ON ASPHERICAL CONFIGURATION LIE GROUPOIDS

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ABSTRACT. The complement of the hyperplanes $\{x_i = x_j\}$, for all $i \neq j$ in M^n, for M an aspherical 2-manifold, is known to be aspherical. Here we consider the situation, when M is a 2-dimensional orbifold. We prove this complement to be aspherical for a class of aspherical 2-dimensional orbifolds, and predict that it should be true in general also. We generalize this question in the category of Lie groupoids, as orbifolds can be identified with a certain kind of Lie groupoids.

CONTENTS

1. INTRODUCTION

Let X be a topological space and $PB_n(X)$ be the configuration space of ordered n-tuples of pairwise distinct points of X. That is,

$$PB_n(X) = \{(x_1, x_2, \ldots, x_n) \in X^n \mid x_i \neq x_j, \text{ for } i \neq j\}.$$

Let M be a connected manifold of dimension ≥ 2, and $n \geq 2$. The Fadell-Neuwirth fibration theorem ([[7], Theorem 3]) says that the projection map $M^n \to M^r$ to the first r coordinates restricts to the following locally trivial fiber bundle map, with fiber homeomorphic to $PB_{n-r}(\widetilde{M})$, where $\widetilde{M} = M - \{r \text{ points}\}$.

$$f(M) : PB_n(M) \to PB_r(M).$$

Our main motivation for this article is the following corollary of the fibration theorem. In the rest of the article we assume $r = n - 1$.

Theorem 1.1. *([7], Corollary 2.2]) Let M be a connected aspherical 2-manifold. Then $PB_n(M)$ is also aspherical.*

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Proof. Note that, the fiber of $f(M)$ is aspherical, since it is homeomorphic to $M - \{(n-1) \text{ points}\}$. Next, since $f(M)$ is a fibration, it induces a long exact sequence of homotopy groups. Therefore, by induction on n, we conclude that $PB_n(M)$ is aspherical. $\square$

It is an important subject to study the homotopy groups, especially the fundamental groups of the configuration spaces of a manifold. Since in dimension ≥ 3, the space $PB_n(M)$ and the product manifold M^n have isomorphic fundamental groups, the dimension 2 case is of much interest. Using the fibration $f(M)$, there are results to compute the higher homotopy groups of the configuration space as well. See [7] for more on this subject.

Orbifolds are also of fundamental importance in algebraic and differential geometry, topology and string theory. In [17] we studied the possibility of extending the Fadell-Neuwirth fibration theorem for orbifolds. However, to define a fibration between orbifolds, we had to consider the category of Lie groupoids. Since an orbifold can be realized as a Lie groupoid ([12]), and there are enough tools in this category to define a fibration. There, we defined two notions (a and b-types) of a fibration ([[17], Definition 2.4]) and the corresponding (a and b-types) configuration Lie groupoids of a Lie groupoid to enable us to state a Fadell-Neuwirth type fibration theorem. For an orbifold M, the b-type configuration Lie groupoid is the correct model to induce the orbifold structure on $PB_n(M)$. We proved that the Fadell-Neuwirth fibration theorem extends in this generality, under some strong hypothesis (c-groupoid). We will show in Proposition 3.1, that this is the best possible extension. For this, we will deduce that the map $f(M)$ is not a a(or b)-type fibration for the a(or b)-type configuration Lie groupoids of Lie groupoids, corresponding to global quotient orbifolds of dimension ≥ 2 with a homological condition, and a non-empty singular set. In fact, for 2-dimensional orbifolds with at least one cone point, $f(M)$ does not induce a long exact sequence of orbifold homotopy groups. See Remark 3.2 for a more precise statement.

Recall that, for a connected aspherical 2-manifold M, by Theorem 1.1, $PB_n(M)$ is an aspherical manifold. Equivalently, the universal cover of $PB_n(M)$ is a contractible manifold.

We define an orbifold M to be *aspherical*, if its universal orbifold cover $\widetilde{M}$ is a contractible orbifold, that is, if the orbifold homotopy groups $\pi_q^{orb}(\widetilde{M})$, for $q \geq 1$, are trivial.

At this point, one may ask if Theorem 1.1 is true for connected 2-dimensional orbifolds. In the following theorem we give an answer to this question, for a class of aspherical 2-dimensional orbifolds. Let $\mathbb{C}^*$ be $\mathbb{C}$ with one puncture.

Theorem 1.2. *Let M be one of the following 2-dimensional orbifolds.*

- *Underlying space is $\mathbb{C}$ with one cone point of order $m \geq 2$.*
- *Underlying space is $\mathbb{C}$ with two cone points of order 2.*
- *Underlying space is $\mathbb{C}^*$ with one cone point of order 2.*

Then $PB_n(M)$ is aspherical.

Here, we make some remark on the reasons behind the three cases in the theorem. The first one relates to the solution of the $K(\pi,1)$-problem of the hyperplane arrangement complement corresponding to the unitary reflection groups $G(m,l,n)$ ([14]), and the finite type Artin groups of type D_n ($m = 2$ case) ([3]). The second and third cases need the solutions of the $K(\pi,1)$-problem for the affine Artin groups of types $\widetilde{D}_n$ and $\widetilde{B}_n$, respectively ([15], [4]). The $K(\pi, 1)$-problem for hyperplane arrangement complements is well known, and there are milestone results in the literature. See [5] for finite type Artin groups, [15] for affine Artin groups, and [2] for finite complex reflection groups.

We predict the following.

