

Generalized extremal branes in AdS/CMT and holographic superconductors

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AdS₄ generalized extremal branes are scrutinized in the context of AdS/CFT. Holographic superconductors are studied as dual objects to AdS₄ generalized extremal branes, whose coefficients of response and transport in the dual condensed matter theory are calculated and discussed. The holographic Weyl anomaly is also addressed. The holographic superconductor bound current is shown to be enhanced by the parameter controlling the family of AdS₄ generalized extremal branes, when compared to the standard AdS₄-Reissner–Nordström black brane results. In the probe limit, the electrical DC conductivity for holographic superconductors with AdS₄ generalized extremal brane dual background is also reported.

I. INTRODUCTION

The AdS/CFT correspondence consists of a solid apparatus where strongly-coupled field theories are studied. Any given field theory, including finite temperature ones, has a hydrodynamical description in the infrared (IR) limit, corresponding to long-length scales. In the anti-de Sitter (AdS) bulk space, a theory of gravity is dual to the conformal field theory (CFT) on the boundary. At finite temperature, the bulk geometry consists of an AdS black brane with an event horizon. The holographic duality conjectures that the CFT at the long-scale regime, on the boundary, must be ruled by the near-horizon limit in the bulk. For instance, any general relativistic black hole presents a spurious fluid on its horizon, consisting of the so-called membrane paradigm, whose low-energy regime is a strongly-coupled field theory [1–3]. Transport coefficients can be computed by implementing gravitational perturbations in the black brane horizon. The shear viscosity of the dual field theory can be read off the absorption cross-section of the graviton by the black brane [4]. Einstein’s equations in the AdS bulk correspond to the Navier–Stokes equations on the AdS boundary, constituting the fluid/gravity correspondence [5].

AdS/CFT states that AdS–Schwarzschild black branes have the same metric as the stack of N D_3 branes, which is dual to finite-temperature $\mathcal{N} = 4$ supersymmetric $SU(N)$ Yang–Mills theory, in the limit of large N and large ’t Hooft coupling [1]. A D_3 brane embedded in the AdS₅ bulk yields

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a bulk Riemann tensor related to the D_3 brane Riemann tensor by the Gauss–Codazzi equations. A constraint to Einstein’s effective field equations within the holographic membrane paradigm is to demand the general-relativistic limit, consisting of a rigid brane with infinite tension as a low-energy limit [6–9]. However, D_3 branes do have finite tension. AdS/CFT and the membrane paradigm were successfully studied in Ref. [10]. The AdS/CFT fermionic sector has been tested in supersymmetric backgrounds [11]. However, a static AdS black hole in the supersymmetric limit can have a naked singularity, which can be avoided by turning on rotation, as in the Reissner–Nordström spacetime. A precise link between braneworld scenarios and AdS/CFT duality has been explored, allowing black branes and their underlying hydrodynamics to be probed by well-known AdS/CFT methods [12]. The AdS_4 –Reissner–Nordström (AdS_4 –RN) black brane plays a prominent role in this procedure, in the context of AdS/CMT (condensed matter theory) correspondence, for being dual to a finite temperature (2+1)-dimensional CFT, describing a conserved $U(1)$ charge in the boundary [13, 14]. The boundary conditions at infinity, in the AdS_4 bulk, correspond to ultraviolet values of couplings in the field theory on the boundary. Several seminal results about AdS_4 –RN black branes and the dual CMT have been seamlessly implemented, in particular concerning strange metals and holographic superconductors as well [15–17].

Superfluids and holographic superconductors can be described by a dual AdS_4 –RN geometry coupled to an AdS_4 scalar field [18, 19]. The local $U(1)$ symmetry of the AdS_4 bulk is associated with a global $U(1)$ symmetry on the AdS_4 boundary. A Higgs-like scalar field in the bulk corresponds to the condensate in the dual field theory [20]. In the AdS_4 –RN black hole near-horizon region, the Abelian symmetry in AdS_4 can be spontaneously broken, since the scalar field effective mass can attain negative values, below the so-called Breitenlohner–Freedman bound [21]. It causes the scalar field to be unstable, undergoing condensation into an atmosphere near the AdS_4 –RN black hole horizon. The AdS_4 –RN black brane is already known to describe strange metals in the holographic duality setup. Holography can relate it to finite-density, strongly-coupled, dual objects in the AdS_4 boundary, representing metallic states of matter at zero temperature [22]. These metallic states in condensed matter are observed experimentally in strongly-correlated systems, encompassing high- T_c superconductors, and constitute a class of non-Fermi-liquids corresponding to holographic strange metals. The description of the simplest kinds of strange metals, through a weakly-coupled dual theory of gravity, can be emulated by the charged AdS_4 –RN black brane [23]. Since the discovery of superconductivity at high temperatures in cuprates, holographic superconductors in AdS/CMT have been playing a prominent role in describing them. Superconductivity phenomena in cuprates cannot be reported by the standard theory based on the Bardeen–Cooper–Schrieffer

microscopic theory of superconductivity, being usually described by a different mechanism.

By promoting the standard AdS_4 –RN black brane metric to AdS_4 generalized black branes, we expect to be able to describe a higher range of holographic superconductor materials. Hence AdS_4 generalized extremal branes may model a larger class of strange metals, whose transport and response coefficients, such as the electrical conductivity, the shear viscosity-to-entropy density ratio, and the bulk viscosity-to-entropy density ratio can be fine-tuned by the parameter that governs the family of AdS_4 generalized extremal brane. This paper is organized as follows: in Sec. II AdS_4 generalized black branes are introduced in the ADM formalism, whose metric can be obtained from the Hamiltonian and momentum constraints. The very existence of a second event Killing horizon yields the parameter β that regulates the family of AdS_4 generalized extremal branes to depend on the $U(1)$ charge. The Maxwell equations are solved, and the magnetic potential is obtained, having an appropriate formula involving the β -dependent chemical potential. Using the GKPW relation, the shear viscosity-to-entropy density and the bulk viscosity-to-entropy density ratios are derived and discussed, yielding a more strict range for β . The conformal anomaly is also addressed. Sec. III is devoted to approaching the holographic superconductor in the AdS_4 generalized black brane setup, described by a complex charged scalar field emulating the Higgs field in Ginzburg–Landau theory, minimally coupled to Einstein–Maxwell theory. The Breitenlohner–Freedman bound is then discussed, and the equations of motion are solved and analyzed. Type-I and type-II superconductors are addressed. For type-II superconductors, a vortex lattice, engendering a supercurrent, is investigated. The bound current will be shown to be enhanced when compared to the AdS_4 –RN black brane case, whereas the supercurrent is not modified. In Sec. IV the electrical DC conductivity of AdS_4 generalized extremal branes at vanishing temperature is also calculated for the dual holographic superconductors, and analyzed as a function of the frequency-to-chemical potential ratio. Finally, concluding remarks and some perspectives for the forthcoming developments are presented in Sec. V.

II. AdS_4 GENERALIZED BLACK BRANES

The AdS_4 –Reissner–Nordström (AdS_4 –RN) black brane background is a well-known solution of the Einstein–Maxwell- AdS_4 theory,

$$S = \frac{1}{16\pi G_4} \int \left(R - 2\Lambda_4 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) \sqrt{-g} d^4x, \quad (1)$$

with cosmological constant $\Lambda_4 = -3/L^2$, where R is the Ricci scalar curvature, and $F_{\mu\nu}$ is the Maxwell electromagnetic field strength [13, 14]. The AdS₄–RN solution with a planar horizon has metric

$$ds_4^2 = \frac{r^2}{L^2} (-f(r)dt^2 + dx^2 + dy^2) + \frac{L^2}{r^2 f(r)} dr^2, \quad (2)$$

with

$$f(r) = 1 - (1 + \mathfrak{q}^2) \left(\frac{r_0}{r}\right)^3 + \mathfrak{q}^2 \left(\frac{r_0}{r}\right)^4. \quad (3)$$

where $\mathfrak{q}$ denotes the black hole charge and r_0 represents the event horizon. The standard AdS₄–RN solution in Eqs. (2, 3) is not valid only in the extremal case, but also in the non-extremal case. The solution also has the electromagnetic field strength component $F_{tz} = \mathfrak{q}r_0/L^2$, in Poincaré coordinates $u = r_0/r$. The extremal limit happens as $\mathfrak{q} \rightarrow \pm\sqrt{3}$, with r_0/L^2 fixed. In the gauge/gravity dictionary, the black brane planar horizon is consistent with a dual QFT on a (2+1)-Minkowski background and is useful for describing condensed matter systems. Also, the AdS₄–RN black brane charge corresponds to a dual field theory with a charge density encoding, e.g., the fluid flow of electrons in metals. The AdS₄–RN black brane is the unique regular solution of (1) with Poncaré symmetry, with electric flux at infinity. Expanding the solution (2, 3) in the near-horizon yields the background topology AdS₂ $\times$ $\mathbb{R}^2$ [24]. The near-horizon extremal geometry is dual to a field theory in the infrared limit.

The AdS/CFT membrane paradigm can set the AdS₄ spacetime as the boundary of an AdS₅ bulk, with cosmological constant Λ , related to the boundary vacuum energy density. The AdS₅ bulk, with electromagnetic field strength, satisfies Einstein’s equations,

$$R_{AB} - \frac{1}{2}R g_{AB} = \Lambda g_{AB} + T_{AB}, \quad (4)$$

for

$$T_{AB} = F_{AC}F_B^C - \frac{1}{4}g_{AB}F^2, \quad (5)$$

where $F^2 = F_{AB}F^{AB}$. Projecting Eq. (4) onto AdS₄, and introducing Gaussian coordinates (x^μ, w) in the AdS₅ bulk yields, at $w = 0$:

$$R_{\mu w} = 0, \quad R = \Lambda_4. \quad (6)$$

For static solutions, Eqs. (6) encode the Hamiltonian and the ADM momentum constraints [25] encompassing the embedding of the AdS₄–RN black brane on the AdS₅ bulk. To apply the ADM

formalism to generate a family of AdS_4 generalized extremal branes, let us emulate the metric in Eq. (2), still considering the temporal component in Eq. (3), however with a radial component deforming $f(r)$, as [26]

$$ds_4^2 = \frac{r^2}{L^2} (-f(r)dt^2 + dx^2 + dy^2) + \frac{L^2}{r^2} n(r)dr^2, \quad (7)$$

The r -coordinate can be identified to the renormalization group flow scale. A family of analytic solutions of the form (7) can be obtained by relaxing the condition $f(r) = -1/n(r)$ in Eq. (2), by fixing $f(r)$ as in Eq. (3) for the AdS_4 -RN spacetime, and finding the most general solutions for the constraints (6). The momentum constraints are identically satisfied by the metric (7) and the Hamiltonian constraint reads

$$\begin{aligned} n(r) & \left[\frac{(r-r_0)n'(r)}{L^4} (-q^2 r_0^3 + r^3 + r^2 r_0 + r r_0^2) ((q^2 + 1) r r_0^3 - 2q^2 r_0^4 + 2r^4) (r - (2\beta + 7)r_0)^3 + 4r^3 \right] \\ & + \frac{4r^4 n'(r)}{n(r)} - \frac{r^3}{[(q^2 + 1) r r_0^3 - q^2 r_0^4 - r^4]^2} [4q^4 r_0^8 - 24(q^2 + 1) r^5 r_0^3 + 32q^2 r^4 r_0^4 [6\beta + (4\beta + 59)q^2 + 21] \\ & + 3(q^2 + 1)^2 r^2 r_0^6 + (2\beta - 27q^2 - 29) r^2 r_0^2 - 8q^2 (q^2 + 1) r r_0^7 + 12r^8] = 0, \end{aligned} \quad (8)$$

where β is a real arbitrary parameter that governs the deformation of the AdS_4 -RN black brane. Hence, the solution

$$n(r) = \frac{1}{f(r)} \left\{ \frac{1 - \frac{r_0}{r}}{1 - \frac{r_0}{r} [1 + \frac{1}{3}(\beta - 1)]} \right\}, \quad (9)$$

satisfies Eq. (8), as long as the temporal component $f(r)$ is given by Eq. (3). Notice that for $\beta \rightarrow 1$, the metric Eq. (7) is precisely the AdS_4 -RN black brane background. Similarly to the AdS_4 -RN black brane, the AdS_4 generalized extremal brane also describes the physics underlying the electric flux in an asymptotically AdS_4 geometry. From now on, the value $L = 1$ will be assumed.

