

# ELEMENTARY PROPERTIES OF FREE LATTICES

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ABSTRACT. We start a systematic analysis of the first-order model theory of free lattices. Firstly, we prove that the free lattices of finite rank are not positively indistinguishable, as there is a positive  $\exists\forall$ -sentence true in  $\mathbf{F}_3$  and false in  $\mathbf{F}_4$ . Secondly, we show that every model of  $\text{Th}(\mathbf{F}_n)$  admits a canonical homomorphism into the profinite-bounded completion  $\mathbf{H}_n$  of  $\mathbf{F}_n$ . Thirdly, we show that  $\mathbf{H}_n$  is isomorphic to the Dedekind-MacNeille completion of  $\mathbf{F}_n$ , and that  $\mathbf{H}_n$  is not positively elementarily equivalent to  $\mathbf{F}_n$ , as there is a positive  $\forall\exists$ -sentence true in  $\mathbf{H}_n$  and false in  $\mathbf{F}_n$ . Finally, we show that  $\text{DM}(\mathbf{F}_n)$  is a retract of  $\text{Id}(\mathbf{F}_n)$  and that for any lattice  $\mathbf{K}$  which satisfies Whitman's condition (W) and which is generated by join prime elements, the three lattices  $\mathbf{K}$ ,  $\text{DM}(\mathbf{K})$ , and  $\text{Id}(\mathbf{K})$  all share the same positive universal first-order theory.

## 1. INTRODUCTION

The model theoretic analysis of free objects (in the sense of universal algebra) has a long tradition, dating back to the 1940's with the work of Tarski and his school. Problems in this areas are often pretty hard and they require advanced technology to be solved. A canonical example of this phenomenon is the solution of Tarski's problem on the elementary equivalence of free groups, which was solved in 2006 independently by Sela [19] and Kharlampovich & Myasnikov [12]: all non-abelian finitely generated free groups are elementarily equivalent, regardless of the number of generators. As for any naturally arising mathematical structure (or class of structures), many are the model-theoretic questions that can be asked about it. In the context of free objects we might argue that the focus has been on the following four fundamental problems:

- (A) (positive) elementary equivalence of free objects of different rank;
- (B) characterization of the finitely generated models elementarily equivalent to a given free object of finite rank;
- (C) decidability of (fragments of) the (positive) first-order theory of a free object;
- (D) analysis of the stability (in the sense of Shelah [20]) properties of a free object.

The model theoretic literature is full of such results, where, once again, probably the most advanced results are on the model theory of free groups. Another important case worth mentioning is the one of free abelian groups. In this case it follows easily from [22] that free abelian groups of different rank are not elementarily equivalent and that free abelian groups of finite rank are superstable. Another important piece of literature is on the model theory of free algebras in the context

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of infinitary logic, cf. in particular the fundamental work of Eklof, Mekler, and Shelah [7, 15].

In universal algebra, among the most natural classes of algebraic structures that occur in nature there are certainly *lattices*, and so, as for any variety of algebras, there are *free lattices*. In the last 30 years or so, the algebraic study of free lattices has reached a very mature state, as witnessed by the canonical reference [8] on this topic (the “blue book”). Despite the widespread interest of model theorists in free objects and despite the advanced development of the theory of free lattices, at the best of our knowledge, very little is known on the model theory of free lattices. We consider this a sad state of affairs and we think of this paper as a starting point for a remedy, hoping that it will sparkle interest.

To start our model theoretic analysis we test what is the situation against Problems (A)-(D) above in the context of free lattices. With regard to (D), as it is easy to see, free lattices fail the stability property (once again, in the sense of Shelah [20]) very badly, and so, although more refined questions can still be meaningfully asked, this might not be the right starting point. Concerning (A), it is easy to see that free lattices of different finite rank can be distinguished by a  $\exists\forall$ -sentence of first-order logic, because the generating set is unique. This leaves us then with the following three questions, where we denote by  $\mathbf{F}_n$  the free lattice of rank  $n$ .

- (A) Are  $\mathbf{F}_n$  and  $\mathbf{F}_m$  elementarily equivalent in *positive* first-order logic?
- (B) Which are the finitely generated lattices elementarily equivalent to  $\mathbf{F}_n$ ?
- (C) Is the first-order theory of  $\mathbf{F}_n$  decidable?

Unfortunately, Question (C) resisted our tries, but we hope that this paper will spark some interest in this fundamental question. Notice that in his celebrated paper [25], Whitman solved the word problem for free lattices, exhibiting a natural algorithmic procedure to decide whether two lattice terms are equivalent (modulo the theory of lattices)<sup>1</sup>. In logical terms this means that the positive universal theory of a free lattice is decidable. Thus, as a starting point toward Question (C), we ask:

**Problem 1.1.** *Let  $3 \leq n \leq \omega$ . Is the  $\forall$ -theory of  $\mathbf{F}_n$  decidable?*

We then move to Questions (A) and (B), in this respect the situation is more favorable. In particular, concerning (A) we were able to show the following:

**Theorem 1.2.** *The free lattices  $\mathbf{F}_n$  (for  $3 \leq n < \omega$ ) are not positively indistinguishable. In fact there is a  $\exists\forall$ -positive sentence true in  $\mathbf{F}_3$  and false in  $\mathbf{F}_4$ .*

Interestingly, our proof does not extend to  $n \geq 4$ . We naturally wonder if this is a limitation of our methods or if there is an intrinsic reason for this. Also, we want to mention that Theorem 1.2 was motivated by the analysis of the positive first-order theory of free semigroups and free monoids from references [6, 18].

Finally, we move to Question (B). This is the venue that inspired the most interesting results of this paper, with applications also to infinitely generated models of  $\text{Th}(\mathbf{F}_n)$ . The crucial result in this direction is the following “Profinite Theorem”.

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<sup>1</sup>As an historical note: it turns out that Skolem in a famous 1920 paper [21] solved the word problem not only for free lattices but for any finitely presented lattice; see [9] for Skolem’s solution. However, this section of Skolem’s paper remained unknown until it was found by Stanley Burris late in the 20th century.

**Theorem 1.3.** *Let  $3 \leq n < \omega$  and  $\mathbf{K} \equiv \mathbf{F}_n$ . Then  $\mathbf{K}$  admits a canonical homomorphism  $h_{\mathbf{K}}$  into the profinite-bounded completion  $\mathbf{H}_n$  of  $\mathbf{F}_n$  (cf. Fact 4.7). Furthermore,  $\mathbf{H}_n$  is isomorphic to the Dedekind-MacNeille completion of  $\mathbf{F}_n$  and  $\mathbf{H}_n \not\equiv \mathbf{F}_n$ , in fact there is a positive  $\forall\exists$ -sentence true in  $\mathbf{H}_n$  and false in  $\mathbf{F}_n$ .*

Theorem 1.3 led us to investigations related to an old question of Grätzer, that is, which first-order conditions are preserved in passing from  $\mathbf{K}$  to  $\text{Id}(\mathbf{K})$ ? An old result of Baker & Hales [2] says that  $\mathbf{K}$  and  $\text{Id}(\mathbf{K})$  share the same positive universal theory. On the other hand, as observed by Funayama in 1944 [10], there are distributive lattices  $\mathbf{K}$  such that  $\text{DM}(\mathbf{K})$  is not distributive, and so in the case of  $\text{DM}(\mathbf{K})$  this preservation of the positive universal theory of  $\mathbf{K}$  is not at all to be taken for granted. In this direction, in our next and final theorem we isolate two properties (satisfied by free lattices) of an arbitrary lattice  $\mathbf{K}$  which ensure that the three lattices  $\mathbf{K}$ ,  $\text{DM}(\mathbf{K})$ , and  $\text{Id}(\mathbf{K})$  all share the same positive universal theory.

