Quantisation of the gauge-invariant models for massive higher-spin bosonic fields

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Abstract

In 2001, Zinoviev proposed a gauge-invariant formulation for a massive bosonic field with spin $s \ge 2$ in a spacetime of constant curvature. In this paper we carry out the Faddeev-Popov quantisation of this theory in *d*-dimensional Minkowski space. We also make use of the Zinoviev theory to derive a generalisation of the Singh-Hagen model for a massive integer-spin field in d > 4 dimensions.

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1 Introduction

Long ago [1-3], the equations describing an on-shell massive field of arbitrary spin s in four dimensions were derived. In the integer-spin case, the equations are

$$\partial^{\nu}\varphi_{\nu\mu_{1}...\mu_{s-1}} = 0$$
, $(\Box - m^{2})\varphi_{\mu_{1}...\mu_{s}} = 0$, (1.1a)

where the dynamical field $\varphi_{\mu_1...\mu_s}$ is symmetric and traceless,

$$\varphi_{(\mu_1\dots\mu_s)} = \varphi_{\mu_1\dots\mu_s} , \qquad \eta^{\nu\rho}\varphi_{\nu\rho\mu_1\dots\mu_{s-2}} = 0 .$$
(1.1b)

In order to realise these equations as Euler-Lagrange equations in a Lagrangian field theory, it was pointed out that certain auxiliary fields are required for s > 1 [3]. The correct set of auxiliary fields and the action principle were found by Singh and Hagen in the bosonic [4] and the fermionic [5] cases. One may think of the integer-spin model of [4] as a higher-spin generalisation of the massive spin-two model proposed by Fierz and Pauli [3],

$$\mathcal{L} = \frac{1}{2}\varphi^{\mu\nu}(\Box - m^2)\varphi_{\mu\nu} + \partial_{\nu}\varphi^{\nu\mu}\partial^{\lambda}\varphi_{\lambda\mu} + \frac{1}{3}\varphi\left\{2\partial_{\mu}\partial_{\nu}\varphi^{\mu\nu} - \left(\Box - 2m^2\right)\varphi\right\}.$$
 (1.2)

Considering a massless limit of the Singh-Hagen model allowed Fronsdal to derive gaugeinvariant formulations for massless higher-spin bosonic fields [6]. It is well known that the massive spin-one model (known as the Proca theory)

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}m^2 A^{\mu}A_{\mu} , \qquad F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} , \qquad (1.3)$$

has a gauge-invariant Stueckelberg reformulation

$$\widetilde{\mathcal{L}} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}\partial^{\mu}\varphi\partial_{\mu}\varphi - \frac{1}{2}m^{2}A^{\mu}A_{\mu} + mA^{\mu}\partial_{\mu}\varphi , \qquad (1.4)$$

which is obtained from (1.3) by replacing $A_{\mu} \to A_{\mu} - m^{-1}\partial_{\mu}\varphi$. By construction, the gauge symmetry of $\widetilde{\mathcal{L}}$ has the form

$$\delta A_{\mu} = \partial_{\mu} \xi , \qquad \delta \varphi = m \xi , \qquad (1.5)$$

with the gauge parameter ξ being arbitrary. This local symmetry allows one to choose the gauge condition $\varphi = 0$, and then $\widetilde{\mathcal{L}}$ turns into (1.3).

The Singh-Hagen model [4] is a non-gauge theory. For various reasons, it is of interest to have its gauge-invariant reformulation being similar to that for the massive spin-one model just discussed. For the spin values s = 2 and s = 3, such reformulations were derived forty years ago by Zinoviev [7]. In 1997, his results were extended by Klishevich and Zinoviev to the case of an arbitrary integer spin in four-dimensional Minkowski space in [8].¹ Finally, a gauge-invariant formulation for massive particles of arbitrary integer spin was constructed by Zinoviev [9] in a *d*-dimensional space of constant curvature. The construction of [9] inspired Metsaev to propose a gauge-invariant formulation for massive totally symmetric fermionic fields in *d*-dimensional (anti-)de Sitter space [10]. The gauge-invariant formulations of [9, 10] have been used for various applications and generalisations, including the frame-like gauge invariant formulation for massive high spin fields [11] and Lagrangian descriptions of massive $\mathcal{N} = 1$ supermultiplets with arbitrary superspin [12, 13].

Recently, gauge-invariant actions for massive arbitrary spin particles in d dimensions have been derived [14] using dimensional reduction of the massless Fronsdal's models in (d + 1)dimensions [6,15]. A precise correspondence with the formulations developed earlier in [9,10] has not been discussed.

To the best of our knowledge, covariant quantisation of the Zinoviev theory has never been studied. This paper is aimed at filling the gap by carrying out the Faddeev-Popov quantisation of the theory in Minkowski space \mathbb{M}^d .

 $^{^{1}}$ It was also shown by the authors of [8] that the Singh-Hagen theory is obtained from their formulation by appropriately fixing the gauge freedom.

This paper is organised as follows. In Section 2 we review the Zinoviev theory in \mathbb{M}^d . Section 3 derives a *d*-dimensional extension of the Singh-Hagen theory.² Sections 4 and 5 are devoted to the quantisation of the massive spin-2 and spin-3 models. The quantisation procedure is then extended in Section 6 to the spin-*s* case.

Throughout this paper, the mostly plus Minkowski metric is used, $\eta_{\mu\nu} = \text{diag}(-1, +1, \dots, +1)$. The dynamical variables of the spin-s Zinoviev's theory are symmetric double traceless gauge fields $\phi_{\mu_1\dots\mu_k}^{(k)}$, with $k = s, s - 1, \dots, 0$. Given such a field $\phi_{\mu_1\dots\mu_k}^{(k)}$, with $k \ge 2$, its trace is denoted

$$\tilde{\phi}_{\mu_3...\mu_k}^{(k)} := \eta^{\rho\sigma} \phi_{\rho\sigma\mu_3...\mu_k}^{(k)} .$$
(1.6)

In the k > 0 case, associated with $\phi_{\mu_1...\mu_k}^{(k)}$ is the symmetric and traceless gauge parameter $\xi_{\mu_1...\mu_{k-1}}^{(k-1)}$. Let $\overline{\psi}^{(k-1)}$ and $\psi^{(k-1)}$ be the symmetric and traceless Faddeev-Popov ghosts corresponding to $\xi^{(k-1)}$. For path integrals, the following shorthand is adopted

$$\int \mathcal{D}(\phi;k) := \int \prod_{j=0}^{k} \mathcal{D}\phi^{(j)} , \qquad (1.7a)$$

$$\int \mathcal{D}(\phi,\psi;k) := \int \Big(\prod_{j=1}^{k} \mathcal{D}\phi^{(j)} \mathcal{D}\overline{\psi}^{(j-1)} \mathcal{D}\psi^{(j-1)} \Big) \mathcal{D}\phi^{(0)} .$$
(1.7b)

In order to evaluate the ghost contributions, the following identity is noted: Faddeev-Popov determinants are realised in terms of the path integral according to the general rule:

$$\det M^{(k)} = \int \mathcal{D}\overline{\psi}^{(k)} \mathcal{D}\psi^{(k)}$$

$$\times \exp\left\{-i\int d^d x \int d^d x' \,\overline{\psi}^{(k)\mu_1\dots\mu_k}(x) M_{\mu_1\dots\mu_k}{}^{\nu_1\dots\nu_k}(x,x') \,\psi^{(k)}_{\nu_1\dots\nu_k}(x')\right\} \,. \tag{1.8a}$$

Here $M^{(k)}$ is an operator acting on the space of symmetric and traceless fields $\psi^{(k)}$,

$$M^{(k)}: \psi^{(k)}_{\mu_1\dots\mu_k}(x) \to \int \mathrm{d}^d x' \, M_{\mu_1\dots\mu_k}{}^{\nu_1\dots\nu_k}(x,x') \, \psi^{(k)}_{\nu_1\dots\nu_k}(x') \; . \tag{1.8b}$$

2 The Zinoviev theory

In this section we briefly review the Zinoviev theory for a massive spin-s field in \mathbb{M}^d . It is described by the action

$$S = \int \mathrm{d}^d x \, \mathcal{L}^{(s)} \,, \qquad \mathcal{L}^{(s)} = \sum_{k=0}^s \mathcal{L}_c(\phi^{(k)}) \,, \qquad (2.1a)$$

²It was recently mentioned [14] that "this formulation only works in four spacetime dimensions."

where the dynamical variables $\phi_{\mu_1...\mu_k}^{(k)}$ are symmetric double traceless fields, and $\mathcal{L}_c(\phi^{(k)})$ has the form

$$\mathcal{L}_c(\phi^{(k)}) = \mathcal{L}_0(\phi^{(k)}) + \mathcal{L}_m(\phi^{(k)}) .$$
(2.1b)

Here the first term on the right is Fronsdal's Lagrangian [6] for a massless spin-k field,

$$\mathcal{L}_{0}(\phi^{(k)}) = -\frac{1}{2}\partial^{\mu}\phi^{(k)\mu_{1}...\mu_{k}}\partial_{\mu}\phi^{(k)}_{\mu_{1}...\mu_{k}} + \frac{k}{2}\partial_{\mu}\phi^{(k)\mu\mu_{2}...\mu_{k}}\partial^{\nu}\phi^{(k)}_{\nu\mu_{2}...\mu_{k}} + \frac{k(k-1)}{4}\partial^{\mu}\tilde{\phi}^{(k)\mu_{3}...\mu_{k}}\partial_{\mu}\tilde{\phi}^{(k)}_{\mu_{3}...\mu_{k}} + \frac{k(k-1)}{2}\partial_{\mu}\partial_{\nu}\phi^{(k)\mu\nu\mu_{3}...\mu_{k}}\tilde{\phi}^{(k)}_{\mu_{3}...\mu_{k}}$$
(2.2)
$$+ \frac{k(k-1)(k-2)}{8}\partial_{\mu}\tilde{\phi}^{(k)\mu\mu_{4}...\mu_{k}}\partial^{\nu}\tilde{\phi}^{(k)}_{\nu\mu_{4}...\mu_{k}} .$$