Asphericity conjecture. *$PB_n(M)$ is aspherical, for a connected aspherical 2-dimensional orbifold M.*

Since a contractible orbifold is a manifold ([11]), the universal orbifold cover $\widetilde{M}$ of a connected aspherical 2-dimensional orbifold, is an aspherical simply connected 2-manifold. Therefore, $\widetilde{M}$ is diffeomorphic to a submanifold of $\mathbb{R}^2$, and $\pi_1^{orb}(M)$ is acting effectively and properly discontinuously on $\widetilde{M}$. We will prove in Proposition 3.3, that the Asphericity conjecture is equivalent to the following statement.

Asphericity conjecture for orbit configuration spaces. *Let H be a discrete group acting effectively and properly discontinuously on a connected and simply connected submanifold $\widetilde{M}$ of $\mathbb{R}^2$. Then, the following orbit configuration space of the action of H on $\widetilde{M}$, is aspherical.*

$$PB_n(\widetilde{M}, H) := \{(x_1, x_2, \ldots, x_n) \in M^n \mid Hx_i \neq Hx_j, \text{ for } i \neq j\}.$$

To define the orbifold homotopy groups of an orbifold, one needs to look at orbifolds as Lie groupoids. In the next section we will see how to study orbifolds in the category of Lie groupoids. Then, we will formulate the above statements in this general setting. Also, we will justify the equivalence of the Asphericity conjecture and its orbit configuration space version. In the final section we give the proof of Theorem 1.2.

2. Lie groupoids and orbifolds

In this paper by a 'manifold' we mean a 'Hausdorff smooth manifold' and by a 'group' we mean a 'discrete group', unless mentioned otherwise. A 'map' is either continuous or smooth, which will be clear from the context. And a 'fibration' would mean a 'Hurewicz fibration'.

We now recall some basics on Lie groupoids and orbifolds. See [1], [12] or [13] for more details.

2.1. **Lie groupoids.** Let $\mathcal{G}$ be a *Lie groupoid* with object space $\mathcal{G}_0$ and morphism space $\mathcal{G}_1$. Let $s, t : \mathcal{G}_1 \to \mathcal{G}_0$ be the source and the target maps defined by $s(\sigma) = x$ and $t(\sigma) = y$, for $\sigma \in mor_{\mathcal{G}}(x, y) \subset \mathcal{G}_1$. Recall that s and t are smooth and submersions.

A *homomorphism* f between two Lie groupoids is a smooth functor which respects all the structure maps. f_0 and f_1 denote the object and morphism level maps of f, respectively. For any $x \in \mathcal{G}_0$, the set $t(s^{-1}(x))$ is called the *orbit* of x. The space $|\mathcal{G}|$ of all orbits with respect to the quotient topology is called the *orbit space* of the Lie groupoid. If $f : \mathcal{G} \to \mathcal{H}$ is a homomorphism between two Lie groupoids, then f induces a map $|f| : |\mathcal{G}| \to |\mathcal{H}|$, making the following diagram commutative. We define $\mathcal{G}$ to be *Hausdorff* if $|\mathcal{G}|$ is Hausdorff, and it is called a *$c-groupoid$* if the quotient map $\mathcal{G}_0 \to |\mathcal{G}|$ is a covering map. Hence a c-groupoid is Hausdorff.

$$\begin{array}{ccc} \mathcal{G}_0 & \xrightarrow{f} & \mathcal{H}_0 \\ \downarrow & & \downarrow \\ |\mathcal{G}| & \xrightarrow{|f|} & |\mathcal{H}|. \end{array}$$

Given a Hausdorff Lie groupoid $\mathcal{G}$, we defined in [[17], Definition 2.8] the *b-configuration Lie groupoid* $PB^b_n(\mathcal{G})$. In this paper we do not use the superscript b as we consider only this configuration Lie-groupoid. Recall that, its object space $PB_n(\mathcal{G})_0$ is the n-tuple of objects of $\mathcal{G}$ with mutually distinct orbits.

$$PB_n(\mathcal{G})_0 = \{(x_1, x_2, \dots, x_n) \in \mathcal{G}_0^n \mid t(s^{-1}(x_i)) \neq t(s^{-1}(x_j)), \text{ for } i \neq j\}.$$

The morphism space $PB_n(\mathcal{G})_1$ is $(s^n, t^n)^{-1}(PB_n(\mathcal{G})_0 \times PB_n(\mathcal{G})_0)$. We also showed in [[17], Lemma 2.9] that the projection to the first $n-1$ coordinates on both the object and morphism spaces define a homomorphism $f(\mathcal{G}) : PB_n(\mathcal{G}) \to PB_{n-1}(\mathcal{G})$.

Let $\mathcal{H}$ and $\mathcal{G}$ be two Lie groupoids and $f : \mathcal{H} \to \mathcal{G}$ be a homomorphism, such that $f_0 : \mathcal{H}_0 \to \mathcal{G}_0$ is a covering map. Then, f is called a *covering homomorphism* of Lie groupoids if $\mathcal{H}_0$ is a left $\mathcal{G}$-space with f_0 equal to the action map, $\mathcal{H}_1 = \mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{H}_0$ and f_1 is the first projection. The source and the target maps of $\mathcal{H}$ coincide with the second projection and the action map, respectively.

Next we recall the important concept of the classifying space of a Lie groupoid, which is required to define algebraic invariants of the Lie groupoid. For a Lie groupoid $\mathcal{G}$, the classifying space $B\mathcal{G}$ is defined as the geometric realization of the simplicial manifold $\mathcal{G}_\bullet$ defined by the following iterated fibered products.

$$\mathcal{G}_k = \mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1 \times_{\mathcal{G}_0} \cdots \times_{\mathcal{G}_0} \mathcal{G}_1.$$

See [[1], p. 25] or [12] for some discussion on this matter, in the context of orbifold Lie groupoids (Example 2.3).