The limit $\lim_{r \rightarrow r_\beta} 1/n(r) = 0$, in Eq. (9), for the (coordinate) singularity

$$r_\beta = \frac{r_0}{3} [2 + \beta], \quad (10)$$

is essential for studying event horizons in AdS_4 generalized extremal branes. To realize whether the singularity r_β is an event horizon, the Killing horizon condition $\xi^\mu \xi_\mu = 0$, for ξ^μ being the associated timelike Killing vector, yields two possible values for β . The first one,

$$\beta = 1, \quad (11)$$

corresponds to the standard AdS₄–RN black brane. The other possibility that is consistent with the identification of r_β with an event horizon is given by [26]

$$\beta = \frac{2\sqrt[3]{2}}{\sqrt[3]{-7 - 27q^2 + 3\sqrt{3}\sqrt{3 + 14q^2 + 27q^4}}} + \frac{1}{\sqrt[3]{2}} \sqrt[3]{7 + 27q^2 - 3\sqrt{3}\sqrt{3 + 14q^2 + 27q^4}} - 3. \quad (12)$$

Eq. (12) expresses the parameter β expressed as a function of the tidal charge q , which plays the role of the charge wrapped by the event horizon. For both the values of β in Eqs. (11, 12), the surface dictated by the respective values of r_β consists of a Killing horizon. Fig. 1 illustrates the parameter β as a function of q .

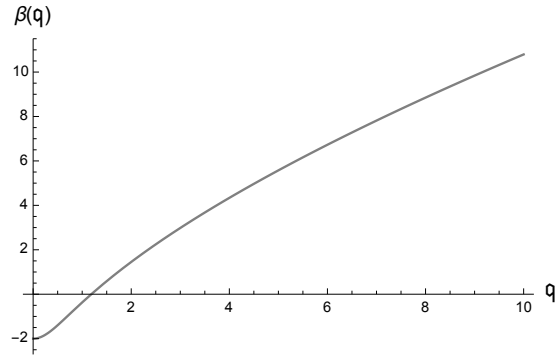


FIG. 1: Plot of the parameter $\beta(q)$ as a function of q , in Eq. (12).

As long as $\beta > -2$, as β is the deformation parameter, the charge q must be a function of β , and not the contrary. Therefore, using the implicit function theorem, one arrives at a more straightforward expression:

$$q(\beta) = \frac{1}{3\sqrt{3}} \sqrt{\beta^3 + 9\beta^2 + 33\beta + 38}. \quad (13)$$

Fig. 2 illustrates the parameter β as a function of q .

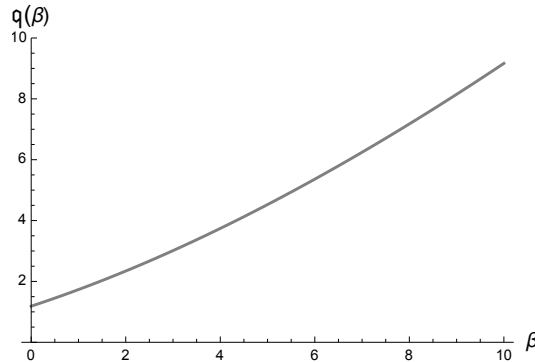


FIG. 2: Plot of the parameter $q(\beta)$ as a function of β , in Eq. (13).

Figs. 1 and 2 illustrates the saturation $\mathfrak{q} = \sqrt{3}$ at $\beta = 1$.

The electromagnetic potential has boundary components represented by the chemical potential and the charge density, to wit, in the near-boundary expansion,

$$A(r) = \mu - \frac{\mathfrak{q}}{r} + \mathcal{O}\left(\frac{1}{r^2}\right). \quad (14)$$

To require a regular gauge connection, one must impose $\lim_{r \rightarrow r_0} A(r) = 0$ at the horizon, implying that $\mu = \frac{\mathfrak{q}}{r_0}$. Otherwise, calculating the holonomy of the electromagnetic potential around the (Euclidean) time loop would yield a non-vanishing value, when the loop collapses at the r_0 horizon, meaning a singular gauge connection. The planar AdS₄-RN black brane is described by two scales: the chemical potential $\mu = \lim_{r \rightarrow \infty} A(r)$ and the horizon radius r_0 . For the AdS₄ generalized extremal brane, the solution to the Maxwell equations $\partial_\mu (\sqrt{-g} F^{\mu\nu}) = 0$, for $F_{\mu\nu} = \partial_{[\mu} A_{\nu]}$, yields

$$A(r) = \alpha(\beta, r) \mathfrak{q} \left(2\sqrt{r-r_0} \sqrt{\beta+2} \sqrt{3r-r_0(\beta+2)} + r(\beta-1) \left[\pi - 2 \arctan \left(\frac{\sqrt{r-r_0(\beta+2)}}{\sqrt{3r-r_0(\beta+2)}} \right) \right] \right), \quad (15)$$

where

$$\alpha(\beta, r) = \frac{1}{2\sqrt{3}r_0\sqrt{\beta+2}r}. \quad (16)$$

One can verify that

$$\lim_{\beta \rightarrow 1} A(r) = \mathfrak{q} \left(\frac{1}{r} - \frac{1}{r_0} \right), \quad (17)$$

corresponding to the AdS₄-RN setup. The chemical potential, μ , of the CFT₃ in the AdS₄ generalized extremal brane boundary, can be obtained by the expansion of Eq. (15), as

$$\mu = -\frac{\mathfrak{q}}{6r_0} \left[6 + \frac{\sqrt{3}(\beta-1)}{\sqrt{2+\beta}} \arctan \left(\sqrt{\frac{\beta+2}{3}} \right) \right]. \quad (18)$$

A range for the charge $\mathfrak{q}$ can be obtained, when the shear viscosity-to-entropy density ratio is calculated for the AdS₄ generalized extremal brane. The Kovtun-Son-Starinets (KSS) result, $\eta/s = 1/4\pi$ [4], will be shown to correspond to $\mathfrak{q} = \sqrt{3}$, by adding an off-diagonal gravitational perturbation $h_{xy}(t)$. The constitutive equation for the stress tensor with the dissipative term is given by

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu + P g^{\mu\nu} - P^{\mu\sigma} P^{\nu\lambda} \left[\eta \left(\nabla_{(\sigma} u_{\lambda)} - \frac{2}{3} g_{\sigma\lambda} \nabla_k u^k \right) + \zeta g_{\sigma\lambda} \nabla_k u^k \right], \quad (19)$$

where ∇_μ represents the covariant derivative with respect to the perturbed metric $g_{\mu\nu}$. The $P^{\mu\nu} := g^{\mu\nu} + u^\mu u^\nu$ is the projection tensor and the shear, η , and the bulk, ζ , viscosities carry dissipative

effects. Kubo's formula for the shear viscosity η can be derived by coupling fictitious gravity to the fluid and then determining the response of $T^{\mu\nu}$ under small gravitational perturbations. The covariant derivative of the velocity field reads $\nabla_\mu u_\nu = \Gamma_{\mu\nu}^t$, assuming $u_k = u_k(t)$. Calculating the response in T^{xy} yields

$$\delta \langle T^{xy} \rangle = -2\eta \Gamma_{xy}^t = -\eta \partial_t h_{xy} , \quad (20)$$

whose Fourier transform is given by $\delta \langle T^{xy}(\omega) \rangle = i\omega\eta h_{xy}$. Comparing this equation to $\delta \langle T^{xy} \rangle = -G_R^{xy,xy} h_{xy}$ yields the Kubo's formula for the shear viscosity [4, 27]

$$\eta = - \lim_{\substack{\omega \rightarrow 0 \\ k \rightarrow 0}} \frac{1}{\omega} \Im G_R^{xy,xy}(\omega, \vec{k}). \quad (21)$$

AdS/CFT duality states that the partition functions of both the gauge and the gravitational theories are equivalent objects, by the so-called GKPW relation [1, 28, 29],

$$\left\langle e^{i \int \varphi^{(0)} \mathcal{O}} \right\rangle = e^{i \bar{S}[\varphi^{(0)}]}. \quad (22)$$

Here $\bar{S}$ denotes the on-shell action, whereas the scalar field in the bulk theory of gravity is denoted by φ . With the definition $u = r_0/r$, one can denote $\varphi^{(0)} = \lim_{u \rightarrow 0} \varphi$. The gravitational theory is the Einstein–Hilbert action and a term describing matter,

$$S = \frac{1}{16\pi G_4} \int \sqrt{-g} (R - 2\Lambda_4) d^4x + S_{\text{MAT}}, \quad (23)$$

where denoting by a prime the differentiation with respect to u ,

$$S_{\text{MAT}} = -\frac{1}{2} \int \sqrt{-g} \nabla_\mu \varphi \nabla^\mu \varphi d^4x = -\frac{1}{2} \int \frac{r_0^3}{u^2} \varphi'^2 dx^4. \quad (24)$$

The last equality in Eq. (24) comes from requiring the scalar field $\varphi = \varphi(u)$ to be static. Integrating by parts and assuming the scalar field fades away as it approaches its null value at the horizon yields the equation of motion $\frac{1}{2} (\varphi'/u^3)' \sim 0$, whose asymptotic behavior is given by

$$\varphi \sim \varphi^{(0)} \left(1 + \varphi^{(1)} u^3 \right). \quad (25)$$

Therefore, the action (24) reduces to a surface term on the AdS boundary. Substituting it in Eq. (25), the on-shell action reads

$$\bar{S}[\varphi^{(0)}] = \frac{3}{2} \int r_0^3 \left(\varphi^{(0)} \right)^2 \varphi^{(1)} d^3x, \quad (26)$$

whereas the one-point function has the form

$$\langle \mathcal{O} \rangle_S = 3r_0^3 \varphi^{(1)} \varphi^{(0)}. \quad (27)$$

Compared to the linear response relation yields the Green function $\lim_{k \rightarrow 0} G_R^{\mathcal{O}, \mathcal{O}}(k) = -3r_0^3 \varphi^{(1)}$.

