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h-FUNCTION, HILBERT-KUNZ DENSITY FUNCTION AND FROBENIUS-POINCARÉ FUNCTION

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Abstract. Given ideals I, J of a noetherian local ring $(R, \mathfrak{m})$ such that $I + J$ is $\mathfrak{m}$-primary and a finitely generated R-module M, we associate an invariant of (M, R, I, J) called the h-function. Our results on h-functions allow extensions of the theories of Frobenius-Poincaré functions and Hilbert-Kunz density functions from the known graded case to the local case, answering a question of V.Trivedi. When J is $\mathfrak{m}$-primary, we describe the support of the corresponding density function in terms of other invariants of (R, I, J). We show that the support captures the F-threshold: $c^J(I)$, under mild assumptions, extending results of V. Trivedi and Watanabe. The h-function encodes Hilbert-Samuel, Hilbert-Kunz multiplicity and F-threshold of the ideal pair involved. Using this feature of h-functions, we provide an equivalent formulation of a conjecture of Huneke, Mustaţă, Takagi, Watanabe; recover a result of Smirnov and Betancourt; prove that a result of Hanes comparing multiplicities, is equivalent to an a priori weaker containment condition on ideals. We also point out that a conjecture of Smirnov-Betancourt as stated is false and suggest a correction which we relate to the conjecture of Huneke et al.

We develop the theory of h-functions in a more general setting which yields a density function for F-signature. A key to many results on h-functions is a 'convexity technique' that we introduce, which in particular proves differentiability of Hilbert-Kunz density functions almost everywhere on $(0, \infty)$, thus contributing to another question of Trivedi.

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1. Introduction

Hilbert-Kunz multiplicity and F-signature are numerical invariants appearing in prime characteristics commutative algebra and algebraic geometry. These quantify severity of singularities at a point of a variety and also relate to other invariants, such as the cardinality of the local fundamental group of the punctured spectrum of a strongly F-regular local ring; see [AE08], [CST18] and Section 2. The theory of Hilbert-Kunz multiplicity in the graded case has witnessed two new generalizations in recent years: the Hilbert-Kunz density function and the Frobenius-Poincaré function. Fix a standard graded ring S in prime characteristic and a homogeneous ideal $\mathfrak{a}$ of finite colength. When $\dim(S) \geq 2$, Trivedi has proven the existence of a compactly supported real valued continuous function $g_{S,\mathfrak{a}}$ of a real variable- called the Hilbert-Kunz density function- whose integral is the Hilbert-Kunz multiplicity $e_{HK}(\mathfrak{a}, S)$; see Section 2 for details. For the pair $(S, \mathfrak{a})$, where $\dim(S)$ is not necessarily at least two, the associated Frobenius-Poincaré function is an entire function in one complex variable, whose value at the origin is the Hilbert-Kunz multiplicity $e_{HK}(\mathfrak{a}, S)$; see Section 2. These two functions not only encode more subtle invariants of $(S, \mathfrak{a})$ than the Hilbert-Kunz multiplicity but also allow application of geometric tools, such as sheaf cohomology on $\mathrm{Proj}(S)$, and tools from homological algebra. Successful applications of the Hilbert-Kunz density functions have resolved Watanabe and Yoshida's conjecture on the values of Hilbert-Kunz multiplicity of quadric hypersurfaces, rationality of Hilbert-Kunz multiplicities and F-thresholds of two dimensional normal rings among other results; see [Tri23], [TW21], [Tri05], [Tri19].

Building extensions of these two theories to the setting of a noetherian local ring is a natural question; see Trivedi's question [Tri18, Question 1.3]. In this article, we extend the theories of Hilbert-Kunz density function and Frobenius-Poincaré function to the local setting. Our extensions are facilitated by a systematic study of a new function, which we call the h-function.

Fix a noetherian local domain $(R, \mathfrak{m})$ of prime characteristic $p > 0$ and Krull dimension d, where the Frobenius endomorphism is a finite map. Fix two ideals I, J of R such that $I + J$ is $\mathfrak{m}$-primary. We prove:

Theorem A: Consider the sequence of functions of a real variable

$$h_{n,I,J}(s) = l\left(\frac{R}{(I^{\lceil sp^n \rceil} + J^{[p^n]})R}\right),$$

where $J^{[p^n]}$ is the ideal generated by $\{f^{p^n} \mid f \in J\}$; and $l(\_\_)$ is the length function.

(1) (Theorem 3.7, Theorem 3.29) There is a real-valued function of a real variable denoted by $h_{I,J}(s)$ such that given an interval $[s_1, s_2] \subseteq \mathbb{R}$, there is a constant C depending only on s_1, s_2 satisfying

$$|h_{I,J}(s) - \frac{h_{n,I,J}(s)}{p^{nd}}| \leq \frac{C}{p^n}, \text{ for all } s \in [s_1, s_2] \text{ and } n \in \mathbb{N}.$$

Consequently, the sequence of functions $\frac{h_{n,I,J}(s)}{p^{nd}}$ converges to $h_{I,J}(s)$ and the convergence is uniform on every compact subset of $\mathbb{R}$.

(2) (Theorem 3.30, Theorem 3.20) Given real numbers $s_2 > s_1 > 0$, there is a constant C'- depending only on s_1, s_2 such that for $x, y \in [s_1, s_2]$,

$$|h_{I,J}(x) - h_{I,J}(y)| \leq C|x - y|.$$

That is, away from zero, $h_{I,J}$ is locally Lipschitz continuous.

The function $h_{I,J}$ is called the h-function associated to the pair (I, J). In fact we prove a version the above theorem for an ideal I and a family of ideals $J_\bullet$ satisfying what we call **Condition C** allowing for applications to other numerical invariants such as F-signature; see Theorem 3.7.

Special instances of this h-function have been considered by different authors: in [Tay18] when both I and J are $\mathfrak{m}$-primary, in [BST13] when R is regular, I is principal and $J = \mathfrak{m}$ to study F-signature of a pair and in [Kos17] in the same set up but in a different context. **Theorem A** generalizes their results. Moreover the techniques involved in our proofs yield uniform convergence which is crucial for us.

In Theorem 3.16, we prove that there is a polynomial $P_1(s)$ of degree $\dim(R/J)$ such that $h_{I,J}(s) \leq P_1(s)$ for all s. Using this polynomial bound we prove existence and holomorphicity of a function $F_{R,I,J}(y)$ on the open lower half complex plane; see Theorem 4.3. We moreover show:

$$F_{R,I,J}(y) = \int_{\mathbb{R}} h_{I,J}(t)e^{-ity}(iy)dt.$$

When J is $\mathfrak{m}$-primary, we prove $F_{R,I,J}(y)$ is entire. When $(R, \mathfrak{m}, J)$ *comes from a graded pair* $(S, \mathfrak{a})$, i.e. $(R, \mathfrak{m})$ is the localization of a standard graded ring S at the homogeneous maximal ideal, I is the homogeneous maximal ideal and J comes from a homogeneous ideal of finite colength $\mathfrak{a}$, $F_{R,I,J}(y)$ coincides with the Frobenius-Poincaré function of the pair $(S, \mathfrak{a})$; see Proposition 6.8,(3). Unlike [Muk22], our treatment allows us to consider Frobenius-Poincaré function of $(S, \mathfrak{a})$, where $\mathfrak{a}$ need not have finite colength; see Proposition 6.8, (2).

Extending the theory of Hilbert-Kunz density functions is more involved. Set

$$f_n(s) = h_{n,I,J}(s + \frac{1}{p^n}) - h_{n,I,J}(s).$$

When $(R, \mathfrak{m}, J)$ comes from a graded pair $(S, \mathfrak{a})$, where $\dim(S) \geq 2$, we point out that the sequence of functions

$$\frac{f_n(s)}{(p^n)^{d-1}}$$

converges uniformly to the Hilbert-Kunz density function of $(S, \mathfrak{a})$; see Theorem 6.6. But for arbitrary ideals I, J of a local ring $(R, \mathfrak{m})$, the pointwise convergence of $f_n(s)/(p^n)^{d-1}$ at every s is not clear. In fact when $I = 0$ the sequence does not converge at $s = 0$, see Example 5.12. In this direction, we relate the convergence of $f_n(s)/(p^n)^{d-1}$ to the differentiability of $h_{I,J}$ at s. In Theorem 5.8 we prove,

If $h_{I,J}(s)$ if differentiable at s, $f_n(s)/(p^n)^{d-1}$ converges to $h'_{I,J}(s)$.

In the direction of differentiability of h, we prove:

Theorem B:(Theorem 5.4,(3),(4)) Let $h_{I,J}$ be as before.

(1) The left and right hand derivative of h exist at all nonzero points.
(2) Outside a countable subset of $(0, \infty)$, h is continuously differentiable.

Theorem B, (2) implies that if $(R, \mathfrak{m})$ is local, then for any R-ideals I, J,$f_n(s)/(p^n)^{d-1}$ converges outside a countable subset of $\mathbb{R}$ and coincides with the derivative of $h_{I,J}(s)$, thus outside this countable set the limiting function $f_n(s)/(p^n)^{d-1}$ yields a well-defined notion of density function. In Theorem 5.4, we actually prove existence of density function more generally for a family satisfying **Condition C**. This generalization in particular yields a density function for F-signature. When $(R, \mathfrak{m}, J)$ comes from a graded pair $(S, \mathfrak{a})$ with $\dim(S) \geq 2$, we prove that the corresponding h-function $h_{R,\mathfrak{m},J}$ is continuously differentiable and the derivative coincides with the Hilbert-Kunz density function that Trivedi defines. We moreover prove the existence and continuity of the density function to the case when $\mathfrak{a}$ does not have finite colength; see Theorem 6.7. Our work shows that h-function is twice differentiable outside a set of measure zero contributing to Trivedi's question about the order of differentiability of the Hilbert-Kunz density function; see [Tri23, Question 1], Remark 5.5.

Theorem B is a consequence of a 'convexity technique' that we introduce. For fixed $s_0 > 0$, in Theorem 5.3, we construct a function $H(s, s_0)$ which we prove to be convex and show that

$$H(s, s_0) = h(s)/c(s) - h(s_0)/c(s_0) + \int_{s_0}^{s} h(t)c'(t)/c^2(t)dt,$$

where $c(s) = s^{\mu-1}/(\mu-1)!$, μ being the cardinality of a set of generators of I. **Theorem B** then follows from general properties of convex functions. The underlying idea of the same convexity argument is used to prove Lipschitz continuity of h-functions stated in **Theorem A**.

The behaviour of $h_{I,J}$ near zero is more subtle. We prove $h_{I,J}$ is continuous at zero if and only if I is nonzero. In fact our result implies,

Theorem C:(Theorem 8.12) Suppose $\dim(R/I) = d'$. Denote the set of minimal primes of R/I of dimension d' by $\mathrm{Assh}(R/I)$. Then

$$\lim_{s \to 0+} \frac{h(s)}{s^{d-d'}} = \frac{1}{(d-d')!} \sum_{P \in \mathrm{Assh}(R/I)} e_{HK}(J, R/P) e(I, R_P),$$

where $e(I, \_\_)$ denotes the Hilbert-Samuel multiplicity with respect to I. In particular, the order of vanishing $h(s)$ at $s = 0$ is $d - d'$. **Theorem C** extends part of [BST13, Theorem 4.6], where R is assumed to be regular, I a principal ideal and $J = \mathfrak{m}$.

The h-function treats different numerical invariants of (R, I, J) on an equal footing. when I is $\mathfrak{m}$-primary, for $s > 0$ and close to zero $h_{I,J}(s) = e(I, R)\frac{s^d}{d!}$; see Lemma 8.3. When J is $\mathfrak{m}$-primary, for large s, $h_{I,J}(s) = e_{HK}(R, J)$. Since $h_{I,J}(s)$ is a non-decreasing function there is a smallest point after which $h_{I,J}(s) = e_{HK}(R, J)$. We describe this 'minimal stable point' of $h_{I,J}(s)$.

Theorem D:(Theorem 8.7, Lemma 8.4)Suppose J is $\mathfrak{m}$-primary. Let $\alpha_{R,I,J} = \sup\{s \in \mathbb{R} \mid s > 0\,, h_{I,J}(s) \neq e_{HK}(J,R)\}$. Consider the sequence of numbers,

$$r_I^J(n) = \max\{t \in \mathbb{N} | I^t \nsubseteq (J^{[p^n]})^*\},$$

where $(J^{[p^n]})^*$ denotes the tight closure of the ideal $(J^{[p^n]})$; see Definition 2.5. Then $(r_I^J(n)/p^n)_n$ is a non-decreasing sequence converging to $\alpha_{R,I,J}$.

Moreover, the density function $f_{R,I,J}$ is zero outside $[0, \alpha_{R,I,J}]$ and nonzero at every point in $(0, \alpha_{R,I,J})$ where it exists; see Corollary 8.8. The above description of $\alpha_{R,I,J}$ resembles that of the well known F-threshold $c^J(I)$; see Definition 8.1. Recall that F-threshold is an invariant extensively studied in prime characteristic singularity theory; see [Hun+08b], [MTW05] and is closely related log canonical threshold via reduction modulo p; see [TW04], [HW02]. In general, $\alpha_{R,I,J}$ is bounded above by $c^J(I)$. We prove, under suitable hypothesis, for example, strong F-regularity at every point of $\mathrm{Spec}(R) - \{\mathfrak{m}\}$, $\alpha_{R,I,J}$ coincides with $c^J(I)$; see Theorem 8.10. Whenever $h_{I,J}$ is differentiable, the support of $\frac{d}{ds}h_{I,J}$- which agrees with the Hilbert-Kunz density function of (R, I, J)- is $[0, \alpha_{R,I,J}]$; see Corollary 8.8. Thus Theorem 8.7, Theorem 8.10 and Corollary 8.8 extend Trivedi and Watanabe's description of the support of Hilbert-Kunz density function when R is graded and strongly F-regular on the punctured spectrum; see Remark 8.9, [TW21, Theorem 4.9]. We do not know whether $\alpha_{R,I,J}$ always coincides with $c^J(I)$; see Question 10.1.