Definition 2.1. The k-th *homotopy group* of $\mathcal{G}$ is defined as the k-th ordinary homotopy group of $B\mathcal{G}$. That is, $\pi_k(\mathcal{G}, \widetilde{x}_0) := \pi_k(B\mathcal{G}, \widetilde{x}_0)$ for $\widetilde{x}_0 \in \mathcal{G}_0$. A Lie groupoid $\mathcal{G}$ is called *aspherical* if $\pi_k(\mathcal{G}, \widetilde{x}_0) = 0$ for all $k \geq 2$ and for all $\widetilde{x}_0 \in \mathcal{G}_0$.

Note that, a homomorphism $f : \mathcal{G} \to \mathcal{H}$ induces a map $Bf : B\mathcal{G} \to B\mathcal{H}$. Also see [1] or [12] for some more on homotopy groups of Lie groupoids.

Now, we recall the concept of equivalence between two Lie groupoids. One consequence of this concept is that an equivalence $f : \mathcal{G} \to \mathcal{H}$ between two Lie groupoids induces a weak homotopy equivalence $Bf : B\mathcal{G} \to B\mathcal{H}$.

A more appropriate notion of equivalence between Lie groupoids is Morita equivalence.

Definition 2.2. Let $f : \mathcal{G} \to \mathcal{H}$ be a homomorphism between Lie groupoids. f is called an *equivalence* if the following conditions are satisfied.

- The following composition is a surjective submersion.

  $$\mathcal{H}_1 \times_{\mathcal{H}_0} \mathcal{G}_0 \xrightarrow{\pi_1} \mathcal{H}_1 \xrightarrow{t} \mathcal{H}_0.$$

  Here $\mathcal{H}_1 \times_{\mathcal{H}_0} \mathcal{G}_0$ is the fibered product, defined by s and f_0.

  $$\begin{array}{ccc} \mathcal{H}_1 \times_{\mathcal{G}_0} \mathcal{G}_0 & \xrightarrow{\pi_1} & \mathcal{H}_1 \\ \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle s} \\ \mathcal{G}_0 & \xrightarrow{f_0} & \mathcal{H}_0. \end{array}$$

- The following commutative diagram is a fibered product of manifolds.

$$\begin{tikzcd} \mathcal{G}_1 \arrow[r, "f_1"] \arrow[d, "(s,t)"] & \mathcal{H}_1 \arrow[d, "(s,t)"] \\ \mathcal{G}_0 \times \mathcal{G}_0 \arrow[r, "f_0\times f_0"] & \mathcal{H}_0 \times \mathcal{H}_0. \end{tikzcd}$$

$\mathcal{G}$ and $\mathcal{H}$ are called *Morita equivalent* if there is a third Lie groupoid $\mathcal{K}$ and two equivalences as follows.

$$\mathcal{G} \longleftarrow \mathcal{K} \longrightarrow \mathcal{H}.$$

Next, we recall a standard example of a Lie groupoid which is relevant for us.

Example 2.3. Let $\widetilde{M}$ be a manifold, and a Lie group H is acting on $\widetilde{M}$ smoothly. Out of this information one constructs a Lie groupoid $\mathcal{G}(\widetilde{M}, H)$ as follows, and call it the *translation Lie groupoid*. Define $\mathcal{G}(\widetilde{M}, H)_0 = \widetilde{M}$, $\mathcal{G}(\widetilde{M}, H)_1 = \widetilde{M}\times H$, $s(x, h) = x$, $t(x, h) = h(x)$, $u(x) = (x, 1)$, $i(x, h) = (x, h^{-1})$ and $(h(x), h') \circ (x, h) = (x, h'h)$, for $h, h' \in H$ and $x \in \widetilde{M}$. When H is the trivial group then $\mathcal{G}(\widetilde{M}, H)$ is called the *unit groupoid*, denoted by $\mathcal{G}(\widetilde{M})$ and is identified with $\widetilde{M}$. In this paper we always consider H to be discrete (unless explicitly mentioned) and is acting effectively and properly discontinuously on $\widetilde{M}$ ([[21], Definition 3.7.1]). Then $\mathcal{G}(\widetilde{M}, H)$ is an example of an *(effective) orbifold Lie groupoid*. In this case $\mathcal{G}(\widetilde{M}, H)$ is also called an orbifold Lie groupoid inducing the orbifold structure on $M = \widetilde{M}/H$. For the more general definition of *orbifold Lie groupoid* see [12] or [[1], Definition 1.38].

Definition 2.4. We call an effective orbifold Lie groupoid of type $\mathcal{G}(\widetilde{M}, H)$ as in Example 2.3, a *translation orbifold Lie groupoid*.

Example 2.5. Let H and $\widetilde{M}$ be as in Example 2.3. Let H' be a subgroup of H and $i : H' \to H$ be the inclusion map. Then the maps $f_0 := id : \widetilde{M} \to \widetilde{M}$ and $f_1 := (id, i) : \widetilde{M} \times H' \to \widetilde{M} \times H$ together define a homomorphism $f : \mathcal{G}(\widetilde{M}, H') \to \mathcal{G}(\widetilde{M}, H)$.

Frequently, in this paper we will be using the following lemma.

Lemma 2.6. *Let $\mathcal{G}$ and $\mathcal{H}$ be two orbifold Lie groupoids, and $f : \mathcal{G} \to \mathcal{H}$ be a covering homomorphism. Then f induces isomorphisms on higher homotopy groups and an injection on the fundamental groups.*

Proof. It is easy to see that a covering homomorphism between two orbifold Lie groupoids induces a covering map on their classifying spaces. The Lemma now follows from standard covering space theory. See [[1], Proposition 2.17]. □

2.2. **Orbifolds as Lie groupoids.** An *orbifold* ([21]) or a *V-manifold* as in [19], is defined as follows.