The second term in (2.1b) is a massive contribution of the following structure:

$$\mathcal{L}_{m}(\phi^{(k)}) = a_{k}\phi^{(k-1)\mu_{2}...\mu_{k}}\partial^{\mu}\phi^{(k)}_{\mu\mu_{2}...\mu_{k}} + b_{k}\tilde{\phi}^{(k)\mu_{3}...\mu_{k}}\partial^{\mu}\phi^{(k-1)}_{\mu\mu_{3}...\mu_{k}} + c_{k}\partial_{\mu}\tilde{\phi}^{(k)\mu\mu_{4}...\mu_{k}}\tilde{\phi}^{(k-1)}_{\mu_{4}...\mu_{k}} + d_{k}\phi^{(k)\mu_{1}...\mu_{k}}\phi^{(k)}_{\mu_{1}...\mu_{k}} + e_{k}\tilde{\phi}^{(k)\mu_{3}...\mu_{k}}\tilde{\phi}^{(k)}_{\mu_{3}...\mu_{k}} - f_{k}\tilde{\phi}^{(k)\mu_{3}...\mu_{k}}\phi^{(k-2)}_{\mu_{3}...\mu_{k}}.$$
(2.3)

It involves several numerical coefficients, all of which are determined by requiring the action (2.1) to be invariant under gauge transformations

$$\delta\phi_{\mu_1\dots\mu_k}^{(k)} = \partial_{(\mu_1}\xi_{\mu_2\dots\mu_k)}^{(k-1)} + \alpha_k\xi_{\mu_1\dots\mu_k}^{(k)} - \frac{k(k-1)}{2}\beta_k\eta_{(\mu_1\mu_2}m\xi_{\mu_3\dots\mu_k)}^{(k-2)} , \qquad (2.4)$$

where α_k and β_k are some coefficients, and $\xi^{(k)}$ are symmetric and traceless gauge parameters.³ Direct calculations show that the coefficients in (2.3) and β 's in (2.4) are fixed in terms of α 's:

$$a_{k} = -k\alpha_{k-1} , \qquad b_{k} = -k(k-1)\alpha_{k-1} , \qquad c_{k} = -\frac{k(k-1)(k-2)}{4}\alpha_{k-1} , \qquad (2.5a)$$

$$2d_k = \frac{2(k+1)(2k+d-3)}{2d+k-4} (\alpha_k)^2 - k(\alpha_{k-1})^2 , \quad k \ge 1 ; \qquad d_0 = \frac{d}{d-2} (\alpha_1)^2 , \quad (2.5b)$$

$$2e_k = -\frac{k(k^2 - 1)(2k + d)}{4(2k + d - 4)}(\alpha_k)^2 + \frac{k^2(k - 1)}{2}(\alpha_{k-1})^2 , \qquad (2.5c)$$

$$2f_k = -k(k-1)\alpha_{k-1}\alpha_{k-2} , \qquad (2.5d)$$

$$\beta_k = \frac{2\alpha_{k-1}}{(k-1)(2k+d-6)} \,. \tag{2.5e}$$

For the coefficients α 's one gets

$$(\alpha_k)^2 = \frac{s(s-k)(s+k+d-3)}{(k+1)(2k+d-2)} (\alpha_{s-1})^2 , \qquad 0 \le k \le s-2 , \qquad (2.6a)$$

for which it is convenient to choose $(\alpha_{s-1})^2$:

$$\frac{(\alpha_{s-1})^2 = \frac{m^2}{s}}{s} \implies (\alpha_k)^2 = \frac{(s-k)(s+k+d-3)}{(k+1)(2k+d-2)}m^2 , \quad 0 \le k \le s-2 .$$
(2.6b)

³As usual, it is assumed in (2.4) that when a term would contain fields of negative rank in the above, that term is ignored.

3 Singh-Hagen model in *d* dimensions

In four dimensions, it was shown by Klishevich and Zinoviev [8] that the Singh-Hagen theory [4] is obtained from the d = 4 version of the theory described in the previous section by appropriately fixing the gauge freedom. Here we extend the analysis of [8] to d dimensions.

In order to derive a *d*-dimensional analogue of the Singh-Hagen theory from Zinoviev's one, it suffices to choose a unitary gauge. To start with, for k > 1 we decompose each double traceless symmetric field $\phi_{\mu_1...\mu_k}^{(k)}$ into a sum of two traceless symmetric fields $\omega_{\mu_1...\mu_k}^{(k)}$ and $\varphi_{\mu_1...\mu_{k-2}}^{(k-2)}$. In terms of these, the field $\phi^{(k)}$ is

$$\phi_{\mu_1\dots\mu_k}^{(k)} = \omega_{\mu_1\dots\mu_k}^{(k)} + \frac{k(k-1)}{2(d+2k-4)} \eta_{(\mu_1\mu_2}\varphi_{\mu_3\dots\mu_k)}^{(k-2)} , \qquad (3.1)$$

with $\tilde{\phi}_{\mu_1\dots\mu_{k-2}}^{(k)} = \varphi_{\mu_1\dots\mu_{k-2}}^{(k-2)}$, with $k = s, \dots, 2$. Then, the gauge freedom (2.4) may be completely fixed by imposing the conditions

$$\omega_{\mu_1\dots\mu_k}^{(k)} = 0 , \qquad k = 0, 1, \dots, s - 1 .$$
(3.2)

The remaining rank-s field will be re-labelled as follows: $\omega_{\mu_1...\mu_s}^{(s)} = \varphi_{\mu_1...\mu_s}^{(s)}$. As a result, upon imposing the gauge condition (3.2) we stay with the following fields

$$\phi_{\mu_1\dots\mu_s}^{(s)} = \varphi_{\mu_1\dots\mu_s}^{(s)} + \frac{s(s-1)}{2(d+2s-4)} \eta_{(\mu_1\mu_2} \varphi_{\mu_3\dots\mu_s)}^{(s-2)} , \qquad (3.3a)$$

$$\phi_{\mu_1\dots\mu_k}^{(k)} = \frac{k(k-1)}{2(d+2k-4)} \eta_{(\mu_1\mu_2} \varphi_{\mu_3\dots\mu_k)}^{(k-2)} , \qquad k \le s-1 .$$
(3.3b)

We now turn to massaging the separate contributions to the Lagrangian $\mathcal{L}^{(s)}$, eq. (2.1), in the gauge (3.2) or, equivalently, (3.3). For the massless Lagrangian $\mathcal{L}_0(\phi^{(s)})$ we obtain

$$\mathcal{L}_{0}(\phi^{(s)}) = \frac{1}{2} \varphi^{(s)\mu_{1}...\mu_{s}} \Box \varphi^{(s)}_{\mu_{1}...\mu_{s}} + \frac{s}{2} \partial_{\mu} \varphi^{(s)\mu\mu_{2}...\mu_{s}} \partial^{\nu} \varphi^{(s)}_{\nu\mu_{2}...\mu_{s}} + \frac{s(s-1)(d+2s-6)}{2(d+2s-4)} \partial_{\mu} \partial_{\nu} \varphi^{(s)\mu\nu\mu_{3}...\mu_{s}} \varphi^{(s-2)}_{\mu_{3}...\mu_{s}} - \frac{s(s-1)(d+2s-6)(d+2s-5)}{4(d+2s-4)^{2}} \varphi^{(s-2)\mu_{3}...\mu_{s}} \Box \varphi^{(s-2)}_{\mu_{3}...\mu_{s}} + \frac{(s-2)(d+2s-6)(d+2s-8)s(s-1)}{8(d+2s-4)^{2}} \partial_{\mu} \varphi^{(s-2)\mu\mu_{4}...\mu_{s}} \partial^{\nu} \varphi^{(s-2)}_{\nu\mu_{4}...\mu_{s}} .$$
(3.4)

For $0 \le k \le s - 1$, the massless Lagrangian $\mathcal{L}_0(\phi^{(k)})$ leads to

$$\mathcal{L}_0(\phi^{(k)}) = -\frac{k(k-1)(d+2k-6)(d+2k-5)}{4(d+2k-4)^2}\varphi^{(k-2)\mu_3\dots\mu_k}\Box\varphi^{(k-2)}_{\mu_3\dots\mu_k}$$

$$+\frac{k(k-1)(k-2)(d+2k-6)(d+2k-8)}{8(d+2k-4)^2}\partial_{\mu}\varphi^{(k-2)\mu\mu_{4}...\mu_{k}}\partial^{\nu}\varphi^{(k-2)}_{\nu\mu_{4}...\mu_{k}}.$$
 (3.5)

Next, we turn to the massive contributions (2.3). For $0 \le k \le s - 1$, we obtain

$$\mathcal{L}_{m}(\phi^{(k)}) = a_{k} \frac{(k-1)(k-2)}{2(d+2k-6)} \varphi^{(k-3)\mu_{4}...\mu_{k}} \partial^{\mu} \varphi^{(k-2)}_{\mu\mu_{4}...\mu_{k}} + b_{k} \frac{k-2}{d+2k-6} \varphi^{(k-2)\mu\mu_{4}...\mu_{k}} \partial_{\mu} \varphi^{(k-3)}_{\mu_{4}...\mu_{k}} + c_{k} \partial_{\mu} \varphi^{(k-2)\mu\mu_{4}...\mu_{k}} \varphi^{(k-3)}_{\mu_{4}...\mu_{k}} + d_{k} \frac{k(k-1)}{2(d+2k-4)} \varphi^{(k-2)\mu_{3}...\mu_{k}} \varphi^{(k-2)}_{\mu_{3}...\mu_{k}}$$
(3.6)
$$+ e_{k} \varphi^{(k-2)\mu_{3}...\mu_{k}} \varphi^{(k-2)}_{\mu_{3}...\mu_{k}} .$$