Whitman's solution to the word problem for free lattices uses the following condition, which holds in free lattices:

$$(W) \quad s \wedge t \leq u \vee v \text{ implies } s \leq u \vee v \text{ or } t \leq u \vee v \text{ or } s \wedge t \leq u \text{ or } s \wedge t \leq v.$$

**Theorem 1.4.** *Let  $\mathbf{K}$  be a lattice satisfying Whitman's condition (W) and which is generated by join prime elements. Then the three lattices  $\mathbf{K}$ ,  $\text{DM}(\mathbf{K})$ , and  $\text{Id}(\mathbf{K})$  all share the same positive universal theory. Furthermore, in the case  $\mathbf{K} = \mathbf{F}_n$ , then  $\text{DM}(\mathbf{K})$  is a retract of  $\text{Id}(\mathbf{K})$  and so, in particular, the first-order positive theory of  $\text{Id}(\mathbf{F}_n)$  is contained in the first-order positive theory of  $\text{DM}(\mathbf{F}_n)$ .*

Motivated by Theorem 1.4, in Corollary 5.5 we show that  $\text{DM}(\mathbf{F}_n)$  and  $\text{Id}(\mathbf{F}_n)$  are *not* elementarily equivalent. But we leave open the following question.

**Question 1.5.** *Are  $\text{DM}(\mathbf{F}_n)$  and  $\text{Id}(\mathbf{F}_n)$  positively elementarily equivalent?*

What we find particularly interesting about Theorem 1.3 is that this theorem is reminiscent of the model theory of free abelian groups. In fact in that case the same thing happens, with the crucial difference, though, that  $\mathbb{Z}^n$  is elementarily equivalent to its profinite completion. We notice that the lattice  $\mathbf{H}_n$  plays a crucial role also in the lattice theoretic literature, in particular in connection with Day's Theorem, see [8, Section 2.7]. The realization that  $\mathbf{H}_n$  is isomorphic to the Dedekind-MacNeille completion of  $\mathbf{F}_n$  was the crucial ingredient in showing that  $\mathbf{H}_n$  and  $\mathbf{F}_n$  are not elementarily equivalent, and in fact this model theoretic question inspired this result, but we believe that this fact is of independent interest and could be further explored by lattice theorists. Furthermore, despite the hopelessness of a classification of the models of  $\text{Th}(\mathbf{F}_n)$  (recall the instability mentioned above), Theorem 1.3 reduces the understanding of models of  $\text{Th}(\mathbf{F}_n)$  to understanding  $\mathbf{H}_n$  and to understanding the equivalence classes induced by  $\ker(h_{\mathbf{K}})$ . In fact the source of instability present in  $\mathbf{F}_n$  reduces to the fact that such equivalence classes can be in general very complicated. On the other hand, under further assumptions on models of  $\text{Th}(\mathbf{F}_n)$  there is hope to prove some positive results. For example, in light of Theorem 1.3 we might say that a lattice  $\mathbf{K} \models \text{Th}(\mathbf{F}_n)$  is *standard* if the map  $h_{\mathbf{K}}$  has range in  $\mathbf{F}_n$ , that is each element of  $\mathbf{K}$  is congruent modulo  $\ker(h_{\mathbf{K}})$  to an element of  $\mathbf{F}_n$  (notice that the equivalence relation  $\ker(h_{\mathbf{K}})$  can be described explicitly, cf. Section 4). In this direction we propose the following conjecture:

**Conjecture 1.6.** *There is no finitely generated standard elementary extension of  $\mathbf{F}_n$ .*

We actually further believe that  $\mathbf{F}_n$  is the *only* finitely generated model of its theory, this property is known in the model theoretic community as first-order rigidity, a property which received quite some attention in recent years, see e.g. the result of Avni-Lubotzky-Meiri showing that irreducible non-uniform higher-rank characteristic zero arithmetic lattices (e.g.  $\mathrm{SL}_n(\mathbb{Z})$  for  $n \geq 3$ ) are first-order rigid [1]. Note that “lattices” in topological groups are not the same as “lattices” in our sense of ordered sets with meet and join.

## 2. NOTATION

We write lattices in boldface letters, so  $\mathbf{L}, \mathbf{K}$ , etc. Given a lattice  $\mathbf{L}$ , we write the sup and inf of  $\mathbf{L}$  as  $\vee$  and  $\wedge$ , but when convenient we switch to the “field notation”, so  $+$  and  $\cdot$  for sup and inf, respectively. We write tuples of elements (or variables) as  $\mathbf{x} = (x_1, \dots, x_n)$ . Given a cardinal number  $\kappa$  we denote by  $\mathbf{F}_\kappa$  the free lattice on  $\kappa$ -many generators. By the language of lattice theory we mean the language  $L = \{\vee, \wedge\}$ . In particular we do not require 0 and 1 to be in the language (as we also consider  $\mathbf{F}_\kappa$  for infinite  $\kappa$  and such lattices do not have max or min).

An element  $a$  of a lattice  $\mathbf{L}$  is *join irreducible* if it is not a proper join, i.e., there do not exist  $b < a$  and  $c < a$  such that  $a = b \vee c$ . Equivalently,  $a$  is join irreducible if it is not the join of a finite nonempty set of elements strictly below  $a$ . In any lattice satisfying (W), no element can be a proper join and a proper meet. Thus in a free lattice every element is either join irreducible or meet irreducible; generators are both.

On the other hand, in any lattice, the least upper bound of  $\{b \in L : b < a\}$  is either  $a$  or the unique largest element  $a_*$  below  $a$ . In the latter case, we say that  $a$  is *completely join irreducible*. Denote the set of completely join irreducible elements of  $\mathbf{L}$  by  $\mathrm{CJI}(\mathbf{L})$ . A join irreducible element in a free lattice need not be completely join irreducible: some are, and some are not.

The terms *meet irreducible* and *completely meet irreducible* are defined dually, along with the notation  $\mathrm{CMI}(\mathbf{L})$ .

## 3. THE POSITIVE THEORY OF FREE LATTICES

Let  $\mathrm{PTh}(\mathbf{L})$  denote the positive first-order theory of  $\mathbf{L}$ . Since  $\mathbf{F}_n$  is a homomorphic image of  $\mathbf{F}_{n+1}$ , we have  $\mathrm{PTh}(\mathbf{F}_n) \supseteq \mathrm{PTh}(\mathbf{F}_{n+1})$ . To distinguish them, we seek a positive sentence  $\pi(\mathbf{x})$  that holds in  $\mathbf{F}_n$  but not in  $\mathbf{F}_{n+1}$ . The aim of this section is to show that we can do this for  $n = 3$ . However, for  $n \geq 4$  it remains open whether there is a positive sentence that holds in  $\mathbf{F}_n$  but not in  $\mathbf{F}_{n+1}$ .

Consider the following positive first-order formulas in the language of lattice theory:

$$\mathrm{NI}(x_1, \dots, x_m) : \quad (\mathrm{OR}_{1 \leq i \leq m} x_i \leq \sum_{j \neq i} x_j) \quad \mathrm{OR} \quad (\mathrm{OR}_{1 \leq i \leq m} x_i \geq \prod_{j \neq i} x_j);$$

$$t(u) : \quad \forall w \ w \leq u;$$

$$b(u) : \quad \forall w \ w \geq u.$$

The last one is more complicated. For a finite set  $X$  and bounded intervals  $I_1, \dots, I_n$ , write  $\mathrm{CI}(X, I_1, \dots, I_n)$  to mean  $X \subseteq \bigcup_{1 \leq j \leq n} I_j$ . The inclusion can be written as  $\&_{x \in X} \mathrm{OR}_{1 \leq j \leq n} (x \in I_j)$ . This is a positive first-order condition as we require each  $I_j$  to be bounded, so e.g.  $x \in I = [c, d]$  gets written out as  $c \leq x \ \& \ x \leq d$ , and so on.

Moreover, for a set  $\{x, y, z\}$  we define the following intervals:

$$\begin{aligned} I^x &= [x + yz, x + (x + y)(x + z)(y + z)] \\ J_x &= [x(xy + xz + yz), x(y + z)] \\ K &= [xy + xz + yz, (x + y)(x + z)(y + z)]. \end{aligned}$$

**Theorem 3.1.** *The following sentence  $\pi_3$  holds in  $\mathbf{F}_3$  but not in  $\mathbf{F}_4$ :*

$$\begin{aligned} &\exists z_1 \exists z_2 \exists z_3 \ t(z_1 + z_2 + z_3) \ \& \ b(z_1 z_2 z_3) \ \& \\ &\forall x_1 \forall x_2 \forall x_3 \forall x_4 \ [\text{NI}(x_1, x_2, x_3, x_4) \ \text{OR} \\ &\text{OR}_{i \neq j} \text{CI}(\{x_1, \dots, x_4\}, I^{z_i}, J_{z_j}, K) \ \text{OR} \\ &\text{OR}_{i \leq 3} \text{CI}(\{x_1, \dots, x_4\}, [z_i, z_i], K)], \end{aligned}$$

where  $I^{z_i}, J_{z_j}, K$  are with respect to the set  $\{z_1, z_2, z_3\}$ .