The η/s ratio is used to delimit the parameter $\beta = \beta(\mathbf{q})$ of generalized AdS₄ black branes. For it, a bulk small gravitational perturbation h_{xy} can be considered, such that $ds^2 = ds_0^2 + 2h_{xy}dx dy$, where ds_0^2 is given by Eq. (7). The response of the energy-momentum tensor on the boundary wits

$$\delta \langle T^{xy} \rangle = i\omega \eta h_{xy}, \quad (28)$$

for $h_{xy}^{(0)}$ being the perturbation to the boundary theory, which is asymptotically related to h_{xy} by the result $g^{xx}h_{xy} \sim h_{xy}^{(0)} \left(1 + h_{xy}^{(1)}u^3\right)$. Eq. (27) yields

$$\delta \langle T^{xy} \rangle = \frac{r_0^3}{16\pi} 3h_{xy}^{(1)}h_{xy}. \quad (29)$$

Comparing Eqs. (28) and Eq. (29) implies that $i\omega\eta = \frac{3r_0^3}{16\pi G_4} h_{xy}^{(1)}$. Taking into account that the entropy density associated with Eq. (7) reads $s = r_\beta^2/4$, implies

$$\frac{\eta}{s} = -\frac{3ir_0}{4\pi \left[1 + \frac{2}{9}(\beta - 1)\right]^2} \frac{h_{xy}^{(1)}}{\omega}. \quad (30)$$

One now must find $h_{xy}^{(1)}$, solving the equation of motion for the 4D massless scalar perturbation $g^{xx}h_{xy}$. Taking $\varphi(u) = \phi(u)e^{-i\omega t}$, the perturbation reads

$$\phi = \phi^{(0)} \left[1 - i\frac{\omega}{r_0} \frac{\mathbf{q}^2 - 3}{|\mathbf{q}^2 - 3|} \int u^2 \sqrt{\left\{ \frac{1-u}{1-u \left[1 + \frac{1}{3}(\beta - 1)\right]} \right\}} du \right], \quad (31)$$

and

$$h_{xy}^{(1)} = -\frac{i\omega}{3r_0} \frac{\mathbf{q}^2 - 3}{|\mathbf{q}^2 - 3|}. \quad (32)$$

When Eq. (32) is superseded into Eq. (30), it yields

$$\frac{\eta}{s} = 9 \left(1 - \frac{2\sqrt[3]{2}}{\alpha_{\mathbf{q}}} + \frac{\alpha_{\mathbf{q}}}{\sqrt[3]{2}} \right)^{-2} \frac{(3 - \mathbf{q}^2)}{|3 - \mathbf{q}^2|}, \quad (33)$$

for $\alpha_{\mathbf{q}} = \sqrt[3]{-7 - 27\mathbf{q}^2 + 3\sqrt{9 + 42\mathbf{q}^2 + 81\mathbf{q}^4}}$, where the $\beta = \beta(\mathbf{q})$ parameter was written as given by Eq. (12).

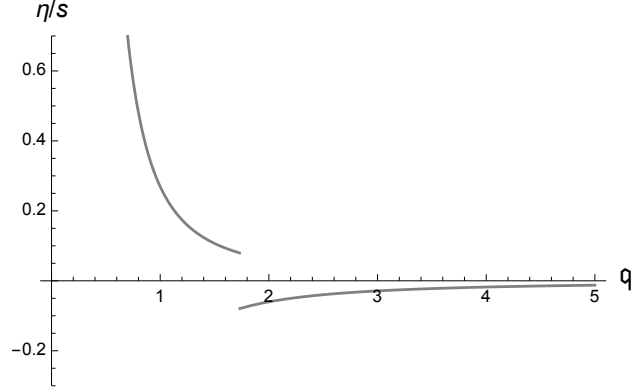


FIG. 3: Plot of η/s as a function of q , as in Eq. (33). One can realize that the η/s ratio changes sign at $q = \sqrt{3}$.

Fig. 3 transport at the extremal horizon as well (via the membrane paradigm), shows that the bound $0 < q < \sqrt{3}$ yields a positive value of the ratio η/s , which assumes the saturated KSS value $\eta/s = 1/4\pi$ when the tidal charge saturates, namely, $q = \sqrt{3}$. The thermodynamically correct range $0 < q < \sqrt{3}$ implies $-2 < \beta < 1$ for the AdS_4 generalized extremal brane, yielding the event horizon r_β to be smaller than r_0 , being an inner horizon. It indeed complies with the interpretation of r_0 as the effective outermost of these event horizons. Therefore, in the next section, the term near-horizon corresponds to $u = r_0/r \rightarrow 1$.

Similarly, the bulk viscosity can be calculated by the Kubo's formula

$$\zeta = \lim_{\substack{\omega \rightarrow 0 \\ k \rightarrow 0}} \frac{1}{\omega} \Im G_R^{PP}(\omega, \vec{k}), \quad (34)$$

where

$$G_R^{PP}(\omega, \vec{k}) = \frac{k_i k_j k_m k_n}{k^4} \left[G_R^{ij,mn}(\omega, \vec{k}) + \frac{1}{3} \delta_{ab} T^{ab} \left(\delta^{i(m} \delta^{j|n)} - \delta^{ij} \delta^{mn} \right) \right] + \frac{1}{3} \delta_{ij} T^{ij} - \frac{4}{3} G_R^{xy,xy}(\omega, \vec{k})$$

is the response function to longitudinal fluctuations [30]. The bulk viscosity-to-entropy density ratio reads, for the AdS_4 generalized extremal brane,

$$\frac{\zeta}{s} = \frac{3}{16} \left(1 - \sqrt[3]{2} \gamma_q + \frac{1}{\sqrt[3]{2} \gamma_q} \right) (3 - q^2)^2, \quad (35)$$

where $\gamma_q = \sqrt[3]{4 + 14q^2 + \sqrt{4 + 14q^2 + 46q^4 + 12q^6}}$, is plotted in Fig. 4, in the allowed range $0 < q < \sqrt{3}$ where $\eta/s > 0$.

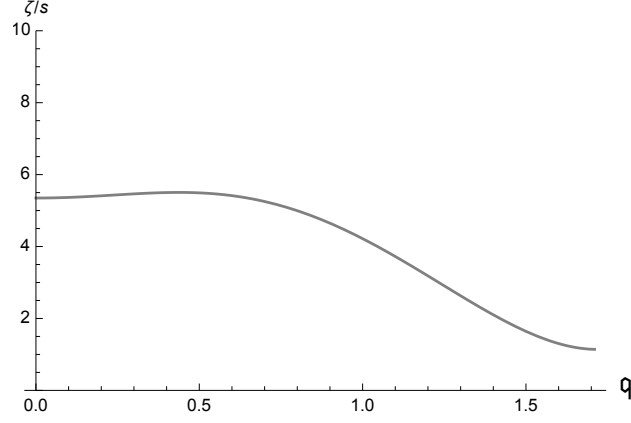


FIG. 4: Plot of ζ/s as a function of q , as in Eq. (35).

The temperature of the AdS_4 generalized extremal brane at the Killing horizon r_β can be evaluated by [26]

$$\kappa^2 = \lim_{r \rightarrow r_\beta} \left[- \frac{(\xi^\mu \nabla_\mu \xi^\lambda)(\xi^\nu \nabla_\nu \xi_\lambda)}{\xi^\rho \xi_\rho} \right]. \quad (36)$$

In this way κ is the surface gravity at the horizon [31], which is related to the temperature of AdS_4 generalized extremal branes by

$$T = \frac{\kappa}{2\pi}. \quad (37)$$

Evaluating (36) at the horizon $r = r_\beta$ yields

$$\kappa^2 = \lim_{r \rightarrow r_\beta} \frac{[2r^4 + (1 + q^2)r_0^3r - 2q^2r_0^4]^2 [3r + (\beta + 2)r_0]}{12r^4(r - r_0)^2(r^3 + r^2r_0 + rr_0^2 - q^2r_0^3)} = 0 \quad (38)$$

Since Eq. (38) is independent of β , it holds for the entire range $-2 < \beta < 1$. A zero-temperature black brane is expected, when extremal AdS_4 generalized extremal branes are regarded. In the extremal case where the electromagnetic field carries all its energy, the extremal AdS_4 -RN black brane corresponds to the zero-temperature state of the boundary [22]. The metric (2) with coefficients (3) define the extremal AdS_4 -RN black brane with the planar horizon, with mass coincident with its underlying electromagnetic energy. Therefore, it no longer sheds energy, owing to charge conservation. Consequently, the Hawking temperature of AdS_4 -RN black branes vanishes as $r \rightarrow r_\beta$ since they are forbidden to radiate due to charge conservation. An analogous process occurs in the family of extremal AdS_4 generalized black branes described by the metric (7), with the metric temporal coefficient $f(r)$ as in Eq. (3), but with metric radial coefficient (9). The associated Hawking temperature equals zero, as $r \rightarrow r_\beta$. The extremal AdS_4 -RN black branes have their whole mass

due to their conserved electric charge. Hence, further reducing the mass becomes impossible, and energy conservation prevents radiation emission. The horizon has a lower limit, having macroscopic size, implying that the extremal AdS₄-RN black brane does carry a finite Bekenstein–Hawking entropy. Therefore, this feature does allow the deformation parameter β to attain the value $\beta = -2$, since for it $r_\beta = 0$ and the black brane charge vanishes, $q = 0$, according to Eq. (10) and Fig. 1. Also, near $\beta = -2$, the horizon r_β would not be macroscopic. Therefore, one can stipulate a reasonable range

$$-1.95 \lesssim \beta < 1, \quad (39)$$

compatible with a macroscopic horizon. In order to study systems in condensed matter, the infrared (IR) limit has a deep interest, corresponding to the near-horizon limit of AdS₄ generalized black branes. Like the standard AdS₄-RN, the near horizon structure is AdS₂ $\times$ $\mathbb{R}^2$ in the range (39).

