Direct sum graph of the subspaces of a finite dimensional vector space over finite fields

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Abstract. In this paper, we introduce a new graph structure, called the *direct sum graph* on a finite dimensional vector space. We investigate the connectivity, diameter and the completeness of $\Gamma_{U\oplus W}(\mathbb{V})$. Further, we find its domination number and independence number. We also determine the degree of each vertex in case the base field is finite and show that the graph $\Gamma_{U\oplus W}(\mathbb{V})$ is not Eulerian. We also show that under some mild conditions the graph $\Gamma_{U\oplus W}(\mathbb{V})$ is triangulated. We determine the clique number of $\Gamma_{U\oplus W}(\mathbb{V})$ for some particular cases. Finally, we find the size, girth, edge-connectivity and the chromatic number of $\Gamma_{U\oplus W}(\mathbb{V})$.

Keywords: Vector space; graph, Maximal cliques; Euler graph.

AMS subject classification: 05C25, 05C69.

1 Introduction

Let $G = (V(G), E(G))$ be a simple graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. For $1 \leq i \neq j \leq n$, we write $v_i \sim v_j$ if v_i is adjacent to v_j in G. The cardinality of $V(G)$ and $E(G)$ is called the *order* and the *size* of G, respectively. If $u \in V(G)$, then $N(u)$ is the set of neighbors of u in G, that is, $N(u) = \{v \in (G) : uv \in E(G)\}$. For any two distinct vertices u and v of G, $d(u,v)$ denotes the length of a shortest path between u and v. Clearly $d(u,u) = 0$ and $d(u,v) = \infty$, if there is no path connecting u and v. The *diameter* of G is defined as $diam(G) = max\{d(u,v) : u,v \in V(G)\}$. A graph G is said to be *complete* if every pair of distinct vertices are adjacent and a complete graph on n vertices is denoted by K_n. A complete subgraph of a graph G is called a clique. A maximal clique is a clique which is maximal with respect to inclusion. The clique number of G, written as $\omega(G)$, is the maximum order of a clique in G. The chromatic number of G, denoted as $\chi(G)$, is the minimum number of colors required to label the vertices so that the adjacent vertices receive different colors. A graph G is said to be *connected* if for any pair of vertices $u, v \in V(G)$, there exists a path between u and v. A subset $\alpha(G)$ of V is said to be independent if no two vertices in that subset are pairwise adjacent. The independence number of a graph is the cardinality of the maximum independent set of the vertices in G. A subset D of V is said to be dominating set if every edge in $V \setminus D$ is adjacent to at least one vertex in D. The dominating number of G, denoted by $\gamma(G)$, is the cardinality of the minimum dominating set in G. A subset D of V is said to be a minimal dominating set if D is a dominating set and no proper subset of D is a dominating set. A graph is said to be Eulerian if it contains a closed walk which traverses all the edges in G exactly once. A graph is said to be triangulated if for any vertex u in V, there exists $v, w \in V$, such that (u, v, w) is a triangle. The girth of a graph is the length of the shortest cycle, if it exists. Otherwise, it is defined as ∞. For undefined terms and concepts, the reader is referred to [7, 19].

In 1988, Beck [1] initiated the study of graphs associated with various algebraic structures by introducing the concept of zero-divisor graphs associated to rings. Redmond [22] extended the study of the zero divisor graph of commutative rings to non-commutative rings. DeMeyer et al. [14] associated graphs to semigroups. Redmond [23] took a new approach to define zero divisor graphs with the help of an ideal of a ring. The association of a graph and vector space has history back in 1958 by Gould [16]. Later, Chen [8] investigated on vector spaces associated with a graph. Carvalho [4] has studied vector space and the Petersen Graph. In the recent past, Manjula [18] used vector spaces and made it possible to use techniques of linear algebra in studying the graph. Intersection graphs associated with subspaces of vector spaces were studied in [3, 5, 6, 17, 24]. Recently, Das [9] introduced the graphs associated with the elements of finite dimensional vector space known as *nonzero component graph* of vector space. More work on this can be seen in [11–13].

Motivated by the above work, we introduce the *direct sum graph* as follows: Let $\mathbb{V}$ be a vector space over a field $\mathcal{F}$ with two subspaces U and W. We say that $\mathbb{V}$ is

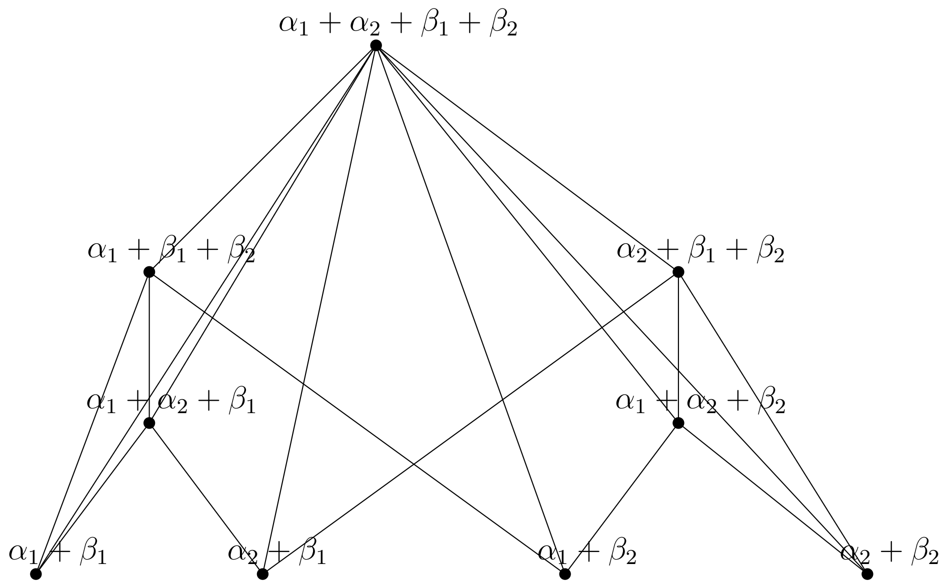

Figure 1: $dim(\mathbb{V}) = 4,\ ,\ dim(U) = 2,\ dim(W) = 2,\ and\ \mathcal{F} = \mathbb{F}_2$

the direct sum of the subspaces U and W, if each vector of $\mathbb{V}$ can always be expressed as the sum of a vector from U and a vector from W, and this expression can only be accomplished in one way (i.e. uniquely). In other words, for every $x \in \mathbb{V}$, there exists vectors $u \in U$, and $w \in W$ such that $x = u + w$ and this representation of v is unique. Let $\mathcal{B}_{\mathbb{V}} = \{\alpha_1, \alpha_2, \dots, \alpha_r + \beta_1 + \beta_2 + \cdots + \beta_s\}$, with $r + s = n$ be a basis of $\mathbb{V}$ and $\mathcal{B}_{\mathbb{U}} = \{\alpha_1, \alpha_2, \dots, \alpha_r\}$ and $\mathcal{B}_{\mathbb{W}} = \{\beta_1, \beta_2, \dots, \beta_s\}$ be basis of U and W respectively. Then any vector $x = u + w$ with unique representation as $\mathbf{x} = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_r\alpha_r + b_1\beta_1 + b_2\beta_2 + \cdots + b_s\beta_s$, is said to have its basic representation with respect to $\mathcal{B}_{\mathbb{U}}$ and $\mathcal{B}_{\mathbb{W}}$. We define the *direct sum graph*, denoted by $\Gamma_{U\oplus W}(\mathbb{V}) = (V, E)$, of a finite dimensional vector space with respect to U and W as follows: $V = \{\mathbf{x} = u + w \in \mathbb{V}\ |u \neq 0 \text{ and } w \neq 0\}$ and for $\mathbf{x_1}, \mathbf{x_2} \in V$, there is an edge between $\mathbf{x_1}$ and $\mathbf{x_2}$, that is, $\mathbf{x_1} \sim \mathbf{x_2}$ or $(\mathbf{x_1}, \mathbf{x_2}) \in E$ if and only if $\mathbf{x_1}$ and $\mathbf{x_2}$ share at least two basis elements one each from $\mathcal{B}_{\mathbb{U}}$ and $\mathcal{B}_{\mathbb{W}}$ having nonzero coefficients in there basic representation. It is necessary to mention here that one of the subspaces is the zero subspace, that is, $\mathbb{U} = 0$(say), that is, $\mathbf{x_1} \sim \mathbf{x_2}$ if and only if $\mathbf{x_1}$ and $\mathbf{x_2}$ share at least one basis element from $\mathcal{B}_{\mathbb{W}}$ having nonzero coefficients in there basic representation. In other words, the *direct sum graph* is the generalization of the *nonzero component graph*. For some examples of the *direct sum graph* $\Gamma_{U\oplus W}(\mathbb{V})$, see Figures 1 and 2.