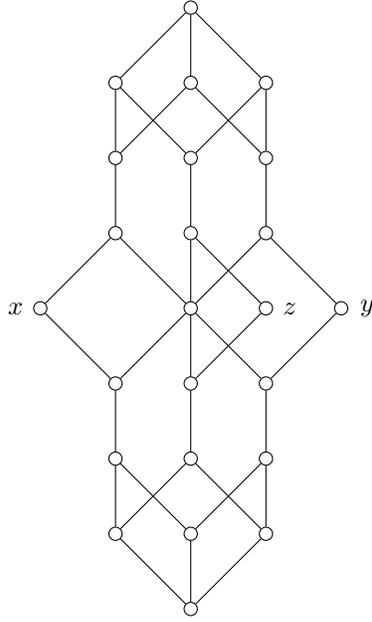
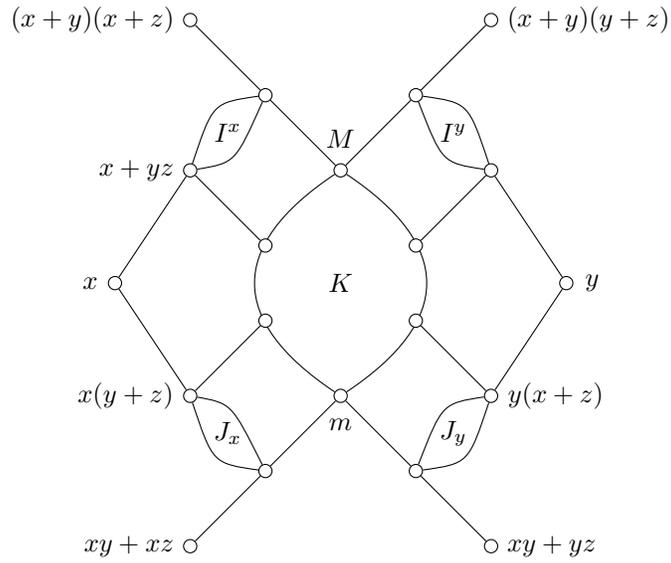
*Proof.* First we show that the sentence holds in  $\mathbf{F}_3$ . Let  $\mathbf{F}_3$  be generated by  $x, y, z$  and take  $z_1 = x, z_2 = y, z_3 = z$ . Let  $x_1, \dots, x_4$  be elements of  $\mathbf{F}_3$ . In the presence of (W),  $\text{NI}(x_1, x_2, x_3, x_4)$  says exactly that those elements do not generate a copy of  $\mathbf{F}_4$  by [8, Corollary 1.12]. On the other hand, Whitman showed that  $\mathbf{F}_3$  contains  $\mathbf{F}_\omega$  [25], see [8, Theorems 1.28 and 9.10]. In particular,  $\mathbf{F}_3$  contains many copies of  $\mathbf{F}_4$ , and the sentence  $\pi_3$  restricts their location.

The free distributive lattice  $\text{FD}_3$  is a bounded lattice. Also, Alan Day's doubling construction preserves the property of being bounded [4, 5], [8, Sec. II.3]. By doubling the elements in  $\text{FD}_3$  that are the join of two atoms, or the meet of two coatoms, we obtain the lattice  $\mathbf{A}$  in Figure 1. Thus the natural homomorphism  $h : \mathbf{F}_3 \rightarrow \mathbf{A}$  is bounded. The algorithm for computing lower and upper bounds for congruence classes of the kernel of a bounded homomorphism, which goes back to Jónsson [11] and McKenzie [14], is given in Theorem 2.3 of [8]. Applying this to the homomorphism  $h$  decomposes  $\mathbf{F}_3$  into a disjoint union the congruence classes of  $\ker h$ . These turn out to be intervals of the following forms (up to permutations of variables):

$$\begin{aligned} [u, u] & \qquad \qquad \qquad (T^u) \\ & \text{for } u = (x + y)(x + z), x + y, x + z, x + y + z \\ [x + yz, x + M] & \qquad \qquad \qquad (I^x) \\ [x, x] & \qquad \qquad \qquad (G_x) \\ [xm, x(y + z)] & \qquad \qquad \qquad (J_x) \\ [v, v] & \qquad \qquad \qquad (B_v) \\ & \text{for } v = xy + xz, xy, xz, xyz \\ [m, M] & \qquad \qquad \qquad (K). \end{aligned}$$

where  $m = xy + xz + yz$  and  $M = (x + y)(x + z)(y + z)$ . Except for the singleton classes  $T^u$  with  $u \in \{x + y, x + z, y + z, x + y + z\}$ , and  $B_v$  with  $v \in \{xy, xz, yz, xyz\}$ , this decomposition is sketched schematically in Figure 2.

The claim is that if  $x_1, \dots, x_4$  generate a copy of  $\mathbf{F}_4$ , then  $x_1, \dots, x_4$  is contained in a union of intervals of the form  $I^x \cup J_y \cup K$  or  $\{x\} \cup K$ , up to permutations of variables. Certainly we cannot have  $x_i \in T^u$ , that is  $x_i = u$ , or  $x_i \in B_v$  for the elements at the top or bottom of  $\mathbf{F}_3$ , as each such  $\uparrow u$  and  $\downarrow v$  contains only finitely

FIGURE 1. Lattice  $\mathbf{A}$  obtained by doubling six elements in  $\mathbf{FD}_3$ FIGURE 2. Schematic of interval decomposition of  $\mathbf{F}_3$ .

many elements, whereas  $\uparrow x_i$  and  $\downarrow x_i$  are infinite. Moreover,  $\{x_1, \dots, x_4\}$  cannot contain elements from both sets of any of the following pairs of intervals:

$$\begin{array}{ccc} \{x\} \& I^x & \{x\} \& J_x & I^x \& J_x \\ \{y\} \& I^x & \{y\} \& J_x & \{y\} \& \{x\} \\ I^y \& I^x & J_y \& J_x & & \end{array}$$

The first line is because it would make some pair  $x_i, x_j$  comparable. For the second line, note if  $t \in I^x$  or  $t = x$ , then  $y + t = x + y$ , and similarly for the third line, if  $u \in I^x$  and  $v \in I^y$ , then  $u + v = x + y$ . But  $x + y$  is a coatom of  $\mathbf{F}_3$ , and you cannot have  $x_i + x_j$  being a coatom, no matter where the remaining  $x_k$ 's lie, since the filter  $\uparrow(x_i + x_j)$  has at least 4 elements. Thus you cannot have the first entries in the 2nd and 3rd lines inhabited, and dually for the second entries, and for the same reason  $\{x_i, x_j\} \neq \{y, x\}$ . That leaves options contained in unions of the form  $I^x \cup J_y \cup K$  or  $\{x\} \cup K$ , as claimed.

Now to show that  $\pi_3$  fails in  $F_4$ . The logical form of  $\pi_3$  is the following:

$$\exists \mathbf{z} \forall \mathbf{x} A \& (B \text{ or } C).$$

Its negation would be the following:

$$\forall \mathbf{z} \exists \mathbf{x} A \rightarrow (\neg B \& \neg C).$$

We will show, switching the quantifiers, that the following holds:

$$\exists \mathbf{x} \forall \mathbf{z} A \rightarrow (\neg B \& \neg C),$$

which is slightly stronger, since it means that  $\mathbf{x}$  is chosen uniformly.