Now, one can better clarify aspects of the β parameter. AdS/CFT relates the electric part of the Weyl tensor, $\mathcal{E}_{\mu\nu}$, representing the propagation of classical gravitational waves in the AdS₅ bulk, to the expectation value $\langle T_{\mu\nu} \rangle$ of the renormalized energy-momentum tensor of conformal fields on the AdS₄. The large- N limit expansion in CFT requires $N \sim 1/(T_4 \ell_p)^2 \gg 1$. In the original Randall–Sundrum braneworld models, the Planck length, ℓ_p (for $8\pi G_4 = \ell_p^2$, where G_4 is the 4D Newton’s constant), is related to the AdS₅ Planck length ℓ_5 by $\ell_p^2 = T_4 \ell_5^3$ [32], where T_4 is the AdS₄ generalized extremal brane tension [33, 34]. In addition, embedding AdS₄ in AdS₅ introduces a normalizable 4D graviton and an ultraviolet (UV) cut-off in the CFT, which is proportional to T_4^{-1} . In the AdS/CFT setup, $\mathcal{E}_\mu^\nu \sim \ell_p^2 \langle T_\mu^\nu \rangle$. Since the electric part of the Weyl tensor is traceless, it implies that $\langle T_\mu^\mu \rangle = 0$, which is valid as long as the conformal symmetry is not anomalous. One can use the GKPW relation to consider the partition function of the dual theory at the boundary. On the other hand, statistical-mechanical arguments state that the partition function is proportional to the free energy, F , which yields the energy density and the pressure. For AdS₄ generalized extremal branes, a conformal anomaly encodes quantum corrections induced by $\beta \neq 1$, to wit

$$\langle T_\mu^\mu \rangle = \left[(4q^2 - 3(q^2 + 1)) (q^2 - 3)^2 ((3\beta + 2) - 3)^2 \right] \left(\frac{3 + \beta}{4 + 3\beta} \right)^2. \quad (40)$$

It complies with the fact that the vanishing of $\langle T_\mu^\mu \rangle$ implies the UV cutoff must be smaller than any physical length scale involved [35, 36]. Also, $\langle T_\mu^\mu \rangle \neq 0$ signs the presence of an intrinsic length. Otherwise, the CFT could be affected by that scale. For the AdS₄ generalized extremal brane, the horizon r_0 is a natural length scale, and one therefore expects that only wave modes in the CFT having wavelength $\ell \ll r_0$, but still larger than T_4^{-1} , can freely propagate [37, 38].

For the AdS_4 generalized extremal brane, the holographic computation of the Weyl anomaly can be implemented [34, 39]. Denoting a and c central charges of the conformal gauge theory (see Eq. (24) of Ref. [40]),

$$\langle T^\mu_\mu \rangle_{\text{CFT}} = \frac{c}{16\pi^2} \left(R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 2R_{\mu\nu} R^{\mu\nu} + \frac{1}{3} R^2 \right) - \frac{a}{16\pi^2} (R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4R_{\mu\nu} R^{\mu\nu} + R^2), \quad (41)$$

where the terms in parentheses are, respectively, the Euler density and the square of the Weyl curvature [41]. The AdS_4 generalized extremal brane at the boundary $u \rightarrow 0$, can have the square of the Weyl curvature to be Taylor-expanded as

$$N^2 \left[\frac{40}{3} + \frac{64}{9} (3\beta - 1)u + \frac{2}{27} (315\beta^2 + 6\beta - 37) u^2 + \frac{8}{81} u^3 [27\beta(\beta(7\beta + 4) - 1) + 189\mathfrak{q}^2 + 179] \right. \\ \left. + \frac{1}{243} u^4 [(3\beta - 1)(36\beta(3\beta(5\beta + 9) + 16) + 1219) + 27(123\beta - 137)\mathfrak{q}^2] \right] + \mathcal{O}(u^5), \quad (42)$$

and the Euler density reading

$$N^2 \left[120 + 64(3\beta - 1)u + 24(9\beta^2 - 1) u^2 + u^3 \left(32\beta(6\beta^2 + 3\beta - 1) + 168\mathfrak{q}^2 + \frac{1448}{9} \right) \right. \\ \left. + u^4 \left(\frac{2}{27} (3\beta - 1)(36\beta(3\beta(5\beta + 7) + 8) + 853) + 2(93\beta - 79)\mathfrak{q}^2 \right) \right] + \mathcal{O}(u^5), \quad (43)$$

where $N^2 = \pi L^3 / 2G_4$. Therefore, the CFT boundary $u \rightarrow 0$ limit implies that

$$\langle T^\mu_\mu \rangle_{\text{CFT}} = \frac{400N^2}{3}, \quad (44)$$

having an analogous result to the AdS_4 -RN black brane. In the next section, we will study the condensed matter state dual to extremal AdS_4 generalized black branes.

III. HOLOGRAPHIC SUPERCONDUCTOR AND AdS_4 GENERALIZED BLACK BRANES

Now, one can turn into the holographic superconductor, with the AdS_4 generalized black brane background. As reported by the Ginzburg–Landau theory, superconductivity relies on the (spontaneous) $U(1)$ symmetry breaking leading to a superfluid states, as long as the $U(1)$ broken symmetry is global. Nevertheless, the superfluid state can be promoted to a superconductor by the weak coupling of a $U(1)$ gauge field in the boundary theory, with a conserved $U(1)$ current. The gauge symmetry in the dual bulk is broken, by a subsequent formation of a Higgs condensate in the bulk [18]. This setup enables to describe superconductivity in the bulk, by adding a charged complex scalar field, ϕ , emulating the Higgs field in Ginzburg–Landau theory, minimally coupled to gravity,

and a Maxwell Yang–Mills field as well. The 4-dimensional s -wave holographic superconductor (HS) in the AdS_4 bulk embedded in an AdS_5 space can be described by

$$S_{\text{BULK}} = \int (R - 2\Lambda_4) \sqrt{-g} d^4x + S_{\text{HS}} , \quad (45a)$$

$$S_{\text{HS}} = -\frac{1}{g_{\text{YM}}^2} \int \sqrt{-g} \left(\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + D^\mu \phi D_\mu \phi^* + m^2 |\phi|^2 \right) d^4x, \quad (45b)$$

where g_{YM} is an effective coupling. Here $F_{\mu\nu} = \partial_{[\mu} A_{\nu]}$ is the $\text{U}(1)$ gauge field strength associated with the $\text{U}(1)$ gauge field A_μ that describes finite charge density. The Maxwell term lies in the part S_{HS} of the action in Eq. (45b) and it does not contribute to the computation of the shear viscosity, η . The bulk scalar field ϕ is the dual object to the order parameter, whose VEV parametrizes the broken $\text{U}(1)$ global symmetry in the boundary. It is worth mentioning that the potential $V(\phi) = m^2 |\phi|^2$ in (45b) refers to the so-called minimal holographic superconductor, where one disregards self-interaction (and higher-order integer powers) of the Higgs scalar. The AdS_4 generalized extremal brane, presented in the previous section, will be used to study the holographic superconductor for the probe limit $g_{\text{YM}} \gg 1$ yielding no backreaction of the scalar and gauge fields onto the AdS_4 generalized extremal brane geometry. Expressing the extrinsic curvature, coming from the AdS_4 space embedded in AdS_5 , by [42]

$$K_{\mu\nu} = -\frac{1}{2} \left[T_{\mu\nu} + \frac{1}{3} (T_4 - T^\alpha_\alpha) g_{\mu\nu} \right], \quad (46)$$

where T_4 is the AdS_4 generalized extremal brane intrinsic tension yields the electric part of the Weyl tensor to be written as

$$E_{\mu\nu} = -\frac{\Lambda}{6} g_{\mu\nu} - \partial_w K_{\mu\nu} + K_\mu^\rho K_{\rho\nu}, \quad (47)$$

where w denotes the Gaussian coordinates along the AdS_5 bulk, as previously mentioned. Therefore, the bulk equations of motion for the matter fields are obtained from the action (45b) and expressed as

$$D_\mu D^\mu \phi - m^2 \phi = 0, \quad (48a)$$

$$\nabla^\mu F_{\mu\nu} - 2\Im(\phi^\dagger D_\nu \phi) = 0, \quad (48b)$$

with covariant derivative $D_\mu \phi = \partial_\mu \phi + iq A_\mu \phi$, where q represents the scalar field charge. Assuming a central scalar field $\phi = \phi(r)$ and an electrostatic potential $A_t \neq 0$, Eq. (48b) yields

$$-m^2 \phi + q^2 f(r) (A^t)^2 \phi = 0, \quad (49)$$

for $f(r)$ being the time component of the AdS_4 generalized extremal brane metric, given by Eq. (3). It suggests that the electrostatic potential plays the role of a negative mass-squared, characterizing the charged scalar field. If it gets large enough, it can stir up instabilities. For $\beta \approx 1$, the AdS_4 generalized black brane (7) with metric terms (3, 9) can be described by a domain wall playing the role of a tunneling barrier, going from a pure AdS_4 bulk, in the UV regime, and the near-horizon $\text{AdS}_2 \times \mathbb{R}^2$ topology, in the IR limit [43]. For any value of β , in the pure AdS_4 bulk case, taking into account the electromagnetic potential in Eq. (15), the term $f(r)(A^t)^2$ in Eq. (49) contributes as

$$\begin{aligned} & \frac{1}{12(\beta+2)R^2} \left\{ q^2(\beta-1) \left[\frac{5}{4(\beta+2)^{3/2}\sqrt{R}(\beta R+2R-1)} - \frac{\beta-1}{2} \right. \right. \\ & \left. \left. + \left((\beta-1) \left(2\text{arctanh} \left(\frac{1}{\sqrt{(\beta+2)R}} \right) + \pi \right) - (\beta+2) \right)^2 \right] \right\} \times \frac{1}{r^2}. \end{aligned} \quad (50)$$

and is subleading to the mass term in the UV limit. The limit $\beta \rightarrow -2$ makes this term diverge, and a secure neighborhood of the value $\beta = -2$ should be avoided, again corroborating with the range (39). As the potential (50) goes as $\propto 1/r^2$ as $r \rightarrow \infty$, the quantum theory is stable and the Breitenlohner–Freedman (BF) bound for the AdS_4 generalized black brane reads¹

$$m^2 \gtrsim -\frac{9}{4}, \quad (51)$$

analogously to the AdS_4 -RN black brane case, irrespectively of the value of β . The vacuum is stable if the BF bound (51) holds. For analyzing the IR limit, $\text{AdS}_2 \times \mathbb{R}^2$, the term $f(r)(A^t)^2$ in Eq. (49) is a constant. For values of $\beta \approx 1$, a second stability criterium exists for the extremal AdS_4 generalized black brane in the IR geometry, namely

$$m^2 \geq -\frac{3}{2} + \frac{q^2 g_{\text{YM}}^2 L^2}{8\pi G_4}. \quad (52)$$

Therefore, the charged scalar field coupled to AdS_4 generalized black brane geometry can be unstable in the IR limit and stable in the UV regime. When the BF bound is violated, in the IR limit, spontaneous pair production sets in. The corresponding scalar field quanta can form a macroscopic ground state condensate, equivalent to the Green's function of the scalar field having its pole transferred to the upper half plane [44]. Near the charged extremal AdS_4 generalized black brane horizon r_β , the electric field is strong enough to make the local vacuum discharge, emulating superradiance. The positively charged quanta escape and constitute AdS_4 generalized

¹ Remember that here we assumed $L = 1$ and, correspondingly, $m^2 L^2 \geq -\frac{9}{4}$, for the most general case.

black brane hair. The AdS_4 geometry plays the role of a near-infinity gravitational potential barrier balancing the repulsive electromagnetic force, developing a charged scalar atmosphere around the AdS_4 generalized black brane. The charge is carried to the atmosphere, decreasing the near-horizon electric field intensity, as long as the pair production phenomenon does not come to an end. In this way, the charged scalar field attains a finite value of amplitude, which is maximal at the near-horizon limit and corresponds to scalar hair.

