Superfluidity of Total Angular Momentum

Yeyang Zhang[1] and Ryuichi Shindou[1,*]

[1] *International Center for Quantum Materials, School of Physics, Peking University, Beijing 100871, China*

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Spontaneous symmetry breaking of a U(1) symmetry leads to superfluidity of a corresponding conserved charge. We generalize the superfluidity to systems with U(1) symmetries acting on both matter fields and two-dimensional spatial coordinates. Such systems can be effectively realized in easy-plane ferromagnetic systems with spin-orbit coupling where the conserved charge is a total angular momentum. We clarify that under a steady injection of spin angular momentum, the superfluid of the total angular momentum shows spacetime oscillations of the spin density and geometry-dependent spin hydrodynamics. We also demonstrate that the steady spin injection destabilizes the superfluid of total angular momentum, causing a dissipation effect in its spin hydrodynamic properties. Our study broadens the comprehension of superfluidity and sheds new light on the interplay between symmetries and phases of matter.

Introduction.—The discovery of superfluidity [1–3] is a milestone in the history of physics. Exotic macroscopic quantum phenomena in superfluids are explained by the condensation of bosonic atoms [4, 5] or neutral Cooper pairs [6]. Spontaneous symmetry breaking (SSB) of a U(1) global gauge symmetry leads to Goldstone modes with gapless and linear dispersions [7–9], which enables dissipationless mass currents. By alternative U(1) symmetries, the superfluidity can be generalized to spin [10–17] and excitonic [18–23] currents.

General relations between Goldstone modes and SSB of continuous symmetries are derived in the literature [24–27], while they mostly considered continuous *internal symmetries* that transform only field operators locally. *Spacetime symmetries* act on both field operators and spacetime coordinates [28], and the symmetries bring about fundamental physical consequences such as the relativistic spin-orbit coupling (SOC). The continuous spacetime symmetries can be spontaneously broken in spinful superfluids in cold-atom systems [29–34]. Nonetheless, it remains largely unexplored how the SSB of the continuous spacetime symmetries affects the hydrodynamic transport of "charges" associated with the broken spacetime symmetries.

In this Letter, we generalize the concept of superfluidity to the SSB of continuous spacetime symmetries. As a physical system, we consider the superfluidity of total angular momentum, where a joint U(1) rotational symmetry of the in-plane spin vector and two-dimensional (2D) spatial coordinates is spontaneously broken. The superfluid of total angular momentum is nothing but a spin superfluid [10–17] in the presence of the SOC. It can be effectively realized in a ferromagnet or a spin-triplet exciton condensate [35–37] with easy-plane spin anisotropy.

We derive an effective field theory of a Goldstone mode in the total-angular-momentum superfluid and solve its classical equation of motion in the presence of a steady injection of spin. We find that the total-angular-momentum superfluid shows spacetime oscillations of spin density and current under the spin injection, which contrasts with conventional spin superfluid without SOC [10–17]. We also uncover unique geometry dependence and non-reciprocity in its hydrodynamic spin transport. When the system is in a circular geometry with finite curvature, the spin hydrodynamics depends on the direction of the spin flow as well as the curvature of the system.

We also show that unlike in the conventional spin superfluid, the steady spin injection destabilizes the total-angular-momentum superfluid. Landau argued that uniform superfluids moving slower than a critical velocity realize states at local minima of energy, so the superfluidity is protected from any dissipative perturbation [13, 38–40]. Based on the same spirit as Landau's argument, we demonstrate that the total-angular-momentum superfluid is *not* an energy-local-minimum state in the presence of the spin injection, and decay processes to lower energy states bring about a dissipation effect in the spin hydrodynamic properties of the superfluid.

Model.—Consider a 2D complex bosonic field $\phi \equiv \phi_x + i\phi_y$, where the 2D real and time-reversally-odd vector field (ϕ_x, ϕ_y) and 2D spatial coordinate (x, y) transform under a joint U(1) rotation around z-direction,

$$\phi \to \phi e^{i\epsilon}, \quad x + iy \to (x + iy)e^{i\epsilon}. \tag{1}$$

The vector field here stands for a spin vector in physical systems. In the presence of the time-reversal symmetry, $\phi \to -\phi^\dagger$, $t \to -t$, $i \to -i$, the SSB of the joint U(1) symmetry is characterized by a real-time field theory of ϕ,

$$\begin{aligned}\mathcal{L}_\phi =& \frac{\eta_1^2}{2}(\partial_t\phi^\dagger)(\partial_t\phi) - \frac{\eta_1^2 c_\perp^2}{2}(\partial_j\phi^\dagger)(\partial_j\phi) - \frac{\eta_1^2 c_z^2}{2}(\partial_z\phi^\dagger)(\partial_z\phi) \\ &- \frac{\alpha\eta_1^2 c_\perp^2}{4}[(\partial_-\phi)^2 + (\partial_+\phi^\dagger)^2] - \frac{U}{2}(\phi^\dagger\phi - \rho_0)^2, \end{aligned} \tag{2}$$

where $\partial_\pm \equiv \partial_x \pm i\partial_y$, $j = x, y$. A global phase of ϕ is properly chosen so that α is real and positive. $\alpha \in \mathbb{R}$ and $\alpha > 0$. We assume $0 < \alpha < 1$ for the stability of

* rshindou@pku.edu.cn

the theory. Ground states for $\rho_0 > 0$ break the U(1) symmetry by uniform field configurations $\phi = \sqrt{\rho_0}e^{i\theta_0}$. In this Letter, we study a classical equation of motion (EOM) of θ around the ground states.

The joint nature of the rotational symmetry results from spin-orbit locking in solid-state materials with SOC. An example of the joint rotational symmetry breaking is the XY ferromagnet phase in a three-dimensional (3D) trigonal or hexagonal lattice. A localized spin model for spin-orbit coupled magnets generally comprises symmetric and antisymmetric exchange interactions,

$$H_{\rm spin} = \frac{1}{2}\sum_{\boldsymbol{i},\boldsymbol{j}}\sum_{\mu,\nu=x,y,z}\Big(\mathcal{J}_{\boldsymbol{ij},\mu\nu} + \mathcal{D}_{\boldsymbol{ij},\mu\nu}\Big)S_{\boldsymbol{i},\mu}S_{\boldsymbol{j},\nu}, \quad (3)$$

with lattice sites $\boldsymbol{i} \equiv (\boldsymbol{i}_\perp, i_z)$, $\boldsymbol{j} \equiv (\boldsymbol{j}_\perp, j_z)$, spin operators $S_{\boldsymbol{i},\mu}$ ($\mu = x, y, z$), $\mathcal{J}_{\boldsymbol{ij},\mu\nu} = \mathcal{J}_{\boldsymbol{ij},\nu\mu}$, and $\mathcal{D}_{\boldsymbol{ij},\mu\nu} = -\mathcal{D}_{\boldsymbol{ij},\nu\mu}$. $\boldsymbol{i}_\perp$ and i_z are xy and z coordinates of the lattice site $\boldsymbol{i}$ on the lattices, respectively. Suppose that $H_{\rm spin}$ has an easy-plane spin anisotropy in the XY spin plane, and undergoes a quantum phase transition of ferromagnetic ordering of the XY spins, $\vec{S}_{\boldsymbol{i},\perp} \equiv (S_{\boldsymbol{i},x}, S_{\boldsymbol{i},y})$. When $H_{\rm spin}$ belongs to a point group of C_{3i}, D_{3d}, C_{3v}, C_{3h}, C_{6h}, C_{6v}, D_{3h} or D_{6h}, spin hydrodynamics of the XY spin near the ferromagnetic transition point is described by Eq. (2), where $\phi(\boldsymbol{r}_i) \equiv S_{\boldsymbol{i},x} + iS_{\boldsymbol{i},y}$ and a 2 by 2 symmetric matrix comprised of $\mathcal{J}_{\boldsymbol{ij},\mu\nu}$ ($\mu, \nu = x, y$) determines the strength of the α term. Specifically, for each bond $(\boldsymbol{i}, \boldsymbol{j})$, the 2 by 2 matrix $\mathcal{J}_{\boldsymbol{ij},\mu\nu}$ has real eigenvalues $\lambda_{\boldsymbol{ij},m}$ and eigenvectors $\boldsymbol{t}_{\boldsymbol{ij},m}$ ($m = 1, 2$). Defining $\Delta\lambda_{\boldsymbol{ij}} \equiv \lambda_{\boldsymbol{ij},1} - \lambda_{\boldsymbol{ij},2}$, $a_{\boldsymbol{ij},\perp} \equiv |\boldsymbol{i}_\perp - \boldsymbol{j}_\perp|$, Ω as the total volume of the material, and $\varepsilon_{\boldsymbol{ij},\perp}$ as an angle between $\boldsymbol{t}_{\boldsymbol{ij},1}$ in the XY spin plane and $\boldsymbol{i}_\perp - \boldsymbol{j}_\perp$ in the xy coordinate plane, α in Eq. (2) is given by a sum of $\Delta\lambda_{\boldsymbol{ij}}$ with a phase $e^{2i\varepsilon_{ij,\perp}}$ over all the bonds [41],

$$\alpha\eta_1^2 c_\perp^2 = \frac{1}{8\Omega}\sum_{\boldsymbol{i},\boldsymbol{j}} a_{\boldsymbol{ij},\perp}^2\,\Delta\lambda_{\boldsymbol{ij}}\,e^{2i\varepsilon_{\boldsymbol{ij},\perp}}. \quad (4)$$

Note that the joint rotational symmetry in solid-state materials with periodic lattices must be discrete due to the lattices. In fact, $\mathcal{L}_\phi$ for the XY ferromaget on the hexagonal lattices generally acquires additional hexagonal easy spin axes within the XY spin plane in the form of $\tilde{c}_6\phi^6 + \text{h.c.}$. Near the ferromagnetic transition point, however, the additional term becomes effectively negligible compared to the α term in a hydrodynamic regime with an intermediate crossover length scale [41].

Another example of Eq. (2) can be found in the triplet exciton condensate phase in semiconductors with Rashba spin-orbit interaction. Suppose that electron energy bands near the conduction-band bottom and valence-band top in the semiconductors can be approximated by a model with continuous rotational symmetry,

$$\begin{aligned} H_{\rm ex,0} = &\int d^3\boldsymbol{r}\boldsymbol{a}^\dagger[(-\frac{\partial_i^2}{2m_0} + \epsilon_{g0})\boldsymbol{\sigma}_0 - \xi_{R0}(i\partial_y\boldsymbol{\sigma}_x - i\partial_x\boldsymbol{\sigma}_y)]\boldsymbol{a} \\ &+ \int d^3\boldsymbol{r}\boldsymbol{b}^\dagger[(\frac{\partial_i^2}{2m_0'} - \epsilon_{g0})\boldsymbol{\sigma}_0 + \xi_{R0}'(i\partial_y\boldsymbol{\sigma}_x - i\partial_x\boldsymbol{\sigma}_y)]\boldsymbol{b} \\ &+ \int d^3\boldsymbol{r}(\Delta_t\boldsymbol{a}^\dagger\boldsymbol{\sigma}_0\boldsymbol{b} + \Delta_t^*\boldsymbol{b}^\dagger\boldsymbol{\sigma}_0\boldsymbol{a}), \end{aligned} \quad (5)$$

with $i = x, y, z$, $\boldsymbol{a} \equiv (a_\uparrow, a_\downarrow)$ and $\boldsymbol{b} \equiv (b_\uparrow, b_\downarrow)$ for spin-$\frac{1}{2}$ electrons in conduction and valence bands, respectively. In the presence of the Rashba interactions ($\xi_{R,0}$, $\xi_{R,0}'$) and spinless inter-band coupling (Δ_t, Δ_t^*), an attractive interaction between conduction electrons and valence holes induces a condensation of the XY components of exciton pairing, $O_\mu \equiv \langle\boldsymbol{b}^\dagger\sigma_j\boldsymbol{a}\rangle$ ($j = x, y$). The spin hydrodynamics of the XY-components can be well described by Eq. (2) with $\phi \propto \text{Re}O_x + i\text{Re}O_y$, where α is determined by the Rashba interactions [41].

Motivated by these physical realizations, we study classical motion around the ground states. Taking $\phi = \sqrt{\rho_0 + \delta\rho}e^{i\theta}$, integrating a gapped amplitude mode $\delta\rho$, and neglecting fluctuations along z, we obtain a two-dimensional (2D) effective field theory for a Goldstone mode θ in the SSB phase,

$$\begin{aligned} \mathcal{L} = &\frac{1}{2}(\partial_t\theta)^2 - \frac{1}{2}(\partial_x\theta)^2[1 - \alpha\cos(2\theta)] \\ &- \frac{1}{2}(\partial_y\theta)^2[1 + \alpha\cos(2\theta)] + \alpha(\partial_x\theta)(\partial_y\theta)\sin(2\theta). \end{aligned} \quad (6)$$

We set $\eta_1 = c_\perp = \rho_0 = 1$ without loss of generality. For a given ground state $\phi = \sqrt{\rho_0}e^{i\theta_0}$, the dispersion of a phase fluctuation $\delta\theta = \theta - \theta_0$ is gapless with a linear dispersion, where velocities are anisotropic and depend on θ_0. Note that the joint U(1) symmetry generally allows higher-order terms in derivatives or fields in the effective theory, while they do not affect the hydrodynamic transport of low-energy excitations near the ground states.

According to Noether's theorem [28, 42], the U(1) continuous spacetime symmetry endows the classical motion with a conserved current of total angular momentum, which can be divided into a spin part (j_μ^s) and an orbital part (j_μ^l),

$$\begin{aligned} j_\mu^s &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\theta)}\Delta\theta, \\ j_\mu^l &= [\delta_{\mu\nu}\mathcal{L} - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\theta)}(\partial_\nu\theta)]\Delta x_\nu, \end{aligned} \quad (7)$$

with $\mu, \nu \in \{t, x, y\}$, $\Delta x_\nu \in \{\Delta t, \Delta x, \Delta y\}$, $\Delta\theta = 1$, and $(\Delta t, \Delta x, \Delta y) = (0, -y, x)$. The two parts are not conserved by themselves, $\partial_\mu j_\mu^s = -\partial_\mu j_\mu^l = G$, where a spin torque G can be defined by the divergence of the spin current. The spin torque (G), spin currents (j_x^s, j_y^s), and a spin angular momentum along z-direction (j_t^s) are given by the following equations [41],

$$\begin{aligned} G &= -\alpha[(\partial_x\theta)^2 - (\partial_y\theta)^2]\sin(2\theta) + 2\alpha(\partial_x\theta)(\partial_y\theta)\cos(2\theta), \\ j_x^s &= -(\partial_x\theta)[1-\alpha\cos(2\theta)] + \alpha(\partial_y\theta)\sin(2\theta), \\ j_y^s &= -(\partial_y\theta)[1+\alpha\cos(2\theta)] + \alpha(\partial_x\theta)\sin(2\theta), \\ s &\equiv j_t^s = \partial_t\theta. \end{aligned} \tag{8}$$

Though the orbital part j_μ^l is non-local, the spin torque G as well as the spin part j_μ^s are local. The locality of the spin torque results from a continuous spacetime translational symmetry of $\mathcal{L}$.

Spin injection and transport.—To illustrate observables of a total-angular-momentum superfluid, consider a uniform spin current j_0 ($j_0 > 0$) injected into one end ($x = 0$) of the superfluid ($0 < x < L$). The spin current passes through the superfluid and flows into a spin non-superfluid at the other end $x = L$ (see Fig. 1(a)) [11, 13]. The non-superfluid "lead" has diffusive spin transport. Hydrodynamic spin transport in the superfluid is determined by a one-dimensional (1D) EOM of the Goldstone mode $\theta(x,t)$ in Eq. (6) with $\partial_y\theta = 0$,

$$\partial_t^2\theta - (\partial_x^2\theta)[1-\alpha\cos(2\theta)] - \alpha(\partial_x\theta)^2\sin(2\theta) = 0. \tag{9}$$

The EOM Eq. (9) will be solved together with proper boundary conditions. To determine the boundary conditions, note that spin transport in the non-superfluid ($x > L$) is described by diffusion equations [11, 13],

$$\frac{\partial s}{\partial t} + \frac{\partial j_x^s}{\partial x} = -\frac{s}{T_1'},\ j_x^s = -D_s\frac{\partial s}{\partial x}, \tag{10}$$

with relaxation time T_1' and a diffusion coefficient D_s. The diffusive spin current is caused by the gradient of the spin density. Due to the relaxation time, the density and current decay exponentially in space for $L > 0$,

$$\begin{aligned} s(x,t) &\equiv \sum_{c\in\mathbb{R}} s_c(x,t) = \sum_{c\in\mathbb{R}} a_c e^{ict} e^{-\sqrt{D_s^{-1}\omega_c}x}, \\ j_x^s(x,t) &\equiv \sum_{c\in\mathbb{R}} j_{x,c}^s(x,t) = \sum_{c\in\mathbb{R}} \sqrt{D_s\omega_c}a_c e^{ict} e^{-\sqrt{D_s^{-1}\omega_c}x}. \end{aligned} \tag{11}$$

Here $\omega_c = ic + \frac{1}{T_1}$, a_c are complex coefficients, and the square roots of $D_s^{-1}\omega_c$ take positive real parts. The spin current is assumed to be continuous at the junction between the superfluid and non-superfluid, and it is proportional to the gradient of an effective local magnetic field felt by the spin density [11],

$$\begin{aligned} & j_x^s(x=L-,t) = j_x^s(x=L+,t) \\ =& -\beta_t[\frac{1}{\chi'}s(x=L+,t) - \frac{1}{\chi}s(x=L-,t)]. \end{aligned} \tag{12}$$

Here χ, χ' are magnetic susceptibilities at $x = L-$ and $x = L+$ respectively, β_t is a response coefficient of the junction, and they are all positive. Eq. (12) imposes a boundary condition (BC) on the spin density and current at $x = L-$ for each frequency c,

$$s_c(x=L-,t) = k_c j_{x,c}^s(x=L-,t), \tag{13}$$

with $k_c \equiv \frac{\chi}{\chi'}\left[D_s(\frac{1}{T_1'}+ic)\right]^{-\frac{1}{2}} + \frac{\chi}{\beta_t}$, $k_{-c} = k_c^*$, and $\mathrm{Re}(k_c) > 0$. The steady injection of spin imposes another boundary condition at $x = 0+$, $j_x^s(x=0+,t) = j_0$ [11]. In the following, the EOM Eq. (9) is solved for $\theta(x,t)$ such that $s(x,t)$ and $j_x^s(x,t)$ satisfy the BCs.

An analytical solution of $\theta(x,t)$ can be obtained perturbatively in the SOC. The solution at the first order consists of three parts,

$$\theta(x,t) = \theta_0(x,t) + \theta_1(x,t) + \theta_2(x,t) + \mathcal{O}(\alpha^2). \tag{14}$$

θ_0 is the zeroth order solution satisfying the EOM and BCs [11, 13],

$$\theta_0(x,t) = k_0 j_0 t - j_0 x, \tag{15}$$

with $k_0 = \frac{\chi}{\chi'}\sqrt{\frac{T_1'}{D_s}} + \frac{\chi}{\beta_t}$. An oscillation is absent at the zeroth order due to the BCs with $\mathrm{Re}(k_c) > 0$. θ_1 and θ_2 are at the first order in α. θ_1 is a special solution of an inhomogeneous linear differential equation,

$$\partial_t^2\theta_1 - \partial_x^2\theta_1 = -\alpha(\partial_x^2\theta_0)\cos(2\theta_0) + \alpha(\partial_x\theta_0)^2\sin(2\theta_0). \tag{16}$$

θ_2 is a solution of a homogeneous linear differential equation such that θ satisfies the BCs at the first order in α,

$$\partial_t^2\theta_2 - \partial_x^2\theta_2 = 0. \tag{17}$$

The solution at the first order oscillates with two spatial wavenumbers, $2j_0$ and $2k_0j_0$, and one temporal frequency $2k_0j_0$ [41],

$$\begin{aligned} \theta(x,t) =& j_0(k_0t - x) - \frac{\alpha}{4(k_0^2-1)}\sin[2j_0(k_0t-x)] \\ &- \frac{\alpha(2k_0^2-1)}{4(k_0^2-1)}\cos(2k_0j_0t)\sin(2k_0j_0x) \\ &+ \alpha\mathrm{Im}(\eta)\cos(2k_0j_0t)\cos(2k_0j_0x) \\ &+ \alpha\mathrm{Re}(\eta)\sin(2k_0j_0t)\cos(2k_0j_0x) + \mathcal{O}(\alpha^2). \end{aligned} \tag{18}$$

η is a constant depending on k_0, $k_{c=2k_0j_0}$, and $2j_0L$. Note that the perturbative solution is divergent and fails near a "resonant" point $k_0 = 1$ [41, 43].

Higher-order solutions can be systematically obtained by the perturbative iteration, where the spin density and current have the same periodicity in time as the first-order solution, $\pi(k_0j_0)^{-1}$. The time periodicity can be detected by a time-resolved measurement of the spin density in the non-superfluid "lead", which depends on the injected spin current (j_0) and properties of the junction (k_0). The higher-order solution has no spatial periodicity in general, while its Fourier-transform in space has two

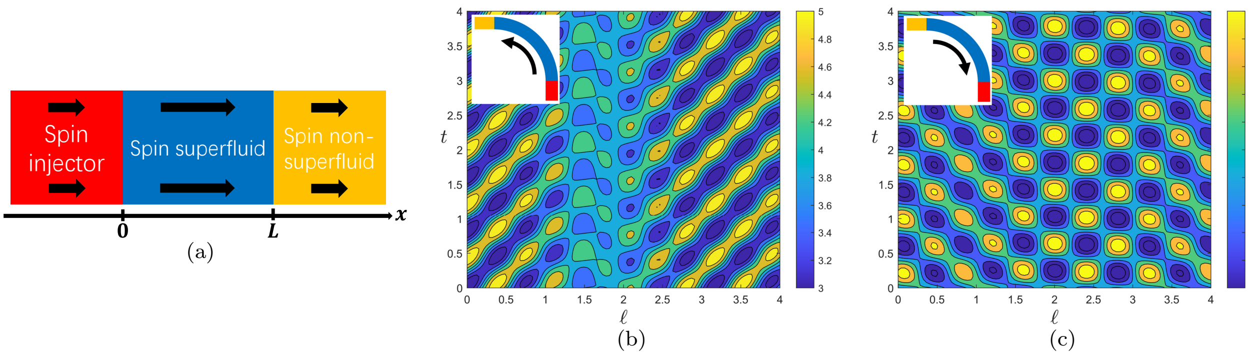

FIG. 1. A spin-injection model. A steady spin current j_0 is injected from a spin injector (red) to the total-angular-momentum superfluid (blue). The spin current passes through the superfluid (blue) and flows into a spin non-superfluid (yellow). The direction of the dc component of the current is indicated by black arrows. (a) A straight geometry. (b) A contour plot of $s(\ell,t)$ in a circular geometry with a positive current $j_0 = 4$. (c) A contour plot of $|s(\ell,t)|$ in a circular geometry with a negative current $j_0 = -4$. $\chi = \chi' = D_s = 1$, $T_1' = 4$, $\beta_t = 2$, $r = 1$, $L = 4$, $\alpha = 0.1$ are used in the contour plots.

major peaks at $2j_0$ and $2k_0 j_0$ as in the first-order solution. The two major wavenumbers can be observed by a local measurement of the spin density in the superfluid.

The spin hydrodynamics under the spin current has a unique geometric effect in a geometry with a finite curvature (Figs. 1(b),1(c)). To see this, suppose that the width of the junction in the circular geometry is small enough that the radius of the junction is taken as a constant r and the field depends only on time and a 1D angular coordinate ϑ. With $(x,y) = r(\cos\vartheta, \sin\vartheta)$, Eq. (6) leads to a 1D Lagrangian [41],

$$\mathcal{L} = \frac{1}{2}(\partial_t\theta)^2 - \frac{1}{2}(\partial_\ell\theta)^2[1 + \alpha\cos(2\theta - \frac{2}{r}\ell)], \tag{19}$$

where $\ell \equiv r\vartheta$. The corresponding EOM under the injected spin current j_0 together with the junction parameter k_0 has a zeroth-order solution, $\theta_0(\ell,t) = k_0 j_0 t - j_0\ell$, and a first-order solution, $\theta_0(\ell,t)+\theta_1(\ell,t)+\theta_2(\ell,t)$. Here θ_1 is a special solution of an inhomogeneous differential equation,

$$\begin{aligned}&\partial_t^2\theta_1 - \partial_\ell^2\theta_1\\ &= -\alpha j_0(j_0 + \frac{2}{r})\sin[2k_0 j_0 t - 2(j_0 + \frac{1}{r})\ell].\end{aligned} \tag{20}$$

θ_1 and θ_2 introduce two wavenumbers, $2j_0 + \frac{2}{r}$ and $2k_0 j_0$, in the observables respectively, where the wavenumber of θ_1 acquires a linear curvature ($\frac{1}{r}$) dependence. Due to the curvature dependence, two injected spin currents with opposite signs ($j_0 \equiv \mathrm{j}_0$ from Fig. 1(b) and $j_0 = -\mathrm{j}_0$ from Fig. 1(c)) lead to different spatial distributions of the observables (non-reciprocal spin hydrodynamics) [41].

Dissipation effect.—In the presence of the Galilean covariance, a uniform superfluid moving slower than the velocity of its Goldstone mode achieves a local energy minimum, so that it is stable against dissipation by local perturbations (e.g. elastic scattering by disorder) [13, 41]. To see the stability of a supercurrent state with the broken U(1) spacetime symmetry, we compare classical energies of the 1D solution $\theta(x,t)$ and its local deformation $\theta(x,t) + \delta\theta(x,t)$. The deformation $\delta\theta(x,t)$ is induced by local perturbations, so the spacetime derivatives of $\delta\theta$ do not contain any uniform component in spacetime. $\theta(x,t)+\delta\theta(x,t)$ as well as $\theta(x,t)$ is a classical solution of Eq. (9), while they do not necessarily share the same boundary conditions. The classical energy in the 1D model can be evaluated from a Hamiltonian,

$$H[\theta] = \int dx\Big\{\frac{1}{2}(\partial_t\theta)^2 + \frac{1}{2}(\partial_x\theta)^2[1 - \alpha\cos(2\theta)]\Big\}. \tag{21}$$

As the classical energies are independent of time, for simplicity, we compare time averages of the energies (with $k_0 \neq 1$) over a large period of time T [41],

$$\begin{aligned}\Delta J \equiv& \lim_{T\to\infty}\frac{1}{T}\Big(\int_0^T dt H[\theta+\delta\theta] - \int_0^T dt H[\theta]\Big)\\ =&\lim_{T\to\infty}\frac{1}{T}\int_0^T dt\int dx\{(\partial_t\theta)(\partial_t\delta\theta) + (\partial_x\theta)(\partial_x\delta\theta)\\ &\times[1-\alpha\cos(2\theta)] + \alpha(\partial_x\theta)^2\sin(2\theta)(\delta\theta)\} + \mathcal{O}((\delta\theta)^2)\\ =&2\lim_{T\to\infty}\int_0^T dt\int dx(\partial_x\theta_2)(\partial_x\delta\theta_0) + \mathcal{O}(\alpha^2\delta\theta, (\delta\theta)^2),\end{aligned} \tag{22}$$

with $\theta + \delta\theta = \theta_0 + \delta\theta_0 + \mathcal{O}(\alpha)$. Terms oscillating in space or time vanish after the spacetime integrals. $\delta\theta_0$, as well as θ_2, is a solution of Eq. (17), and both are given by linear superpositions of $e^{iq(t-x)}$ and $e^{iq(x+t)}$ over q. Thus, for a given $\theta_2 \neq 0$, one can always choose $\delta\theta_0$ such that the spacetime integral of the right-hand side of Eq. (22) remains non-zero and negative, i.e. $\delta J < 0$. This means that the supercurrent state is classically unstable toward other states, and energy always dissipates by the local perturbations. The instability results from the absence of the Galilean covariance. The superflow state is distinct from the ground state by the spacetime oscillation feature, and the energy of the superflow state can be lowered by excitations $\delta\theta_0$ which match the oscillation periodicity.

The dissipation effect on the spin hydrodynamics can be studied through an addition of the simplest time-reversal breaking term, $-T_1^{-1}\partial_t\theta$, into the classical EOM [41]. With finite T_1^{-1}, the zeroth-order solution of spin current acquires linear spatial decay [11, 13], which contrasts with the exponential decay in the non-superfluid [13]. Thereby, the hydrodynamic feature of spin transport survives against dissipation.

Summary.— In this Letter, we generalize the U(1) internal symmetry in conventional superfluid theories into the U(1) spacetime symmetry. Due to the joint symmetry, the supercurrent state shows geometry-dependent spacetime oscillations, and it is unstable against the dissipation effect. Our study paves the way for further exploration of multiple spacetime symmetries and their coupling with internal symmetries.

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SUPPLEMENTAL MATERIAL FOR "SUPERFLUIDITY OF TOTAL ANGULAR MOMENTUM"

In this Supplemental Material, we provide detailed derivations and discussions that support the results in the main text. In Sec. I, we discuss how the theory with the U(1) spacetime symmetry can be a good approximation for a certain quantum spin model on a lattice. We also derive the U(1) theory from a microscopic model of exciton condensations in semiconductors. In Sec. II, we derive the spin and orbital parts of Noether's current and verify their conservation relations. In Sec. III, we solve the spacetime distribution of the Goldstone mode in the spin-injection model with the different geometries. In Sec. IV, we demonstrate the possibility of energy dissipation by using a stability analysis against local perturbations. In Sec. V, we study the effects of dissipation on the classical motion. In Sec. VI, we discuss some special parameter points in the spin-injection model. In Appendix A, we use the same stability analysis as in Sec. III and derive the Landau criterion in a conventional superfluid. We use this classic and simple example, to demonstrate the validity of our stability analysis. In Appendix B, we present how to construct a local deformation of classical solutions of the equation of motion (EOM).

I. MICROSCOPIC MODELS

In this section, we discuss physical realizations of the U(1) spacetime symmetry in two physical models; (i) an XY spin model for spin-orbit coupled magnets near a critical point, and (ii) a triplet excitonic model for semiconductors. The U(1) spacetime symmetry is a joint continuous rotational symmetry that acts on both internal matter field and two-dimensional spatial coordinates. The joint nature of the symmetry results from a locking between the rotation of the matter field and that of the spatial coordinate. Such locking is ubiquitous in solid-state materials with relativistic spin-orbit interaction, where spin or an interband component of spin forms the matter field. In solid-state materials with periodic lattices, the spatial rotation must be discrete due to the lattices, so the joint rotation symmetry is also discrete; the U(1) spacetime symmetry can not be an exact symmetry and it is valid only approximately. Nonetheless, for some solid-state systems, the approximation becomes effective where the difference between discrete and continuous joint rotations becomes irrelevant.