Our applications of the theory of h-functions in Section 9 highlights its feature of capturing different numerical invariants such as F-threshold, Hilbert-Kunz and Hilbert-Samuel multiplicity simultaneously. In Section 9.1 we discuss Watanabe's question comparing Hilbert-Samuel and Hilbert-Kunz multiplicity. This question has been affirmatively confirmed by Hanes. Our theory of h-functions shows that Watanabe's question is equivalent to, a priori, a weaker problem about containment of ideals; see Proposition 9.1, Remark 9.2. In a different direction, we show that even a coarse approximation of an h-function recovers Smirnov and Betancourt's result comparing Hilbert-Kunz multiplicity of a ring and its quotient by part of a system of parameters; see Theorem 9.12. We point out that their conjecture motivating their aforesaid result is false; see Proposition 9.8. So we propose a corrected version in Conjecture 9.10. Indeed Proposition 9.9 shows that a special case of this corrected version is equivalent to another conjecture of Huneke, Mustaţă, Takagi and Watanabe; the latter has been already verified when the rings and ideals involved are graded. We show that the general case of Huneke et al.'s conjecture is equivalent to a question about h-functions; see Proposition 9.13.

Some questions regarding h-functions and the resulting density function are listed in Section 10.

Notation and conventions: All rings are commutative and noetherian. The symbol p denotes a positive prime number. Unless otherwise said, the pair $(R, \mathfrak{m})$ denotes a noetherian local ring R- not necessarily a domain- with maximal ideal $\mathfrak{m}$. By saying $(R, \mathfrak{m})$ is graded, we mean R is a standard graded ring with homogeneous maximal ideal $\mathfrak{m}$. When $(R, \mathfrak{m})$ is assumed to be graded, R-modules and ideals are always assumed to be $\mathbb{Z}$-graded. We assume R has characteristic p and R is F-finite, i.e. the Frobenius endomorphism of R is finite. We index the sequences of numbers and functions by n. Whenever the letter q appears in such a sequence, q denotes p^n. For an ideal $I \subset R$, $I^{[p^n]}$ or $I^{[q]}$ denotes the ideal generated by $\{f^q \mid f \in I\}$ and is called the q or p^n-th *Frobenius power* of I. The operator $l_R(\_\_)$ or simply $l(\_\_)$ denotes the length function. For an

R-module M, F^n_*M denotes the R-module whose underlying abelian group is M, but the R-action comes from restriction scalars through the iterated Frobenius morphism $F^n : R \to R$.

2. Background material

Let $(R, \mathfrak{m})$ be a noetherian local or graded ring, J be an $\mathfrak{m}$-primary ideal, M be a finitely generated R-module. Although the germ of Hilbert-Kunz multiplicity was present in Kunz's seminal work [Kun69], its existence was not proven until Monsky's work:

Theorem 2.1. *(see [Mon83]) There is a real number denoted by $e_{HK}(J, M)$ such that,*

$$l(\frac{M}{J^{[p^n]}M}) = e_{HK}(J, M)(p^n)^{\dim(M)} + O((p^n)^{\dim(M)-1}).$$

The number $e_{HK}(J, M)$ is called the Hilbert-Kunz multiplicity of M with respect to J.

Smaller values of $e_{HK}(R, \mathfrak{m})$ predicts milder singularity of $(R, \mathfrak{m})$; see for e.g. [AE08, Cor 3.6], [Man04]. It is imperative to consider Hilbert-Kunz multiplicity with respect to arbitrary ideals, for e.g. to realize F-signature (see Example 3.10)- an invariant characterizing strong F-regularity of $(R, \mathfrak{m})$- in terms of Hilbert-Kunz multiplicity; see [PT18, Cor 6.5]. We refer the readers to [Hun13], [Muk23, Chapter 2] and the references there in for surveying the state of art.

When $(R, \mathfrak{m})$ is graded, Trivedi's Hilbert-Kunz density function refines the notion of Hilbert-Kunz multiplicity:

Theorem 2.2. *(see [Tri18]) Let $(R, \mathfrak{m})$ be graded, J be a finite colength homogeneous ideal, M be a finitely generated $\mathbb{Z}$-graded R-module. Consider the sequence of functions of a real variable s,*

$$\tilde{g}_{n,M,J}(s) = l([\frac{M}{J^{[q]}M}]_{\lfloor sq \rfloor}).$$

(1) *There is a compact subset of $\mathbb{R}$ containing the supports of all $\tilde{g}_n$'s.*
(2) *If $\dim(M) \geq 1$, there is a function-denoted by $\tilde{g}_{M,J}$- such that $(\frac{1}{q})^{\dim(M)-1}\tilde{g}_{n,M,J}(s)$ converges pointwise to $\tilde{g}_{M,J}(s)$ for all $s \in \mathbb{R}$.*
(3) *When $\dim(M) \geq 2$, the above convergence is uniform and $\tilde{g}_{M,J}$ is continuous.*
(4)
$$e_{HK}(J, M) = \int_0^\infty \tilde{g}_{M,J}(s)ds.$$

Definition 2.3. The function $\tilde{g}_{M,J}$ is called the *Hilbert-Kunz density function* of (M, J).

For a graded ring $(R, \mathfrak{m})$, the *Frobenius-Poincaré* function produces another refinement of the Hilbert-Kunz multiplicity. Frobenius-Poincaré functions are essentialy a limiting function of the Hilbert series of $\frac{M}{J^{[q]}M}$ in the variable e^{-iy}, see [Muk22, Rmk 3.6].

Theorem 2.4. *(see [Muk22]) Let $(R, \mathfrak{m})$ be graded, M be a finitely generated $\mathbb{Z}$-graded R-module, J be a finite colength homogeneous ideal. Consider the sequence of entire functions on $\mathbb{C}$*

$$G_{n,M,J}(y) = (\frac{1}{q})^{\dim(M)} l([\frac{M}{J^{[q]}M}]_j)e^{-iyj/q}.$$

(1) *The sequence of functions $G_{n,M,J}(y)$ converges to an entire function $G_{M,J}(y)$ [1] on $\mathbb{C}$. The convergence is uniform on every compact subset of $\mathbb{C}$.*

[1] Note the difference in notation from [Muk22].

(2)
$$G_{M,J}(0) = e_{HK}(J, M).$$

The last theorem holds for any graded ring which are not necessarily standard graded. For the notion of Hilbert-Kunz density function in the non-standard graded setting, see [TW22]. By [Muk23, Theorem 8.3.2], for a standard graded $(R, \mathfrak{m})$ of Krull dimension at least one, the holomorphic Fourier transform of $\tilde{g}_{M,J}$ is $G_{M,J}$, i.e.

$$G_{M,J}(y) = \int_0^\infty \tilde{g}_{M,J}(s) e^{-iys} ds.$$

Thus when $\dim(M) \geq 2$, the Hilbert-Kunz density function and the Frobenius-Poincaré function determine each other; see [Muk23, Rmk 8.2.4]. Both Hilbert-Kunz density function and Frobenius-Poincaré function capture more subtle graded invariants of (M, J) than the Hilbert-Kunz multiplicity. For example, when R is two dimensional and normal, J is generated by forms of the same degree, $\tilde{g}_{R,J}$ and $G_{R,J}$ determine and are determined by slopes and ranks of factors in the Harder-Narasimhan filtration of a syzygy bundle associated to J on $\mathrm{Proj}(R)$; see [Tri05], [Bre07], [Tri18, Example 3.3], [Muk22, Chap 6]. For other results on Hilbert-Kunz density functions and Frobenius-Poincaré functions, see the reference section of [Muk23]. These two functions and the Hilbert-Kunz multiplicity of (R, J) detects J up to its tight closure. Recall:

Definition 2.5. ([HH90, Def 3.1]) Let A be a ring of characteristic $p > 0$. We say $x \in A$ is in the tight closure of an ideal I if there is a c not in any minimal primes of A such that $cx^{p^n} \in I^{[p^n]}$ for all large n. The elements in the tight closure of I form an ideal, denoted by I^*.

Theorem 2.6. *([Hun13, Prop 5.4, Thm 5.5], [HH90, Thm 8.17]) Let $I \subseteq J$ be two ideals in $(R, \mathfrak{m})$.*
1. *If $I^* = J^*$, $e_{HK}(I, R) = e_{HK}(J, R)$.*
2. *Conversely, when R is formally equidimensional, i.e. all the minimal primes of the completion $\hat{R}$ have the same dimension, $e_{HK}(I, R) = e_{HK}(J, R)$ implies $I^* = J^*$. Therefore, when $(R, \mathfrak{m})$ is a graded ring where all the minimal primes have the same dimension, $\tilde{g}_{I,R} = \tilde{g}_{J,R}$ or $G_{I,R} = G_{J,R}$ implies $I^* = J^*$.*

3. h-FUNCTION

Given ideals I, J of a local ring $(R, \mathfrak{m})$ such that $I + J$ is $\mathfrak{m}$-primary and a finitely generated R-module M, we assign a real-valued function $h_{M,I,J}$ of a real variable, which we refer to as the corresponding *h-function.* The existence and continuity of $h_{M,I,J}$ is proven in Section 3.4. When R is additionally a domain and $M = R$, given an ideal I and a family of ideals $\{J_n\}_{n \in \mathbb{N}}$- satisfying what we call **Condition C** below- in Section 3.1, we associate a corresponding h-function which is continuous on $\mathbb{R}_{>0}$.

3.1. h-functions of a domain.

Definition 3.1. Let $I_\bullet = \{I_n\}_{n \in \mathbb{N}}$ be a family of ideals of the F-finite local ring R.
1. $I_\bullet$ is called a *weak p-family* if there exists $c \in R$ not contained in any minimal primes of maximal dimension of R such that $cI_n^{[p]} \in I_{n+1}$.
2. $I_\bullet$ is called a *weak p^{-1}-family* if exists a nonzero $\phi \in \mathrm{Hom}_R(F_*R, R)$ such that $\phi(F_*I_{n+1}) \subset I_n$.
3. A *big p-family (resp. big p^{-1}-family)* is a weak p (resp. p^{-1})-family $I_\bullet$ such that there is an $\alpha \in \mathbb{N}$ for which $\mathfrak{m}^{[p^{n+\alpha}]} \subseteq I_n$ for all n.

A family of ideals where (1) holds with $c = 1$ and $\mathfrak{m}^{[p^n]} \subseteq I_n$, has been called a p-family of ideals; see [HJ18]. Notions of p and p^{-1}-families provide an abstract framework for proving existence of asymptotic numerical invariants:

Theorem 3.2. *(see [PT18, Theorem 4.3]) Let $(R, \mathfrak{m}, k)$ be an F-finite local domain of dimension d, $\{I_n\}_{n\in\mathbb{N}}$ a sequence of ideals such that $\mathfrak{m}^{[p^n]} \subset I_e$ for all $n \in \mathbb{N}$.*

1. *If there exists a nonzero $c \in R$ such that $cI_n^{[p]} \subset I_{n+1}$ for all $n \in \mathbb{N}$, then $\eta = \lim_{e\to\infty} 1/p^{nd} l_R(R/I_n)$ exists, and there exists a positive constant C that only depends on c such that $\eta - 1/p^{nd} l_R(R/I_n) \le C/p^n$ for all $n \in \mathbb{N}$.*
2. *If there exists a nonzero $\phi \in \mathrm{Hom}_R(F_*R, R)$ such that $\phi(F_*I_{n+1}) \subset I_n$ for all $e \in \mathbb{N}$, then $\eta = \lim_{n\to\infty} 1/p^{nd} l_R(R/I_n)$ exists, and there exists a positive constant C that only depends on ϕ such that $1/p^{nd} l_R(R/I_n) - \eta \le C/p^n$ for all $n \in \mathbb{N}$.*
3. *If the conditions in (1) and (2) are both satisfied then there exists a constant C that only depends on c and ϕ such that $|1/p^{nd} l_R(R/I_n) - \eta| \le C/p^n$.*

Lemma 3.3. *Let $(R, \mathfrak{m})$ be a local domain. Let I_n, J_n be two weak p-families, then so is the family $I_n + J_n$. If I_n, J_n are two weak p^{-1}-families, then so is the family $I_n + J_n$. When one of the families are big (p or p^{-1}), then so is their sum.*

Proof. Suppose there are nonzero elements c_1, c_2 such that $c_1 I_n^{[p]} \subset I_{n+1}$ and $c_2 J_n^{[p]} \subset J_{n+1}$, then $c = c_1c_2$ is still nonzero and satisfies $cI_n^{[p]} \subset I_{n+1}$, $cJ_n^{[p]} \subset J_{n+1}$. So $c(I_n + J_n)^{[p]} \subset I_{n+1} + J_{n+1}$. Suppose there are nonzero elements $\phi_1, \phi_2 \in \mathrm{Hom}_R(F_*R, R)$, such that $\phi_1(F_*I_{n+1}) \subset I_n$ and $\phi_2(F_*J_{n+1}) \subset J_n$. For $\phi \in \mathrm{Hom}_R(F_*R, R)$ and $r \in R$, define $F_*r \cdot \phi \in \mathrm{Hom}_R(F_*R, R)$ by the formula $(F_*r \cdot \phi)(F_*s) = \phi(F_*(rs))$. This puts an F_*R-module structure on $\mathrm{Hom}_R(F_*R, R)$, which turns out to be a torsion free module of rank one. So the F_*R-submodules of $\mathrm{Hom}_R(F_*R, R)$ generated by ϕ_1 and ϕ_2 have a nonzero intersection, or in other words, there exist nonzero $c_1, c_2 \in R$ and a nonzero element $\phi \in \mathrm{Hom}_R(F_*R, R)$ such that $\phi = \phi_1(F_*(c_1\cdot)) = \phi_2(F_*(c_2\cdot))$. Thus, $\phi(F_*I_{n+1}) \subset I_n$ and $\phi(F_*J_{n+1}) \subset J_n$. So $\phi(F_*(I_{n+1} + J_{n+1})) \subset I_n + J_n$.