Let us suppose that $m^2 < 0$, above the BF bound (51). In the decoupling limit $8\pi G_4 g_{\text{YM}}^2 L^2 \gg 1$ of weakly-coupled gravity, the scalar and gauge sectors become decoupled from gravity and do not contribute to the AdS_4 curvature. Therefore, all the analytical and numerical methods can be used in the fixed background of the AdS_4 generalized extremal brane with metric (7), with coefficients (3, 9). Using the notation

$$\varphi = \frac{\phi}{u}, \quad (53a)$$

$$V(u) = \frac{1}{u^2} \left(m^2 + 2\sqrt{f(u)n(u)} - \frac{(f'(u)n(u) + n'(u)f(u))u}{2\sqrt{f(u)n(u)}} \right), \quad (53b)$$

in the $A_u = 0$ gauge the static bulk equations (48a, 48b) read

$$\left[-\sqrt{f(u)n(u)}\partial_u^2 - \partial_x^2 - \partial_y^2 + 2|\varphi|^2 \right] A_t = 0, \quad (54a)$$

$$\left[-\partial_u(\sqrt{f(u)n(u)}\partial_u) - \partial_x^2 - \partial_y^2 + 2|\varphi|^2 \right] A_i - 2\Im(\varphi^\dagger \partial_i \varphi) + \partial_i \left(\delta^{jk} \partial_j A_k \right) = 0, \quad (54b)$$

$$\left[-\partial_u(\sqrt{f(u)n(u)}\partial_u) - \frac{A_t^2}{\sqrt{f(u)n(u)}} - \delta^{ij} D_i D_j + V(u) \right] \varphi = 0, \quad (54c)$$

$$\partial_u \left[(\delta^{ij} \partial_i A_j) \right] - 2\Im(\varphi^\dagger \partial_u \varphi) = 0, \quad (54d)$$

The CFT two-point correlation function for scalar operators with scaling dimension can be expressed in terms of the ratio of the coefficients $\phi^{(\pm)}$ of the leading and sub-leading asymptotes of the corresponding AdS massive scalar waves. In the near-boundary region one can express $\langle \mathcal{O}(-k) \mathcal{O}(k) \rangle = \phi^{(+)} / \phi^{(-)}$, for the bare (UV) scalar operator $\mathcal{O}(x)$ of the CFT with conformal scaling dimension Δ . Therefore, both the electromagnetic potential and the scalar field with

$$A_\mu \approx \mathbf{A}_\mu + A_\mu^{(+)} u, \quad (55a)$$

$$\phi \approx \phi^{(-)} u^{\Delta_-} + \phi^{(+)} u^{\Delta_+}, \quad (55b)$$

with scaling dimension $\Delta_\pm = \frac{3}{2} \pm \sqrt{\frac{9}{4} + m^2}$, set by the mass of the AdS scalar wave. In Eq. (55a) $A_t = \mu$ is the chemical potential, and $A_t^{(+)}$ represents the charge density $\langle \rho \rangle$, whereas $\mathbf{A}_i$ is the vector part of the electromagnetic potential and $A_i^{(+)}$ represents the current density $\langle J_i \rangle$. In Eq.

(55b) the term $\phi^{(+)}$ represents the order parameter $\langle \mathcal{O} \rangle$, whereas $\phi^{(-)}$ is the external source for the order parameter [22]. According to the standard AdS/CFT,

$$\langle J_\mu \rangle = \frac{1}{g_{\text{YM}}} \lim_{u \rightarrow 0} F_{u\mu}. \quad (56a)$$

The limit $g_{\text{YM}}^2 \rightarrow 1$ is assumed in what follows, for the sake of simplicity.

A. Small values of the magnetic field

The limit $\beta \rightarrow 1$ in Eq. (11) recovers the AdS₄ black brane solution, matching the well-known results in Ref. [45]. One can argue whether magnetic fields can percolate into the holographic superconductor. For a small magnetic field, the results for the AdS₄ generalized extremal brane are analogous to the standard AdS₄-RN black brane solution. One can consider the system to be at zero temperature, exploring the $T = 0$ part of the phase diagram, since the focus hereon will comprise the magnetic field, the critical values of the magnetic field, H_{c1} and H_{c2} , the Meissner effect, and creation of vortices. The $T = 0$ temperature is consistent with a uniform scalar condensate $\varphi_0 = \varphi_0(u)$ satisfying the equations of motion. One can then apply a magnetic field perturbatively, considering $A_y = A_y(x, u)$ and $B_z = \partial_x A_y$. Therefore, taking into account the AdS₄ generalized extremal brane metric coefficients (3, 9), the bulk Maxwell equations can be expressed as

$$\left[-\partial_u(\sqrt{f(u)n(u)}\partial_u) + k^2 + 2|\varphi_0|^2 \right] \dot{A}_y = 0, \quad (57)$$

where

$$\dot{A}_y(k, u) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-ikx} A_y(x, u) dx. \quad (58)$$

For $r \gg r_\beta$, the solution of Eq. (57) reads

$$\begin{aligned} \dot{A}_y(k, u) = & c_1 U \left(-\frac{9\beta^2 - 6\beta - 4k^2 - 8|\varphi_0|^2 - 15}{(\beta+1)(3\beta-1)}, \frac{2(3\beta-7)}{3\beta-1} + 1, \frac{3}{2}u(\beta+1) - \frac{9(\beta+1)}{3\beta-1} \right) \\ & \times \exp \left(\frac{-9\beta}{2(1-3\beta)^2} \left[(3\beta^2 + 2\beta - 1)u + (5-3\beta) \log((1-3\beta)u+6) + (3\beta-5) \log(3\beta u - u + 6) \right] \right) \\ & + c_2 L_{\frac{\frac{2(3\beta-7)}{3\beta-1}}{-4k^2+9\beta^2-6\beta-8|\varphi_0|^2-15}}^{\frac{2(3\beta-7)}{3\beta-1}} \left(\frac{3}{2}u(\beta+1) - \frac{9(\beta+1)}{3\beta-1} \right) \\ & \times \exp \left(\frac{3}{2(1-3\beta)} \left[(3\beta^2 + 2\beta - 1)u + (5-3\beta) \log((1-3\beta)u+6) + (3\beta-5) \log(3\beta u - u + 6) \right] \right), \end{aligned} \quad (59)$$

where $U[a, b, z]$ denotes the confluent hypergeometric function and $L_m^n[z]$ is the generalized Laguerre polynomial, with c_1 and c_2 integration constants.

One of the theoretical explanations of the Meissner effect, which is the physical property that makes the condensate repel magnetic flux, comes from the London equation, which states that the magnetic field strength decays inside the superconductor. The Meissner effect distinguishes superconductors from superfluids. It happens in superconductors with gauged symmetry. Under the usual Dirichlet boundary condition, no Meissner effect sets in: the magnetic field percolates the holographic superconductor. Eq. (57) can be integrated as

$$\sqrt{f(u)n(u)}\partial_u\mathring{A}_y = -\int_u^1 (k^2 + 2|\varphi_0|^2) \mathring{A}_y(u') du'. \quad (60)$$

Eq. (60) can be then integrated to yield

$$\mathring{A}_y = \mathring{A}_y \left[1 - k^2 \int_0^u \frac{du'}{1 + u' + u'^2} - \int_0^u \left(\int_{u'}^1 2|\varphi_0|^2 du'' + \dots \right) \frac{du'}{\sqrt{f(u')n(u')}} \right]. \quad (61)$$

A regular solution exists since the u' -integral involves the $1/\sqrt{f(u')n(u')}$ term, whose eventual divergence at the horizon can be avoided, by the vanishing of the u'' -integral at the horizon. The first term in Eq. (61) represents the magnetic induction $\mathring{\mathbf{B}} = iq\mathring{A}_y$. Therefore, a solution for the magnetic induction exists even for non-vanishing values of the uniform condensate. It implies that the lower critical magnetic field for the holographic superconductor under the Dirichlet boundary condition equals zero. The terms of Eq. (61) encode the current,

$$\langle \mathring{J}_y \rangle = \lim_{u \rightarrow 0} \partial_u \mathring{A}_y = \mathring{A}_y \left(-k^2 - 2 \int_0^1 |\varphi_0|^2 du + \dots \right). \quad (62)$$

The second term in Eq. (62) represents the supercurrent, namely, the superconducting current in a superconductor. However, in the absence of the Ampère law, $\nabla \times \mathbf{B} = e^2 \mathbf{J}$, at the boundary, no Meissner effect exists. The first term in Eq. (62) can be interpreted as the bound current producing a diamagnetic current, even in the standard electromagnetic theory with no scalar field φ_0 . By changing the boundary condition and imposing the (holographic) semiclassical equation $\partial_j F^{ij} = e^2 \langle J^i \rangle$ yields [45]

$$\left[k^2 + e^2 \left(k^2 + 2 \int_0^1 |\varphi_0|^2 du \right) \right] \mathring{A}_y = 0, \quad (63)$$

implying that the magnetic potential, and consequently, the magnetic induction, equals zero. This result emulates the Ginzburg–Landau theory. To get a non-trivial solution, an external source has to be added to the holographic semiclassical equation,

$$\partial_j F^{ij} = e^2 \langle J^i \rangle + e^2 J_{\text{EXT}}^i. \quad (64)$$

Eq. (64) can be rewritten as [45]

$$k^2 \mathring{A}^y = \mu_m \mathring{J}_{\text{EXT}}^y, \quad (65)$$

considering an effective magnetic permeability $\mu_m = \frac{e^2}{1+e^2}$. Eq. (65) is the Ampère law $\nabla \times \mathbf{B} = \mu_m \mathbf{J}$. Hence, bound current can shift the vacuum value $\mu_0 = e^2$ to μ_m , which is related to the magnetic susceptibility χ_m as

$$\chi_m = \frac{\mu_m}{\mu_0} - 1 = -\frac{e^2}{1+e^2} < 0, \quad (66a)$$

constituting a diamagnetic current. Denoting $I = \int_0^1 du |\varphi_0|^2$, the supercurrent can be addressed and the semiclassical equation becomes

$$k^2 \mathring{A}_y = \mu_m \left(-2I \mathring{A}_y + \mathring{J}_y^{\text{ext}} \right). \quad (67)$$

The case $I \neq 0$ implies that $\mathring{A}_y \propto e^{-x/\lambda}$, where $\lambda^{-1/2} = 2\mu_m I$, for λ denoting the magnetic penetration length, preventing the Meissner effect in the so-called type-II superconductors. For type-II superconductors the magnetic field percolates the superconductor, and vortexes can be evinced. Type-I superconducting materials are ideal diamagnets, and there is a critical magnetic induction above which superconductivity evanesces; the Meissner effect prevents the magnetic field from entering the superconductor. On the other hand, type-II superconductors have two critical magnetic fields, H_{c1} and H_{c2} . The first one labels the critical magnetic field for which the first vortex can permeate the superconductor. The second one regulates situations where the vortexes are maximally stored inside the superconductor, in the sense that introducing one more magnetic flux breaks the superconductor [46]. Nowhere near the vortex, the condensate can be constant, when the magnetic induction is small. One can then consider planar polar coordinates $ds^2 = dr^2 + r^2 d\phi^2$ and the equation governing $A_\phi = A_\phi(u, r)$ reads [45]

$$\partial_u \left(\sqrt{f(u)n(u)} \partial_u A_\phi \right) + r \partial_r \left(\frac{1}{r} \partial_r A_\phi \right) - 2|\varphi_0|^2 A_\phi = 0. \quad (68)$$

