Urn models, Markov chains and random walks in cosmological topologically massive gravity at the critical point

Yannick Mvondo-She[1,2]

[1]National Institute of Theoretical and Computational Sciences, School of Physics and Mandelstam Institute for Theoretical Physics, University of the Witwatersrand, Johannesburg, Wits, 2050, South Africa
[2]DSI-NRF Centre of Excellence in Mathematical and Statistical Sciences (CoE-MaSS), South Africa
vondosh7@gmail.com

Abstract

We discuss stochastic processes in the logarithmic sector of critical cosmological topologically massive gravity. By applying a result obtained in our previous works, we show that the logarithmic sector can be modelled as an urn scheme, conceptualizing the random process occurring in the theory as an evolutionary process with a dynamical state space described in terms of the content of the urns. The urn process is then identified as the celebrated Hoppe urn model. In this context, the "special" ball in this Pólya-like urn construction finds a nice interpretation as a point-like defect and gives a concrete perspective of defect propagation in the logarithmic sector of the theory. A Markov process encoded in the partition function is described. It is further shown that the structure of the Markov chain consisting of a sample space that is the set of permutations of n elements, leads to a specific description of the chain in terms of a random walk on the symmetric group. We suggest that a possible holographic dual of LCFT could be a theory that takes into account non-equilibrium phenomena.

Contents

1 Introduction

A special model of gravity in three dimensions has been the subject of considerable interest over the past fifteen years. Under the acronym CCTMG, critical cosmological topologically massive gravity has appeared to display remarkable properties, in particular when the Brown–Henneaux boundary conditions are relaxed

[1]. Indeed, within this framework, the theory has been shown to exhibit a new physical degree of freedom that behaves as a complex logarithmic function, the so-called logarithmic primary mode

$$\psi_{\mu\nu}^{new} := \lim_{\mu l \to 1} \frac{\psi_{\mu l}^{M}(\mu l) - \psi_{\mu l}^{L}}{\mu l - 1} = \{-\ln\cosh\rho - i\tau\}\, \psi_{\mu l}^{L}, \tag{1}$$

which under the action of the $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ AdS_3 isometry group generates descendant single particle logarithmic modes, and whose logarithmic branches extend the single particle class of solutions to a logarithmic multi-particle sector.

The appearance of the new field brought a host of features into the theory. One exotic feature of this particular class of gravity is the non-unitarity of the theory, which arises through the emergence of Jordan cells. The latter being a defining property of logarithmic (L) CFTs [2, 3, 4], CCTMG was conjectured to be holographically dual to $c = 0$ LCFTs, particularly effective in describing systems with (quenched) disorder [5], and was called log gravity [6].

A major result was obtained in the derivation of CCTMG's 1-loop partition function [7], which was shown to agree with the partition function of an LCFT up to single particle. The original expression of the partition function reads

$$Z_{\mathrm{CCTMG}}(q,\bar{q}) = \prod_{n=2}^{\infty} \frac{1}{|1-q^n|^2} \prod_{m=2}^{\infty} \prod_{\bar{m}=0}^{\infty} \frac{1}{1-q^m \bar{q}^{\bar{m}}}, \qquad \text{with} \quad q = e^{2i\pi\tau}.\bar{q} = e^{-2i\pi\bar{\tau}}. \tag{2}$$

From the identification of the first product as the three-dimensional gravity partition function $Z_{0,1}$ in [8], we have the convention

$$Z_{\mathrm{CCTMG}}(q,\bar{q}) = Z_{\mathrm{gravity}}(q,\bar{q}) \cdot Z_{\log}(q,\bar{q}), \tag{3}$$

with contributions

$$Z_{\mathrm{gravity}}(q,\bar{q}) = \prod_{n=2}^{\infty} \frac{1}{|1-q^n|^2}, \qquad \text{and} \qquad Z_{\log}(q,\bar{q}) = \prod_{m=2}^{\infty} \prod_{\bar{m}=0}^{\infty} \frac{1}{1-q^m \bar{q}^{\bar{m}}}. \tag{4}$$