To prove the 'big'ness, assume that there is an α such that $\mathfrak{m}^{[p^{n+\alpha}]} \subseteq I_n$. Then we have $\mathfrak{m}^{[p^{n+\alpha}]} \subseteq I_n + J_n$. □

Condition C: Let $(R, \mathfrak{m})$ be an F-finite local ring, I is an ideal and $J_\bullet = \{J_n\}_{n\in\mathbb{N}}$ be a family of ideals in R. We say $I, J_\bullet$ satisfies **Condition C** if

1. The family $J_\bullet$ is weakly p and also weakly p^{-1}.
2. For each real number t, there is an α such that $\mathfrak{m}^{[p^{\alpha+n}]} \subseteq I^{\lceil tq\rceil} + J_n$ for all n.

Condition C provides the right framework where we can prove existence of h-functions; see Theorem 3.7.

Definition 3.4. Let $(R, \mathfrak{m})$ be a local or graded ring. Let I be an ideal and $J_\bullet = \{J_n\}_{n\in\mathbb{N}}$ be a family of ideals in R- homogeneous when R is graded, such that $I + J_n$ is $\mathfrak{m}$-primary for all n. For a finitely generated R-module M (homogeneous when R is graded) and $s \in \mathbb{R}$, set

1. $h_{n,M,I,J_\bullet}(s) = l(\frac{M}{(I^{\lceil sq\rceil} + J_n)M})$.
2. For an integer d, set
$$h_{n,M,I,J_\bullet,d}(s) = \frac{1}{q^d} l(\frac{M}{(I^{\lceil sq\rceil} + J_n)M}).$$
3. We denote the limit of the sequence of numbers $h_{n,M,I,J_\bullet,d}(s)$, whenever it exists, by $h_{M,I,J_\bullet,d}(s)$.

Whenever one or more of the parameters $M, I, J_\bullet$ is clear from the context, we suppress those from $h_{n,M,I,J_\bullet}(s)$, $h_{n,M,I,J_\bullet,d}(s)$ or $h_{M,I,J_\bullet,d}(s)$. In the absence of an explicit d, it should be understood that $d = \dim(M)$. When $J_n = J^{[p^n]}$ for some ideal J, $h_{n,M,I,J}, h_{n,M,I,J,d}, h_{M,I,J}$ stand for $h_{n,M,I,J_\bullet}$, $h_{n,M,I,J_\bullet,d}$ and $h_{M,I,J_\bullet,d}$ respectively.

Remark 3.5. (1) With the notational conventions and suppression of parameters declared above, $h_{n,M,I,J}$ stands for both $l(\frac{M}{(I^{\lceil sq \rceil}+J_n)M})$ and $\frac{1}{q^{\dim(M)}}l(\frac{M}{(I^{\lceil sq \rceil}+J_n)M})$. But in the article, it is always clear from the context what $h_{n,M,I,J}$ denotes. So we do not introduce further conventions.
(2) When $(R, \mathfrak{m})$ is graded, M, I and $J_\bullet$ are homogeneous, $h_{n,M,I,J} = h_{n,M_\mathfrak{m},IR_\mathfrak{m},J_\mathfrak{m}}$. So once we prove statements involving h_n's in the local setting, the corresponding statements in the graded setting follow.

The following comparison between ordinary powers and Frobenius powers is used throughout this article:

Lemma 3.6. *Let R be a ring of characteristic $p > 0$, J be a nonzero R-ideal generated by μ elements, $k \in \mathbb{N}$, and $q = p^n$ is a power of p. Then $J^{q(\mu+k-1)} \subset (J^{[q]})^k \subset J^{qk}$.*

Proof. The second containment is trivial. We prove the first containment. Let $J = (a_1, ..., a_\mu)$, then $J^{q(\mu+k-1)}$ is generated by $a_1^{u_1}...a_\mu^{u_\mu}$ where $\sum u_i = q(\mu + k - 1)$. Let $a = a_1^{u_1}...a_\mu^{u_\mu}$, $v_i = \lfloor u_i/q \rfloor$ and $b = a_1^{v_1}...a_\mu^{v_\mu}$, then since $qv_i \le u_i$, b^q divides a. Now $qv_i \ge u_i - q + 1$, so $\sum qv_i \ge q(\mu + k - 1) + (-q + 1)\mu = q(k - 1) + \mu > q(k - 1)$, so $\sum v_i \ge k$. This means $b \in J^k$ and $a \in J^{k[q]} = J^{[q]k}$. $\square$

Theorem 3.7. *Let $(R, \mathfrak{m}, k)$ be an F-finite local domain of dimension d. Let $J_\bullet$ be a family of ideals such that there is a nonzero $c \in R$ and $\phi \in \mathrm{Hom}_R(F_*R, R)$ satisfying $cJ_n^{[p]} \subseteq J_{n+1}$ and $\phi(F_*J_{n+1}) \subseteq J_n$. Let I be an ideal such that for each $s \in \mathbb{R}$, there is an integer α such that $\mathfrak{m}^{[p^{n+\alpha}]} \subseteq I^{\lceil sq \rceil} + J_n$ for all n. Set $I_n(s) = I^{\lceil sq \rceil} + J_n$.*

1. *Fix $t \in \mathbb{R}$. Choose $\alpha \in \mathbb{N}$ such that $\mathfrak{m}^{[p^{n+\alpha}]} \subseteq I^{\lceil tq \rceil} + J_n$ for all n. Then there exists a positive constant C depending only on c, ϕ, I, α and independent of the specific choice of $J_\bullet$, such that for any $s \in (-\infty, t]$,*
$$h_{R,I,J_\bullet,d}(s) = \lim_{n\to\infty} 1/p^{nd} l_R(R/I_n(s)) \textit{ exists, and}$$
$$|1/p^{nd} l_R(R/I_n(s)) - h_{R,I,J_\bullet,d}(s)| \le C/p^n \textit{ for all } n \in \mathbb{N}. \tag{3.1}$$
2. *Given choices $I, J_\bullet$ and $t \in \mathbb{R}$, one can choose C depending only on t, such that Equation (3.1) holds on $[0, t]$.*
3. *On every bounded subset of $\mathbb{R}$, the sequence of functions $h_{n,I,J_\bullet,d}(s)$ converges uniformly to $h_{R,I,J_\bullet}(s)$.*

Proof. (1) When $I = 0$, $I_n(s) = J_n$, so everything follows from Theorem 3.2.

We assume I is nonzero for the rest of the proof. Note $cI_n(s)^{[p]} = cI^{\lceil sq \rceil [p]} + cJ_n^{[p]} \subseteq cI^{\lceil sq \rceil p} + cJ_n^{[p]} \subseteq I^{\lceil spq \rceil} + J_{n+1}$ as $\lceil sq \rceil p \ge \lceil sqp \rceil$. So
$$cI_n(s)^{[p]} \subseteq I_{n+1}(s)\ . \tag{3.2}$$
Suppose I is generated by μ-many elements. Then
$$I^{\lceil spq \rceil} \subseteq I^{\lceil sq \rceil p - p} \subseteq I^{[p](\lceil sq \rceil - \mu)}; \text{ see Lemma 3.6.}$$
Fix a nonzero $r \in (I^\mu)^{[p]}$. Then the last containment implies,

$$\phi(F_*rF_*I_{n+1}(s)) = \phi(F_*(rI^{\lceil spq \rceil})) + \phi(F_*(rJ_{n+1})) \subseteq \phi(F_*(I^{\lceil sq \rceil [p]})) + J_n \subseteq I_n(s) \text{ for all } s \in \mathbb{R}\ . \tag{3.3}$$

Equation (3.2) and Equation (3.3) imply that, for all s, the nonzero elements $c \in R$ and $\phi(F_*r \cdot \_\_) \in \mathrm{Hom}_R(F_*R, R)$ endow $I_n(s)$ with weakly p and p^{-1}-family structures, respectively. The ideal $\mathfrak{m}^{[p^{n+\alpha}]}$ is contained in $I_n(t)$ and hence in $I_n(s)$ for $s \le t$. The rest follows by applying Theorem 3.2 to the family $I_{n+\alpha}(s)$ for every $s \le t$. The feasibility of choosing C depending only on c, ϕ, α and r also follows from Theorem 3.2. Since $r \in (I^{\mu})^{[p]}$ can be chosen depending only on I, the choice of C depends only on c, ϕ, α and I.

(2) Once I, J_n satisfying the hypothesis is given and $t \in \mathbb{R}$ is given, c, ϕ, α can be chosen depending only on I, J_n, t.

(3) Every bounded subset of $\mathbb{R}$ is contained in some interval $(-\infty, t]$. The dependence of C only on I, J_n and t implies (3). $\square$

The domain assumption is made in the above theorem just so that we can apply Theorem 3.2.

Lemma 3.8. *Suppose I and $J_\bullet$ satisfy the hypothesis of Theorem 3.7. Suppose there is an integer r such that $I^{rp^n} \subseteq J_n$ for all n. Then $h_{n,I,J_\bullet}(s)$ and $h_{I,J_\bullet,d}$ are constant on $[r, \infty)$.*

The next two propositions produce examples of an ideal I and ideal family $J_\bullet$ satisfying **Condition C**. For specific choices of $J_\bullet$ and I, the corresponding functions $h_{I,J_\bullet,d}$ encode widely studied invariants of a prime characteristic ring such as Hilbert-Kunz multiplicity, F-signature, and F-threshold. We do not assume R is a domain in the next two examples.

Proposition 3.9. *Let $J_\bullet$ be a family of ideals which is a big p and also p^{-1}-family. For any ideal I, I, $J_\bullet$ satisfy **Condition C**.*

Proof. Since $J_\bullet$ is big, there is an α such that $\mathfrak{m}^{[p^{n+\alpha}]} \subseteq J_n$. Thus for every $s \in \mathbb{R}$, $\mathfrak{m}^{[p^{n+\alpha}]} \subseteq I^{\lceil sq \rceil} + J_n$. $\square$

When R is a domain, a big p, p^{-1}-family $J_\bullet$ thus produces an h-function. Thanks to Lemma 3.8 such an $h_{I,J_\bullet}$ is eventually constant.

Example 3.10. Examples of $J_\bullet$ which are both big p and also p^{-1} include $J_n = J^{[p^n]}$, where J is an $\mathfrak{m}$-primary ideal. Another example of interest is when J_n is the sequence of ideals defining F-signature of $(R, \mathfrak{m})$ which we now recall. Set $p^\alpha = [k : k^p]$. Take

$$J_n = \{x \in R \mid \phi(x) \in \mathfrak{m}, \text{ for all } \phi \in \mathrm{Hom}_R(F_*^n R, R)\}.$$

Then $p^{\alpha n} l(R/I_n)$ coincides with the free rank of $F_*^n R$: the maximal rank of a free module M such that there is an R-module surjection $F_*^n R \to M$; see [Tuc12, Prop 4.5]. The family J_n is both weakly p and p^{-1}, and J_n contains $\mathfrak{m}^{[p^n]}$. Thanks to Theorem 3.2, the limit

$$s(R) := \lim_{n \to \infty} (\frac{1}{q})^{\dim(R)} l(\frac{R}{J_n})$$

exists. The number $s(R)$ measuring the asymptotic growth of the free rank of $F_*^n R$ is called the F-signature of R. The ring $(R, \mathfrak{m})$ is strongly F-regular if and only if $s(R)$ is positive; see [AL03, Theorem 0.2]. When R is a domain, for any nonzero ideal I, we have $h_{I,J_\bullet}(s) = s(R)$ for large s. The continuity, left-right differentiability of such $h_{I,J_\bullet}$ are consequences of Theorem 5.4.

The examples of h-functions produced by the result below are central to extending theories of Frobenius-Poincaré and Hilbert-Kunz density functions to the local setting.

Proposition 3.11. *For any pair of ideals I, J such that $I + J$ is $\mathfrak{m}$-primary, the ideal I and the family $J_n = J^{[p^n]}$ satisfies **Condition C**.*

Proof. For any nonzero $c \in R$ and nonzero $\phi \in \mathrm{Hom}_R(F_*R, R)$, $c(J^{[p^n]})^{[p]} \subseteq J^{[p^{n+1}]}$ and $\phi(F_*J^{[p^{n+1}]}) \subset J^{[p^n]}$. So the family $J_n = J^{[p^n]}$ is weakly p and p^{-1} Since $I+J$ is $\mathfrak{m}$-primary, given a real number s, $\mathfrak{m}^{[p^\alpha]} \subseteq I^{\lceil s\rceil}+J$ for some α. Then $\mathfrak{m}^{[p^{\alpha+n}]} \subseteq (I^{\lceil s\rceil}+J)^{[p^n]} \subseteq I^{\lceil sq\rceil}+J^{[q]}$. So $I^{\lceil sq\rceil} + J^{[q]}$ is a big p and p^{-1}-family. □

For two $\mathfrak{m}$-primary ideals I, J, in [Tay18] Taylor considers s-multiplicity (function) which, up to multiplication by a positive number depending on s, coincides with the corresponding $h_{I,J}$. When $J_n = J^{[q]}$, our proof of the existence of h-function in Theorem 3.7 is not only different from the proof of Theorem 2.1 of [Tay18], but also still valid when both I and J are not necessarily $\mathfrak{m}$-primary. Moreover, in Theorem 3.7, the flexibility of choosing C depending only on ϕ and c is a byproduct of our proof; this flexibility is crucial in Theorem 3.13 and later.