A. Easy-plane ferromagnetic spin model

To see the effectiveness of the U(1) theory in magnetic systems, let us consider a XY ferromagnetic spin system in three-dimensional (3D) lattices that is symmetric under C_n rotation around a z axis ($n = 3, 4, 6, \cdots$) and time reversal. We will first impose a spatial inversion symmetry; at the end of this subsection, we will show that the inversion symmetry is not necessary to derive the U(1) theory for some cases. We suppose that the spin system is a spin-orbit coupled magnet with an easy-plane spin anisotropy (an XY plane being the easy plane), and it undergoes a continuous phase transition from a disordered phase to a ferromagnetic ordered phase of XY components of spins, $S_{\boldsymbol{i},x}$ and $S_{\boldsymbol{i},y}$. In this subsection, we will discuss the effective symmetry of spin hydrodynamics near the phase transition point.

The second-order phase transition in the XY ferromagnet with the C_n rotation can be described by a partition function $\mathcal{Z}_n$ ($n = 3, 4, 6, \cdots$) with a Ginzburg-Landau (GL) action for a two-dimensional complex variable $\phi(\boldsymbol{x})$ ($\hbar = 1$),

$$\mathcal{Z}_n = \int \mathcal{D}\phi\mathcal{D}\phi^\dagger \exp[-\mathcal{S}_{n,\phi}], \quad \mathcal{S}_{n,\phi} = \int d^3\boldsymbol{r} \int_0^{1/k_BT} d\tau \ s_n[\phi(\boldsymbol{r}, \tau)]. \tag{I.1}$$

Here the internal field $\phi(\boldsymbol{r})$ is a spatial average of $S_{\boldsymbol{i},x} + iS_{\boldsymbol{i},y}$ with respect to a lattice site $\boldsymbol{i}$ over some hydrodynamic volume element. i is the complex unit. $\boldsymbol{r}$ is a spatial coordinate of the hydrodynamic volume element. The transition can happen either at the zero-temperature $T = 0$ critical point (quantum critical point) or at finite-temperature $T \neq 0$ critical point (classical critical point). In this section, we consider the spin hydrodynamics near the $T = 0$ quantum critical point. The time-reversal symmetry means the absence of an external magnetic (Zeeman) field in the spin model, and the ferromagnetic order breaks the symmetry spontaneously. The time-reversal symmetry in the model requires the dynamical exponent z at the quantum critical point to be one, $z = 1$ (see below).

For $n = 4$, the spin system is defined on a 3D tetragonal lattice with a C_4 rotational symmetry around the z axis, such as a layered square lattice. For $n = 3$ or $n = 6$, the system is defined on a 3D trigonal or hexagonal lattice with a C_3 or C_6 rotation, e.g. layered honeycomb or triangle lattices. In the following, we first employ a symmetry argument to determine the form of the GL action $s_n[\phi]$ for $n = 4$, $n = 3$, and $n = 6$ and show that for the $n = 3$ or $n = 6$ case, the U(1) joint rotational symmetry is an effective symmetry of the GL action for the xy spins near the critical point, while for the $n = 4$ case, the effective symmetry remains discrete (a $\mathbb{Z}_4$ joint rotational symmetry). To this end, note that under the C_n rotation around the z axis, the complex field $\phi(\boldsymbol{r})$ of the xy spins, and the 3D spatial coordinate, $\boldsymbol{r} \equiv (x, y, z)$, are rotated together due to the spin-orbit locking,

$$\boldsymbol{r} \to \boldsymbol{r}' = (x', y', z), \quad \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\left(\frac{2\pi}{n}\right) & -\sin\left(\frac{2\pi}{n}\right) \\ \sin\left(\frac{2\pi}{n}\right) & \cos\left(\frac{2\pi}{n}\right) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad \phi(\boldsymbol{r}) \equiv S_x(\boldsymbol{r}) + iS_y(\boldsymbol{r}) \to \phi'(\boldsymbol{r}') = \phi(\boldsymbol{r})e^{i\frac{2\pi}{n}}. \tag{I.2}$$

The spatial inversion changes the sign of the coordinate vector, while the time-reversal changes the sign of ϕ_x, ϕ_y, and i. These symmetries constrain forms of the actions for the ϕ field.

1. GL action for XY ferromagnets with C_4 rotation

The symmetries of the joint C_4 rotation around z and time reversal allow the following terms and their complex conjugates in the GL action,

$$|\phi|^2 \equiv \phi^\dagger\phi, \quad i[\phi^\dagger\partial_z\phi - (\partial_z\phi^\dagger)\phi], \quad (\partial_i\phi^\dagger)(\partial_i\phi) \quad (i = x, y, z), \quad \phi\partial_-^2\phi, \quad \phi\partial_+^2\phi, \quad (|\phi|^2)^2, \quad \phi^4, \quad \cdots, \tag{I.3}$$

with $\partial_\pm \equiv \partial_x \pm i\partial_y$. Higher-order terms in ϕ, higher-order spatial gradient terms, and total-derivative terms are omitted as '$\cdots$'. The higher-order ϕ terms are irrelevant near the critical point where the amplitude of ϕ becomes smaller. The higher-order spatial gradient terms are irrelevant in the hydrodynamic regime where the volume element over which the spin operator is averaged becomes larger. Here the time reversal forbids odd-order terms of ϕ. The spatial inversion further forbids $i[\phi^\dagger\partial_z\phi - (\partial_z\phi^\dagger)\phi]$ from the action. Accordingly, the GL functional form allowed by the symmetries is given by

$$\begin{aligned} \mathcal{Z}_{n=4} &= \int \mathcal{D}\phi\mathcal{D}\phi^\dagger \exp[-\mathcal{S}_{n=4,\phi}], \\ \mathcal{S}_{n=4,\phi} &\equiv \int d^3\boldsymbol{r} \int_0^{\hbar/k_\mathrm{B}T} d\tau \Big\{ \frac{\eta_1^2}{2}(\partial_\tau\phi^\dagger)(\partial_\tau\phi) + \frac{\eta_1^2 c_\perp^2}{2}(\partial_j\phi^\dagger)(\partial_j\phi) + \frac{\eta_1^2 c_z^2}{2}(\partial_z\phi^\dagger)(\partial_z\phi) + \frac{\eta_1^2 c_\perp^2}{4}[\alpha(\partial_-\phi)^2 + \alpha^*(\partial_+\phi^\dagger)^2 \\ &\qquad + \beta(\partial_+\phi)^2 + \beta^*(\partial_-\phi^\dagger)^2] + \frac{U}{2}(\phi^\dagger\phi - \rho_0)^2 + \frac{1}{2}[\tilde{c}_4\phi^4 + \tilde{c}_4^*(\phi^\dagger)^4] + \cdots \Big\}, \end{aligned} \tag{I.4}$$

with $j = x, y$, imaginary time τ, real numbers ρ_0, η_1, c, U, and complex numbers α, β, $\tilde{c}_4$. Here the first-order time-derivative of ϕ, such as $\phi^\dagger\partial_\tau\phi$, is forbidden by the time reversal symmetry. The second-order time-derivative term is induced from the spin Berry phase by an integration of the z-component spin. Note that due to the time-reversal symmetry, the dynamical exponent z around the $T = 0$ quantum critical point becomes one, namely $z = 1$.

A partition function $\mathcal{Z}_{n=4}$ describes the phase transition from the disordered phase ($\rho_0 < 0$) to the ordered phase of the xy spins ($\rho_0 > 0$). In the ordered phase, a phase of ϕ is locked into four minima determined by the $\tilde{c}_4$ term. At the quantum critical point, the terms with higher-order gradient and/or ϕ terms become irrelevant in the long-wavelength limit, and the effective symmetry of the GL action at the critical point is determined by a gapless free theory part,

$$\begin{aligned} \overline{\mathcal{S}}_{n=4,\phi} \equiv \int d^3\boldsymbol{r} \int_0^{\hbar/k_\mathrm{B}T} d\tau \Big\{ &\frac{\eta_1^2}{2}(\partial_\tau\phi^\dagger)(\partial_\tau\phi) + \frac{\eta_1^2 c_\perp^2}{2}(\partial_j\phi^\dagger)(\partial_j\phi) + \frac{\eta_1^2 c_z^2}{2}(\partial_z\phi^\dagger)(\partial_z\phi) \\ &+ \frac{\eta_1^2 c_\perp^2}{4}[\alpha(\partial_-\phi)^2 + \alpha^*(\partial_+\phi^\dagger)^2 + \beta(\partial_+\phi)^2 + \beta^*(\partial_-\phi^\dagger)^2] \Big\}. \end{aligned} \tag{I.5}$$

Importantly, though the α term in the second line is symmetric under the U(1) spacetime symmetry, $\partial_\pm \to \partial'_\pm = e^{\pm i\epsilon}\partial_\pm$, $\phi \to \phi' = e^{i\epsilon}\phi$ for $\forall\epsilon$, the β term is symmetric only under the joint $\mathbb{Z}_4$ rotational symmetry,

$$\partial_\pm \to \partial'_\pm = e^{\pm i\frac{\pi}{2}}\partial_\pm, \quad \phi \to \phi' = e^{i\frac{\pi}{2}}\phi, \quad \phi^\dagger \to (\phi')^\dagger = e^{-i\frac{\pi}{2}}\phi^\dagger. \tag{I.6}$$

Due to the β term, the effective symmetry at the critical point remains discrete for the $n = 4$ case.

2. GL action for XY ferromagnets with C_3 or C_6 rotation

For the hexagonal crystal family, on the contrary, the joint C_3 or C_6 rotational symmetry forbids the β term, so the corresponding gapless free theory does have the U(1) spacetime symmetry. To this end, we analyze the terms allowed in the action. The symmetries of the C_6 rotation around z, spatial inversion, and time reversal allow the following terms and their complex conjugates in the action,

$$|\phi|^2, \quad (\partial_i\phi^\dagger)(\partial_i\phi) \quad (i=x,y,z), \quad \phi\partial_-^2\phi, \quad (|\phi|^2)^2, \quad \phi^3\partial_+^2\phi, \quad \phi^6, \quad \cdots. \tag{I.7}$$

When the C_6 rotation is substituted by the C_3 rotation, Eq. (I.7) also exhausts all symmetry-allowed terms apart from higher-order ϕ terms, higher-order spatial-gradient terms, and total derivative terms. In fact, the C_3 rotational symmetry alone allows ϕ^3 and $\phi\partial_+\phi$, while the ϕ^3 term is prohibited by the time-reversal symmetry and $\phi\partial_+\phi$ is a total derivative term. Thus, the partition function $\mathcal{Z}_{n=3,6}$ near the critical point is given by

$$\begin{aligned}\mathcal{Z}_{n=3,6} &= \int \mathcal{D}\phi\mathcal{D}\phi^\dagger \exp[-\mathcal{S}_{n=3,6,\phi}] \\ \mathcal{S}_{n=3,6,\phi} &\equiv \int d^3\boldsymbol{r}\int_0^{\hbar/k_{\mathrm{B}}T} d\tau\Big\{\frac{\eta_1^2}{2}(\partial_\tau\phi^\dagger)(\partial_\tau\phi) + \frac{\eta_1^2c_\perp^2}{2}(\partial_j\phi^\dagger)(\partial_j\phi) + \frac{\eta_1^2c_z^2}{2}(\partial_z\phi^\dagger)(\partial_z\phi) + \frac{\eta_1^2c_\perp^2}{4}[\alpha(\partial_-\phi)^2 + \alpha^*(\partial_+\phi^\dagger)^2] \\ &\quad + \frac{\eta_1^2c_\perp^2}{4}[\tilde{\alpha}_6\phi^3(\partial_+^2\phi) + \tilde{\alpha}_6^*(\phi^\dagger)^3(\partial_-^2\phi^\dagger)] + \frac{U}{2}(\phi^\dagger\phi-\rho_0)^2 + \frac{1}{2}[\tilde{c}_6\phi^6 + \tilde{c}_6^*(\phi^\dagger)^6] + \cdots\Big\},\end{aligned} \tag{I.8}$$

where $\rho_0 > 0$ and $\rho_0 < 0$ correspond to ordered and disordered phases, respectively. Importantly, a gapless free theory part $\overline{\mathcal{S}}_{n=3,6,\phi}$ of the action,

$$\overline{\mathcal{S}}_{n=3,6,\phi} \equiv \int d^3\boldsymbol{r}\int_0^{\hbar/k_{\mathrm{B}}T} d\tau\Big\{\frac{\eta_1^2}{2}(\partial_\tau\phi^\dagger)(\partial_\tau\phi) + \frac{\eta_1^2c_\perp^2}{2}(\partial_j\phi^\dagger)(\partial_j\phi) + \frac{\eta_1^2c_z^2}{2}(\partial_z\phi^\dagger)(\partial_z\phi) + \frac{\eta_1^2c_\perp^2}{4}[\alpha(\partial_-\phi)^2 + \alpha^*(\partial_+\phi^\dagger)^2]\Big\}, \tag{I.9}$$

is symmetric under the U(1) spacetime symmetry,

$$\partial_\pm \to \partial'_\pm = e^{\pm i\epsilon}\partial_\pm, \quad \phi\to\phi' = e^{i\epsilon}\phi, \quad \phi^\dagger \to (\phi')^\dagger = e^{-i\epsilon}\phi^\dagger, \quad \text{for} \quad \forall\,\epsilon. \tag{I.10}$$

This contrasts with the free theory for the $n=4$ case which is symmetric only under the joint discrete rotational symmetry.

In the ordered phase ($\rho_0 > 0$) for the C_6 case, a phase of ϕ is locked into six minima by the $\tilde{c}_6$ term. The $\tilde{c}_6$ and $\tilde{\alpha}_6$ terms reduce the symmetry of the whole action into a joint discrete ($\mathbb{Z}_6$) rotational symmetry,

$$\partial_\pm \to \partial'_\pm = e^{\pm i\frac{\pi}{3}}\partial_\pm, \quad \phi\to\phi' = e^{i\frac{\pi}{3}}\phi, \quad \phi^\dagger\to(\phi')^\dagger = e^{-i\frac{\pi}{3}}\phi^\dagger. \tag{I.11}$$

In the ordered phase for the C_3 case, the phase of ϕ is locked into three minima by the $\tilde{c}_6$ term and other higher-ordered terms omitted as '$\cdots$' in Eq. (I.8). Nonetheless, unlike the β-term in $\mathcal{Z}_{n=4}$, $\tilde{\alpha}_6$ and $\tilde{c}_6$ terms as well as the higher-order terms are *irrelevant* in the long-wavelength limit at the quantum critical point, since their scaling dimensions at the critical point are all negative.

The scaling dimensions of $\tilde{\alpha}_6$ and $\tilde{c}_6$ terms at the critical point, y_{α_6} and y_{c_6}, can be evaluated from a dimensional analysis of the gapless free theory part at $T=0$; $y_{\alpha_6} = 2-D = -2$ and $y_{c_6} = 6-2D = -2$ with $D = 3+1$. Scaling dimensions of the higher-order ϕ terms and higher-order spatial gradient terms are also negative and smaller than y_{α_6} and y_{c_6}. When the hydrodynamic volume element becomes larger, the terms with negative scaling dimensions get smaller at the critical point. Thanks to their irrelevance at the critical point, the GL action respects effectively the U(1) spacetime symmetry in the long-wavelength limit. In other words, the spin hydrodynamics at the critical point becomes U(1) spacetime symmetric more effectively for larger hydrodynamic volume element. The hydrodynamic regime with the effective U(1) symmetry has a lower crossover boundary in its length scale; in order that the hydrodynamics has the effective U(1) spacetime symmetry, the length scale Λ of the volume element should be greater than a certain crossover length $\Lambda_{c,1}$,

$$\Lambda \gg \Lambda_{c,1}. \tag{I.12}$$

The crossover length scale is dependent on $\tilde{c}_6$, $\tilde{\alpha}_6$, and other higher-order terms that manifest the joint discrete rotational symmetries. In the C_6 case, for example, $\Lambda_{c,1}$ is primarily dependent on $\tilde{c}_6$, $\tilde{\alpha}_6$ and their scaling dimensions with the following scalings,

$$\Lambda_{c,1} \propto |\tilde{c}_6|^{\frac{1}{|y_{c_6}|}} \quad \text{or} \quad |\tilde{\alpha}_6|^{\frac{1}{|y_{\alpha_6}|}}. \tag{I.13}$$

When the system is in the ordered phase but close to the critical point ($\rho_0 \gtrsim 0$), the hydrodynamics regime with the effective U(1) symmetry has also an upper bound in its length scale,

$$\Lambda_{c,2} \gg \Lambda \gg \Lambda_{c,1}. \tag{I.14}$$

The upper bound is because the ground state for $\rho_0 > 0$ breaks the joint rotational symmetry spontaneously, and in this sense, $\tilde{c}_6$, $\tilde{\alpha}_6$, and the other higher-order terms manifesting the discrete symmetry are *dangerously* irrelevant. In typical renormalization group (RG) flow trajectory, they get smaller around a saddle-point fixed point for the critical point upon the increase of the length scale, while in the very long wavelength limit, they become larger again around another fixed point that describes an ordered phase with broken joint U(1) symmetry (a Nambu-Goldstone fixed point). The upper bound $\Lambda_{c,2}$ defines a length scale for this upturn behavior of the dangerously irrelevant scaling variables. Generally, $\Lambda_{c,2}$ has a complicated scaling form of ρ, as it also depends on scaling of the coupling constants around the Nambu-Goldstone fixed point. Nonetheless, $\Lambda_{c,2}$ is always greater than the lower bound, $\Lambda_{c,2} \gg \Lambda_{c,1}$, for smaller ρ_0. In the C_6 case, for example, $\Lambda_{c,2} \gg \Lambda_{c,1}$ is sastified when ρ_0, $\tilde{c}_6$, and $\tilde{\alpha}_6$ are in the following regimes,

$$\rho_0^{\frac{|y_{c_6}|}{y_{\rho_0}}} \ll \frac{1}{|\tilde{c}_6|}, \quad \rho_0^{\frac{|y_{\alpha_6}|}{y_{\rho_0}}} \ll \frac{1}{|\tilde{\alpha}_6|}. \tag{I.15}$$

Here y_{ρ_0} is the scaling dimension of ρ_0 around the critical point; $y_{\rho_0} = 2$.

3. *Continuum limit of generic XY ferromagnetic spin models in the 3D hexagonal crystal family*

The above argument is solely based on the symmetry and scaling arguments, suggesting that *any* XY ferromagnetic spin models with the C_3 or C_6 rotational, spatial inversion, and time reversal symmetries has the effective U(1) spacetime symmetry near the quantum critical point, if the models undergo the continuous phase transition of the ferromagnetic ordering. In the following, we will argue this by deriving explicitly a continuum limit of *generic* XY ferromagnetic spin models with the symmetries.

Exchange interactions in spin-orbit coupled magnets generally comprise of symmetric part $\mathcal{J}_{\boldsymbol{ij},\mu\nu} = \mathcal{J}_{\boldsymbol{ij},\nu\mu}$ and antisymmetric part $\mathcal{D}_{\boldsymbol{ij},\mu\nu} = -\mathcal{D}_{\boldsymbol{ij},\nu\mu}$,

$$\mathcal{H} = \frac{1}{2}\sum_{\boldsymbol{i},\boldsymbol{j}}\sum_{\mu,\nu} S_{\boldsymbol{i},\mu}\Big(\mathcal{J}_{\boldsymbol{ij},\mu\nu} + \mathcal{D}_{\boldsymbol{ij},\mu\nu}\Big)S_{\boldsymbol{j},\nu}, \tag{I.16}$$

with spin vector $\boldsymbol{S_i} \equiv (S_{\boldsymbol{i},x}, S_{\boldsymbol{i},y}, S_{\boldsymbol{i},z})$. We first consider that the spins live on the 3D hexagonal lattice with C_6 rotational and spatial inversion symmetries, namely the lattice belongs to either C_{6h} or D_{6h} point group. Further discussion of other possibilities of point groups will be provided below. Here, the exchange interactions are not only limited to those between the nearest neighboring sites on the lattice, but they can also be between further neighboring sites.

Near the transition point of the ferromagnetic ordering of the XY spins, the Z component of the spins fluctuates rapidly in space and time, so that one can legitimately integrate out the Z component, yielding effective spin models for the XY spins,

$$\begin{aligned}\mathcal{H}^{\mathrm{eff}} &= \frac{1}{2}\sum_{\boldsymbol{i},\boldsymbol{j}}\sum_{\mu,\nu=x,y} S_{\boldsymbol{i},\mu}\Big(J_{\boldsymbol{ij},\mu\nu} + D_{\boldsymbol{ij},\mu\nu}\Big)S_{\boldsymbol{j},\nu} \\ &= \frac{1}{2}\sum_{\boldsymbol{i},\boldsymbol{j}}\bigg\{\begin{pmatrix} S_{\boldsymbol{i},x} & S_{\boldsymbol{i},y}\end{pmatrix}\begin{pmatrix} J_{\boldsymbol{ij},xx} & J_{\boldsymbol{ij},xy} \\ J_{\boldsymbol{ij},yx} & J_{\boldsymbol{ij},yy}\end{pmatrix}\begin{pmatrix} S_{\boldsymbol{j},x} \\ S_{\boldsymbol{j},y}\end{pmatrix} + D_{\boldsymbol{ij},xy}\big[S_{\boldsymbol{i},x}S_{\boldsymbol{j},y} - S_{\boldsymbol{i},y}S_{\boldsymbol{j},x}\big]\bigg\},\end{aligned} \tag{I.17}$$

with 2 by 2 symmetric and antisymmetric interactions, $J_{\boldsymbol{ij},\mu\nu} = J_{\boldsymbol{ij},\nu\mu}$ and $D_{\boldsymbol{ij},\mu\nu} = -D_{\boldsymbol{ij},\nu\mu}$ for $\mu,\nu = x,y$. The effective exchange interactions in Eq. (I.17) as well as the exchange interactions in Eq. (I.16) respect the joint C_6 rotational symmetry and inversion symmetry. In the following, we show that due to the C_6 rotational symmetry, the continuum limit of the symmetric exchange interactions in the effective spin models always take the same form as in Eq. (I.8),

$$\frac{1}{2}\sum_{\boldsymbol{i},\boldsymbol{j}}\sum_{\mu,\nu=x,y} S_{\boldsymbol{i},\mu}J_{\boldsymbol{ij},\mu\nu}S_{\boldsymbol{j},\nu} \simeq \int d\boldsymbol{r}^3\bigg\{\frac{r}{2}|\phi|^2 + \frac{\eta_1^2c_z^2}{2}\partial_z\phi^\dagger\partial_z\phi + \frac{\eta_1^2c_\perp^2}{2}\sum_{i=x,y}\partial_i\phi^\dagger\partial_i\phi + \frac{\eta_1^2c_\perp^2}{4}\big[\alpha(\partial_-\phi)^2 + \alpha^*(\partial_+\phi^\dagger)^2\big] + \cdots\bigg\}. \tag{I.18}$$

To see this, note first that any bond of two spin sites, $(\boldsymbol{i},\boldsymbol{j})$, in a sum of Eq. (I.17) has 5 other bonds in the sum that are derived from the first bond $(\boldsymbol{i},\boldsymbol{j})$ by the C_6 rotation, i.e. $(C_6^n(\boldsymbol{i}),C_6^n(\boldsymbol{j}))$ $(n=1,2,\cdots,5)$. Due to the joint C_6 rotation symmetry, $J_{C_6^n(\boldsymbol{i})C_6^n(\boldsymbol{j}),\cdots}$ and $J_{\boldsymbol{ij},\cdots}$ are related by the C_6 spin rotation around z;

$$\begin{pmatrix} J_{C_6(\boldsymbol{i})C_6(\boldsymbol{j}),xx} & J_{C_6(\boldsymbol{i})C_6(\boldsymbol{j}),xy} \\ J_{C_6(\boldsymbol{i})C_6(\boldsymbol{j}),yx} & J_{C_6(\boldsymbol{i})C_6(\boldsymbol{j}),yy} \end{pmatrix} = \begin{pmatrix} \cos\frac{\pi}{3} & \sin\frac{\pi}{3} \\ -\sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{pmatrix} \begin{pmatrix} J_{\boldsymbol{ij},xx} & J_{\boldsymbol{ij},xy} \\ J_{\boldsymbol{ij},yx} & J_{\boldsymbol{ij},yy} \end{pmatrix} \begin{pmatrix} \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{pmatrix}. \tag{I.19}$$

Then, by using a gradient expansion, $S_{\boldsymbol{j},\mu} = S_{\boldsymbol{i},\mu} + (\boldsymbol{j}-\boldsymbol{i})_\lambda \partial_\lambda S_{\boldsymbol{i},\mu} + \frac{1}{2}(\boldsymbol{j}-\boldsymbol{i})_\lambda(\boldsymbol{j}-\boldsymbol{i})_\epsilon \partial_\lambda\partial_\epsilon S_{\boldsymbol{i},\mu} + \cdots$, one can explicitly show that a sum of the symmetric exchange interactions over the six bonds reduce to the same form of the continuum limit as Eq. (I.18) up to the second order in the gradient expansion;

$$\begin{aligned} \frac{1}{2}\sum_{n=0,1,\cdots,5} & \begin{pmatrix} S_{C_6^n(\boldsymbol{i}),x} & S_{C_6^n(\boldsymbol{i}),y} \end{pmatrix} \begin{pmatrix} J_{C_6^n(\boldsymbol{i})C_6^n(\boldsymbol{j}),xx} & J_{C_6^n(\boldsymbol{i})C_6^n(\boldsymbol{j}),xy} \\ J_{C_6^n(\boldsymbol{i})C_6^n(\boldsymbol{j}),yx} & J_{C_6^n(\boldsymbol{i})C_6^n(\boldsymbol{j}),yy} \end{pmatrix} \begin{pmatrix} S_{C_6^n(\boldsymbol{j}),x} \\ S_{C_6^n(\boldsymbol{j}),y} \end{pmatrix} \\ & \simeq \frac{3}{2}(\lambda_{\boldsymbol{ij},1}+\lambda_{\boldsymbol{ij},2})\phi^\dagger\phi + \frac{3a^2_{\boldsymbol{ij},\perp}}{16}\Big\{ 2(\lambda_{\boldsymbol{ij},1}+\lambda_{\boldsymbol{ij},2})\,\phi^\dagger(\partial_x^2+\partial_y^2)\phi \\ & \qquad + \sum_{m=1,2}\lambda_{\boldsymbol{ij},m} e^{2i(\varphi_{\boldsymbol{ij},\perp}-\psi_{\boldsymbol{ij},\perp,m})}\,\phi(\partial_x - i\partial_y)^2\phi + \text{c.c.}\Big\} \\ & \qquad + \frac{3a^2_{\boldsymbol{ij},z}}{4}(\lambda_{\boldsymbol{ij},1}+\lambda_{\boldsymbol{ij},2})\,\phi^\dagger\partial_z^2\phi + \ldots \end{aligned} \tag{I.20}$$

Here ϕ on the right-hand side is from $\phi(\boldsymbol{r_i}) \equiv S_{\boldsymbol{i},x} + iS_{\boldsymbol{i},y}$ in Eq. (I.17), and the higher-order derivative and total derivative terms are omitted. We also regard that $\boldsymbol{i}$ and $C_6^n(\boldsymbol{i})$ $(n=1,\cdots,5)$ are the same for the argument of ϕ, because their differences (if exist) can be controlled by the microscopic length. $a_{\boldsymbol{ij},\perp}$ and $a_{\boldsymbol{ij},z}$ are the spatial length of the bond $(\boldsymbol{i},\boldsymbol{j})$ within the xy plane and along z axis, respectively; $a_{\boldsymbol{ij},\perp} \equiv |\boldsymbol{i}_\perp - \boldsymbol{j}_\perp|$, $a_{\boldsymbol{ij},z} \equiv |i_z - j_z|$, with $\boldsymbol{i} = (\boldsymbol{i}_\perp, i_z)$ and $\boldsymbol{j} = (\boldsymbol{j}_\perp, j_z)$. $\varphi_{\boldsymbol{ij},\perp}$ is the angle between $\boldsymbol{j}_\perp - \boldsymbol{i}_\perp$ and the x axis. $\lambda_{\boldsymbol{ij},m}$ and $\boldsymbol{t}_{\boldsymbol{ij},m}$ are real-valued eigenvalues and eigenvectors of the 2 by 2 symmetric matrix $J_{\boldsymbol{ij}}$ $(m=1,2)$. $\psi_{\boldsymbol{ij},\perp,m}$ is the angle between $\boldsymbol{t}_{\boldsymbol{ij},m}$ and the x-axis in the xy plane. As $\boldsymbol{t}_{\boldsymbol{ij},1}$ and $\boldsymbol{t}_{\boldsymbol{ij},2}$ are orthogonal to each other, $\psi_{\boldsymbol{ij},2} = \psi_{\boldsymbol{ij},1} + \pi/2$ and $\sum_{m=1,2}\lambda_{\boldsymbol{ij},m}e^{2i(\varphi_{\boldsymbol{ij},\perp}-\psi_{\boldsymbol{ij},\perp,m})} = (\lambda_{\boldsymbol{ij},1}-\lambda_{\boldsymbol{ij},2})e^{2i(\varphi_{\boldsymbol{ij},\perp}-\psi_{\boldsymbol{ij},\perp,1})}$. A sum of Eq. (I.20) over different types of bonds leads to Eq. (I.18), where α is simply given by the sum of $a^2_{\boldsymbol{ij},\perp}\sum_{m=1,2}\lambda_{\boldsymbol{ij},m}e^{2i(\varphi_{\boldsymbol{ij},\perp}-\psi_{\boldsymbol{ij},\perp,m})}$. Note that in the absence of the spin-orbit interaction, $J_{\boldsymbol{ij}}$ are always proportional to the unit matrix, where $\lambda_{\boldsymbol{ij},1} = \lambda_{\boldsymbol{ij},2}$, $(\lambda_{\boldsymbol{ij},1}-\lambda_{\boldsymbol{ij},2})e^{2i(\varphi_{\boldsymbol{ij},\perp}-\psi_{\boldsymbol{ij},\perp,1})} = 0$ for any bond $(\boldsymbol{i},\boldsymbol{j})$, and α vanishes.