Definition 2.7. Let M be a paracompact Hausdorff topological space. Assume for each $x \in M$, there is a connected open neighborhood $U_x \subset M$ of x satisfying the following conditions.

- There is a connected open set $\widetilde{U}_x$ in some $\mathbb{R}^n$ and a finite group G_x of diffeomorphisms of $\widetilde{U}_x$. Furthermore, there is a G_x-equivariant map $\phi_x : \widetilde{U}_x \to M$ such that the induced map $[\phi_x] : \widetilde{U}_x/G_x \to M$ is a homeomorphism onto U_x.

Then, $(\widetilde{U}_x, G_x, \phi_x)$ is called a *chart* and M is called an *orbifold*, with *underlying space* M. Given a chart (U_x, G_x, ϕ_x), the group G_x can be shown to be unique, and is called the *local group* at x. If the local group at x is trivial, then x is called a *smooth* or a *regular* point, otherwise it is a *singular* point or a *singularity*.

Assume dimension of M is 2. Let (U_x, G_x, ϕ_x) be a chart such that, G_x is finite cyclic of order q, acting by rotation around the origin $(0,0) \in \widetilde{U}_x \subset \mathbb{R}^2$ by an angle $\frac{2\pi}{q}$ and $\phi_x((0,0)) = x$. Then, x is called a *cone point* of *order* q. Also, there are two other types of singularities, called *reflector lines* and *corner reflectors*. In this dimension, it is known that the underlying space is homeomorphic to a 2-dimensional manifold. The genus of the underlying space M is called the *genus* of the orbifold M. See [20] for more details.

Example 2.8. An orbifold Lie groupoid $\mathcal{G}$ gives an orbifold structure on $|\mathcal{G}|$. See [[1], Proposition 1.44].

We now have the following useful lemma.

Lemma 2.9. *Let $\mathcal{G}$ and $\mathcal{H}$ be two orbifold Lie groupoids and $f : \mathcal{G} \to \mathcal{H}$ be a homomorphism. Then, f is a covering homomorphism of Lie groupoids if and only if $|f| : |\mathcal{G}| \to |\mathcal{H}|$ is an orbifold covering map.*

Proof. See [[1], p. 40] for a proof. □

Example 2.10. If a group H acts effectively and properly discontinuously on a manifold $\widetilde{M}$, and H' is a subgroup of H, then the homomorphism $\mathcal{G}(\widetilde{M}, H') \to \mathcal{G}(\widetilde{M}, H)$ (Example 2.5) is a covering homomorphism. In particular, if a finite group H acts effectively on a manifold $\widetilde{M}$, then $\mathcal{G}(\widetilde{M}) \to \mathcal{G}(\widetilde{M}, H)$ is a covering homomorphism.

It is well known that two orbifold Lie groupoids induce equivalent orbifold structures on M if and only if they are Morita equivalent ([13], [12]).

In our situation of translation orbifold Lie groupoids we see in the following lemma, that when two translation orbifold Lie groupoids are Morita equivalent then in the Morita equivalence, the third orbifold Lie groupoid also can be chosen to be a translation orbifold Lie groupoid. We need this lemma for the proof of Theorem 3.5.

Lemma 2.11. *Let two translation orbifold Lie groupoids $\mathcal{G}(M_1, H_1)$ and $\mathcal{G}(M_2, H_2)$ are inducing equivalent orbifold structures on M. Then there is a third translation orbifold Lie groupoid $\mathcal{G}(M_3, H_3)$, which is equivalent to both $\mathcal{G}(M_1, H_1)$ and $\mathcal{G}(M_2, H_2)$.*

Proof. Let $p_1 : (M_1, m_1) \to (M, m)$ and $p_2 : (M_2, m_2) \to (M, m)$ be the orbifold covering projections, with groups of covering transformations H_1 and H_2, respectively. Here $m \in M$ is a smooth point. Consider the orbifold covering M_3 of M corresponding to the subgroup

$$K := (p_1)_*(\pi_1(M_1, m_1)) \cap (p_2)_*(\pi_1(M_2, m_2)) < \pi_1^{orb}(M, m).$$

Let H_3 be the group of covering transformation of the orbifold covering map $p_3 : M_3 \to M$.

Then, we will establish the following diagram.

$$\mathcal{G}(M_1, H_1) \longleftarrow \mathcal{G}(M_3, H_3) \longrightarrow \mathcal{G}(M_2, H_2).$$

We will first define the arrows and then show that they are, in fact, homomorphisms and equivalences.

We just need to check it for one of these arrows, since the same proof will work for the other one as well.

Denote $\mathcal{G}(M_3, H_3)$ by $\mathcal{G}$ and $\mathcal{G}(M_1, H_1)$ by $\mathcal{H}$. Let $p : (M_3, m_3) \to (M_1, m_1)$ be the covering map corresponding to the subgroup $(p_1)_*^{-1}(K)$ of $\pi_1(M_1, m_1)$, and let

$$\rho : H_3 = \pi_1^{orb}(M, m)/p_{3*}(\pi_1(M_3, m_3)) \longrightarrow \pi_1^{orb}(M, m)/p_{1*}(\pi_1(M_1, m_1)) = H_1$$

be the quotient homomorphism. Note that, p is a genuine covering map of manifolds.

Then define $f_0 = p$ and $f_1 = (p, \rho) : M_3 \times H_3 \to M_1 \times H_1$. The following commutative diagram shows that $f : \mathcal{G} \to \mathcal{H}$ is a homomorphism, where $h_3 \in H_3$.

$$\begin{tikzcd}
M_3 \arrow[rr, "h_3"] \arrow[d, "p"] & & M_3 \arrow[d, "p"] \\
M_1 \arrow[rr, "\rho(h_3)"] \arrow[dr, "p_1"] & & M_1 \arrow[dl, "p_1"'] \\
& M &
\end{tikzcd}$$

Now we check that f is an equivalence. See Definition 2.2.