We take  $x_1, \dots, x_4$  to be the standard generators of  $\mathbf{F}_4$ , for which  $\text{NI}(x_1, \dots, x_4)$  fails, and intend to show that there do not exist  $z_1, z_2, z_3$  with  $z_1 + z_2 + z_3 = 1$  and  $z_1 z_2 z_3 = 0$  satisfying - up to symmetry - one of the following two conditions:

$$\text{CI}(\{x_1, \dots, x_4\}, I^{z_1}, J_{z_2}, K) \text{ or } \text{CI}(\{x_1, \dots, x_4\}, [z_1, z_1], K).$$

Case 1.  $\text{CI}(\{x_1, \dots, x_4\}, I^{z_1}, J_{z_2}, K)$  holds.

Then the least element of  $J_{z_2}$ , which is  $z_2(z_1 z_2 + z_1 z_3 + z_2 z_3)$ , is  $0 = x_1 x_2 x_3 x_4$ . *A fortiori*  $z_1 z_2 + z_2 z_3 = 0$ , whence  $z_1 z_2 = 0 = z_2 z_3$ . By  $(\text{SD}_\wedge)$ ,  $z_2(z_1 + z_3) = 0$ . But that is the top element of  $J_{z_2}$ , so there are no  $x_i$ 's in that interval. Dually, there are no  $x_j$ 's in  $I^{z_1}$ . Thus they are all in  $K = [z_1 z_2 + z_1 z_3 + z_2 z_3, (z_1 + z_2)(z_1 + z_3)(z_2 + z_3)]$ . But then  $z_1 z_2 + z_1 z_3 + z_2 z_3 = 0$ . Again all 3 joinands are 0, and applying  $(\text{SD}_\wedge)$  twice (in its more general form,  $u = ab = cd$  implies  $u = (a + c)(a + d)(b + c)(b + d)$ , cf. [8, Theorem 1.21]) we get  $(z_1 + z_2)(z_1 + z_3)(z_2 + z_3) = 0$ . However,  $(z_1 + z_2)(z_1 + z_3)(z_2 + z_3) = 1$  since it is the top of  $K$  and all the  $x_j$ 's are in  $K$ . Therefore, that is a contradiction.

Case 2.  $\text{CI}(\{x_1, \dots, x_4\}, [z_1, z_1], K)$  holds.

W.l.o.g.  $z_1 = x_1$  and  $x_2, x_3, x_4 \in K$ , else we revert to the previous case where everything is in  $K$ . If, say,  $z_2 \leq z_3$ , then the bottom of  $K$  is  $z_1 z_2 + z_1 z_3 + z_2 z_3 = z_2 + z_1 z_3$ . Thus  $z_2$  and  $x_1 z_3$  are both below  $x_2 x_3 x_4$ , and since each  $x_j$  is meet prime in  $\mathbf{F}_4$ , we get  $z_3 \leq x_2 x_3 x_4$ . That contradicts  $z_1 + z_2 + z_3 = 1 = x_1 + \dots + x_4$ . Hence the  $z_j$ 's are incomparable and distinct. On the other hand, from  $z_1 z_2 z_3 = 0$  we get  $\{x_2, x_3, x_4\} \gg \{z_2, z_3\}$ , and dually from  $z_1 + z_2 + z_3 = 1$  we get  $\{x_2, x_3, x_4\} \ll \{z_2, z_3\}$ . ( $A \gg B$  if  $\forall a \exists b a \geq b$ ;  $C \ll D$  is dual; these are not symmetric.) Thus each of  $x_2, x_3, x_4$  is above some  $z_j$  and below some  $z_k$ , with  $\{j, k\} \subseteq \{2, 3\}$ . Assume say  $x_2 \geq z_2$ . By  $\{x_2, x_3, x_4\} \ll \{z_2, z_3\}$  either  $x_2 \leq z_2$  or  $x_2 \leq z_3$ . The latter gives

$z_3 \leq x_2 \leq z_2$ , contradicting the argument above. Thus  $x_2 = z_2$ . Similarly  $x_3 = z_3$ . That makes  $x_4 \geq z_2$  and  $x_4 \geq z_3$  both impossible.

Hence, both cases lead to a contradiction, and we are done.  $\blacksquare$

#### 4. THE PROFINITE-BOUNDED COMPLETION OF $\mathbf{F}_n$

**Definition 4.1.** Let  $\mathbf{L}$  be a lattice, we say that  $a \in \mathbf{L}$  is doubly prime if it is both join prime and meet prime, that is for every  $b, c \in \mathbf{L}$ , the following hold:

- (1)  $a \leq b \vee c$  implies  $a \leq b$  or  $a \leq c$ ;
- (2)  $b \wedge c \leq a$  implies  $b \leq a$  or  $c \leq a$ .

**Fact 4.2.** Let  $a$  be an element in a free lattice  $\mathbf{F}(X)$ . The following are equivalent:

- (1)  $a \in X$ ,
- (2)  $a \in \mathbf{F}(X)$  is doubly prime (cf. Definition 4.1).

**Definition 4.3.** We say that the a model  $A$  is prime if it embeds elementarily in every model of its first-order theory. We say that  $A$  is minimal if it has no proper elementary substructures.

**Fact 4.4** ([17, Proposition 5.1]). Let  $A$  be a countable structure. Then  $A$  is a prime model of its theory iff, for every  $0 < n < \omega$ , each orbit under the natural action of  $\text{Aut}(A)$  on  $A^n$  is first-order definable without parameters in  $A$ .

**Lemma 4.5.** Let  $\kappa$  be a cardinal.

- (1) If  $\kappa$  is finite, then  $\mathbf{F}_\kappa$  is a prime and minimal model of its theory.
- (2) If  $\kappa = \omega$ , then  $\mathbf{F}_\kappa$  is a prime model of its theory but it is not minimal.
- (3) If  $\kappa > \omega$ , then  $\mathbf{F}_\omega$  embeds elementarily in  $\mathbf{F}_\kappa$ .

*Proof.* We first prove that if  $\kappa \leq \aleph_0$ , then  $\mathbf{F}_\kappa = \mathbf{F}(X)$  (so  $|X| = \kappa$ ) is a prime model of its theory. To this extent, by Fact 4.4 it suffices to show that for every  $n < \omega$  and for every  $n$ -tuple  $\bar{a}$  of elements of  $\mathbf{F}_\kappa$ , the  $\text{Aut}(\mathbf{F}_\kappa)$ -orbit of  $\bar{a} = (a_1, \dots, a_n)$  is first-order definable in  $\mathbf{F}_\kappa$  without parameters (notice that under our assumptions  $\mathbf{F}_\kappa$  is countable). For every  $1 \leq i \leq n$ , let  $t_i(\bar{x})$  be a term in the variables  $\bar{x} \subseteq X$  (adding variables possibly not actually occurring in  $t_i(\bar{x})$  we can assume that  $\bar{x}$  is the same for all the  $i$ 's) such that  $t_i^{\mathbf{F}_\kappa}(\bar{x}) = a_i$ . Then  $\bar{b} = (b_1, \dots, b_n)$  is in the  $\text{Aut}(\mathbf{F}_\kappa)$ -orbit of  $\bar{a}$  iff there is a tuple  $\bar{y}$  of the same length as  $\bar{x}$  of doubly prime elements of  $\mathbf{F}_\kappa$  such that for every  $1 \leq i \leq n$  we have  $t_i^{\mathbf{F}_\kappa}(\bar{y}) = b_i$ , and by Fact 4.2 this is first-order. This proves that  $\mathbf{F}_\kappa$  is prime for  $\kappa \leq \aleph_0$ . The claim about the minimality and not minimality of  $\mathbf{F}_\kappa = \mathbf{F}(X)$  is clear, as if  $|X| = \aleph_0$  is infinite, then the sublattice generated by an infinite proper subset of  $X$  is also a prime model of its theory, while if  $X$  is finite this is not the case, as the existence of exactly  $n < \omega$  doubly prime elements is expressible in first order logic. Finally, concerning (3), it suffices to show that for every finite subset  $\{a_1, \dots, a_m\}$  of  $\mathbf{F}_\omega$  and every element  $b \in \mathbf{F}_\kappa$  there exists an automorphism of  $\mathbf{F}_\kappa$  which fixes  $\{a_1, \dots, a_m\}$  and maps  $b$  into  $\mathbf{F}_\omega$ , and this is easy to see (this is a well-known general fact).  $\blacksquare$

We now introduce the crucial notions of lower and upper bounded lattices. Our treatment of the subject will be brief, for more see e.g. [8, Chapter II] or [16].