Throughout this paper, the subspaces U and W of $\mathbb{V} = U \oplus W$ are non trivial and

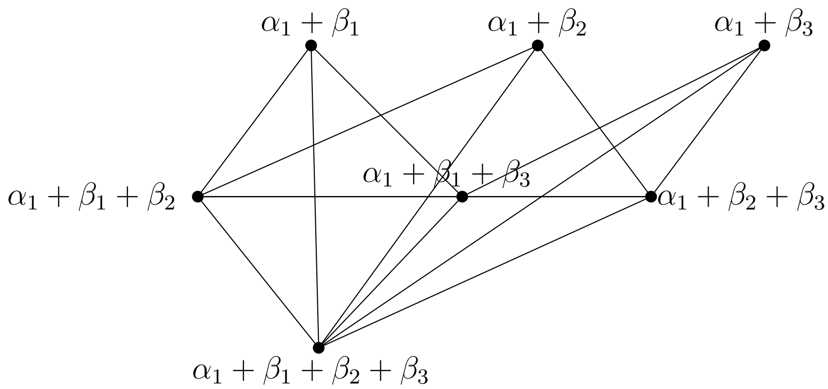

Figure 2: $dim(\mathbb{V}) = 3,\ dim(U) = 1,\ dim(W) = 2,\ and\ \mathcal{F} = \mathbb{F}_2$

$dim(\mathbb{V}) > 1$.

The paper is organized as follows. In Section 2, we determine the order, size and edge connectivity of $\Gamma_{U\oplus W}(\mathbb{V})$. In Section 3, we show that $\Gamma_{U\oplus W}(\mathbb{V})$ is not Eulerian. In addition, we find the conditions under which the graph $\Gamma_{U\oplus W}(\mathbb{V})$ is triangulated. In Section 4, we find clique number and the chromatic number of $\Gamma_{U\oplus W}(\mathbb{V})$.

2 Properties of $\Gamma_{U\oplus W}(\mathbb{V})$

We begin this section by investigating some basic properties like connectedness, completeness, independence number and domination number of $\Gamma_{U\oplus W}(\mathbb{V})$.

Theorem 2.1 *Let $\mathbb{V} = U \oplus W$ be a vector space over a field $\mathcal{F}$. Then $\Gamma_{U\oplus W}(\mathbb{V})$ is connected and $diam(\Gamma_{U\oplus W}) = 2$, if $dim(\mathbb{V}) \geq 3$. If $dim(\mathbb{V}) = 2$, then $diam(\Gamma_{U\oplus W}) = 1$.*

Proof. Assume that $dim(\mathbb{V}) = 2$, U and W are non trivial subspaces of $\mathbb{V}$ that is $dim(U) = dim(W) = 1$. It is easy to see that any two vertices $\mathbf{x}, \mathbf{y} \in \mathbf{V}$ are adjacent. Hence, $\Gamma_{U\oplus W}(\mathbb{V})$ is a complete graph. Now, assume that $dim(\mathbb{V}) = n \geq 3$. Let $Dim(U) = r$ and $Dim(W) = s$, with $r, s \geq 1$ and $r + s = n$. Depending on the choice of a subspace U and W, we will break the proof in two cases:

Case(i) Assume that $Dim(U) = 1$. Then $Dim(W) = n - 1$, that is, W is a hyperspace. Let $\mathbf{x}, \mathbf{y} \in \mathbf{V}$ such that $\mathbf{x} = u + v$ and $\mathbf{y} = u' + v'$. If $\mathbf{x}$ and $\mathbf{y}$ are adjacent, then $d(\mathbf{x}, \mathbf{y}) = 1$. Let $\mathbf{x}$ and $\mathbf{y}$ bee not adjacent. Since $dim(U) = 1$, so u and u' share non-zero coefficient in their basic representation, but v and v' do not share any non-zero coefficient in their basic representation, that is, there exists β_i, β_j $(\beta_i \neq \beta_j)$, which have non-zero coefficient in the basic representation of v and v', respectively. Consider $\mathbf{z} = \mathbf{x} + \mathbf{y} = u + u' + v + v'$. Clearly, $\mathbf{x} \sim \mathbf{z}$ and $\mathbf{z} \sim \mathbf{y}$. Hence, $d(\mathbf{x}, \mathbf{y}) = 2$. Thus, $\Gamma_{U \oplus W}(\mathbb{V})$ is connected and $diam(\Gamma_W) = 2$.

Case(ii) Assume that $Dim(U) = r$ and $Dim(W) = s$, with $r, s \geq 2$ and $r + s = n$. Let $\mathbf{x}, \mathbf{y} \in \mathbf{V}$ such that $\mathbf{x} = u + v$ and $\mathbf{y} = u' + v'$. If $\mathbf{x}$ and $\mathbf{y}$ are adjacent, then $d(\mathbf{x}, \mathbf{y}) = 1$. If $\mathbf{x}$ and $\mathbf{y}$ are not adjacent, then both u, u' and v, v' do not share a non-zero coefficient in their basic representation, that is, there exists α_i, α_j and β_i, β_j ($\alpha_i \neq \alpha_j$ and $\beta_i \neq \beta_j$), which have non-zero coefficient in the basic representation of u, u' and v, v', respectively. Consider $\mathbf{z} = \mathbf{x} + \mathbf{y} = u + u' + v + v'$. Clearly, $\mathbf{x} \sim \mathbf{z}$ and $\mathbf{z} \sim \mathbf{y}$. Hence, $d(\mathbf{x}, \mathbf{y}) = 2$. Thus, $\Gamma_{U \oplus W}(\mathbb{V})$ is connected and $diam(\Gamma_W) = 2$. ■

Theorem 2.2 *Let $\mathbb{V} = U \oplus W$ be a vector space over a field $\mathcal{F}$. Then $\Gamma_{U \oplus W}(\mathbb{V})$ is complete if and only if $dim(\mathbb{V}) = 2$.*

Proof. Assume that $\Gamma_{U \oplus W}(\mathbb{V})$ is complete. We will show that $dim(\mathbb{V}) = 2$. Suppose on the contrary that $dim(\mathbb{V}) > 2$. Let $Dim(U) = r$ and $Dim(W) = s$, with $r \geq 1$, $s \geq 2$ and $r + s = n$. Let $\mathbf{x}, \mathbf{y} \in \mathbf{V}$ such that $\mathbf{x} = \alpha_i + \beta_j$ and $\mathbf{y} = \alpha_i + \beta_k$, where $j \neq k$. Clearly, $\mathbf{x} \nsim \mathbf{z}$, a contradiction. Thus, $dim(\mathbb{V}) = 2$.

Conversely, suppose that $dim(\mathbb{V}) = 2$. So U and W are non trivial subspaces of $\mathbb{V}$. In other words, $dim(U) = dim(W) = 1$. Any two vertices $\mathbf{x}$ and $\mathbf{y}$ are of the form $\mathbf{x} = a_i\alpha_1 + b_i\beta_1$ and $\mathbf{y} = a_i'\alpha_1 + b_i'\beta_1$. Therefore $\mathbf{x} \sim \mathbf{y}$. Hence, $\Gamma_{U \oplus W}(\mathbb{V})$ is a complete graph. ■

Theorem 2.3 *Let $\mathbb{V} = U \oplus W$ be a vector space over a field $\mathcal{F}$. Then the domination number of $\Gamma_{U \oplus W}(\mathbb{V})$ is 1.*

Proof. Assume that $Dim(U) = r$ and $Dim(W) = s$, with $r + s = n$. Let $\{\alpha_1, \alpha_2, \ldots, \alpha_r\}$ and $\{\beta_1, \beta_2, \ldots, \beta_s\}$ be the basis of U and W, respectively. It is easy to see that the vertex $\alpha_1 + \alpha_2 + \cdots + \alpha_r + \beta_1 + \beta_2 + \cdots + \beta_s$ is adjacent to any other vertex of $\mathbf{V}$. Thus the domination number of $\Gamma_{U \oplus W}(\mathbb{V})$ is 1. ■

Remark 2.4 *The set $\{\alpha_1 + \beta_1, \alpha_1 + \beta_2, \ldots, \alpha_1 + \beta_r, \alpha_2 + \beta_1, \alpha_2 + \beta_2, \ldots, \alpha_r + \beta_s\}$ is a minimal dominating set of $\Gamma_{U \oplus W}(\mathbb{V})$. Now, the question arises what is the maximum possible number of vertices in a minimal dominating set. The answer is given as $dim(U)dim(W)$ in the next theorem.*