The first condition in the definition of an equivalence says that the composition

$$\mathcal{H}_1 \times_{\mathcal{H}_0} \mathcal{G}_0 \xrightarrow{\pi_1} \mathcal{H}_1 \xrightarrow{t} \mathcal{H}_0$$

should be a surjective submersion. In our situation it takes the following form.

$$(M_1 \times H_1) \times_{M_1} M_3 \xrightarrow{\pi_1} (M_1 \times H_1) \xrightarrow{t} M_1.$$

The fibered product is with respect to s (first projection) and p. Since p and t are both surjective submersions, it follows that $t \circ \pi_1$ is a surjective submersion.

Next, we have to check the second condition in the definition of an equivalence, that is, we have to show that the following diagram is a fibered product in the category of manifolds.

$$\begin{tikzcd}
\mathcal{G}_1 \arrow[r, "f_1"] \arrow[d, "(s,t)"] & \mathcal{H}_1 \arrow[d, "(s,t)"] \\
\mathcal{G}_0 \times \mathcal{G}_0 \arrow[r, "f_0 \times f_0"] & \mathcal{H}_0 \times \mathcal{H}_0.
\end{tikzcd}$$

In this case, the diagram takes the form.

$$\begin{tikzcd}
M_3 \times H_3 \arrow[r, "(p,\rho)"] \arrow[d, "(s,t)"] & M_1 \times H_1 \arrow[d, "(s,t)"] \\
M_3 \times M_3 \arrow[r, "(p,p)"] & M_1 \times M_1.
\end{tikzcd}$$

Recall that, here $(s, t)(m_3, h_3) = (m_3, h_3(m_3))$ for $(m_3, h_3) \in M_3 \times H_3$, and similarly, $(s, t)(m_1, h_1) = (m_1, h_1(m_1))$ for $(m_1, h_1) \in M_1 \times H_1$.

To check that the above diagram is a fibered product, we have to complete the following commutative diagram by defining the dashed arrow l, such that the whole diagram commutes. Here A is a manifold and the maps i, j and k are smooth. The two lower right hand side triangles are given to be commutative, that is, $(s, t) \circ i = k = (p, p) \circ j$. For $a \in A$ let $i(a) = (m_1(a), h_1(a))$ and $j(a) = (m_3(a), m_3'(a))$. Then, we get $k(a) = (p(m_3(a)), p(m_3'(a))) = (m_1(a), h_1(a)(m_1(a)))$.

$$\begin{tikzcd}
M_3 \times H_3 \arrow[rr, "(p,\rho)"] \arrow[dd, "(s,t)"] & & M_1 \times H_1 \arrow[dd, "(s,t)"] \\
& A \arrow[ul, dashed, "l"'] \arrow[ur, "i"] \arrow[dl, "j"] \arrow[dr, "k"'] & \\
M_3 \times M_3 \arrow[rr, "(p,p)"] & & M_1 \times M_1.
\end{tikzcd}$$

Next, consider the following diagram. The unique map $h_3(a)$ is obtained by lifting the composition of the horizontal maps using lifting criterion of covering space theory. Now, we define $l(a) = (m_3(a), h_3(a))$. This completes the proof that f is an equivalence of Lie groupoids.

$$\begin{tikzcd}
& & (M_3, m'_3(a)) \arrow[d, "p"] \\
(M_3, m_3(a)) \arrow[r, "p"] \arrow[urr, dashed, "h_3(a)"] & (M_1, m_1(a)) \arrow[r, "h_1(a)"] & (M_1, p(m'_3(a))).
\end{tikzcd}$$

This completes the proof of the lemma. □

We are now ready to recall the following definition.

Definition 2.12. Let $\mathcal{G}$ be an orbifold Lie groupoid inducing an orbifold structure on M. Then, the k-th *orbifold homotopy group* $\pi_k^{orb}(M, x_0)$ of M is defined as the homotopy group $\pi_k(\mathcal{G}, \widetilde{x}_0)$ for some $\widetilde{x}_0 \in \mathcal{G}_0$ lying above a base point $x_0 \in M$. M is called *aspherical* if $\mathcal{G}$ is aspherical.

The fundamental group of an orbifold Lie groupoid $\mathcal{G}$ inducing the orbifold structure on M, is also called the *orbifold fundamental group* of the orbifold M. This group is identified with the standard definition of orbifold fundamental group of M (see [21]). Hence, by Lemmas 2.6 and 2.9, we get the following lemma.

Lemma 2.13. *An orbifold covering map induces isomorphisms on higher homotopy groups, and an injection on orbifold fundamental groups.*

An useful immediate corollary of the lemma is the following.

Corollary 2.14. *Let M be a connected orbifold and $p : \widetilde{M} \to M$ be an orbifold covering map. Assume that $\widetilde{M}$ is connected and has no singular points. Then $p_* : \pi_k(\widetilde{M}) \to \pi_k^{orb}(M)$ is an isomorphism for all $k \geq 2$.*

In general an orbifold need not be the quotient of a manifold by an effective and properly discontinuous action of a discrete group. For example, the sphere with one cone point and the sphere with two cone points of different orders are examples of closed 2-dimensional orbifolds, which are not covered by manifolds. See [21]. Also see Proposition 1.54 and Conjecture 1.55 in [1] for some more general discussion.

It is standard to call a connected orbifold M *good* or *developable* if there is a manifold $\widetilde{M}$ and an orbifold covering map $\widetilde{M} \to M$.

We will need the following extension of the main theorem of [16], which says that a connected 2-dimensional orbifold with finitely generated and infinite orbifold fundamental group is good.