The ansatz $A_\phi = U(u)R(r)$ yields

$$\frac{1}{U} \frac{d}{du} \left(\sqrt{f(u)n(u)} \frac{d}{du} U(u) \right) - 2|\varphi_0|^2 U(u) = -\frac{r}{R} \frac{d}{dr} \left(\frac{1}{r} \frac{d}{dr} R(r) \right) = -\frac{1}{\lambda^2}, \quad (69)$$

where λ stands for a separation constant. The second equation in (69) describes the superconducting vortex, whose arbitrary solution can be written as a linear combination of the first and second kind Bessel functions, as

$$R(r) = c_1 r J_1 \left(\frac{r}{\lambda} \right) + c_2 r Y_1 \left(\frac{r}{\lambda} \right), \quad (70)$$

with c_1 and c_2 integration constants. Asymptotically, one can realize that

$$\lim_{r \rightarrow \infty} R(r) \propto \sqrt{r} e^{-r/\lambda}. \quad (71)$$

Thus, λ in the vortex case represents the magnetic penetration length. The first equation in (69) can be expressed as

$$\sqrt{f(u)n(u)} \partial_u U = \int_u^1 (\lambda^{-2} - 2|\varphi_0|^2) U du', \quad (72a)$$

which can be integrated, leading to

$$U = \mathcal{U} \left\{ 1 + \int_0^u \frac{du'}{\sqrt{f(u')n(u')}} \int_{u'}^1 (1/\lambda^2 - 2|\varphi_0|^2) du'' + \dots \right\}. \quad (72b)$$

For $u \ll 1$, the current is given by

$$\langle J_\phi \rangle = \frac{1}{r^2} \partial_u A_\phi. \quad (73)$$

This result for the AdS_4 generalized extremal brane equals the one for the AdS_4 -RN black brane [45]. Imposing the semiclassical equation, $\nabla^j F_{\phi j} = e^2 \langle J_\phi \rangle$, it yields

$$\lambda^2 = \frac{1 + e^2}{2e^2 I}, \quad (74)$$

which is analogous to the standard AdS_4 -RN black brane case.

B. Higher values of the magnetic field near H_{c2}

Strengthening the magnetic field, more vortexes can be created, constituting a vortex lattice engendering a supercurrent [47]. The superconducting state is thoroughly disrupted at the upper critical magnetic field H_{c2} [45]. Different from the current induced by a magnetic field, which can be computed by the linear response to the magnetic potential, the supercurrent consists of the dissipationless flow of the paired electrons. However, as in the standard holographic superconductor, there is no Maxwell equation on the AdS boundary. Therefore, the magnetic field can enter the superconductor not only at vortex cores. The holographic semiclassical equation was shown in Ref. [45] to imply the Meissner effect for the AdS_4 -RN black brane. The case for the AdS_4 generalized extremal brane will be addressed. Near the upper critical magnetic field, the scalar field, representing the condensate, is residual. Hence, one can expand any matter field as a ε -series, where ε denotes the deviation parameter with respect to the critical point. The condensate and

the temporal and spatial components of the magnetic potential are, respectively, expanded as

$$\varphi(\vec{x}, u) = \varepsilon \varphi^{(1)} + \cdots , \quad (75a)$$

$$A_t(\vec{x}, u) = A_t^{(0)} + \varepsilon^2 A_t^{(2)} + \cdots , \quad (75b)$$

$$A_i(\vec{x}, u) = A_i^{(0)} + \varepsilon^2 A_i^{(2)} + \cdots . \quad (75c)$$

At zeroth order, the static bulk equations (54) read, for the AdS₄ generalized extremal brane,

$$\left(\sqrt{f(u)n(u)} \partial_u^2 + \partial_x^2 + \partial_y^2 \right) A_t^{(0)} = 0, \quad (76a)$$

$$\left[\partial_u (\sqrt{f(u)n(u)} \partial_u) + \partial_x^2 + \partial_y^2 \right] A_i^{(0)} - \partial_i \left(\delta^{jk} \partial_j A_k^{(0)} \right) = 0, \quad (76b)$$

$$\partial_u \left(\delta^{jk} \partial_j A_k^{(0)} \right) = 0, \quad (76c)$$

Therefore, the Maxwell equations yield

$$A_t^{(0)} = \mu(1 - u), \quad A_x^{(0)} = 0, \quad A_y^{(0)} = Hx . \quad (77)$$

where H denotes the critical homogeneous magnetic field and μ is, as previously, the chemical potential. At first order, the bulk scalar equation (54) can be written as

$$\left[-\partial_u (\sqrt{f(u)n(u)} \partial_u) + \frac{m^2}{u^4} - \frac{\mu^2(1-u)^2}{\sqrt{f(u)n(u)}} - \partial_x^2 - (\partial_y - iHx)^2 \right] \varphi^{(1)} = 0 . \quad (78)$$

Using the ansatz $\varphi^{(1)} = e^{iky} \chi_k(x) \rho(u)$ yields

$$\left[-\partial_u (\sqrt{f(u)n(u)} \partial_u) + \frac{m^2}{u^4} - \frac{\mu^2(1-u)^2}{\sqrt{f(u)n(u)}} \right] \rho(u) = -E \rho(u) , \quad (79a)$$

$$\left[-\partial_x^2 + H^2 \left(x - \frac{k}{H} \right)^2 \right] \chi_k(x) = E \chi_k(x) , \quad (79b)$$

where E is a separation constant. By denoting

$$\zeta = \sqrt{H} \left(x - \frac{k}{H} \right) , \quad (80)$$

the regular solution of Eq. (79b) can be written in terms of the Hermite function, H_n , as

$$\chi_k(z) = e^{-\zeta^2/2} H_n(z) , \quad (81)$$

corresponding to the quantized eigenvalue $E = E_n = (2n + 1)H$. The droplet solution yields a vortex lattice and corresponds to putting $n = 0$, being consistent with

$$\chi_k(x) = \exp \left[-\frac{H}{2} \left(x - \frac{k}{H} \right)^2 \right] . \quad (82)$$

The general solution wits

$$\varphi^{(1)} = \rho_0(u) \Sigma(x, y) , \quad (83)$$

where $\Sigma(x, y) = \int_{\mathbb{R}} C(k) e^{iky} \chi_k(x) dk$, and ρ_0 is the solution of Eq. (79a), with $E = H$. One can obtain the vortex lattice solution by suitably choosing $C(k)$. The first order solution Eq. (83) satisfies

$$2\Im \left[(\varphi^{(1)})^\dagger D_i^{(0)} \varphi^{(1)} \right] = -\varepsilon_i^j \partial_j |\varphi^{(1)}|^2 , \quad (84)$$

for ε_{ij} denoting the 2-index Levi-Civita tensor [47].

Now, the Maxwell equation, in second order, is given by

$$- \left[\partial_u (\sqrt{f(u)n(u)} \partial_u) + \partial_x^2 + \partial_y^2 \right] A_i^{(2)} + \varepsilon_i^j \partial_j |\varphi^{(1)}|^2 + \partial_i \left(\delta^{jk} \partial_j A_k^{(2)} \right) = 0 , \quad (85a)$$

$$\partial_u \left(\delta^{jk} \partial_j A_k^{(2)} \right) = 0. \quad (85b)$$

Eq. (85b) states that the term $\delta^{jk} \partial_j A_k^{(2)}$ does not depend on u and it can be chosen to vanish, as a gauge transformation maintaining the $A_u = 0$ gauge. In momentum space,

$$\left(-\partial_u (\sqrt{f(u)n(u)} \partial_u) + k^2 \right) \mathring{A}_i^{(2)} + i\varepsilon_i^j k_j |\mathring{\varphi}^{(1)}|^2 = 0, \quad (86)$$

where $|\mathring{\varphi}^{(1)}|^2$ denotes the Fourier transformation of $|\varphi^{(1)}|^2$.

To show the Meissner effect, it is enough to take the long-wavelength $k \rightarrow 0$ limit. One uses the tortoise coordinate $u_\star$, defined for the AdS_4 generalized extremal brane as

$$du_\star := \frac{du}{\sqrt{f(u)n(u)}}, \quad (87)$$

or, explicitly,

$$u_\star = \frac{1}{(\beta+2)^2(u-1)} \sqrt{\frac{u-1}{3(\beta+2)u-9}} \times \left[(\beta+2)(u-1)((\beta+2)u-3) - (\beta-1)^{3/2} \sqrt{\beta+2} \sqrt{u-1} \sqrt{\frac{3(u-1)}{\beta-1} + u \sinh^{-1} \left(\frac{\sqrt{u-1}}{\sqrt{\frac{\beta-1}{\beta+2}}} \right)} \right]. \quad (88)$$

For $\beta = 1$, the $u_\star \rightarrow \infty$ limit does correspond to the horizon. Then,

$$\left(-\frac{d^2}{du_\star^2} + k^2 \sqrt{f(u_\star)n(u_\star)} \right) \mathring{A}_i^{(2)} + i\varepsilon_i^j k_j \sqrt{f(u)n(u)} |\mathring{\varphi}^{(1)}|^2 = 0. \quad (89)$$

Using the bulk Green's function,

$$\left(-\frac{d^2}{du_\star^2} + k^2 \sqrt{f(u_\star)n(u_\star)} \right) G(u_\star, u'_\star) = \delta(u_\star - u'_\star), \quad (90)$$

the solution of Eq. (89) can be formally written as

$$\mathring{A}_i^{(2)} = \mathbf{a}_i - i\varepsilon_i^j k_j \int_0^\infty G(u_\star, u'_\star) \sqrt{f(u'_\star)n(u'_\star)} |\mathring{\varphi}^{(1)}|^2(u'_\star) du'_\star. \quad (91)$$

Imposing the boundary conditions $\lim_{u_\star \rightarrow 0} G(u_\star, u'_\star) = 0$ and $\lim_{u_\star \rightarrow \infty} \partial_\star G(u_\star, u'_\star) = 0$, the term $\mathbf{a}_i$ in Eq. (91) consists of the homogeneous solution of the PDE

$$\left(-\frac{d^2}{du_\star^2} + k^2 \sqrt{f(u_\star)n(u_\star)} \right) \mathbf{a}_i = 0, \quad (92)$$

if boundary conditions regular at the horizon are imposed as well as $\lim_{u \rightarrow 0} \mathbf{a}_i = \mathring{A}_i^{(2)}$. In the limit $u \rightarrow 0$, the tortoise-like coordinate $u_\star$ reads

$$\lim_{u \rightarrow 0} u_\star = -\frac{1}{(\beta+2)^2} \left[\beta + \sqrt{\frac{\beta+2}{3-3\beta}} (\beta-1)^{3/2} \sin^{-1} \left(\sqrt{\frac{\beta+2}{\beta-1}} \right) + 2 \right]. \quad (93)$$

One can construct the homogeneous solution by the k -expansion satisfying the boundary conditions, with $u \approx 0$,

$$\mathbf{a}_i \sim \mathring{A}_i^{(2)} (1 - k^2 u + \dots). \quad (94)$$

The expansion in Eq. (94) was obtained for the AdS₄-RN black brane. For the AdS₄ generalized extremal brane, one can try to emulate this result. The general solution of Eq. (92), for $u \ll 1$, can be written as

$$\mathbf{a}_i(k, \beta) = \left(c_1 D_{\xi_+(k, \beta)} + c_2 i^{1/2} D_{\xi_-(k, \beta)} \right) \sqrt{k} \sqrt[4]{\beta^2 + 2\beta - 3} \left(\frac{1}{\sqrt[4]{6}} u + \frac{\sqrt[4]{\frac{8}{3}}}{\beta + 3} \right) \quad (95)$$

where c_1 and c_2 are integration constants, and

$$\xi_\pm(k, \beta) = \pm \frac{\sqrt{6}k(5\beta + 19)i \mp 3(\beta + 3)\sqrt{\beta^2 + 2\beta - 3}}{6(\beta + 3)\sqrt{\beta^2 + 2\beta - 3}} \quad (96)$$

are the order of the parabolic cylinder Weber functions $D_{\xi_\pm}(z)$. The profile of the solution (95) is depicted in what follows, for some values of c_1 and c_2 . Fig. 5 illustrates the case $c_1 = 1$ and $c_2 = 0.1$. For these values, the field $\mathbf{a}_i$ is a monotonically increasing function of β , irrespectively of the value of k .

