \mathcal{O}_p^\times$, we then have

$$\begin{pmatrix} & & 1\\ & 1 & \\ 1 & & \tau\end{pmatrix} A_n = \begin{pmatrix} 1 & & \tau^{-1}\\ & 1 & \\ & & 1\end{pmatrix} A_n \begin{pmatrix} \overline{\tau}^{-1} & & \\ & 1 & \\ p^{2n} & & \tau\end{pmatrix} \in I_pA_nI_p;$$

when $\tau = 0$, we have $JA_n \in I_pJA_nI_p$. Combining these discussions, we obtain

$$\bigcup_{\substack{\tau\in\mathcal{O}_p/N\mathcal{O}_p\\ \delta\in\mathcal{O}_p/N\mathcal{O}_p\\ \tau+\overline{\tau}+\delta\overline{\delta}=0}} I_p \begin{pmatrix} & & 1\\ & 1 & \overline{\delta}\\ 1 & -\delta & \tau\end{pmatrix} A_nI_p \subseteq I_pA_nI_p \bigcup I_pJA_nI_p. \tag{A.17}$$

Let $\delta \in \mathcal{O}_p^\times$. Then for $\tau \in \mathcal{O}_p$ such that $\tau+\overline{\tau}+\delta\overline{\delta} = 0$, one has $\tau \in \mathcal{O}_p^\times$. Then

$$\begin{pmatrix} & & 1\\ & 1 & \overline{\delta}\\ 1 & -\delta & \tau\end{pmatrix} A_nJ = \begin{pmatrix} \overline{\tau}^{-1} & -\delta\overline{\tau}^{-1} & 1\\ & -1 & -\overline{\delta}\\ & & \tau\end{pmatrix} A_n \begin{pmatrix} & & 1\\ & -\overline{\tau}\tau^{-1} & -p^n\overline{\delta}\tau^{-1}\\ 1 & -p^n\delta\tau^{-1} & p^{2n}\tau^{-1}\end{pmatrix}.$$

Denote by $\mathrm{LHS}_{\delta,\tau}^{(2)}$ the left hand side of the above identity. Note that

$$\begin{pmatrix} \overline{\tau}^{-1} & -\delta\overline{\tau}^{-1} & 1\\ & -1 & -\overline{\delta}\\ & & \tau\end{pmatrix} \in I_p, \quad \begin{pmatrix} & & 1\\ & -\overline{\tau}\tau^{-1} & -p^n\overline{\delta}\tau^{-1}\\ 1 & -p^n\delta\tau^{-1} & p^{2n}\tau^{-1}\end{pmatrix} \in JI_p.$$

Then we have $\mathrm{LHS}_{\delta,\tau}^{(2)} \in I_pA_nJI_p$. Suppose, on the other hand, $\delta = 0$. Then $\tau+\overline{\tau} = 0$. When $\tau \in \mathcal{O}_p^\times$, we then have

$$\begin{pmatrix} & & 1\\ & 1 & \\ 1 & & \tau\end{pmatrix} A_nJ = \begin{pmatrix} \overline{\tau}^{-1} & & 1\\ & 1 & \\ & & \tau\end{pmatrix} A_nJ \begin{pmatrix} 1 & & p^{2n}\tau^{-1}\\ & 1 & \\ & & 1\end{pmatrix} \in I_pA_nJI_p;$$

when $\tau = 0$, we have $JA_nJ \in I_pJA_nJI_p$. Combining these discussions, we obtain

$$\bigcup_{\substack{\tau\in\mathcal{O}_p/N\mathcal{O}_p\\ \delta\in\mathcal{O}_p/N\mathcal{O}_p\\ \tau+\overline{\tau}+\delta\overline{\delta}=0}} I_p \begin{pmatrix} & & 1\\ & 1 & \overline{\delta}\\ 1 & -\delta & \tau\end{pmatrix} A_nJI_p \subseteq I_pA_nJI_p \bigcup I_pJA_nJI_p. \tag{A.18}$$

It the follows from (A.17), (A.18) and definition of U_3 that

$$G(\mathbb{Z}_p)A_nG(\mathbb{Z}_p) \subseteq I_pA_nI_p \bigcup I_pA_nJI_p \bigcup I_pJA_nI_p \bigcup I_pJA_nJI_p \subseteq G(\mathbb{Z}_p)A_nG(\mathbb{Z}_p),$$

namely,

$$G(\mathbb{Z}_p)A_nG(\mathbb{Z}_p) = I_pA_nI_p \bigcup I_pA_nJI_p \bigcup I_pJA_nI_p \bigcup I_pJA_nJI_p.$$

Moreover, by Lemma A.5, the union is in fact disjoint. As a consequence, we obtain (A.14). ☐

For some inert primes not dividing N, we will also need another closely related double coset decomposition.