Substituting in the values for a_k , b_k and c_k from (2.5), upon integration by parts we get

$$\mathcal{L}_{m}(\phi^{(k)}) = \alpha_{k-1} \frac{k(k-1)(k-2)(d+2k-8)}{4(d+2k-6)} \varphi^{(k-3)\mu_{4}...\mu_{k}} \partial^{\mu} \varphi^{(k-2)}_{\mu\mu_{4}...\mu_{k}} + \left(d_{k} \frac{k(k-1)}{2(d+2k-4)} + e_{k} \right) \varphi^{(k-2)\mu_{3}..\mu_{k}} \varphi^{(k-2)}_{\mu_{3}..\mu_{k}} .$$
(3.7)

On the other hand, the massive contribution $\mathcal{L}_m(\phi^{(s)})$ is

$$\mathcal{L}_{m}(\phi^{(s)}) = d_{s}\varphi^{(s)\mu_{1}\dots\mu_{s}}\varphi^{(s)}_{\mu_{1}\dots\mu_{s}} + \alpha_{s-1}\frac{s(s-1)(s-2)(d+2s-8)}{4(d+2s-6)}\varphi^{(s-3)\mu_{4}\dots\mu_{s}}\partial^{\mu}\varphi^{(s-2)}_{\mu\mu_{4}\dots\mu_{s}} + \left(d_{s}\frac{s(s-1)}{2(d+2s-4)} + e_{s}\right)\varphi^{(s-2)\mu_{3}\dots\mu_{s}}\varphi^{(s-2)}_{\mu_{3}\dots\mu_{s}}.$$
(3.8)

Finally, it remains to substitute the contributions (3.4-3.8) in the full Lagrangian (2.1) as well as to make use of (2.5) and (2.6) to end up with

$$\begin{aligned} \mathcal{L}^{(s)} &= \frac{1}{2} \varphi^{(s)\mu_1...\mu_s} (\Box - m^2) \varphi^{(s)}_{\mu_1...\mu_s} + \frac{1}{2} s \partial^{\nu} \varphi^{(s)}_{\nu\mu_2...\mu_s} \partial_{\lambda} \varphi^{(s)\lambda\mu_2...\mu_s} \\ &+ \frac{s(s-1)(d+2s-6)}{2(d+2s-4)} \partial_{\mu} \partial_{\nu} \varphi^{(s)\mu\nu\mu_3...\mu_s} \varphi^{(s-2)}_{\mu_3...\mu_s} \\ &- \frac{s(s-1)(d+2s-6)(d+2s-5)}{4(d+2s-4)^2} \varphi^{(s-2)\mu_3...\mu_s} \left(\Box - \frac{d+2s-4}{d+2s-6}m^2\right) \varphi^{(s-2)}_{\mu_3...\mu_s} \\ &+ \frac{s(s-1)(s-2)(d+2s-6)(d+2s-8)}{8(d+2s-4)^2} \partial_{\mu} \varphi^{(s-2)\mu\mu_4...\mu_s} \partial^{\nu} \varphi^{(s-2)}_{\nu\mu_4...\mu_s} \\ &- \sum_{q=3}^{s} \left(\frac{(s-q+2)(s-q+1)(d+2s-2q-2)}{4(d+2s-2q)}\right) \\ &\times \left[\frac{(d+2s-2q-1)}{2(d+2s-2q)} \varphi^{(s-q)\mu_{q+1}...\mu_s} \left(\Box - \frac{(d+2s-2q)q(d+2s-q-3)}{2(d+2s-2q-2)(d+2s-2q-1)}m^2\right) \varphi^{(s-q)}_{\mu_{q+1}...\mu_s} \\ &- \frac{(s-q)(d+2s-2q-4)}{2(d+2s-2q)} \partial_{\mu} \varphi^{(s-q)\mu\mu_{q+2}...\mu_s} \partial^{\nu} \varphi^{(s-q)}_{\nu\mu_{q+2}...\mu_s} \end{aligned}$$

$$-\sqrt{\frac{(s-q+3)(q-2)(d+2s-q-1)}{(d+2s-2q+2)}}m\varphi^{(s-q)\mu_{q+1}\dots\mu_{s}}\partial^{\mu}\varphi^{(s-q+1)}_{\mu\mu_{q+1}\dots\mu_{s}}\right] .$$

This Lagrangian defines the Singh-Hagen model in d dimensions, which has so far been described in the literature only in the d = 4 case [4], see below.

Let us analyse the equation of motion corresponding to (3.9). It is useful to adopt the Singh-Hagen notation $\{\ldots\}_{S.T.}$, which denotes the symmetric and traceless component of the term within the brackets. The equation of motion for the field $\varphi^{(s)}$ is

$$(-\Box + m^2)\varphi_{\mu_1\dots\mu_s}^{(s)} + s\partial^{\nu} \{\partial_{\mu_1}\varphi_{\nu\mu_2\dots\mu_s}^{(s)}\}_{\text{S.T.}} = \frac{s(s-1)(d+2s-6)}{2(d+2s-4)} \{\partial_{\mu_1}\partial_{\mu_2}\varphi_{\mu_3\dots\mu_s}^{(s-2)}\}_{\text{S.T.}} \quad (3.10a)$$

The equation of motion for the field $\varphi^{(s-2)}$ is

$$\frac{s(s-1)(d+2s-6)(d+2s-5)}{(d+2s-4)^2} \left(\Box - \frac{d+2s-4}{d+2s-6} m^2 \right) \varphi_{\mu_3\dots\mu_s}^{(s-2)}
+ \frac{s(s-1)(s-2)(d+2s-6)(d+2s-8)}{2(d+2s-4)^2} \partial^{\nu} \{ \partial_{\mu_3} \varphi_{\nu\mu_4\dots\mu_s}^{(s-2)} \}_{\text{S.T.}}$$
(3.10b)

$$+ \frac{(s-1)(s-2)(d+2s-4)\sqrt{s}}{2(d+2s-6)} m \{ \partial_{\mu_3} \varphi_{\mu_4\dots\mu_s}^{(s-3)} \}_{\text{S.T.}} = \frac{s(s-1)(d+2s-6)}{(d+2s-4)} \partial^{\nu} \partial^{\lambda} \varphi_{\nu\lambda\mu_3\dots\mu_s}^{(s)} .$$

Finally, the equation of motion for the fields $\varphi^{(s-q)}$, with $3 \leq q \leq s$, are the following:

$$\begin{split} & \left(\frac{(s-q+2)(s-q+1)(d+2s-2q-2)}{4(d+2s-2q)}\right) \\ \times \left[\frac{(d+2s-2q-1)}{(d+2s-2q)} \left(\Box - \frac{(d+2s-2q)q(d+2s-q-3)}{2(d+2s-2q-2)(d+2s-2q-1)}m^2\right)\varphi_{\mu_{q+1}\dots\mu_s}^{(s-q)} \right. \\ & \left. + \frac{(s-q)(d+2s-2q-4)}{(d+2s-2q)}\partial^{\nu} \{\partial_{\mu_{q+1}}\varphi_{\nu\mu_{q+2}\dots\mu_s}^{(s-q)}\}_{\text{S.T.}} \right. \end{split} \tag{3.10c} \\ & \left. - \sqrt{\frac{(s-q+3)(q-2)(d+2s-q-1)}{(d+2s-2q+2)}}m\partial^{\mu}\varphi_{\mu\mu_{q+1}\dots\mu_s}^{(s-q+1)} \right] \\ = - \left(\frac{(s-q+3)(s-q+2)(d+2s-2q)}{4(d+2s-2q+2)}\right) \\ \times \sqrt{\frac{(s-q+4)(q-3)(d+2s-q)}{(d+2s-2q)}}m\{\partial_{\mu_{q+1}}\varphi_{\mu_{q+2}\dots\mu_s}^{(s-q-1)}\}_{\text{S.T.}} . \end{split}$$

After some algebra, it may be seen that these equations yield $\varphi^{(s-q)} = 0$ for $s \ge q \ge 2$, while the field $\varphi^{(s)}$ obeys the Fierz-Pauli equations (1.1a). As a result, the number of on-shell degrees of freedom is

$$n(d,s) = \frac{2s+d-3}{d-3} \binom{d+s-4}{s} , \qquad (3.11)$$

which reduces to 2s + 1 degrees of freedom in four spacetime dimensions. In three spacetime dimensions, (3.11) should be replaced with n(3, s) = 2.

As an instructive check, it is worth comparing (3.9) directly with the Singh-Hagen model [4] formulated in \mathbb{M}^4 . Choosing d = 4 in (3.9) yields

$$\mathcal{L} = \frac{1}{2} \varphi^{(s)\mu_{1}\dots\mu_{s}} (\Box - m^{2}) \varphi^{(s)}_{\mu_{1}\dots\mu_{s}} + \frac{1}{2} s \partial^{\nu} \varphi^{(s)}_{\nu\mu_{2}\dots\mu_{s}} \partial_{\lambda} \varphi^{(s)\lambda\mu_{2}\dots\mu_{s}} + \frac{(s-1)^{2}}{2} \partial_{\mu} \partial_{\nu} \varphi^{(s)\mu\nu\mu_{3}\dots\mu_{s}} \varphi^{(s-2)}_{\mu_{3}\dots\mu_{s}} - \frac{(s-1)^{2}(2s-1)}{8s} \varphi^{(s-2)\mu_{3}\dots\mu_{s}} \left(\Box - \frac{s}{s-1}m^{2}\right) \varphi^{(s-2)}_{\mu_{3}\dots\mu_{s}} + \frac{(s-1)^{2}(s-2)^{2}}{8s} \partial_{\mu} \varphi^{(s-2)\mu\mu_{4}\dots\mu_{s}} \partial^{\nu} \varphi^{(s-2)}_{\nu\mu_{4}\dots\mu_{s}} - \sum_{q=3}^{s} \left[\frac{(s-q+1)^{2}(2s-2q+3)}{8(s-q+2)} \varphi^{(s-q)\mu_{q+1}\dots\mu_{s}} \left(\Box - \frac{(s-q+2)k(2s-q+1)}{2(s-q+1)(2s-2q+3)}m^{2}\right) \varphi^{(s-q)}_{\mu_{q+1}\dots\mu_{s}} - \frac{(s-q+1)^{2}(s-q)^{2}}{8(s-q+2)} \partial_{\mu} \varphi^{(s-q)\mu\mu_{q+2}\dots\mu_{s}} \partial^{\nu} \varphi^{(s-q)}_{\nu\mu_{q+2}\dots\mu_{s}} - \frac{(s-q+1)^{2}}{4} \sqrt{\frac{(q-2)(2s-q+3)}{2}} m \varphi^{(s-q)\mu_{q+1}\dots\mu_{s}} \partial^{\mu} \varphi^{(s-q+1)}_{\mu\mu_{q+1}\dots\mu_{s}} \right] .$$