Theorem 2.5 *Let $\mathbb{V} = U \oplus W$ be a vector space over a field $\mathcal{F}$. If $\mathbf{D} = \{x_1, x_2, \ldots, x_k\}$ is a minimal dominating set of $\Gamma_{U \oplus W}(\mathbb{V})$, then $k \leq dim(U)dim(W)$.*

Proof. Assume that $dim(U) = r$ and $dim(W) = s$, with $r + s = n$. Let $\{\alpha_1, \alpha_2, \ldots, \alpha_r\}$ and $\{\beta_1, \beta_2, \ldots, \beta_s\}$ be the basis of U and W, respectively. Let $\mathbf{D} = \{x_1, x_2, \ldots, x_k\}$ be a minimal dominating set of $\Gamma_{U\oplus W}(\mathbb{V})$. Construct $\mathbf{D}_i = \mathbf{D}\setminus\{x_i\}$, for $1 \leq i \leq k$. Indeed, $\mathbf{D}_i$ is not a dominating set. In other words, corresponding to every $\mathbf{D}_i$, there exists $\eta_i \in \Gamma_{U\oplus W}(\mathbb{V})$, which is not adjacent to any element of $\mathbf{D}_i$ but adjacent to x_i. Since, $\eta_i \neq 0$, there exit α_p^i and β_q^i such that η_i has non-zero component along α_p^i and β_q^i. Now, as η_i is not adjacent to any element of $\mathbf{D}_i$, so is $\alpha_p^i + \beta_q^i$. Thus, for all $i \in \{1, 2, \ldots, k\}$, there exits $\alpha_p^i + \beta_q^i$ such that $\alpha_p^i + \beta_q^i \sim x_i$, but $\alpha_p^i + \beta_q^i \nsim x_j$, for all $j \neq i$.

To complete the proof, it only suffices to show that $\alpha_p^i + \beta_q^i \neq \alpha_p^j + \beta_q^j$ for $i \neq j$. Suppose on contrary $\alpha_p^i + \beta_q^i = \alpha_p^j + \beta_q^j$ for $i \neq j$. Since $\alpha_p^i + \beta_q^i \sim x_i$ and $\alpha_p^i + \beta_q^i = \alpha_p^j + \beta_q^j$, it follows that $\alpha_p^j + \beta_q^j \sim x_i$, which contradicts $\alpha_p^i + \beta_q^i \nsim x_j$, for all $j \neq i$. Hence, $\alpha_p^i + \beta_q^i \neq \alpha_p^j + \beta_q^j$ for $i \neq j$. i.e, all $\alpha_p^i + \beta_q^i$'s are distinct, and the total number of such combinations are rs, it follows that $k \leq dim(U)dim(W)$. ■

From Theorem 2.2, it is clear that $\Gamma_{U\oplus W}(\mathbb{V})$ is complete if and only if $dim(\mathbb{V}) = 2$. Now, the question arises what is the independence number of $\Gamma_{U\oplus W}(\mathbb{V})$ if $dim(\mathbb{V}) \geq 3$. The answer is given in the next theorem.

Theorem 2.6 *Let $\mathbb{V} = U \oplus W$ be a vector space over a field $\mathcal{F}$ with $dim(\mathbb{V}) \geq 3$. Then the independence number of $\Gamma_{U\oplus W}(\mathbb{V})$ is $dim(U)dim(W)$.*

Proof. Since $\{\alpha_1+\beta_1, \alpha_1+\beta_2, \ldots, \alpha_1+\beta_r, \alpha_2+\beta_1, \alpha_2+\beta_2, \ldots, \alpha_r+\beta_s\}$ is an independent set in $\Gamma_{U\oplus W}(\mathbb{V})$, the independence number of $\Gamma_{U\oplus W}(\mathbb{V}) \geq dim(U)dim(W)$. It suffices to show that the independence number of $\Gamma_{U\oplus W}(\mathbb{V}) \leq dim(U)dim(W)$. Suppose to the contrary that $\{x_1, x_2, \ldots, x_k\}$ is an independent set in $\Gamma_{U\oplus W}(\mathbb{V})$ with $k > dim(U)dim(W)$, that is, $x_i \nsim x_j$ for $1 \leq i \neq j \leq k$. Also, for $x_i \in \mathbf{V}$, for all $i \in \{1, 2, \ldots, k\}$, there exits α_p^i, β_q^i having non zero coefficient along α_p^i and β_q^i in its basic representation. Claim that $\alpha_p^i + \beta_q^i \neq \alpha_p^j + \beta_q^j$ for $i \neq j$. Assume that $\alpha_p^i + \beta_q^i = \alpha_p^j + \beta_q^j$ for $i \neq j$. Since $\alpha_p^i + \beta_q^i \sim x_i$ and $\alpha_p^i + \beta_q^i = \alpha_p^j + \beta_q^j$, it follows that $\alpha_p^j + \beta_q^j \sim x_i$. But $\alpha_p^j + \beta_q^j \sim x_j$, which contradicts $x_i \nsim x_j$, for all $i \neq j$. Hence, $\alpha_p^i + \beta_q^i \neq \alpha_p^j + \beta_q^j$ for $i \neq j$. As there are exactly $dim(U)dim(W)$ distinct $\alpha_i + \beta_j$'s, it follows that $k \leq dim(U)dim(W)$, which is a contradiction. Hence, the independence number of $\Gamma_{U\oplus W}(\mathbb{V})$ is $dim(U)dim(W)$. ■

Now, we will find the degree of each vertex of $\Gamma_{U\oplus W}(\mathbb{V})$ if the base field is finite. Let $\mathcal{F} = \mathbb{F}_q$ be a field with q elements and $\mathbb{V} = U \oplus W$ be an n dimensional vector space over $\mathcal{F}$. Let $dim(U) = r$ and $dim(W) = s$, with $r + s = n$. Let $\{\alpha_1, \alpha_2, \ldots, \alpha_r\}$ and $\{\beta_1, \beta_2, \ldots, \beta_s\}$

be basis of U and W, respectively. It is to be noted that

$$\begin{aligned} & N(\alpha_{i_1} + \alpha_{i_2} + \cdots + \alpha_{i_l} + \beta_{j_1} + \beta_{j_2} + \cdots + \beta_{j_m}) \\ = \quad & N(c_1\alpha_{i_1} + c_2\alpha_{i_2} + \cdots + c_l\alpha_{i_l} + d_1\beta_{j_1} + d_2\beta_{j_2} + \cdots + d_m\beta_{j_m}), \end{aligned}$$

where $1 \leq l \leq r$ and $1 \leq m \leq s$ and $c_i \neq 0 \neq d_j \in \mathbb{F}$.

Theorem 2.7 *Let $\mathbb{V} = U \oplus W$ be an n dimensional vector space over a field $\mathcal{F}$ with q elements. Suppose $\{\alpha_1, \alpha_2, \ldots, \alpha_r\}$ and $\{\beta_1, \beta_2, \ldots, \beta_s\}$ are the basis of U and W, respectively, with $r+s = n$. Let $\Gamma_{U\oplus W}(\mathbb{V})$ be a graph associated with $\mathbb{V} = U \oplus W$ Then, the degree of the vertex $c_1\alpha_{i_1} + c_2\alpha_{i_2} + \cdots + c_l\alpha_{i_l} + d_1\beta_{j_1} + d_2\beta_{j_2} + \cdots + d_m\beta_{j_m}$, where $c_1c_2\ldots c_ld_1d_2\ldots d_m \neq 0$, is $(q^l - 1)(q^m - 1)q^{n-(l+m)} - 1$.*

Proof. First fix β_{j_1}. The number of vertices with α_{i_1} and β_{j_1}, as a nonzero component in the basic representation (including $\alpha_{i_1} + \beta_{j_1}$ itself) is $(q-1)^2q^{r+s-2}$. Therefore,

$$deg(\alpha_{i_1} + \beta_{j_1}) = (q-1)^2q^{r+s-2} - 1.$$

The number of the vertices with (α_{i_1} and β_{j_1}) or (α_{i_2} and β_{j_1}) as the non-zero component is equal to the number of vertices with (α_{i_1} and β_{j_1}) as the non-zero component + the number of vertices with (α_{i_2} and β_{j_1}) as the non-zero component $-$ the number of vertices with both ($\alpha_{i_1}, \alpha_{i_2}$ and β_{j_1}) as the non-zero component

$$\begin{aligned} &= (q-1)^2q^{r+s-2} + (q-1)^2q^{r+s-2} - (q-1)^3q^{r+s-3} \\ &= 2(q-1)^2q^{r+s-2} - (q-1)^3q^{r+s-3} \\ &= \big(2(q-1)q^{r-1} - (q-1)^2q^{r-2}\big)(q-1)q^{s-1} \\ &= (q^2-1)q^{r-2}(q-1)q^{s-1}. \end{aligned}$$