Lemma A.7. *Let p be an inert prime. We have for $n \geq 1$ we have*

$$G(\mathbb{Z}_p)A_nG(\mathbb{Z}_p) = A_nG(\mathbb{Z}_p) \sqcup \bigsqcup_{\substack{\delta\,(\mathrm{mod}\,p^n)\\ \tau\,(\mathrm{mod}\,p^{2n}),\ \tau+\overline{\tau}=0}} \gamma(\delta,\tau)A_nG(\mathbb{Z}_p)$$

with

$$\gamma(\delta,\tau)=\begin{pmatrix}\tau & \delta & 1\\ -\overline{\delta} & 1 & \\ 1 & & \end{pmatrix}$$

Proof. We have

$$G(\mathbb{Z}_p)A_nG(\mathbb{Z}_p)=G(\mathbb{Z}_p)A_{-n}G(\mathbb{Z}_p)=A_{-n}A_n.G(\mathbb{Z}_p).A_{-n}.G(\mathbb{Z}_p).$$

Let $K_{2,1}(p^n)$ be intersection

$$A_n.G(\mathbb{Z}_p).A_{-n}\cap G(\mathbb{Z}_p).$$

We have

$$K_{2,1}(p^n)=G(\mathbb{Z}_p)\cap\begin{pmatrix}\mathcal{O}_p & \mathcal{O}_p & \mathcal{O}_p\\ p^n\mathcal{O}_p & \mathcal{O}_p & \mathcal{O}_p\\ p^{2n}\mathcal{O}_p^0 & p^n\mathcal{O}_p & \mathcal{O}_p\end{pmatrix}$$

where

$$\mathcal{O}_p^0=\{z\in\mathcal{O}_p,\ \mathrm{tr}(z)=0\}.$$

We have the following decomposition

$$G(\mathbb{Z}_p)=K_{2,1}(p^n)\sqcup\bigsqcup_{\substack{\delta\,(\mathrm{mod}\,p^n)\\ \tau\,(\mathrm{mod}\,p^{2n}),\ \tau+\overline{\tau}=0}}\gamma(\delta,\tau)K_{2,1}(p^n).\tag{A.19}$$

Let $N,\overline{N},A\subset G(\mathbb{Q}_p)$ be respectively the upper triangular nilpotent subgroup, the lower triangular nilpotent subgroup and the diagonal torus. From the Iwahori decomposition we have

$$\begin{aligned}K_{2,1}(p^n)&=(K_{2,1}(p^n)\cap\overline{N}).(K_{2,1}(p^n)\cap A).(K_{2,1}(p^n)\cap N)\\ &=(K_{2,1}(p^n)\cap\overline{N}).(G(\mathbb{Z}_p)\cap A).(G(\mathbb{Z}_p)\cap N)\end{aligned}$$

Let

$$G(\mathbb{Z}_p)A_nG(\mathbb{Z}_p)=K_{2,1}(p^n)A_nG(\mathbb{Z}_p)\cup\bigcup_{\substack{\delta\,(\mathrm{mod}\,p^n)\\ \tau\,(\mathrm{mod}\,p^{2n}),\ \tau+\overline{\tau}=0}}\gamma(\delta,\tau)K_{2,1}(p^n)A_nG(\mathbb{Z}_p).$$

since

$$K_{2,1}(p^n)A_n=A_nA_{-n}K_{2,1}(p^n)A_n\subset A_nG(\mathbb{Z}_p)$$

we have

$$G(\mathbb{Z}_p)A_nG(\mathbb{Z}_p)=A_nG(\mathbb{Z}_p)\cup\bigcup_{\substack{\delta\,(\mathrm{mod}\,p^n)\\ \tau\,(\mathrm{mod}\,p^{2n}),\ \tau+\overline{\tau}=0}}\gamma(\delta,\tau)A_nG(\mathbb{Z}_p)$$

and the disjointness in (A.19) implies the disjointness of this union. $\square$

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