3.2. **Growth and $\mathfrak{m}$-adic continuity of h-function.** Next, we investigate how $h_{n,I,J_\bullet}(s)$ changes when the I or $J_\bullet$ is replaced by another ideal or ideal family which is $\mathfrak{m}$-adically close to the initial one. The results we prove are used later in Section 6, for example, to prove continuity of Hilbert-Kunz density function $\tilde{g}_{M,J}$ for non $\mathfrak{m}$-primary J; see Theorem 6.7.

Lemma 3.12. *Let R be a noetherian local ring, I, J be two R-ideals such that $I+J$ is $\mathfrak{m}$-primary. Let I', J' be two ideals such that $I \subset I'$, $J \subset J'$. Then $h_{n,M,I,J}(s) \geq h_{n,M,I',J'}(s)$.*

Proof. If $I \subset I'$, $J \subset J'$ then $(I^{\lceil sp\rceil} + J^{[p]})M \subset (I'^{\lceil sp\rceil} + J'^{[p]})M$, so $l(M/(I^{\lceil sp\rceil} + J^{[p]})M) \geq l(M/(I'^{\lceil sp\rceil} + J'^{[p]})M)$, which just means $h_{n,M,I,J}(s) \geq h_{n,M,I',J'}(s)$. □

Theorem 3.13. *Let $(R, \mathfrak{m})$ be a noetherian local ring. Assume $I, J_\bullet$ satisfy **Condition C**.*
(1) Fix $s_0 \in \mathbb{R}$. We can choose t depending only on $I, J_\bullet, s_0$ such that for any ideals $J \subset \mathfrak{m}^t$, $I \subset I'$, and all n,

$$h_{n,M,I',J_\bullet}(s) = h_{n,M,I',J_\bullet+J^{[p^n]}}(s) \text{ for } s \leq s_0.$$

(2) Assume $J_\bullet$ is both big p and p^{-1}-family. There exists a constant c such that for any ideals $I' \subset \mathfrak{m}^t$, $t \in \mathbb{N}$ and $s \in \mathbb{R}$,

$$h_{n,M,I,J_\bullet}(s - c/t) \leq h_{n,M,I+I',J_\bullet}(s) \leq h_{n,M,I,J_\bullet}(s) \leq h_{n,M,I+I_t,J_\bullet}(s + c/t).$$

(3) Fix $s_0 > 0$. There exists a t_0 and a constant c, both only depending on $s_0, I, J_\bullet$ such that for any $t \geq t_0$, $I_t \subseteq \mathfrak{m}^t$,

$$h_{n,M,I,J_\bullet}(s - c/t) \leq h_{n,M,I+I_t,J_\bullet}(s) \leq h_{n,M,I,J_\bullet}(s) \leq h_{n,M,I+I_t,J_\bullet}(s + c/t),$$

for $s \leq s_0$.

Proof. (1) Let t be the smallest integer such that $\mathfrak{m}^{t[q]} \subset I^{\lceil s_0 q\rceil} + J_n$ for all n. By the previous lemma, it suffices to consider the case where $J = \mathfrak{m}^t$. So for $I \subseteq I'$,

$$I'^{\lceil sq\rceil} + J_n = I'^{\lceil sq\rceil} + J_n + \mathfrak{m}^{t[q]} \text{ for } s \leq s_0 \text{ and all } n \in \mathbb{N},$$

proving the desired statement.

(2) Since $J_\bullet$ is a big family, we can choose t_0 such that $\mathfrak{m}^{t_0[q]} \subseteq J_n$ for all n. We may also assume $I' = \mathfrak{m}^t$. Let $\mathfrak{m}$ be generated by μ-elements, set $\epsilon_t = t_0\mu/t$. Then $\mathfrak{m}^{t\lceil \epsilon_t q\rceil} \subseteq \mathfrak{m}^{t_0\mu q} \subseteq \mathfrak{m}^{t_0[q]} \subset J_n$ for all n. So

$$(I+\mathfrak{m}^t)^{\lceil sq\rceil} = \sum_{0\leq j\leq \lceil sq\rceil} I^{\lceil sq\rceil - j}\mathfrak{m}^{tj} \subset I^{\lceil sq\rceil - \lceil \epsilon_t q\rceil} + \mathfrak{m}^{t\lceil \epsilon_t q\rceil} \subset I^{\lceil sq\rceil - \lceil \epsilon_t q\rceil} + J_n \subseteq I^{\lceil (s - t_0\mu/t)q\rceil} + J_n$$

Thus we have

$$l(M/(I^{\lceil (s-t_0\mu/t)q\rceil}+J_n)M)\leq l(M/((I+\mathfrak{m}^t)^{\lceil sq\rceil}+J_n)M)\leq l(M/(I^{\lceil sq\rceil}+J_n)M).$$

So taking $c=t_0\mu$ verifies the first two inequalities. These equalities are independent of s, so we may replace s by $s+c/t$ to get the third inequality.

(3) By (1) we can choose t_1 depending on $s_0, I, J_\bullet$ so that $h_{n,M,I',J_\bullet+\mathfrak{m}^{t_1 [q]}}(s)=h_{n,M,I',J_\bullet}(s)$ whenever $I\subset I'$ and $s\leq s_0+1$. By (2), we can choose c depending on $J_\bullet$ and $\mathfrak{m}^{t_1}$ such that

$$h_{n,M,I,J_\bullet+\mathfrak{m}^{t_1[q]}}(s-\frac{c}{t})\leq h_{n,M,I+I_t,J_\bullet+\mathfrak{m}^{t_1[q]}}(s)\leq h_{n,M,I,J_\bullet+\mathfrak{m}^{t_1[q]}}(s)\leq h_{n,M,I+I_t,J_\bullet+\mathfrak{m}^{t_1[q]}}(s+\frac{c}{t}),$$

for $I_t\subseteq m^t$. Take $t_0=c$. Since for $t\geq t_0$ and $s\leq s_0$, $s+\frac{c}{t}\leq s_0+1$, the above chain of inequalities imply

$$h_{n,M,I,J_\bullet}(s-c/t)\leq h_{n,M,I+I_t,J_\bullet}(s)\leq h_{n,M,I,J_\bullet}(s)\leq h_{n,M,I+I_t,J_\bullet}(s+c/t).$$

□

Assertion (1) of the theorem above allows us to replace $J_\bullet$ by a big p and p^{-1}-family in questions involving local structure of h-functions. This observation is repeatedly used later; see Theorem 6.7.

Next we prove that the sequence $h_{n,I,J_\bullet,d}(s)$ is uniformly bounded on every compact subset. When $J_\bullet=J^{[p^n]}$ for some J, we refine the bound to show that $h_{n,I,J_\bullet,d}(s)$ is bounded above by a polynomial of degree $\dim(R/J)$ in Theorem 3.16. The uniform (in n) polynomial bound on h_n is used in the extension of the theory of Frobenius-Poincaré functions in Lemma 4.1 and Theorem 4.3.

Lemma 3.14. *In a local ring $(R,\mathfrak{m})$, let $I, J_\bullet$ satisfy **Condition C**. Let M be a finitely generated R-module. Given $s_0\in\mathbb{R}$, there is a constant C depending only on s_0 such that*

$$h_{n,M,I,J_\bullet}(s)\leq Cq^d$$

for all $s\leq s_0$ and all $n\in\mathbb{N}$.

Proof. Choose α such that $\mathfrak{m}^{[p^{n+\alpha}]}\subseteq I^{\lceil s_0q\rceil}+J_n$. So for $s\leq s_0$,

$$h_{n,M,I,J_\bullet}(s)\leq l(\frac{M}{\mathfrak{m}^{[p^{n+\alpha}]}M})\leq Cq^d.$$

The last ineuqality is a consequence of [Mon83]. □

Remark 3.15. Given a noetherian local ring $(R,\mathfrak{m},k)$ containing $\mathbb{F}_p$ and a field extension $k\subseteq L$, denote by S the $\mathfrak{m}$-adic completion of $L\otimes_k\hat{R}$. Here $\hat{R}$ is the $\mathfrak{m}$-adic completion of R which can be treated as a k-algebra thanks to the existence of a coefficient field of $\hat{R}$; see [Sta23, tag 0323]. The residue field of the local ring S is isomorphic to L. The natural map $R\to S$ is faithfully flat. Now given a finite length R-module M, $l_R(M)=l_S(S\otimes_R M)$. We use this observation to make simplifying assumption on the residue field of R.

Theorem 3.16. *Let $(R,\mathfrak{m})$ be a noetherian local ring of dimension d, I, J be two R-ideals such that $I+J$ is $\mathfrak{m}$-primary. Assume I is generated by μ elements, M is generated by ν elements, and $d'=\dim R/J$. Then:*

(1) *There exist a polynomial $P_1(s)$ of degree d' such that for any $s\geq 0$,*

$$\frac{l(M/(I^{\lceil sq\rceil}+J^{[q]})M)}{l(R/\mathfrak{m}^{[q]})}\leq P_1(s).$$

Moreover if $d'>0$, the leading coefficient of P_1 can be taken to be $\frac{\nu e(I,R/J)}{d'!}$

(2) *There exist a polynomial $P_2(s)$ such that*

$$\frac{l(M/(I^{\lceil sq\rceil}+J^{[q]})M)}{q^d} \leq P_2(s).$$

In other words, $h_{n,M,d}(s)/q^d \leq P_2(s)$.

(3) *There exists a polynomial P_3 of degree d' and leading coefficient $\frac{\nu e(I,R/J)e_{HK}(R)}{d'!}$ such that for any $s \geq 0$,*

$$\overline{\lim}_{n\to\infty}\frac{l(M/(I^{\lceil sq\rceil}+J^{[q]})M)}{q^d} \leq P_3(s).$$

Proof. We may assume that the residue field is perfect using Remark 3.15.

(1) Suppose M is generated by ν many elements. Then

$$\begin{aligned} l(M/(I^{\lceil sq\rceil}+J^{[q]})M) &\leq \nu l(R/I^{\lceil sq\rceil}+J^{[q]}) \\ &\leq \nu l(R/(I^{\lceil s\rceil})^{[q]}+J^{[q]}) \\ &\leq \nu l(F^n_*R/(I^{\lceil s\rceil}+J)F^n_*R) \\ &\leq \nu \mu_R(F^n_*R) l(R/I^{\lceil s\rceil}+J) \end{aligned}$$

Let P_0 be the Hilbert-Samuel polynomial for the I-adic filtration on R/J; P_0 has degree d' and leading coefficient $\frac{\nu e(I,R/J)}{d'!} > 0$. Since P_0 is a polynomial with positive leading coefficient, we can fix s_0 such that for $s \geq s_0$, $l(R/I^{\lceil s\rceil}+J) = P_0(\lceil s\rceil)$ and P_0 is non-decreasing. Thus for $s \geq s_0$,

$$l(R/I^{\lfloor s\rfloor}+J) \leq P_0(s+1).$$

When R/J has Krull dimension zero, we can take the constant polynomial $P_1(s) = l(R/J) < \infty$, then $l(R/I^{\lceil s\rceil}+J) \leq P_1(s)$ for all s. When R/J has positive Krull dimension, we can add a suitable positive constant to $P_0(s+1)$ to get a P_1 so that $l(R/I^{\lfloor s\rfloor}+J) \leq P_1(s)$ on $[0, s_0+1]$ and thus on $\mathbb{R}$.

(2) Since $\lim_{n\to\infty} l(R/\mathfrak{m}^{[q]})/q^d = e_{HK}(R)$ exists,

$$C = \sup_n l(R/\mathfrak{m}^{[q]})/q^d$$

exists. So for any n, $l(R/\mathfrak{m}^{[q]})/q^d \leq C$, and $P_2 = CP_1$ satisfies (2).

(3)

$$\begin{aligned} &\overline{\lim}_{n\to\infty}\frac{l(M/(I^{\lceil sq\rceil}+J^{[q]})M)}{q^d} \\ \leq\ &\overline{\lim}_{n\to\infty}\frac{l(M/(I^{\lceil sq\rceil}+J^{[q]})M)}{l(R/\mathfrak{m}^{[q]})}\overline{\lim}_{n\to\infty}\frac{l(R/\mathfrak{m}^{[q]})}{q^d} \\ \leq\ &e_{HK}(R)P_1(s). \end{aligned}$$

So $P_3 = e_{HK}(R)P_1$ works.

□

3.3. **Lipschitz continuity of h-functions: Application of a 'convexity technique'.** Proving continuity of $h_{R,I,J_\bullet}$- when R is a domain is more involved than proving its existence. In this subsection, we develop results aiding the proof of Lipschitz continuity of $h_{R,I,J_\bullet}$ proven in Theorem 3.20. When $J_n = J^{[q]}$, these results are used to prove existence and continuity of the h-function of a finitely generated module in Theorem 3.29, by reducing the problem to the case where R is reduced. The key result which allows these reduction steps is Theorem 3.19. We prove Theorem 3.19 by utilizing the monotonicity of a certain numerical function. This technique of using the monotonicity which we call the 'convexity technique' is repeatedly used later for example to prove left and right differentiability of the h-function among other properties. The required monotonicity result appears in Lemma 3.17. This is an adaptation and generalization of Boij-Smith's result in [BS15] which is suitable for our purpose.