Subsequent efforts to organize the multi-log excitations in a systematic way were undertaken, producing interesting results [9, 10]. The log partition function was shown to be a τ-function of the Kadomtsev–Petviashvili (KP) hierarchy [11], expressed in terms of Schur polynomials $\chi_R(\mathcal{G}_1, \ldots, \mathcal{G}_n)$, which form a basis of the vector space of power series in variables with the coordinate sequence $(\mathcal{G}_k)_{k=1}^{n}$ as

$$\mathcal{G}_k\,(q,\bar{q}) = \frac{1}{|1-q^k|^2}, \tag{5}$$

and which are parametrized by Young diagrams $R = \overbrace{\square \cdots \square}^{n}$ corresponding to the one-part partition $n^1 \vdash n$. The Schur polynomials playing a key role in the representation theory of the symmetric group, their appearance in the log partition function indicates the enumeration of product decomposition of permutations. The resulting expression of the log partition function as the generating function of the cycle decomposition of any permutation π with exactly j_k cycles of length k in the symmetric group S_n of all permutations on integers $1, \ldots, n$ is

$$Z_{log}\,(\mathcal{G}_1, \ldots, \mathcal{G}_n) = 1 + \sum_{n=1}^{\infty} \chi_{\underbrace{\square \ldots \square}_{n}} \cdot \left(q^2\right)^n \tag{6a}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{1}{n!} Z_n\,(\mathcal{G}_1, \ldots, \mathcal{G}_n)\,\left(q^2\right)^n, \tag{6b}$$

with

$$Z_n(\mathcal{G}_1, \ldots, \mathcal{G}_n) = \sum_{\pi \in S_n} \prod_{k=1}^{n} \mathcal{G}_k^{j_k}. \tag{7}$$

Eventually, the logarithm of the log partition function is a solution of the KP I integrable hierarchy, signaling the presence of solitons unstable with respect to transverse perturbations in the theory.

An interesting feature originally observed in CCTMG is the unstable aspect caused by the logarithmic mode, which renders the theory sensible. It was suggested in [1] that the instability could be an artifact of perturbation theory. This argument was shown to be encoded in the Hopf algebraic structure of the log partition function [12], which as a composition of functions, can be endowed with the coproduct of a Faà di Bruno Hopf algebra expected to appear in any perturbation theory.

Generating functions that provide solutions to integrable hierarchies constitute a vast family which includes generating functions of Hurwitz numbers. The log partition function was shown to fall within that category, as a generating function of Hurwitz numbers too, by expressing the one-part Schur polynomial as

$$\chi_{\underbrace{\square \cdots \square}_{n}} = \sum_{\substack{\sum_{k=1}^{n} k j_k = n \\ n \geq 1 \\ j_k \geq 0}} \left\{ H^{\bullet}_{0 \xrightarrow{n} 0} \left[\left([k]^{j_k}\right)_{k=1}^{n}, \left([k]^{j_k}\right)_{k=1}^{n} \right] \right\} \prod_{k=1}^{n} \mathcal{G}_k^{j_k}, \tag{8}$$

with the disconnected Hurwitz numbers given in term of the sequence $\left([k]^{j_k}\right)_{k=1}^{n} = \left([1]^{j_1}, \ldots, [n]^{j_n}\right)$ as

$$H^{\bullet}_{0 \xrightarrow{n} 0} \left[\left([k]^{j_k}\right)_{k=1}^{n}, \left([k]^{j_k}\right)_{k=1}^{n} \right] = \prod_{k=1}^{n} \frac{1}{j_k!(k)^{j_k}}, \qquad \sum_{k=1}^{n} k j_k = 1, \tag{9}$$

where the sequence $\left([k]^{j_k}\right)_{k=1}^{n}$ associated to the monomials $\prod_{k=1}^{n} \mathcal{G}_k^{j_k}$ are such that $[k]^{j_k} = \overbrace{k, \ldots . k}^{j_k \text{ times}}$.

A novel interpretation of the log partition function was established through the connection between its random combinatorial structures and a classical mutation model with numerous applications in mathematical population genetics, called the infinite-alleles model [13]. Such interpretation was possible by realizing that the Hurwitz numbers in the log partition function constitute a set of variables determined by a stochastic process, whose probability distribution is governed by a closed-form sampling formula for the infinite-alleles model called the Ewens sampling formula, with parameter $\theta = 1$ in our case. The model is based on the fact that genes are taken as DNA sequences with a large number of sequence possibilities, the configuration of a sample of size n is specified by an n-tuple $\mathbf{j} = (j_k)_{k=1}^{n} \vdash n$, where j_k denotes the number of allelic types that appear exactly k times in the sample, and mutations engender new alleles (gene types or DNA sequences) not already present in the existing population.