Therefore, $deg(\alpha_{i_1} + \alpha_{i_2} + \beta_{j_1}) = (q^2-1)q^{r-2}(q-1)q^{s-1} - 1$.
Similarly, for finding the degree of $\alpha_{i_1} + \alpha_{i_2} + \alpha_{i_3} + \beta_{j_1}$, the number of vertices with α_{i_1}andβ_{j_1} or α_{i_2}andβ_{j_1} or α_{i_3}andβ_{j_1} as the non-zero component is equal to

$$\begin{aligned} &= \binom{3}{1}(q-1)^2q^{r+s-2} - \binom{3}{2}(q-1)^3q^{r+s-3} + \binom{3}{3}(q-1)^4q^{r+s-4} \\ &= \left[\binom{3}{1}(q-1)q^{r-1} - \binom{3}{2}(q-1)^2q^{r-2} + \binom{3}{3}(q-1)^3q^{r-3}\right](q-1)q^{s-1} \\ &= (q^3-1)q^{r-3}(q-1)q^{s-1}. \end{aligned}$$

Hence, $deg(\alpha_{i_1}+\alpha_{i_2}+\alpha_{i_3}+\beta_{j_1}) = (q^3-1)q^{r-3}(q-1)q^{s-1}-1$.
Proceeding in this way, we get

$$deg(\alpha_{i_1}+\alpha_{i_2}+\cdots+\alpha_{i_l}+\beta_{j_1}) = (q^l-1)q^{r-l}(q-1)q^{s-1}-1.$$

Now, run β_{j_p} from $1 \leq p \leq m$, by the above arguments, it is easy to see that

$$deg(\alpha_{i_1}+\alpha_{i_2}+\cdots+\alpha_{i_l}+\beta_{j_1}+\beta_{j_2}+\cdots+\beta_{j_m}) = (q^l-1)q^{r-l}(q^m-1)q^{s-m}-1.$$

Hence,

$$deg(\alpha_{i_1}+\alpha_{i_2}+\cdots+\alpha_{i_l}+\beta_{j_1}+\beta_{j_2}+\cdots+\beta_{j_m}) = (q^l-1)(q^m-1)q^{n-(l+m)}-1.$$

■

3 $\Gamma_{U\oplus W}(\mathbb{V})$ is not Eulerian but triangulated

In this section, we first find the minimum degree and the edge connectivity of $\Gamma_{U\oplus W}(\mathbb{V})$. The main aim of this section is to show that the graph $\Gamma_{U\oplus W}(\mathbb{V})$ is not Eulerian but triangulated. To do this, we require the minimum degree of $\Gamma_{U\oplus W}(\mathbb{V})$, which is obtained as follows.

Theorem 3.1 *Let $\mathbb{V} = U \oplus W$ be an n dimensional vector space over a field $\mathcal{F}$ with q elements. Then the minimum degree δ of $\Gamma_{U\oplus W}(\mathbb{V})$ is $(q-1)^2q^{n-2}-1$.*

Proof. From Theorem 2.7, the degree of the vertex $c_1\alpha_{i_1}+c_2\alpha_{i_2}+\cdots+c_l\alpha_{i_l}+d_1\beta_{j_1}+d_2\beta_{j_2}+\cdots+d_m\beta_{j_m}$, where $c_1c_2\ldots c_ld_1d_2\ldots d_m \neq 0$, is $(q^l-1)(q^m-1)q^{n-(l+m)}-1$. Thus the degree will be minimized if $l = m = 1$ and hence $\delta = (q-1)^2q^{n-2}-1$. ■

Corollary 3.2 *Let $\mathbb{V} = U \oplus W$ be an n dimensional vector space over a field $\mathcal{F}$ with q elements. Then the edge connectivity of $\Gamma_{U\oplus W}(\mathbb{V})$ is $(q-1)^2q^{n-2}-1$.*

Proof. From Theorem 2.2, if $dim(\mathbb{V}) = 2$, then $\Gamma_{U\oplus W}(\mathbb{V})$ is complete. Thus, the degree of every vertex is same. Therefore, the edge connectivity of $\Gamma_{U\oplus W}(\mathbb{V})$ is $(q-1)^2q^{n-2}-1$. On the other hand, if $dim(\mathbb{V}) \geq 3$, then $diam(\Gamma_{U\oplus W}(\mathbb{V})) = 2$, by Theorem 2.1. Therefore, its edge connectivity is equal to its minimum degree (see [21],). i.e, $(q-1)^2q^{n-2}-1$. ■

Now, we show that the graph $\Gamma_{U\oplus W}(\mathbb{V})$ is not Eulerian.

Theorem 3.3 *Let $\mathbb{V} = U \oplus W$ be an n dimensional vector space over a field $\mathcal{F}$ with q elements. Then $\Gamma_{U\oplus W}(\mathbb{V})$ is not an Eulerian.*

Proof. If $q = 2$, by Theorem 2.7, every vertex is of odd degree. Again, if q is odd prime, $\Gamma_{U\oplus W}(\mathbb{V})$ is not Eulerian, since, in this case the degree of every vertex is odd. Thus the graph is not Eulerian in any case. ■

In order to determine whether the graph is triangulated or not, we need to find the order and size of $\Gamma_{U\oplus W}(\mathbb{V})$ and the minimum degree of a vertex.

Theorem 3.4 *Let $\mathbb{V} = U \oplus W$ be an n dimensional vector space over a field $\mathcal{F}$ with q elements. Then the order of $\Gamma_{U\oplus W}(\mathbb{V})$ is $(q^r-1)(q^s-1)$ and size M of $\Gamma_W(\mathbb{V}_\alpha)$ is*

$$\frac{\left(q^{2r}-(2q-1)^r\right)\left(q^{2s}-(2q-1)^s\right)-(q^r-1)(q^s-1)}{2}.$$

Proof. It is easy to observe that the order of $\Gamma_{U\oplus W}(\mathbb{V})$ is $(q^r-1)(q^s-1)$.
It is easy to see that the number of vectors having exactly l α_i's and m, β_j's with nonzero coefficient in its basic representation are $\binom{r}{l}(q-1)^l\binom{s}{m}(q-1)^m$, where $1 \le l \le r$ and $1 \le m \le s$. By Theorem 2.7 and noting the fact that the sum of the degrees of all the vertices in $\Gamma_{U\oplus W}(\mathbb{V})$ is equal to $2M$, we have

$$\begin{aligned}
2M &= \sum_{l=1}^{r}\sum_{m=1}^{s}\binom{r}{l}(q-1)^l\binom{s}{m}(q-1)^m\left((q^l-1)(q^m-1)q^{n-(l+m)}-1\right)\\
&= \sum_{l=1}^{r}\binom{r}{l}(q-1)^l(q^l-1)q^{r-l}\sum_{m=1}^{s}\binom{s}{m}(q-1)^m(q^m-1)q^{s-m}\\
&-\sum_{l=1}^{r}\binom{r}{l}(q-1)^l\sum_{m=1}^{s}\binom{s}{m}(q-1)^m\\
&= \sum_{l=1}^{r}\binom{r}{l}(q-1)^l(q^r-q^{r-l})\sum_{m=1}^{s}\binom{s}{m}(q-1)^m(q^s-q^{s-m}\\
&-\sum_{l=1}^{r}\binom{r}{l}(q-1)^l\sum_{m=1}^{s}\binom{s}{m}(q-1)^m
\end{aligned}$$

That is,

$$\begin{aligned}
2M &= \left(\sum_{l=1}^{r}\binom{r}{l}(q-1)^l q^r - \sum_{l=1}^{r}\binom{r}{l}(q-1)^l q^{r-l}\right)\left(\sum_{m=1}^{s}\binom{s}{m}(q-1)^m q^s\right.\\
&\quad \left.-\sum_{m=1}^{s}\binom{s}{m}(q-1)^m q^{s-m}\right) - \sum_{l=1}^{r}\binom{r}{l}(q-1)^l \sum_{m=1}^{s}\binom{s}{m}(q-1)^m\\
&= \big(q^r(q^r-1) - [(q+q-1)^r - q^r]\big)\big(q^s(q^s-1) - [(q+q-1)^s - q^s]\big)\\
&\quad - [(q-1+1)^r - 1][(q-1+1)^r - 1]\\
&= \big(q^{2r} - (2q-1)^r\big)\big(q^{2s} - (2q-1)^s\big) - (q^r-1)(q^s-1).
\end{aligned}$$

Hence the result follows. ■

The following theorem shows that $\Gamma_{U\oplus W}(\mathbb{V})$ is triangulated.