Let us rescale the fields $\varphi^{(s-2)}$ and $\varphi^{(s-q)}$, with $3 \le q \le s$, as follows:

$$\begin{aligned} \varphi_{\mu_{3}\dots\mu_{s}}^{(s-2)} &\longrightarrow \frac{2s}{2s-1}\varphi_{\mu_{3}\dots\mu_{s}}^{(s-2)} , \\ \varphi_{\mu_{q+1}\dots\mu_{s}}^{(s-q)} &\longrightarrow \frac{2(s-1)}{(s-q+1)}\sqrt{\frac{s(s-q+2)}{(2s-1)(2s-2q+3)}} \\ &\times \left(\prod_{j=2}^{q-1}\sqrt{\frac{(j-1)(s-j)^{2}(s-j+2)(2s-j+2)}{2(s-j+1)(2s-2j+1)(2s-2j+3)}}\right)\varphi_{\mu_{q+1}\dots\mu_{s}}^{(s-q)} . \end{aligned}$$

$$(3.13a)$$

This yields

$$\mathcal{L} = \frac{1}{2} \varphi^{(s)\mu_{1}...\mu_{s}} (\Box - m^{2}) \varphi^{(s)}_{\mu_{1}...\mu_{s}} + \frac{1}{2} s \partial^{\nu} \varphi^{(s)}_{\nu\mu_{2}...\mu_{s}} \partial_{\lambda} \varphi^{(s)\lambda\mu_{2}...\mu_{s}}
+ \frac{s(s-1)^{2}}{2s-1} \left\{ \partial_{\mu} \partial_{\nu} \varphi^{(s)\mu\nu\mu_{3}...\mu_{s}} \varphi^{(s-2)}_{\mu_{3}...\mu_{s}} - \frac{1}{2} \varphi^{(s-2)\mu_{3}...\mu_{s}} \left(\Box - \frac{s}{s-1} m^{2}\right) \varphi^{(s-2)}_{\mu_{3}...\mu_{s}}
+ \frac{(s-2)^{2}}{2(2s-1)} \partial_{\mu} \varphi^{(s-2)\mu\mu_{4}...\mu_{s}} \partial^{\nu} \varphi^{(s-2)}_{\nu\mu_{4}...\mu_{s}}
- \sum_{q=3}^{s} \left(\prod_{j=2}^{q-1} \frac{(j-1)(s-j)^{2}(s-j+2)(2s-j+2)}{2(s-j+1)(2s-2j+1)(2s-2j+3)}\right)
\times \left[\frac{1}{2} \varphi^{(s-q)\mu_{q+1}...\mu_{s}} \left(\Box - \frac{q(s-q+2)(2s-q+1)}{2(s-q+1)(2s-2q+3)} m^{2}\right) \varphi^{(s-q)}_{\mu_{q+1}...\mu_{s}}\right]$$
(3.14)

$$-\frac{(s-q)^2}{2(2s-2q+3)}\partial_{\mu}\varphi^{(s-q)\mu\mu_{q+2}...\mu_s}\partial^{\nu}\varphi^{(s-q)}_{\nu\mu_{q+2}...\mu_s} - m\varphi^{(s-q)\mu_{q+1}...\mu_s}\partial^{\mu}\varphi^{(s-q+1)}_{\mu\mu_{q+1}...\mu_s}\bigg]\bigg\} ,$$

which is exactly the Lagrangian derived by Singh and Hagen [4].

4 Quantisation: Spin s = 2

The Zinoviev theory (2.1) is an irreducible gauge theory (following the terminology of the Batalin-Vilkovisky formalism [16]) and can be quantised à la Faddeev and Popov [17]. Before considering its quantisation in the general $s \ge 2$ case, it is worth investigating how the process is carried out for the simplest s = 2 and s = 3 values.

In the spin s = 2 case, the action (2.1) is

$$S = \int d^{d}x \left\{ -\frac{1}{2} \partial^{\mu} \phi^{(2)\nu\lambda} \partial_{\mu} \phi^{(2)}_{\nu\lambda} + \partial_{\mu} \phi^{(2)\mu\lambda} \partial^{\nu} \phi^{(2)}_{\nu\lambda} + \frac{1}{2} \partial^{\mu} \tilde{\phi}^{(2)} \partial_{\mu} \tilde{\phi}^{(2)} + \partial^{\mu} \partial^{\mu} \phi^{(2)}_{\mu\nu} \partial^{\mu} \phi^{(2)}_{\mu\nu} - \frac{1}{2} \partial^{\mu} \phi^{(0)} \partial_{\mu} \phi^{(0)} + a_{2} \phi^{(1)\nu} \partial^{\mu} \phi^{(2)}_{\mu\nu} + b_{2} \tilde{\phi}^{(2)} \partial^{\mu} \phi^{(1)}_{\mu} + d_{2} \phi^{(2)\mu\nu} \phi^{(2)}_{\mu\nu} + e_{2} (\tilde{\phi}^{(2)})^{2} + a_{1} \phi^{(0)} \partial^{\mu} \phi^{(1)}_{\mu} - f_{2} \tilde{\phi}^{(2)} \phi^{(0)} + d_{1} \phi^{(1)\mu} \phi^{(1)}_{\mu} + d_{0} (\phi^{(0)})^{2} \right\} ,$$

$$(4.1)$$

and the gauge transformation (2.4) reads

$$\delta\phi_{\mu\nu}^{(2)} = \frac{1}{2} \left(\partial_{\mu}\xi_{\nu}^{(1)} + \partial_{\nu}\xi_{\mu}^{(1)} \right) - \beta_{2}\eta_{\mu\nu}\xi^{(0)} , \qquad (4.2a)$$

$$\delta \phi_{\mu}^{(1)} = \partial_{\mu} \xi^{(0)} + \alpha_1 \xi_{\mu}^{(1)} , \qquad (4.2b)$$

$$\delta \phi^{(0)} = \alpha_0 \xi^{(0)} .$$
 (4.2c)

To carry out the Faddeev-Popov scheme, suitable gauge conditions are required. We choose the following gauge-fixing functions:

$$\Xi_{\mu}^{(1)} - \chi_{\mu}^{(1)} = 2\partial^{\nu}\phi_{\mu\nu}^{(2)} - \partial_{\mu}\tilde{\phi}^{(2)} - m\sqrt{2}\phi_{\mu}^{(1)} - \chi_{\mu}^{(1)} , \qquad (4.3a)$$

$$\Xi^{(0)} - \chi^{(0)} = \partial^{\mu} \phi^{(1)}_{\mu} - \frac{m}{\sqrt{2}} \tilde{\phi}^{(2)} - \frac{2d-2}{d-2} \frac{m^2}{\alpha_0} \phi^{(0)} - \chi^{(0)} , \qquad (4.3b)$$

where $\chi^{(1)}$ and $\chi^{(0)}$ are background fields. The explicit expressions for $\Xi^{(1)}$ and $\Xi^{(0)}$ have been chosen so that their gauge variations are

$$\delta \Xi^{(1)}_{\mu} = (\Box - m^2) \xi^{(1)}_{\mu} , \qquad (4.4a)$$

$$\delta \Xi^{(0)} = (\Box - m^2) \xi^{(0)} . \tag{4.4b}$$

With these, we have the partition function

$$Z^{(2)} = \int \mathcal{D}(\phi; 2) \Delta^{(1)} \Delta^{(0)} \delta\left[\Xi^{(1)}_{\mu} - \chi^{(1)}_{\mu}\right] \delta\left[\Xi^{(0)} - \chi^{(0)}\right] e^{iS} , \qquad (4.5)$$

where the objects $\Delta^{(1)}$ and $\Delta^{(0)}$ are the Faddeev-Popov determinants,

$$\Delta^{(0)} = \det\left(\frac{\delta\Xi^{(0)}(x)}{\delta\xi^{(0)}(x')}\right) = \det\left[(\Box - m^2)\delta^d(x - x')\right] , \qquad (4.6a)$$

$$\Delta^{(1)} = \det\left(\frac{\delta\Xi^{(1)}_{\mu}(x)}{\delta\xi^{(1)}_{\nu}(x')}\right) = \det\left[\delta_{\mu}{}^{\nu}(\Box - m^2)\delta^d(x - x')\right].$$
(4.6b)