From the investigations done up to now, the log sector can be understood as an integrable model of random growth with a diffusion process on a singular manifold. In this model, solitons unstable with respect to modes of transverse perturbations are broken up in fragments of (defects) soliton clusters realizing the disordered landscape. In this work, building on the knowledge obtained, we make further probes into the log sector.

In section 2, we shown that the log sector is a dynamical sample space that can be modelled using an urn scheme. An urn model can be described as a system of one or more urns containing balls of different colors [14]. As a mathematical model, it represents a useful abstract tool for conceptualizing and modeling random processes through the action of drawing and returning rules, which control the evolution of the sample space of the urn (the contents of the urn). Of interest to us, a particular class of urn models called Pólya urn models consists of an initial single urn whose evolution follows a general method of ball drawing and replacement [15]. We first show that the log sector can be described as an urn process whose evolution is built on the action of linear partial differential operators already constructed in [9] as a generator of the Heisenberg-Weyl algebra. The description is made more explicit by identifying the evolutionary stochastic process of the log sector to the celebrated Hoppe urn model [16], a Pólya-like urn model whose construction was motivated by the idea of obtaining a simple and intuitive way to conceptualize the Ewens sampling formula. The mutation process described by the Ewens sampling formula is reproduced in the Hoppe urn model by introducing a special ball, the "mutator", which initiates the evolutionary process as the only ball in a single urn. Throughout the process, a ball is then withdrawn at random. If the mutator is withdrawn, it is placed back together with a ball of a new colour in the urn. If a ball of any other colour is drawn, one follows the mechanism of Polya's urn, and the selected ball is placed back in the urn together with another ball of the same color. The mutator ball is ignored in describing the urn configuration as it is always present with a prescribed θ-weight as a device for generating mutations. Two inferences can be drawn from this sequential construction scheme. Firstly, constructing the log sector in this way, the weight $\theta = 1$ of the mutator becomes a bias parameter which allows us to interpret the mutator as a defect, and give further evidence of the presence of a diffusing point-like defect in the generation of the disorder landscape in the theory. Secondly, the random clustering process in the evolution of the log multiparticle sector is represented as a Markov chain on irreducible representations of S_n, with clusters counted in sequences of permutations. We show in our case, that the Markov property is implied by the property that the genus zero double Hurwitz numbers with branching corresponding to two identical partitions of n over zero and infinity that is generated by the log partition function satisfies a recurrence relation.

In section 3, we argue that the stochastic process in the log sector can be further described as a random walk on irreducible representations of S_n generated by the Ewens sampling formula. Random walks on finite groups are special types of Markov chains, which in the case of S_n, correspond to random walks on permutations. In the particular case of the log sector, the study the evolution of the cycles in the random permutation, leads to a random transposition walk on S_n. Indeed, the presence in the theory of double Hurwitz numbers in genus zero which not only enumerate the number of $\mathbb{CP}^1 \mapsto \mathbb{CP}^1$ branched covers with two branching points under a specific branching prescription but can also be interpreted as the number of ways of factorizing permutations with given cycle type into transpositions in the symmetric group S_n is a strong indication that the random walk on S_n should be a random transposition walk.

In section 4, we finally summarize our results, and give an outlook.

2 Urn models, defects and Markov Chains

In this section, from the information encoded in the log partition function, we show that the logarithmic sector of CCTMG can be constructed using an urn scheme. We first show that the urn model description can be realized algebraically via the Heisenberg–Weyl algebra. In our case, the latter was obtained in our previous work [9]. We then identify the correct urn model to be the celebrated Hoppe urn model, leading us to a concrete inference of the presence of defects in the theory as mentioned in [13], as well as to a description of the Markov process associated to the Hoppe urn model using a recurrence relation on the Hurwitz numbers appearing in the log partition function.

2.1 From the log partition function to urn models

A partition of a positive integer n can be represented as a sum of positive integers $n = n_1 + n_2 + \ldots + n_m$, and be described equivalently as a sequence $n_{(1)} \geq n_{(2)} \geq \ldots \geq n_{(m)}$, and in terms of a multiplicity sequence $(j_k)_{k=1}^n$ by defining $j_k = \#\{i|\lambda_i = k\}$ as the number of times integer k appears in the unordered set of so-called "occupancy numbers" $\{n_1, \ldots, n_m\}$, such that $\sum_{i=1}^n j_i = m$ and $\sum_{i=1}^n i j_i = n$. These equivalent descriptions exhibit the well-known bijection between the set of partitions of n and the conjugacy classes of the symmetric group S_n, established through the fact that the group S_n can be divided into conjugacy classes according to a cycle structure specified by the length k of a cycle and the number j_k of such a cycle. As a result, there is a one-to-one correspondence between an integer partition and a permutation $\pi \in S_n$ given by its cycle structure.