The continuum limit of the antisymmetric exchange interaction yields the first-order spatial gradient terms,

$$D_{\boldsymbol{ij},xy}(S_{\boldsymbol{i},x}S_{\boldsymbol{j},y} - S_{\boldsymbol{i},y}S_{\boldsymbol{j},x}) = iD_{\boldsymbol{ij},xy}(\boldsymbol{i}-\boldsymbol{j})_\mu\big(\phi^\dagger\partial_\mu\phi - (\partial_\mu\phi^\dagger)\phi\big) + \mathcal{O}(\partial^3). \tag{I.21}$$

In the presence of the spatial inversion, they are cancelled by its inversion symmetric counterpart;

$$D_{I(\boldsymbol{i})I(\boldsymbol{j}),xy}(S_{I(\boldsymbol{i}),x}S_{I(\boldsymbol{j}),y} - S_{I(\boldsymbol{i}),y}S_{I(\boldsymbol{j}),x}) = -iD_{\boldsymbol{ij},xy}(\boldsymbol{i}-\boldsymbol{j})_\mu\big(\phi^\dagger\partial_\mu\phi - (\partial_\mu\phi^\dagger)\phi\big) + \mathcal{O}(\partial^3).$$

with $D_{I(\boldsymbol{i})I(\boldsymbol{j}),xy} = D_{\boldsymbol{ij},xy}$. Thus, the antisymmetric interaction gives only higher-order gradient terms in the continuum limit for the GL action; they are all irrelevant in the sense that their scaling dimensions around the critical point are negative.

Near the quantum critical point of the ferromagnetic order of the xy spin, the systems effectively have the U(1) spacetime symmetry. Note that apart from the C_{6h} and D_{6h} point groups, the hexagonal crystal family (including the trigonal crystal system and the hexagonal crystal system) has 10 other point groups: C_3, C_{3i}, D_3, C_{3v}, D_{3d} from the trigonal crystal system and C_6, C_{3h}, D_6, C_{6v}, D_{3h} from the hexagonal crystal system. The GL action for C_3, D_3, C_6, and D_6 has an additional term, $i\gamma[\phi^\dagger\partial_z\phi - (\partial_z\phi^\dagger)\phi]$, that comes from the antisymmetric exchange interaction in Eq. (I.21). Such a term is prohibited for C_{3i}, D_{3d}, C_{6h}, and D_{6h} because there is the inversion symmetry. The term is also prohibited for C_{3v}, C_{3h}, C_{6v}, and D_{3h}. Although there is no inversion symmetry, for C_{3v}, C_{6v}, and D_{3h}, there is a vertical mirror symmetry that reflects only one component of the in-plane spin vector; for C_{3h} and D_{3h}, there is a horizontal mirror symmetry which makes the term opposite. In conclusion, a continuum limit of *generic* XY spin models in the hexagonal crystal family with a spatial inversion or rotoinversion symmetry as well as the time-reversal symmetry is described by Eq. (I.8). For the C_3, D_3, C_6, and D_6 cases, the first order z-derivative term can be eliminated by re-definitions of ϕ and ρ_0; $\phi_{\text{new}} = e^{-izA_z}\phi_{\text{old}}$ and $-U(\rho_0)_{\text{new}} = -U(\rho_0)_{\text{old}} - (\eta_1^2 c_z^2 A_z^2)/2$ with $\gamma = \eta_1^2 c_z^2 A_z/2$. If A_z is commensurate to the phase locking by the $\tilde{c}_6$ term, $3A_z a_z = \mathbb{Z}\pi$ with an integer $\mathbb{Z}$ $(a_{\boldsymbol{ij},z} = a_z)$, the partition function has no magnetic frustration and it describes the continuous phase transition from a disordered phase $(\rho_0 < 0)$ to the XY ferromagnetic order phase with a spin-helix along z $(\rho_0 > 0)$. As the first order z-derivative term also respects the U(1) spacetime symmetry, the systems near the transition point have also the effective U(1) spacetime symmetry for these cases.

4. Hydrodynamics in an intermediate length scale near the quantum critical point

An analytic continuation of Eq. (I.8) at $T=0$ ($\tau \to it$) leads to the following real-time complex field theory $\tilde{\mathcal{L}}_{\phi}$,

$$\begin{aligned}\tilde{\mathcal{L}}_{\phi} =& \frac{\eta_1^2}{2}(\partial_t\phi^{\dagger})(\partial_t\phi) - \frac{\eta_1^2 c_{\perp}^2}{2}(\partial_i\phi^{\dagger})(\partial_i\phi) - \frac{\eta_1^2 c^2}{4}[\alpha(\partial_-\phi)^2 + \alpha^*(\partial_+\phi^{\dagger})^2] \\ &- \frac{\eta_1^2 c^2}{4}[\tilde{\alpha}_6(\partial_+\phi)^2\phi^2 + \tilde{\alpha}_6^*(\partial_-\phi^{\dagger})^2(\phi^{\dagger})^2] - \frac{U}{2}(\phi^{\dagger}\phi - \rho_0)^2 - \frac{1}{2}[\tilde{c}_6\phi^6 + \tilde{c}_6^*(\phi^{\dagger})^6],\end{aligned} \tag{I.22}$$

where we take classical solutions of ϕ independent of z, so the term of $(\partial_z\phi^{\dagger})(\partial_z\phi)$ is negligible. Here, without loss of generality, let us take α, $\tilde{\alpha}_6$ and $\tilde{c}_6$ to be real, and assume that a coupling between the phase mode θ and the amplitude mode $\rho = \phi^{\dagger}\phi$ can be neglected. Then, we obtain an effective theory of the phase mode θ,

$$\begin{aligned}\tilde{\mathcal{L}} =& \frac{\eta_1^2\rho_0}{2}(\partial_t\theta)^2 - \frac{\eta_1^2 c_{\perp}^2\rho_0}{2}(\partial_x\theta)^2[1 - \alpha\cos(2\theta) - \tilde{\alpha}_6\rho_0^2\cos(4\theta)] \\ &- \frac{\eta_1^2 c_{\perp}^2\rho_0}{2}(\partial_y\theta)^2[1 + \alpha\cos(2\theta) + \tilde{\alpha}_6\rho_0\cos(4\theta)] \\ &+ \eta_1^2 c_{\perp}^2\rho_0(\partial_x\theta)(\partial_y\theta)[\alpha\sin(2\theta) - \tilde{\alpha}_6\rho_0\sin(4\theta)] - \tilde{c}_6\rho_0^3\cos(6\theta).\end{aligned} \tag{I.23}$$

As in Eq. (I.8), terms with higher-order derivatives or higher order in ρ_0 are neglected in Eq. (I.23). Eq. (I.23) is symmetric under the joint $\mathbb{Z}_6$ rotation,

$$\theta \to \theta + \frac{n\pi}{3}, \quad \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \cos\left(\frac{n\pi}{3}\right) & -\sin\left(\frac{n\pi}{3}\right) \\ \sin\left(\frac{n\pi}{3}\right) & \cos\left(\frac{n\pi}{3}\right) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \tag{I.24}$$

while in the absence of $\tilde{\alpha}_6$ and $\tilde{c}_6$, it is symmetric under the joint U(1) rotation;

$$\theta \to \theta + \epsilon, \quad \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \cos\epsilon & -\sin\epsilon \\ \sin\epsilon & \cos\epsilon \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \tag{I.25}$$

for $\forall\epsilon$.

The U(1) theory becomes a good approximation theory for Eq. (I.23), when ρ_0 approaches zero before $\partial_\mu\theta$ ($\mu = t, x, y$) approach zero. This is the case for the 3D spin model Eq. (I.14) in the intermediate length scale near the quantum critical point. Thereby, the internal field ϕ is introduced as a spatial average of $S_{\boldsymbol{i},x} + iS_{\boldsymbol{i},y}$ over some hydrodynamic volume element. When the length scale of the volume element increases within the intermediate length scale, $\Lambda_{c,1} < \Lambda < \Lambda_{c,2}$, a scaling of ρ_0 and $\partial_\mu\theta$ is controlled by the quantum critical point; ρ_0 gets small faster than $\partial_\mu\theta$, and the approximation becomes better. On the other hand, when the length scale of the element becomes larger than the upper bound $\Lambda_{c,2}$, another scaling law from the Nambu-Goldstone fixed point kicks in, and the $\tilde{c}_6$ and $\tilde{\alpha}_6$ terms become relevant again [44, 45]. Besides, for a two-dimensional (2D) quantum spin model, although the $\tilde{c}_6$ term is dangerously marginal instead of dangerously irrelevant from simple dimensional counting, as long as a bare value of $\tilde{c}_6$ is small enough, there is still an intermediate length scale where the U(1) theory is applicable. To summarize, the U(1) theory is effective near the quantum critical point only when θ fluctuates over a length in the intermediate length scale. When θ fluctuates more slowly than $\Lambda_{c,2}$, $\partial_\mu\theta$ becomes smaller than a small but finite ρ_0, and the $\tilde{c}_6$ and $\tilde{\alpha}_6$ terms dominate over the others, giving a large contribution to the equation of motion.

B. Spin-triplet exciton model

As another example of solid-state materials where the U(1) theory of spin dynamics is applicable, we consider semiconductors with electron excitations near a conduction-band bottom and hole excitations near a valence-band top around a high-symmetric k point, e.g. the Γ point. Near the band top and bottom, suppose the kinetic-energy bands can be approximately described by a rotational-symmetric continuous theory. The theory with relativistic spin-orbit interaction is expected to have joint continuous rotational symmetry.

To be specific, we consider a condensate of spin-triplet excitons in a two-dimensional semiconductor model with Rashba-type spin-orbit interactions. We consider a 2D model for simplicity. It may be regarded as an effective model for 3D. The semiconductor model is given by

$$
\begin{aligned}
H_{\rm ex} &= \int d^2\boldsymbol{r}\boldsymbol{a}^{\dagger}[(-\frac{\partial_i^2}{2m_0}+\epsilon_{g0})\boldsymbol{\sigma}_0+\xi_{R0}(-i\partial_y\boldsymbol{\sigma}_x+i\partial_x\boldsymbol{\sigma}_y)]\boldsymbol{a} \\
&+\int d^2\boldsymbol{r}\boldsymbol{b}^{\dagger}[(\frac{\partial_i^2}{2m_0'}-\epsilon_{g0})\boldsymbol{\sigma}_0+\xi_{R0}'(i\partial_y\boldsymbol{\sigma}_x-i\partial_x\boldsymbol{\sigma}_y)]\boldsymbol{b}+\int d^2\boldsymbol{r}(\Delta_t\boldsymbol{a}^{\dagger}\boldsymbol{\sigma}_0\boldsymbol{b}+\Delta_t^*\boldsymbol{b}^{\dagger}\boldsymbol{\sigma}_0\boldsymbol{a}) \\
&+\frac{g_{s0}}{2}\sum_{\sigma,\sigma'=\uparrow,\downarrow}\int d^2\boldsymbol{r}(a^{\dagger}_{\sigma}a^{\dagger}_{\sigma'}a_{\sigma'}a_{\sigma}+b^{\dagger}_{\sigma}b^{\dagger}_{\sigma'}b_{\sigma'}b_{\sigma}+2\xi_1 a^{\dagger}_{\sigma}b^{\dagger}_{\sigma'}b_{\sigma'}a_{\sigma}),
\end{aligned}
\tag{I.26}
$$

with $i=x,y$. Here $\boldsymbol{a}$ and $\boldsymbol{b}$ are spin-$\frac{1}{2}$ electron annihilation operators near the Γ point in the conduction band and valence band, respectively. We suppose inter-band interaction is smaller than intra-band interaction, namely $0<\xi_1<1$. Due to the attraction between electrons and holes ($\xi_1 g_{s0}$), the quasiparticles form bound states inside a band gap (ϵ_{g0}). The bound states have spin-singlet component and spin-triplet components. In the presence of Rashba interaction (ξ_{R0},ξ'_{R0}) and inter-band "spinless" hopping (Δ_t), the in-plane component of the spin-triplet states undergoes Bose-Einstein condensation at $q=0$. In the following, we will show that this condensation is described by Eq. (2) in the main text (without the c_z term). we will derive Eq. (2) in the main text from Eq. (I.26). For simplicity, we take the electron band and the hole band in a symmetric form, $m_0=m_0'$, $\xi_{R0}=\xi'_{R0}$, while the derivation can be generalized into the case with $m_0 xi_{R0}=m_0'\xi'_{R0}$. The derivation can be also applicable to a three-dimensional model with a finite effective mass along z. Due to the Rashba interaction (ξ_{R0}) and interband tunneling (Δ_t), the system has only a U(1) rotational symmetry and a time-reversal symmetry,

$$
\boldsymbol{a}\rightarrow e^{-i\epsilon\boldsymbol{\sigma}_z/2}\boldsymbol{a},\quad \boldsymbol{b}\rightarrow e^{-i\epsilon\boldsymbol{\sigma}_z/2}\boldsymbol{b},\quad \left(\begin{array}{c} x \\ y \end{array}\right)\rightarrow\left(\begin{array}{cc} \cos\epsilon & -\sin\epsilon \\ \sin\epsilon & \cos\epsilon \end{array}\right)\left(\begin{array}{c} x \\ y \end{array}\right),
\tag{I.27}
$$

$$
\boldsymbol{a}\rightarrow i\boldsymbol{\sigma}_y\boldsymbol{a},\quad \boldsymbol{b}\rightarrow i\boldsymbol{\sigma}_y\boldsymbol{b},\quad t\rightarrow -t,\quad i\rightarrow -i.
\tag{I.28}
$$

The quadratic part of the Hamiltonian Eq. (I.26) is diagonalized,

$$
\begin{aligned}
H_{\rm ex} = &\sum_{\boldsymbol{k}}\{a^{\dagger}_{\boldsymbol{k},\sigma_{\boldsymbol{k}}}[\frac{(|\boldsymbol{k}|-k_R)^2}{2m_0}+E_{g0}]a_{\boldsymbol{k},\sigma_{\boldsymbol{k}}}-b^{\dagger}_{\boldsymbol{k},\sigma_{\boldsymbol{k}}}[\frac{(|\boldsymbol{k}|-k_R)^2}{2m_0}+E_{g0}]b_{\boldsymbol{k},\sigma_{\boldsymbol{k}}}\} \\
&+\sum_{\boldsymbol{k}}[\Delta_t a^{\dagger}_{\boldsymbol{k},\sigma_{\boldsymbol{k}}}b_{\boldsymbol{k},\sigma_{\boldsymbol{k}}}+\Delta_t^* b^{\dagger}_{\boldsymbol{k},\sigma_{\boldsymbol{k}}}a_{\boldsymbol{k},\sigma_{\boldsymbol{k}}}] \\
&+\frac{g_s}{2}\sum_{\sigma,\sigma'=\uparrow,\downarrow}\int d^2\boldsymbol{r}(a^{\dagger}_{\sigma}a^{\dagger}_{\sigma'}a_{\sigma'}a_{\sigma}+b^{\dagger}_{\sigma}b^{\dagger}_{\sigma'}b_{\sigma'}b_{\sigma}+2a^{\dagger}_{\sigma}b^{\dagger}_{\sigma'}b_{\sigma'}a_{\sigma}),
\end{aligned}
\tag{I.29}
$$

where

$$
k_R=m_0\xi_{R0},\quad E_{g0}=\epsilon_{g0}-\frac{k_R^2}{2m_0},
\tag{I.30}
$$

$\boldsymbol{a}_{\boldsymbol{k}}$ and $\boldsymbol{b}_{\boldsymbol{k}}$ are Fourier transforms of $\boldsymbol{a}(\boldsymbol{r})$ and $\boldsymbol{b}(\boldsymbol{r})$, $\sigma_{\boldsymbol{k}}$ denotes up spin along the direction of $\widehat{\boldsymbol{z}}\times\widehat{\boldsymbol{k}}$, $\widehat{\boldsymbol{k}}\equiv\frac{\boldsymbol{k}}{|\boldsymbol{k}|}$. Here we discard the down-spin bands of the conduction and valence bands, because they are higher in energy and they do not constitute low-energy exciton levels. Since excitons are formed by electrons and holes around the Γ point, we neglect $|\boldsymbol{k}|$-dependence of the hybridization coefficients of the conduction and valence bands,

$$
a_{\sigma}=\alpha_{\sigma}\cos\Theta-\beta_{\sigma}e^{i\Phi}\sin\Theta,\quad b_{\sigma}=\alpha_{\sigma}e^{-i\Phi}\sin\Theta+\beta_{\sigma}\cos\Theta,
\tag{I.31}
$$

where

$$
E_{g0}=\sqrt{E_{g0}^2+|\Delta_t|^2}\cos2\Theta,\quad \Delta_t=\sqrt{E_{g0}^2+|\Delta_t|^2}e^{i\Phi}\sin2\Theta,
\tag{I.32}
$$

and we used

$$
\left(\begin{array}{cc} \cos2\Theta & e^{i\Phi}\sin2\Theta \\ e^{-i\Phi}\sin2\Theta & -\cos2\Theta \end{array}\right)=\left(\begin{array}{cc} \cos\Theta & -e^{i\Phi}\sin\Theta \\ e^{i\Phi}\sin\Theta & \cos\Theta \end{array}\right)\left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right)\left(\begin{array}{cc} \cos\Theta & e^{i\Phi}\sin\Theta \\ -e^{-i\Phi}\sin\Theta & \cos\Theta \end{array}\right).
\tag{I.33}
$$

Taking Eq. (I.33) into the interaction term and taking $\Phi = 0$ for simplicity, we get

$$\begin{aligned}
&g_{s0}(a^\dagger_\sigma a^\dagger_{\sigma'} a_{\sigma'} a_\sigma + b^\dagger_\sigma b^\dagger_{\sigma'} b_{\sigma'} b_\sigma + \xi_1 a^\dagger_\sigma b^\dagger_{\sigma'} b_{\sigma'} a_\sigma + \xi_1 b^\dagger_\sigma a^\dagger_{\sigma'} a_{\sigma'} b_\sigma)\\
=&g_{s0}[(\alpha^\dagger_\sigma \beta^\dagger_{\sigma'} \beta_{\sigma'} \alpha_\sigma + \beta^\dagger_\sigma \alpha^\dagger_{\sigma'} \alpha_{\sigma'} \beta_\sigma)(\cos^2\Theta \sin^2\Theta + \cos^2\Theta \sin^2\Theta + \xi_1 \cos^4\Theta + \xi_1 \sin^4\Theta)\\
&+ (\alpha^\dagger_\sigma \beta^\dagger_{\sigma'} \alpha_{\sigma'} \beta_\sigma + \beta^\dagger_\sigma \alpha^\dagger_{\sigma'} \beta_{\sigma'} \alpha_\sigma)(\cos^2\Theta \sin^2\Theta + \cos^2\Theta \sin^2\Theta - \xi_1 \cos^2\Theta \sin^2\Theta - \xi_1 \cos^2\Theta \sin^2\Theta)\\
&+ (\alpha^\dagger_\sigma \alpha^\dagger_{\sigma'} \beta_{\sigma'} \beta_\sigma + \beta^\dagger_\sigma \beta^\dagger_{\sigma'} \alpha_{\sigma'} \alpha_\sigma)(\cos^2\Theta \sin^2\Theta + \cos^2\Theta \sin^2\Theta - \xi_1 \cos^2\Theta \sin^2\Theta - \xi_1 \cos^2\Theta \sin^2\Theta) + ...]\\
\equiv&g_s(\alpha^\dagger_\sigma \beta^\dagger_{\sigma'} \beta_{\sigma'} \alpha_\sigma + \beta^\dagger_\sigma \alpha^\dagger_{\sigma'} \alpha_{\sigma'} \beta_\sigma) + wg_s(\alpha^\dagger_\sigma \beta^\dagger_{\sigma'} \alpha_{\sigma'} \beta_\sigma + \beta^\dagger_\sigma \alpha^\dagger_{\sigma'} \beta_{\sigma'} \alpha_\sigma)\\
&+ wg_s(\alpha^\dagger_\sigma \alpha^\dagger_{\sigma'} \beta_{\sigma'} \beta_\sigma + \beta^\dagger_\sigma \beta^\dagger_{\sigma'} \alpha_{\sigma'} \alpha_\sigma) + ... \qquad \text{(I.34)}
\end{aligned}$$

Here we only keep terms in exciton-pairing channels in the basis of α and β, $\alpha^\dagger_\sigma \beta^\dagger_{\sigma'} \beta_{\sigma'} \alpha_\sigma$, $\alpha^\dagger_\sigma \alpha^\dagger_{\sigma'} \beta_{\sigma'} \beta_\sigma$, and $\beta^\dagger_\sigma \beta^\dagger_{\sigma'} \alpha_{\sigma'} \alpha_\sigma$. Neglected terms contain also hybridization between excitons and intraband collective modes, $\alpha^\dagger_\sigma \alpha^\dagger_{\sigma'} \alpha_{\sigma'} \beta_\sigma$, which in the absence of the time-reversal symmetry leads to an additional cubic term $(\partial_+\phi)(\phi^\dagger)^2$ in Eq. (2) in the main text. The hybridization and other neglected terms can be safely omitted as the intraband collective modes are gapped excitations in the semiconductor. The Hamiltonian can be rewritten by the new basis,

$$\begin{aligned}
H_{\mathrm{ex}} =& \sum_{\boldsymbol{k}} \{\alpha^\dagger_{\boldsymbol{k},\sigma_{\boldsymbol{k}}} [\frac{(|\boldsymbol{k}| - k_R)^2}{2m} + E_g] \alpha_{\boldsymbol{k},\sigma_{\boldsymbol{k}}} - \beta^\dagger_{\boldsymbol{k},\sigma_{\boldsymbol{k}}} [\frac{(|\boldsymbol{k}| - k_R)^2}{2m} + E_g] \beta_{\boldsymbol{k},\sigma_{\boldsymbol{k}}}\}\\
&+ g_s \sum_{\sigma,\sigma'=\uparrow,\downarrow} \int d^2\boldsymbol{r} (\alpha^\dagger_\sigma \beta^\dagger_{\sigma'} \beta_{\sigma'} \alpha_\sigma + w \alpha^\dagger_\sigma \beta^\dagger_{\sigma'} \alpha_{\sigma'} \beta_\sigma + \frac{w}{2} \alpha^\dagger_\sigma \alpha^\dagger_{\sigma'} \beta_{\sigma'} \beta_\sigma + \frac{w}{2} \beta^\dagger_\sigma \beta^\dagger_{\sigma'} \alpha_{\sigma'} \alpha_\sigma), \qquad \text{(I.35)}
\end{aligned}$$

where

$$\begin{aligned}
\sqrt{[\frac{(|\boldsymbol{k}| - k_R)^2}{2m_0} + E_{g0}]^2 + |\Delta_t|^2} \approx& \sqrt{E^2_{g0} + |\Delta_t|^2} + \frac{E_{g0}}{\sqrt{E^2_{g0} + |\Delta_t|^2}} \frac{(|\boldsymbol{k}| - k_R)^2}{2m_0}\\
=& E_g + \frac{(|\boldsymbol{k}| - k_R)^2}{2m}, \qquad \text{(I.36)}
\end{aligned}$$

$$g_s = \frac{g_{s0}}{2}[\sin^2(2\Theta) + \xi_1 + \xi_1 \cos^2(2\Theta)], \quad w = \frac{(1 - \xi_1)\sin^2(2\Theta)}{\sin^2(2\Theta) + \xi_1[1 + \cos^2(2\Theta)]}, \qquad \text{(I.37)}$$

with $0 < w < 1$. Exciton operators are defined by $O_\mu = \boldsymbol{b}^\dagger \boldsymbol{\sigma}_\mu \boldsymbol{a}$ where $\mu = 0, x, y, z$. In terms of a completeness relation

$$\frac{1}{2} \sum_\mu (\sigma_\mu)_{\alpha\beta} (\sigma_\mu)_{\gamma\delta} = \delta_{\alpha\delta} \delta_{\beta\gamma}, \qquad \text{(I.38)}$$

the interaction terms are decomposed as follows,

$$\sum_{\sigma,\sigma'} \alpha^\dagger_\sigma \beta^\dagger_{\sigma'} \beta_{\sigma'} \alpha_\sigma = -\sum_{\sigma,\sigma'} \alpha^\dagger_\sigma \beta_{\sigma'} \beta^\dagger_{\sigma'} \alpha_\sigma = -\frac{1}{2} \sum_\mu O^\dagger_\mu O_\mu, \qquad \text{(I.39)}$$

$$\sum_{\sigma,\sigma'} \alpha^\dagger_\sigma \beta^\dagger_{\sigma'} \alpha_{\sigma'} \beta_\sigma = \sum_{\sigma,\sigma'} \alpha^\dagger_\sigma \beta_\sigma \beta^\dagger_{\sigma'} \alpha_{\sigma'} = O^\dagger_0 O_0, \qquad \text{(I.40)}$$

$$\sum_{\sigma,\sigma'} \alpha^\dagger_\sigma \alpha^\dagger_{\sigma'} \beta_{\sigma'} \beta_\sigma \sim -\sum_{\sigma,\sigma'} \alpha^\dagger_\sigma \beta_{\sigma'} \alpha^\dagger_{\sigma'} \beta_\sigma + \sum_{\sigma,\sigma'} \alpha^\dagger_\sigma \beta_\sigma \alpha^\dagger_{\sigma'} \beta_{\sigma'} = -\frac{1}{2} \sum_\mu O^\dagger_\mu O^\dagger_\mu + O^\dagger_0 O^\dagger_0. \qquad \text{(I.41)}$$

In Eq. (I.41), we decompose the interaction in two different channels, so we do not divide the result by two. Adding

Eqs. (I.39-I.41) together, we get

$$\begin{aligned}
&g_s\sum_{\sigma,\sigma'}(\alpha^\dagger_\sigma\beta^\dagger_{\sigma'}\beta_{\sigma'}\alpha_\sigma+w\alpha^\dagger_\sigma\beta^\dagger_{\sigma'}\alpha_{\sigma'}\beta_\sigma+\frac{w}{2}\alpha^\dagger_\sigma\alpha^\dagger_{\sigma'}\beta_{\sigma'}\beta_\sigma+\frac{w}{2}\beta^\dagger_\sigma\beta^\dagger_{\sigma'}\alpha_{\sigma'}\alpha_\sigma)\\
=&g_s[-\frac{1}{2}\sum_\mu O^\dagger_\mu O_\mu+wO^\dagger_0O_0-\frac{w}{4}\sum_\mu(O^\dagger_\mu O^\dagger_\mu+O_\mu O_\mu)+\frac{w}{2}(O^\dagger_0O^\dagger_0+O_0O_0)]\\
=&-\frac{g_s}{4}[\sum_r(2O^\dagger_rO_r+wO^\dagger_rO^\dagger_r+wO_rO_r)+(2-4w)O^\dagger_0O_0-wO^\dagger_0O^\dagger_0-wO_0O_0]\\
=&-\frac{g_s}{2}[\sum_r(1+w\mathrm{cos}2q_r)P_r^2+(1-2w-w\mathrm{cos}2q_0)P_0^2],
\end{aligned}\tag{I.42}$$

where $r=x,y,z$, $O_\mu\equiv P_\mu e^{iq_\mu}$. Note that due to w, the U(1) symmetry of the four-component exciton field reduces to a $\mathbb{Z}_2$ symmetry. Since $0<w<1$, $q_0=\pm\frac{\pi}{2}$ and $q_r=0,\pi$ are preferred by the interaction. Fluctuations of q_μ around the minima are gapped excitations, so they can be safely neglected. This leads to

$$\begin{aligned}
&g_s\sum_{\sigma,\sigma'}(\alpha^\dagger_\sigma\beta^\dagger_{\sigma'}\beta_{\sigma'}\alpha_\sigma+w\alpha^\dagger_\sigma\beta^\dagger_{\sigma'}\alpha_{\sigma'}\beta_\sigma-\frac{w}{2}\alpha^\dagger_\sigma\alpha^\dagger_{\sigma'}\beta_{\sigma'}\beta_\sigma-\frac{w}{2}\beta^\dagger_\sigma\beta^\dagger_{\sigma'}\alpha_{\sigma'}\alpha_\sigma)\\
=&-\frac{g_s}{2}[\sum_r(1+w)P_r^2+(1-w)P_0^2]=-\frac{g_s}{2}[\sum_r(1+w)\Big(\frac{O_r-O^\dagger_r}{2i}\Big)^2+(1-w)\Big(\frac{O_0+O^\dagger_0}{2}\Big)^2].
\end{aligned}\tag{I.43}$$

By the Hubbard-Stratonovich transformation, we can introduce real exciton fields ϕ_μ,

$$\begin{aligned}
&\mathrm{exp}\{\int d\tau d^2\boldsymbol{r}\frac{g_s}{2}[\sum_r(1+w)P_r^2+(1-w)P_0^2]\}=\int\mathcal{D}\phi_\mu\mathrm{exp}\{-\int d\tau d^2\boldsymbol{r}[-\sum_r i\phi_r(O^\dagger_r-O_r)\\
&-\phi_0(O^\dagger_0+O_0)+\sum_r\frac{2}{g_s(1+w)}\phi_r^2+\frac{2}{g_s(1-w)}\phi_0^2]\},
\end{aligned}\tag{I.44}$$

where ϕ_r and ϕ_0 have the physical meanings of $\frac{g_s(1+w)}{2}\langle P_r\rangle$ and $\frac{g_s(1-w)}{2}\langle P_0\rangle$, respectively. Since $0<w<1$, the interaction term (Eq. (I.43)) favors the triplet excitons (ϕ_r) over the singlet excitons (ϕ_0). The quadratic part of Eq. (I.35) also lifts the four-fold degeneracy of ϕ_μ, while mass terms for ϕ_x and ϕ_y remain degenerate.