Lemma 3.17. *Let $(R, \mathfrak{m})$ be a noetherian local ring, I be an $\mathfrak{m}$-primary ideal generated by μ elements, M be a finitely generated R-module, S be the polynomial ring of μ-variables over $R/\mathfrak{m}$. Then the function $i \to l(I^iM/I^{i+1}M)/l(S_i)$ is decreasing for $i \geq 0$.*

Proof. Consider the associated graded ring $\mathrm{gr}_I(R)$. Since I is generated by a set of μ elements, as a graded ring $\mathrm{gr}_I(R)$ is a quotient of the standard graded polynomial ring $R/I[T_1, ..., T_\mu]$ over R/I. Recall $S = \frac{R}{\mathfrak{m}}[T_1, ..., T_\mu]$. Since M/IM is Artinian, there exists a filtration

$$0 = N_0 \subset N_1 \subset ... \subset N_l = M/IM, \text{ such that } N_{j+1}/N_j = R/\mathfrak{m} \text{ for } 0 \leq j \leq l-1.$$

Let M_j be the $\mathrm{gr}_I(R)$-submodule of $\mathrm{gr}_I(M)$ spanned by N_j. Then M_{j+1}/M_j is annihilated by $\mathfrak{m}\mathrm{gr}_I(R)$. So it is naturally a $\mathrm{gr}_I(R)/\mathfrak{m}\mathrm{gr}_I(R)$-module, hence is an S-module, and it is generated in degree 0. So by Theorem 1.1 of [BS15], for any $i \geq 0$,

$$l(M_{j+1}/M_j)_i/l(S_i) \geq l(M_{j+1}/M_j)_{i+1}/l(S_{i+1}).$$

Since truncation at degree i is an exact functor from $\mathrm{gr}_I(R)$-modules to R-modules, taking sum over $0 \leq j \leq l-1$ we get $l(M_l)_i/l(S_i) \geq l(M_l)_{i+1}/l(S_{i+1})$. Since $M_l = \mathrm{gr}_I(R)N_l = \mathrm{gr}_I(M)$, we are done. □

When I is a principal ideal, the above lemma manifests into the following easily verifiable result.

Example 3.18. Let R be a noetherian local ring, f be an element in R such that R/fR has finite length. Then for any $j \geq i \geq 0$, $l(f^iR/f^{i+1}R) \geq l(f^jR/f^{j+1}R)$. This means that the function $i \to l(R/f^iR)$ is convex on $\mathbb{N}$; see Definition 5.2.

Theorem 3.19. *Let R be a noetherian local ring, M be a finitely generated module of dimension d. Suppose $I, J_\bullet$ satisfy* ***Condition C****. Fix $0 < s_1 < s_2 < \infty \in \mathbb{R}$. Then there is a constant C and a power $q_0 = p^{n_0}$ that depend on s_1, s_2, but independent of n such that for any $s_1 \leq s \leq s_2 - 1/q$ and $q \geq q_0$*

$$l\left(\frac{(I^{\lceil sq \rceil} + J_n)M}{(I^{\lceil sq \rceil + 1} + J_n)M}\right) \leq Cq^{d-1}$$

In other words, whenever $s_1 \leq s \leq s_2 - 1/q$ and $q \geq q_0$,

$$|h_{n,M}(s + 1/q) - h_{n,M}(s)| \leq Cq^{d-1}.$$

Proof. We may assume $s_1, s_2 \in \mathbb{Z}[1/p]$. Otherwise, since $\mathbb{Z}[1/p]$ is dense in $\mathbb{R}$, we can choose $s_1' \in (0, s_1) \cap \mathbb{Z}[1/p]$, $s_2' \in (s_2, \infty) \cap \mathbb{Z}[1/p]$ and replace s_1, s_2 by s_1', s_2'. Choose $s_3 \in \mathbb{Z}[1/p]$ such that $0 < s_3 < s_1$ and choose q_0 such that $s_1q_0, s_2q_0, s_3q_0 \in \mathbb{Z}$. Let I be generated by a set of μ many elements. Applying Lemma 3.17 to the module M/J_nM we know for any $0 \leq t \leq \lceil sq \rceil$,

$$\frac{l\left(\frac{I^{\lceil sq \rceil}(M/J_nM)}{I^{\lceil sq \rceil + 1}(M/J_nM)}\right)}{\binom{\mu + \lceil sq \rceil - 1}{\mu - 1}} \leq \frac{l\left(\frac{I^t(M/J_nM)}{I^{t+1}(M/J_nM)}\right)}{\binom{\mu + t - 1}{\mu - 1}}.$$

Rewritten, the above inequality yields

$$\frac{l(\frac{(I^{\lceil sq\rceil}+J_n)M}{(I^{\lceil sq\rceil+1}+J_n)M})}{\binom{\mu+\lceil sq\rceil-1}{\mu-1}} \leq \frac{l(\frac{(I^t+J_n)M}{(I^{t+1}+J_n)M})}{\binom{\mu+t-1}{\mu-1}}.$$

Thus for $s_1 \leq s \leq s_2 - \frac{1}{q}$ and $q \geq q_0$,

$$\begin{aligned}
(\lceil sq\rceil - s_3q)l(\frac{(I^{\lceil sq\rceil}+J_n)M}{(I^{\lceil sq\rceil+1}+J_n)M}) &\leq \binom{\mu+\lceil sq\rceil-1}{\mu-1}\sum_{t=s_3q}^{\lceil sq\rceil-1}\frac{l(\frac{(I^t+J_n)M}{(I^{t+1}+J_n)M})}{\binom{\mu+t-1}{\mu-1}} \\
&\leq \frac{\binom{\mu+\lceil sq\rceil-1}{\mu-1}}{\binom{\mu+s_3q-1}{\mu-1}} l(\frac{(I^{\lceil sq\rceil}+J_n)M}{(I^{s_3q}+J_n)M}) \\
&\leq \frac{\binom{\mu+\lceil sq\rceil-1}{\mu-1}}{\binom{\mu+s_3q-1}{\mu-1}}[l(\frac{M}{(I^{\lceil sq\rceil}+J_n)M}) - l(\frac{M}{(I^{s_3q}+J_n)M})] \\
&\leq \frac{\binom{\mu+s_2q-1}{\mu-1}}{\binom{\mu+s_3q-1}{\mu-1}}[l(\frac{M}{(I^{s_2q}+J_n)M}) - l(\frac{M}{(I^{s_3q}+J_n)M})].
\end{aligned}$$

Therefore for $s_1 \leq s \leq s_2 - \frac{1}{q}$ and $q \geq q_0$,

$$l(\frac{(I^{\lceil sq\rceil}+J_n)M}{(I^{\lceil sq\rceil+1}+J_n)M}) \leq \frac{1}{s_1q - s_3q}\frac{\binom{\mu+s_2q-1}{\mu-1}}{\binom{\mu+s_3q-1}{\mu-1}}[l(\frac{M}{(I^{s_2q}+J_n)M}) - l(\frac{M}{(I^{s_3q}+J_n)M})] \leq Cq^{d-1}.$$

By Lemma 3.14, we can choose a constant C' depending only on s_2 such that for $s \leq s_2$,

$$l(\frac{M}{(I^{sq}+J_n)M}) \leq C'q^d.$$

Since $\binom{\mu+s_2q-1}{\mu-1}/\binom{\mu+s_3q-1}{\mu-1}$ is bounded above by a constant depending on s_1, s_2, s_3 and s_3 depends only on s_2, we can choose C depending only on s_1, s_2 such that for all n and $q \geq q_0$,

$$l(\frac{(I^{\lceil sq\rceil}+J_n)M}{(I^{\lceil sq\rceil+1}+J_n)M}) \leq Cq^{d-1}.$$

Here C is a constant only depending on s_1, s_2, s_3, and s_3 depends only on s_1. □

Therefore, whenever $h_{M,I,J_\bullet}$ exists, it is locally Lipschitz continuous away from zero.

Theorem 3.20. *Let I be an ideal and $J_\bullet$ be a family of ideals satisfying **Condition C** in a domain $(R, \mathfrak{m})$ of Krull dimension d. Given real numbers $0 < s_1 < s_2$, there is a constant C depending only in s_1, s_2 such that for any $x, y \in [s_1, s_2]$,*

$$|h_R(x) - h_R(y)| \leq C|x-y|$$

Proof. Given s_1, s_2 as above and x, y in $[s_1, s_2]$, by Theorem 3.19, we can choose a constant C depending only on s_1, s_2 such that

$$|h_{n,R}(x) - h_{n,R}(y)| = |h_{n,R}(\frac{\lceil qx\rceil}{q}) - h_{n,R}(\frac{\lceil qy\rceil}{q})| \leq C|\frac{\lceil qx\rceil}{q} - \frac{\lceil qy\rceil}{q}|q^d \text{ for all } n.$$

Divide both sides by q^d and take limit as n approaches infinity. Since for any real number s, $\frac{h_n(s)}{q^d}$ and $\lceil qs\rceil/q$ converge to $h_R(s)$ and s respectively,

$$|h_R(x) - h_R(y)| \leq C|x-y|.$$

□

Lemma 3.21. *Assume the residue field of R is perfect and M is a module of dimension d. For each integer $n_0 \geq 0$ and fixed $0 < s_1 < s_2 < \infty \in \mathbb{R}$, there is a constant C independent of n such that*

$$|h_{n+n_0,M,I,J}(s) - h_{n,F_*^{n_0}M,I,J}(s)| \leq Cq^{d-1}$$

for any $s_1 \leq s \leq s_2$.

Proof. For any q_0, $\lceil sqq_0 \rceil \leq \lceil sq \rceil q_0 \leq \lceil sqq_0 \rceil + q_0$. We have,

$$\begin{aligned}
&|h_{n+n_0,M,I,J}(s) - h_{n,F_*^{n_0}M,I,J,d}(s)| \\
&= |l(M/(I^{\lceil sqq_0 \rceil} + J^{[qq_0]})M) - l(F_*^{n_0}M/(I^{\lceil sq \rceil} + J^{[q]})F_*^{n_0}M)| \\
&= |l(M/(I^{\lceil sqq_0 \rceil} + J^{[qq_0]})M) - l(M/(I^{\lceil sq \rceil [q_0]} + J^{[qq_0]})M)| \\
&= (l(I^{\lceil sqq_0 \rceil} + J^{[qq_0]})M/(I^{\lceil sq \rceil q_0} + J^{[qq_0]})M) + l(I^{\lceil sq \rceil q_0} + J^{[qq_0]})M/(I^{\lceil sq \rceil [q_0]} + J^{[qq_0]})M)) \ .
\end{aligned}$$

Note that $1/q_0 \lceil sqq_0 \rceil \geq sq \geq \lceil sq \rceil - 1$, so $\lceil sq \rceil q_0 \leq \lceil sqq_0 \rceil + q_0$, so $I^{\lceil sqq_0 \rceil + q_0} \subset I^{\lceil sq \rceil q_0}$. Suppose I is generated by μ elements, then by Lemma 3.6, $I^{\lceil sq \rceil q_0} \subset I^{(\lceil sq \rceil - \mu + 1)[q_0]}$. Now by Theorem 3.19, we can choose a constant C depending only on s_1, s_2 but independent of q such that for all $s \in [s_1, s_2]$,

$$\begin{aligned}
&l(\frac{(I^{\lceil sqq_0 \rceil} + J^{[qq_0]})M}{(I^{\lceil sq \rceil q_0} + J^{[qq_0]})M}) + l(\frac{(I^{\lceil sq \rceil q_0} + J^{[qq_0]})M}{(I^{\lceil sq \rceil [q_0]} + J^{[qq_0]})M}) \\
&\leq l(\frac{(I^{\lceil sqq_0 \rceil} + J^{[qq_0]})M}{(I^{\lceil sqq_0 \rceil + q_0} + J^{[qq_0]})M}) + l(\frac{(I^{(\lceil sq \rceil - \mu + 1)[q_0]} + J^{[qq_0]})M}{(I^{\lceil sq \rceil [q_0]} + J^{[qq_0]})M}) \leq Cq^{d-1} \ .
\end{aligned}$$

□

The lemma above allows us to replace M by $F_*^{n_0}M$ in the proof of the existence of $h_{M,I,J,d}$. Since we may replace R by $R/\text{ann}(F_*^{n_0}M)$ and for large enough n_0, $\text{ann}(F_*^{n_0}M)$ contains the nilradical of R, we may assume R is reduced while proving the existence of $h_{M,I,J}$.

Corollary 3.22. Assume the residue field of R is perfect. For each $n_0 \geq 0$, $h_{M,I,J,d}(s)$ exists if and only if $h_{F_*^{n_0}M,I,J,d}(s)$ exists, and if they both exist then

$$q_0^d h_{M,I,J,d}(s) = h_{F_*^{n_0}M,I,J,d}(s).$$

3.4. **h-function of a module.** For a noetherian local ring $(R, \mathfrak{m})$, R-ideals I, J such that $I + J$ is $\mathfrak{m}$-primary and a finitely generated R-module M, we prove the existence of $h_{M,I,J}$ in Theorem 3.29 and prove the local Lipschitz continuity of $h_{M,I,J}$ in Theorem 3.30. First, we prove preparatory results to reduce the problem of the existence of $h_{M,I,J}$ to the situation where $M = R$ and R is a domain. Recall:

Definition 3.23. Set $\text{Assh}R = \{P \in \text{Spec}R : \dim R = \dim R/P\}$.