Proposition 2.15. *Let M be a connected 2-dimensional orbifold, with infinite orbifold fundamental group. Then M is good.*

Proof. First, recall that a 2-dimensional orbifold has three different types of singular sets: cone points, reflector lines and corner reflectors (See [[20], p. 422]). The points on the reflector lines and corner reflectors contributes to the boundary of the underlying space, called *orbifold boundary*. Hence, after taking a double of the underlying space along this orbifold boundary components, and then applying [[16], Lemma 2.1], we get an orbifold double cover of M which has only cone points.

Therefore, we can assume that the orbifold M has only cone points. Also, we replace each manifold boundary component with a puncture. Clearly, this does not affect the orbifold fundamental group of M.

Note that, if $\pi_1^{orb}(M)$ is finitely generated, then the proposition follows from [[16], Theorem 1.1]. If it is infinitely generated, then M either has infinite genus or has infinitely many punctures or infinitely many cone points.

Hence, we can write M as an infinite increasing union of orbifolds of the type $M(r_i, s_i)$, $i \in \mathbb{N}$. Each $M(r_i, s_i)$ has finite genus, r_i number of punctures and s_i number of cone points. Furthermore, we can assume that $M(r_i, s_i)$ has infinite orbifold fundamental group. Then clearly, $M(r_i, s_i)$ is aspherical, since they are all good orbifolds with infinite orbifold fundamental groups. Hence, a direct limit argument shows that M is also aspherical, since the orbifold homotopy groups are covariant functors. Therefore, by Lemma 2.13, the universal orbifold covering $\widetilde{M}$ of M has all the orbifold homotopy groups trivial. We now apply [11] to conclude that $\widetilde{M}$ is a manifold. Hence M is a good orbifold. □

3. Asphericity

In this section we state the Asphericity conjecture and Theorem 1.2, in the general set up of the category of Lie groupoids.

If M is an orbifold, then $PB_n(M)$ is an orbifold, since it is an open set in the product orbifold M^n.

Let H be a group acting effectively and properly discontinuously on a connected manifold $\widetilde{M}$. Then the quotient $M = \widetilde{M}/H$ has an orbifold structure.

Consider the space $PB_n(\widetilde{M}, H)$ of n-tuples of points of $\widetilde{M}$ with pairwise distinct orbits, defined in the Introduction. Then, H^n acts effectively and properly discontinuously on $PB_n(\widetilde{M}, H)$, with quotient, the orbifold $PB_n(M)$. Hence, $PB_n(\mathcal{G}(\widetilde{M}, H))$, the configuration Lie groupoid of n points of $\mathcal{G}(\widetilde{M}, H)$ is the corresponding translation orbifold Lie groupoid $\mathcal{G}(PB_n(\widetilde{M}, H), H^n)$. Recall that, for a good orbifold M, we can have many configuration Lie groupoids associated to different regular orbifold coverings of M. Also, clearly given two such orbifold coverings, the corresponding configuration Lie groupoids will induce equivalent orbifold structures on $PB_n(M)$.

In [[17], Theorem 2.10] we proved that the homomorphism $f(\mathcal{G}) : PB_n(\mathcal{G}) \to PB_{n-1}(\mathcal{G})$ is a b-fibration for a c-groupoid $\mathcal{G}$. We had also shown that, this is the best possible class of Lie groupoids to which this fibration result can be proven. In the following proposition, we deduce a more general statement.

A surjective map $g : X \to Y$ is called a *quasifibration*, if $g : (X, g^{-1}(y)) \to (Y, y)$ is a weak homotopy equivalence, for all $y \in Y$ ([6]). Hence, a fibration is a quasifibration, and also a quasifibration induces a long exact sequence of homotopy groups.

Proposition 3.1. *Let H be a finite group, acting effectively on a connected manifold $\widetilde{M}$ of dimension $m \geq 2$, with at least one fixed point. Assume that the integral homology group $H_m(\widetilde{M}, \mathbb{Z})$ is finitely generated. Then the projection map $f_0 : PB_n(\widetilde{M}, H) \to PB_{n-1}(\widetilde{M}, H)$ is not a quasifibration.*

Proof. By hypothesis, there is a point $s \in \widetilde{M}$, so that the isotropy group $H_s = \{h \in H \mid hs = s\}$ is nontrivial. There is a neighbourhood $U_s \subset \widetilde{M}$ of s preserved by H_s and $hU_s \cap U_s = \emptyset$ for all $h \in H - H_s$. Such a neighbourhood exists, see the proof of [[21], Proposition 5.2.6]. Since regular points are dense in $\widetilde{M}/H$, there is a point $s' \in U_s$ which has trivial isotropy group. That is, s' corresponds to a regular point and s corresponds to a singular point on $\widetilde{M}/H$.

Choose a point $x = (s, x_2, \ldots, x_{n-1}) \in PB_{n-1}(\widetilde{M}, H)$, such that $Hx_i \neq Hs'$ for all $i = 2, 3, \ldots, n-1$. Let $y = (s', x_2, \ldots, x_{n-1})$. Note that $|Hs| < |Hs'|$. Then it follows that $f_0^{-1}(x)$ and $f_0^{-1}(y)$ are obtained from $\widetilde{M}$, by removing $|Hs|$ and $|Hs'|$ number of points, respectively. Hence, they have non-isomorphic integral homology groups in dimension m, since $H_m(\widetilde{M}, \mathbb{Z})$ is finitely generated. Therefore, $f_0^{-1}(x)$ and $f_0^{-1}(y)$ are not weak homotopy equivalent. On the other hand, for a quasifibration over a path connected space any two fibers are weak homotopy equivalent ([9], chap. 4, p. 479). Therefore, f_0 is not a quasifibration. □

The examples in the proposition above are the primary reasons why the Fadell-Neuwirth fibration theorem does not extend to orbifolds, and more generally to Lie groupoids.