The log partition function generates counts of random permutations weighted by the number of cycles, where a permutation $\pi \in S_n$ of $\{1, \ldots, n\}$ is decomposed as a product of cycles with π chosen uniformly with probability $1/n!$ and distributed according to the Ewens sampling formula, by encoding a permutation of n with exactly j_k cycles of length (or size) k in the product

$$\prod_{k=1}^{n} \mathcal{G}_k^{j_k}. \tag{10}$$

The above information captured by Eq. (10) can be associated to the elements (balls) of an urn as follows. Consider an urn that consists of elements $\{\bullet\ \bullet\ \bullet\ \bullet\ \bullet\ \bullet\ \bullet\ \bullet\}$ of length (or size) $n = 8$. If we arrange the colors in clusters, and denote the clusters with ket notation as states, the states $|\bullet\bullet\bullet\bullet\rangle$, $|\bullet\bullet\rangle$ and $|\bullet\bullet\rangle$

each appearing with multiplicity one can be encoded as $\mathcal{G}^1_{\bullet\bullet\bullet\bullet}\mathcal{G}^1_{\bullet\bullet}\mathcal{G}^1_{\bullet\bullet}$. Finally, if we are only interested in the cluster size (i.e, the cycle length), we have the equivalence

$$\mathcal{G}^1_{\bullet\bullet\bullet\bullet}\mathcal{G}^1_{\bullet\bullet}\mathcal{G}^1_{\bullet\bullet} \equiv \mathcal{G}^1_4\mathcal{G}^1_2\mathcal{G}^1_2 = \mathcal{G}^1_4\mathcal{G}^2_2, \tag{11}$$

where the partition of $n = 8$ appears as $1\cdot4+2\cdot2$. This equivalence enables us to proceed with the description of the log sector as an urn model.

2.2 Urn models and Heisenberg algebra

Enumeration problems in the probabilistic evolution of an urn process can be encoded in generating functions and represented in terms of polynomials and differential operators. Using an operator representation, a direct correspondence between the algebra of urn processes and the algebra of differential operators was established in [17]. At the centre of the correspondence is the Heisenberg–Weyl algebra. We use this formalism to show how the generic case of a single-mode urn model can be constructed in terms of appropriate differential operators.

We first consider an urn $\mathcal{U}_n$ containing n balls. such that the content of the urn obey two elementary operations

$\mathcal{X}$: putting a ball into the urn, i.e. $\mathcal{U}_n \mapsto \mathcal{U}_{n+1}$
$\mathcal{D}$: withdrawing a ball from the urn, i.e. $\mathcal{U}_n \mapsto \mathcal{U}_{n-1}$

We then represent the urn $\mathcal{U}_n$ by the polynomial $Z_n(\mathcal{G}_1, \dots, \mathcal{G}_n)$, and the elementary operations $\mathcal{X}$ and $\mathcal{D}$ by multiplication $\hat{X}$ and derivative $\hat{D}$ operators respectively. If we consider the operators

$$\hat{X} = \mathcal{G}_1 + \sum_{k=1}^{\infty} k\mathcal{G}_{k+1}\frac{\partial}{\partial\mathcal{G}_k}, \tag{12a}$$

$$\hat{D} = \frac{\partial}{\partial\mathcal{G}_1} \tag{12b}$$

that we constructed in [9], then the Heisenberg–Weyl algebra generated by the operators $\hat{X}$ and $\hat{D}$ satisfying the canonical commutation relation $\left[\hat{D}, \hat{X}\right] = 1$ manifests itself through the actions of multiplication and derivative operators on the polynomial $Z_n(\mathcal{G}_1, \dots, \mathcal{G}_n)$ as

$$\hat{X}Z_n = Z_{n+1}, \tag{13a}$$

$$\hat{D}Z_n = nZ_{n-1}. \tag{13b}$$

The interpretation of the correspondence between the algebra of urn processes and the algebra of differential operator both of which realize the Heisenberg–Weyl algebra is that, given an urn $\mathcal{U}_n$ containing n balls,

– there is only *one* way of putting a ball into the urn ($\mathcal{X}$),
– there are n possible ways of withdrawing a ball from the urn ($\mathcal{D}$).