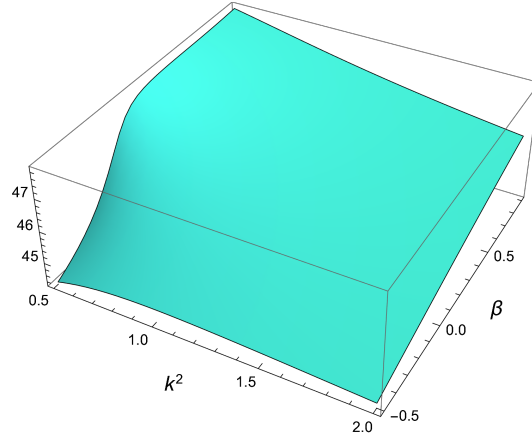


FIG. 5: Plot of $a_i = a_i(k, \beta)$ in Eq. (95), for $c_1 = 1$ and $c_2 = 0.1$.

One can choose $c_1 = 10$ and $c_2 = -10$, which is portrayed in Fig. 6. For these values, the field a_i is a monotonically increasing function of k^2 , irrespectively of the value of β , and a monotonically increasing function of β for $k \gtrsim 0.8$.

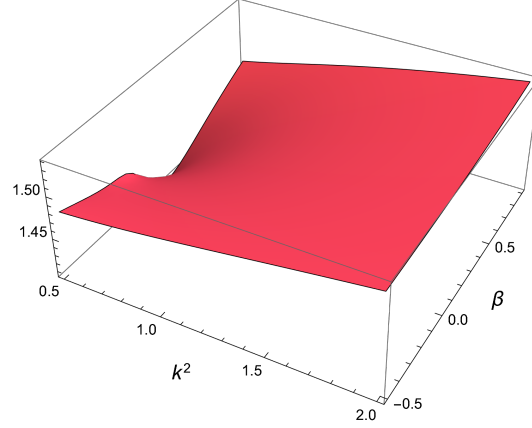


FIG. 6: Plot of $a_i = a_i(k, \beta)$ in Eq. (95), for $c_1 = 10$ and $c_2 = -10$.

Now, the choice $c_1 = 5$ and $c_2 = -10$ yields Fig. 7.

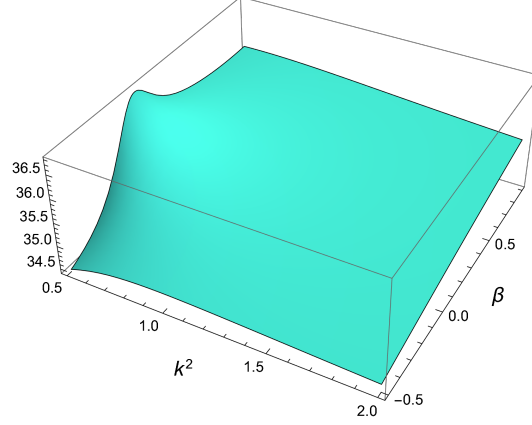


FIG. 7: Plot of $a_i = a_i(k, \beta)$ in Eq. (95), for $c_1 = 10$ and $c_2 = -10$.

One can construct the homogeneous solution by the k -expansion

$$a_i(k, \beta) = a_0(\beta) + k^2 a_2(\beta) + k^4 a_4(\beta) + \dots, \quad (97)$$

where the coefficients a_{2k} , which can be read off Eq. (95) and Figs. 5 – 7, depend of the parameter β in the black brane. When $\beta \rightarrow 1$, one recovers the results for the AdS₄-RN black brane. No, realizing that since the term $i\varepsilon_i{}^j k_j \sqrt{f(u)n(u)} |\dot{\phi}^{(1)}|^2$ is proportional to k , it is enough to consider the Green's function at $k \approx 0$,

$$-\partial_{u_*}^2 G(u_*, u'_*) = \delta(u_* - u'_*). \quad (98)$$

The Green's function can be obtained from two homogeneous solutions,

$$g_1 = u_* , \quad g_2 = 1 , \quad (99)$$

respectively satisfying the boundary condition at the AdS boundary and the horizon. Therefore, the Green's function reads

$$G(u_*, u'_*) = \begin{cases} g_1(u_*)g_2(u'_*) = u'_* & (u'_* < u_* < \infty) \\ g_1(u'_*)g_2(u_*) = u_* & (0 < u_* < u'_*) \end{cases}$$

Thus, up to order $\mathcal{O}(k^3)$, one can write

$$\mathring{A}_i^{(2)} = a_i - i u_* \varepsilon_i{}^j k_j \int_{u_*}^{\infty} \sqrt{f(u'_*)n(u'_*)} |\dot{\phi}^{(1)}|^2 du'_* - i \varepsilon_i{}^j k_j \int_0^{u_*} u'_* \sqrt{f(u'_*)n(u'_*)} |\dot{\phi}^{(1)}|^2 du'_*. \quad (100)$$

The effective current is given by

$$\begin{aligned} \langle \mathring{J}_i \rangle &= \frac{\partial a_i}{\partial u_*} - \varepsilon_i{}^j k_j \int_0^{\infty} du'_* \sqrt{f(u'_*)n(u'_*)} |\dot{\phi}^{(1)}|^2 \\ &= \left[1 + \frac{1}{36} (1 - \beta) \right] \mathring{J}_i^n + \mathring{J}_i^s . \end{aligned} \quad (101)$$

where $\mathring{J}_i^n = -k^2 \mathring{A}_i^{(2)}$ is the bound current and the second term of Eq. (101) is the supercurrent. The bound current is enhanced by the parameter $\beta(\mathbf{q})$ controlling the AdS_4 generalized black brane solutions. For $\beta \rightarrow 1$, the results for the AdS_4 -RN black brane are recovered. Therefore, we conclude that the bound current can be enhanced up to $\sim 8.2\%$ when compared to the AdS_4 -RN black brane case, for the saturation occurring for the lowest allowed value in the range $-1.95 \lesssim \beta \leq 1$, for the extremal AdS_4 generalized extremal brane.

IV. DC CONDUCTIVITY OF ADS_4 GENERALIZED EXTREMAL BRANES

As a final result, one can use the linear response to compute the electrical DC conductivity as a function of the frequency, in the dual CFT, of the holographic superconductor in the AdS_4 generalized extremal brane background. For it, the fluctuations of the gauge potential A_i in the bulk have to be evaluated. The Maxwell equations at $k \approx 0$, with a stationary time dependence $e^{-i\omega t}$, can be expressed as

$$u^4 \partial_u^2 A_i + \left(2u^3 + \frac{(f'(u)n(u) + n'(u)f(u))}{2\sqrt{f(u)n(u)}} \right) \partial_u A_i + \left(\frac{\omega^2}{f^2(u)} + \frac{m^2 \phi^2(u)}{\sqrt{f(u)n(u)}} \right) A_i = 0. \quad (102)$$

One can solve Eq. (102) when ingoing wave boundary conditions at the horizon are used [18, 48]. The asymptotic behavior, for $u \rightarrow 0$, takes the form

$$\delta A_i = \delta A_i^{[0]} + A_i^{[1]} u + \mathcal{O}(u^2), \quad (103)$$

for the dual source and expectation value for the current being given by $\delta A_i^{[0]} \sim A_i$, $A_i^{[1]} \sim \langle J^x \rangle$, whereas the electric field component reads $E_i = \lim_{u \rightarrow 0} \partial_t (\delta A_i) = i\omega \delta A_i^{[0]}$. In the AdS_4 context of the holographic superconductor, the term $-\delta A_i^{[0]}$ can be interpreted as the superfluid velocity, whereas $A_i^{[1]}$ is the supercurrent [49]. Therefore, the conductivity reads, by Ohm's law,

$$\sigma(\omega) = \frac{\langle J^i \rangle}{E^i} = -i \frac{\delta A_i^{[1]}}{\omega \delta A_i^{[0]}}. \quad (104)$$

In the near-boundary limit, $u \rightarrow 0$, one can expand the complex charged scalar field, playing the role of the Higgs field in Ginzburg–Landau theory, as [18]

$$\phi(u) = \phi_1 u + \phi_2 u^2. \quad (105)$$

One can identify ϕ_1 to the source and $\langle \mathcal{O}_2 \rangle \propto \phi_2$ the condensate $\langle \mathcal{O}_2 \rangle$. Although the range $-1.95 \lesssim \beta < 1$ is formally allowed, the most relevant cases rely on realizing the AdS_4 generalized extremal brane as a deformation of the AdS_4 -RN black brane, and the parameter β as a perturbation.

As a consequence the values near $\beta \sim 1$ are relevant for comparing to the well-known transport coefficients associated with the AdS₄-RN black brane. Fig. 8 shows a superconducting gap in the

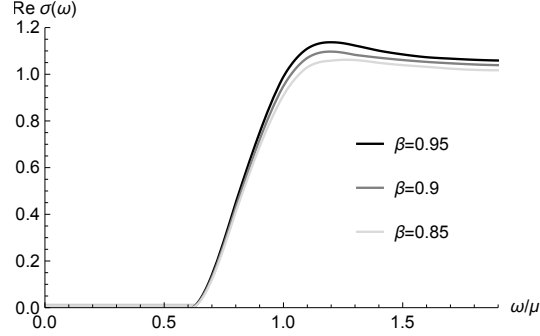


FIG. 8: Real part of the DC conductivity, $\text{Re } \sigma(\omega)$, at vanishing temperature for holographic superconductors with AdS₄ generalized extremal brane background, as a function of the frequency-to-chemical potential ratio, for values of β deforming the AdS₄-RN black brane.