**Definition 4.6.** Let  $\mathbf{K}$  and  $\mathbf{L}$  be lattices. A homomorphism  $f : \mathbf{K} \rightarrow \mathbf{L}$  is said to be lower bounded if for every  $a \in \mathbf{L}$ , the set  $\{u \in \mathbf{K} : f(u) \geq a\}$  is either empty or has a least element. A finitely generated lattice  $\mathbf{L}$  is called lower bounded if every

homomorphism  $f : \mathbf{K} \rightarrow \mathbf{L}$ , where  $\mathbf{K}$  is finitely generated, is lower bounded. Let  $D_0(\mathbf{L})$  denote the set of join prime elements of  $\mathbf{L}$ , i.e., those elements which have no nontrivial join-cover (cf. [8, pg. 29]). For  $k > 0$ , let  $a \in D_k(\mathbf{L})$  if every nontrivial join-cover  $V$  of  $a$  has a refinement  $U \subseteq D_{k-1}(\mathbf{L})$  which is also a join-cover of  $a$ . Then a finitely generated lattice  $\mathbf{L}$  is lower bounded if and only if  $\bigcup_{k < \omega} D_k(\mathbf{L}) = \mathbf{L}$ . Observe that, from the definition,  $D_0(\mathbf{L}) \subseteq D_1(\mathbf{L}) \subseteq D_2(\mathbf{L}) \subseteq \dots$ . Thus, if  $\mathbf{L}$  is lower bounded and  $a \in \mathbf{L}$ , we define  $\rho(a)$ , the  $D$ -rank of  $a$ , to be the least integer  $k$  such that  $a \in D_k(\mathbf{L})$ . Every finitely generated lower bounded lattice has the minimal join-cover refinement property [8, Cor. 2.19], which implies that every element is a finite join of join irreducible elements. Thus we can define the  $D$ -rank of a lower bounded lattice  $L$  to be  $\sup\{\rho(a) : a \in J(\mathbf{L})\}$ . The notions of upper bounded homomorphism, upper bounded lattice, etc. are defined dually. In the upper case we write  $D^{\text{op}}$ -rank of  $a$ , to distinguish the two notions.

**Fact 4.7** ([16]). *Let  $k < \omega$ . The class of lattices all of whose finitely generated sublattices are lower and upper bounded of  $D$ -rank and  $D^{\text{op}}$ -rank  $\leq k$  (cf. Definition 4.6) forms a variety  $\mathcal{V}_k$ . We denote by  $\mathbf{B}_{(n,k)}$  the free object of rank  $n$  in the variety  $\mathcal{V}_k$  and with  $h_k$  the canonical homomorphism of  $\mathbf{F}_n$  onto  $\mathbf{B}_{(n,k)}$ . Also, for  $k < \ell < \omega$ , the variety  $\mathcal{V}_k$  is contained in the variety  $\mathcal{V}_\ell$ , and so, for fixed  $n < \omega$ , we let  $f_{(k,\ell)}$  be the canonical homomorphism from  $\mathbf{B}_{(n,\ell)}$  onto  $\mathbf{B}_{(n,k)}$ . Further,  $(\mathbf{B}_{(n,k)}, f_{(k,\ell)} : \ell \leq k < \omega)$  is an inverse system.*

Finally, we denote by  $\mathbf{H}_n$  the inverse limit of the inverse system  $(\mathbf{B}_{(n,k)}, f_{(k,\ell)} : \ell \leq k < \omega)$ . The lattice  $\mathbf{F}_n$  embeds canonically into  $\mathbf{H}_n$ . We refer to  $\mathbf{H}_n$  as the profinite-bounded completion of  $\mathbf{F}_n$ .

**Lemma 4.8.** *Let  $\mathbf{K}$  be an elementary extension of  $\mathbf{F}_n = \mathbf{F}(X)$  with  $X = \{x_1, \dots, x_n\}$  and let  $\mathbf{L}$  be a finite, upper and lower bounded lattice generated by  $\{a_1, \dots, a_n\}$ . Then the homomorphism  $f : \mathbf{F}_n \rightarrow \mathbf{L}$  such that  $x_i \mapsto a_i$  extends canonically to a homomorphism  $\hat{f}^{\mathbf{K}} = \hat{f} : \mathbf{K} \rightarrow \mathbf{L}$ . Further, in the context of Fact 4.7,  $(\mathbf{K}, \hat{h}_k^{\mathbf{K}} : k < \omega)$  is a cone of the inverse system  $(\mathbf{B}_{(n,k)}, f_{(k,\ell)} : \ell \leq k < \omega)$ , and thus there is a homomorphism  $h : \mathbf{K} \rightarrow \mathbf{H}_n$  which commutes with  $(\mathbf{B}_{(n,k)}, f_{(k,\ell)} : \ell \leq k < \omega)$ .*

*Proof.* As  $L$  is bounded, necessarily  $f : \mathbf{F}_n \rightarrow \mathbf{L}$  is bounded, i.e., the kernel of  $f$  partitions  $\mathbf{F}_n$  into bounded congruence classes, that is, every congruence class  $f^{-1}(a)/\ker(f)$ , for  $a \in \mathbf{L}$ , has a least element  $\beta(a)$  and a greatest element  $\alpha(a)$ . Thus, the equivalence classes of this partition are intervals of the form  $[\beta(a), \alpha(a)]$  for  $a \in \mathbf{L}$ , and for  $u \in \mathbf{F}_n$  we have that  $f(u) = a$  iff  $\beta(a) \leq u \leq \alpha(a)$ . Again, the algorithm for computing lower and upper bounds  $\beta$  and  $\alpha$  for congruence classes of the kernel of a bounded homomorphism is given in Theorem 2.3 of [8].

As  $\mathbf{K}$  is an elementary extension of  $\mathbf{F}_n$ , this also determines a partition of  $\mathbf{K}$ , let  $\psi_{\mathbf{K}}$  be the corresponding equivalence relation on  $\mathbf{K}$ . Define then  $\hat{f}^{\mathbf{K}} = \hat{f} : \mathbf{K} \rightarrow \mathbf{L}$  as  $\hat{f}(u) = a$  iff  $u \in a/\psi_{\mathbf{K}}$ . We claim that  $\hat{f}$  is a homomorphism. We show that  $\hat{f}$  preserves joins, a dual argument works for meets. To this extent, remember that  $\beta(c) \vee \beta(d) = \beta(c \vee d)$ , and  $\alpha(c) \vee \alpha(d) \leq \alpha(c \vee d)$ . Thus if  $u, v \in \mathbf{K}$  and  $\beta(c) \leq u \leq \alpha(c)$  and  $\beta(d) \leq v \leq \alpha(d)$ , then  $\beta(c \vee d) \leq u \vee v \leq \alpha(c \vee d)$ . Finally, the ‘‘further part’’ of the lemma is easy as  $\mathbf{H}_n$  is the inverse limit of  $(\mathbf{B}_{(n,k)}, f_{(k,\ell)} : \ell \leq k < \omega)$ . ■

**Example 4.9.** *In Figure 3 we see how  $\mathbf{F}_3$  (and thus any elementary extension  $\mathbf{K}$  of  $\mathbf{F}_3$ ) gets partitioned into the pentagon  $\mathbf{N}_5$  via the natural map of  $\mathbf{F}_3$  onto  $\mathbf{N}_5$ . Note that the pentagon is in the variety  $\mathcal{V}_1$ , since  $D_1(\mathbf{N}_5) = \mathbf{N}_5$ .*

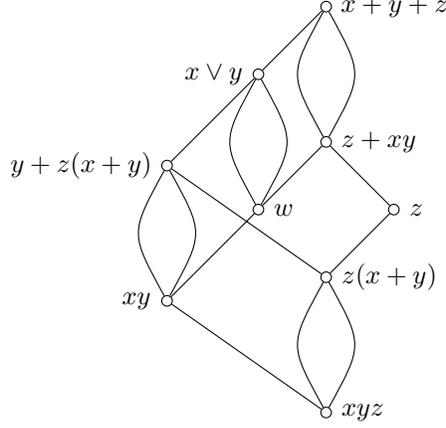


FIGURE 3. The bounded congruence classes of the natural map of  $\mathbf{F}_3$  onto the pentagon  $\mathbf{N}_5$ , where we let  $w = x(z + xy)$ .