Since the partition function (4.5) is independent of the background fields $\chi^{(1)}$ and $\chi^{(0)}$, we can average over them with a convenient weight of the form

$$\exp\left\{-\frac{\mathrm{i}}{2}\int \mathrm{d}^{d}x \,\left[\frac{\chi^{(1)\mu}\chi^{(1)}_{\mu}}{\omega_{1}} + \frac{(\chi^{(0)})^{2}}{\omega_{0}}\right]\right\} , \qquad (4.7)$$

where ω_1 and ω_0 are constants that will be chosen in such a way as to diagonalise the action as well as cause all divergence terms to vanish. Doing so, (4.5) becomes

$$Z^{(2)} = \int \mathcal{D}(\phi; 2) \Delta^{(1)} \Delta^{(0)} \\ \times \exp\left\{-\frac{i}{2} \int d^d x \left[\frac{1}{\omega_1} \left(2\partial^{\nu}\phi^{(2)}_{\mu\nu} - \partial_{\mu}\tilde{\phi}^{(2)} - m\sqrt{2}\phi^{(1)}_{\mu}\right)^2 + \frac{1}{\omega_0} \left(\partial^{\mu}\phi^{(1)}_{\mu} - \frac{m}{\sqrt{2}}\tilde{\phi}^{(2)} - \frac{2d-2}{d-2}\frac{m^2}{\alpha_0}\phi^{(0)}\right)^2\right]\right\} e^{iS} .$$

$$(4.8)$$

Substituting the classical action (4.1) in (4.8) yields

$$Z^{(2)} = \int \mathcal{D}(\phi; 2) \Delta^{(1)} \Delta^{(0)} \exp\left\{ i \int d^d x \left[-\frac{1}{2} \partial^\mu \phi^{(2)\nu\lambda} \partial_\mu \phi^{(2)}_{\nu\lambda} + \left(1 - \frac{2}{\omega_1} \right) \partial_\mu \phi^{(2)\mu\lambda} \partial^\nu \phi^{(2)}_{\nu\lambda} + \frac{1}{2} \left(1 - \frac{1}{\omega_1} \right) \partial^\mu \tilde{\phi}^{(2)} \partial_\mu \tilde{\phi}^{(2)} + \left(1 - \frac{2}{\omega_1} \right) \partial^\mu \partial^\nu \phi^{(2)}_{\mu\nu} \tilde{\phi}^{(2)} - \frac{1}{2} \partial^\mu \phi^{(1)\nu} \partial_\mu \phi^{(1)}_{\nu}$$

$$+ \left(d_{1} - \frac{m^{2}}{\omega_{1}}\right)\phi^{(1)\mu}\phi^{(1)}_{\mu} + \frac{1}{2}\left(1 - \frac{1}{\omega_{0}}\right)\partial^{\mu}\phi^{(1)}_{\mu}\partial^{\nu}\phi^{(1)}_{\nu} - \frac{1}{2}\partial^{\mu}\phi^{(0)}\partial_{\mu}\phi^{(0)} + \left(a_{2} + \frac{2\sqrt{2}m}{\omega_{1}}\right)\phi^{(1)\nu}\partial^{\mu}\phi^{(2)}_{\mu\nu} + \left(b_{2} + \frac{m}{\sqrt{2}\omega_{0}} + \frac{\sqrt{2}m}{\omega_{1}}\right)\tilde{\phi}^{(2)}\partial^{\mu}\phi^{(1)}_{\mu}$$

$$+ d_{2}\phi^{(2)\mu\nu}\phi^{(2)}_{\mu\nu} + \left(e_{2} - \frac{m^{2}}{4\omega_{0}}\right)(\tilde{\phi}^{(2)})^{2} - \left(f_{2} + \frac{\sqrt{2}(d-1)}{d-2}\frac{m^{3}}{\alpha_{0}\omega_{0}}\right)\tilde{\phi}^{(2)}\phi^{(0)} + \left(a_{1} + \frac{2d-2}{d-2}\frac{m^{2}}{\alpha_{0}\omega_{0}}\right)\phi^{(0)}\partial^{\mu}\phi^{(1)}_{\mu} + \left(d_{0} - \frac{(2d-2)^{2}}{2(d-2)^{2}}\frac{m^{4}}{(\alpha_{0})^{2}\omega_{0}}\right)(\phi^{(0)})^{2}\right] \right\} .$$

$$+ wire how the weak wave$$

To get our desired result, we choose

$$\omega_1 = 2 , \qquad \omega_0 = 1 .$$
 (4.10)

With such a choice, making use of (2.5) and (2.6b) gives

$$Z^{(2)} = \int \mathcal{D}(\phi; 2) \Delta^{(1)} \Delta^{(0)} \exp\left\{ i \int d^d x \left[\frac{1}{2} \phi^{(2)\mu\nu} \left(\Box - m^2 \right) \phi^{(2)}_{\mu\nu} - \frac{1}{4} \tilde{\phi}^{(2)} \left(\Box - m^2 \right) \tilde{\phi}^{(2)} + \frac{1}{2} \phi^{(1)\mu} \left(\Box - m^2 \right) \phi^{(1)}_{\mu} + \frac{1}{2} \phi^{(0)} \left(\Box - m^2 \right) \phi^{(0)}_{\mu} \right] \right\} .$$

$$(4.11)$$

It only remains to recast the ghost contributions in terms of path integrals. In accordance with (1.8) we rewrite (4.6a) and (4.6b) as follows:

$$\Delta^{(0)} = \int \mathcal{D}\overline{\psi}^{(0)} \mathcal{D}\psi^{(0)} \exp\left[-i\int d^d x \ \overline{\psi}^{(0)} (\Box - m^2)\psi^{(0)}\right], \qquad (4.12a)$$

$$\Delta^{(1)} = \int \mathcal{D}\overline{\psi}^{(1)} \mathcal{D}\psi^{(1)} \exp\left[-i\int d^d x \ \overline{\psi}^{(1)\mu} (\Box - m^2)\psi^{(1)}_{\mu}\right] .$$
(4.12b)

With these, (4.11) becomes

$$Z^{(2)} = \int \mathcal{D}(\phi, \psi; 2) \exp\left\{ i \int d^d x \left[\frac{1}{2} \phi^{(2)\mu\nu} \left(\Box - m^2 \right) \phi^{(2)}_{\mu\nu} - \frac{1}{4} \tilde{\phi}^{(2)} \left(\Box - m^2 \right) \tilde{\phi}^{(2)} - \overline{\psi}^{(1)\mu} \left(\Box - m^2 \right) \psi^{(1)\mu}_{\mu} + \frac{1}{2} \phi^{(1)\mu} \left(\Box - m^2 \right) \phi^{(1)}_{\mu} - \overline{\psi}^{(0)} \left(\Box - m^2 \right) \psi^{(0)} + \frac{1}{2} \phi^{(0)} \left(\Box - m^2 \right) \phi^{(0)} \right] \right\} ,$$

$$(4.13)$$

which is the fully diagonalised partition function for the spin-2 case.

5 Quantisation: Spin s = 3

A similar procedure is carried out for the spin-3 field. The corresponding classical action is

$$S = \int \mathrm{d}^d x \, \left\{ -\frac{1}{2} \partial^\mu \phi^{(3)\nu\lambda\rho} \partial_\mu \phi^{(3)}_{\nu\lambda\rho} + \frac{3}{2} \partial_\mu \phi^{(3)\mu\lambda\rho} \partial^\nu \phi^{(3)}_{\nu\lambda\rho} + \frac{3}{2} \partial^\mu \tilde{\phi}^{(3)\nu} \partial_\mu \tilde{\phi}^{(3)}_{\nu} \right\}$$

$$+ 3\partial_{\mu}\partial_{\nu}\phi^{(3)\mu\nu\lambda}\tilde{\phi}^{(3)}_{\lambda} + \frac{3}{4}\partial_{\mu}\tilde{\phi}^{(3)\mu}\partial_{\nu}\tilde{\phi}^{(3)\nu} + a_{3}\phi^{(2)\nu\lambda}\partial^{\mu}\phi^{(3)}_{\mu\nu\lambda} + b_{3}m\tilde{\phi}^{(3)\nu}\partial^{\mu}\phi^{(2)}_{\mu\nu} + c_{3}\partial_{\mu}\tilde{\phi}^{(3)\mu}\tilde{\phi}^{(2)} + d_{3}\phi^{(3)\mu\nu\lambda}\phi^{(3)}_{\mu\nu\lambda} + e_{3}\tilde{\phi}^{(3)\mu}\tilde{\phi}^{(3)}_{\mu} - f_{3}\tilde{\phi}^{(3)\mu}\phi^{(1)}_{\mu} - \frac{1}{2}\partial^{\mu}\phi^{(2)\nu\lambda}\partial_{\mu}\phi^{(2)}_{\nu\lambda} + \partial_{\mu}\phi^{(2)\mu\lambda}\partial^{\nu}\phi^{(2)}_{\nu\lambda}$$
(5.1)
$$+ \frac{1}{2}\partial^{\mu}\tilde{\phi}^{(2)}_{\mu}\partial_{\mu}\tilde{\phi}^{(2)} + \partial^{\mu}\partial^{\nu}\phi^{(2)}_{\mu\nu}\tilde{\phi}^{(2)} - \frac{1}{2}\partial^{\mu}\phi^{(1)\nu}\partial_{\mu}\phi^{(1)}_{\nu} + \frac{1}{2}\partial^{\mu}\phi^{(1)}_{\mu}\partial_{\nu}\phi^{(1)}_{\nu} - \frac{1}{2}\partial^{\mu}\phi^{(0)}\partial_{\mu}\phi^{(0)} + a_{2}\phi^{(1)\nu}\partial^{\mu}\phi^{(2)}_{\mu\nu} + b_{2}\tilde{\phi}^{(2)}\partial^{\mu}\phi^{(1)}_{\mu} + d_{2}\phi^{(2)\mu\nu}\phi^{(2)}_{\mu\nu} + e_{2}(\tilde{\phi}^{(2)})^{2} - f_{2}\tilde{\phi}^{(2)}\phi^{(0)} + a_{1}m\phi^{(0)}\partial^{\mu}\phi^{(1)}_{\mu} + d_{1}\phi^{(1)\mu}\phi^{(1)}_{\mu} + d_{0}(\phi^{(0)})^{2} \right\} .$$