Theorem 3.5 *Let $\mathbb{V} = U \oplus W$ be an n dimensional vector space over a field $\mathcal{F}$ with q elements.*

(i) If $n = 2$, $dim(W) = n-1$ and $q = 2$, then $\Gamma_{U\oplus W}(\mathbb{V})$ is a trivial graph.

(ii) If $n = 2$, $dim(W) = n-1$ and $q \neq 2$, then $\Gamma_{U\oplus W}(\mathbb{V})$ is triangulated.

(iii) If $n = 3$, $dim(W) = n-1$ and $q = 2$, then $\Gamma_{U\oplus W}(\mathbb{V})$ contains no cycle.

(iv) If $n = 3$, $dim(W) = n-1$ and $q \neq 2$, then $\Gamma_{U\oplus W}(\mathbb{V})$ is triangulated.

(v) If $n \geq 4$, $dim(W) = n-1$ and $q \geq 2$, then $\Gamma_{U\oplus W}(\mathbb{V})$ is triangulated.

(vi) If $n \geq 4$, $dim(W) \leq n-2$ and $q \geq 2$, then $\Gamma_{U\oplus W}(\mathbb{V})$ is triangulated.

Proof. (i). For $n = 2$ and $q = 2$, the vertex set $\mathbf{V}$ is $\{\alpha_1 + \beta_1\}$. Thus, $\Gamma_{U\oplus W}(\mathbb{V})$ is a trivial graph. (ii). For $n = 2$ and $q \neq 2$, $\Gamma_{U\oplus W}(\mathbb{V})$ is a complete graph and therefore triangulated. (iii). For $n = 3$ and $q = 2$, the vertex set $\mathbf{V}$ is $\{\alpha_1+\beta_1, \alpha_1+\beta_2, \alpha_1+\beta_1+\beta_2\}$. Clearly, $\Gamma_{U\oplus W}(\mathbb{V})$ contains no cycle. (iv). For $n = 3$ and $q \neq 2$, there exists $a \in \mathcal{F}\setminus\{0,1\}$. For any arbitrary $\alpha_1 + \beta_1 + \beta_2$, there exist two vertices either$\{\alpha_1 + a\beta_1 + a\beta_2, \alpha_1 + \beta_1\}$ or $\{\alpha_1 + a\beta_1 + a\beta_2, \alpha_1 + \beta_2\}$ such that $\alpha_1+\beta_1+\beta_2 \sim \alpha_1+a\beta_1+a\beta_2 \sim \alpha_1+\beta_1 \sim \alpha_1+\beta_1+\beta_2$ and $\alpha_1 + \beta_1 + \beta_2 \sim \alpha_1 + a\beta_1 + a\beta_2 \sim \alpha_1 + \beta_2 \sim \alpha_1 + \beta_1 + \beta_2$ is a cycle of length three. Hence $\Gamma_{U\oplus W}(\mathbb{V})$ is triangulated. (v). For $n \geq 4$ and $q \geq 2$, every vertex is of the form $c_1\alpha_1 + d_1\beta_1 + d_2\beta_2 + \cdots + \cdots + d_k\beta_k$, where $c_1 \neq 0, d_1, d_2, d_k \in \mathcal{F}$ and $1 \leq k \leq n-1$. Consider an arbitrary vertex $x = c_1\alpha_1 + d_1\beta_{i_1}$, there exists $y = c_1\alpha_1 + d_1\beta_{i_1} + d_2\beta_{i_2}$ and

$z = c_1\alpha_1 + d_1\beta_{i_1} + d_2\beta_{i_1} + d_3\beta_{i_1}$ such that $x \sim y \sim z \sim x$ is a cycle of length three. Hence $\Gamma_{U\oplus W}(\mathbb{V})$ is triangulated. (vi) For $n \geq 4$ and $q \geq 2$, every vertex is of the form $c_1\alpha_{i_1} + c_2\alpha_{i_2} + \cdots + c_l\alpha_{i_l} + d_1\beta_{j_1} + d_2\beta_{j_1} + \cdots + d_m\beta_{j_m}$, where $l + m = n$. Consider any arbitrary vertex $x = c_1\alpha_{i_1} + c_2\alpha_{i_2} + \cdots + c_l\alpha_{i_l} + d_1\beta_{j_1} + d_2\beta_{j_1} + \cdots + d_m\beta_{j_m}$. There exists $y = c_p\alpha_{i_p} + c_q\alpha_{i_q} + d_p\beta_{j_p} + d_q\beta_{j_q}$ and $z = c_p\alpha_{i_p} + d_p\beta_{j_p}$ such that x, y and z share at least two basis vectors having nonzero coefficients in there basic representation. In other words $x \sim y \sim z \sim x$ is a cycle of length three. Hence $\Gamma_{U\oplus W}(\mathbb{V})$ is triangulated. ∎

Now, we obtain the girth of $\Gamma_{U\oplus W}(\mathbb{V})$.

Theorem 3.6 *Let $\mathbb{V} = U \oplus W$ be an n dimensional vector space over a field $\mathcal{F}$ with q elements. Then girth*

$$gr(\Gamma_W(\mathbb{V}_\alpha)) = \begin{cases} \infty & \text{if n=2, dim(W)= n-1 and q=2} \\ 3 & \text{if n=2, dim(W)= n-1 and } q \neq 2 \\ \infty & \text{if n=3, dim(W)= n-1 and q= 2} \\ 3 & \text{if n=3, dim(W)= n-1 and } q \neq 2 \\ 3 & \text{if } n \geq 4\text{, dim(W)= n-1 and } q \geq 2 \\ 3 & \text{if } n \geq 4\text{, dim(W)} \leq \text{ n-2 and } q \geq 2. \end{cases}$$

Proof. The result follows from Theorem 3.5. ∎

4 Maximal Cliques in $\Gamma_{U\oplus W}(\mathbb{V})$

In this section, we find the clique number of $\Gamma_{U\oplus W}(\mathbb{V})$. For $x \in \Gamma_{U\oplus W}(\mathbb{V})$, let S_x (skeleton of x) be the set of α_i's and β_j's with nonzero coefficients in the basic representation of x with respect to the basis of $\mathbb{V}$. It is to be noted that two distinct vertices may have same skeleton. Moreover, if $x = c_1\alpha_{i_1} + c_2\alpha_{i_2} + \cdots + c_{l-1}\alpha_{i_{l-1}} + d_1\beta_{j_1} + d_2\beta_{j_1} + \cdots + d_{m-1}\beta_{j_{m-1}}$ and $y = c_1\alpha_{i_1} + c_2\alpha_{i_2} + \cdots + c_l\alpha_{i_l} + d_1\beta_{j_1} + d_2\beta_{j_1} + \cdots + d_m\beta_{j_m}$, then $S_x \subset S_y$. Also, $2 \leq |S_x| \leq n$, for all $x \in \Gamma_{U\oplus W}(\mathbb{V})$. Let $\mathbf{M}$ be a maximal clique in $\Gamma_{U\oplus W}(\mathbb{V})$ and S_x^U and S_x^W be the skeleton of elements in subspaces U and W of $\mathbb{V}$. Define $S(\mathbf{M}) = \{S_x = S_x^U \cup S_x^W : x \in \mathbf{M}\}$ and $S[\mathbf{M}] = \{|S_x| == S_x^U + S_x^W : S_x \in S(\mathbb{M})\}$. Since $S[\mathbf{M}] \neq \phi$, by well-ordering principle, it has a least element, say $k_1 + k_2 = k$. Then, there exist some $x^* \in \mathbf{M}$ with $|S_x^*| = k_1 + k_2$, where $x^* = c_1\alpha_{i_1} + c_2\alpha_{i_2} + \cdots + c_{k_1}\alpha_{i_{k_1}} + d_1\beta_{j_1} + d_2\beta_{j_1} + \cdots + d_{k_2}\beta_{j_{k_2}}$. Depending upon the choice of $k_1 \leq \frac{r}{2}$, $k_2 \leq \frac{s}{2}$ or $k_1 > \frac{r}{2}$, $k_2 > \frac{r}{2}$, we show that there exists four types of maximal cliques in $\Gamma_{U\oplus W}(\mathbb{V})$.