**Notation 4.10.** As in the proof of 4.8, letting  $\mathbf{K}$  be an elementary extension of  $\mathbf{F}_n$  we denote by  $\psi = \psi_{\mathbf{K}}$  the congruence induced by the homomorphism  $h : \mathbf{K} \rightarrow \mathbf{H}_n$ .

**Proposition 4.11.** In the context of Lemma 4.8 and Notation 4.10:

- (1)  $\psi = \bigcap_{k < \omega} \varphi_k$ , where  $\varphi_k$  is the kernel of the homomorphism  $\hat{h}_k^{\mathbf{K}} : \mathbf{K} \rightarrow \mathbf{B}_{(n,k)}$ ;
- (2)  $\psi \upharpoonright_{\mathbf{F}_n} = \Delta$  (the identity).

*Proof.* (1) is clear, (2) is because of [8, Theorem 4.1], which is Day's theorem that free lattices are weakly atomic [4]; see also [8, Section II.7].  $\blacksquare$

Let us elaborate on Day's theorem, which plays a crucial role in the arguments below. For each  $w \in \mathbf{F}_n$  there is a congruence  $\varphi_w$  which is maximal with respect to the property that  $(v, w) \notin \varphi$  for any  $v < w$ . It turns out that  $\mathbf{F}_n/\varphi_w$  is a finite lower bounded lattice of  $D$ -rank  $\rho(w)$ . Moreover, a join irreducible element  $w$  is completely join irreducible if and only if  $\varphi_w$  is both lower and upper bounded. In that case,  $\mathbf{F}_n/\varphi_w$  is in  $\mathcal{V}_n$  where  $n$  is the complexity of  $w$ , that is,  $w \in X^{\wedge(\vee\wedge)^n}$ .

Day's theorem says that the completely join irreducible elements of  $\mathbf{F}_n$  are join-dense, i.e., if  $u \not\leq v$  in  $\mathbf{F}_n$  then there exists a completely join irreducible element  $w$  with  $w \leq u$  and  $w \not\leq v$ . Thus every element in a free lattice is the join of the completely join irreducible elements below it.

**Fact 4.12.** As in the proof of 4.8, letting  $h_k : \mathbf{F}_n \rightarrow \mathbf{B}_{(n,k)}$  be the canonical homomorphism and letting  $\alpha_k(u)$  and  $\beta_k(u)$  be, respectively, the greatest and least element of the equivalence class  $u/\ker(h_k)$ , then to every element  $u \in \mathbf{F}_n$  we can associate sequences  $(\beta_k(u) : k < \omega), (\alpha_k(u) : k < \omega) \in (\mathbf{F}_n)^\omega$  such that:

$$(\star) \quad \beta_0(u) \leq \beta_1(u) \leq \dots \leq u \leq \dots \leq \alpha_1(u) \leq \alpha_0(u).$$

In fact, to every element  $\mathbf{c} = (c_k : k < \omega) \in \mathbf{H}_n$ , letting  $\alpha_k(\mathbf{c})$  be the greatest element of  $h_k^{-1}(c_k) \subseteq \mathbf{F}_n$  and  $\beta_k(\mathbf{c})$  be the least element of  $h_k^{-1}(c_k) \subseteq \mathbf{F}_n$  we have:

$$(\star\star) \quad \beta_0(\mathbf{c}) \leq \beta_1(\mathbf{c}) \leq \dots \leq \dots \leq \alpha_1(\mathbf{c}) \leq \alpha_0(\mathbf{c}).$$

For  $\mathbf{a} \in \mathbf{F}_n$ , this double sequence corresponds to the sequence  $(\star)$  identifying  $\mathbf{F}_n$  with its canonical embedding into  $\mathbf{H}_n$  (cf. what has been said at the end of 4.7). As a piece of notation, given  $\mathbf{c} \in H$  we also write  $\mathbf{c} = (\mathbf{b}, \mathbf{a})$ , where we let:

$$\mathbf{b} = (\beta_j(\mathbf{c}) = b_j : j < \omega) \quad \text{and} \quad \mathbf{a} = (\alpha_j(\mathbf{c}) = a_j : j < \omega).$$

With respect to this identification, the lattice order of  $\mathbf{H}_n$  can be characterized as:

$$\mathbf{c} \leq \mathbf{c}' \Leftrightarrow \mathbf{b} \leq \mathbf{b}' \Leftrightarrow b_j \leq b'_j \text{ for all } j < \omega$$

$$\mathbf{c} \leq \mathbf{c}' \Leftrightarrow \mathbf{a} \leq \mathbf{a}' \Leftrightarrow a_j \leq a'_j \text{ for all } j < \omega.$$

An element  $w$  in a lattice  $\mathbf{L}$  is *lower atomic* if for all  $u < w$  there exists  $v$  such that  $u \leq v < w$ . The dual condition is called *upper atomic*, and  $w$  is *totally atomic* if it is both lower and upper atomic.

For an element  $w$  of a finitely generated free lattice  $\mathbf{F}_n$ , there is a one-to-one correspondence between the lower covers of  $w$  and its completely join irreducible canonical joinands. If every canonical joinand is completely join irreducible, then  $w$  is lower atomic. The dual statements hold for upper atomic and completely meet irreducible canonical meetands. See Theorem 3.5 and Corollary 3.8 of [8], expanded in Theorem 3.26 and Corollary 3.27.

**Fact 4.13.** *In the context of 4.12, there are four types of elements in  $\mathbf{F}_n$ :*

- (1) *the ones such that the  $\beta$ 's and the  $\alpha$ 's are eventually constant;*
- (2) *the ones such that the  $\beta$ 's are eventually constant but the  $\alpha$ 's are not;*
- (3) *the ones such that the  $\alpha$ 's are eventually constant but the  $\beta$ 's are not;*
- (4) *the ones such that neither the  $\beta$ 's nor the  $\alpha$ 's are eventually constant.*

*Furthermore, the four types above admit the following algebraic description:*

- (1')  *$c \in \mathbf{F}_n$  is as in (1) iff  $c$  is totally atomic;*
- (2')  *$c \in \mathbf{F}_n$  is as in (2) iff  $c$  is lower atomic but not upper atomic;*
- (3')  *$c \in \mathbf{F}_n$  is as in (3) iff  $c$  is upper atomic but not lower atomic;*
- (4')  *$c \in \mathbf{F}_n$  is as in (4) iff  $c$  is neither upper nor lower atomic.*

*Notice that by [8, Chapter 6] the number of elements of type (1') is finite.*

Concerning the Dedekind-MacNeille completion (below) see e.g. [3, pp. 165-169].

**Notation 4.14.** *Let  $\mathbf{P}$  be a poset. We denote by  $\text{DM}(\mathbf{P})$  the Dedekind-MacNeille completion of  $\mathbf{P}$ , i.e., the set of subsets of  $P$  such that  $C = C^{u\ell}$ , where for  $D \subseteq P$ :*

$$D^u = \{p \in P : \forall d \in D, d \leq p\} \quad \text{and} \quad D^\ell = \{p \in P : \forall d \in D, p \leq d\}.$$

$\text{DM}(\mathbf{P})$  is ordered by inclusion. Recall that  $\text{CJI}(\mathbf{P})$  (resp.  $\text{CMI}(\mathbf{P})$ ) denotes the set of completely join irreducible (resp. completely meet irreducible) elements of  $\mathbf{P}$ . For  $k < \omega$ , we let  $\text{CJI}_k(\mathbf{P})$  denote the set of completely join irreducible elements of  $\mathbf{P}$  of  $D$ -rank  $\leq k$ . We shorten completely join irreducible with CJI and, when clear from the context, we write  $\text{CJI}_k$  and CJI instead of  $\text{CJI}_k(\mathbf{P})$  and  $\text{CJI}(\mathbf{P})$ .