Remark 3.2. In [8], Flechsig pointed out that the short exact sequence proved in [17] is, in fact, a four-term exact sequence. That is, the kernel of the homomorphism $f(M)_* : \pi_1^{orb}(PB_n(M)) \to \pi_1^{orb}(PB_{n-1}(M))$ is not isomorphic to the orbifold fundamental group of a generic fiber (that is, fiber over a smooth point) of $f(M)$, if M is a genus zero 2-dimensional orbifold with at least one puncture and at least one cone point. See [18] for more on this matter. Therefore, together with Theorem 1.2, for such M we conclude that $f(M)$ is not a quasifibration of orbifolds, that is, it does not induce a long exact sequence of orbifold homotopy groups. Furthermore, using [[18], Theorem 2.2], the same would not be true in general also if the Asphericity conjecture has a positive answer.

Let $\mathcal{C}$ be the class of all connected 2-dimensional orbifolds with $\pi_1^{orb}(M)$ infinite. Let $M \in \mathcal{C}$. Then, by Proposition 2.15, M is a good orbifold. For convenience we denote by $\mathcal{G}_M$, a translation orbifold Lie groupoid $\mathcal{G}(\widetilde{M}, H)$, inducing the orbifold structure on M. Since M is good, there are many such translation orbifold Lie groupoids.

The following proposition justifies the equivalence between the Asphericity conjecture and its orbit configuration space version, of the Introduction.

Proposition 3.3. *Let $M \in \mathcal{C}$ and consider a translation orbifold Lie groupoid $\mathcal{G}(\widetilde{M}, H)$, such that $M = \widetilde{M}/H$. Then, $PB_n(M)$ is aspherical if and only if the manifold $PB_n(\widetilde{M}, H)$ is aspherical.*

Proof. Note that, by our convention (Example 2.3) H is acting on $\widetilde{M}$ effectively and properly discontinuously, so that $M = \widetilde{M}/H$. Therefore, the quotient map

$PB_n(\widetilde{M}, H) \to PB_n(M)$ is an orbifold covering map. Hence by Corollary 2.13, $PB_n(\widetilde{M}, H)$ is aspherical if and only if $PB_n(M)$ is aspherical. $\square$

Corollary 3.4. *Let M be as in the statement of Theorem 1.2. Then, for any translation orbifold Lie groupoid $\mathcal{G}_M$, $PB_n(\mathcal{G}_M)$ is aspherical.*

The above corollary is equivalent to Theorem 1.2, but stated in the category of Lie groupoids. The advantage of this statement is that we can now state the Asphericity conjecture in a wider context.

Recall that, we defined a Lie groupoid $\mathcal{G}$ to be Hausdorff if $|\mathcal{G}|$ is Hausdorff.

Asphericity Problem. *Consider an aspherical Hausdorff Lie groupoid $\mathcal{G}$, such that $\mathcal{G}_0$ is connected and 2-dimensional, then $PB_n(\mathcal{G})$ is aspherical.*

Next, consider the homomorphism $f(\mathcal{G}_M) : PB_n(\mathcal{G}_M) \to PB_{n-1}(\mathcal{G}_M)$, for $M \in \mathcal{C}$.

We end this section by giving a functorial relationship between the homomorphisms $f(\mathcal{G}_M)_*$ and $f(\mathcal{H}_M)_*$, for two translation orbifold Lie groupoids $\mathcal{G}_M$ and $\mathcal{H}_M$, respectively.

Theorem 3.5. *For $M \in \mathcal{C}$ and for any two translation orbifold Lie groupoids $\mathcal{G}_M$ and $\mathcal{H}_M$, we have the following commutative diagram for all q, where the horizontals maps are isomorphisms.*

$$\begin{tikzcd}
\pi_q(PB_n(\mathcal{G}_M)) \arrow[r] \arrow[d, "f(\mathcal{G}_M)_*"] & \pi_q(PB_n(\mathcal{H}_M)) \arrow[d, "f(\mathcal{H}_M)_*"] \\
\pi_q(PB_{n-1}(\mathcal{G}_M)) \arrow[r] & \pi_q(PB_{n-1}(\mathcal{H}_M)).
\end{tikzcd}$$

Proof. The statement is to relate the identification of homotopy groups of the different translation orbifold Lie groupoids inducing the orbifold structure on $PB_n(M)$, and the corresponding homomorphism $f(\mathcal{G}_M)_*$, via a commutative diagram.

Since the two translation orbifold Lie groupoids $\mathcal{G}_M$ and $\mathcal{H}_M$ induce the same orbifold structure on M, by Lemma 2.11 there is another translation orbifold Lie groupoid $\mathcal{K}_M$ and a diagram of equivalences.

$$\mathcal{G}_M \longleftarrow \mathcal{K}_M \longrightarrow \mathcal{H}_M.$$

Since all the orbifold Lie groupoids we are considering are of translation type, it is easy to see that the above diagram induces the following diagram of equivalences. For, the morphism space of $PB_n(\mathcal{G}(\widetilde{M}, H))$ is nothing but $PB_n(\mathcal{G}(\widetilde{M}, H))_0 \times H^n = PB_n(\widetilde{M}, H) \times H^n$, for any translation orbifold Lie groupoid $\mathcal{G}(\widetilde{M}, H)$. See Definition 2.2.

$$PB_n(\mathcal{G}_M) \longleftarrow PB_n(\mathcal{K}_M) \longrightarrow PB_n(\mathcal{H}_M).$$

Hence, we get the following commutative diagram of homomorphisms.

$$\begin{tikzcd}
PB_n(\mathcal{G}_M) \arrow[d, "f(\mathcal{G}_M)"] & PB_n(\mathcal{K}_M) \arrow[l] \arrow[r] \arrow[d, "f(\mathcal{K}_M)"] & PB_n(\mathcal{H}_M) \arrow[d, "f(\mathcal{H}_M)"] \\
PB_{n-1}(\mathcal{G}_M) & PB_{n-1}(\mathcal{K}_M) \arrow[l] \arrow[r] & PB_{n-1(}\mathcal{H}_M).
\end{tikzcd}$$

In the diagram all the horizontal homomorphisms are equivalences, and hence induce weak homotopy equivalences on the classifying spaces.