Due to the adjustment of the conduction band and valence band, $m_0\xi_{R0}=m'_0\xi'_{R0}$, one can expect that momentum-energy dispersions of the exciton bands have minima at $q=0$, so condensation of the exciton fields happen at the zero momentum. In the following, we keep track of all the four components, ϕ_μ ($\mu=0,x,y,z$), in the derivation of Eq. (2) in the main text, to see whether and when ϕ_x and ϕ_y achieve the lowest energy (smallest mass at $q=0$) among others. Fermion fields can be integrated out,

$$\int\mathcal{D}[\boldsymbol{a}^\dagger,\boldsymbol{b}^\dagger,\boldsymbol{a},\boldsymbol{b}]\mathrm{exp}\big[-\begin{pmatrix}\boldsymbol{a}^\dagger & \boldsymbol{b}^\dagger\end{pmatrix}\boldsymbol{G}^{-1}\begin{pmatrix}\boldsymbol{a}\\ \boldsymbol{b}\end{pmatrix}\big]=\det(\boldsymbol{G}^{-1})=e^{\mathrm{Trln}(\boldsymbol{G}^{-1})},\tag{I.45}$$

where

$$\boldsymbol{G}^{-1}=\boldsymbol{G}_0^{-1}+\boldsymbol{G}_\phi,\tag{I.46}$$

$$\begin{aligned}
\boldsymbol{G}_0=&\begin{pmatrix} g^a_{0,k}\boldsymbol{P}_{\sigma_{\boldsymbol{k}}} & 0\\ 0 & g^b_{0,k}\boldsymbol{P}_{\sigma_{\boldsymbol{k}}}\end{pmatrix}\\
\equiv&\begin{pmatrix}[-i\omega_n+\frac{(|\boldsymbol{k}|-k_R)^2}{2m}+E_g]^{-1}\frac{\boldsymbol{\sigma}_0+\boldsymbol{\sigma}_{\hat{\boldsymbol{z}}\times\hat{\boldsymbol{k}}}}{2} & 0\\ 0 & [-i\omega_n-\frac{(|\boldsymbol{k}|-k_R)^2}{2m}-E_g]^{-1}\frac{\boldsymbol{\sigma}_0+\boldsymbol{\sigma}_{\hat{\boldsymbol{z}}\times\hat{\boldsymbol{k}}}}{2}\end{pmatrix},
\end{aligned}\tag{I.47}$$

$$\boldsymbol{G}_\phi=\begin{pmatrix}0 & -i\sum_r\phi_r\boldsymbol{\sigma}_r-\phi_0\boldsymbol{\sigma}_0\\ i\sum_r\phi_r\boldsymbol{\sigma}_r-\phi_0\boldsymbol{\sigma}_0 & 0\end{pmatrix}.\tag{I.48}$$

$\boldsymbol{G}_0^{-1}$ and $\boldsymbol{G}_\phi$ are block-diagonal in the momentum-frequency space and the coordinate space, respectively. $\boldsymbol{G}_0$ is diagonal in spin along $\hat{\boldsymbol{z}}\times\hat{\boldsymbol{k}}$ and its diagonal element is zero for the down spin,

$$\boldsymbol{P}_{\sigma_{\boldsymbol{k}}}=\frac{1}{2}(\boldsymbol{\sigma}_0+\boldsymbol{\sigma}_{\hat{\boldsymbol{z}}\times\hat{\boldsymbol{k}}})=\frac{1}{2}(\boldsymbol{\sigma}_0-\boldsymbol{\sigma}_x\mathrm{sin}\theta_{\hat{\boldsymbol{k}}}+\boldsymbol{\sigma}_y\mathrm{cos}\theta_{\hat{\boldsymbol{k}}}).\tag{I.49}$$

The integration leads to an effective theory of the exciton fields,

$$\mathcal{S}_\phi[\phi_\mu] = -\mathrm{Trln}(\mathbf{1}+\boldsymbol{G}_0\boldsymbol{G}_\phi) + \frac{2}{g_s}\int d\tau d^2\boldsymbol{r}(\sum_r \frac{1}{1+w}\phi_r^2 + \frac{1}{1-w}\phi_0^2), \tag{I.50}$$

$$-\mathrm{Trln}(\mathbf{1}+\boldsymbol{G}_0\boldsymbol{G}_\phi) = \frac{1}{2}\mathrm{Tr}(\boldsymbol{G}_0\boldsymbol{G}_\phi\boldsymbol{G}_0\boldsymbol{G}_\phi) + \frac{1}{4}\mathrm{Tr}(\boldsymbol{G}_0\boldsymbol{G}_\phi\boldsymbol{G}_0\boldsymbol{G}_\phi\boldsymbol{G}_0\boldsymbol{G}_\phi\boldsymbol{G}_0\boldsymbol{G}_\phi) + ... \tag{I.51}$$

Note that "Tr" stands for traces over both spacetime and spin indices, while "tr" is trace over only spin indices (see below). To determine the form of the effective theory, we use the following relations,

$$\boldsymbol{P}_{\sigma_{\boldsymbol{k}}}\boldsymbol{\sigma}_z\boldsymbol{P}_{\sigma_{\boldsymbol{k}}} = 0, \tag{I.52}$$

$$\boldsymbol{P}_{\sigma_{\boldsymbol{k}}}\boldsymbol{\sigma}_x\boldsymbol{P}_{\sigma_{\boldsymbol{k}}} = \boldsymbol{P}_{\sigma_{\boldsymbol{k}}}(\boldsymbol{\sigma}_{\widehat{\boldsymbol{k}}}\mathrm{cos}\theta_{\widehat{\boldsymbol{k}}} - \boldsymbol{\sigma}_{\widehat{\boldsymbol{z}}\times\widehat{\boldsymbol{k}}}\mathrm{sin}\theta_{\widehat{\boldsymbol{k}}})\boldsymbol{P}_{\sigma_{\boldsymbol{k}}} = -\mathrm{sin}\theta_{\widehat{\boldsymbol{k}}}\boldsymbol{P}_{\sigma_{\boldsymbol{k}}}\boldsymbol{\sigma}_{\widehat{\boldsymbol{z}}\times\widehat{\boldsymbol{k}}} = -\mathrm{sin}\theta_{\widehat{\boldsymbol{k}}}\boldsymbol{\sigma}_{\widehat{\boldsymbol{z}}\times\widehat{\boldsymbol{k}}}\boldsymbol{P}_{\sigma_{\boldsymbol{k}}}, \tag{I.53}$$

$$\mathrm{tr}(\boldsymbol{P}_{\sigma_{\boldsymbol{k}}}\boldsymbol{\sigma}_0\boldsymbol{P}_{\sigma_{\boldsymbol{k}}}\boldsymbol{\sigma}_0) = \mathrm{tr}(\boldsymbol{P}^2_{\sigma_{\boldsymbol{k}}}) = \mathrm{tr}(\boldsymbol{P}_{\sigma_{\boldsymbol{k}}}) = 1, \tag{I.54}$$

$$\begin{aligned}
&\mathrm{tr}(\boldsymbol{P}_{\sigma_{\boldsymbol{k}}}\boldsymbol{\sigma}_x\boldsymbol{P}_{\sigma_{\boldsymbol{k+q}}}\boldsymbol{\sigma}_x) = \frac{1}{4}\mathrm{tr}(\boldsymbol{\sigma}_x\boldsymbol{\sigma}_x) + \frac{1}{4}\mathrm{tr}(\boldsymbol{\sigma}_{\widehat{\boldsymbol{z}}\times\widehat{\boldsymbol{k}}}\boldsymbol{\sigma}_x\boldsymbol{\sigma}_{\widehat{\boldsymbol{z}}\times\widehat{\boldsymbol{k+q}}}\boldsymbol{\sigma}_x)\\
=&\frac{1}{2} + \frac{1}{4}\mathrm{tr}[(-\mathrm{sin}\theta_{\widehat{\boldsymbol{k}}}\boldsymbol{\sigma}_x + \mathrm{cos}\theta_{\widehat{\boldsymbol{k}}}\boldsymbol{\sigma}_y)\boldsymbol{\sigma}_x(-\mathrm{sin}\theta_{\widehat{\boldsymbol{k+q}}}\boldsymbol{\sigma}_x + \mathrm{cos}\theta_{\widehat{\boldsymbol{k+q}}}\boldsymbol{\sigma}_y)\boldsymbol{\sigma}_x]\\
=&\frac{1}{2}(1 + \mathrm{sin}\theta_{\widehat{\boldsymbol{k}}}\mathrm{sin}\theta_{\widehat{\boldsymbol{k+q}}} - \mathrm{cos}\theta_{\widehat{\boldsymbol{k}}}\mathrm{cos}\theta_{\widehat{\boldsymbol{k+q}}}) = \frac{1}{2}[1 - \mathrm{cos}(\theta_{\widehat{\boldsymbol{k}}} + \theta_{\widehat{\boldsymbol{k+q}}})],
\end{aligned} \tag{I.55}$$

$$\mathrm{tr}(\boldsymbol{P}_{\sigma_{\boldsymbol{k}}}\boldsymbol{\sigma}_y\boldsymbol{P}_{\sigma_{\boldsymbol{k+q}}}\boldsymbol{\sigma}_y) = \frac{1}{2}[1 + \mathrm{cos}(\theta_{\widehat{\boldsymbol{k}}} + \theta_{\widehat{\boldsymbol{k+q}}})], \tag{I.56}$$

$$\begin{aligned}
&\mathrm{tr}(\boldsymbol{P}_{\sigma_{\boldsymbol{k}}}\boldsymbol{\sigma}_x\boldsymbol{P}_{\sigma_{\boldsymbol{k+q}}}\boldsymbol{\sigma}_y) = \frac{1}{4}\mathrm{tr}(\boldsymbol{\sigma}_{\widehat{\boldsymbol{z}}\times\widehat{\boldsymbol{k}}}\boldsymbol{\sigma}_x\boldsymbol{\sigma}_{\widehat{\boldsymbol{z}}\times\widehat{\boldsymbol{k+q}}}\boldsymbol{\sigma}_y)\\
=&\frac{1}{4}\mathrm{tr}[(-\mathrm{sin}\theta_{\widehat{\boldsymbol{k}}}\boldsymbol{\sigma}_x + \mathrm{cos}\theta_{\widehat{\boldsymbol{k}}}\boldsymbol{\sigma}_y)\boldsymbol{\sigma}_x(-\mathrm{sin}\theta_{\widehat{\boldsymbol{k+q}}}\boldsymbol{\sigma}_x + \mathrm{cos}\theta_{\widehat{\boldsymbol{k+q}}}\boldsymbol{\sigma}_y)\boldsymbol{\sigma}_y]\\
=&-\frac{1}{2}(\mathrm{sin}\theta_{\widehat{\boldsymbol{k}}}\mathrm{cos}\theta_{\widehat{\boldsymbol{k+q}}} + \mathrm{cos}\theta_{\widehat{\boldsymbol{k}}}\mathrm{sin}\theta_{\widehat{\boldsymbol{k+q}}}) = -\frac{1}{2}\mathrm{sin}(\theta_{\widehat{\boldsymbol{k}}} + \theta_{\widehat{\boldsymbol{k+q}}}),
\end{aligned} \tag{I.57}$$

$$\mathrm{tr}(\boldsymbol{P}_{\sigma_{\boldsymbol{k}}}\boldsymbol{\sigma}_y\boldsymbol{P}_{\sigma_{\boldsymbol{k+q}}}\boldsymbol{\sigma}_x) = -\frac{1}{2}\mathrm{sin}(\theta_{\widehat{\boldsymbol{k}}} + \theta_{\widehat{\boldsymbol{k+q}}}), \tag{I.58}$$

$$\begin{aligned}
&\mathrm{tr}(\boldsymbol{P}_{\sigma_{\boldsymbol{k}}}\boldsymbol{\sigma}_x\boldsymbol{P}_{\sigma_{\boldsymbol{k}}}\boldsymbol{\sigma}_x\boldsymbol{P}_{\sigma_{\boldsymbol{k}}}\boldsymbol{\sigma}_x\boldsymbol{P}_{\sigma_{\boldsymbol{k}}}\boldsymbol{\sigma}_x) = \mathrm{tr}[(\boldsymbol{P}_{\sigma_{\boldsymbol{k}}}\boldsymbol{\sigma}_x\boldsymbol{P}_{\sigma_{\boldsymbol{k}}})^4]\\
=&\mathrm{tr}[(\boldsymbol{P}_{\sigma_{\boldsymbol{k}}}\boldsymbol{\sigma}_{\widehat{\boldsymbol{z}}\times\widehat{\boldsymbol{k}}}\boldsymbol{\sigma}_{\widehat{\boldsymbol{z}}\times\widehat{\boldsymbol{k}}}\boldsymbol{P}_{\sigma_{\boldsymbol{k}}})^2]\mathrm{sin}^4\theta_{\widehat{\boldsymbol{k}}} = \mathrm{tr}(\boldsymbol{P}_{\sigma_{\boldsymbol{k}}})\mathrm{sin}^4\theta_{\widehat{\boldsymbol{k}}} = \mathrm{sin}^4\theta_{\widehat{\boldsymbol{k}}}.
\end{aligned} \tag{I.59}$$

As the exciton field $\phi_\mu(\boldsymbol{q})$ (Fourier transform of $\phi_\mu(\boldsymbol{r})$) at the zero momentum ($\boldsymbol{q}=0$) is expected to have the smallest energy, we expand the effective theory in terms of small $\boldsymbol{q}$. The zeroth order in $\boldsymbol{q}$ gives the mass (M_μ) and the quartic term (U) from the first and the second terms in Eq. (I.51), respectively. Let us first calculate the masses for the four exciton components ($\mu = 0, x, y, z$),

$$\begin{aligned}
&\frac{1}{2}\mathrm{Tr}(\boldsymbol{G}_0\boldsymbol{G}_\phi\boldsymbol{G}_0\boldsymbol{G}_\phi) \supset \sum_\mu \int d\tau d^2\boldsymbol{r}\phi_\mu^2[\frac{1}{\beta L^2}\sum_k g^a_{0,k}g^b_{0,k}\mathrm{tr}(\boldsymbol{P}_{\sigma_{\boldsymbol{k}}}\boldsymbol{\sigma}_\mu\boldsymbol{P}_{\sigma_{\boldsymbol{k}}}\boldsymbol{\sigma}_\mu)]\\
=&\int d\tau d^2\boldsymbol{r}\phi_0^2(\frac{1}{\beta L^2}\sum_k g^a_{0,k}g^b_{0,k}) + \frac{1}{2}\int d\tau d^2\boldsymbol{r}(\phi_x^2+\phi_y^2)(\frac{1}{\beta L^2}\sum_k g^a_{0,k}g^b_{0,k}),
\end{aligned} \tag{I.60}$$

where we used Eqs. (I.52,I.54-I.56) and $\sum_{\boldsymbol{k}}\cos(2\theta_{\widehat{\boldsymbol{k}}}) = \sum_{\boldsymbol{k}}\sin(2\theta_{\widehat{\boldsymbol{k}}}) = 0$. Taking Eqs. (I.50,I.60) together, we get

$$\frac{1}{2}\sum_\mu M_\mu\phi_\mu^2 = \frac{2}{g_s}(\sum_r \frac{1}{1+w}\phi_r^2 + \frac{1}{1-w}\phi_0^2) - \frac{D_0}{2}(\phi_x^2+\phi_y^2+2\phi_0^2), \tag{I.61}$$

where

$$D_0 = -\frac{1}{\beta L^2}\sum_k g^a_{0,k} g^b_{0,k} = -\frac{1}{\beta L^2}\sum_{n,\boldsymbol{k}} \frac{1}{i\omega_n - \xi_{\boldsymbol{k}}}\frac{1}{i\omega_n + \xi_{\boldsymbol{k}}}$$
$$= -\frac{1}{L^2}\sum_{\boldsymbol{k}} \frac{n_F(\xi_{\boldsymbol{k}}) - n_F(-\xi_{\boldsymbol{k}})}{2\xi_{\boldsymbol{k}}} = \int \frac{d^2\boldsymbol{k}}{(2\pi)^2}\frac{\tanh(\frac{1}{2}\beta\xi_{\boldsymbol{k}})}{2\xi_{\boldsymbol{k}}} > 0, \tag{I.62}$$

$$\xi_{\boldsymbol{k}} \equiv \frac{(|\boldsymbol{k}| - k_R)^2}{2m} + E_g, \quad n_F(\xi_{\boldsymbol{k}}) \equiv \frac{1}{e^{\beta\xi_{\boldsymbol{k}}} + 1}, \tag{I.63}$$

β is the inverse of temperature. So we have

$$M_z = \frac{4}{g_s(1+w)} > 0, \quad M_0 = \frac{4}{g_s(1-w)} - 2D_0, \quad M_i = \frac{4}{g_s(1+w)} - D_0, \tag{I.64}$$

where $i = x, y$. $M_i < 0 < M_0$ is realized by

$$\frac{1}{1+w} < \frac{1}{4} g_s D_0 < \frac{1}{2(1-w)}. \tag{I.65}$$

Given that $w > \frac{1}{3}$, the condition can be realized by proper g_s, β, and E_g. Given that the condition is satisfied, we henceforth consider the effective theory only of $\phi = \phi_x + i\phi_y$ and neglect the other exciton components as they are gapped modes.

Eq. (2) in the main text takes the following form in the imaginary-time representation,

$$\mathcal{L}_{\phi,E} = \frac{\eta_1^2}{2}(\partial_\tau \phi^\dagger)(\partial_\tau \phi) + \frac{\eta_1^2 c_\perp^2}{2}(\partial_i \phi^\dagger)(\partial_i \phi) + \frac{\eta_1^2 c_\perp^2}{4}[\alpha(\partial_- \phi)^2 + \alpha^*(\partial_+ \phi^\dagger)^2] + \frac{U}{2}(\phi^\dagger\phi - \rho_0)^2, \tag{I.66}$$

where the mass term was already obtained,

$$2U\rho_0 = M_i = D_0 - \frac{4}{g_s(1+w)}. \tag{I.67}$$

To determine U, we calculate the quartic term in ϕ_x from the second term of Eq. (I.51), using Eq. (I.59) and $\sum_{\boldsymbol{k}} \sin^4 \theta_{\widehat{\boldsymbol{k}}} = \frac{3}{8}\sum_{\boldsymbol{k}}$,

$$\frac{1}{4}\mathrm{Tr}[(\boldsymbol{G}_0 \boldsymbol{G}_\phi)^4] \supset \frac{1}{\beta L^2}\sum_{q_1,q_2,q_3} \phi_x(q_1)\phi_x(q_2)\phi_x(q_3)\phi_x(-q_1-q_2-q_3)\{\frac{1}{2\beta L^2}\sum_k (g^a_{0,k} g^b_{0,k})^2 \mathrm{tr}[(\boldsymbol{P}_{\sigma_{\boldsymbol{k}}}\boldsymbol{\sigma}_x)^4]\}$$
$$= \int d\tau d^2\boldsymbol{r}\phi_x^4\{\frac{1}{2\beta L^2}\sum_k (g^a_{0,k} g^b_{0,k})^2 \mathrm{tr}[(\boldsymbol{P}_{\sigma_{\boldsymbol{k}}}\boldsymbol{\sigma}_x)^4]\} = \frac{3}{8}\int d\tau d^2\boldsymbol{r}\phi_x^4[\frac{1}{2\beta L^2}\sum_k (g^a_{0,k} g^b_{0,k})^2]. \tag{I.68}$$

Thus, U is given by

$$U = \frac{3}{8\beta L^2}\sum_k (g^a_{0,k} g^b_{0,k})^2 = \frac{3}{8}\int \frac{d^2\boldsymbol{k}}{(2\pi)^2}\{\frac{d}{dz}[\frac{1}{e^{\beta z}+1}\frac{1}{(z-\xi_{\boldsymbol{k}})^2}]|_{z=-\xi_{\boldsymbol{k}}} + \frac{d}{dz}[\frac{1}{e^{\beta z}+1}\frac{1}{(z+\xi_{\boldsymbol{k}})^2}]|_{z=\xi_{\boldsymbol{k}}}]$$
$$= \frac{3}{8}\int \frac{d^2\boldsymbol{k}}{(2\pi)^2}\frac{1}{(2\xi_{\boldsymbol{k}})^2}[\frac{\tanh(\frac{1}{2}\beta\xi_{\boldsymbol{k}})}{\xi_{\boldsymbol{k}}} - \frac{\beta}{1+\cosh(\beta\xi_{\boldsymbol{k}})}] > 0. \tag{I.69}$$

To determine the coefficients of the lowest-order gradient terms, we define $\boldsymbol{\sigma}_\pm = \frac{1}{2}(\boldsymbol{\sigma}_x \pm i\boldsymbol{\sigma}_y)$, $\phi = \phi_x + i\phi_y$, $\phi^\dagger = \phi_x - i\phi_y$, and set $\phi_0 = \phi_z = 0$ in $\boldsymbol{G}_\phi$ in Eq. (I.48). Note also that

$$\boldsymbol{G}_\phi = \begin{pmatrix} 0 & -i(\phi\boldsymbol{\sigma}_- + \phi^\dagger\boldsymbol{\sigma}_+) \\ i(\phi\boldsymbol{\sigma}_- + \phi^\dagger\boldsymbol{\sigma}_+) & 0 \end{pmatrix}, \tag{I.70}$$

$$\mathrm{tr}(\boldsymbol{P}_{\sigma_{\boldsymbol{k}}}\boldsymbol{\sigma}_+ \boldsymbol{P}_{\sigma_{\boldsymbol{k}+\boldsymbol{q}}}\boldsymbol{\sigma}_-) = \frac{1}{8}[1 - \cos(\theta_{\widehat{\boldsymbol{k}}} + \theta_{\widehat{\boldsymbol{k}+\boldsymbol{q}}}) + 1 + \cos(\theta_{\widehat{\boldsymbol{k}}} + \theta_{\widehat{\boldsymbol{k}+\boldsymbol{q}}})] = \frac{1}{4}, \tag{I.71}$$

$$\begin{aligned}
\mathrm{tr}(\boldsymbol{P}_{\sigma_{\boldsymbol{k}}}\boldsymbol{\sigma}_+\boldsymbol{P}_{\sigma_{\boldsymbol{k+q}}}\boldsymbol{\sigma}_+) =& \frac{1}{8}[1-\cos(\theta_{\widehat{\boldsymbol{k}}}+\theta_{\widehat{\boldsymbol{k+q}}})-1-\cos(\theta_{\widehat{\boldsymbol{k}}}+\theta_{\widehat{\boldsymbol{k+q}}})\\
&-i\sin(\theta_{\widehat{\boldsymbol{k}}}+\theta_{\widehat{\boldsymbol{k+q}}})-i\sin(\theta_{\widehat{\boldsymbol{k}}}+\theta_{\widehat{\boldsymbol{k+q}}})] = -\frac{1}{4}e^{i(\theta_{\widehat{\boldsymbol{k}}}+\theta_{\widehat{\boldsymbol{k+q}}})},
\end{aligned} \tag{I.72}$$

$$\mathrm{tr}(\boldsymbol{P}_{\sigma_{\boldsymbol{k}}}\boldsymbol{\sigma}_-\boldsymbol{P}_{\sigma_{\boldsymbol{k+q}}}\boldsymbol{\sigma}_-) = -\frac{1}{4}e^{-i(\theta_{\widehat{\boldsymbol{k}}}+\theta_{\widehat{\boldsymbol{k+q}}})}. \tag{I.73}$$

In terms of Eq. (I.70), the first term of Eq. (I.51) is given by

$$\begin{aligned}
\frac{1}{2}\mathrm{Tr}(\boldsymbol{G}_0\boldsymbol{G}_\phi\boldsymbol{G}_0\boldsymbol{G}_\phi) =& \frac{1}{2\beta L^2}\sum_{k,q}(g^a_{0,k}g^b_{0,k+q}+g^b_{0,k}g^a_{0,k+q})[\phi^\dagger_q\phi_q\mathrm{tr}(\boldsymbol{P}_{\sigma_{\boldsymbol{k}}}\boldsymbol{\sigma}_+\boldsymbol{P}_{\sigma_{\boldsymbol{k+q}}}\boldsymbol{\sigma}_-)\\
&+\phi_{-q}\phi_q\mathrm{tr}(\boldsymbol{P}_{\sigma_{\boldsymbol{k}}}\boldsymbol{\sigma}_-\boldsymbol{P}_{\sigma_{\boldsymbol{k+q}}}\boldsymbol{\sigma}_-)]+\mathrm{H.c.}
\end{aligned} \tag{I.74}$$

By an expansion of small $q\equiv(\boldsymbol{q},i\omega_m)$, we get

$$\eta_1^2 = -\frac{1}{4\beta L^2}\partial^2_{i\omega_m}|_{q=0}\sum_k(g^a_{0,k}g^b_{0,k+q}+g^a_{0,k+q}g^b_{0,k}) = -\frac{1}{2\beta L^2}\partial^2_{i\omega_m}|_{q=0}\sum_k g^a_{0,k-\frac{q}{2}}g^b_{0,k+\frac{q}{2}}, \tag{I.75}$$

$$\eta_1^2c^2 = \frac{1}{4\beta L^2}\partial^2_{q_x}|_{q=0}\sum_k(g^a_{0,k}g^b_{0,k+q}+g^a_{0,k+q}g^b_{0,k}) = \frac{1}{2\beta L^2}\partial^2_{q_x}|_{q=0}\sum_k g^a_{0,k-\frac{q}{2}}g^b_{0,k+\frac{q}{2}}, \tag{I.76}$$

$$\begin{aligned}
\eta_1^2c^2\alpha =& \frac{1}{4\beta L^2}\partial^2_{q_x}|_{q=0}\sum_k(g^a_{0,k}g^b_{0,k+q}+g^a_{0,k+q}g^b_{0,k})e^{-i(\theta_{\widehat{\boldsymbol{k}}}+\theta_{\widehat{\boldsymbol{k+q}}})}\\
=& \frac{1}{2\beta L^2}\partial^2_{q_x}|_{q=0}\sum_k g^a_{0,k-\frac{q}{2}}g^b_{0,k+\frac{q}{2}}e^{-i(\theta_{\widehat{\boldsymbol{k}-\frac{\boldsymbol{q}}{2}}}+\theta_{\widehat{\boldsymbol{k}+\frac{\boldsymbol{q}}{2}}})},
\end{aligned} \tag{I.77}$$

where we used

$$\sum_k g^a_{0,k}g^b_{0,k+q} = \sum_k g^a_{0,k-q}g^b_{0,k} = \sum_k g^a_{0,k-\frac{q}{2}}g^b_{0,k+\frac{q}{2}}. \tag{I.78}$$

Equivalently, we can also use

$$\sum_k\frac{1}{2}(g^a_{0,k})''g^b_{0,k} = \sum_k\frac{1}{2}g^a_{0,k}(g^b_{0,k})'' = \sum_k[\frac{1}{8}(g^a_{0,k})''g^b_{0,k}+\frac{1}{8}g^a_{0,k}(g^b_{0,k})''-\frac{1}{4}(g^a_{0,k})'(g^b_{0,k})'], \tag{I.79}$$

or

$$\sum_k(g^a_{0,k})''g^b_{0,k} = \sum_k g^a_{0,k}(g^b_{0,k})'' = -\sum_k(g^a_{0,k})'(g^b_{0,k})'. \tag{I.80}$$

Here primes and double primes denote first-order and second-order derivatives with respect to one of the spacetime components of k. Note that Eqs. (I.78,I.80) are valid given that associated integrals vanish or are sufficiently small in the ultraviolet regime (large k region). Using them, we can determine η_1 and $\eta_1c_\perp$ as follows,

$$\begin{aligned}
\eta_1^2 =& \frac{1}{2\beta L^2}\sum_k(\partial_{i\omega_m}|_{q=0}g^a_{0,k+q})(\partial_{i\omega_m}|_{q=0}g^b_{0,k+q})\\
=& \frac{1}{2\beta L^2}\sum_k\frac{1}{(i\omega_n-\xi_{\boldsymbol{k}})^2}\frac{1}{(i\omega_n+\xi_{\boldsymbol{k}})^2} = \frac{1}{2\beta L^2}\sum_k(g^a_{0,k}g^b_{0,k})^2 = \frac{4U}{3}>0,
\end{aligned} \tag{I.81}$$

$$\begin{aligned}
\eta_1^2c_\perp^2 =& -\frac{1}{2\beta L^2}\sum_k(\partial_{q_x}|_{q=0}g^a_{0,k+q})(\partial_{q_x}|_{q=0}g^b_{0,k+q})\\
=& \frac{1}{2\beta L^2}\sum_k\frac{1}{(i\omega_n-\xi_{\boldsymbol{k}})^2}\frac{1}{(i\omega_n+\xi_{\boldsymbol{k}})^2}\frac{(|\boldsymbol{k}|-k_R)^2k_x^2}{m^2|\boldsymbol{k}|^2}\\
=& \frac{1}{2}\int\frac{d^2\boldsymbol{k}}{(2\pi)^2}\frac{1}{(2\xi_{\boldsymbol{k}})^2}[\frac{\tanh(\frac{1}{2}\beta\xi_{\boldsymbol{k}})}{\xi_{\boldsymbol{k}}}-\frac{\beta}{1+\cosh(\beta\xi_{\boldsymbol{k}})}]\frac{(|\boldsymbol{k}|-k_R)^2}{2m^2}>0.
\end{aligned} \tag{I.82}$$

To determine the coefficient of the spin-coordinate coupling term ($\eta_1^2 c_\perp^2 \alpha$), We can use a similar trick as Eqs. (I.78,I.80) to simplify Eq. (I.77),

$$\eta_1^2 c_\perp^2 \alpha = -\frac{1}{2\beta L^2}\sum_k \partial_{q_x}|_{q=0}[g^a_{0,k+q}e^{-i\theta_{\widehat{\boldsymbol{k+q}}}}]\partial_{q_x}|_{q=0}[g^b_{0,k+q}e^{-i\theta_{\widehat{\boldsymbol{k+q}}}}], \tag{I.83}$$

where

$$\partial_{q_x}|_{q=0}e^{-i\theta_{\widehat{\boldsymbol{k+q}}}} = -ie^{-i\theta_{\widehat{\boldsymbol{k}}}}\partial_{q_x}|_{q=0}\mathrm{arctan}\frac{k_y}{k_x+q_x} = ie^{-i\theta_{\widehat{\boldsymbol{k}}}}\frac{k_y}{k_x^2+k_y^2} = ie^{-i\theta_{\widehat{\boldsymbol{k}}}}\frac{\sin\theta_{\widehat{\boldsymbol{k}}}}{|\boldsymbol{k}|}. \tag{I.84}$$

Then we have

$$\begin{aligned}
\eta_1^2 c^2\alpha =& -\frac{1}{2\beta L^2}\sum_k[-(g^a_{0,k})^2\frac{(|\boldsymbol{k}|-k_R)\cos\theta_{\widehat{\boldsymbol{k}}}}{m}+ig^a_{0,k}\frac{\sin\theta_{\widehat{\boldsymbol{k}}}}{|\boldsymbol{k}|}][(g^b_{0,k})^2\frac{(|\boldsymbol{k}|-k_R)\cos\theta_{\widehat{\boldsymbol{k}}}}{m}+ig^b_{0,k}\frac{\sin\theta_{\widehat{\boldsymbol{k}}}}{|\boldsymbol{k}|}]e^{-2i\theta_{\widehat{\boldsymbol{k}}}}\\
=& -\frac{1}{2\beta L^2}\sum_k[-(g^a_{0,k})^2\frac{|\boldsymbol{k}|-k_R}{2m}+g^a_{0,k}\frac{1}{2|\boldsymbol{k}|}][(g^b_{0,k})^2\frac{|\boldsymbol{k}|-k_R}{2m}+g^b_{0,k}\frac{1}{2|\boldsymbol{k}|}]\\
=& -\frac{1}{8\beta L^2}\sum_k\{-(g^a_{0,k}g^b_{0,k})^2\frac{(|\boldsymbol{k}|-k_R)^2}{m^2}+g^a_{0,k}g^b_{0,k}\frac{1}{|\boldsymbol{k}|^2}+[g^a_{0,k}(g^b_{0,k})^2-(g^a_{0,k})^2g^b_{0,k}]\frac{|\boldsymbol{k}|-k_R}{m|\boldsymbol{k}|}\},
\end{aligned} \tag{I.85}$$

where

$$\begin{aligned}
\sum_n[g^a_{0,k}(g^b_{0,k})^2-(g^a_{0,k})^2g^b_{0,k}] =& \sum_n[\frac{1}{-i\omega_n+\xi_{\boldsymbol{k}}}(\frac{1}{-i\omega_n-\xi_{\boldsymbol{k}}})^2-(\frac{1}{-i\omega_n+\xi_{\boldsymbol{k}}})^2\frac{1}{-i\omega_n-\xi_{\boldsymbol{k}}}]\\
=&\frac{(i\omega_n+\xi_{\boldsymbol{k}})-(i\omega_n-\xi_{\boldsymbol{k}})}{(i\omega_n+\xi_{\boldsymbol{k}})^2(i\omega_n-\xi_{\boldsymbol{k}})^2} = \sum_n 2\xi_{\boldsymbol{k}}(g^a_{0,k}g^b_{0,k})^2,
\end{aligned} \tag{I.86}$$

$$\frac{|\boldsymbol{k}|-k_R}{m|\boldsymbol{k}|}[2\xi_{\boldsymbol{k}}-\frac{(|\boldsymbol{k}|-k_R)|\boldsymbol{k}|}{m}] = \frac{|\boldsymbol{k}|-k_R}{m|\boldsymbol{k}|}[2E_g-\frac{(|\boldsymbol{k}|-k_R)k_R}{m}]. \tag{I.87}$$