Theorem 4.1 *Let $\mathbf{M}$ be a maximal clique in $\Gamma_{U\oplus W}(\mathbb{V})$. If $k_1 + k_2$ is the least element of $S[\mathbf{M}]$ with $k_1 \leq \frac{r}{2}$ and $k_2 \leq \frac{s}{2}$, then $\mathbf{M} \in M^{i,j}_{k_1,k_2}$ with $1 \leq k_1 \leq \frac{r}{2}$, $1 \leq k_2 \leq \frac{s}{2}$, $i \in \{1, 2, \ldots, r\}$ and $j \in \{1, 2, \ldots, r\}$, where $M^{i,j}_{k_1,k_2} = \{x \in \Gamma_{U\oplus W}(\mathbb{V}) : \alpha_i, \beta_j \in S_x \text{and } |S_x| \geq k\}$ and*

$$|\mathbf{M}| = (q-1)^2 \sum_{i=k_1-1}^{r-1} \sum_{j=k_2-1}^{s-1} \binom{r-1}{i}\binom{s-1}{j}(q-1)^{i+j}$$

Proof. Let $x = u + w \in \mathbf{M}$, where $u \in U$ and $w \in W$. Since $k_1 \leq \frac{r}{2}$, by Erdos-Ko-Rado theorem [15], the maximum number of pairwise-intersecting k_1-subsets in U is $\binom{r-1}{k_1-1}$ and the maximum is achieved only if each k_1-subset contains a fixed element, say α_i. Now, the number of u's in U with $|S_u| = k_1, k_1 + 1, k_1 + 2, \ldots, r$ and $\alpha_i \in S_u$ are

$$\binom{r-1}{k_1-1}(q-1)^{k_1}, \binom{r-1}{k_1}(q-1)^{k_1+1}, \binom{r-1}{k_1+1}(q-1)^{k_1+2}, \ldots, \binom{r-1}{r-1}(q-1)^{r}, \quad (4.1)$$

respectively.
Using the similar argument, for $k_2 \leq \frac{s}{2}$, the number of W's in W with $|S_w| = k_2, k_2+1, k_2+2, \ldots, s$ and $\beta_j \in S_w$ are

$$\binom{s-1}{k_2-1}(q-1)^{k_2}, \binom{s-1}{k_2}(q-1)^{k_2+1}, \binom{s-1}{k_2+1}(q-1)^{k_2+2}, \ldots, \binom{s-1}{s-1}(q-1)^{s}, \quad (4.2)$$

respectively. As $\mathbf{M}$ is a maximal clique, and minimum of S_x, for $x \in \mathbf{M}$ is $k_1 + k_2$ and $S_x \cap S_y \neq \phi$, so $\mathbf{M} = \{x \in \Gamma_{U\oplus W}(\mathbb{V}) : \alpha_i, \beta_j \in S_x \text{and } |S_x| \geq k_1 + k_2\}$.
Now, the number of x's in $\mathbf{M}$ with $|S_x| = k_1 + k_2$ and $\alpha_i, \beta_j \in S_x$ is

$$\binom{r-1}{k_1-1}(q-1)^{k_1}\binom{s-1}{k_2-1}(q-1)^{k_2}.$$

The number of x's in $\mathbf{M}$ with $|S_x| = k_1 + k_2 + 1$ and $\alpha_i, \beta_j \in S_x$ are

$$\binom{r-1}{k_1-1}(q-1)^{k_1}\binom{s-1}{k_2}(q-1)^{k_2+1} + \binom{r-1}{k_1}(q-1)^{k_1+1} + \binom{s-1}{k_2-1}(q-1)^{k_2}.$$

Similarly, for $|S_x| = k_1 + k_2 + 2, \ldots, r+s$ and $\alpha_i, \beta_j \in S_x$, we can find the number of x's in $\mathbf{M}$ by the help of equations (4.1) and (4.2). Therefore, we have

$$\begin{aligned}
|\mathbf{M}| &= \binom{r-1}{k_1-1}(q-1)^{k_1}\binom{s-1}{k_2-1}(q-1)^{k_2} + \binom{r-1}{k_1-1}(q-1)^{k_1}\binom{s-1}{k_2}(q-1)^{k_2+1} \\
&+ \binom{r-1}{k_1}(q-1)^{k_1+1} + \binom{s-1}{k_2-1}(q-1)^{k_2} + \cdots + \binom{r-1}{r-1}(q-1)^{r}\binom{s-1}{s-1}(q-1)^{s} \\
&= \binom{r-1}{k_1-1}(q-1)^{k_1}\Bigg[\binom{s-1}{k_2-1}(q-1)^{k_2} + \binom{s-1}{k_2}(q-1)^{k_2+1} \\
&\quad + \cdots + \binom{s-1}{s-1}(q-1)^{s}\Bigg] + \binom{r-1}{k_1}(q-1)^{k_1+1}\Bigg[\binom{s-1}{k_2-1}(q-1)^{k_2} \\
&\quad + \binom{s-1}{k_2}(q-1)^{k_2+1} + \cdots + \binom{s-1}{s-1}(q-1)^{s}\Bigg] + \ldots \\
&\quad + \binom{r-1}{r-1}(q-1)^{r}\Bigg[\binom{s-1}{k_2-1}(q-1)^{k_2} + \binom{s-1}{k_2}(q-1)^{k_2+1} \\
&\quad + \cdots + \binom{s-1}{s-1}(q-1)^{s}\Bigg] \\
&= \sum_{i=k_1-1}^{r-1}\binom{r-1}{i}(q-1)^{i+1}\sum_{j=k_2-1}^{s-1}\binom{s-1}{j}(q-1)^{j+1} \\
&= (q-1)^2\sum_{i=k_1-1}^{r-1}\sum_{j=k_2-1}^{s-1}\binom{r-1}{i}\binom{s-1}{j}(q-1)^{i+j}.
\end{aligned}$$

It is to be noted that for same value of k_1, k_2 and by fixing different α_i's and β_j's, we get different maximal cliques. Since these maximal cliques depends both on k_1, k_2 and α_i, β_j, we get a family of maximal cliques $M^{i,j}_{k_1,k_2} = \{x \in \Gamma_{U\oplus W}(\mathbb{V}) : \alpha_i, \beta_j \in S_x \text{and } |S_x| \geq k\}$, where $1 \leq k_1 \leq \frac{r}{2}, 1 \leq k_2 \leq \frac{s}{2}$, $i \in \{1, 2, \ldots, r\}$ and $j \in \{1, 2, \ldots, r\}$ and $\mathbf{M} \in M^{i,j}_{k_1,k_2}$. ■

Theorem 4.2 *Let* $\mathbf{M}$ *be a maximal clique in* $\Gamma_{U\oplus W}(\mathbb{V})$*. If* $k_1 + k_2$ *is the least element of* $S[\mathbf{M}]$ *with* $k_1 \leq \frac{r}{2}$ *and* $k_2 > \frac{s}{2}$*, then* $\mathbf{M} \in M^{i,j}_{k_1,k_2}$ *with* $1 \leq k_1 \leq \frac{r}{2}, k_2 = \lfloor \frac{s}{2} \rfloor + 1$*,* $i \in \{1, 2, \ldots, r\}$ *and* $j \in \{1, 2, \ldots, r\}$*, where* $M^{i,j}_{k_1,k_2} = \{x \in \Gamma_{U\oplus W}(\mathbb{V}) : \alpha_i, \beta_j \in S_x \mathit{and} |S_x| = |S_u| + |S_w| \geq k\}$ *and*

$$|\mathbf{M}| = (q-1)\sum_{i=k_1-1}^{r-1}\sum_{j=k_2}^{s}\binom{r-1}{i}\binom{s}{j}(q-1)^{i+j}.$$