It is invariant under the gauge transformations

$$\delta\phi_{\mu\nu\lambda}^{(3)} = \frac{1}{3} \left(\partial_{\mu}\xi_{\nu\lambda}^{(2)} + \partial_{\nu}\xi_{\mu\lambda}^{(2)} + \partial_{\lambda}\xi_{\mu\nu}^{(2)} \right) - \beta_3 \left(\eta_{\mu\nu}\xi_{\lambda}^{(1)} + \eta_{\lambda\mu}\xi_{\nu}^{(1)} + \eta_{\nu\lambda}\xi_{\mu}^{(1)} + \eta_{\nu\lambda}\xi_{\mu}^{(1)} \right) , \qquad (5.2a)$$

$$\delta\phi_{\mu\nu}^{(2)} = \frac{1}{2} \left(\partial_{\mu}\xi_{\nu}^{(1)} + \partial_{\nu}\xi_{\mu}^{(1)} \right) + \alpha_{2}\xi_{\mu\nu}^{(2)} - \beta_{2}\eta_{\mu\nu}\xi^{(0)} , \qquad (5.2b)$$

$$\delta\phi_{\mu}^{(1)} = \partial_{\mu}\xi^{(0)} + \alpha_{1}\xi_{\mu}^{(1)} , \qquad (5.2c)$$

$$\delta\phi^{(0)} = \alpha_0 \xi^{(0)} . (5.2d)$$

To quantise the theory, we choose the following gauge-fixing functions:

$$\Xi_{\mu\nu}^{(2)} - \chi_{\mu\nu}^{(2)} = 3\partial^{\lambda}\phi_{\mu\nu\lambda}^{(3)} - 3\partial_{(\mu}\tilde{\phi}_{\nu)}^{(3)} - \sqrt{3}m\phi_{\mu\nu}^{(2)} + \frac{\sqrt{3}}{d}\eta_{\mu\nu}m\tilde{\phi}^{(2)} - \chi_{\mu\nu}^{(2)} , \qquad (5.3a)$$

$$\Xi_{\mu}^{(1)} - \chi_{\mu}^{(1)} = 2\partial^{\nu}\phi_{\mu\nu}^{(2)} - \sqrt{3}m\tilde{\phi}_{\mu}^{(3)} - \partial_{\mu}\tilde{\phi}^{(2)} - 2m\sqrt{\frac{d+1}{d}\phi_{\mu}^{(1)} - \chi_{\mu}^{(1)}}, \qquad (5.3b)$$

$$\Xi^{(0)} - \chi^{(0)} = \partial^{\mu} \phi^{(1)}_{\mu} - m \sqrt{\frac{d+1}{d}} \tilde{\phi}^{(2)} - m \sqrt{\frac{3d}{d-2}} \phi^{(0)} - \chi^{(0)} , \qquad (5.3c)$$

where $\chi^{(2)}_{\mu\nu}$, $\chi^{(1)}_{\mu}$ and $\chi^{(0)}$ are background fields. Here both $\Xi^{(2)}_{\mu\nu}$ and $\chi^{(2)}_{\mu\nu}$ are symmetric and traceless. The gauge-fixing functions (5.3) have been chosen so that $\Xi^{(2)}_{\mu\nu}$ varies as

$$\delta \Xi_{\mu\nu}^{(2)} = (\Box - m^2) \xi_{\mu\nu}^{(2)} , \qquad (5.4)$$

while the gauge variations $\delta \Xi^{(1)}$ and $\delta \Xi^{(0)}$ are given by (4.4a) and (4.4b), respectively. The partition function is then

$$Z^{(3)} = \int \mathcal{D}(\phi; 3) \Delta^{(2)} \Delta^{(1)} \Delta^{(0)} \delta \left[\Xi^{(2)}_{\mu\nu} - \chi^{(2)}_{\mu\nu} \right] \delta \left[\Xi^{(1)}_{\mu} - \chi^{(1)}_{\mu} \right] \delta \left[\Xi^{(0)} - \chi^{(0)} \right] e^{iS} , \qquad (5.5)$$

where once again $\Delta^{(0)}$ and $\Delta^{(1)}$ are ghost contributions of spin 0 and 1 while $\Delta^{(2)}$ is the ghost contribution of spin 2.

Since the partition function (5.5) is independent of the background fields $\chi^{(2)}$, $\chi^{(1)}$ and $\chi^{(0)}$, we can average over them with a convenient weight of the form

$$\exp\left\{-\frac{\mathrm{i}}{2}\int \mathrm{d}^{d}x \left[\frac{\chi^{(2)\mu\nu}\chi^{(2)}_{\mu\nu}}{\omega_{2}} + \frac{\chi^{(1)\mu}\chi^{(1)}_{\mu}}{\omega_{1}} + \frac{(\chi^{(0)})^{2}}{\omega_{0}}\right]\right\}$$
(5.6)

Upon doing so, the relation (5.5) turns into

$$Z^{(3)} = \int \mathcal{D}(\phi; 3) \Delta^{(2)} \Delta^{(1)} \Delta^{(0)} \exp\left\{-\frac{i}{2} \int d^{d}x \\ \times \left[\frac{1}{\omega_{2}} \left(3\partial_{\lambda}\phi^{(3)\mu\nu\lambda} - 3\partial^{(\mu}\tilde{\phi}^{(3)\nu)} - \sqrt{3}m\phi^{(2)\mu\nu} + \frac{\sqrt{3}}{d}\eta^{\mu\nu}m\tilde{\phi}^{(2)}\right)^{2} \\ + \frac{1}{\omega_{1}} \left(2\partial^{\nu}\phi^{(2)}_{\mu\nu} - \sqrt{3}m\tilde{\phi}^{(3)}_{\mu} - \partial_{\mu}\tilde{\phi}^{(2)} - 2m\sqrt{\frac{d+1}{d}}\phi^{(1)}_{\mu}\right)^{2} \\ + \frac{1}{\omega_{0}} \left(\partial^{\mu}\phi^{(1)}_{\mu} - m\sqrt{\frac{d+1}{d}}\tilde{\phi}^{(2)} - m\sqrt{\frac{3d}{d-2}}\phi^{(0)}\right)^{2}\right]\right\} e^{iS} .$$
(5.7)

This is then combined with (5.1) to get

$$Z^{(3)} = \int \mathcal{D}(\phi; 3) \Delta^{(2)} \Delta^{(1)} \Delta^{(0)} \exp\left\{ i \int d^{d}x \left[-\frac{1}{2} \partial^{\mu} \phi^{(3)\nu\lambda\rho} \partial_{\mu} \phi^{(3)}_{\nu\lambda\rho} + \left(\frac{3}{2} - \frac{9}{2\omega_{2}} \right) \partial_{\mu} \phi^{(3)\mu\lambda\rho} \partial^{\nu} \phi^{(3)}_{\nu\lambda\rho} \right. \\ \left. + \left(\frac{3}{2} - \frac{9}{4\omega_{2}} \right) \partial^{\mu} \tilde{\phi}^{(3)\nu} \partial_{\mu} \tilde{\phi}^{(3)}_{\nu} + \left(3 - \frac{9}{\omega_{2}} \right) \partial_{\mu} \partial_{\nu} \phi^{(3)\mu\nu\lambda} \tilde{\phi}^{(3)}_{\lambda} + \left(\frac{3}{4} - \frac{9}{4\omega_{2}} \right) \partial_{\mu} \tilde{\phi}^{(3)\mu} \partial_{\nu} \tilde{\phi}^{(3)\nu} \\ \left. + \left(a_{3} + \frac{3\sqrt{3}m}{\omega_{2}} \right) \phi^{(2)\nu\lambda} \partial^{\mu} \phi^{(3)}_{\mu\nu\lambda} + \left(b_{3} + \frac{2\sqrt{3}m}{\omega_{1}} + \frac{3\sqrt{3}m}{\omega_{2}} \right) \tilde{\phi}^{(3)\nu} \partial^{\mu} \phi^{(2)}_{\mu\nu\nu} \\ \left. + \left(c_{3} + \frac{\sqrt{3}m}{\omega_{1}} \right) \partial_{\mu} \tilde{\phi}^{(3)\mu} \tilde{\phi}^{(2)} + \frac{m^{2}}{2} \phi^{(3)\mu\nu\lambda} \phi^{(3)}_{\mu\nu\lambda} + \left(e_{3} - \frac{3m^{2}}{2\omega_{1}} \right) \tilde{\phi}^{(3)\mu} \tilde{\phi}^{(3)}_{\mu} \\ \left. - \left(f_{3} + \frac{2m^{2}}{\omega_{1}} \sqrt{\frac{3(d+1)}{d}} \right) \tilde{\phi}^{(3)\mu} \phi^{(1)}_{\mu} - \frac{1}{2} \partial^{\mu} \phi^{(2)\nu\lambda} \partial_{\mu} \phi^{(2)}_{\nu\lambda} + \left(1 - \frac{2}{\omega_{1}} \right) \partial_{\mu} \phi^{(2)\mu\lambda} \partial^{\nu} \phi^{(2)}_{\nu\lambda} \\ \left. + \left(\frac{1}{2} - \frac{1}{2\omega_{0}} \right) \partial^{\mu} \phi^{(1)}_{\mu} \partial_{\nu} \phi^{(1)}_{\nu} + \frac{1}{2} \partial^{\mu} \phi^{(0)} \partial_{\mu} \phi^{(0)} + \left(a_{2} + \frac{4m}{\omega_{1}} \sqrt{\frac{d+1}{d}} \right) \phi^{(1)\nu} \partial^{\mu} \phi^{(2)}_{\mu\nu} \\ \left. + \left(b_{2} + \frac{m}{\omega_{0}} \sqrt{\frac{d+1}{d}} + \frac{2m}{\omega_{1}} \sqrt{\frac{d+1}{d}} \right) \tilde{\phi}^{(2)} \partial^{\mu} \phi^{(1)}_{\mu} - \frac{3}{2\omega_{2}} m^{2} \phi^{(2)\mu\nu} \phi^{(2)}_{\mu\nu} \right\}$$

$$+ \left(e_2 - \frac{3m^2}{2d\omega_2} - \frac{(d+1)m^2}{2d\omega_0}\right)(\tilde{\phi}^{(2)})^2 - \left(f_2 + \frac{m^2}{\omega_0}\sqrt{\frac{3(d+1)}{d-2}}\right)m^2\tilde{\phi}^{(2)}\phi^{(0)} \\ + \left(a_1 + \frac{m}{\omega_0}\sqrt{\frac{3d}{d-2}}\right)\phi^{(0)}\partial^{\mu}\phi^{(1)}_{\mu} + \left(d_1 - \frac{2(d+1)m^2}{d\omega_1}\right)\phi^{(1)\mu}\phi^{(1)}_{\mu} \\ + \left(d_0 - \frac{3dm^2}{2(d-2)\omega_0}\right)(\phi^{(0)})^2\right]\right\} .$$