In terms of Eqs. (I.62,I.69,I.86,I.87), we finally determine $\eta_1^2 c^2\alpha$ as follows

$$\begin{aligned}
\eta_1^2 c_\perp^2\alpha =&\frac{1}{2}\int\frac{d^2\boldsymbol{k}}{(2\pi)^2}\{\frac{\tanh(\frac{1}{2}\beta\xi_{\boldsymbol{k}})}{2\xi_{\boldsymbol{k}}}\frac{1}{4|\boldsymbol{k}|^2}\\
&+\frac{1}{(2\xi_{\boldsymbol{k}})^2}[\frac{\tanh(\frac{1}{2}\beta\xi_{\boldsymbol{k}})}{\xi_{\boldsymbol{k}}}-\frac{\beta}{1+\cosh(\beta\xi_{\boldsymbol{k}})}][\frac{(|\boldsymbol{k}|-k_R)k_R}{m}-2E_g]\frac{|\boldsymbol{k}|-k_R}{4m|\boldsymbol{k}|}\}.
\end{aligned} \tag{I.88}$$

To evaluate α and $c_\perp$, note first that the omission of the down-spin bands in Eq. (I.29) is justified when

$$1 \ll \beta E_g \ll \beta\frac{k_R^2}{2m}. \tag{I.89}$$

The condition also implies that when the temperature is low enough, electrons and holes are excited only around k_R. This naturally lets us introduce an "ultraviolet" cutoff k_g in the integral over $|\boldsymbol{k}|$ in Eqs. (I.82,I.88),

$$\eta_1^2 c_\perp^2 = \int\frac{d^2\boldsymbol{k}}{(2\pi)^2}\frac{1}{16\xi_{\boldsymbol{k}}^3}\frac{(|\boldsymbol{k}|-k_R)^2}{m^2} = \int_{k_R-k_g}^{k_R+k_g}\frac{dk}{2\pi}\frac{k}{16\xi_k^3}\frac{(k-k_R)^2}{m^2}, \tag{I.90}$$

$$\begin{aligned}
\eta_1^2 c_\perp^2\alpha =& \int\frac{d^2\boldsymbol{k}}{(2\pi)^2}\frac{1}{16\xi_{\boldsymbol{k}}^3|\boldsymbol{k}|^2}\{\frac{k_R|\boldsymbol{k}|(|\boldsymbol{k}|-k_R)^2}{2m^2}-\frac{E_g|\boldsymbol{k}|(|\boldsymbol{k}|-k_R)}{m}+[\frac{(|\boldsymbol{k}|-k_R)^2}{2m}+E_g]^2\}\\
=&\int_{k_R-k_g}^{k_R+k_g}\frac{dk}{2\pi}\frac{1}{16\xi_k^3 k}\{\frac{(k-k_R)^2(k^2+k_R^2)}{4m^2}-\frac{E_g k_R(k-k_R)}{m}+E_g^2\}.
\end{aligned} \tag{I.91}$$

The cutoff k_g satifies that $k_g = \mathcal{O}(\sqrt{2mE_g}) \ll k_R$. Note also that without the cutoff the integral in Eq. (I.88) has the logarithmic divergence at $k = 0$. With the cutoff, We finally obtain

$$\eta_1^2 c_\perp^2 E_g = \int_{-k_g}^{k_g} \frac{dk}{2\pi} \frac{(k_R+k)E_g}{16} (\frac{k^2}{2m} + E_g)^{-3} \frac{k^2}{m^2} = \int_0^{k_g} \frac{dk}{\pi} \frac{k_R E_g}{16} (\frac{k^2}{2m} + E_g)^{-3} \frac{k^2}{m^2} = \mathcal{O}(\frac{k_R}{k_g}), \tag{I.92}$$

$$\begin{aligned}\alpha\eta_1^2 c_\perp^2 E_g =& \int_{-k_g}^{k_g} \frac{dk}{2\pi} \frac{(k_R+k)^{-1}E_g}{16} (\frac{k^2}{2m} + E_g)^{-3} \{\frac{k[(k_R+k)^2 + k_R^2]}{4m^2} - \frac{E_g k_R k}{m} + E_g^2\} \\ =& \frac{1}{2}\eta_1^2 c^2 E_g + \mathcal{O}(1) < \frac{1}{2}\eta_1^2 c^2 E_g + |\mathcal{O}(\frac{k_R}{k_g})|.\end{aligned} \tag{I.93}$$

A comparison between Eq. (I.92) and Eq. (I.93) suggests that $|\alpha| = \frac{1}{2} < 1$ in the limit of Eq. (I.89). $\alpha = \mathcal{O}(1)$ is due to the large spin-orbit-coupling limit, while $\alpha \ll \mathcal{O}(1)$ for smaller spin-orbit coupling. Nonetheless, the competition between different components of excitons will be more complicated in the smaller spin-orbit coupling case. $\eta_1^2 c_\perp^2 E_g \gg 1$ is consistent with the physical picture of Eq. (I.89). In the large spin-orbit coupling limit, We can also simplify Eqs. (I.62,I.81) and all the coefficients in Eq. (2) in the main text,

$$D_0 = \frac{4}{g_s(1+w)} + 2U\rho_0 = \int \frac{d^2\boldsymbol{k}}{(2\pi)^2} \frac{1}{2\xi_{\boldsymbol{k}}} = \int_0^{k_g} \frac{dk}{\pi} \frac{1}{2} (\frac{k^2}{2m} + E_g)^{-1}, \tag{I.94}$$

$$\eta_1^2 = \frac{4U}{3} = \int \frac{d^2\boldsymbol{k}}{(2\pi)^2} \frac{1}{8\xi_{\boldsymbol{k}}^3} = \int_0^{k_g} \frac{dk}{\pi} \frac{1}{8} (\frac{k^2}{2m} + E_g)^{-3}. \tag{I.95}$$

II. OBSERVABLES AND CONSERVATION RELATIONS

In this section, we derive the spin (j_μ^s) and orbital (j_μ^l) parts of Noether's current in Eqs. (7,8) in the main text and verify that the total angular momentum is conserved. We start with the classical effective theory Eq. (6) in the main text,

$$\mathcal{L} = \frac{1}{2}(\partial_t\theta)^2 - \frac{1}{2}(\partial_x\theta)^2[1 - \alpha\cos(2\theta)] - \frac{1}{2}(\partial_y\theta)^2[1 + \alpha\cos(2\theta)] + \alpha(\partial_x\theta)(\partial_y\theta)\sin(2\theta). \tag{II.1}$$

The theory has a U(1) spacetime symmetry,

$$\theta \to \theta + \epsilon\Delta\theta = \theta + \epsilon, \quad x \to x + \epsilon\Delta x = x - \epsilon y, \quad y \to y + \epsilon\Delta y = y + \epsilon x, \quad t \to t + \epsilon\Delta t = t. \tag{II.2}$$

With the continuous symmetry Eq. (II.2), Noether's theorem gives a conserved current,

$$j_\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\theta)}\Delta\theta + [\delta_{\mu\nu}\mathcal{L} - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\theta)}(\partial_\nu\theta)]\Delta x_\nu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\theta)}\Delta\theta + T_{\mu\nu}\Delta x_\nu, \tag{II.3}$$

where $\mu, \nu \in \{t, x, y\}$, $\Delta x_\nu \in \{\Delta t, \Delta x, \Delta y\}$. $T_{\mu\nu} \equiv \delta_{\mu\nu}\mathcal{L} - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\theta)}(\partial_\nu\theta)$ is a stress-energy tensor. The conserved current obeys $\partial_\mu j_\mu = 0$ as long as an equation of motion is satisfied,

$$\partial_\mu[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\theta)}] - \frac{\partial\mathcal{L}}{\partial\theta} = 0. \tag{II.4}$$

The equation of motion is given by

$$\begin{aligned}&\partial_t^2\theta - (\partial_x^2\theta)[1 - \alpha\cos(2\theta)] - 2\alpha(\partial_x\theta)^2\sin(2\theta) - (\partial_y^2\theta)[1 + \alpha\cos(2\theta)] + 2\alpha(\partial_y\theta)^2\sin(2\theta) \\ &+ 2\alpha(\partial_x\partial_y\theta)\sin(2\theta) + 4\alpha(\partial_x\theta)(\partial_y\theta)\cos(2\theta) + \alpha[(\partial_x\theta)^2 - (\partial_y\theta)^2]\sin(2\theta) - 2\alpha(\partial_x\theta)(\partial_y\theta)\cos(2\theta) \\ =&\partial_t^2\theta - (\partial_x^2\theta)[1 - \alpha\cos(2\theta)] - (\partial_y^2\theta)[1 + \alpha\cos(2\theta)] + 2\alpha(\partial_x\partial_y\theta)\sin(2\theta) \\ &- \alpha[(\partial_x\theta)^2 - (\partial_y\theta)^2]\sin(2\theta) + 2\alpha(\partial_x\theta)(\partial_y\theta)\cos(2\theta) = 0.\end{aligned} \tag{II.5}$$

The theory has spatial and temporal translational symmetries, which imposes a conservation rule on the stress-energy tensor,

$$\partial_\mu T_{\mu\nu} = \partial_\mu[\delta_{\mu\nu}\mathcal{L} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu\theta)}(\partial_\nu\theta)] = 0. \tag{II.6}$$

The total angular momentum current of Eq. (II.3) can be divided into spin angular momentum current j_s that does not depend on $T_{\mu\nu}$, and orbital angular momentum current j_l that depends on $T_{\mu\nu}$.

$$j^s_\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu\theta)}\Delta\theta, \quad j^l_\mu = [\delta_{\mu\nu}\mathcal{L} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu\theta)}(\partial_\nu\theta)]\Delta x_\nu. \tag{II.7}$$

Let us focus on the spin part. Spin angular momentum density is

$$s = j^s_t = \partial_t\theta, \tag{II.8}$$

and corresponding spin currents are

$$j^s_x = -(\partial_x\theta)[1-\alpha\cos(2\theta)] + \alpha(\partial_y\theta)\sin(2\theta), \tag{II.9}$$

$$j^s_y = -(\partial_y\theta)[1+\alpha\cos(2\theta)] + \alpha(\partial_x\theta)\sin(2\theta). \tag{II.10}$$

The spin density and current are zero at equilibrium, $j^s_\mu = 0$. The spin angular momentum is not conserved,

$$\partial_\mu j^s_\mu = G = -\partial_\mu j^l_\mu. \tag{II.11}$$

Local sources of the spin angular momentum are given by spin torque G. The torque represents the mutual conversion between orbital and spin angular momenta. Taking the equation of motion Eq. (II.5) into Eqs. (II.8-II.10), we get the spin torque as follows,

$$\begin{aligned} G =&\partial_\mu j^s_\mu = \partial^2_t\theta - (\partial^2_x\theta)[1-\alpha\cos(2\theta)] - (\partial^2_y\theta)[1+\alpha\cos(2\theta)] - 2(\partial_x\theta)^2\alpha\sin(2\theta) + 2(\partial_y\theta)^2\alpha\sin(2\theta) \\ &+ 2\alpha(\partial_x\partial_y\theta)\sin(2\theta) + 4\alpha(\partial_x\theta)(\partial_y\theta)\cos(2\theta) \\ =& -\alpha[(\partial_x\theta)^2 - (\partial_y\theta)^2]\sin(2\theta) + 2\alpha(\partial_x\theta)(\partial_y\theta)\cos(2\theta). \end{aligned} \tag{II.12}$$

The orbital-angular-momentum density and current are given by

$$l = j^l_t = y(\partial_t\theta)(\partial_x\theta) - x(\partial_t\theta)(\partial_y\theta), \tag{II.13}$$

$$\begin{aligned} j^l_x =& -y\big\{\frac{1}{2}(\partial_t\theta)^2 - \frac{1}{2}(\partial_y\theta)^2[1+\alpha\cos(2\theta)] + \frac{1}{2}(\partial_x\theta)^2[1-\alpha\cos(2\theta)]\big\} \\ &+ x\big\{(\partial_x\theta)(\partial_y\theta)[1-\alpha\cos(2\theta)] - \alpha(\partial_y\theta)^2\sin(2\theta)\big\}, \end{aligned} \tag{II.14}$$

$$\begin{aligned} j^l_y =& x\big\{\frac{1}{2}(\partial_t\theta)^2 - \frac{1}{2}(\partial_x\theta)^2[1-\alpha\cos(2\theta)] + \frac{1}{2}(\partial_y\theta)^2[1+\alpha\cos(2\theta)]\big\} \\ &- y\big\{(\partial_x\theta)(\partial_y\theta)[1+\alpha\cos(2\theta)] - \alpha(\partial_x\theta)^2\sin(2\theta)\big\}. \end{aligned} \tag{II.15}$$

The orbital angular momentum density and current depends explicitly on spatial coordinates, and they depend on a choice of the origin for the spatial coordinates. Besides, the equation of motion Eq. (II.5) gives $\partial^2_t\theta$ instead of $\partial_t\theta$, while j^s_t as well as $\partial_t j^s_t$ contains $\partial_t\theta$. Nonetheless, we can verify the continuity equation Eq. (II.11) directly, using Eqs. (II.5,II.13-II.15). $\partial_\mu j^l_\mu$ is formally given by a term that has no explicit dependence on x and y, and terms that depend explicitly and linearly on the spatial coordinates. The latter terms vanish thanks to Eq. (II.5);

$$\begin{aligned} &\frac{\partial(\partial_\mu j^l_\mu)}{\partial y}\Big|_{x,\theta,\partial_\mu\theta} = (\partial^2_t\theta)(\partial_x\theta) + (\partial_t\theta)(\partial_x\partial_t\theta) \\ &- (\partial_t\theta)(\partial_x\partial_t\theta) + (\partial_y\theta)(\partial_x\partial_y\theta)[1+\alpha\cos(2\theta)] - \alpha(\partial_y\theta)^2(\partial_x\theta)\sin(2\theta) \\ &- (\partial_x\theta)(\partial^2_x\theta)[1-\alpha\cos(2\theta)] - \alpha(\partial_x\theta)^2(\partial_x\theta)\sin(2\theta) \\ &- (\partial_x\theta)(\partial^2_y\theta)[1+\alpha\cos(2\theta)] - (\partial_x\partial_y\theta)(\partial_y\theta)[1+\alpha\cos(2\theta)] + 2\alpha(\partial_x\theta)(\partial_y\theta)^2\sin(2\theta) \\ &+ 2\alpha(\partial_x\theta)(\partial_x\partial_y\theta)\sin(2\theta) + 2\alpha(\partial_x\theta)^2(\partial_y\theta)\cos(2\theta) \\ =&(\partial^2_t\theta)(\partial_x\theta) - (\partial_x\theta)(\partial^2_x\theta + \partial^2_y\theta) \\ &+ \alpha(\partial_x\theta)\sin(2\theta)[(\partial_y\theta)^2 - (\partial_x\theta)^2 + 2(\partial_x\partial_y\theta)] \\ &+ \alpha(\partial_x\theta)\cos(2\theta)[(\partial_x\theta)^2 - (\partial_y\theta)^2 + 2(\partial_x\theta)(\partial_y\theta)] = 0, \end{aligned} \tag{II.16}$$

$$
\begin{aligned}
&\frac{\partial(\partial_\mu j^l_\mu)}{\partial x}\Big|_{y,\theta,\partial_\mu\theta} = -(\partial_t^2\theta)(\partial_y\theta) - (\partial_t\theta)(\partial_y\partial_t\theta) \\
&+ (\partial_t\theta)(\partial_y\partial_t\theta) - (\partial_x\theta)(\partial_x\partial_y\theta)[1-\alpha\cos(2\theta)] - \alpha(\partial_x\theta)^2(\partial_y\theta)\sin(2\theta) \\
&+ (\partial_y\theta)(\partial_y^2\theta)[1+\alpha\cos(2\theta)] - \alpha(\partial_y\theta)^2(\partial_y\theta)\sin(2\theta) \\
&+ (\partial_y\theta)(\partial_x^2\theta)[1-\alpha\cos(2\theta)] + (\partial_x\partial_y\theta)(\partial_x\theta)[1-\alpha\cos(2\theta)] + 2\alpha(\partial_y\theta)(\partial_x\theta)^2\sin(2\theta) \\
&- 2\alpha(\partial_y\theta)(\partial_x\partial_y\theta)\sin(2\theta) - 2\alpha(\partial_y\theta)^2(\partial_x\theta)\cos(2\theta) \\
=& -(\partial_t^2\theta)(\partial_y\theta) + (\partial_y\theta)(\partial_x^2\theta + \partial_y^2\theta) \\
&+ \alpha(\partial_y\theta)\sin(2\theta)[(\partial_x\theta)^2 - (\partial_y\theta)^2 - 2(\partial_x\partial_y\theta)] \\
&+ \alpha(\partial_y\theta)\cos(2\theta)[(\partial_y\theta)^2 - (\partial_x\theta)^2 - 2(\partial_x\theta)(\partial_y\theta)] = 0.
\end{aligned}
\tag{II.17}
$$

The former term is nothing but $-G$,

$$
\begin{aligned}
&\partial_\mu j^l_\mu - \frac{\partial(\partial_\mu j^l_\mu)}{\partial x}\Big|_{y,\theta,\partial_\mu\theta} - \frac{\partial(\partial_\mu j^l_\mu)}{\partial y}\Big|_{x,\theta,\partial_\mu\theta} \\
=&(\partial_x\theta)(\partial_y\theta)[1-\alpha\cos(2\theta)] - \alpha(\partial_y\theta)^2\sin(2\theta) - (\partial_x\theta)(\partial_y\theta)[1+\alpha\cos(2\theta)] + \alpha(\partial_x\theta)^2\sin(2\theta) \\
=&\alpha[(\partial_x\theta)^2 - (\partial_y\theta)^2]\sin(2\theta) - 2\alpha(\partial_x\theta)(\partial_y\theta)\cos(2\theta).
\end{aligned}
\tag{II.18}
$$

Thus, the total angular momentum is indeed conserved,

$$
\partial_\mu j^l_\mu = \alpha[(\partial_x\theta)^2 - (\partial_y\theta)^2]\sin(2\theta) - 2\alpha(\partial_x\theta)(\partial_y\theta)\cos(2\theta) = -G. \tag{II.19}
$$

III. SOLUTIONS FOR THE SPIN-INJECTION MODEL

In this section, we solve $\theta(x,t)$ in the spin-injection model, Eq. (9) in the main text, together with the boundary condition, Eq. (13) in the main text and $j^s_x(x=0,t) = j_0$. We consider a general junction parameter k_0 except $k_0 = 1$ (straight geometry), $k_0^2 = (1+\frac{1}{j_0 r})^2$ (circular geometry), and $k_0 = 0$, while leaving discussions about solutions at these special parameter points for Sec. VI.

A. Straight geometry without curvature

For the straight geometry without the curvature (Fig. 1(a) in the main text), let us consider the equation of motion in the one-dimensional system,

$$
\partial_t^2\theta = (\partial_x^2\theta)[1-\alpha\cos(2\theta)] + \alpha(\partial_x\theta)^2\sin(2\theta). \tag{III.1}
$$

The boundary conditions are given by

$$
j^s_x(0,t) = j_0, \tag{III.2}
$$

$$
s_c(L,t) = k_c j^s_{x,c}(L,t), \tag{III.3}
$$

with $k_c \equiv \frac{\chi}{\chi'}\big[D_s\big(\frac{1}{T_1'} + ic\big)\big]^{-1/2} + \frac{\chi}{\beta_t}$. Here "$c$" stands for the frequency of spin density and current at $x = L$, and Eq. (III.3) is imposed for each frequency component of the density and current. The density and current are given by θ (Eq. (8) in the main text),

$$
s = \partial_t\theta, \quad j^s_x = -(\partial_x\theta)[1-\alpha\cos(2\theta)], \quad j^s_y = \alpha\partial_x\theta\sin(2\theta), \quad G = -\alpha(\partial_x\theta)^2\sin(2\theta). \tag{III.4}
$$

We solve the equation of motion by a perturbative expansion of α. With $\theta(x,t) = \theta_0(x,t) + \mathcal{O}(\alpha)$, the zeroth order is

$$
\partial_t^2\theta_0 = \partial_x^2\theta_0. \tag{III.5}
$$

The general solution of Eq. (III.5) is

$$\theta_0(x,t) = At + Bx + F_0 + \sum_{c\in\mathbb{R},c\neq 0}[F_c e^{ic(t-x)} + F'_c e^{ic(t+x)}], \tag{III.6}$$

where A, B, F_c, F'_c are constants. Eq. (III.2) leads to

$$\theta_0(x,t) = At - j_0 x + F_0 + \sum_{c\in\mathbb{R},c\neq 0} 2F_c\cos(cx)e^{ict}. \tag{III.7}$$

Since $\mathrm{Re}(k_c) > 0$, Eq. (III.3) requires $F_c = 0$ for $c \neq 0$, and the zeroth order takes the following form,

$$\theta_0(x,t) = s(x,t)t - j^s_x(x,t)x + F(0) = k_0 j_0 t - j_0 x + F_0. \tag{III.8}$$

Here F_0 can be absorbed by a time translation so we take $F_0 = 0$. Note that without the spin-orbit coupling ($\alpha = 0$), the spin density and current are static,

$$j^s_x(0 < x < L) = j_0 + \mathcal{O}(\alpha), \quad s(0 < x < L) = k_0 j_0 + \mathcal{O}(\alpha). \tag{III.9}$$

Upon a substitution of Eq. (III.9) into Eq. (III.1) and an expansion of Eq. (III.1) in α, the first-order correction to the solution is given by an inhomogeneous linear differential equation. Thereby, the first-order solution has two parts, θ_1 and θ_2,

$$\theta(x,t) = \theta_0(x,t) + \theta_1(x,t) + \theta_2(x,t) + \mathcal{O}(\alpha^2), \tag{III.10}$$

and θ_1 is a special solution of the inhomogenous equation,

$$\partial_t^2\theta_1 - \partial_x^2\theta_1 = -\alpha(\partial_x^2\theta_0)\cos(2\theta_0) + \alpha(\partial_x\theta_0)^2\sin(2\theta_0) = \alpha j_0^2\sin(2k_0 j_0 t - 2j_0 x). \tag{III.11}$$

$\theta_2(x,t)$ is a solution of the homogeneous differential equation,

$$\partial_t^2\theta_2 - \partial_x^2\theta_2 = 0. \tag{III.12}$$

With $\theta_1(x,t)$ and $\theta_2(x,t)$, the spin density and current should satisfy the BCs up to the first order in α.

Thanks to the linear x and t-dependence of $\theta_0(x,t)$ and $k_0 \neq 1$, we can find a special solution,

$$\theta_1(x,t) = -\frac{\alpha}{4(k_0^2-1)}\sin(2k_0 j_0 t - 2j_0 x). \tag{III.13}$$

$\theta_2(x,t)$ takes the same form as Eq. (III.6). With these solutions, the spin density and current are given by the following up to the first order in α,

$$s = k_0 j_0 - \frac{k_0 j_0 \alpha}{2(k_0^2-1)}\cos(2k_0 j_0 t - 2j_0 x) + \partial_t\theta_2 + \mathcal{O}(\alpha^2), \tag{III.14}$$

$$\begin{aligned} j^s_x =& j_0 - \alpha j_0\cos(2k_0 j_0 t - 2j_0 x) - \frac{j_0\alpha}{2(k_0^2-1)}\cos(2k_0 j_0 t - 2j_0 x) - \partial_x\theta_2 + \mathcal{O}(\alpha^2) \\ =& j_0 - \frac{j_0\alpha(2k_0^2-1)}{2(k_0^2-1)}\cos(2k_0 j_0 t - 2j_0 x) - \partial_x\theta_2 + \mathcal{O}(\alpha^2). \end{aligned} \tag{III.15}$$

In order that Eqs. (III.14,III.15) satisfy the BCs, $\theta_2(x,t)$ must have the same frequency as $\theta_1(x,t)$,

$$\theta_2(x,t) = \alpha g e^{2ik_0 j_0(t-x)} + \alpha g' e^{2ik_0 j_0(t+x)} + \text{c.c.}. \tag{III.16}$$

Here g and g' are complex constants. By the same reasoning as in the text below Eq. (III.7), other frequency components in $\theta_2(x,t)$ vanish. This leads to

$$\begin{aligned} s =& \frac{1}{2}k_0 j_0 + \alpha k_0 j_0 e^{2ik_0 j_0 t}[2ige^{-2ik_0 j_0 x} - \frac{e^{-2ij_0 x}}{4(k_0^2-1)}] \\ &+ 2i\alpha k_0 j_0 g' e^{2ik_0 j_0(t+x)} + \mathcal{O}(\alpha^2) + \text{c.c.}, \end{aligned} \tag{III.17}$$

$$j_x^s = \frac{1}{2}j_0 + \alpha j_0 e^{2ik_0j_0t}[2ik_0 g e^{-2ik_0j_0x} - \frac{(2k_0^2-1)e^{-2ij_0x}}{4(k_0^2-1)}] - 2i\alpha k_0 j_0 g' e^{2ik_0j_0(t+x)} + \mathcal{O}(\alpha^2) + \text{c.c.}. \tag{III.18}$$

The boundary conditions Eqs. (III.2,III.3) require

$$\alpha j_0[2ik_0 g - \frac{2k_0^2-1}{4(k_0^2-1)}] - 2i\alpha k_0 j_0 g' = 0, \tag{III.19}$$

$$\begin{aligned}&\alpha j_0[2ik_0 g e^{-2ik_0j_0L} - \frac{e^{-2ij_0L}}{4(k_0^2-1)}] + 2i\alpha j_0 k_0 g' e^{2ik_0j_0L}\\ =&k\alpha j_0[2ik_0 g e^{-2ik_0j_0L} - \frac{(2k_0^2-1)e^{-2ij_0L}}{4(k_0^2-1)}] - 2ik\alpha k_0 j_0 g' e^{2ik_0j_0L},\end{aligned} \tag{III.20}$$

with

$$k_0 = \frac{\chi}{\chi'}(\frac{D_s}{T_1'})^{-\frac{1}{2}} + \frac{\chi}{\beta_t}, \tag{III.21}$$

$$k \equiv k_{2k_0j_0} = \frac{\chi}{\chi'}\left[D_s(\frac{1}{T_1'} + 2ik_0j_0)\right]^{-\frac{1}{2}} + \frac{\chi}{\beta_t}, \tag{III.22}$$

and $k_{-c} = (k_c)^*$ for a real number c. Eqs. (III.19,III.20) can be simplified,

$$2ik_0(g - g') = \frac{2k_0^2-1}{4(k_0^2-1)}, \tag{III.23}$$

$$2ik_0(ge^{-2ik_0j_0L} + g'e^{2ik_0j_0L}) - 2ikk_0(ge^{-2ik_0j_0L} - g'e^{2ik_0j_0L}) = \frac{1-(2k_0^2-1)k}{4(k_0^2-1)}e^{-2ij_0L}. \tag{III.24}$$

The two equations Eqs. (III.23,III.24) determine the two coefficients, g and g',

$$\begin{aligned}&\begin{pmatrix} 1 & -1 \\ (1-k)e^{-ik_0\beta_L} & (1+k)e^{ik_0\beta_L}\end{pmatrix}\begin{pmatrix} g \\ g'\end{pmatrix}\\ =&\frac{1}{8ik_0(k_0^2-1)}\begin{pmatrix} 2k_0^2-1 \\ (1-(2k_0^2-1)k)e^{-i\beta_L}\end{pmatrix},\end{aligned} \tag{III.25}$$

where $\beta_L \equiv 2j_0L$. The solution of the equations is

$$g = \frac{(2k_0^2-1)(1+k)e^{ik_0\beta_L} + [1-(2k_0^2-1)k]e^{-i\beta_L}}{(1+k)e^{ik_0\beta_L} + (1-k)e^{-ik_0\beta_L}}\frac{1}{8ik_0(k_0^2-1)}, \tag{III.26}$$

$$g' = \frac{[1-(2k_0^2-1)k]e^{-i\beta_L} - (2k_0^2-1)(1-k)e^{-ik_0\beta_L}}{(1+k)e^{ik_0\beta_L} + (1-k)e^{-ik_0\beta_L}}\frac{1}{8ik_0(k_0^2-1)}. \tag{III.27}$$

From Eqs. (III.8,III.10,III.13,III.16), we get

$$\begin{aligned}\theta(x,t) =&j_0(k_0t - x) - \frac{\alpha}{4(k_0^2-1)}\sin[2j_0(k_0t-x)]\\ &+ 2\alpha\mathrm{Re}(g)\cos[2k_0j_0(t-x)] + 2\alpha\mathrm{Re}(g')\cos[2k_0j_0(t+x)]\\ &- 2\alpha\mathrm{Im}(g)\sin[2k_0j_0(t-x)] - 2\alpha\mathrm{Im}(g')\sin[2k_0j_0(t+x)] + \mathcal{O}(\alpha^2).\end{aligned} \tag{III.28}$$

The solution has one frequency ($2k_0j_0$) in time and two wavenumbers ($2j_0$, $2k_0j_0$) in space. The solution Eq. (III.28) can be also rewritten as

$$\begin{aligned}&\theta(x,t) = j_0(k_0t - x) - \frac{\alpha}{4(k_0^2-1)}\sin[2j_0(k_0t-x)]\\ &+ 2\alpha\mathrm{Re}(g+g')\cos(2k_0j_0t)\cos(2k_0j_0x) + 2\alpha\mathrm{Re}(g-g')\sin(2k_0j_0t)\sin(2k_0j_0x)\\ &- 2\alpha\mathrm{Im}(g+g')\sin(2k_0j_0t)\cos(2k_0j_0x) + 2\alpha\mathrm{Im}(g-g')\cos(2k_0j_0t)\sin(2k_0j_0x) + \mathcal{O}(\alpha^2).\end{aligned} \tag{III.29}$$

According to Eqs. (III.23,III.26,III.27), Eq. (III.29) is nothing but Eq. (18) in the main text,

$$\begin{aligned}\theta(x,t) =&j_0(k_0t-x) - \frac{\alpha}{4(k_0^2-1)}\sin[2j_0(k_0t-x)] - \frac{\alpha(2k_0^2-1)}{4(k_0^2-1)}\cos(2k_0j_0t)\sin(2k_0j_0x)\\ &+\alpha\mathrm{Im}(\eta)\cos(2k_0j_0t)\cos(2k_0j_0x) + \alpha\mathrm{Re}(\eta)\sin(2k_0j_0t)\cos(2k_0j_0x) + \mathcal{O}(\alpha^2),\end{aligned} \tag{III.30}$$

where

$$\eta \equiv 2i(g+g') = \frac{(2k_0^2-1)[(1+k)e^{ik_0\beta_L} - (1-k)e^{-ik_0\beta_L}] + 2[1-(2k_0^2-1)k]e^{-i\beta_L}}{4k_0(k_0^2-1)[(1+k)e^{ik_0\beta_L} + (1-k)e^{-ik_0\beta_L}]}. \tag{III.31}$$

Higher-order solutions can be obtained by the same perturbative iteration method. In the solution, the spin density and current have the same periodicity in time as the first-order solution, $\pi(k_0j_0)^{-1}$. This is because the inhomogeneous terms at every order keep the same discrete time translational symmetry as that for the first order. For irrational k_0, the solution is not periodic in the space coordinate x because of the superpositions of the two wavenumbers. Nonetheless, the Fourier-transform in the space has two major peaks at $2j_0$ and $2k_0j_0$.