Proof. Let $x = u + w \in \mathbf{M}$, where $u \in U$ and $w \in W$. Since $k_1 \leq \frac{r}{2}$, following the same arguments as in Theorem **??**, the number of u's in U with $|S_u| = k_1, k_1+1, k_1+2, \ldots, r$ and $\alpha_i \in S_u$ are

$$\binom{r-1}{k_1-1}(q-1)^{k_1}, \binom{r-1}{k_1}(q-1)^{k_1+1}, \binom{r-1}{k_1+1}(q-1)^{k_1+2}, \ldots, \binom{r-1}{r-1}(q-1)^{r},$$

respectively. Now, for $k_2 > \frac{s}{2}$, by Erdos-Ko-Rado theorem [15], the maximum number of pairwise-intersecting k_2-subsets in V is $\binom{s}{k_2}$ and the maximum is achieved only if each k_2-subset contains a fixed element, say β_j. The number of w's in W with $|S_w| = k_2, k_2 + 1, k_2+2, \ldots, s$ and $\beta_j \in S_w$ are

$$\binom{s}{k_2}(q-1)^{k_2}, \binom{s}{k_2+1}(q-1)^{k_2+1}, \binom{s}{k_2+2}(q-1)^{k_2+2}, \ldots, \binom{s}{s}(q-1)^{s},$$

respectively. As $\mathbf{M}$ is a maximal clique, and minimum of S_x for $x \in \mathbf{M}$ is $k_1 + k_2$ and $S_x \cap S_y \neq \phi$, so $\mathbf{M} = \{x \in \Gamma_{U \oplus W}(\mathbb{V}) : \alpha_i, \beta_j \in S_x \text{and } |S_x| \geq k_1 + k_2\}$.
Now, following the same procedure as in the proof of Theorem 4.1, we have

$$\begin{aligned}
|\mathbf{M}| &= \binom{r-1}{k_1-1}(q-1)^{k_1}\binom{s}{k_2}(q-1)^{k_2} + \binom{r-1}{k_1-1}(q-1)^{k_1}\binom{s}{k_2+1}(q-1)^{k_2+1} \\
&\quad + \binom{r-1}{k_1}(q-1)^{k_1+1} + \binom{s}{k_2}(q-1)^{k_2} + \cdots + \binom{r-1}{r-1}(q-1)^{r}\binom{s}{s}(q-1)^{s} \\
&= \binom{r-1}{k_1-1}(q-1)^{k_1}\left[\binom{s}{k_2}(q-1)^{k_2} + \binom{s}{k_2+1}(q-1)^{k_2+1}\right. \\
&\quad \left. + \cdots + \binom{s}{s}(q-1)^{s}\right] + \binom{r-1}{k_1}(q-1)^{k_1+1}\left[\binom{s}{k_2}(q-1)^{k_2}\right. \\
&\quad \left. + \binom{s}{k_2+1}(q-1)^{k_2+1} + \cdots + \binom{s}{s}(q-1)^{s}\right] + \ldots \\
&\qquad + \binom{r-1}{r-1}(q-1)^{r}\left[\binom{s}{k_2}(q-1)^{k_2} + \binom{s}{k_2+1}(q-1)^{k_2+1}\right. \\
&\qquad \left. + \cdots + \binom{s}{s}(q-1)^{s}\right] \\
&\qquad = \sum_{i=k_1-1}^{r-1}\binom{r-1}{i}(q-1)^{i+1}\sum_{j=k_2}^{s}\binom{s}{j}(q-1)^{j} \\
&\qquad = (q-1)\sum_{i=k_1-1}^{r-1}\sum_{j=k_2}^{s}\binom{r-1}{i}\binom{s}{j}(q-1)^{i+j}.
\end{aligned}$$

It is to be noted that for same value of k_1, k_2 and by fixing different α_i's and β_j's, we get different maximal cliques. Since these maximal cliques depend both on k_1, k_2 and α_i, β_j, we get a family of maximal cliques $M^{i,j}_{k_1,k_2} = \{x \in \Gamma_{U\oplus W}(\mathbb{V}) : \alpha_i, \beta_j \in S_x \text{and } |S_x| = |S_u| + |S_w| \geq k_1 + k_2\}$, where $1 \leq k_1 \leq \frac{r}{2}, k_2 > \frac{s}{2}$, $i \in \{1, 2, \ldots, r\}$ and $j \in \{1, 2, \ldots, r\}$ and $\mathbf{M} \in M^{i,j}_{k_1,k_2}$. Now, as $k_2 > \frac{s}{2}$, $\{w \in W : |S_w| \geq k_2 + 1\} \subset \{w \in W : |S_w| \geq k_2\}$. Maximality of $\mathbf{M}$ is obtained by minimizing k_2 provided $k_2 > \frac{s}{2}$, that is $k_2 = \lfloor \frac{s}{2} \rfloor + 1$. Thus, we get a family of maximal cliques $M^{i,j}_{k_1}$, where $1 \leq k_1 \leq \frac{r}{2}$, $i \in \{1, 2, \ldots, r\}$ and $j \in \{1, 2, \ldots, r\}$ and $\mathbf{M} \in M^{i,j}_{k_1}$. ∎

Theorem 4.3 *Let $\mathbf{M}$ be a maximal clique in $\Gamma_{U\oplus W}(\mathbb{V})$. If $k_1 + k_2$ is the least element of $S[\mathbf{M}]$ with $k_1 > \frac{r}{2}$ and $k_2 \leq \frac{s}{2}$, then $\mathbf{M} \in M^{i,j}_{k_1,k_2}$ with $k_1 = \lfloor \frac{r}{2} \rfloor + 1$, $1 \leq k_2 \leq \frac{s}{2}$, $i \in \{1, 2, \ldots, r\}$ and $j \in \{1, 2, \ldots, r\}$, where $M^{i,j}_{k_1,k_2} = \{x \in \Gamma_{U\oplus W}(\mathbb{V}) : \alpha_i, \beta_j \in S_x \text{and} |S_x| = |S_u| + |S_w| \geq k\}$ and*

$$|\mathbf{M}| = (q-1) \sum_{i=k_1}^{r} \sum_{j=k_2-1}^{s-1} \binom{r}{i} \binom{s-1}{j} (q-1)^{i+j}.$$

Proof. The proof is similar to that of Theorems 4.1 and 4.2. ∎

Theorem 4.4 *Let $\mathbf{M}$ be a maximal clique in $\Gamma_{U\oplus W}(\mathbb{V})$. If $k_1 + k_2$ is the least element of $S[\mathbf{M}]$ with $k_1 > \frac{r}{2}$ and $k_2 > \frac{s}{2}$, then $k_1 = \lfloor \frac{r}{2} \rfloor + 1$ and $k_2 = \lfloor \frac{s}{2} \rfloor + 1$ and $\mathbf{M} = \{x \in \Gamma_{U\oplus W}(\mathbb{V}) : |S_x| = |S_u| + |S_w|,\ |S_u| \geq \lfloor \frac{r}{2} \rfloor + 1$ and $|S_w| \geq \lfloor \frac{s}{2} \rfloor + 1\}$ and*

$$|\mathbf{M}| = \sum_{i=\lfloor \frac{r}{2} \rfloor+1}^{r} \sum_{j=\lfloor \frac{s}{2} \rfloor+1}^{s} \binom{r}{i} \binom{s}{j} (q-1)^{i+j}.$$

Proof. Let $x = u + w \in \mathbf{M}$, where $u \in U$ and $w \in W$. Since $k_1 > \frac{r}{2}$ and $k_2 > \frac{s}{2}$, by following the same arguments as in Theorems 4.1 and 4.2, the number of u's in U with $|S_u| = k_1, k_1+1, k_1+2, \ldots, r$ and $\alpha_i \in S_u$ are

$$\binom{r}{k_1}(q-1)^{k_1}, \binom{r}{k_1+1}(q-1)^{k_1+1}, \binom{r}{k_1+2}(q-1)^{k_1+2}, \ldots, \binom{r}{r}(q-1)^{r},$$

Similarly, the number of W's in W with $|S_w| = k_2, k_2+1, k_2+2, \ldots, s$ and $\beta_j \in S_w$ are

$$\binom{s}{k_2}(q-1)^{k_2}, \binom{s}{k_2+1}(q-1)^{k_2+1}, \binom{s}{k_2+2}(q-1)^{k_2+2}, \ldots, \binom{s}{s}(q-1)^{s},$$

respectively. As $\mathbf{M}$ is a maximal clique, and minimum of S_x for $x \in \mathbf{M}$ is $k_1 + k_2$ and $S_x \cap S_y \neq \phi$, so $\mathbf{M} = \{x \in \Gamma_{U\oplus W}(\mathbb{V}) : \alpha_i, \beta_j \in S_x \text{and } |S_x| \geq k_1 + k_2\}$.