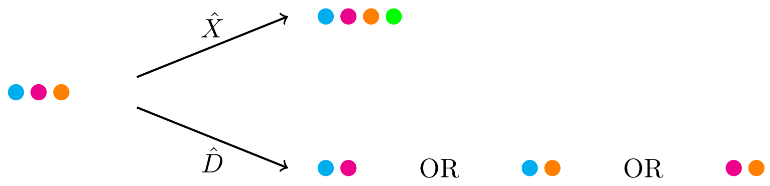

Figure 1: Multiplication and derivation operations from an urn containing three balls.

This can be illustrated by Fig. (1). In what follows, we will see how the above urn model representation takes a well define description, in terms of the celebrated Hoppe urn model.

2.3 Hoppe's urn model description of the log sector

Hoppe's urn model was introduced for the first time in [16] within the framework of Pólya urn models, and with the distinctive feature that whereas Pólya urns scheme of ball drawing and replacement is defined via a finite set of admissible colors, the Hoppe urn model in contrast admits an infinite set of colors.

Hoppe's urn scheme was motivated by the biological phenomenon of the evolution of the partition of alleles according to the Ewens sampling formula, with new alleles represented as the aforementioned infinite set of colors arising via mutation realized by a fixed color. The evolutionary process starts with a single urn containing only the ball with the fixed "black" color, called the mutator as it engenders the mutation. As a ball is withdrawn at random, if the mutator is withdrawn, it is placed back together with a ball of a new colour in the urn. If a ball of any other colour is drawn, one follows the mechanism of Polya's urn, and the selected ball is placed back in the urn together with another ball of the same color. The mutator ball is ignored in describing the urn configuration as it is always present with a prescribed weight $\theta = 1$ and can be treated as a device that generates balls of new colors.

The Hoppe's urn sequential construction scheme is illustrated up to order $n = 3$ in Fig (2), along with the multiplication operator action giving the algebraic correspondence of the probabilistic evolution of the urn process encoded in the polynomials of the log partition function.

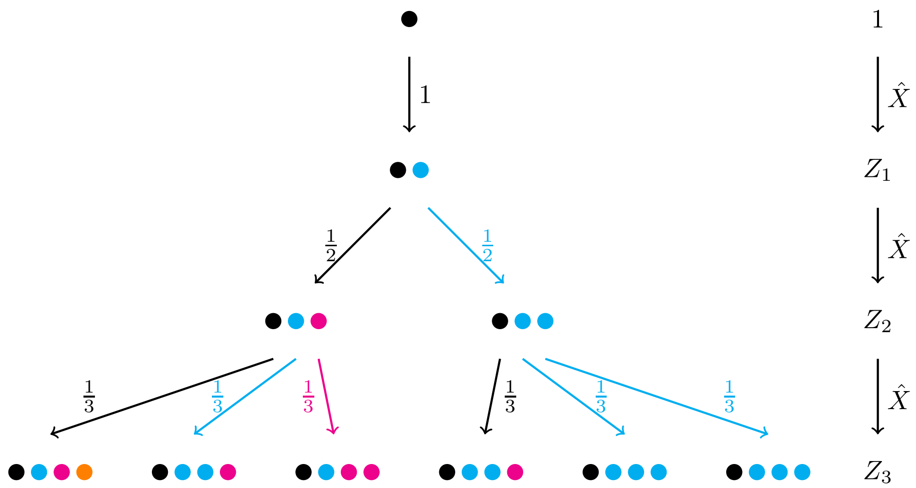

Figure 2: Log sector evolution in terms of sample paths and multiplication operation $\hat{X}$ (i.e, adding a ball) in Hoppe's urn up to order $n = 3$.

From the evolution of cluster distributions in the log partition function according to the Ewens sampling formula and the description of the log sector as Hoppe's urn model, we infer the following two points.