DC conductivity for values $\omega/\mu \approx 0.62$, irrespectively the value of β . It complies with the properties of holographic superconductors, since wave states having energy smaller than the energy gap set by the order parameter, cannot be filled. From the quantitative point of view, the real part of the DC conductivity of the holographic superconductor in Fig. 8 approaches the one for the AdS₄-RN black brane when $\beta \rightarrow 1$. The lower the value of β , which controls the AdS₄ generalized black brane (7) with metric terms (3, 9), the lower the maxima of the real part of the DC conductivity are, and the lower its asymptotic value as a function of ω/μ . The value of ω/μ for which the superconducting gap sets in has no significant alteration concerning the AdS₄-RN black brane, irrespectively of the value of β . The imaginary part of the DC conductivity of the holographic

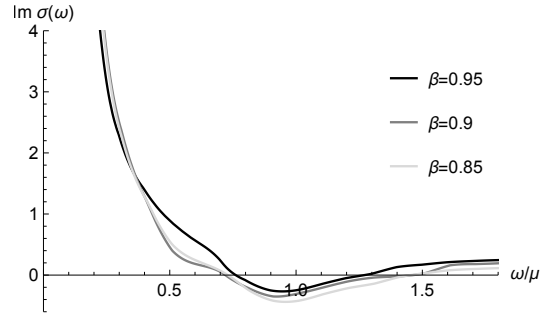


FIG. 9: Imaginary part of the DC conductivity, $\text{Im } \sigma(\omega)$, at vanishing temperature for holographic superconductors with AdS₄ generalized extremal brane background, as a function of the frequency-to-chemical potential ratio, for values of β deforming the AdS₄-RN black brane.

superconductor in Fig. 9 for the AdS₄ generalized extremal brane is qualitatively similar to the

AdS₄-RN black brane. However, the lower the value of β , the lower its asymptotic value as a function of ω/μ and the higher the value of $\text{Im } \sigma(\omega)$, for fixed values of $\omega/\mu \gtrsim 0.35$.

V. CONCLUDING REMARKS AND PERSPECTIVES

AdS₄ generalized extremal branes were studied as the gravitational dual system to holographic superconductors. Transport and response coefficients, such as the shear viscosity-to-entropy density ratio, η/s , and the bulk viscosity-to-entropy density ratio, ζ/s , were obtained through gravitational perturbations, using the GKPW relation and the respective Kubo's formulæ for the shear and bulk viscosities. They were reported and discussed in the light of the parameter β that governs the family of AdS₄ generalized extremal branes, whose charge q is also a function of β , as long as a second event Killing horizon exists. Maxwell equations were solved, to obtain an explicit expression for the magnetic potential involving the β -dependent chemical potential. Using the GKPW relation, the η/s and ζ/s ratios were derived and discussed, yielding a more strict range for β . The holographic computation of the Weyl anomaly associated with AdS₄ generalized extremal branes was implemented, with the near-boundary limit having an analogous result to the AdS₄-RN black brane, as expected. The holographic superconductor in the AdS₄ generalized black brane background, in the probe limit, was considered and implemented with a complex charged scalar field playing the role of the Higgs field in Ginzburg–Landau theory, minimally coupled to Einstein–Maxwell–AdS₄ theory. The Breitenlohner–Freedman bound was analyzed and the equations of motion, resulting from the variational principle applied to the Higgs–Einstein–Maxwell–AdS₄ action, were solved and discussed, with applications to type-I and type-II superconductors. The case of type-II superconductors was discussed, with a vortex lattice formed by the magnetic field percolating the superconductor. The regimes of small and high values of the magnetic field were approached. We showed that the bound current strengthens by a numerical factor dependent of β , which can be up $\sim 8.2\%$ greater when compared to the AdS₄-RN black brane case, when the range (39), for which η/s is positive, is regarded. Ref. [50] investigated the low-energy collective excitations in holographic superconductors with dynamical Maxwell fields at the boundary. Quasinormal modes (QNMs) computations corroborated the characteristic features of the Anderson–Higgs mechanism, the Higgs modes, and the plasma frequency-gapped modes were derived, showing to be consistent with the Ginzburg–Landau theory. Eq. (74) is consistent with these QNMs, particularly in the vicinity of the phase transition temperature where the probe-limit analysis is effective.

The linear response theory was also employed, to calculate the electrical DC conductivity of

the holographic superconductor as a function of the frequency in the dual CFT associated with the AdS_4 generalized black brane background. A superconducting gap in the real part of the DC conductivity occurs for frequencies $\omega = 0.62\mu$, where μ is the chemical potential, irrespectively the value of β . For the analysis involving the DC conductivity, the range $0.85 \lesssim \beta \lesssim 1$ was considered, implementing the AdS_4 generalized extremal brane as a deformation of the AdS_4 -RN black brane. For the real part of the DC conductivity, the lower the value of β , the lower its maxima are, and the lower its asymptotic value as a function of ω/μ . We conclude that the parameter β , carrying the AdS_4 generalized extremal brane charge as in Eq. (12), plotted in Fig. 1, can fine-tune transport and response coefficients of holographic superconductors, when AdS_4 generalized black branes are taken into account. One can try to implement AdS_4 generalized extremal branes in Einstein–dilaton gravity, whose dual holographic superconductors were reported in Ref. [51].

One can introduce dynamical gauge fields when Neumann boundary conditions are imposed on the AdS_4 boundary. A gauge symmetry emerges with dual CFT_3 having a spectrum containing a massless gauge field. We can study the effects of the dynamical gauge field in the vortex lattice configurations, emulating to AdS_4 generalized extremal branes the relevant results of Refs. [52], where it is known to significantly affect the energetics and phase transitions. As a perspective, Ref. [36] reported a functional measure on both sides of AdS/CFT , proposing corrections to transport and response coefficients in second-order relativistic hydrodynamics. The decay rate of sound waves, the energy density, the relaxation time, the pressure, and the bulk viscosity, as well as conformal traceless tensor fields have been shown to support relevant quantum corrections. They all achieve an imaginary component that carries the instability of the strongly-coupled fluid flows in the CFT on the boundary. The latest experimental data was used to explore quantum gravity with the quark-gluon plasma. One can use the functional measure, encoding effects of quantum gravity, to study AdS_4 generalized extremal branes and the dual holographic superconductors. Analog gravity models based upon hydrodynamics have been studied [53], and we expect to emulate them for AdS_4 generalized extremal branes. Besides, soft-hair excitations, in the duality between generalized Navier–Stokes equations and soft-hairy horizons established in Ref. [54], may be used to probe holographic superconductors. Quantum hair can be also studied in the context of AdS_4 generalized extremal branes [55].

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Appendix A: Generalized extremal brane as an exact solution of higher-order curvature terms

The parameter β appeared as a consequence of the momentum and Hamiltonian constraint in the ADM-like protocol that led to Eq. (8). However, Refs. [56, 57] studied respectively cubic gravity and Lee-Wick gravity. Ref. [58] also addressed cubic terms involving Ricci and Riemann terms as an attempt to 2-loop quantum corrections to 4-dimensional gravity. For the Ricci and Einstein cubic gravity to encompass terms that are not topological, alternatively to the embedding protocol, the generalized extremal brane described by the metric (2), with metric coefficients (3, 9), is an exact solution of the equations of motion coming from the action

$$S = \int d^4x \sqrt{-g} \left(f_3(R, R_{\mu\nu}, R_{\mu\nu\rho\sigma}) - \frac{L^2}{4} F_{\mu\nu} F^{\mu\nu} \right) + \overbrace{\lim_{u \rightarrow 0} \int d^4x \sqrt{h} K}^{I_{\text{Gibbons-Hawking}}} + S_{\text{c.t.}}, \quad (\text{A1})$$

where

$$\begin{aligned} f_3 = & (R - 2\Lambda_4) + \beta_1 G_{\mu\nu} \square R^{\mu\nu} \\ & + \beta_2 \left(-\frac{7}{20} R^3 + \frac{7}{5} R R_{\mu\nu} R^{\mu\nu} - \frac{7}{3} R_\nu^\mu R_\rho^\nu R_\mu^\rho + 14 R_{\mu\nu}^{\rho\sigma} R_{\rho\sigma}^{\alpha\beta} R_{\alpha\beta}^{\rho\sigma} \right. \\ & \quad \left. - 4 R_{\mu\nu\rho\sigma} R_{\alpha}^{\mu\nu\rho} R^{\sigma\alpha} - \frac{7}{20} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} R + 4 R_{\mu\nu\rho\sigma} R^{\mu\rho} R^{\nu\sigma} \right) \\ & + \beta_3 \left(\nabla_\mu R_{\rho\sigma} \nabla^\mu R^{\rho\sigma} + \nabla_\mu R_{\rho\sigma} \nabla^\sigma R^{\mu\rho} + \nabla_\mu R \nabla^\mu R + \nabla_\mu R_{\rho\sigma\tau\xi} \nabla^\rho R^{\mu\sigma\tau\xi} \right), \end{aligned} \quad (\text{A2})$$

with Gibbons–Hawking term

$$\begin{aligned} \lim_{u \rightarrow 0} \int d^4x \sqrt{h} K = & -\frac{4}{u^3 ((1+2\beta)u^3 + 2)^2} \sqrt{-\frac{(3u^7 - 5u^3 + 2)((\beta-1)u^5 - u^3 + 1)}{(1+2\beta)u^4 - 2}} \\ & \times u^4 \left[-32\beta + u^2 (2\beta + u^2 (56\beta + 9(\beta-1)(2+\beta)u^{10} + 6(1+2\beta)u^7 \right. \\ & \quad \left. - 5(2\beta^2 + \beta - 5)u^5 + 24u^3 - (\beta+2)(\beta+1)u^2 - 3) + 4) + 8 \right], \end{aligned} \quad (\text{A3})$$

and counterterm (c.t.) given by $\sim u^{-3} \sqrt{f(u)n(u)}$.

Using Ref. [59], the rational coefficients accompanying each one of the cubic terms make such terms neither trivial nor topological in four dimensions. Ref. [60] showed that the 4-dimensional Lee–Wick term, whose coefficient is β_1 in the action (A1, A2), is superrenormalizable. Ref. [61] excluded static and spherically symmetric black holes in Einsteinian cubic gravity with unsuppressed higher-order curvature terms where the temporal and radial metric components are equivalent to each other. Our metric (2), with coefficients (3, 9) evades this equivalence, deforms the AdS_4 -Reissner–Nordström geometry and it is induced by cubic curvature terms and other higher-curvature terms beyond General Relativity. I am now studying the linear stability of the generalized extremal branes against odd-parity perturbations. Up to now, the results I have obtained comprise the cubic terms in the action (A1, A2), whose coefficient is β_2 , having three propagating degrees of freedom in the odd-parity sector. One dynamical perturbation behaves as a soft ghost mode. It is worth mentioning that General Relativity has one dynamical perturbation. Therefore we want to explore this soft ghost mode better, also using the results in Ref. [62]. About the terms in the action (A1, A2), whose coefficient is β_3 , their renormalizability must still be explored, which is still an intricate task. Therefore, in this setup, the parameter $\beta = h(\beta_1, \beta_2, \beta_3)$, where h is a rational function, is a rational combination of the coefficients β_1, β_2 , and β_3 , respectively encoding quantum gravity/stringy effects beyond General Relativity. To extend Ref. [58] as an attempt to identify cubic gravity and the terms accompanying the coefficient β_3 in the action (A1, A2) as 2-loop quantum corrections to 4-dimensional gravity, one must further investigate the terms in the action (A2), whose coefficient is β_3 . The final answer remains unknown up to now.

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