Therefore, following the same procedure as in the proof of Theorem 4.1, we have

$$\begin{aligned}|\mathbf{M}| &= \sum_{i=k_1}^{r}\binom{r}{i}(q-1)^i\sum_{j=k_2}^{s}\binom{s}{j}(q-1)^j\\ &= \sum_{i=k_1}^{r}\sum_{j=k_2}^{s}\binom{r}{i}\binom{s}{j}(q-1)^{i+j}.\end{aligned}$$

Now, as $k_1 > \frac{r}{2}$ and $k_2 > \frac{s}{2}$, $\{x \in \Gamma_{U\oplus W}(\mathbb{V}) : |S_x| \geq k_1 + k_2 + 1\} \subset \{x \in \Gamma_{U\oplus W}(\mathbb{V}) : |S_x| \geq k_1 + k_2\}$. Maximality of $\mathbf{M}$ is obtained by minimizing k_1 and k_2 provided $k_1 > \frac{r}{2}$ and $k_2 > \frac{s}{2}$, that is, $k_1 = \lfloor\frac{r}{2}\rfloor + 1$ and $k_2 = \lfloor\frac{s}{2}\rfloor + 1$. Thus,

$$|\mathbf{M}| = \sum_{i=\lfloor\frac{r}{2}\rfloor+1}^{r}\sum_{j=\lfloor\frac{s}{2}\rfloor+1}^{s}\binom{r}{i}\binom{s}{j}(q-1)^{i+j}.$$

■

Remark 4.5 *Note that $|M^{i,j}_{k_1,k_2}|$ is maximum when $k_1 = k_2 = 1$ in case $k_1 \leq \frac{r}{2}$ and $k_2 \leq \frac{s}{2}$. So, combining Theorems 4.1, 4.2, 4.3 and 4.4, it is easy to see that the clique number of $\Gamma_{U\oplus W}(\mathbb{V})$ is*

$$\omega(\Gamma_W(\mathbb{V}_\alpha)) = max\Bigg\{(q-1)q^{r-1}(q-1)q^{s-1}, (q-1)q^{r-1}\sum_{j=\lfloor\frac{s}{2}\rfloor+1}^{s}\binom{s}{j}(q-1)^j,$$
$$\sum_{i=\lfloor\frac{r}{2}\rfloor+1}^{r}\binom{r}{i}(q-1)^i(q-1)q^{s-1}, \sum_{i=\lfloor\frac{r}{2}\rfloor+1}^{r}\binom{r}{i}(q-1)^i\sum_{j=\lfloor\frac{s}{2}\rfloor+1}^{s}\binom{s}{j}(q-1)^j.\Bigg\}$$

and it depends on q, r, s.

Corollary 4.6 *If $\mathcal{F} = \mathbb{F}_2$ and $n \geq 4$, then the clique number $\omega(\Gamma_{U\oplus W}(\mathbb{V})) = 2^{n-2}$.*

Proof. If $q = 2$, then $(q-1)q^{r-1} = q^{r-1}$. Moreover, it is easy to see that for $q = 2$ and $p = 2m$ or $p = 2m+1$,

$$\sum_{j=\lfloor\frac{p}{2}\rfloor+1}^{p}\binom{p}{j}(q-1)^j < 2^{p-1}$$

.

Using Remark 4.5, we get $\omega(\Gamma_{U\oplus W}(\mathbb{V})) = 2^{n-2}$. ■

Corollary 4.7 *If $\mathcal{F} = \mathbb{F}_2$, $n \geq 4$, and $\chi(\Gamma_{U\oplus W}(\mathbb{V}))$ is the chromatic number of $\Gamma_{U\oplus W}(\mathbb{V})$, then*

$$2^{n-2} \leq \chi(\Gamma_{U\oplus W}(\mathbb{V})) \leq \frac{q^n - (q^r + q^s + rs)}{2} + 2^{n-3} + 1$$

Proof. For any graph G, $\omega(G) \leq \chi(G)$. Therefore, first part of inequality follows from corollary 4.6. For the other inequality, use the following result of [2]

$$\chi(G) \leq \frac{\omega(G) + |\mathbf{V}| + 1 - \alpha(\Gamma_W)}{2},$$

where $\alpha(\Gamma_W)$ is the independence number of G. Thus, by using Theorems 2.2 and 3.6, we have

$$\begin{aligned}\chi(\Gamma_W(\mathbb{V}_\alpha)) &\leq \frac{2^{n-2} + (q^r - 1)(q^s - 1) + 1 - rs}{2} \\ &= \frac{q^n - (q^r + q^s + rs)}{2} + 2^{n-3} + 1.\end{aligned}$$

■

5 Conclusion

In this work, we have introduced the direct sum graph $\Gamma_{U\oplus W}(\mathbb{V})$ on a finite dimensional vector space $\mathbb{V}$. We investigated some basic properties like connectedness, completeness, independence number and domination number of $\Gamma_{U\oplus W}(\mathbb{V})$. For $n \geq 3$ and $q > 2$, we have shown that $\Gamma_{U\oplus W}(\mathbb{V})$ is triangulated. Also, we also provided the exact value of the minimum degree and the edge connectivity of $\Gamma_{U\oplus W}(\mathbb{V}))$. Moreover, we have seen that $\Gamma_{U\oplus W}(\mathbb{V})$ is not Eulerian. Further, we have found the clique number and the chromatic number of $\Gamma_{U\oplus W}(\mathbb{V})$. For future research, we can think of investigating the following. (i) When $\Gamma_{U\oplus W}(\mathbb{V})$ is regular, (ii) whether $\Gamma_{U\oplus W}(\mathbb{V})$ is a line graph, (iii) characterize line graphs of $\Gamma_{U\oplus W}(\mathbb{V})$, (iv) to find genus of $\Gamma_{U\oplus W}(\mathbb{V})$.

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References

[1] I. Beck, Coloring of commutative rings, *J. Algebra* **116(1)** (1988) 208-226.

[2] R. C. Brigham and R. D. Dutton, *A compilation of relations between graph invariants*, Networks **15** (1985) 73-107.

[3] P. J. Cameron and S. Ghosh, *The power graph of a finite group*, Discrete Math. **311** (2011) 1220-1222.

[4] D. Carvalho, H. Marcelo and C. H. C. Little, *Vector spaces and the Petersen graph*, Elect. J. Comb. **15(1)** (2008), R9.

[5] I. Chakrabarty, S. Ghosh, T. K. Mukherjee and M. K. Sen, *Intersection graphs of ideals of rings*, Discrete Math. **309(17)** (2009) 5381-5392.

[6] I. Chakrabarty, S. Ghosh and M. K. Sen, *Undirected power graphs of semigroup*s, Semigroup Forum **78** (2009) 410-426.

[7] G. Chartrand and P. Zhang, Introduction to Graph Theory, *Tata McGraw Hill, Edition New Delhi (2006).*

[8] W. Chen, *On vector spaces associated with a graph*, SIAM J. Appl. Math. **20** (1971) 526-529.

[9] A. Das, *Non-zero component graph of a finite dimensional vector space,* Comm. Algebra **44** (2016) 3918-3926.

[10] A. Das, *On nonzero component graph of a vector space over finite fields,* J. Algebra Appl. **16(1)** (2017) 1750007.

[11] A. Das, *Subspace inclusion graph of a vector space,* Comm. Algebra **44** (2016) 4724-4731.

[12] A. Das, *Non-zero component graph of a finite dimensional vector space,* Linear Multinear Algebra **65** (2017) 1276-1287.

[13] A. Das, *On Subspace inclusion graph of a vector space,* Linear Multilinear Algebra **66(3)** (2018) 554-564.

[14] F. DeMeyer, T. McKenzie and K. Schneider, *The zero-divisor graph of a commutative semigroup*, Semigroup Forum **65** (2002) 206-214.

[15] P. Erdos, C. Ko and R. Rado, *Intersection theorems for system of finite sets,* Q.J. Math. Oxford Ser. **12** (1961) 313-320.

[16] R. Gould, *Graphs and vector spaces*, Studies in Applied Mathematics **37** (1958), 193-214.

[17] N. Jafari Rad and S. H. Jafari, *Results on the intersection graphs of subspaces of a vector space*, (2011) https://arxiv.org/abs/1105.0803

[18] V. Manjula, *Vector Space of a Graph*, Int. Jour. of Math. and Comp. Res. **2** (2014) 348-352.

[19] S. Pirzada, *An Introduction to Graph Theory*, Universities Press, Orient BlackSwan, Hyderabad (2012).

[20] S. Pirzada, and B.A. Wani, *Subspace based nonzero component graph of a finite dimensional vector space*, Asian European J. Math. (2022) 2250214. https://doi.org/10.1142/S179355712250214X.

[21] J. Plesnik, *Critical graphs of given diameter*, Acta Fac. Rerum Natur. Univ. Comenian. Math. **30** (1975) 71-93.

[22] S. P. Redmond, *The zero-divisor graph of a noncommutative ring*, Int. J. Commutative Rings **4** (2002) 203-211.

[23] S. P. Redmond, *An deal based zero divisor graph of a commutative ring*, Comm. Algebra **31** (2003) 4425-4443.

[24] Y. Talebi, M. S. Esmaeilifar and S. Azizpour, *A kind of intersection graph of vector space*. J. Discrete Math. Sc. Cryp. **12(6)** (2009) 681-689.