It may be seen that the coefficients in (5.6) that diagonalise the action and remove all terms with divergences (such as $\partial_{\mu}\phi^{(3)\mu\lambda\rho}\partial^{\nu}\phi^{(3)}_{\nu\lambda\rho}$) are:

$$\omega_2 = 3 , \qquad \omega_1 = 2 , \qquad \omega_0 = 1 .$$
 (5.9)

Taking into account the relations (2.5) and integrating by parts, one arrives at

$$Z^{(3)} = \int \mathcal{D}(\phi; 3) \Delta^{(2)} \Delta^{(1)} \Delta^{(0)} \exp\left\{ i \int d^{d}x \left[\frac{1}{2} \phi^{(3)\mu\nu\lambda} \left(\Box - m^{2} \right) \phi^{(3)}_{\mu\nu\lambda} - \frac{3}{4} \tilde{\phi}^{(3)\mu} \left(\Box - m^{2} \right) \tilde{\phi}^{(3)}_{\mu} + \frac{1}{2} \phi^{(2)\mu\nu} \left(\Box - m^{2} \right) \phi^{(2)}_{\mu\nu} - \frac{1}{4} \tilde{\phi}^{(2)} \left(\Box - m^{2} \right) \tilde{\phi}^{(2)} - \frac{1}{2} \phi^{(1)\mu} \left(\Box - m^{2} \right) \phi^{(1)}_{\mu} + \frac{1}{2} \phi^{(0)} \left(\Box - m^{2} \right) \phi^{(0)}_{\mu} \right] \right\} .$$
(5.10)

Finally, it only remains to massage the ghost contributions. The spin 0 and 1 ghost contributions are just (4.12a) and (4.12b) as before. The spin 3 contribution is given by

$$\Delta^{(2)} = \det\left(\frac{\delta\Xi^{(2)}_{\mu\nu}(x)}{\delta\xi^{(2)}_{\lambda\rho}(x')}\right) = \det\left[\hat{\delta}_{\mu\nu}{}^{\lambda\rho}(\Box - m^2)\delta^d(x - x')\right], \qquad (5.11)$$

where $\hat{\delta}_{\mu\nu}{}^{\lambda\rho}$ is the Kronecker delta on the space of symmetric traceless second-rank tensors,

$$\hat{\delta}_{\mu\nu}{}^{\lambda\rho} = \delta_{(\mu}{}^{\lambda}\delta_{\nu)}{}^{\rho} - \frac{1}{d}\eta_{\mu\nu}\eta^{\lambda\rho} .$$
(5.12)

Upon making use of (1.8), one obtains

$$\Delta^{(2)} = \int \mathcal{D}\overline{\psi}^{(2)} \mathcal{D}\psi^{(2)} \exp\left[-i\int d^d x \ \overline{\psi}^{(2)\mu\nu} \left(\Box - m^2\right)\psi^{(2)}_{\mu\nu}\right] .$$
(5.13)

With this, the partition (5.10) takes the form

$$Z^{(3)} = \int \mathcal{D}(\phi, \psi; 3) \exp\left\{ i \int d^{d}x \left[\frac{1}{2} \phi^{(3)\mu\nu\lambda} \left(\Box - m^{2} \right) \phi^{(3)}_{\mu\nu\lambda} - \frac{3}{4} \tilde{\phi}^{(3)\mu} \left(\Box - m^{2} \right) \tilde{\phi}^{(3)}_{\mu} - \overline{\psi}^{(2)\mu\nu} \left(\Box - m^{2} \right) \psi^{(2)}_{\mu\nu} + \frac{1}{2} \phi^{(2)\mu\nu} \left(\Box - m^{2} \right) \phi^{(2)}_{\mu\nu} - \frac{1}{4} \tilde{\phi}^{(2)} \left(\Box - m^{2} \right) \tilde{\phi}^{(2)} - \overline{\psi}^{(1)\mu} \left(\Box - m^{2} \right) \psi^{(1)}_{\mu} + \frac{1}{2} \phi^{(1)\mu} \left(\Box - m^{2} \right) \phi^{(1)}_{\mu} - \overline{\psi}^{(0)} \left(\Box - m^{2} \right) \psi^{(0)} + \frac{1}{2} \phi^{(0)} \left(\Box - m^{2} \right) \phi^{(0)}_{\mu} \right\} .$$
(5.14)

It is seen that the gauge-fixed action is fully diagonalised.

6 Quantisation: Arbitrary integer spin s

In the spin-s case, the gauge-invariant action has the form (2.1). The corresponding gauge freedom is given by the transformations (2.4) where $0 \le k \le s$.

To quantise the theory, we introduce the following symmetric and traceless gauge-fixing functions:

$$\Xi_{\mu_{1}...\mu_{k}}^{(k)} - \chi_{\mu_{1}...\mu_{k}}^{(k)} = (k+1)\partial^{\mu}\phi_{\mu\mu_{1}...\mu_{k}}^{(k+1)} - \frac{(k+2)(k+1)}{2}\alpha_{k+1}\tilde{\phi}_{\mu_{1}...\mu_{k}}^{(k+2)} . \\ - \frac{k(k+1)}{2}\partial_{(\mu_{1}}\tilde{\phi}_{\mu_{2}...\mu_{k})}^{(k+1)} - \frac{(k+1)k(d+2k-4)}{2}\beta_{k+1}\phi_{\mu_{1}...\mu_{k}}^{(k)} \\ + \frac{(k+1)k^{2}(k-1)}{4}\beta_{k+1}\eta_{(\mu_{1}\mu_{2}}\tilde{\phi}_{\mu_{3}...\mu_{k})}^{(k)} - \chi_{\mu_{1}...\mu_{k}}^{(k)} , \qquad (6.1)$$

where $\chi^{(k)}$ is a background symmetric traceless field. The gauge fixing function $\Xi^{(k)}$ has been chosen such that its variation is

$$\delta \Xi^{(k)}_{\mu_1 \dots \mu_k} = (\Box - m^2) \xi^{(k)}_{\mu_1 \dots \mu_k} .$$
(6.2)

The partition function is given by

$$Z^{(s)} = \int \mathcal{D}(\phi; s) \left(\prod_{k=0}^{s-1} \Delta^{(k)} \delta \left[\Xi^{(k)} - \chi^{(k)} \right] \right) e^{iS} .$$
 (6.3)

Since the partition function is independent of the background fields $\chi^{(k)}$, we can average over them with a convenient weight of the form

$$\prod_{k=0}^{s-1} \exp\left\{-\frac{\mathrm{i}}{2} \int \mathrm{d}^d x \left[\frac{\chi^{(k)\mu_1\dots\mu_k}\chi^{(k)}_{\mu_1\dots\mu_k}}{\omega_k}\right]\right\} , \qquad (6.4)$$

where ω_k are constants chosen such that they diagonalise the action. This gives

$$Z^{(s)} = \int \mathcal{D}(\phi; s) \left(\prod_{k=0}^{s-1} \Delta^{(k)} \exp\left\{ -\frac{\mathrm{i}}{2} \int \mathrm{d}^d x \left[\frac{\Xi^{(k)\mu_1\dots\mu_k} \Xi^{(k)}_{\mu_1\dots\mu_k}}{\omega_k} \right] \right\} \right) \mathrm{e}^{\mathrm{i}S} . \tag{6.5}$$

One finds that $\Xi^{(k)\mu_1\dots\mu_k}\Xi^{(k)}_{\mu_1\dots\mu_k}$ is expanded fully as

$$\frac{(k+1)^2(k+2)^2}{4}(\alpha_{k+1})^2\tilde{\phi}^{(k+2)\mu_1\dots\mu_k}\tilde{\phi}^{(k+2)}_{\mu_1\dots\mu_k} + (k+1)^2\partial_\mu\phi^{(k+1)\mu\mu_1\dots\mu_k}\partial^\nu\phi^{(k+1)}_{\nu\mu_1\dots\mu_k} + \frac{k(k+1)^2(k-1)}{4}\partial^\mu\tilde{\phi}^{(k+1)\nu\mu_3\dots\mu_k}\partial_\nu\tilde{\phi}^{(k+1)}_{\mu\mu_3\dots\mu_k}$$

$$+\frac{(k+1)^{2}k^{2}(d+2k-4)^{2}}{4}(\beta_{k+1})^{2}\phi^{(k)\mu_{1}...\mu_{k}}\phi^{(k)}_{\mu_{1}...\mu_{k}} -\frac{(k+1)^{2}k^{3}(k-1)(d+2k-4)}{8}(\beta_{k+1})^{2}\tilde{\phi}^{(k)\mu_{3}...\mu_{k}}\tilde{\phi}^{(k)}_{\mu_{3}...\mu_{k}}$$

$$-(k+2)(k+1)^{2}\alpha_{k+1}\tilde{\phi}^{(k+2)\mu_{1}...\mu_{k}}\partial^{\mu}\phi^{(k+1)}_{\mu\mu_{1}...\mu_{k}} +\frac{(k+2)(k+1)^{2}k}{2}\alpha_{k+1}\tilde{\phi}^{(k+2)\mu_{1}...\mu_{k}}\partial_{\mu_{1}}\tilde{\phi}^{(k+1)}_{\mu_{2}...\mu_{k}} +\frac{(k+2)(k+1)^{2}k(d+2k-4)}{2}\alpha_{k+1}\beta_{k+1}\tilde{\phi}^{(k+2)\mu_{1}...\mu_{k}}\phi^{(k)}_{\mu_{1}...\mu_{k}} +\frac{(k+1)^{2}\partial_{\mu}\phi^{(k+1)\mu\mu_{1}...\mu_{k}}\partial_{\mu_{1}}\tilde{\phi}^{(k+1)}_{\mu_{2}...\mu_{k}} +(k+1)^{2}k(d+2k-4)\beta_{k+1}\partial_{\mu}\phi^{(k+1)\mu\mu_{1}...\mu_{k}}\phi^{(k)}_{\mu_{1}...\mu_{k}} +\frac{(k+1)^{2}k^{2}(d+2k-4)}{2}\beta_{k+1}\partial^{\mu_{1}}\tilde{\phi}^{(k+1)\mu_{2}...\mu_{k}}\phi^{(k)}_{\mu_{1}...\mu_{k}} .$$