**Theorem 4.15.** *For  $\mathbf{F}_n$  with  $n$  finite,  $\mathbf{H}_n \cong \text{DM}(\mathbf{F}_n)$ .*

*Proof.* For a sequence  $\mathbf{c} = (\mathbf{b}, \mathbf{a})$  in  $\mathbf{H}_n$ , let us set notation:

$$\begin{aligned} A &= \{a_j : j \in \omega\} \\ B &= \{b_j : j \in \omega\} \\ A^\ell &= \bigcap \downarrow a_j \\ \downarrow B &= \bigcup \downarrow b_j \end{aligned}$$

**Lemma 4.16.**  $\downarrow B \cap \text{CJI} = A^\ell \cap \text{CJI}$ .

*Proof.* First we show  $\downarrow B \cap \text{CJI} \subseteq A^\ell$ . Consider  $b_j$  and arbitrary  $k$ , w.l.o.g.  $k \geq j$  as the  $a_j$ 's are descending. Then  $b_j \leq b_k \leq a_k$ . Next  $A^\ell \cap \text{CJI} \subseteq \downarrow B$ . Let  $w \in \text{LHS}$  of rank  $j$ . Then  $w \leq a_j$  implies  $w = h_j(w) \leq h_j(a_j)$ , whence  $w = \beta h_j(w) \leq \beta h_j(a_j) = b_j$ . ■

**Lemma 4.17.** *Let  $I$  and  $J$  be DM-closed ideals of  $\mathbf{F}_n$ . If  $I \cap \text{CJI} = J \cap \text{CJI}$ , then  $I = J$ .*

*Proof.* DM-closed ideals, which are of the form  $D^\ell$ , are closed under joins that exist in  $\mathbf{F}_n$ . On the other hand, by Day's theorem, every element of  $\mathbf{F}_n$  is a (possibly infinite) join of CJI elements. ■

Now we set about establishing the isomorphism of Theorem 4.15. Define a map  $\varphi : \mathbf{H}_n \rightarrow \text{DM}(\mathbf{H}_n)$  by  $\varphi(\mathbf{c}) = A^\ell$  for each  $\mathbf{c} = (\mathbf{b}, \mathbf{a})$  in  $\mathbf{H}_n$ . Recall that  $\mathbf{c} \leq \mathbf{c}' \Leftrightarrow \mathbf{b} \leq \mathbf{b}'$  by Fact 4.12. Then:

- (1)  $B \subseteq A^\ell$  and  $A^\ell \in \text{DM}(\mathbf{F}_n)$ .
- (2)  $\varphi$  is order-preserving:  $\mathbf{b} \leq \mathbf{b}'$  means  $b_j \leq b'_j$  for all  $j$ . It follows that  $a_j \leq a'_j$  for all  $j$ , so  $A^\ell \subseteq (A')^\ell$ .
- (3)  $\varphi$  is 1-to-1:  $\mathbf{b} \not\leq \mathbf{b}'$  implies there exists  $j$  with  $b_j \not\leq b'_j$ . Then by Day's theorem [4] there is a CJI  $u$  such that  $u \leq b_j$  and  $u \not\leq b'_j$ , whence  $h_j(b'_j) \not\leq u$ . Thus  $a'_j \not\leq u$ , as in general  $h_j(u) \leq v$  iff  $u \leq \alpha(v)$ , whence  $a'_j \not\leq b_j$ .
- (4)  $\varphi$  is onto: let  $I = I^{u^\ell}$  be a DM-closed ideal. Define a sequence  $b_j = \bigvee (I \cap \text{CJI}_j)$ , which makes sense because  $\text{CJI}_j$  is finite. Because every element of  $\mathbf{F}_n$  is a join of CJI elements and  $B = \{b_j : j \in \omega\} \subseteq I$ , we have  $I \cap \text{CJI} = \downarrow B \cap \text{CJI}$ . By Lemma 4.16 this yields  $I \cap \text{CJI} = A^\ell \cap \text{CJI}$ , whence by Lemma 4.17,  $I = A^\ell = \varphi(\mathbf{c})$  is in the range of  $\varphi$ . ■

**Lemma 4.18.** *Let  $\mathbf{L}_0$  be a lattice admitting a fixed-point free polynomial  $p(x, \mathbf{b})$ , with  $\mathbf{b}$  a finite sequence in  $\mathbf{L}_0$  and let  $\mathbf{L}_1$  be a complete lattice containing  $\mathbf{L}_0$ . Then there is a positive  $\forall\exists$ -sentence true in  $\mathbf{L}_1$  and false in  $\mathbf{L}_0$ .*

*Proof.* For  $j = 0, 1$ , let  $p_j : \mathbf{L}_j \rightarrow \mathbf{L}_j$  be such that  $a \mapsto p(a, \mathbf{b})$  (i.e., the corresponding polynomial function). Then, recalling that  $\mathbf{L}_1$  is a complete lattice, and recalling also the choice of  $p(x, \mathbf{b})$ , the following two things happen:

$$\mathbf{L}_1 \models \forall \mathbf{b} \exists x (p(x, \mathbf{b}) = x) \quad \text{and} \quad \mathbf{L}_0 \not\models \forall \mathbf{b} \exists x (p(x, \mathbf{b}) = x).$$

Indeed, let  $A_1 = \{a \in \mathbf{L}_1 : a \leq p(a, \mathbf{b})\}$  and  $a_1 = \bigvee A_1$ . It is straightforward that:

$$p(a_1, \mathbf{b}) = a_1.$$

This last equality is the Tarski fixed-point theorem; see [13, 23] and the discussion in Section I.5 of [8]. ■

**Theorem 4.19.** *Let  $3 \leq n < \omega$ . Then  $\mathbf{H}_n$  is not elementarily equivalent to  $\mathbf{F}_n$ , in fact there is a positive  $\forall\exists$ -sentence true in  $\mathbf{H}_n$  and false in  $\mathbf{F}_n$ .*

*Proof.* This follows immediately from Lemmas 4.15 and 4.18 and the existence of fixed-point free polynomials in  $\mathbf{F}_n$ , see e.g. [8, Section I.5]. ■

## 5. PROOF OF THEOREM 1.4

The ideals of a lattice  $\mathbf{L}$ , ordered by set inclusion, form a complete lattice  $\text{Id}(\mathbf{L})$ . Likewise the filters, ordered by set inclusion, form a complete lattice  $\text{Fil}(\mathbf{L})$ . (If  $\mathbf{L}$  has a least element 0, we normally take  $\{0\}$  to be the least ideal; if not then the empty set is contained in  $\text{Id}(\mathbf{L})$ . Dually for  $\text{Fil}(\mathbf{L})$ . The filter lattice is often ordered by *reverse set inclusion*; for current purposes, set inclusion is more natural. That makes the join of two filters  $F \vee G$  to be the filter generated by  $F \cup G$ , directly analogous to the join of two ideals in  $\text{Id}(\mathbf{L})$ . Recall that  $\text{DM}(\mathbf{F}_n) \cong \mathbf{H}_n$  by Theorem 4.15, so in what follows we identify the two objects.

**Theorem 5.1.** *Let  $\kappa : \text{Id}(\mathbf{F}_n) \rightarrow \text{DM}(\mathbf{F}_n)$  via  $\kappa(I) = I^{u\ell}$ . Then the map  $\kappa$  witnesses that  $\mathbf{H}_n$  is a retract of the ideal lattice  $\text{Id}(\mathbf{F}_n)$ .*

Our path to Theorem 5.1 includes a more general result, i.e., our Theorem 1.4. To this extent, let us start by recalling some basic facts.

**Lemma 5.2.** *Let  $\mathbf{L}$  be a lattice. Then:*

- (1) *For any  $D \subseteq L$ ,  $D^u \in \text{Fil}(\mathbf{L})$ .*
- (2) *An ideal  $I$  of  $\mathbf{L}$  is DM-closed, i.e.,  $\kappa(I) = I$ , if and only if  $I = F^\ell$  for some filter  $F$  of  $\mathbf{L}$ , where  $\kappa : \text{Id}(\mathbf{L}) \rightarrow \text{DM}(\mathbf{L})$  is defined as  $\kappa(I) = I^{u\ell}$ .*

**Lemma 5.3.** *Let  $\mathbf{K}$  be a lattice satisfying the following assumptions:*

- (1)  *$\mathbf{K}$  satisfies (W);*
- (2)  *$\mathbf{K}$  is generated by a set  $X$  of join prime elements.*

*Let  $F, G$  be filters of  $\mathbf{K}$ . Then:*

- (i)  $F^\ell \vee G^\ell = (F \cap G)^\ell$ ;
- (ii)  $F^\ell \cap G^\ell = (F \vee G)^\ell$ .