Lemma 3.24. *[Mon83, Proof of Lemma 1.3]If M, N are two R-modules such that $M_P \cong N_P, \forall P \in \text{Assh}R$. Then there is an exact sequence*

$$0 \to N_1 \to M \to N \to N_2 \to 0$$

such that $\dim N_1, \dim N_2 \leq \dim(R) - 1$. Moreover it breaks up into two short exact sequences:

$$0 \to N_1 \to M \to N_3 \to 0$$
$$0 \to N_3 \to N \to N_2 \to 0.$$

Lemma 3.25. *Let $N \subset M$ be two R-modules of finite length, and take $a \in R$, then $l(M/aM) \geq l(N/aN)$.*

Proof. Consider the commutative diagram,

$$\begin{tikzcd}
0 \arrow[r] & 0 :_N a \arrow[r] \arrow[d] & N \arrow[r, "a"] \arrow[d] & N \arrow[r] \arrow[d] & \frac{N}{aN} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & 0 :_M a \arrow[r] & M \arrow[r, "a"] & M \arrow[r] & \frac{M}{aM} \arrow[r] & 0
\end{tikzcd}$$

We see the map $0 :_N a \to 0 :_M a$ is injective. By the additivity of length on exact sequences we see $l(M/aM) = l(0 :_M a) \geq l(0 :_N a) = l(N/aN)$. □

Lemma 3.26. *Let $(R, \mathfrak{m}, k)$ be a local ring of dimension d. Suppose $I, J_\bullet$ satisfy* ***Condition C****, and M is a module of dimension $d' \leq d-1$. Fix $s_0 \in \mathbb{R}$. Then there are constants C_1, C_2 depending on s_0 but independent of n such that $l(\mathrm{Tor}_0^R(R/(I^{\lceil sq\rceil}+J_n), M)) \leq C_1 q^{d-1}$ and $l(\mathrm{Tor}_1^R(R/(I^{\lceil sq\rceil}+J_n), M)) \leq C_2 q^{d-1}$ for any $s \leq s_0$. Moreover if $J_\bullet$ is big, C_1, C_2 can be chosen independent of s.*

Proof. Since $I, J_\bullet$ satisfy **Condition C**, we can find an $\mathfrak{m}$-primary ideal J such that for $s \leq s_0$, $J^{[q]} \subseteq I^{\lceil sq\rceil} + J_n$ for all n. As $M/J^{[q]}M$ surjects onto $\mathrm{Tor}_0^R(R/(I^{\lceil sq\rceil}+J_n), M)$, and we can find a constant C_1, such that $l(M/J^{[q]}M) \leq C_1 q^{\dim M}$, $l(\mathrm{Tor}_0^R(R/I^{\lceil sq\rceil}+J^{[q]}, M)) \leq C_1 q^{d-1}$.

To see the bound on Tor_1, for a fixed $s \leq s_0$, consider the exact sequence:

$$0 \to (I^{\lceil sq\rceil}+J_n)/J^{[q]} \to R/J^{[q]} \to R/(I^{\lceil sq\rceil}+J^{[q]}) \to 0$$

So by the long exact sequence of Tor, it suffices to show that we can choose C_2 satisfying

$$l(\mathrm{Tor}_1^R(R/J^{[q]}, M)) \leq C_2 q^{d-1} \,\text{and}\, l(\frac{I^{\lceil sq\rceil}+J_n}{J^{[q]}} \otimes M) \leq C_2 q^{d-1}.$$

Choosing a C_2 satisfying the first inequality is possible thanks to [HMM04, Lemma 1.1]. For the remaining inequality, by taking a prime cyclic filtration of M, we may assume $M = R/P$ for some $P \in \mathrm{Spec}(R)$ with $\dim M \leq \dim R - 1$. In this case, $P \notin \mathrm{Assh}(R)$. So we can choose $b \in P$ such that $\dim R/bR \leq \dim R - 1$. Apply Lemma 3.25 to $I^{\lceil sq\rceil} + J^{[q]}/J^{[q]} \subset R/J^{[q]}$, we see that we can enlarge C_2 independently of s and q so that

$$\begin{aligned} l(l(\frac{I^{\lceil sq\rceil}+J_n}{J^{[q]}} \otimes_R R/P) &\leq l(l(\frac{I^{\lceil sq\rceil}+J_n}{J^{[q]}} \otimes_R R/bR) \\ &\leq l(R/J^{[q]} \otimes_R R/bR) = l(R/bR + J^{[q]}) \leq C_2 q^{d-1}. \end{aligned}$$

If $J_\bullet$ is big then for large s, $I^{\lceil sq\rceil} \subset J_n$ for all n. So we can find constant D such that for every n, $l(\mathrm{Tor}_0^R(R/(I^{\lceil sq\rceil}+J_n), M))$ and $l(\mathrm{Tor}_1^R(R/(I^{\lceil sq\rceil}+J_n), M))$ are constant for $s \geq D$. So C_1, C_2 can be chosen independent of $s \in \mathbb{R}$. So we are done. □

Lemma 3.27. *Let M, N be two finitely generated R-modules that are isomorphic at $P \in \mathrm{Assh}R$. Then for any $t > 0$, there is a constant C depending on M, I, J, t but independent of n such that for any $s < t$*

$$|h_{n,M,d}(s) - h_{n,N,d}(s)| \leq C/q$$

Moreover if J is $\mathfrak{m}$-primary, then C can be chosen independently of t.

Proof. By Lemma 3.24, there is an exact sequence

$$0 \to N_1 \to M \to N \to N_2 \to 0$$

such that $\dim N_1, \dim N_2 \leq d-1$. And it breaks up into two short exact sequences:

$$0 \to N_1 \to M \to N_3 \to 0$$

$$0 \to N_3 \to N \to N_2 \to 0$$

Now by the long exact sequence of Tor we get

$$|l(M/(I^{\lceil sq \rceil} + J^{[q]})M) - l(N_3/(I^{\lceil sq \rceil} + J^{[q]})N_3)| \leq l(N_1/(I^{\lceil sq \rceil} + J^{[q]})N_1),$$

$$\begin{aligned} &|l(\frac{N_3}{(I^{\lceil sq \rceil} + J^{[q]})N_3}) - l(\frac{N}{(I^{\lceil sq \rceil} + J^{[q]})N})| \\ &\leq l(\frac{N_2}{(I^{\lceil sq \rceil} + J^{[q]})N_2}) + l(\mathrm{Tor}_1^R(\frac{R}{I^{\lceil sq \rceil} + J^{[q]}}, N_2)) \ . \end{aligned}$$

Thus by Lemma 3.26, there is a constant C such that

$$|l(M/(I^{\lceil sq \rceil} + J^{[q]})M) - l(N/(I^{\lceil sq \rceil} + J^{[q]})N)| \leq Cq^{d-1},$$

and divide both sides by q^d to get the conclusion. □

Lemma 3.28. *Let $(R, \mathfrak{m}, k)$ be a local ring, I, J be two ideals such that $I+J$ is $\mathfrak{m}$-primary, and M be a finitely generated R-module of dimension d. For any $0 < s_1 < s_2 < \infty$, there is a constant C depending on M, I, J, s_1, s_2 but independent of n such that for any $s_1 \leq s \leq s_2$*

$$|h_{n+1,M,d}(s) - h_{n,M,d}(s)| \leq C/q$$

Proof. We may assume that the residue field is perfect using Remark 3.15. Choose sufficiently large n_0 such that $R/\mathrm{ann}(F_*^{n_0}M)$ is reduced. The positive constants C_1, C_2, C_3 chosen below depends only on M, I, J, s_1, s_2 and is independent of n. By Lemma 3.21,

$$|h_{n+n_0,M,I,J}(s) - h_{n,F_*^{n_0}M,I,J}(s)| \leq C_1 q^{d-1}$$

and

$$|h_{n+n_0+1,M,I,J}(s) - h_{n+1,F_*^{n_0}M,I,J}(s)| \leq C_1 q^{d-1}$$

So it suffices to prove existence of a suitable C such that

$$|h_{n+1,F_*^{n_0}M,d}(s) - h_{n,F_*^{n_0}M,d}(s)| \leq C/q.$$

Replacing M by $F_*^{n_0}M$ and R by $R/\mathrm{ann}(F_*^{n_0}M)$, so we may assume R is reduced. In this case it suffices to prove

$$|h_{n+1,M,I,J}(s) - h_{n,F_*M,I,J}(s)| \leq C_2 q^{d-1}.$$

Thanks to the reducedness of R, $R_P = R_P/PR_P$ is a field and M_P is free over R_P for any $P \in \mathrm{Assh}(R)$. Applying equation 2.2 of [Kun76] to the domain R/P, we see the localizations of $M^{\oplus p^d}$ and F_*M are isomorphic at all $P \in \mathrm{Assh}R$. So by Lemma 3.27,

$$|h_{n,F_*M,I,J}(s) - p^d h_{n,M,I,J}(s)| \leq C_3 q^{d-1}.$$

Thus one can choose a C' which depends only on M, I, J, s_1, s_2 such that for all $s \in [s_1, s_2]$ and $n \in \mathbb{N}$,

$$|h_{n+1,M,I,J}(s) - p^d h_{n,M,I,J}(s)| \leq C' q^{d-1}.$$

Dividing by $(pq)^d$ and letting $C = C'/p^d$, we get

$$|h_{n+1,M,I,J,d}(s) - h_{n,M,I,J,d}(s)| \leq C/q.$$

□

Theorem 3.29. *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring, I, J be two R-ideals such that $I + J$ is $\mathfrak{m}$-primary, and M is a finitely generated R-module. Then for every $s \in \mathbb{R}$,*

$$\lim_{n\to\infty} \frac{1}{q^{\dim(M)}} h_{n,M,I,J}(s) = h_{M,I,J}(s)$$

exists. Moreover the convergence is uniform on $[s_1, s_2]$ for any $0 < s_1 < s_2 < \infty$.

Proof. By replacing R by $R/\mathrm{ann}(M)$, we may assume $\dim(M) = \dim(R)$. Given s_1, s_2 as in the statement, it follows from Lemma 3.28 that $h_{n,M,I,J}(s)/q^{\dim(M)}$ is uniformly Cauchy on $[s_1, s_2]$. So the theorem follows. □

We also have:

Theorem 3.30. *Let $(R, \mathfrak{m}, k)$ be a local ring of dimension d, I, J be two R-ideals such that $I + J$ is $\mathfrak{m}$-primary, and M be a finitely generated R-module. Then:*

1. *$h_M(s)$ is Lipschitz continuous on $[s_1, s_2]$ for any $0 < s_1 < s_2 < \infty$. Consequently, it is continuous on $(0, \infty)$.*
2. *$h_M(s)$ is increasing. It is 0 on $(-\infty, 0]$. It is continuous if and only if it is continuous at 0, if and only if $\lim_{s\to 0^+} h_M(s) = 0$. The limit $\lim_{s\to 0^+} h_M(s)$ always exists and is nonnegative.*
3. *Assume J is $\mathfrak{m}$-primary. Then for $s >> 0$, $h_{n,M}(s) = e_{HK}(J, M)$ is a constant.*
4. *There is a polynomial $P(s)$ of degree $\dim R/J$ such that $h_M(s) \le P(s)$ on $\mathbb{R}$.*

Proof.
(1) An argument similar to that in the proof of Theorem 3.20 with R replaced by M and $J_n = J^{[q]}$ yields a proof. The difference is that when $J_n = J^{[q]}$, we know the existence of $h_{M,I,J}$.

(2) If $s_1 \le s_2$, then $\lceil s_1 q\rceil \le \lceil s_2 q\rceil$, so $I^{\lceil s_2 q\rceil} \subset I^{\lceil s_1 q\rceil}$. This implies
$$l(M/(I^{\lceil s_1 q\rceil} + J^{[q]})M) \le l(M/(I^{\lceil s_2 q\rceil} + J^{[q]})M),$$
which is just
$$h_{n,M}(s_1) \le h_{n,M}(s_2).$$
So after dividing $p^{n\dim M}$ and let $n \to \infty$, we get $h_M(s_1) \le h_M(s_2)$. This implies $h_M(s)$ is increasing; so in particular the limit $\lim_{s\to 0^+} h_M(s)$ always exists and is at least $h_M(0)$. If $s \le 0$, then $\lceil sq\rceil \le 0$, so $I^{\lceil sq\rceil} = R$. Thus $M/(I^{\lceil s_1 q\rceil} + J^{[q]})M = 0$ and $h_{n,M}(s) = 0$ for any n, so $h_M(s) = 0$. So $h_M(s)$ is continuous on $(-\infty, 0)$ and $(0, \infty)$, and $\lim_{s\to 0^-} h_M(s) = 0 = h_M(0)$, so we get (2).

(3) Let J be generated by μ elements. For $s >> 0$, $I^{\lfloor s/\mu\rfloor} \subset J$. So $I^{\lceil sq\rceil} \subset I^{\lfloor s/\mu\rfloor q\mu} \subset J^{q\mu} \subset J^{[q]}$, so $h_{n,M}(s) = l(M/J^{[q]}M)$ and $h_M(s) = \lim_{n\to\infty} \frac{l(M/J^{[q]}M)}{q^d} = e_{HK}(J, M)$.

(4) This is a corollary of Theorem 3.16 and Theorem 3.29.

□

We record the associativity formula for h-function Lemma 3.27.

Proposition 3.31. *Let M be a d-dimensional finitely generated R-module. Let $P_1, P_2, \dots,$ P_t be the d-dimensional minimal primes in the support of M. Then,*

$$h_{M,I,J,d}(s) = \sum_{j=1}^{t} l_{R_{P_j}}(M_{P_j}) h_{R/P_j, IR/P_j, JR/P_j, d}(s).$$

Proof. By replacing R by $R/\mathrm{ann}(M)$, we can assume $\dim(R) = d$. We can always assume R is reduced. Indeed, since R is noetherian, we can choose e_0 such that the image of the nilradical of R under the e_0-th iteration of the Frobenius is zero. Now by Corollary 3.22, we can replace M by some $F^{e_0}_* M$ and pass to the reduced case. Once R is reduced, the two modules

$$M \text{ and } \bigoplus_{j=1}^{t}(\frac{R}{P_j})^{l_{R_{P_j}}(M_{P_j})}$$

are isomorphic after localizing at each of the primes P_j's. So the result follows from Lemma 3.27.