$$(6.6)$$

These terms can be split into massless and massive parts, allowing one to look at how they modify the massive and massless Lagrangian contributions separately. Denoting the new contributions with a prime, the effect of (6.6) on (2.2) is

$$\mathcal{L}_{0}^{\prime}(\phi^{(k)}) = -\frac{1}{2}\partial^{\mu}\phi^{(k)\mu_{1}...\mu_{k}}\partial_{\mu}\phi^{(k)}_{\mu_{1}...\mu_{k}} + \frac{k}{2}\partial_{\mu}\phi^{(k)\mu\mu_{2}...\mu_{k}}\partial^{\nu}\phi^{(k)}_{\nu\mu_{2}...\mu_{k}} + \frac{k(k-1)}{4}\partial^{\mu}\tilde{\phi}^{(k)\mu_{3}...\mu_{k}}\partial_{\mu}\tilde{\phi}^{(k)}_{\mu_{3}...\mu_{k}} + \frac{k(k-1)}{2}\partial_{\mu}\partial_{\nu}\phi^{(k)\mu\nu\mu_{3}...\mu_{k}}\tilde{\phi}^{(k)}_{\mu_{3}...\mu_{k}} + \frac{k(k-1)(k-2)}{8}\partial_{\mu}\tilde{\phi}^{(k)\mu\mu_{4}...\mu_{k}}\partial^{\nu}\tilde{\phi}^{(k)}_{\nu\mu_{4}...\mu_{k}} + \frac{(k-1)k^{2}}{2\omega_{k-1}}\partial_{\mu}\phi^{(k)\mu\nu\mu_{3}...\mu_{k}}\partial_{\nu}\tilde{\phi}^{(k)}_{\mu_{3}...\mu_{k}} - \frac{(k-1)k^{2}(k-2)}{8\omega_{k-1}}\partial^{\mu}\tilde{\phi}^{(k)\mu_{3}...\mu_{k}}\partial_{\nu}\tilde{\phi}^{(k)}_{\mu_{4}...\mu_{k}} - \frac{k^{2}}{2\omega_{k-1}}\partial_{\mu}\phi^{(k)\mu\mu_{2}...\mu_{k}}\partial^{\nu}\phi^{(k)}_{\nu\mu_{2}...\mu_{k}} - \frac{(k-1)k^{2}}{8\omega_{k-1}}\partial^{\mu}\tilde{\phi}^{(k)\mu_{3}...\mu_{k}}\partial_{\mu}\tilde{\phi}^{(k)}_{\mu_{3}...\mu_{k}}.$$
(6.7)

To cancel out all off diagonal and divergence contributions, one needs to choose

$$\omega_k = k + 1 . \tag{6.8}$$

With such a choice, (6.7) becomes

$$\mathcal{L}_{0}^{\prime}(\phi^{(k)}) = -\frac{1}{2}\partial^{\mu}\phi^{(k)\mu_{1}...\mu_{k}}\partial_{\mu}\phi^{(k)}_{\mu_{1}...\mu_{k}} + \frac{k(k-1)}{8}\partial^{\mu}\tilde{\phi}^{(k)\mu_{3}...\mu_{k}}\partial_{\mu}\tilde{\phi}^{(k)}_{\mu_{3}...\mu_{k}} + \frac{k(k-1)}{2}\partial_{\mu}\partial_{\nu}\phi^{(k)\mu\nu\mu_{3}...\mu_{k}}\tilde{\phi}^{(k)}_{\mu_{3}...\mu_{k}} + \frac{(k-1)k}{2}\partial_{\mu}\phi^{(k)\mu\nu\mu_{3}...\mu_{k}}\partial_{\nu}\tilde{\phi}^{(k)}_{\mu_{3}...\mu_{k}} + \frac{k(k-1)(k-2)}{8}\partial_{\mu}\tilde{\phi}^{(k)\mu\mu_{4}...\mu_{k}}\partial^{\nu}\tilde{\phi}^{(k)}_{\nu\mu_{4}...\mu_{k}} ,$$
(6.9)

and therefore

$$\int d^d x \mathcal{L}'_0(\phi^{(k)}) = \int d^d x \left[\frac{1}{2} \phi^{(k)\mu_1 \dots \mu_k} \Box \phi^{(k)}_{\mu_1 \dots \mu_k} - \frac{k(k-1)}{8} \tilde{\phi}^{(k)\mu_3 \dots \mu_k} \Box \tilde{\phi}^{(k)}_{\mu_3 \dots \mu_k} \right] .$$
(6.10)

One now must check if this value of ω_k diagonalises the massive Lagrangian contributions. Indeed, by combining the massive terms in (6.6) with (2.3) and (6.8), as well as substituting (2.5) and (2.6b), one has

$$\mathcal{L}'_{m}(\phi^{(k)}) = \left[-\frac{1}{2} m^{2} \phi^{(k)\mu_{1}\dots\mu_{k}} \phi^{(k)}_{\mu_{1}\dots\mu_{k}} + \frac{k(k-1)}{8} m^{2} \tilde{\phi}^{(k)\mu_{3}\dots\mu_{k}} \tilde{\phi}^{(k)}_{\mu_{3}\dots\mu_{k}} \right] .$$
(6.11)

Thus, the entire action is diagonalised by choosing (6.8). With this, (6.5) becomes

$$Z^{(s)} = \int \mathcal{D}(\phi; s) \left(\prod_{k=0}^{s-1} \Delta^{(k)} \right) \\ \times \exp \left\{ \sum_{k=0}^{s} \left[i \int d^{d}x \left(\frac{1}{2} \phi^{(k)\mu_{1}...\mu_{k}} (\Box - m^{2}) \phi^{(k)}_{\mu_{1}...\mu_{k}} - \frac{k(k-1)}{8} \tilde{\phi}^{(k)\mu_{3}...\mu_{k}} (\Box - m^{2}) \tilde{\phi}^{(k)}_{\mu_{3}...\mu_{k}} \right) \right] \right\} .$$
(6.12)

The Faddeev-Popov determinants $\Delta^{(k)}$ are given by

$$\Delta^{(k)} = \det\left(\frac{\delta \Xi^{(k)}_{\mu_1...\mu_k}(x)}{\delta \xi^{(k)}_{\nu_1...\nu_k}(x')}\right) = \det\left[\hat{\delta}_{\mu_1...\mu_k}{}^{\nu_1...\nu_k}(\Box - m^2)\delta^d(x - x')\right],$$
(6.13)

where $\hat{\delta}_{\mu_1...\mu_k}^{\nu_1...\nu_k}$ denotes the Kronecker delta on the space of symmetric traceless rank-k tensors. Making use of (1.8) gives

$$\Delta^{(k)} = \int \mathcal{D}\overline{\psi}^{(k)} \mathcal{D}\psi^{(k)} \exp\left[-i \int d^d x \,\overline{\psi}^{(k)\mu_1\dots\mu_k} (\Box - m^2)\psi^{(k)}_{\mu_1\dots\mu_k}\right] \,. \tag{6.14}$$

The full partition function is obtained by inserting (6.14) in (6.12), yielding

$$Z^{(s)} = \int \mathcal{D}(\phi, \psi; s) \exp\left\{ i \int d^{d}x \left[\frac{1}{2} \phi^{(s)\mu_{1}\dots\mu_{s}} (\Box - m^{2}) \phi^{(s)}_{\mu_{1}\dots\mu_{s}} - \frac{s(s-1)}{8} \tilde{\phi}^{(s)\mu_{3}\dots\mu_{s}} (\Box - m^{2}) \tilde{\phi}^{(s)}_{\mu_{3}\dots\mu_{s}} + \sum_{k=0}^{s-1} \left(\frac{1}{2} \phi^{(k)\mu_{1}\dots\mu_{k}} (\Box - m^{2}) \phi^{(k)}_{\mu_{1}\dots\mu_{k}} - \frac{k(k-1)}{8} \tilde{\phi}^{(k)\mu_{3}\dots\mu_{k}} (\Box - m^{2}) \tilde{\phi}^{(k)}_{\mu_{3}\dots\mu_{k}} - \overline{\psi}^{(k)\mu_{1}\dots\mu_{k}} (\Box - m^{2}) \psi^{(k)}_{\mu_{1}\dots\mu_{k}} \right) \right] \right\} .$$
(6.15)

The structure of (6.15) is similar to the effective action for massive antisymmetric tensor field models in d dimensions [18, 19]. It is obvious from (6.15) that the massive propagators are remarkably simple in the Feynman-like gauge, which we have used. An alternative quantisation scheme is to make use of the unitary gauge (3.3), in which the gauge freedom is absent and the theory is described by the *d*-dimensional Singh-Hagen model (3.9). The corresponding propagators for the massive integer-spin fields in four dimensions were derived in [20].

Making use of (6.15), one can count the number of degrees of freedom, n(d, s), using the relation $Z^{(s)} = \det^{-n(d,s)/2} (\Box - m^2)$. A symmetric rank-k tensor in d dimensions has

$$\binom{d+k-1}{k} \tag{6.16}$$

independent components. In the case of a traceless symmetric tensor, this should be reduced by the number of independent components of a symmetric rank-(k-2) tensor. Given a double traceless symmetric tensor, (6.16) should be reduced by the number of of a symmetric rank-(k-4) tensor. The total degrees of freedom are counted as

$$\sum_{k=0}^{s} \binom{d+k-1}{k} - \sum_{k=4}^{s} \binom{d+k-1}{k} - 2\left(\sum_{k=0}^{s-1} \binom{d+k-1}{k} - \sum_{k=2}^{s-1} \binom{d+k-1}{k}\right), \quad (6.17)$$

which, after some algebra, simplifies to n(d, s) given by eq. (3.11).

All fields in (6.15) have one and the same kinetic operator, $(\Box - m^2)$. This degeneracy is characteristic of \mathbb{M}^4 . In the case of an (anti-)de Sitter background, the structure of kinetic operators will depend on the rank of a field, and the partition function will depend on the spacetime curvature. This will be discussed elsewhere.

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