Now, note that the homomorphisms $f(\mathcal{G}_M)$, $f(\mathcal{K}_M)$ and $f(\mathcal{H}_M)$ all induce the same projection map $PB_n(M) \to PB_{n-1}(M)$.

The theorem now follows, by applying the homotopy functor on the above diagram. □

4. Proof

In this section we prove Theorem 1.2, which supports the Asphericity conjecture.

Proof of Theorem 1.2. **Case 1.** Let M be the complex plane with a cone point of order $m \geq 2$ at the origin. That is, the underlying space of M is $\mathbb{C}$ and the cyclic group (say H) of order m is acting on $\mathbb{C}$ by rotation about the origin, by an angle $\frac{2\pi}{m}$, to get the orbifold M as the quotient.

First note that, in this case the universal orbifold cover of M is $\widetilde{M} = \mathbb{C}$. Then, note that $PB_n(\widetilde{M}, H)$ is the following hyperplane arrangement complement.

$$PB_n(\widetilde{M}, H) = \{(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n \mid z_i^m \neq z_j^m,\ i \neq j\}.$$

In [[14], p5] it was shown that $PB_n(\widetilde{M}, H)$ fibers over the manifold $PB_{n-1}(\mathbb{C}^*)$. For $m = 2$ it was established in [3]. The fibration $B : PB_n(\widetilde{M}, H) \to PB_{n-1}(\mathbb{C}^*)$ is defined by $z_j \mapsto z_n^m - z_j^m$, for $j = 1, 2, \ldots, n-1$. By Theorem 1.1, $PB_{n-1}(\mathbb{C}^*)$ is aspherical. Therefore, again using the long exact sequence of homotopy groups, and by an induction on n, we get that $PB_n(\widetilde{M}, H)$ is aspherical. Hence, by Corollary 2.14 $PB_n(M)$ is aspherical.

Case 2. Assume M has $\mathbb{C}$ as the underlying space, with two cone points at 0 and $\frac{1}{2}$ of order 2 each.

Consider the hyperplane arrangement complement corresponding to the affine Artin group of type $\widetilde{D}_n$ ([3]).

$$P\widetilde{D}_n(\mathbb{C}) := \{(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n \mid z_i \pm z_j \notin \mathbb{Z},\ i \neq j\}.$$

We proceed to show that there is an orbifold covering map $P\widetilde{D}_n(\mathbb{C}) \to PB_n(M)$. We define this map in the following diagram.

$$\begin{tikzcd} P\widetilde{D}_n(\mathbb{C}) \arrow[r, "E"] \arrow[dr] & X \arrow[d, "Q"] \\ & PB_n(M) \end{tikzcd}$$

Here $X = \{(w_1, w_2, \ldots, w_n) \in (\mathbb{C}^*)^n \mid z_i \neq z_j^{\pm 1},\ i \neq j\}$. The map E is the restriction of the n-fold product of the exponential map $z \mapsto \exp(2\pi i z)$, and Q is the restriction of the n-fold product of the map $q : \mathbb{C}^* \to M$, defined by

$$q(w) = \frac{1}{4}\left(1 - \frac{1 + w^2}{2w}\right).$$

E is a genuine covering map and Q is a 2^n-sheeted orbifold covering map. Since q is a 2-fold orbifold covering map as q sends the branch point $+1$ to 0 and -1 to $\frac{1}{2}$, and it is of degree 2 around these points. Therefore, $Q \circ E : P\widetilde{D}_n(\mathbb{C}) \to PB_n(M)$ is an orbifold covering map.

On the other hand recently, in [15], it was proved that $P\widetilde{D}_n(\mathbb{C})$ is aspherical. Therefore, by Corollary 2.14 $PB_n(M)$ is also aspherical.

Case 3. Assume M has $\mathbb{C} - \{1\}$ as the underlying space with 0 a cone point of order 2.

Consider the following hyperplane arrangement complement.

$$W = \{w \in \mathbb{C}^n \mid w_i \neq \pm w_j, \text{ for all } i \neq j; w_k \neq \pm 1, \text{ for all } k\}.$$

In [[4], §3] the following homeomorphism is observed.

$$\mathbb{C}^* \times W \simeq X := \{x \in \mathbb{C}^{n+1} \mid x_i \neq \pm x_j, \text{ for all } i \neq j; x_1 \neq 0\}.$$
$$(\lambda, w_1, w_2, \ldots, w_n) \mapsto (\lambda, \lambda w_1, \ldots, \lambda w_n)$$

In [[4], Lemma 3.1] it is then proved that the hyperplane arrangement complement X is simplicial, in the sense of [5]. Hence, again by [5] X is aspherical. Therefore, W is aspherical.

Next, note that there is the following finite sheeted orbifold covering map.

$$W \to PB_n(M).$$
$$(w_1, w_2, \ldots, w_n) \mapsto (w_1^2, w_2^2, \ldots, w_n^2)$$

Thus, $PB_n(M)$ is also aspherical.

This completes the proof of the theorem. □

Remark 4.1. For a direct proof of asphericity of X, see [10].

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