2.3.1 Defect

We previously argued [13] that in the log partition function expansion over rooted trees, soliton clusters are pinned by point-defects represented by the roots as the pinning sites. The content of the urns following Hoppe's scheme corresponds to the graphical representation of the log partition function in terms of rooted trees [12]. In the context of the log sector multiparticle evolution, we then interpret the mutator balls present in the urns of Hoppe's model as the point-defects responsible for the diffusion of the soliton clusters that realize the disorder landscape. Our assertion is justified by the fact that biased urn models can be applied to situations where items are sampled with bias, and one can estimate the number of certain items, for example defective items in a production, from biased sampling. Clearly here, the evolution of the point-like defects in the urn model description of the log sector can be understood from the distribution of cycle counts according to the θ-biased permutation with parameter $\theta = 1$.

2.3.2 Markov process

Markov chains come from the study of randomly perturbed dynamical systems, and arise in various models of non-equilibrium statistical mechanics. They are therefore useful in our study of the log sector.

A Markov chain is a stochastic model that describes a process where transitions between states are governed by probability distributions. It can be defined as a sequence of random variables whose dependence on each other is characterized by the fundamental Markov property, which says that the future state of a given system is conditioned by the past only through the present state of the system. From the Hoppe's urn description of the log sector, we see the appearance of a random grouping process in which units (the balls in the urns) associate in (color-)clusters. We show below that the evolution of the random clustering process can be explained as a Markov process.

The random state of an n-particle in the log sector is described by the sequence $(j_k)_{k=1}^n \in \mathbb{Z}_+^\infty$, in which j_k denotes the number of clusters of size k satisfying $\sum_{k=1}^n kj_k = 1$. Let $\left(J_k^{(n)}\right)_{k=1}^n$ be a sequence of random variables with possible realization in the sequence of values $(j_k)_{k=1}^n$. Clearly, the only possible realization of the sequence $\left(J_k^{(n)}\right)_{k=1}^n$ is the one that partitions an n-particle state into j_1 clusters of size 1, j_2 clusters of size 2, and so on. In other words, the sequence $\left(J_k^{(n)}\right)_{k=1}^n$ is a random integer partition representing the cycle structure of the random permutation $\Pi^{(n)}$ of $[n]$. Equivalently, the two sequences $\left([k]^{j_k}\right)_{k=1}^n$ in the double Hurwitz numbers of Eq. (9) are partitions of the integer n. We therefore reformulate a well-known result with the original aspect being that it involves the genus zero double Hurwitz numbers with branching corresponding to two identical partitions over zero and infinity, and propose that the process $\left(J_k^{(n)}\right)_{k=1}^n$ is a Markov process with state distribution

$$H^\bullet_{0 \xrightarrow{n} 0}\left\{\left[\left(J_k^{(n)}\right)_{k=1}^n, \left(J_k^{(n)}\right)_{k=1}^n\right] \mid \left(J_k^{(n)}\right)_{k=1}^n = (j_k)_{k=1}^n\right\} = \prod_{k=1}^n \frac{1}{j_k!(k)^{j_k}}, \tag{14}$$

that starts at the identity partition for $n = 1$, and evolves in $\mathbb{Z}_+^\infty$ with transition probabilities

$$H^\bullet_{0 \xrightarrow{n+1} 0}\left[(j_1+1, j_2, \ldots), (j_1+1, j_2, \ldots)\right] = \frac{1}{1+n}, \tag{15a}$$

$$H^\bullet_{0 \xrightarrow{n+1} 0}\left[(j_1, \ldots, j_k-1, j_{k+1}+1, \ldots), (j_1, \ldots, j_k-1, j_{k+1}+1, \ldots)\right] = \frac{kj_k}{1+n}, \tag{15b}$$

where the transition in Eq. (15a) occurs when $n+1$ starts a new cycle, and transition in Eq. (15b) takes place when the element is inserted in an existing cycle of $\Pi^{(n)}$.

In particular, the probability in Eq. (14) given by the genus zero double Hurwitz numbers with branching corresponding to two identical partitions over zero and infinity can be shown by induction on n using the recurrence relation (also found in [18])

$$\begin{aligned}(1+n)H^\bullet_{0 \xrightarrow{n+1} 0}\left[(j_1, j_2, \ldots), (j_1, j_2, \ldots)\right] &= H^\bullet_{0 \xrightarrow{n} 0}\left[(j_1-1, j_2, \ldots), (j_1-1, j_2, \ldots)\right] \\ + \sum_{k=1}^n k\left(j_k+1\right) H^\bullet_{0 \xrightarrow{n} 0}&\left[(j_1, \ldots, j_k-1, j_{k+1}+1, \ldots), (j_1, \ldots, j_k-1, j_{k+1}+1, \ldots)\right],\end{aligned} \tag{16}$$

which indicates the Markov property.