B. Circular geometry with curvature

For the one-dimensional circular geometry with a finite radius r of the curvature (Figs. 1(b,c) in the main text), the Lagrangian is generalized as follows

$$\begin{aligned}\mathcal{L} =&\frac{1}{2}(\partial_t\theta)^2 - \frac{1}{2r^2}(\partial_\vartheta\theta)^2 + \frac{\alpha}{2r^2}(\partial_\vartheta\theta)^2[\cos(2\theta)(\sin^2\vartheta - \cos^2\vartheta) - 2\sin(2\theta)\sin\theta\cos\theta]\\ =&\frac{1}{2}(\partial_t\theta)^2 - \frac{1}{2}(\partial_\ell\theta)^2[1+\alpha\cos(2\theta - \frac{2}{r}\ell)],\end{aligned} \tag{III.32}$$

with a one-dimensional coordinate $\ell \equiv r\vartheta$, and

$$\partial_r\theta(x,y) \equiv \partial_r(r\cos\vartheta, r\sin\vartheta) = 0, \quad \partial_x = -\frac{1}{r}(\sin\vartheta)\partial_\vartheta, \quad \partial_y = \frac{1}{r}(\cos\vartheta)\partial_\vartheta. \tag{III.33}$$

The Lagrangian gives the classical equation of motion in the one-dimensional system,

$$\begin{aligned}&\partial_t^2\theta - (\partial_\ell^2\theta)[1+\alpha\cos(2\theta-\frac{2}{r}\ell)] + 2\alpha(\partial_\ell\theta)\sin(2\theta-\frac{2}{r}\ell)[(\partial_\ell\theta)-\frac{1}{r}] - \alpha(\partial_\ell\theta)^2\sin(2\theta-\frac{2}{r}\ell)\\ =&\partial_t^2\theta - (\partial_\ell^2\theta)[1+\alpha\cos(2\theta-\frac{2}{r}\ell)] + \alpha(\partial_\ell\theta)[(\partial_\ell\theta)-\frac{2}{r}]\sin(2\theta-\frac{2}{r}\ell) = 0,\end{aligned} \tag{III.34}$$

and the spin density and current,

$$s = \partial_t\theta, \quad j_\ell^s = -(\partial_\ell\theta)[1+\alpha\cos(2\theta-\frac{2}{r}\ell)]. \tag{III.35}$$

The boundary conditions are imposed on the spin density and current,

$$j_\ell^s(0,t) = j_0, \quad s_c(L,t) = k_c j_{\ell,c}^s(L,t). \tag{III.36}$$

The boundary condition at $\ell = L$ is imposed on every frequency (c) component of the density and current, and $k_c = \frac{\chi}{\chi'}\left[D_s\left(\frac{1}{T_1'}+ic\right)\right]^{-1/2} + \frac{\chi}{\beta_t}$.

The zeroth-order solution of the EOM that satisfies the BCs is given by

$$\theta_0(\ell,t) = k_0j_0t - j_0\ell. \tag{III.37}$$

In the perturbative iteration method, the first-order solution comprises of $\theta_1(\ell,t)$ and $\theta_2(\ell,t)$. $\theta_1(\ell,t)$ is a special solution of the inhomogeneous linear differential equation, Eq. (20) in the main text, while $\theta_2(\ell,t)$ is a solution of the homogeneous linear differential equation. For $k_0^2 \neq (1+\frac{1}{j_0r})^2$, we find the special solution,

$$\theta_1(\ell,t) = \frac{\alpha(1+\frac{2}{j_0r})}{4[k_0^2-(1+\frac{1}{j_0r})^2]}\sin[2k_0j_0t - 2(j_0+\frac{1}{r})\ell], \tag{III.38}$$

together with

$$\theta_2(\ell,t) = At + B\ell + F_0 + \sum_{c\in\mathbb{R},c\neq 0}\left[F_c e^{ic(t-\ell)} + F'_c e^{ic(t+\ell)}\right]. \tag{III.39}$$

A substitution of $\theta = \theta_0 + \theta_1 + \theta_2 + \mathcal{O}(\alpha^2)$ into Eq. (III.35) leads to

$$s = k_0 j_0 + \frac{k_0 j_0 \alpha(1+\frac{2}{j_0 r})}{2[k_0^2 - (1+\frac{1}{j_0 r})^2]}\cos[2k_0 j_0 t - 2(j_0 + \frac{1}{r})\ell] + \partial_t\theta_2 + \mathcal{O}(\alpha^2), \tag{III.40}$$

$$\begin{aligned} j^s_\ell =& j_0 + \frac{\alpha j_0}{2}\frac{2k_0^2 - 2(1+\frac{1}{j_0 r})^2 + (1+\frac{1}{j_0 r})(1+\frac{2}{j_0 r})}{k_0^2 - (1+\frac{1}{j_0 r})^2}\cos[2k_0 j_0 t - 2(j_0+\frac{1}{r})\ell] - \partial_x\theta_2 + \mathcal{O}(\alpha^2) \\ =& j_0 + \frac{j_0\alpha[2k_0^2 - (1+\frac{1}{j_0 r})]}{2[k_0^2 - (1+\frac{1}{j_0 r})^2]}\cos[2k_0 j_0 t - 2(j_0+\frac{1}{r})\ell] - \partial_x\theta_2 + \mathcal{O}(\alpha^2). \end{aligned} \tag{III.41}$$

In order that Eqs.(III.40,III.41) satisfy the BCs, $\theta_2(\ell,t)$ must have the same frequency as $\theta_1(\ell,t)$;

$$\theta_2(\ell,t) = \alpha g e^{2ik_0 j_0(t-\ell)} + \alpha g' e^{2ik_0 j_0(t+\ell)} + \text{c.c.}. \tag{III.42}$$

The complex constants, g and g', are determined by

$$\begin{aligned} &\begin{pmatrix} 1 & -1 \\ (1-k)e^{-ik_0\beta_L} & (1+k)e^{ik_0\beta_L} \end{pmatrix}\begin{pmatrix} g \\ g' \end{pmatrix} \\ =& \frac{1}{8ik_0[k_0^2 - (1+\frac{1}{j_0 r})^2]}\begin{pmatrix} 2k_0^2 - (1+\frac{1}{j_0 r}) \\ \{1+\frac{2}{j_0 r} - [2k_0^2 - (1+\frac{1}{j_0 r})]k\}e^{-i(1+\frac{1}{j_0 r})\beta_L} \end{pmatrix}, \end{aligned} \tag{III.43}$$

where k is given by Eq. (III.22).

$\theta_1(\ell,t)$ vanishes when $j_0 = -\frac{2}{r}$. Even when $\theta_1(\ell,t) = 0$, $\theta_2(\ell,t) \neq 0$ in general. The non-zero $\theta_2(\ell,t)$ comes from an $\mathcal{O}(\alpha)$ contribution of j^s_ℓ in Eq. (III.35). For $j_0 = -\frac{2}{r}$ and $k_0 = \frac{1}{2}$, both $\theta_1(\ell,t)$ and $\theta_2(\ell,t)$ reduce to zero, and $\theta_0(\ell,t)$ becomes an "exact" solution satisfying the BCs. However, the exactness is not protected by the symmetry of the theory, and there will be a finite $\theta_1(\ell,t)$ when higher-order expansion terms are considered in Eq. (6) in the main text.

Besides, Eqs. (III.25,III.43) always have unique solutions for g and g', because

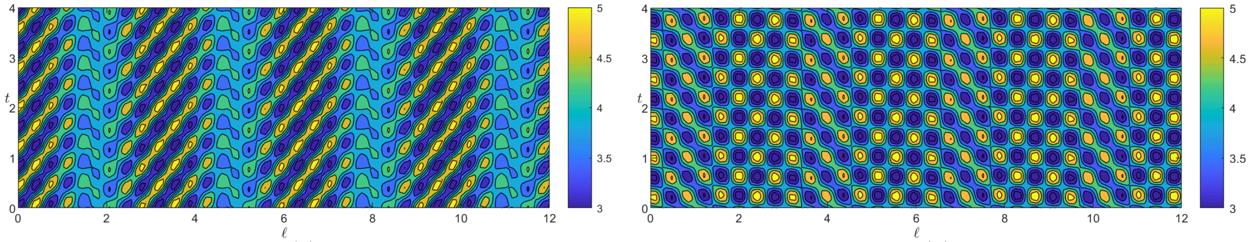
$$(1+k)e^{ik_0\beta_L} + (1-k)e^{-ik_0\beta_L} = 0 \tag{III.44}$$

or

$$k = \frac{e^{-ik_0\beta_L} + e^{ik_0\beta_L}}{e^{-ik_0\beta_L} - e^{ik_0\beta_L}} = \frac{i}{\tan(2k_0\beta_L)} \tag{III.45}$$

contradicts with $\mathrm{Re}(k) > 0$. The solutions of Eq. (III.25) and Eq. (III.43) are divergent at $k_0 = 1$ (straight) and $k_0^2 = (1+\frac{1}{j_0 r})^2$ (circular), respectively. Physically, the divergence could be avoided by finite dissipation time T_1. A more detailed discussion on the divergence is given in Sec. VI.

The radius (r) dependence of $\theta(\ell,t)$ leads to the non-reciprocity of the hydrodynamic spin transport. We show the non-reciprocity in Figs. 1(b,c) in the main text. The non-reciprocity is essentially from $\theta_1(\ell,t)$, as the spatial wavelength of $\theta_1(\ell,t)$ depends on the radius r; it also comes from $\theta_2(\ell,t)$ as $\theta_2(\ell,t)$ is different for two opposite currents to satisfy the boundary conditions. From the figures, we can see that $\theta(\ell,t)$ is periodic along t because $\theta_1(\ell,t)$ and $\theta_2(\ell,t)$ share the same temporal frequency; the structure of $\theta(\ell,t)$ along ℓ is more complicated because $\theta_1(\ell,t)$ and $\theta_2(\ell,t)$ give two different spatial wavenumbers. To show the spatial structure of $\theta(\ell,t)$ more clearly, we plot them in a larger range of ℓ (see Fig. 2 in this Supplemental Material), although physically ℓ should not be greater than L, and L should not be greater than $2\pi r$.



(a) (b)

FIG. 2. Contour plots of $\theta(\ell,t)$ in a larger spatial range of ℓ with the same parameters in Figs. 1(b,c) in the main text. Fig. (a) and Fig. (b) above correspond to Fig. 1(b) and Fig. 1(c) in the main text, respectively.

IV. POSSIBILITY OF DISSIPATION

In this section, we study the stability of the superfluid state with a finite supercurrent in the U(1) spacetime theory. The Lagrangian Eq. (6) in the main text leads to a classical equation of motion, Eq. (II.5). A solution of the EOM was obtained under boundary conditions of a finite current (Sec. III). The solution characterizes the supercurrent state. To study the stability of the supercurrent state, we compare a classical energy of the solution $\theta(x,y,t)$ with an energy of other solutions of the EOM with different BCs, say $\theta(x,y,t)+\delta\theta(x,y,t)$. Here, we consider that $\delta\theta$ is a deformation induced by spatially local perturbations. Thus, the spacetime derivatives of $\delta\theta(x,y,t)$ do not contain any uniform component in spacetime. The classical energy is evaluated by a Hamiltonian that corresponds to the Lagrangian Eq. (II.1),

$$H[\theta]=\int d^2\boldsymbol{r}\Big\{\frac{1}{2}(\partial_t\theta)^2+\frac{1}{2}(\partial_x\theta)^2\big[1-\alpha\cos(2\theta)\big]+\frac{1}{2}(\partial_y\theta)^2\big[1+\alpha\cos(2\theta)\big]-\alpha(\partial_x\theta)(\partial_y\theta)\sin(2\theta)\Big\}. \tag{IV.1}$$

The solution for the supercurrent state with broken U(1) spacetime symmetry depends on time, e.g. Eq. (18) in the main text, while the Hamiltonian of θ and $\theta+\delta\theta$ are conserved, i.e. time-independent. Thus, for clarity of calculation, we compare the "time averages" of the classical energies over a large period of time T,

$$\Delta J=\lim_{T\to\infty}\frac{1}{T}\Big(\int_0^T H[\theta+\delta\theta]dt-\int_0^T H[\theta]dt\Big). \tag{IV.2}$$

When the classical energy of θ is lower than $\theta+\delta\theta$ for arbitrary small $\delta\theta$, the supercurrent state of $\theta(x,t)$ is stable against the local perturbation. If it is not for some $\delta\theta$, the supercurrent state is no longer stable, and it must experience energy dissipation. Effects of the energy dissipation can be included as finite relaxation time into the classical EOM (see Sec. V). To demonstrate the validity of our method used in this section, we also apply the same method to a conventional superfluid moving at a finite velocity and derive its Landau criterion (see Appendix A).

As explained above, $\delta\theta$ is a deformation induced by the local perturbations, and the spacetime derivatives of $\delta\theta$ are considered to be always zero on average. The locality of $\delta\theta$ is crucial in the following argument. For example, a finite average of the space derivative of $\delta\theta$ changes a uniform current, and such $\delta\theta$ should be excluded from the local deformation. This is because even for the conventional superfluid, the classical energy with a smaller velocity of the supercurrent will always decrease. In the derivation of the Landau criterion in Appendix A, only single excitations with (k,ω) are considered; perturbation that lowers the average velocity is excluded implicitly.

In this section, we apply the stability analysis to the total-angular-momentum superfluid in the spin-injection model. We consider a general value of the junction parameter k_0, except for $k_0=1$ (straight geometry), $k_0^2=(1+\frac{1}{j_0 r})^2$ (circular geometry) and $k_0=0$, while leaving discussions about some of these points for Sec. VI. In the following, let us study the straight geometry case.

In the one-dimensional spin injection model, θ depends only on x and t; $\theta(x,t)$ and $\theta(x,t)+\delta\theta(x,t)$. The energy difference between θ and $\theta+\delta\theta$ will be evaluated order by order in powers of the SOC (α). To this end, we expand θ and $\delta\theta$ in powers of α,

$$\theta=\theta_0(x,t)+\theta_1'(x,t)+\mathcal{O}(\alpha^2), \tag{IV.3}$$

$$\delta\theta(x,t)=\delta\theta_0(x,t)+\delta\theta_1(x,t)+\mathcal{O}(\alpha^2), \tag{IV.4}$$

where

$$\theta_1'(x,t) = \theta_1(x,t) + \theta_2(x,t), \tag{IV.5}$$

$\delta\theta_1(x,t) = \mathcal{O}(\alpha)$. $\delta\theta_0(x,t)$, $\theta_0(x,t)$, and $\theta_2(x,t)$ are solutions of Eq. (III.5). Since the spacetime derivatives of $\delta\theta(x,t)$ is not uniform, $\delta\theta_0(x,t)$, as well as $\theta_2(x,t)$, is given by linear superpositions of $e^{iq(x-t)}$ and $e^{iq(x+t)}$ over q, e.g.

$$\delta\theta_0(x,t) = \frac{1}{\sqrt{L}}\sum_q [\delta d_q e^{iq(x-t)} + \delta d_q' e^{iq(x+t)}], \tag{IV.6}$$

with the system length L. In Appendix B, we give a perturbation theory in the SOC (α) that determines the higher order corrections (e.g. $\delta\theta_1$) for a given $\delta\theta_0$ in the form of Eq. (IV.6).

Given the α-expansions of θ and $\theta + \delta\theta$, we now evaluate their energy difference order by order in the power of α. We expand ΔJ in Eq. (IV.2),

$$\Delta J = \delta J + \frac{1}{2}\delta^2 J + \mathcal{O}((\delta\theta)^2). \tag{IV.7}$$

Here δJ and $\delta^2 J$ are at the first and second order in $\delta\theta$, respectively. The first-order variation is (Eq. (22) in the main text)

$$\begin{aligned}
\delta J =& \frac{1}{T}\int_0^T dt \int dx\{(\partial_t\theta)(\partial_t\delta\theta) + (\partial_x\theta)(\partial_x\delta\theta)[1-\alpha\cos(2\theta)] + \alpha(\partial_x\theta)^2\sin(2\theta)(\delta\theta)\} \\
=& \frac{1}{T}\int_0^T dt \int dx[(\partial_t\theta_0)(\partial_t\delta\theta_0) + (\partial_t\theta_1')(\partial_t\delta\theta_0) + (\partial_t\theta_0)(\partial_t\delta\theta_1) \\
&+ (\partial_x\theta_0)(\partial_x\delta\theta_0) + (\partial_x\theta_1')(\partial_x\delta\theta_0) + (\partial_x\theta_0)(\partial_x\delta\theta_1) \\
&- \alpha(\partial_x\theta_0)(\partial_x\delta\theta_0)\cos(2\theta_0) + \alpha(\partial_x\theta_0)^2\sin(2\theta_0)(\delta\theta_0)] + \mathcal{O}(\alpha^2) \\
=& \frac{1}{T}\int_0^T dt \int dx\{(\partial_t\theta_1')(\partial_t\delta\theta_0) + (\partial_x\theta_1')(\partial_x\delta\theta_0) + \alpha[(\partial_x^2\theta_0)\cos(2\theta_0) - (\partial_x\theta_0)^2\sin(2\theta_0)]\delta\theta_0\} + \mathcal{O}(\alpha^2) \\
=& \frac{1}{T}\int_0^T dt \int dx[(\partial_t\theta_1')(\partial_t\delta\theta_0) + (\partial_x\theta_1')(\partial_x\delta\theta_0) + (-\partial_t^2\theta_1 + \partial_x^2\theta_1)\delta\theta_0] + \mathcal{O}(\alpha^2) \\
=& \frac{2}{T}\int_0^T dt \int dx(\partial_x\theta_2)(\partial_x\delta\theta_0) + \mathcal{O}(\alpha^2),
\end{aligned} \tag{IV.8}$$

where we neglect boundary contributions in the right-hand side, e.g.

$$\begin{aligned}
\frac{1}{T}\int_0^T (\partial_t\theta_0)(\partial_t\delta\theta_1) &= \frac{k_0 j_0}{T}\int_0^T (\partial_t\delta\theta_1) = \mathcal{O}(T^{-1}), \\
\int dx(\partial_x\theta_0)(\partial_x\delta\theta_1) &= -j_0\int dx(\partial_x\delta\theta_1) = \mathcal{O}(1) \ll \mathcal{O}(L).
\end{aligned} \tag{IV.9}$$

From the 3rd line to the 4th line, we use Eq. (III.11). From the 4th line to the last line, we neglect terms that contain θ_1 and $\delta\theta_0$, because for $k_0 \neq 1$, θ_1 and $\delta\theta_0$ have different velocities (ratios between the frequency and wavenumber), and their product must vanish under the spacetime integral. The second-order variation is

$$\begin{aligned}
\frac{1}{2}\delta^2 J =& \int dtdx\{\frac{1}{2}(\partial_t\delta\theta)^2 + \frac{1}{2}(\partial_x\delta\theta)^2[1-\alpha\cos(2\theta)] - \alpha(\delta\theta)^2[(\partial_x\theta)^2\cos(2\theta) + (\partial_x^2\theta)\sin(2\theta)]\} \\
=& \int dtdx\{\frac{1}{2}(\partial_t\delta\theta)^2 + \frac{1}{2}(\partial_x\delta\theta)^2 - \frac{\alpha}{2}(\partial_x\delta\theta_0)^2\cos(2\theta_0) \\
&- \alpha(\delta\theta_0)^2[(\partial_x\theta_0)^2\cos(2\theta_0) + (\partial_x^2\theta_0)\sin(2\theta_0)]\} + \mathcal{O}(\alpha^2) \\
=& \int dtdx\{\frac{1}{2}(\partial_t\delta\theta)^2 + \frac{1}{2}(\partial_x\delta\theta)^2 \\
&- \frac{\alpha}{2}(\partial_x\delta\theta_0)^2\cos(2\theta_0) - \alpha(\delta\theta_0)^2(\partial_x\theta_0)^2\cos(2\theta_0)\} + \mathcal{O}(\alpha^2).
\end{aligned} \tag{IV.10}$$

Note that $\frac{1}{2}\delta^2 J \geq 0$ at $\mathcal{O}(\alpha)$. This is because the leading order term ($\mathcal{O}(1)$-term) is positive semi-definite, and negative contributions come from $\mathcal{O}(\alpha)$ terms. Besides, under the spacetime integral, the $\mathcal{O}(\alpha)$ terms can be nonzero only if $\delta\theta_0$ in Eq. (IV.6) comprises of (more than) two Fourier components, q_1, q_2, $\cdots$, and an oscillation function from $(\delta\theta_0)^2$ and that from $\cos(2\theta_0) = \cos(2j_0k_0t - 2j_0x)$ cancel each other, e.g.

$$q_1 - q_2 = \pm j_0 k_0, \quad q_1 + q_2 = \pm j_0. \tag{IV.11}$$

In the presence of such components in $\delta\theta_0$, however, the leading order term is positive definite.

$\delta\theta_0$, as well as θ_2, is a solution of Eq. (III.5); both are given by linear superpositions of $e^{iq(t+x)}$ and $e^{iq(t-x)}$ over q. Thus, for given $\theta_2 \neq 0$, one can always choose $\delta\theta_0$ such that the spacetime integral of $(\partial_x\theta_2)(\partial_x\delta\theta_0)$ remains non-zero and negative, $\delta J < 0$. This suggests that the superflow state is stable only when $\theta_2(x,t) = 0$, while it is not stable when $\theta_2(x,t) \neq 0$.

The same conclusion holds true in the spin-injection model with the circular geometry. In the one-dimensional spin-injection model with finite curvature, the Hamiltonian is given by

$$\mathcal{H} = \int d\ell \Big\{ \frac{1}{2}(\partial_t\theta)^2 + \frac{1}{2}(\partial_\ell\theta)^2[1 + \alpha\cos(2\theta - \frac{2}{r}\ell)] \Big\}, \tag{IV.12}$$

where θ and $\theta + \delta\theta$ depend only on ℓ and t. Their energy difference ΔJ can be expanded in the powers of small local deformation $\delta\theta$. The first- and second-order variations of the energy in $\delta\theta$ are

$$\delta J = \frac{1}{T}\int_0^T dt \int d\ell \Big\{ (\partial_t\theta)(\partial_t\delta\theta) + (\partial_\ell\theta)(\partial_\ell\delta\theta)[1 + \alpha\cos(2\theta - \frac{2}{r}\ell)] - \alpha(\partial_\ell\theta)^2\sin(2\theta - \frac{2}{r}\ell)(\delta\theta) \Big\}, \tag{IV.13}$$

$$\begin{aligned}\frac{1}{2}\delta^2 J =& \frac{1}{T}\int_0^T dt \int d\ell \Big\{ \frac{1}{2}(\partial_t\delta\theta)^2 + \frac{1}{2}(\partial_\ell\delta\theta)^2 \\ &+ \alpha(\partial_\ell\delta\theta)^2\cos(2\theta - \frac{2}{r}\ell) + \alpha(\delta\theta)^2\{[(\partial_\ell\theta)^2 - \frac{2}{r}(\partial_\ell\theta)]\cos(2\theta - \frac{2}{r}\ell) + (\partial_\ell^2\theta)\sin(2\theta - \frac{2}{r}\ell)\} \Big\},\end{aligned} \tag{IV.14}$$

respectively. Eqs. (IV.13,IV.14) have a similar structure as Eqs. (IV.8,IV.10), respectively. For $k_0^2 \neq (1 + \frac{1}{j_0 r})^2$ (resonance point), one can use the α-expansion of θ and $\delta\theta$, and the expressions support the same conclusion in the circular geometry case; $\delta J < 0$ for some $\delta\theta_0$ and $\frac{1}{2}\delta^2 J \geq 0$.

In summary, contrary to the conventional superfluid with $\theta_1 = \theta_2 = 0$, the supercurrent state with the broken U(1) spacetime symmetry is classically unstable toward other states, and it must experience the energy dissipation by local perturbation. Physically speaking, the difference in the stabilities between these two types of superfluids comes from the fact that the spin-injection boundary condition does not break the U(1) symmetry of the conventional superfluid, but it breaks the U(1) spacetime symmetry of the total-angular-momentum superfluid; under the U(1) spacetime rotation, the whole junction should also be rotated. Effects of the energy dissipation can be included as finite relaxation time T^{-1}, while the motion of θ with vanishing or small T_1^{-1} can be realized only in a superclean limit.

A conventional superfluid described by a non-relativistic complex field has a critical velocity given by the Landau criterion. Below the critical velocity, a supercurrent is stable. The Landau criterion should be derived from a theory of the complex field instead of an effective theory of a Goldstone mode. This is because when the velocity approaches the critical value, a low-energy condition is already violated. Our analyses only study the stability in the low-energy limit where a non-relativistic complex field and a relativistic complex field both lead to a Goldstone mode with linear dispersion.

V. EFFECTS OF DISSIPATION IN THE CLASSICAL EOM

Note first that the classical equation, Eq. (II.5), as well as its 1D descendants, Eqs. (III.1,III.34), are all invariant under the time-reversal operation; $t \to -t$, and $\theta \to \theta + \pi$. In the previous section, we demonstrate that the classical energy of the spin supercurrent state is higher than other states due to the finite α. This suggests that the supercurrent state decays into other states with lower energy. Such an energy-non-conserving decay process generally breaks the time-reversal symmetry of the classical equation. To study the effect of the decay process into the spin hydrodynamics predicted in Sec. III, we include the simplest time-reversal-breaking term, $\partial_t\theta$, into the classical equation;

$$\begin{aligned}&\partial_t^2\theta - (\partial_x^2)\theta[1 - \alpha\cos(2\theta)] - (\partial_y^2\theta)[1 + \alpha\cos(2\theta)] + 2\alpha(\partial_x\partial_y\theta)\sin(2\theta) \\ &- \alpha[(\partial_x\theta)^2 - (\partial_y\theta)^2]\sin(2\theta) + 2\alpha(\partial_x\theta)(\partial_y\theta)\cos(2\theta) = -\frac{1}{T_1}\partial_t\theta.\end{aligned} \tag{V.1}$$

From the symmetry point of view, one could also add other time-reversal breaking terms that respect the U(1) spacetime symmetry but breaks the time-reversal symmetry, e.g.

$$\cdots = -\frac{1}{T_1}\partial_t\theta - \frac{1}{T_2}\partial_t\theta \times (\partial_x^2\theta + \partial_y^2\theta) - \frac{1}{T_3}\partial_t\theta \times \left\{(\partial_x^2\theta - \partial_y^2\theta)\cos(2\theta) + 2\partial_x\partial_y\theta\sin(2\theta)\right\} + \cdots \tag{V.2}$$

Nonetheless, the first term on the right-hand side always dominates the others in the hydrodynamic regime, since the physical variable θ changes much more slowly than any microscopic length scales in the hydrodynamic regime, and in this sense, the other terms in Eq. (V.2) are higher-order spatial gradient terms than the first term in Eq. (V.2). In this section, we will solve Eq. (V.1) or its 1D descendant in the spin-injection model with the straight geometry,

$$\partial_t^2\theta - (\partial_x^2\theta)[1 - \alpha\cos(2\theta)] - \alpha(\partial_x\theta)^2\sin(2\theta) = -\frac{1}{T_1}\partial_t\theta. \tag{V.3}$$

$\theta_0(x,t)$ with the dissipation term was previously solved by Ref. [11, 13]. It satisfies

$$\partial_t^2\theta_0 + \frac{1}{T_1}\partial_t\theta_0 = \partial_x^2\theta_0. \tag{V.4}$$

The general solution (up to a constant F_0) of Eq. (V.4) is

$$\theta_0(x,t) = At + \frac{A}{2T_1}x^2 + Bx + \sum_{c\in\mathbb{R}, c\neq 0}[F_c e^{ict - i\kappa_c x} + F'_c e^{ict + i\kappa_c x}], \tag{V.5}$$

where A, B, F_c, F'_c are constants and

$$\kappa_c = \sqrt{c^2 + i\frac{c}{T_1}}. \tag{V.6}$$

The boundary conditions Eqs. (III.2,III.3) are satisfied by

$$B = -j_0, \quad A = -k_0(\frac{A}{T_1}L + B), \quad F_c = F'_c = 0, \tag{V.7}$$

which leads to

$$A = (1 + \frac{k_0 L}{T_1})^{-1}k_0 j_0. \tag{V.8}$$

This gives the zeroth order solution of the EOM with the BCs,

$$\theta_0(x,t) = \frac{T_1 k_0 j_0}{T_1 + k_0 L}t - j_0[1 - \frac{k_0 x}{2(T_1 + k_0 L)}]x \equiv \tilde{k}_0 j_0 t - j_0 h(x)x, \tag{V.9}$$

$$j_x^s(0 < x < L) = \frac{T_1 + k_0(L - x)}{T_1 + k_0 L}j_0 + \mathcal{O}(\alpha), \quad s(0 < x < L) = \frac{T_1 k_0 j_0}{T_1 + k_0 L} + \mathcal{O}(\alpha). \tag{V.10}$$

In the conventional spin superfluid with $T_1^{-1} \neq 0$, the spin density and the spin current are static. Different from the dissipationless case ($T^{-1} = 0$), the spin current decreases linearly in the one-dimensional coordinate x, while the spin density is uniform in x.