□

We analyse how $h_{R,I,J}$ depends on different closure operations of ideals. We refer to [HS06] for results on integral closure of ideals.

Proposition 3.32. *Let $(R, \mathfrak{m})$ be a noetherian local ring of dimension d.*

(1) *Let I, J be ideals such that $I + J$ is $\mathfrak{m}$-primary. Let J^* be the tight closure of J. Then $h_{R,I,J} = h_{R,I,J^*}$.*
(2) *Assume R is a domain. Let $I, J_\bullet$ satisfy **Condition C**[2]. Let $\overline{I}$ be the integral closure of I. Then $h_{R,I,J_\bullet} = h_{R,\overline{I},J_\bullet}$.*

Proof. We first prove (1).

$$\frac{1}{q^d}h_{n,R,I,J}(s) - \frac{1}{q^d}h_{n,R,I,J^*}(s) = \frac{1}{q^d}l(\frac{I^{\lceil sq\rceil} + (J^*)^{[q]}}{I^{\lceil sq\rceil} + (J)^{[q]}}) \leq \frac{1}{q^d}l(\frac{(J^*)^{[q]}}{J^{[q]}}) \ . \tag{3.4}$$

Since R is noetherian there is an element $c \in R$ which is not any minimal primes of R such that $c(J^*)^{[q]} \subseteq J^{[q]}$ for all q. Fix a choice of r-many generators of J^*, the q-th powers of these generate $(J^*)^{[q]}$. Thus the length of $(J^*)^{[q]}/J^{[q]}$ is bounded above by $rl(R/(c, J^{[q]}))$. Since the Krull dimension of $\frac{R}{cR}$ is at most $d-1$, $l(R/(c, J^{[q]})) = O(q^{d-1})$ by Lemma 3.14. Thus taking limit as q approaches infinity in Equation (3.4), we conclude $h_{R,I,J}(s) = h_{R,I,J^*}(s)$.

For (2), recall that there is a natural number n_0 such that

$$I^{n+n_0} \subseteq \overline{I}^{n+n_0} \subseteq I^n,$$

for all natural numbers n. Thus for a positive real number s and q large enough,

$$\frac{1}{q^d}h_{n,R,I,J_\bullet}(s - \frac{n_0}{q}) \leq \frac{1}{q^d}h_{n,R,\overline{I},J_\bullet}(s) \leq \frac{1}{q^d}h_{n,R,I,J_\bullet}(s).$$

Thus for a positive real number s, $h_{R,\overline{I},J_\bullet}(s) = h_{R,I,J_\bullet}(s)$; see Theorem 3.20. The desired equality at zero follows from definition. □

4. Frobenius-Poincaré function in the local setting

We prove the existence of Frobenius-Poincaré functions in the local setting. Given an ideal I and a family $J_\bullet$ and a finitely generated R-module M, set

$$f_{n,M,I,J_\bullet}(s) = h_{n,M,I,J_\bullet}(s + \frac{1}{q}) - h_{n,M,I,J_\bullet}(s).$$

When $J_n = J^{[q]}$, $f_{n,M,I,J}(s)$ represents $f_{n,M,I,J_\bullet}(s)$. We drop one or more parameters in $f_{n,M,I,J_\bullet}$ when there is no resulting confusion. For the rest of this article, we denote the imaginary part a complex number y by $\Im y$ and the open lower half complex plane by Ω, i.e. $\Omega = \{y \in \mathbb{C} \mid \Im y < 0\}$.

[2]We do not need the domain assumption when $J_n = J^{[q]}$.

Lemma 4.1. *Let $(R, \mathfrak{m}, k)$ be a local ring of dimension d, I, J be two R-ideal, $I+J$ is $\mathfrak{m}$-primary, and M be a finitely generated R-module. Consider the function defined by the infinite series*

$$F_{n,M,I,J}(y) := \sum_{j=0}^{\infty} f_{n,M,I,J}(j/q)e^{-iyj/q}$$

Then $F_{n,M,I,J}(y)$ defines a holomorphic function on Ω. We often drop one or more parameters in $F_{n,M,I,J}$ when there is no chance of confusion.

Proof. There is a polynomial P such that $f_{n,M}(s) \leq h_{n,M}(s+1) \leq P(s)$ for any s; see Theorem 3.16, Theorem 3.30, assertion (2). Thus

$$|f_{n,M,R,I,J}(j/q)e^{-iyj/q}| \leq P(j/q)e^{j\Im y/q}.$$

Since for fixed $\epsilon > 0$, the series $\sum_{0 \leq j < \infty} P(j/q)e^{-j\epsilon/q}$ converges, on the region where $\Im y < -\epsilon$, the sequence of functions $\sum_{j=0}^{\infty} f_{n,M,R,I,J}(j/q)e^{-iyj/q}$ converges uniformly. The limit function is thus holomorphic [Ahl79, Theorem 1, Chap 5]. Taking union over all $\epsilon > 0$, we see $F_{n,M}(y)$ exists and is holomorphic on Ω. $\square$

Remark 4.2. For a big p, p^{-1}-family $J_\bullet$, the analogous $F_{n,M,I,J_\bullet}(y)$ defined using $f_{n,M,I,J_\bullet}$ is entire since the corresponding sum is a finite sum of entire functions.

Now, we want to check the convergence of $(F_{n,M,I,J}(y)/q^{\dim(M)})_n$ whenever it exists. We will be repeatedly using the dominated convergence: if a sequence of measurable functions f_n converges to f pointwise on a measurable set Σ and there is a measurable function g such that $|f_n| \leq g$ on Σ for any n and $\int_\Sigma |g| < \infty$, then $\int_\Sigma |f_n - f|$ converges to 0, so in particular $\int_\Sigma f_n$ converges to $\int_\Sigma f$.

Theorem 4.3. *Let $(R, \mathfrak{m}, k)$ be a local ring, I, J be two R-ideal, $I+J$ is $\mathfrak{m}$-primary, and M be a finitely generated R-module of dimension d.*
(1) Assume J is $\mathfrak{m}$-primary. Then $F_{M,I,J}(y) = \lim_{n\to\infty} F_{n,M}(y)/p^{nd}$ exists for all $y \in \mathbb{C}$. This convergence is uniform on any compact set of $\mathbb{C}$. Suppose $h_M(s)$ is constant for $s \geq C$, then $F_{M,I,J}(y) = \int_0^C h_M(t)iye^{-iyt}dt + h_M(C)e^{-iyC}$.
(2) Assume J is not necessarily $\mathfrak{m}$-primary. Then for every $y \in \Omega$, $F_{n,M}(y)/p^{nd}$ converges to

$$F_{M,I,J}(y) = \int_0^\infty h_M(t)e^{-iyt}iydt.$$

Moreover, this convergence is uniform on any compact subset of Ω and $F_M(y) := F_{M,I,J}(y)$ is holomorphic on Ω.

Proof.

(1) Since J is $\mathfrak{m}$-primary, then $h_M(s) = h_M(C)$ for some fixed $C > 0$ and any $s \geq C$; see Lemma 3.8 and Proposition 3.31. Then,

$$\begin{aligned}
F_{n,M}(y) &= \sum_{j=0}^{\infty} f_{n,M}(j/q)e^{-iyj/q} \\
&= \sum_{j=0}^{\infty}(h_{n,M}((j+1)/q) - h_{n,M}(j/q))e^{-iyj/q} \\
&= \sum_{j=0}^{Cq-1}(h_{n,M}((j+1)/q) - h_{n,M}(j/q))e^{-iyj/q} \\
&= \sum_{j=0}^{Cq-1} h_{n,M}(j/q)(e^{-iy(j-1)/q} - e^{-iy(j)/q}) + h_{n,M}(C)e^{-iy(C-\frac{1}{q})} \\
&= \sum_{j=0}^{Cq-1} h_{n,M}(j/q)e^{-iyj/q}(e^{iy/q} - 1) + h_{n,M}(C)e^{-iy(C-\frac{1}{q})} \\
&= \int_0^C h_{n,M}(t)e^{-iy\lceil tq\rceil/q}q(e^{iy/q} - 1)dt + h_{n,M}(C)e^{-iy(C-\frac{1}{q})}\ .
\end{aligned}$$

Fix a compact subset K of $\mathbb{C}$. Given $\delta > 0$, choose $b > 0$ such that for all $y \in K$, $t \in \mathbb{R}$ and $n \in \mathbb{N}$

$$\int_0^b (\frac{1}{q^d}|h_{n,M}(t)e^{-iy\lceil tq\rceil/q}q(e^{iy/q} - 1)| + |h_M(t)e^{-iyt}(iy)|)dt \leq \frac{\delta}{2}.$$

We have

$$|\frac{1}{q^d}F_{n,M}(y) - \int_0^C h_M(t)e^{-iyt}(iy)dt - h_M(C)e^{-iyC}|$$

$$\leq \int_0^C |h_{n,M,d}(t)e^{-iy\lceil tq\rceil/q}q(e^{iy/q} - 1) - h(y)iye^{-iyt}|dt + |h_{n,M}(C)e^{-iy(C-\frac{1}{q})} - h_M(C)e^{-iyC}|$$

$$\leq \int_0^b (|h_{n,M,d}(t)e^{-iy\lceil tq\rceil/q}q(e^{iy/q} - 1)| + |h_M(t)e^{-iyt}(iy)|)dt$$

$$+ \int_b^C |h_{n,M,d}(t)e^{-iy\lceil tq\rceil/q}q(e^{iy/q} - 1) - h(y)iye^{-iyt}|dt + |h_{n,M}(C)e^{-iy(C-\frac{1}{q})} - h_M(C)e^{-iyC}|.$$

Moreover for $y \in K$, there is a constant C' independent of n such that for all $t \in [b, C]$

$$|h_{n,M,d}(\lfloor tq\rfloor/q) - h_M(t)| \leq C'/q \,\text{and}\, |e^{-iy\lfloor tq\rfloor/q}q(e^{iy/q} - 1) - e^{iyt}(iy)| \leq C'/q.$$

Thus we can choose N_0 such that for all $n \geq N_0$ and $y \in K$,

$$|\frac{1}{q^d}F_{n,M}(y) - \int_0^C h_M(t)e^{-iyt}(iy)dt - h_M(C)e^{-iyC}| \leq \delta.$$

This proves the desired uniform convergence.

(2)We prove uniform convergence of $F_{n,M}/q^d$ to the integral on every compact subset of Ω; the holomorphicity of F_M is then a consequence of [Ahl79, Theorem1, Chap 5]. We have

$$
\begin{aligned}
F_{n,M}(y) &= \sum_{j=0}^{\infty} f_{n,M}(j/q)e^{-iyj/q} \\
&= \sum_{j=0}^{\infty}(h_{n,M}((j+1)/q) - h_{n,M}(j/q))e^{-iyj/q} \\
&= \sum_{j=0}^{\infty} h_{n,M}(j/q)(e^{-iy(j-1)/q} - e^{-iy(j)/q}) \\
&= \sum_{j=0}^{\infty} h_{n,M}(j/q)e^{-iyj/q}(e^{iy/q} - 1) \\
&= \int_0^{\infty} h_{n,M}(t)e^{-iy\lceil tq\rceil/q}q(e^{iy/q} - 1)dt \ .
\end{aligned}
$$

The rearrangements leading to the second and third equality are possible thanks to the absolute convergences implied by Theorem 3.16. Fix any compact $K \subseteq \Omega$. Using triangle inequality, we get

$$
\begin{aligned}
&|h_{n,d}(t)e^{-iy\frac{\lfloor tq\rfloor}{q}}q(e^{iy/q}-1) - h(t)e^{-iyt}(iy)| \\
\leq\ & |h_{n,d}(t) - h(t)||e^{-iy\frac{\lceil tq\rceil}{q}}q(e^{iy/q}-1)| + |h(t)||e^{-iy\frac{\lceil tq\rceil}{q}} - e^{-iyt}||q(e^{iy/q}-1)| \\
& +|h(t)||e^{-iyt}||q(e^{iy/q}-1) - iy| \\
=\ & |h_{n,d}(t) - h(t)||e^{-iy\frac{\lceil tq\rceil}{q}}q(e^{iy/q}-1)| + |h(t)e^{-iyt}||e^{-iy(\frac{\lceil tq\rceil}{q}-t)} - 1||q(e^{iy/q}-1)| \\
& +|h(t)e^{-iyt}||q(e^{iy/q}-1) - iy| \ .
\end{aligned}
$$

It follows from the power series expansion of e^z at zero and the boundedness of K that there are constants C_1, C_2 such that for all $y \in K$, $t \in \mathbb{R}$ and $n \in \mathbb{N}$

$$
|q(e^{iy/q}-1)| \leq C_1|y|,\ |q(e^{iy/q}-1) - iy| \leq C_2\frac{|y|^2}{q},\ |e^{-iy(\frac{\lceil tq\rceil}{q}-t)} - 1| \leq C_1|y(\frac{\lceil tq\rceil}{q} - t)|.
$$

Choose $\epsilon > 0$ such that $K \subseteq \{y \in \mathbb{C} \,|\, \Im y < -\epsilon\}$. Using the comparisons above, we get for all $y \in K$, $t \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$
\begin{aligned}
&|h_{n,d}(t)e^{-iy\frac{\lceil tq\rceil}{q}}q(e^{iy/q}-1) - h(t)e^{-iyt}(iy)| \\
\leq\ & |h_{n,d}(t) - h(t)|e^{-\epsilon t}C_1|y| + |h(t)e^{-\epsilon t}|C_1^2|y|^2|\frac{\lceil tq\rceil}{q} - t| + |h(t)e^{\epsilon t}|C_2\frac{|y|^2}{q} \\
\leq\ & |h_{n,d}(t) - h(t)|e^{-\epsilon t}C_1|y| + |h(t)e^{-\epsilon t}|C_1^2\frac{|y|^2}{q} + |h(t)e^{-\epsilon t}|C_2\frac{|y|^2}{q} \ .
\end{aligned}
$$

Taking integral on $\mathbb{R}_{\geq 0}$, we get for $y \in K$ and all $n \in \mathbb{N}$

$$
\begin{aligned}
&|\frac{1}{q^d}F_{n,M}(y) - F_{M,I,J}(y)| \\
\leq\ & C_1|y|\int_0^{\infty}|h_{n,d}(t) - h(t)|e^{-\epsilon t}dt + (C_1^2 + C_2)\frac{|y|^2}{q}\int_0^{\infty}|h(t)|e^{-\epsilon t}dt \ .
\end{aligned}
$$

Thanks to Theorem 3.16, (2), we can choose a polynomial $P_2 \in \mathbb{R}[t]$ such that $|h_{n,d}(t)| \leq |P_2(t)|$ for all n and $t \in \mathbb{R}$. Since $|P_2(t)e^{-\epsilon t}|$ is integrable on $\mathbb{R}_{\geq 0}$, by dominated convergence

$$\lim_{n\to\infty} \int_0^\infty |h_{n,d}(t) - h(t)|e^{-\epsilon t} dt = 0.$$

Using this in the last inequality implies uniform convergence of $\frac{1}{q^d} F_{n,M}(y)$ to $F_{M,I,J}(y)$ on K. □

Remark 4.4. Suppose $h_M(y)$ is constant for $y \geq C$. Since for $y \in \Omega$, $h_M(C)e^{-iyC}$ converges to zero as y approaches infinity, the two descriptions of h_M in this case match on Ω. When $J_\bullet$ is both big p and p^{-1}, our argument actually produces a corresponding entire function $F_{M,I,J_\bullet}(y)$.