3 Random transposition walk on the symmetric group

This section brings a further link between geometric and stochastic aspects of the symmetric group S_n via the Hurwitz numbers. In the previous section, we have shown that the permutations arising in the evolution of the distribution of clusters in the n-particle log sector are not just a sequence of independent random variables, but rather a Markov chain. A special class of Markov chains are random walks on finite groups. In our case, we have a Markov chain generated by conjugacy classes on S_n under the Ewens distribution that corresponds to a type of random walk called random transposition walk.

In our quest of having a better understanding of the log sector in CCTMG, we have obtained various interpretations of the log partition function, which include the enumeration of branched coverings of Riemann surfaces. The earliest attempts to count such coverings go back to Hurwitz's work on enumerating n-sheeted branched covers of the Riemann sphere with d-branched points, that according to the Riemann existence theorem are in one-to-one correspondence with the possible ways of writing the identity permutation in S_n as a product of d transpositions. Precisely, for an even integer $d \geq 0$, the disconnected Hurwitz number $H_{n,d}$ is the number of factorisations of the identity of S_n into d transpositions

$$H_{n,d} = \#\left\{\tau_1, \ldots, \tau_l \in S_n^l,\ \tau_1, \cdots, \tau_d = Id, \text{each } \tau_i \text{ is a transposition}\right\}, \tag{17}$$

is the number of factorisations of the identity of S_n into d transpositions. The presence of Hurwitz numbers which can be interpreted as the number of ways of factorizing permutations with given cycle type into transpositions in S_n is a sign of a diffusing random transposition walk in the log sector of CCTMG.

4 Summary and outlook

In this work, a further step was taken in the study of the log sector of CCTMG, thanks to a correspondence between the distribution of clusters in the n-particle states and an allelic distribution in mathematical genetics both given by the Ewens sampling theory. The results are summarized below.

- We showed that the log sector can be realized as a generic one-mode urn model whose evolution is generated by adding a ball in using the multiplication generator of a Heisenberg–Weyl algebra constructed in our previous work [9], in terms of differential operators acting on the polynomial components of the log partition function.
- From the above algebraic aspect of urn model realization, we further showed that the Ewens fragmentation process encoded in the cluster-size distribution partition function Z_{log} leads to a Hoppe urn model description of the log sector.
- The θ-ball in Hoppe's urn model was interpreted as a (θ-biased) defect, giving a perspective of the dynamics of defect diffusion in the theory that is complementary to the rooted tree expansion of Z_{log} [12], where the roots were identified as defects [13]. The content of the urns in Hoppe's urn model description is in one-to-one correspondence with the rooted trees, and the mutator ball in the urns corresponds to the roots.
- The Markovian property of partition structures in Hoppe's work [16, 19] was shown by the fact the genus zero double Hurwitz numbers with branching corresponding to two identical partitions of n over zero and infinity that is generated by the log partition function satisfies a recurrence relation.
- The structure of the Markov chain consisting of a state space is the set of permutation on S_n lead to a random transposition walk interpretation of the Markov process, due to a state distribution given by the Hurwitz numbers.

The primary motivation that lead us to write this paper is the desire to investigate the evolution of a metastable behavior in the log sector by means of Markov chains and random walks. Indeed, Markov chains are a crucial tool in the study of metastable dynamical systems. We hope further progress in that direction will be made in the future.

The log partition function describe the log sector of CCTMG as a random dynamical system of collective fields that exhibit an evolutionary stochastic behavior. We have in Hoppe's urn description of the log sector the interesting result that the color labelling of the balls in the urns corresponds to a labelling by age. This parallel between age and order of appearance of novel colors in the n-particle evolution causes us to envisage the possibility that an LCFT dual to CCTMG could be one that takes in consideration nonequilibrium phenomena.

Acknowledgements The author would like to thank the anonymous referee in the review process of [13] for drawing his attention to the investigation of mestability in the log sector of CCTMG. The author is grateful for the time spent at the 13th Joburg Workshop on String Theory, where this work approached its conclusion. This work is supported by the South African Research Chairs initiative of the Department of Science and Technology and the National Research Foundation. The support of the DSI-NRF Centre of Excellence in Mathematical and Statistical Sciences (CoE-MaSS) towards this research is hereby acknowledged. Opinions expressed and conclusions arrived at, are those of the author and are not necessarily to be attributed to the CoE.

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