Due to nonlinear x-dependence of $\theta_0(x,t)$, the perturbative analyses in the SOC (α) becomes harder. To obtain the solution of the EOM analytically, we consider a limit that a phase accumulation γ is small when the spatial dependence of the current is small,

$$\gamma \equiv [-\frac{dh(x)}{dx}L]j_0 L = \frac{k_0 j_0 L^2}{2(T_1 + k_0 L)} \ll 1. \tag{V.11}$$

The small γ limit can be achieved by a small dissipation term or a short propagation distance. The zeroth order solution in the small γ and α limit is

$$\theta_0(x,t) = \tilde{k}_0 j_0(1 - \frac{k_0 L}{T_1})t - j_0 h(x)x = \tilde{k}_0 j_0 t - j_0 x + \mathcal{O}(\gamma). \tag{V.12}$$

We will solve $\theta(x,t)$ up to the first order in α or in γ, namely $\mathcal{O}(\alpha,\gamma)$. Thereby, we keep the zeroth order of $\theta_0(x,t)$ when solving $\theta_1(x,t)$ and $\theta_2(x,t)$. Eq. (III.11) is slightly modified and becomes

$$\partial_t^2\theta_1+\frac{1}{T_1}\partial_t\theta_1-\partial_x^2\theta_1=\alpha j_0^2\mathrm{sin}(2\tilde{k}_0j_0t-2j_0x)=\frac{\alpha j_0^2}{2i}e^{2i\tilde{k}_0j_0t-2ij_0x}+\text{c.c.}. \tag{V.13}$$

Eq. (V.13) has a special solution,

$$\begin{aligned}\theta_1(x,t)=&\frac{\alpha j_0^2e^{2i\tilde{k}_0j_0t-2ij_0x}}{2i(-4\tilde{k}_0^2j_0^2-\frac{2i}{T_1}\tilde{k}_0j_0+4j_0^2)}+\text{c.c.}=-\frac{\alpha j_0T_1e^{2i\tilde{k}_0j_0t-2ij_0x}}{8ij_0T_1(\tilde{k}_0^2-1)+4\tilde{k}_0}+\text{c.c.}\\ \equiv&\alpha g_0e^{2ij_0(\tilde{k}_0t-x)}+\text{c.c.},\end{aligned} \tag{V.14}$$

where

$$g_0=-\frac{j_0T_1}{8ij_0T_1(\tilde{k}_0^2-1)+4\tilde{k}_0}. \tag{V.15}$$

Eq. (III.16) holds true, while Eqs. (III.17,III.18) are modified,

$$\begin{aligned}s=&\frac{1}{2}\tilde{k}_0j_0+\alpha\tilde{k}_0j_0e^{2i\tilde{k}_0j_0t}[2ige^{-2i\tilde{k}_0j_0x}-\frac{j_0T_1e^{-2ij_0x}}{4j_0T_1(\tilde{k}_0^2-1)-2i\tilde{k}_0}]\\ &+2i\alpha\tilde{k}_0j_0g'e^{2i\tilde{k}_0j_0(t+x)}+\mathcal{O}(\alpha^2,\gamma^2,\alpha\gamma)+\text{c.c.},\end{aligned} \tag{V.16}$$

$$\begin{aligned}j_x^s=&\frac{1}{2}j_0+\alpha j_0e^{2i\tilde{k}_0j_0t}[2i\tilde{k}_0ge^{-2i\tilde{k}_0j_0x}-\frac{j_0T_1e^{-2ij_0x}}{4j_0T_1(\tilde{k}_0^2-1)-2i\tilde{k}_0}-\frac{e^{-2ij_0x}}{2}]\\ &-2i\alpha\tilde{k}_0j_0g'e^{2i\tilde{k}_0j_0(t+x)}+o(\alpha,\gamma,\gamma j_0L)+\text{c.c.}\\ =&\frac{1}{2}j_0+\alpha j_0e^{2i\tilde{k}_0j_0t}\{2i\tilde{k}_0ge^{-2i\tilde{k}_0j_0x}-\frac{[j_0T_1(2\tilde{k}_0^2-1)-i\tilde{k}_0]e^{-2ij_0x}}{4j_0T_1(\tilde{k}_0^2-1)-2i\tilde{k}_0}\}\\ &-2i\alpha\tilde{k}_0j_0g'e^{2i\tilde{k}_0j_0(t+x)}+\mathcal{O}(\alpha^2,\gamma^2,\alpha\gamma)+\text{c.c.}.\end{aligned} \tag{V.17}$$

The boundary conditions Eqs. (III.2,III.3) leads to the following secular equation (cf. Eq. (III.25)),

$$\begin{aligned}&\begin{pmatrix}1 & -1\\ (1-k)e^{-i\tilde{k}_0\beta_L} & (1+k)e^{i\tilde{k}_0\beta_L}\end{pmatrix}\begin{pmatrix}g\\ g'\end{pmatrix}\\ =&\frac{1}{8i\tilde{k}_0j_0T_1(\tilde{k}_0^2-1)+4\tilde{k}_0^2}\begin{pmatrix}j_0T_1(2\tilde{k}_0^2-1)-i\tilde{k}_0\\ (j_0T_1-j_0T_1(2\tilde{k}_0^2-1)k+i\tilde{k}_0k)e^{-i\beta_L}\end{pmatrix}.\end{aligned} \tag{V.18}$$

The solution of Eq. (V.18) is

$$g=\frac{[j_0T_1(2\tilde{k}_0^2-1)-i\tilde{k}_0](1+k)e^{i\tilde{k}_0\beta_L}+[j_0T_1-j_0T_1(2\tilde{k}_0^2-1)k+i\tilde{k}_0k]e^{-i\beta L}}{[(1+k)e^{i\tilde{k}_0\beta_L}+(1-k)e^{-i\tilde{k}_0\beta_L}][8i\tilde{k}_0j_0T_1(\tilde{k}_0^2-1)+4\tilde{k}_0^2]}, \tag{V.19}$$

$$g'=\frac{[j_0T_1-j_0T_1(2\tilde{k}_0^2-1)k+i\tilde{k}_0k]e^{-i\beta_L}-[j_0T_1(2\tilde{k}_0^2-1)-i\tilde{k}_0](1-k)e^{-i\tilde{k}_0\beta L}}{[(1+k)e^{i\tilde{k}_0\beta_L}+(1-k)e^{-i\tilde{k}_0\beta_L}][8i\tilde{k}_0j_0T_1(\tilde{k}_0^2-1)+4\tilde{k}_0^2]}. \tag{V.20}$$

Similar to Eq. (III.28), the solution of $\theta(x,t)$ is

$$\begin{aligned}\theta(x,t)=&\tilde{k}_0j_0t-j_0[1-\frac{k_0x}{2(T_1+k_0L)}]x\\ &+2\alpha\mathrm{Re}(g_0)\mathrm{cos}[2j_0(\tilde{k}_0t-x)]-2\alpha\mathrm{Im}(g_0)\mathrm{sin}[2j_0(\tilde{k}_0t-x)]\\ &+2\alpha\mathrm{Re}(g)\mathrm{cos}[2\tilde{k}_0j_0(t-x)]+2\alpha\mathrm{Re}(g')\mathrm{cos}[2\tilde{k}_0j_0(t+x)]\\ &-2\alpha\mathrm{Im}(g)\mathrm{sin}[2\tilde{k}_0j_0(t-x)]+2\alpha\mathrm{Im}(g')\mathrm{sin}[2\tilde{k}_0j_0(t+x)]+\mathcal{O}(\alpha^2,\gamma^2,\alpha\gamma),\end{aligned} \tag{V.21}$$

where g_0, g, g' are given by Eqs. (V.15,V.19,V.20). k_0, k, and $\tilde{k}_0$ are given by Eqs. (III.21,III.22,V.9). Compared to Eqs. (III.25,III.43), Eqs. (V.19,V.20,V.21) have no divergence due to finite T_1. The situation is analogous to periodically driven harmonic oscillators, where the dissipation removes divergence due to resonance [43]. Note also that the solution Eq. (V.21) has a periodicity in time, $\pi(\tilde{k}_0 j_0)^{-1}$, and two characteristc wavelengths, $2j_0$, and $2\tilde{k}_0 j_0$. A solution at higher order in γ has all spatial Fourier components, while it is still periodic in time with the same periodicity.

VI. SPECIAL PARAMETER POINTS IN THE SPIN-INJECTION MODEL

In this section, we study some special parameter points in the spin-injection model without the spin relaxation term, where solutions in Sec. III do not apply directly and need careful investigations.

A. $k_0 = 1$ (straight geometry) and $k_0^2 = (1 + \frac{1}{j_0 r})^2$ (circular geometry)

Consider $k_0 = 1$ in the straight geometry and $k_0^2 = (1+\frac{1}{j_0 r})^2$ in the circular geometry. Naive substitutions of $k_0 = 1$ into Eqs. (III.13,III.25) and of $k_0^2 = (1+\frac{1}{j_0 r})^2$ into Eqs. (III.38,III.43) lead to divergences in θ_1 and θ_2, respectively. It seems that the divergences in θ_1 and θ_2 cancel each other. For example, Eqs. (III.30,III.31) at $k_0 = 1$ become

$$\eta|_{k_0=1} = \frac{[(1+k)e^{i\beta_L} - (1-k)e^{-i\beta_L}] + 2(1-k)e^{-i\beta_L}}{4(k_0^2-1)[(1+k)e^{i\beta_L} + (1-k)e^{-i\beta_L}]} = \frac{1}{4(k_0^2-1)}, \tag{VI.1}$$

$$\begin{aligned} \theta(x,t)|_{k_0=1} =& j_0(t-x) - \frac{\alpha}{4(k_0^2-1)}\sin[2j_0(t-x)] - \frac{\alpha}{4(k_0^2-1)}\cos(2j_0 t)\sin(2j_0 x) \\ &+ \frac{\alpha}{4(k_0^2-1)}\sin(2j_0 t)\cos(2j_0 x) + \mathcal{O}(\alpha^2) \\ =& j_0(t-x) + \mathcal{O}(\alpha^2), \end{aligned} \tag{VI.2}$$

where the final result of $\theta(x,t)$ is apparently finite. However, Eq. (VI.2) is not a solution to the equation of motion Eq. (III.1). In fact, with light-cone coordinates

$$\xi = t - x, \quad \zeta = t + x, \tag{VI.3}$$

Eq. (III.11) at $k_0 = 1$ becomes

$$[(\partial_\xi + \partial_\zeta)^2 - (\partial_\xi - \partial_\zeta)^2]\theta_1 = 4\partial_\xi\partial_\zeta\theta_1 = \alpha j_0^2 \sin(2j_0\xi). \tag{VI.4}$$

Eq. (VI.4) has a special solution which is not consistent to Eq. (VI.2),

$$\theta_1 = -\frac{\alpha\zeta}{16}\cos(2j_0\xi) = -\frac{\alpha(t+x)}{16}\cos[2j_0(t-x)]. \tag{VI.5}$$

Note that $|\theta_1|$ in Eq. (VI.5) is not bounded for large $\zeta = t + x$. This indicates that the perturbation with respect to α becomes invalid at $k_0 = 1$, leading to the discrepancy. The divergences at $k_0 = 1$ can be regarded as the resonance of the inhomogeneous linearized differential equation [43], and one can expect that the SOC has non-perturbative effects around $k_0 = 1$.

To understand the origin of the non-perturbative effect of α, let us consider a set of solutions of Eq. (III.1) that depends on x and t only through $x - vt$. For the later comparison to a special solution developed in Sec. III, $\theta_0 + \theta_1 + \mathcal{O}(\alpha^2)$, let v to be k_0,

$$\theta(x,t) = \theta(x - k_0 t), \quad \theta' \equiv \partial_x\theta = -\frac{1}{k_0}\partial_t\theta. \tag{VI.6}$$

Here, the prime denotes an x-derivative. Eq. (III.1) effectively becomes an ordinary differential equation,

$$(k_0^2 - 1)\theta'' = -\alpha\theta''\cos(2\theta) + \alpha\theta'^2\sin(2\theta). \tag{VI.7}$$

To solve this equation, use its analogy to one-dimensional classical mechanics, where the phase $\theta(x)$ as a function of x corresponds to a one-dimensional coordinate as a function of time. The classical mechanics for the one-dimensional coordinate has a Lagrangian whose variation gives Eq. (VI.7) as a classical EOM,

$$L_{\rm 1D} = \frac{1}{2}(k_0^2 - 1)\theta'^2 + \frac{\alpha}{2}\theta'^2 \cos(2\theta). \tag{VI.8}$$

The classical mechanics has a canonical momentum conjugate to the coordinate,

$$\pi = \frac{\partial L_{\rm 1D}}{\partial \theta'} = (k_0^2 - 1)\theta' + \alpha\theta' \cos(2\theta), \tag{VI.9}$$

as well as a conserved Hamiltonian,

$$H_{\rm 1D} = \pi\theta' - L_{\rm 1D} = \frac{1}{2}(k_0^2 - 1)\theta'^2 + \frac{\alpha}{2}\theta'^2 \cos(2\theta). \tag{VI.10}$$

Utilizing the x-independence ("time"-independence) of $H_{\rm 1D}$, we can solve the EOM from Eq. (VI.10),

$$\frac{d\theta}{dx} = \pm\sqrt{\frac{2H_{\rm 1D}}{k_0^2 - 1 + \alpha\cos(2\theta)}}. \tag{VI.11}$$

Its formal solution is given by

$$\pm(x - x_0) = \int_{\theta(x_0)}^{\theta(x)} \sqrt{\frac{k_0^2 - 1 + \alpha\cos(2\theta)}{2H_{\rm 1D}}} d\theta. \tag{VI.12}$$

With Eq. (VI.6), we get a set of (1+1)-dimensional solutions,

$$\pm(x - k_0 t - C_0) = \int_{\theta(x_0,t_0)}^{\theta(x,t)} \sqrt{\frac{k_0^2 - 1 + \alpha\cos(2\theta)}{2H_{\rm 1D}}} d\theta, \tag{VI.13}$$

where $C_0 = x_0 - k_0 t_0$.

Eq. (VI.13) is inclusive of those perturbative solutions in Sec. III that depend on x and t only through $x - k_0 t$, i.e. $\theta_0 + \theta_1$ given by Eqs. (III.8,III.13). Namely, when $\alpha \ll |k_0^2 - 1|$, we can apply an expansion in α,

$$\pm(x - k_0 t - C_0) = \int_{\theta(x_0,t_0)}^{\theta(x,t)} \sqrt{\frac{k_0^2 - 1}{2H_{\rm 1D}}} [1 + \frac{\alpha}{2(k_0^2 - 1)} \cos(2\theta)] d\theta + \mathcal{O}(\alpha^2). \tag{VI.14}$$

From this, we obtain

$$\begin{aligned} \theta(x,t) =& \pm\sqrt{\frac{2H_{\rm 1D}}{k_0^2 - 1}}(x - k_0 t - C_0') - \frac{\alpha}{4(k_0^2 - 1)}\sin(2\theta) + \mathcal{O}(\alpha^2) \\ =& \theta_0 - \frac{\alpha}{4(k_0^2 - 1)}\sin(2\theta_0) + \mathcal{O}(\alpha^2), \end{aligned} \tag{VI.15}$$

where

$$\theta_0 = \pm\sqrt{\frac{2H_{\rm 1D}}{k_0^2 - 1}}(x - k_0 t - C_0'), \tag{VI.16}$$

$$C_0' = C_0 \mp \sqrt{\frac{k_0^2 - 1}{2H_{\rm 1D}}} \{\theta(x_0, t_0) + \frac{\alpha}{4(k_0^2 - 1)} \sin[2\theta(x_0, t_0)]\}. \tag{VI.17}$$

Note that due to the absence of $\theta_2(x,t)$ in its α-expansion, Eq. (VI.15) does not satisfy the boundary conditions in the spin-injection model in general.

Nonetheless, Eq. (VI.13) is still useful to see that the expansion in α is invalid when $|k_0^2-1|<\mathcal{O}(\alpha)$. Take $k_0=1$ in Eq. (VI.13) as an example. Thereby, the sign of $\cos(2\theta)$ is conserved from Eq. (VI.10). Then, the integral of Eq. (VI.13) gives

$$\pm\sqrt{\frac{2H_{1\mathrm{D}}}{\alpha}}(x-k_0t-C_0)=E(\theta(x,t),2)-E(\theta(x_0,t_0),2), \tag{VI.18}$$

where $E(\theta,m)$ is the second-kind elliptic integral. This special solution can be made independent from α because α can be absorbed into another parameter $H_{1\mathrm{D}}$. The α-independence as well as the conserved sign of $\cos(2\theta)$ are not consistent with $\theta_0(x,t)+\theta_1(x,t)$ in Eqs. (III.8,III.13). This suggests that the expansion in α is invalid, particularly at $k_0=1$.

B. $k_0=0$ (straight geometry)

Another special parameter point is $k_0=0$. $k_0=0$ is a limit where χ in Eq. (III.21) is much smaller than $\chi'\sqrt{\frac{D_s}{T_1'}}$ and β_t, and k_c goes to zero for any c;

$$k_c\equiv\frac{\chi}{\chi'}\sqrt{\frac{1}{D_s(\frac{1}{T_1'}+ic)}}+\frac{\chi}{\beta}\to 0. \tag{VI.19}$$

We first consider $k_0=0$ in the straight geometry ($r^{-1}=0$). θ_0 at $k_0=0$ has no time dependence in Eq. (III.8), so that the phase F_0 in Eq. (III.8) cannot be absorbed into the time. Meanwhile, $s(x=L-)=0$ because $k_c=0$ for any c, and $s(x)\equiv\partial_t\theta=0$ can be always satisfied by a time-independent θ. Thus, we have only to make Eqs. (III.8,III.10) with an additional F_0 to satisfy the other boundary condition, $j_x^s(x=0+)=j_0$. Firstly, let us choose $\theta_0=-j_0x+F_0$ that satisfies the boundary condition. The substitution into Eq. (III.11) gives $\theta_1=\frac{\alpha}{4}\sin(-2j_0x+F_0)$. To satisfy the boundary condition up to the 1st order in α (see also Eq. (III.4)),

$$j_x^s(x=0+)=j_0-\frac{j_0\alpha}{2}\cos(2F_0)-\partial_x\theta_2+\mathcal{O}(\alpha^2)=j_0, \tag{VI.20}$$

we have

$$\theta_2(x)=-\frac{j_0\alpha}{2}\cos(2F_0)x. \tag{VI.21}$$

Here, $|\theta_2(x)|$ for large x is not bounded, where the perturbation in α breaks down. An alternative way to get a consistent perturbative solution is to require $\theta_2(x)=0$ and take $\theta_0(x)=-\tilde{j}_0x+F_0$ where $\tilde{j}_0\neq j_0$. Then we have

$$j_x^s(x=0+)=\tilde{j}_0-\frac{\tilde{j}_0\alpha}{2}\cos(2F_0)-\partial_x\theta_2+\mathcal{O}(\alpha^2)=j_0. \tag{VI.22}$$

From this, we get

$$\tilde{j}_0=j_0[1-\frac{\alpha}{2}\cos(2F_0)]^{-1}=j_0[1+\frac{\alpha}{2}\cos(2F_0)]+\mathcal{O}(\alpha^2). \tag{VI.23}$$

At $\mathcal{O}(\alpha)$, this solution is equivalent to absorbing $\theta_2(x)$ in Eq. (VI.21) into $\theta_0(x)$. Note also that from Eq. (IV.8), $\theta_2(x)=0$ implies that the steady (but not uniform) current without spin accumulation has no energy dissipation. However, a higher-order calculation in α suggests that δJ can be non-zero and negative. This is because $\delta\theta_1$ and θ_1 can share identical Fourier components (see Eq. (B.35) in Appendix B).

C. $k_0=0$ (circular geometry)

Let us next consider the circular geometry with $k_0=0$. In a similar way as in the previous section, take $\theta_2(\ell)=0$ and $\theta_0(\ell)=-\tilde{j}_0\ell+F_0$ with $\tilde{j}_0\neq j_0$. Then, similar to Eq. (20) in the main text, we obtain

$$-\partial_\ell^2\theta_1=-\alpha\tilde{j}_0(\tilde{j}_0+\frac{2}{r})\sin(-2\tilde{j}_0\ell-\frac{2}{r}\ell+2F_0). \tag{VI.24}$$

When $\tilde{j}_0 \neq -\frac{1}{r}$, Eq. (VI.24) leads to

$$\theta_1(\ell) = \frac{\alpha \tilde{j}_0(\tilde{j}_0 + \frac{2}{r})}{4(\tilde{j}_0 + \frac{1}{r})^2}\sin(2\tilde{j}_0\ell + \frac{2}{r}\ell - 2F_0). \tag{VI.25}$$

Note that when $\tilde{j}_0 = -\frac{2}{r}$, $\theta_1(\ell) = 0$, and $\theta(\ell) = \frac{2\ell}{r} + F_0$ ($F_0 \in \mathbb{R}$) becomes an “exact” solution of Eq. (III.34). The solution is not exact in the presence of higher-order derivatives in the model Eq. (6) in the main text.

From Eq. (VI.25) and $\theta_2 = 0$, the boundary condition at $\ell = 0+$ reads

$$\begin{aligned} j_\ell^s(\ell = 0+) = j_0 =& \tilde{j}_0 + \alpha\tilde{j}_0\cos(2F_0) - \frac{\alpha\tilde{j}_0(\tilde{j}_0 + \frac{2}{r})}{2(\tilde{j}_0 + \frac{1}{r})}\cos(2F_0) + \mathcal{O}(\alpha^2) \\ =& \tilde{j}_0 + \frac{\alpha\tilde{j}_0^2}{2(\tilde{j}_0 + \frac{1}{r})}\cos(2F_0) + \mathcal{O}(\alpha^2). \end{aligned} \tag{VI.26}$$

So we have

$$\tilde{j}_0 = j_0 - \frac{\alpha\tilde{j}_0^2}{2(\tilde{j}_0 + \frac{1}{r})}\cos(2F_0) + \mathcal{O}(\alpha^2). \tag{VI.27}$$

As in the previous case, the steady current without spin accumulation has an energy dissipation.

1. $k_0 = 0$, $\tilde{j}_0 = -\frac{1}{r}$ *(circular geometry)*

When $\tilde{j}_0 = -\frac{1}{r}$, Eq. (VI.24) leads to

$$\theta_1 = -\frac{\ell^2}{2}\alpha\tilde{j}_0(\tilde{j}_0 + \frac{2}{r})\sin(2F_0) = \frac{\ell^2\alpha}{2r^2}\sin(2F_0). \tag{VI.28}$$

When $F_0 \neq \frac{N}{2}\pi$, the perturbation of α becomes invalid for larger ℓ, where $|\theta_1(\ell)|$ is not bounded. When $F_0 = \frac{N}{2}\pi$, $\theta_1(\ell) = 0$, and the boundary condition requires

$$j_0 = \tilde{j}_0[1 + \alpha(-1)^N] = -\frac{1}{r}[1 + \alpha(-1)^N]. \tag{VI.29}$$

This solution ($\theta(\ell, t) = \frac{\ell}{r} + \frac{N\pi}{2}$) can exist only at $k_0 = 0$, while it is not continuously connected to a solution at finite small k_0 (see below).

2. $k_0 \to 0$, $j_0 = -\frac{1}{r} + \mathcal{O}(\alpha)$ *(circular geometry)*

To see that the solution Eq. (VI.29) is not continuously connected to a perturbative solution at finite small k_0, let us keep a finite k_0 and choose $j_0 = -\frac{1}{r} + \mathcal{O}(\alpha)$ in Sec. III; $\theta_0(\ell) = \frac{1}{r}(k_0 t - \ell) + \mathcal{O}(\alpha)$. In the perturbation theory, we should neglect $\mathcal{O}(\alpha^2)$ contributions to θ_1, so θ_1 is not affected by the $\mathcal{O}(\alpha)$ component in j_0. Then Eq. (20) in the main text becomes

$$\partial_t^2\theta_1 - \partial_\ell^2\theta_1 = -\frac{\alpha}{r^2}\sin(-\frac{2k_0}{r}t), \tag{VI.30}$$

which leads to a solution,

$$\theta_1 = -\frac{\alpha}{4k_0^2}\sin(\frac{2k_0}{r}t). \tag{VI.31}$$

When $k_0 \to 0$, the solution has the divergence. Physically speaking, when $k_0 \to 0$, F_0 changes slowly with respect to time, so we cannot fix the phase F_0 in Eq. (VI.28).

A. APPENDIX: DERIVATION OF THE LANDAU CRITERION

In this section, we use the same framework as in Sec. IV and derive the Landau criterion. A similar argument can be found in Ref. [13], while the derivation here is more formal than Ref. [13]. We begin with a one-dimensional superfluid model,

$$\tilde{\mathcal{L}}_\phi = i\hbar\phi^\dagger\partial_t\phi - \frac{\hbar^2}{2m}(\partial_x\phi^\dagger)(\partial_x\phi) - \frac{U}{2}(\phi^\dagger\phi)^2 + \mu\phi^\dagger\phi, \tag{A.1}$$

with its classical equation of motion,

$$i\hbar\partial_t\phi = (-\frac{\hbar^2}{2m}\partial_x^2 + U\phi^\dagger\phi - \mu)\phi. \tag{A.2}$$

Thanks to the Galilean covariance of the Lagrangian Eq. (A.1), our discussion will be easier than Sec. IV, and we do not need to expand the variation of the motion as in Sec. IV. Namely, using the Galilean covariance, we can directly obtain two motions that are close to each other, and compare their energies from the corresponding Hamiltonian,

$$\tilde{H}_\phi[\phi] = \int dx[\frac{\hbar^2}{2m}(\partial_x\phi^\dagger)(\partial_x\phi) + \frac{U}{2}(\phi^\dagger\phi)^2 - \mu\phi^\dagger\psi]. \tag{A.3}$$

Consider a steady flow $\phi_0(x,t)$,

$$\phi_0(x,t) = \sqrt{\rho_0}\exp\left[\frac{i}{\hbar}(mvx - \frac{mv^2}{2}t)\right]. \tag{A.4}$$

Let us assume that the following motion $\phi(x,t)$, as well as $\phi_0(x,t)$, satisfies the EOM, Eq. (A.2),

$$\phi(x,t) = \phi'(x',t')\exp\left[\frac{i}{\hbar}(mvx - \frac{mv^2}{2}t)\right] = \phi'(x-vt,t)\exp\left[\frac{i}{\hbar}(mvx - \frac{mv^2}{2}t)\right], \tag{A.5}$$

with $x' = x - vt$ and $t' = t$. Here $|\phi'(x',t') - \sqrt{\rho_0}| \ll \sqrt{\rho_0}$ and $\phi(x,t)$ is close to $\phi_0(x,t)$. Using a Galilean transformation,

$$x' = x - vt, \quad t' = t, \tag{A.6}$$

$$\partial_x = \partial_{x'}, \quad \partial_t = \partial_{t'} - v\partial_x, \tag{A.7}$$

we can see that $\phi'(x',t')$ must satisfy a similar equation as Eq. (A.2),

$$(i\hbar\partial_t + \frac{1}{2}mv^2)\phi'(x',t') = [\frac{1}{2m}(-i\hbar\partial_x + mv)^2 + U\phi'^\dagger\phi' - \mu]\phi'(x',t'), \tag{A.8}$$

$$i\hbar\partial_{t'}\phi'(x',t') = (-\frac{\hbar^2}{2m}\partial_{x'}^2 + U\phi'^\dagger\phi' - \mu)\phi'(x',t'). \tag{A.9}$$

Now we compare the energies of $\phi(x,t)$ and $\phi_0(x,t)$. The energy of $\phi_0(x,t)$ is,

$$\tilde{H}_\phi[\phi_0] = \int dx[\frac{\hbar^2}{2m}(\partial_x\phi_0^\dagger)(\partial_x\phi_0) + \frac{U}{2}(\phi_0^\dagger\phi_0)^2 - \mu\phi_0^\dagger\phi_0] = E_0 + \frac{1}{2}mv^2Q_0, \tag{A.10}$$

where

$$E_0 = \int dx(\frac{U}{2}\rho_0^2 - \mu\rho_0) = -\frac{\mu}{2}\int dx\rho_0, \quad Q_0 = \int dx\rho_0. \tag{A.11}$$

The energy of $\phi(x,t)$ is,

$$\begin{aligned}\tilde{H}_\phi[\phi] &= \int dx[\frac{\hbar^2}{2m}(\partial_x\phi^\dagger)(\partial_x\phi) + \frac{U}{2}(\phi^\dagger\phi)^2 - \mu\phi^\dagger\phi] \\ &= \int dx\phi'^\dagger(x-vt,t)[\frac{1}{2m}(-i\hbar\partial_x + mv)^2 + \frac{U}{2}\phi'^\dagger\phi' - \mu]\phi'(x-vt,t) \\ &= \int dx\phi'^\dagger(x,t)[\frac{1}{2m}(-i\hbar\partial_x + mv)^2 + \frac{U}{2}\phi'^\dagger\phi' - \mu]\phi'(x,t) \\ &= \int dx\phi'^\dagger(x,t)[-\frac{\hbar^2}{2m}\partial_x^2 - iv\hbar\partial_x + \frac{1}{2}mv^2 + \frac{U}{2}\phi'^\dagger\phi' - \mu]\phi'(x,t) \\ &= \tilde{H}_\phi[\phi'] + vP_\phi[\phi'] + \frac{1}{2}mv^2Q_\phi[\phi'],\end{aligned} \tag{A.12}$$

where

$$P_\phi[\phi'] = \int dx \phi'^\dagger(x,t)(-i\hbar\partial_x)\phi'(x,t), \quad Q_\phi[\phi'] = \int dx \phi'^\dagger(x,t)\phi'(x,t). \tag{A.13}$$

Thus, the energy difference between $\phi(x,t)$ and $\phi_0(x,t)$ is

$$\Delta E_\phi[\phi,\phi_0] = \tilde{H}_\phi[\phi] - \tilde{H}_\phi[\phi_0] = (\tilde{H}_\phi[\phi'] - E_0) + vP_\phi[\phi'] + \frac{1}{2}mv^2(Q_\phi[\phi'] - Q_0). \tag{A.14}$$

For ϕ' that satisfies the equation of motion Eq. (A.9), energy $H_\phi[\phi']$, momentum $P_\phi[\phi']$, and U(1) charge $Q_\phi[\phi']$ must be all conserved. This is because from Eq. (A.9), $\phi'(x,t)$ is a solution of Eq. (A.2).