*Hence  $\text{DM}(\mathbf{K})$  is a sublattice of  $\text{Id}(\mathbf{K})$ .*

*Proof.* Part (ii) is general nonsense. The other direction being trivial, we need that  $F^\ell \cap G^\ell \subseteq (F \vee G)^\ell$ . If  $w \in F^\ell \cap G^\ell$ , then  $w \leq f$  for all  $f \in F$ , and  $w \leq g$  for all  $g \in G$ . Therefore  $w \leq f \wedge g$  for any  $f \in F, g \in G$ , and these are the generators for the filter generated by  $F \cup G$ .

For part (i) we need to show the nontrivial direction  $F^\ell \vee G^\ell \supseteq (F \cap G)^\ell$ . Following a standard method of proof and using the contrapositive inclusion, let:

$$\mathbf{S} = \{w \in K : \text{for all } F, G \in \text{Fil}(\mathbf{K}) : w \notin F^\ell \vee G^\ell \text{ implies } w \notin (F \cap G)^\ell\}.$$

We will prove that  $\mathbf{S}$  is a sublattice of  $\mathbf{K}$  containing the generating set  $X$ , which means  $\mathbf{S} = \mathbf{K}$  and the statement is true.

If  $w \in X$  and  $w \notin F^\ell \vee G^\ell$ , then in particular  $w \notin F^\ell \cup G^\ell$ . Then there exist  $f_1 \in F$  and  $g_1 \in G$  such that  $w \not\leq f_1$  and  $w \not\leq g_1$ . As  $w$  is join prime,  $w \not\leq f_1 \vee g_1$  which is in  $F \cap G$ . Hence  $w \notin (F \cap G)^\ell$ . Therefore  $X \subseteq \mathbf{S}$ .

If  $w = w_1 \vee w_2$  with  $w_1, w_2 \in \mathbf{S}$  and  $w \notin F^\ell \vee G^\ell$ , then one join, say  $w_1$ , is such that  $w_1 \notin F^\ell \vee G^\ell$ . Since  $w_1 \in \mathbf{S}$  we have  $w_1 \notin (F \cap G)^\ell$ , which is an ideal. Hence  $w_1 \vee w_2 \notin (F \cap G)^\ell$ , and so  $\mathbf{S}$  is closed under joins.

Now assume  $w = w_1 \wedge w_2$  with  $w_1, w_2 \in S$  and  $w \notin F^\ell \vee G^\ell$ . Then  $w_1 \notin F^\ell \vee G^\ell$  so as  $w_1 \in S$  we get  $w_1 \notin (F \cap G)^\ell$ , whence there exists  $h_1 \in F \cap G$  with  $w_1 \not\leq h_1$ . Likewise there exists  $h_2 \in F \cap G$  such that  $w_2 \not\leq h_2$ . Also  $w \notin F^\ell$  whence  $w \not\leq f_1$  for some  $f_1 \in F$ . Similarly  $w \not\leq g_1$  for some  $g_1 \in G$ .

Now we claim that by (W), we have

$$w_1 \wedge w_2 = w \not\leq (f_1 \wedge h_1 \wedge h_2) \vee (g_1 \wedge h_1 \wedge h_2) \in F \cap G.$$

Note that  $f_1 \wedge h_1 \wedge h_2 \in F$  and  $g_1 \wedge h_1 \wedge h_2 \in G$ , so their join is in  $F \cap G$ . Thus  $w \notin (F \cap G)^\ell$ , as desired, and so  $\mathbf{S}$  is closed under meets.

In view of Lemma 5.2, (i) and (ii) show that the join and meet of two DM-closed ideals is another DM-closed ideal. Thus  $\text{DM}(\mathbf{K})$  is a sublattice of  $\text{Id}(\mathbf{K})$ .  $\blacksquare$

A classic result of K. Baker and A. Hales [2] deals with the connection between a lattice and its ideal lattice:

**Lemma 5.4.** *Let  $\mathbf{L}$  be a lattice. Then*

- (1)  $\text{Id}(\mathbf{L}) \in \text{HSU}(\mathbf{L})$ ;
- (2)  $\text{Id}(\mathbf{L})$  and  $\mathbf{L}$  share the same universal positive theory;
- (3) if  $\mathbf{L}$  satisfies (W), then  $\text{Id}(\mathbf{L})$  satisfies (W).

Lemma 5.3 and 5.4 already give the first part of our Theorem 1.4. Thus, to establish the rest of Theorem 1.4, we are only left to show Theorem 5.1, i.e., that  $\kappa : \text{Id}(\mathbf{F}_n) \rightarrow \text{DM}(\mathbf{F}_n)$  is a homomorphism, which obviously fixes DM-closed sets. For this we need Lemma 5.3 and the corresponding statements for  $I^u$  and  $J^u$  where  $I$  and  $J$  are ideals, with  $u$  and  $\ell$  interchanged. Both versions apply to free lattices because the elements in the generating set are both join and meet prime. To this extent, we calculate:

$$\begin{aligned} \kappa(I \vee J) &= (I \vee J)^{u\ell} && \text{by definition} \\ &= (I^u \cap J^u)^\ell && \text{by (ii) dual} \\ &= I^{u\ell} \vee J^{u\ell} && \text{by (i)} \\ &= \kappa(I) \vee \kappa(J) && \text{by definition,} \end{aligned}$$

$$\begin{aligned} \kappa(I \wedge J) &= (I \wedge J)^{u\ell} && \text{by definition} \\ &= (I^u \vee J^u)^\ell && \text{by (i) dual} \\ &= I^{u\ell} \cap J^{u\ell} && \text{by (ii)} \\ &= \kappa(I) \wedge \kappa(J) && \text{by definition.} \end{aligned}$$

We denote by  $\text{SD}_\vee$  semidistributivity with respect to  $\vee$  and similarly for  $\wedge$ .

**Corollary 5.5.** *The ideal lattice  $\text{Id}(\mathbf{F}_n)$  and the profinite-bounded completion  $\text{DM}(\mathbf{F}_n) \cong \mathbf{H}_n$  have the following properties:*

- (1)  $\text{Id}(\mathbf{F}_n)$  satisfies (W) and  $\text{SD}_\vee$ , but fails  $\text{SD}_\wedge$  for  $n \geq 3$ ;
- (2)  $\mathbf{H}_n$  satisfies (W) and both semidistributive laws.

*Proof.* Whitman's condition for  $\text{Id}(\mathbf{F}_n)$  follows from Lemma 5.4. Join semidistributivity of  $\text{Id}(\mathbf{F}_n)$  is proved by F. Wehrung in Corollary 5.4 of [24]; see the comment immediately after the corollary. By Theorem 5.1 these properties of  $\text{Id}(\mathbf{F}_n)$  are inherited by its retract  $\mathbf{H}_n$ . Moreover, the construction of  $\mathbf{H}_n$  is self-dual, so it also satisfies  $\text{SD}_\wedge$ .

It remains to show that  $\text{Id}(\mathbf{F}_n)$  fails meet semidistributivity. In  $\mathbf{F}_3$ , let

$$\begin{array}{ll} y_0 = y & z_0 = z \\ y_{k+1} = y + xz_k & z_{k+1} = z + xy_k \end{array}$$

In  $\text{Id}(\mathbf{F}_3)$ , let  $X = \downarrow x$ ,  $Y = \bigcup_{k \geq 0} \downarrow y_k$ , and  $Z = \bigcup_{k \geq 0} \downarrow z_k$ . Then  $X \wedge Y = X \wedge Z < X \wedge (Y \vee Z)$ , since  $x(y + z)$  is in the latter but not the first two. ■

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