The average velocity of $\phi(x,t)$ is not small, while $\phi'(x,t)$ can be assumed at the near-equilibrium limit [40]. In this limit, the EOM of $\phi' = \sqrt{\rho_0 + \delta\rho'}e^{i\theta'}$ can be described by a wave equation of $\delta\rho'$ and θ',

$$\partial_t^2\theta'(x,t) - \frac{\rho_0 U}{m}\partial_x^2\theta'(x,t) = 0, \tag{A.15}$$

$$\delta\rho'(x,t) = -\frac{\hbar}{U}\partial_t\theta'(x,t). \tag{A.16}$$

Thereby, $\theta'(x,t)$ is given by a superposition of oscillations,

$$\theta'(x,t) = \frac{1}{\sqrt{L}}\sum_q [f_q e^{iq(x-v_c t)} + f'_q e^{iq(x+v_c t)}], \quad v_c = \sqrt{\frac{\rho_0 U}{m}}. \tag{A.17}$$

In the near-equilibirum limit of ϕ', we evaluate the energy difference in the leading order in small f_q and f'_q,

$$\begin{aligned}\tilde{H}_\phi[\phi'] - E_0 =& \int dx[\frac{\hbar^2\rho_0}{2m}(\partial_x\theta')^2 + \frac{U}{2}(\delta\rho')^2] = \int dx[\frac{\hbar^2\rho_0}{2m}(\partial_x\theta')^2 + \frac{\hbar^2}{2U}(\partial_t\theta')^2] \\ =& \frac{\hbar^2\rho_0}{m}\sum_q q^2(|f_q|^2 + |f'_q|^2),\end{aligned} \tag{A.18}$$

$$P_\phi[\phi'] = \int dx \hbar\delta\rho'(\partial_x\theta') = -\frac{\hbar^2}{U}\int dx(\partial_t\theta')(\partial_x\theta') = \frac{\hbar^2\rho_0}{mv_c}\sum_q q^2(|f_q|^2 - |f'_q|^2), \tag{A.19}$$

$$Q_\phi[\phi'] - Q_0 = \int dx\delta\rho' = 0. \tag{A.20}$$

Taking Eqs. (A.18-A.20) into Eq. (A.14), we have

$$\Delta E_\phi = \frac{\hbar^2\rho_0}{m}\sum_q q^2[(1 + \frac{v}{v_c})|f_q|^2 + (1 - \frac{v}{v_c})|f'_q|^2]. \tag{A.21}$$

To make $\Delta E_\phi \geq 0$ for any (small) f_q and f'_q, we obtain the Landau criterion,

$$|v| \leq v_c = \sqrt{\frac{\rho_0 U}{m}}. \tag{A.22}$$

B. APPENDIX: LOCAL DEFORMATIONS OF A CLASSICAL SOLUTION OF THE EOM

The superfluid state with a finite supercurrent is characterized by the solution $\theta(x,t)$ of the classical EOM in the one-dimensional spin-injection model (e.g. with the straight geometry). In Sec. IV, we introduced its local deformation $\theta(x,t) + \delta\theta(x,t)$ as another solution of the EOM with different boundary conditions. We regarded that $\delta\theta(x,t)$, as well as $\theta(x,t)$, can be determined perturbatively in the SOC (α). At the zeroth order in SOC, $\theta + \delta\theta$, as well as θ, is a solution of $\partial_t^2\theta - \partial_x^2\theta = 0$, and so is $\delta\theta$. Since $\delta\theta(x,t)$ is a local deformation and its spacetime derivatives should not contain any uniform components in space, the zeroth order of $\delta\theta(x,t)$ must be given by Eq. (IV.6);

$$\delta\theta_0(x,t) = \frac{1}{\sqrt{L}}\sum_q \left[\delta d_q e^{iq(x-t)} + \delta d_q' e^{iq(x+t)}\right]. \tag{B.1}$$

In this appendix, for a given form of Eq. (IV.6) as the zeroth order, we will show how to determine the first order of $\delta\theta(x,t)$;

$$\delta\theta(x,t) = \delta\theta_0(x,t) + \delta\theta_1(x,t) + \mathcal{O}(\alpha^2). \tag{B.2}$$

We first give a general framework to determine $\delta\theta$. $\theta + \delta\theta$, as well as θ, is a local minimum of the action, $S = \int d^3r\mathcal{L} \equiv \int dt d^2r\mathcal{L}$, and $\delta\theta$ is infinitesimally small. Thus, we take a $\delta\theta$-variation of S;

$$S[\theta] \equiv S_{xx}[\theta] + S_{yy}[\theta] + S_{xy}[\theta] + \frac{1}{2}\int d^3r(\partial_t\theta)^2, \tag{B.3}$$

$$S_{xx} \equiv -\frac{1}{2}\int d^3r(\partial_x\theta)^2[1-\alpha\cos(2\theta)],\ S_{yy} \equiv \frac{1}{2}\int d^3r(\partial_y\theta)^2[1+\alpha\cos(2\theta)],\ S_{xy} \equiv \alpha(\partial_x\theta)(\partial_y\theta)\sin(2\theta). \tag{B.4}$$

The first-order variation just gives the equation of motion Eq. (II.5),

$$\delta S = \delta S_{xx} + \delta S_{yy} + \delta S_{xy} - \int d^3r(\delta\theta)(\partial_t^2\theta), \tag{B.5}$$

$$\delta S_{xx} \equiv \int d^3r(\delta\theta)\{(\partial_x^2\theta)[1-\alpha\mathrm{cos}(2\theta)] + \alpha(\partial_x\theta)^2\mathrm{sin}(2\theta)\}, \tag{B.6}$$

$$\delta S_{yy} \equiv \int d^3r(\delta\theta)\{(\partial_y^2\theta)[1+\alpha\mathrm{cos}(2\theta)] - \alpha(\partial_y\theta)^2\mathrm{sin}(2\theta)\}, \tag{B.7}$$

$$\delta S_{xy} \equiv -2\alpha\int d^3r(\delta\theta)[(\partial_x\partial_y\theta)\mathrm{sin}(2\theta) + (\partial_x\theta)(\partial_y\theta)\mathrm{cos}(2\theta)]. \tag{B.8}$$

δS vanishes since θ is an extremum or a saddle point of S. The second-order variation $\delta^2 S$ determines small deformation $\delta\theta$ in such a way that $\theta+\delta\theta$ is an extremum or a saddle point of S. S_{xx} gives

$$\begin{aligned}\delta^2 S_{xx} =& \int d^3r(\delta\theta)\{(\partial_x^2\delta\theta)[1-\alpha\mathrm{cos}(2\theta)] + 2\alpha(\partial_x^2\theta)\mathrm{sin}(2\theta)(\delta\theta)\} \\ &+ 2\alpha(\partial_x\theta)(\partial_x\delta\theta)\mathrm{sin}(2\theta) + 2\alpha(\partial_x\theta)^2\mathrm{cos}(2\theta)(\delta\theta)\} \\ =& -\int d^3r[(\partial_x\delta\theta)^2 + \alpha\mathrm{cos}(2\theta)(\delta\theta)\partial_x^2(\delta\theta) - 2\alpha\mathrm{sin}(2\theta)(\partial_x^2\theta)(\delta\theta)^2 \\ &- 2\alpha(\partial_x\theta)(\delta\theta)(\partial_x\delta\theta)\mathrm{sin}(2\theta) - 2\alpha(\partial_x\theta)^2\mathrm{cos}(2\theta)(\delta\theta)^2],\end{aligned} \tag{B.9}$$

where

$$\begin{aligned}\int d^3r\alpha\mathrm{cos}(2\theta)(\delta\theta)\partial_x^2(\delta\theta) =& -\int d^3r\alpha\partial_x[\mathrm{cos}(2\theta)(\delta\theta)](\partial_x\delta\theta) \\ =\int d^3r[2\alpha\mathrm{sin}(2\theta)(\partial_x\theta)(\delta\theta)(\partial_x\delta\theta) - \alpha\mathrm{cos}(2\theta)(\partial_x\delta\theta)^2].&\end{aligned} \tag{B.10}$$

Taking Eq. (B.10) into Eq. (B.9), we get

$$\delta^2 S_{xx} = -\int d^3r\{(\partial_x\delta\theta)^2[1-\alpha\mathrm{cos}(2\theta)] - 2\alpha(\delta\theta)^2[\mathrm{cos}(2\theta)(\partial_x\theta)^2 + \mathrm{sin}(2\theta)(\partial_x^2\theta)]\}. \tag{B.11}$$

Similarly, we get from S_{yy}

$$\delta^2 S_{yy} = -\int d^3r\{(\partial_y\delta\theta)^2[1+\alpha\cos(2\theta)] + 2\alpha(\delta\theta)^2[\cos(2\theta)(\partial_y\theta)^2 + \sin(2\theta)(\partial_y^2\theta)]\}. \tag{B.12}$$

S_{xy} gives

$$\begin{aligned}\delta^2 S_{xy} =& -2\alpha\int d^3r(\delta\theta)[(\partial_x\partial_y\delta\theta)\sin(2\theta) + 2(\delta\theta)\cos(2\theta)(\partial_x\partial_y\theta)\\ &+(\partial_x\delta\theta)(\partial_y\theta)\cos(2\theta) + (\partial_y\delta\theta)(\partial_x\theta)\cos(2\theta) - 2(\delta\theta)\sin(2\theta)(\partial_x\theta)(\partial_y\theta)],\end{aligned} \tag{B.13}$$

where

$$\begin{aligned}-\alpha\int d^3r(\delta\theta)(\partial_x\partial_y\delta\theta)\sin(2\theta) =& \alpha\int d^3r(\partial_y\delta\theta)[(\partial_x\delta\theta)\sin(2\theta) + 2(\delta\theta)\cos(2\theta)(\partial_x\theta)]\\ =&\alpha\int d^3r(\partial_x\delta\theta)[(\partial_y\delta\theta)\sin(2\theta) + 2(\delta\theta)\cos(2\theta)(\partial_y\theta)].\end{aligned} \tag{B.14}$$

Taking Eq. (B.14) into Eq. (B.13), we get from S_{xy}

$$\delta^2 S_{xy} = 2\alpha\int d^3r\{(\partial_x\delta\theta)(\partial_y\delta\theta)\sin(2\theta) + 2(\delta\theta)^2[\sin(2\theta)(\partial_x\theta)(\partial_y\theta) - \cos(2\theta)(\partial_x\partial_y\theta)]\}. \tag{B.15}$$

Besides, we have

$$-\delta\int d^3r(\delta\theta)(\partial_t^2\theta) = \int d^3r(\delta\partial_t\theta)^2. \tag{B.16}$$

Combining Eqs. (B.11,B.12,B.15,B.16) together, we obtain the second-order variation of S with respect to small $\delta\theta$,

$$\begin{aligned}S_{\delta\theta} = \frac{1}{2}\delta^2 S = \int d^3r\{&\frac{1}{2}(\partial_t\delta\theta)^2 - \frac{1}{2}(\partial_x\delta\theta)^2[1-\alpha\cos(2\theta)]\\ &-\frac{1}{2}(\partial_y\delta\theta)^2[1+\alpha\cos(2\theta)] + \alpha(\partial_x\delta\theta)(\partial_y\delta\theta)\sin(2\theta)\\ &+\alpha(\delta\theta)^2\{2\sin(2\theta)(\partial_x\theta)(\partial_y\theta) + \cos(2\theta)[(\partial_x\theta)^2 - (\partial_y\theta)^2]\\ &+\sin(2\theta)(\partial_x^2\theta - \partial_y^2\theta) - 2\cos(2\theta)(\partial_x\partial_y\theta)\}\}.\end{aligned} \tag{B.17}$$

Here θ is a solution of the classical EOM. Given such θ, we have only to find those $\delta\theta$ that makes $\delta S_{\delta\theta}[\delta\theta] = 0$.

In the following, we focus on the one-dimensional solution ($\partial_y\theta = \partial_y\delta\theta = 0$) for simplicity, and neglect the integral over y, while the following derivation can be generalized to $\partial_y\delta\theta \neq 0$. The action becomes in the one-dimensional model

$$\begin{aligned}K \equiv& S_{\delta\theta}|_{\partial_y\theta=\partial_y\delta\theta=0}\\ =&\int_{-\infty}^{\infty}dt\int_L dx\,\{\frac{1}{2}(\partial_t\delta\theta)^2 - \frac{1}{2}(\partial_x\delta\theta)^2[1-\alpha\cos(2\theta)] + \alpha(\delta\theta)^2[(\partial_x\theta)^2\cos(2\theta) + (\partial_x^2\theta)\sin(2\theta)]\}.\end{aligned} \tag{B.18}$$

We substitute into K a perturbative solution of θ in α, e.g. Eqs. (III.28), and expand K in powers of α. This gives

$$K = K_0 + K_1 + K_2 + \mathcal{O}(\alpha^3), \tag{B.19}$$

where $K_n = \mathcal{O}(\alpha^n)$. K is a quadratic function of $\delta\theta$. In terms of a Fourier transform of $\delta\theta_{q,\omega}$,

$$\delta\theta_{q,\omega} = \frac{1}{\sqrt{L}}\int_L dx\int_{-\infty}^{\infty}dt\;\delta\theta(x,t)\,e^{-iqx+i\omega t}, \tag{B.20}$$

the quadratic function can be characterized by matrix elements among wavenumber q and frequency ω;

$$K = \frac{1}{L}\sum_{q,q'}\int\frac{d\omega}{2\pi}\int\frac{d\omega'}{2\pi}\,\delta\theta^\dagger_{q,\omega}K_{q,\omega;q',\omega'}[\theta]\,\delta\theta_{q',\omega'}. \tag{B.21}$$

Let us take a solution $\theta(x,t)$ for the spin-injection model with the straight geometry as an example. We first take a part of $\theta(x,t)$ with only one spatial wavelength from Eqs. (III.8,III.13),

$$\theta(x,t) = \theta_0(x,t) + \theta_1(x,t) + \mathcal{O}(\alpha^2), \tag{B.22}$$

$$\theta_0(x,t) = -j_0 x + j_0 k_0 t, \quad \theta_1(x,t) = \frac{\alpha}{4(k_0^2-1)}\sin(2j_0 x - 2j_0 k_0 t). \tag{B.23}$$

An inclusion of $\theta_2(x,t)$ shall be given later. The zeroth order of K is given by

$$K_0 = \frac{1}{2}\int dt \int_L dx\,[(\partial_t\delta\theta)^2 - (\partial_x\delta\theta)^2] = \frac{1}{2}\sum_q \int \frac{d\omega}{2\pi}\,\delta\theta^\dagger_{q,\omega}(\omega^2 - q^2)\delta\theta_{q,\omega}, \tag{B.24}$$

A substitution of Eq. (B.23) into $\mathcal{O}(\alpha)$-terms in Eq. (B.18) gives K_1 and K_2,

$$\begin{aligned}
&\frac{\alpha}{2}(\partial_x\delta\theta)^2\cos(2\theta_0+2\theta_1) + \alpha(\delta\theta)^2(\partial_x\theta_0+\partial_x\theta_1)^2\cos(2\theta_0+2\theta_1)\\
&\quad+\alpha(\delta\theta)^2(\partial_x^2\theta_0+\partial_x^2\theta_1)\sin(2\theta_0+2\theta_1)\\
&\qquad=\frac{\alpha}{2}(\partial_x\delta\theta)^2\cos(2\theta_0) + \alpha(\delta\theta)^2(\partial_x\theta_0)^2\cos(2\theta_0)\\
&\qquad\quad-\alpha(\partial_x\delta\theta)^2\sin(2\theta_0)\theta_1 - 2\alpha(\delta\theta)^2(\partial_x\theta_0)(\partial_x\theta_1)\cos(2\theta_0)\\
&\qquad\quad-2\alpha(\delta\theta)^2(\partial_x\theta_0)^2\sin(2\theta_0)\theta_1 - \alpha(\delta\theta)^2(\partial_x^2\theta_1)\sin(2\theta_0) + \mathcal{O}(\alpha^3)\\
&\qquad=\frac{\alpha}{2}(\partial_x\delta\theta)^2\cos(2j_0x - 2k_0j_0t) + \alpha j_0^2(\delta\theta)^2\cos(2j_0x-2j_0k_0t)\\
&\qquad\quad+\frac{\alpha^2(\partial_x\delta\theta)^2}{4(k_0^2-1)}\sin^2(2j_0x-2j_0k_0t) - \frac{\alpha^2 j_0^2(\delta\theta)^2}{k_0^2-1}\cos^2(2j_0x-2j_0k_0t)\\
&\qquad\quad+\frac{\alpha^2 j_0^2(\delta\theta)^2}{2(k_0^2-1)}\sin^2(2j_0x-2j_0k_0t) + \frac{\alpha^2 j_0^2(\delta\theta)^2}{k_0^2-1}\sin^2(2j_0x-2j_0k_0t) + \mathrm{O}(\alpha^3).
\end{aligned} \tag{B.25}$$

Equivalently, we have

$$\begin{aligned}
K_1 =&\alpha\int dt\int_L dx\,[\frac{1}{2}(\partial_x\delta\theta)^2\cos(2j_0x-2j_0k_0t) + j_0^2(\delta\theta)^2\cos(2j_0x-2j_0k_0t)]\\
=&\alpha\int dt\int_L dx\,[\frac{1}{4L}\sum_{q,q'}\int_{\omega,\omega'} qq'\delta\theta^\dagger_{q,\omega}\delta\theta_{q',\omega'} + \frac{j_0^2}{2L}\sum_{q,q'}\int_{\omega,\omega'}\delta\theta^\dagger_{q,\omega}\delta\theta_{q',\omega'}]e^{-iqx+i\omega t}e^{iq'x-i\omega' t}e^{2ij_0x-2ij_0k_0t} + \mathrm{H.c.}\\
=&\frac{\alpha}{4}\sum_q\int_\omega \delta\theta^\dagger_{q+2j_0,\omega+2j_0k_0}\delta\theta_{q,\omega}[q(q+2j_0)+2j_0^2] + \mathrm{H.c.}\\
=&\frac{\alpha}{4}\sum_q\int_\omega \delta\theta^\dagger_{q+j_0,\omega+j_0k_0}\delta\theta_{q-j_0,\omega-j_0k_0}(q^2+j_0^2)e^{-2ij_0k_0t} + \mathrm{H.c.},
\end{aligned} \tag{B.26}$$

$$\begin{aligned}
K_2 =&\alpha^2\int dt\int_L dx\left\{\frac{(\partial_x\delta\theta)^2}{8(k_0^2-1)}[1-\cos(4j_0x-4j_0k_0t)] - \frac{j_0^2(\delta\theta)^2}{k_0^2-1}\cos(4j_0x-4j_0k_0t)\right.\\
&\left.+\frac{j_0^2(\delta\theta)^2}{4(k_0^2-1)}[1-\cos(4j_0x-4j_0k_0t)]\right\}\\
=&\frac{\alpha^2}{8(k_0^2-1)}\sum_q\int\frac{d\omega}{2\pi}\,\delta\theta^\dagger_{q,\omega}\delta\theta_{q,\omega}q^2 + \frac{\alpha^2 j_0^2}{4(k_0^2-1)}\sum_q\int\frac{d\omega}{2\pi}\,\delta\theta^\dagger_{q,\omega}\delta\theta_{q,\omega}\\
&-\frac{\alpha^2}{16(k_0^2-1)}\sum_q\int\frac{d\omega}{2\pi}\left\{\delta\theta^\dagger_{q+2j_0,\omega+2j_0k_0}\delta\theta_{q-2j_0,\omega-2j_0k_0}\left[(q^2-4j_0^2)+8j_0^2+2j_0^2\right]e^{-4ij_0k_0t} + \mathrm{H.c.}\right\}\\
=&\frac{\alpha^2}{8(k_0^2-1)}\sum_q\int\frac{d\omega}{2\pi}\,\delta\theta^\dagger_{q,\omega}\delta\theta_{q,\omega}(2j_0^2+q^2)\\
&-\frac{\alpha^2}{16(k_0^2-1)}\sum_q\int\frac{d\omega}{2\pi}\left[\delta\theta^\dagger_{q+2j_0,\omega+2j_0k_0}\delta\theta_{q-2j_0,\omega-2j_0k_0}(q^2+6j_0^2) + \mathrm{H.c.}\right].
\end{aligned} \tag{B.27}$$

Taking Eqs. (B.24,B.26,B.27) into Eq. (B.20), we obtain the matrix elements,

$$\frac{1}{2\pi}K_{q+p,\omega+\nu;q-p,\omega-\nu} = (\omega^2-q^2)\delta_{p,0}\delta(\nu) + \frac{\alpha(q^2+j_0^2)}{2}(\delta_{p,j_0}\delta(\nu-j_0k_0)+\delta_{p,-j_0}\delta(\nu+j_0k_0))$$
$$-\frac{\alpha^2}{8(k_0^2-1)}[-2(2j_0^2+q^2)\delta_{p,0}\delta(\nu)+(q^2+6j_0^2)(\delta_{p,2j_0}\delta(\nu-2j_0k_0)+\delta_{p,-2j_0}\delta(\nu+2j_0k_0))]+\mathcal{O}(\alpha^3). \tag{B.28}$$

To find $\delta\theta$ that satisfies $\delta S_{\delta\theta}[\delta\theta]=0$, we only have to find an eigenmode of K in Eq. (B.21) that belongs to zero eigenvalue ("eigenenergy"). At the zeroth order in α, eigenmodes of K are characterized by q and ω, and the "zero-energy" eigenmodes are obtained by setting ω to be q (on-shell condition). When α is included perturbatively, eigenmodes at q and ω hybridize with eigenmodes at $q\pm 2j_0$ and $\omega\pm 2j_0k_0$ as well as eigenmodes at $q\pm 4j_0$ and $\omega\pm 4j_0k_0$ in terms of off-diagonal mixing terms. Due to the off-diagonal mixing terms, eigenmodes of K are characterized by q and ω modulo $2j_0$ and $2j_0k_0$ respectively, and $(q,\omega)\in[-j_0,j_0]\times[-k_0j_0,k_0j_0]$ plays a role of a first Brillouin zone. In the Brillouin zone, eigenmodes at the same (q,ω) are distinguished by a band index n,

$$K=\frac{1}{2}\sum_{n\in\mathbb{N}}\sum_{-j_0\le q<j_0}\int_{-k_0j_0}^{k_0j_0}\frac{d\omega}{2\pi}\,\delta\varphi^\dagger_{q,\omega,n}\Lambda_{q,\omega,n}\delta\varphi_{q,\omega,n}+\mathcal{O}(\alpha^3), \tag{B.29}$$

with

$$\delta\theta_{q+2j_0m_1,\omega+2k_0j_0m_2}=\sum_{n\in\mathbb{N}}c_{q,\omega,n;m_1,m_2}\delta\varphi_{q,\omega,n}, \tag{B.30}$$

$m_1\in\mathbb{Z}$, $m_2\in\mathbb{Z}$. Here, $c_{q,\omega,n;m_1,m_2}$ is analogous to the periodic part of a Bloch wavefunction in the band theory. From Eq. (B.28) $K_{q,\omega;q,\omega'}$ is real symmetric, so that $c_{q,\omega,n;m_1,m_2}$ are real. An eigenstate of the lowest energy, say $\delta\varphi_{q,\omega,n=0}$, must approach $\delta\theta_{q,\omega}$ in the limit of $\alpha\to 0$. Such lowest eigenmode ($n=0$) is calculated up to the 2nd order in α as follows,

$$\begin{aligned}
c_{q,\omega,0;m_1,m_2}=&\delta_{m_1,0}\delta_{m_2,0}-\frac{\alpha[\frac{(q+j_0)^2+j_0^2}{2}]}{(\omega+2j_0k_0)^2-(q+2j_0)^2-\omega^2+q^2}\delta_{m_1,1}\delta_{m_2,1}\\
&-\frac{\alpha[\frac{(q-j_0)^2+j_0^2}{2}]}{(\omega-2j_0k_0)^2-(q-2j_0)^2-\omega^2+q^2}\delta_{m_1,-1}\delta_{m_2,-1}+\mathcal{O}(\alpha^2)\\
=&\delta_{m_1,0}\delta_{m_2,0}-\alpha\frac{1+\mathcal{O}(q,\omega)}{4(k_0^2-1)}\delta_{m_1,1}\delta_{m_2,1}-\alpha\frac{1+\mathcal{O}(q,\omega)}{4(k_0^2-1)}\delta_{m_1,-1}\delta_{m_2,-1}+\mathcal{O}(\alpha^2).
\end{aligned} \tag{B.31}$$

A corresponding "eigenenergy" is calculated up to the 2nd order,

$$\begin{aligned}
\Lambda_{q,\omega,0}=&\omega^2-q^2+\frac{\alpha^2(2j_0^2+q^2)}{4(k_0^2-1)}\\
&-\frac{\alpha^2[\frac{(q+j_0)^2+j_0^2}{2}]^2}{(\omega+2j_0k_0)^2-(q+2j_0)^2-\omega^2+q^2}-\frac{\alpha^2[\frac{(q-j_0)^2+j_0^2}{2}]^2}{(\omega-2j_0k_0)^2-(q-2j_0)^2-\omega^2+q^2}+\mathcal{O}(\alpha^3)\\
=&\omega^2-q^2+\frac{\alpha^2(2j_0^2+q^2)}{4(k_0^2-1)}-\frac{\alpha^2(2j_0^2+2qj_0+q^2)^2}{4[4j_0^2(k_0^2-1)+4qj_0-4\omega j_0k_0]}\\
&-\frac{\alpha^2(2j_0^2-2qj_0+q^2)^2}{4[4j_0^2(k_0^2-1)-4qj_0+4\omega j_0k_0]}+\mathcal{O}(\alpha^3),
\end{aligned} \tag{B.32}$$

namely

$$\begin{aligned}
\Lambda_{q,\omega,0}=&\omega^2+\frac{\alpha^2j_0^2}{2}\frac{1}{k_0^2-1}-q^2[1-\frac{\alpha^2}{4(k_0^2-1)}]\\
&-\frac{\alpha^2}{16j_0^2(k_0^2-1)}(4j_0^4+8qj_0^3+8q^2j_0^2)[1-\frac{q-\omega k_0}{j_0(k_0^2-1)}+\frac{(q-\omega k_0)^2}{j_0^2(k_0^2-1)^2}+\mathcal{O}((q+\omega k_0)^3)]\\
&-\frac{\alpha^2}{16j_0^2(k_0^2-1)}(4j_0^4-8qj_0^3+8q^2j_0^2)[1+\frac{q-\omega k_0}{j_0(k_0^2-1)}+\frac{(q-\omega k_0)^2}{j_0^2(k_0^2-1)^2}+\mathcal{O}((q+\omega k_0)^3)]+\mathcal{O}(\alpha^3)\\
=&\omega^2+\frac{\alpha^2j_0^2}{2}\frac{1}{k_0^2-1}-q^2[1-\frac{\alpha^2}{4(k_0^2-1)}]-\frac{\alpha^2j_0^2}{2(k_0^2-1)}+\mathcal{O}(\alpha^2\omega^2,\alpha^2q^2,\alpha^2\omega q,\alpha^3)\\
=&\omega^2-q^2[1-\frac{\alpha^2}{4(k_0^2-1)}]+\mathcal{O}(\alpha^2\omega^2,\alpha^2q^2,\alpha^2\omega q,\alpha^3).
\end{aligned} \tag{B.33}$$

The lowest energy band indicates that $\delta\theta$ evaluated on shell, $\Lambda_{q,\omega,0} = 0$, behaves like a gapless classical wave. This is because the original theory, Eq. (6) in the main text, has a spacetime translational symmetry. Thus, for any θ, one can choose $\delta\theta$ as a translation of θ, and such $\delta\theta$ does not change the Lagrangian. For a general k_0 (off the resonance point; $k_0 \neq 1$), α can be treated perturbatively, and the classical wave up to the 2nd order in α has a well-defined (i.e. real-valued) velocity v,

$$1 - \frac{\alpha^2}{4(k_0^2 - 1)} = 1 + \mathcal{O}(\alpha^2) \equiv v^2 > 0. \tag{B.34}$$

By evaluating the eigenmode on shell ($|\omega| = |q| + \mathcal{O}(\alpha^2)|q|$), we finally determine the first-order $\delta\theta_1$ for an arbitrary form of $\delta\theta_0$ given by Eq. (B.1),

$$\begin{aligned}\delta\theta_1(x,t) = \frac{1}{\sqrt{L}} \sum_q^{|q|<j_0} \int_{-k_0 j_0}^{k_0 j_0} d\omega \Big[\delta(q-\omega) d_q + \delta(q+\omega) d'_q\Big] \\ \times \Big[-\alpha \frac{1+\mathcal{O}(q)}{4(k_0^2-1)} e^{i(q+2j_0)x - i(\omega+2j_0k_0)t} - \alpha \frac{1+\mathcal{O}(q)}{4(k_0^2-1)} e^{i(q-2j_0)x - i(\omega-2j_0k_0)t}\Big].\end{aligned} \tag{B.35}$$

Finally, let us include $\theta_2(x,t)$ into Eq. (B.22),

$$\theta(x,t) = \theta_0(x,t) + \theta_1(x,t) + \theta_2(x,t) + \mathcal{O}(\alpha^2), \tag{B.36}$$

$$\begin{aligned}\theta_2(x,t) =& 2\alpha\mathrm{Re}(g)\cos[2k_0j_0(t-x)] + 2\alpha\mathrm{Re}(g')\cos[2k_0j_0(t+x)] \\ &- 2\alpha\mathrm{Im}(g)\sin[2k_0j_0(t-x)] - 2\alpha\mathrm{Im}(g')\sin[2k_0j_0(t+x)].\end{aligned} \tag{B.37}$$

Eq. (B.27) has an additional $\mathcal{O}(\alpha^2)$ contribition,

$$\begin{aligned}\Delta K_2 = \int dtdx[&-\alpha(\partial_x\delta\theta)^2\sin(2\theta_0)\theta_2 + 2\alpha(\delta\theta)^2(\partial_x\theta_0)(\partial_x\theta_2)\cos(2\theta_0) \\ &- 2\alpha(\delta\theta)^2(\partial_x\theta_0)^2\sin(2\theta_0)\theta_2 + \alpha(\delta\theta)^2(\partial_x^2\theta_2)\sin(2\theta_0)].\end{aligned} \tag{B.38}$$

For $k_0 \neq 1$, ΔK_2 contributes only to off-diagonal matrix elements of $K_{q,\omega;q,\omega}$, so that it changes neither Eq. (B.33) nor Eq. (B.35) at their respective sub-leading order.