Definition 4.5. Let I, J be two ideals in $(R, \mathfrak{m})$ such that $I + J$ is $\mathfrak{m}$-primary. For a finitely generated R-module M, the function $F_{M,I,J}(y)$ is called the *Frobenius-Poincaré function* of (M, I, J).

We drop one or more parameters from $F_{M,I,J}$ when there is no possible source of confusion.

The next result directly follows from Proposition 3.31.

Corollary 4.6. Let M, N be two R-modules such that $\dim M = \dim N = \dim R$ and their localizations are isomorphic at all $P \in \mathrm{Assh}R$. Then $F_M(y) = F_N(y)$.

Proof. This is true because $h_M(s) = h_N(s)$. □

5. Existence of density function in the local setting

In this section, we discuss the extension of the theory of Hilbert-Kunz density function in the local setting.

Definition 5.1. Let I be an ideal and $J_\bullet$ be a family of ideals in $(R, \mathfrak{m})$ satisfying **Condition C**. For a finitely generated R-module M and $s \in \mathbb{R}$, recall

$$f_{n,M,I,J_\bullet}(s) = h_{n,M,I,J_\bullet}(s + \frac{1}{q}) - h_{n,M,I,J_\bullet}(s) = l(\frac{(I^{\lceil sq \rceil} + J_n)M}{(I^{\lceil sq \rceil + 1} + J_n)M}).$$

Whenever $((\frac{1}{p^n})^{\dim(M)-1} f_{n,M,I,J_\bullet}(s))_n$ converges, we call the limit the *density function* of $(M, I, J_\bullet)$ at s and denote the limit by $f_{M,I,J_\bullet}(s)$. Whenever $f_{M,I,J_\bullet}(s)$ exists for all $s \in \mathbb{R}$, the resulting function $f_{M,I,J_\bullet}$ is called the *density function* of $(M, I, J_\bullet)$.

We often drop one or more parameters from $f_{n,M,I,J_\bullet}(s), f_{M,I,J_\bullet}(s), f_{M,I,J_\bullet}$ whenever those are clear from the context.

In Theorem 5.8, we relate the existence of $f_{M,I,J_\bullet}(s)$ to the differentiability of $h_{M,I,J_\bullet}$ at s-whenever $h_{M,I,J_\bullet}$ exists. We show that $h_{M,I,J_\bullet}$ is always left and right differentiable everywhere on the real line. The new ingredient is our 'convexity technique'. The h-function being Lipschitz continuous is differentiable outside a set of measure zero. But our method shows that the h-function is continuously differentiable outside a countable set. Recall:

Definition 5.2. Let S be a subset of $\mathbb{R}$. We call a function $\lambda : S \to \mathbb{R}$ to be *convex* if for elements of S, $s_1 < s_2 \leq t_1 < t_2$,

$$\frac{\lambda(s_2) - \lambda(s_1)}{s_2 - s_1} \geq \frac{\lambda(t_2) - \lambda(t_1)}{t_2 - t_1}.$$

Convexity is a notion that appears naturally in mathematical analysis. For references on convex functions, see [NP06].

Let $I, J_\bullet, M$ be as above. Now we lay the groundwork for the construction of the convex function $\mathcal{H}(s, s_0)$ in Theorem 5.3. Fix μ such that I is generated by μ-many elements. Set $M_q = M/J_nM$ and S to be the polynomial ring in μ many variables over $R/\mathfrak{m}$. Given a compact interval $[a, b] \subseteq (0, \infty)$, thanks to Theorem 3.19 we can choose C such that for all $x \in [a, b]$ and $n \in \mathbb{N}$

$$\frac{I^{\lceil xq \rceil} M_q}{I^{\lceil xq \rceil + 1} M_q} = h_n(x + \frac{1}{q}) - h_n(x) \leq C q^{\dim(M)-1}.$$

Recall,

$$l(S_{\lceil xq \rceil}) = \binom{\mu + \lceil xq \rceil - 1}{\mu - 1} = 1/(\mu-1)!(\lceil xq \rceil)^{\mu-1} + O(\lceil xq \rceil^{\mu-2}).$$

Fix $s_0 \in \mathbb{R}$. Taking cues from these two estimates, for $s > s_0$ we define

$$\mathcal{H}_n(s, s_0) = \sum_{j=\lceil s_0 q \rceil}^{\lceil sq \rceil - 1} q^{\mu - \dim(M) - 1} l(I^j M_q / I^{j+1} M_q)/l(S_j) \ . \tag{5.1}$$

Theorem 5.3. *Let $I, J_\bullet$ in the local ring $(R, \mathfrak{m})$ satisfy **Condition C**, M be a finitely generated R-module of Krull dimsnion d, I be generated by a set of μ elements. Set $M_q = M/J_nM$, fix $s_0 \in \mathbb{R}_{>0}$. Consider the two situations:*

(A) *R is a domain and $M = R$.*

(B) *$J_n = J^{[q]}$ for some ideal J such that $I + J$ is $\mathfrak{m}$-primary and M is any finitely generated R-module.*

Set $c(s) = \frac{s^{\mu-1}}{(\mu-1)!}$, where μ is such that I admits a set of generators consisting of μ elements. In the context of (A) or (B)[3]*, set*

$$\mathcal{H}(s, s_0) = h_{M,I,J_\bullet}(s)/c(s) - h_{M,I,J_\bullet}(s_0)/c(s_0) + \int_{s_0}^{s} h_{M,I,J_\bullet}(t) c'(t)/c^2(t) dt.$$

(1) *On any compact subset of (s_0, ∞), $\mathcal{H}_n(s, s_0)$ uniformly converges to $\mathcal{H}(s, s_0)$.*

(2) *The function $\mathcal{H}(s, s_0)$ is a convex function on (s_0, ∞).*

Proof. (1) Let $\mathcal{H}_n(s, s_0)$ be as in Equation (5.1). We have

$$\begin{aligned}
\mathcal{H}_n(s, s_0) &= \sum_{j=\lceil s_0 q \rceil}^{\lceil sq \rceil - 1} q^{\mu-d-1} l(I^j M_q / I^{j+1} M_q)/l(S_j) \\
&= \sum_{j=\lceil s_0 q \rceil}^{\lceil sq \rceil - 1} q^{\mu-d-1} (l(M_q/I^{j+1}M_q) - l(M_q/I^j M_q))/l(S_j) \\
&= q^{\mu-d-1} l(M_q/I^{\lceil sq \rceil} M_q)/l(S_{\lceil sq \rceil - 1}) - q^{\mu-d-1} l(M_q/I^{\lceil s_0 q \rceil} M_q)/l(S_{\lceil s_0 q \rceil}) \\
&\quad + \sum_{j=\lceil s_0 q \rceil + 1}^{\lceil sq \rceil - 1} q^{\mu-d-1} l(M_q/I^j M_q)(1/l(S_{j-1}) - 1/l(S_j)) \ .
\end{aligned}$$

Since we are in the context of (A) or (B), $q^{\mu-d-1} l(M_q/I^{\lceil sq \rceil} M_q)/l(S_{\lceil sq \rceil - 1})$ converges to $h(s)/c(s)$ and $q^{\mu-d-1} l(M_q/I^{\lceil s_0 q \rceil} M_q)/l(S_{\lceil s_0 q \rceil})$ converges to $h(s_0)/c(s_0)$. Also,

[3] $h_{M,I,J_\bullet}$ exists in the context of (A) or (B)

$$\sum_{j=\lceil s_0q\rceil+1}^{\lceil sq\rceil-1} q^{\mu-d-1}l(M_q/I^jM_q)(1/l(S_{j-1})-1/l(S_j))$$
$$=\int_{s_0}^{s-1/q}\frac{l(M_q/I^{\lceil tq\rceil}M_q)}{q^d}\left(\frac{1}{l(S_{\lceil tq\rceil-1})}-\frac{1}{l(S_{\lceil tq\rceil})}\right)(q^\mu)dt\ .$$

When q approaches infinity, $\frac{l(M_q/I^{\lceil tq\rceil}M_q)}{q^d}$ converges to $h_M(t)$, and $(\frac{1}{l(S_{\lceil tq\rceil-1})}-\frac{1}{l(S_{\lceil tq\rceil})})(q^\mu)$ converges to $c'(t)/c^2(t)$. Also, all these convergence are uniform on any compact subset of $(0,\infty)$. So we get a uniform convergence (uniform on s) on any compact subset of (s_0,∞):

$$\int_{s_0}^{s-1/q}\frac{l(M_q/I^{\lfloor tq\rfloor}M_q)}{q^d}\left(\frac{1}{l(S_{\lfloor tq\rfloor-1})}-\frac{1}{l(S_{\lfloor tq\rfloor})}\right)(q^\mu)dt$$
$$\to\int_{s_0}^{s}h(t)c'(t)/c^2(t)dt.$$

This proves that $\mathcal{H}_n(s,s_0)$ converges to $\mathcal{H}(s,s_0)$ and the convergence is uniform on any compact subset of (s_0,∞).

(2) We claim $\mathcal{H}_n$ is convex on $1/p^n\mathbb{Z}\cap(s_0,\infty)$. To this end, it suffices to show

$$\mathcal{H}_n(\frac{i+1}{p^n},s_0)-\mathcal{H}_n(\frac{i}{p^n},s_0)\geq\mathcal{H}_n(\frac{i+2}{p^n},s_0)-\mathcal{H}_n(\frac{i+1}{p^n},s_0).$$

By definition, this is equivalent to showing

$$l(I^iM_q/I^{i+1}M_q)/l(S_i)\geq l(I^{i+1}M_q/I^{i+2}M_q)/l(S_{i+1}),$$

which follows from Lemma 3.17. This convexity of $\mathcal{H}_n(s,s_0)$ implies the convexity of the limit function $\mathcal{H}(s,s_0)$ on $(s_0,\infty)\cap\mathbb{Z}[1/p]$. Therefore for $s_1<s_2\leq t_1<t_2$ in $(s_0,\infty)\cap\mathbb{Z}[1/p]$,

$$\frac{H(s_2,s_0)-H(s_1,s_0)}{s_2-s_1}\geq\frac{H(t_2,s_0)-H(t_1,s_0)}{t_2-t_1}.$$

Since $\mathcal{H}(s,s_0)$ is continuous on (s_0,∞), $(s,t)\to H(t,s_0)-H(s,s_0)/(t-s)$ is continuous. Moreover as $\mathbb{Z}[1/p]\cap(s_0,\infty)$ is dense in (s_0,∞), the slope inequality defining a convex function (see Definition 5.2) holds for $\mathcal{H}(s,s_0)$ for points in (s_0,∞). $\square$

Theorem 5.4. *With notations set in the statement of Theorem 5.3, set $\mathcal{H}(s)=\mathcal{H}(s,s_0)$. Denote the left and right derivative of a function λ at $s\in\mathbb{R}$ by $\lambda'_-(s)$ and $\lambda'_+(s)$ respectively. In the context of situation (A) or (B) stated in Theorem 5.3,*

(1) *Outside a countable subset of (s_0,∞), the derivative of $\mathcal{H}$ exists and is also continuous. The left and right derivative of $\mathcal{H}$ exists everywhere on (s_0,∞). The second derivative of $\mathcal{H}$ exists almost everywhere, i.e. outside a subset of Lebesgue measure zero of (s_0,∞).*
(2) *The left and right derivatives of $\mathcal{H}$ are both decreasing in terms of s. We have $\mathcal{H}'_+(s)\leq\mathcal{H}'_-(s)$, and if $s_1<s_2$, $\mathcal{H}'_-(s_2)\leq\mathcal{H}'_+(s_1)$.*
(3) *Outside a countable subset of $(0,\infty)$, the derivative of h exists and is also continuous. The left and right derivative of h exists everywhere on $(0,\infty)$. The second derivative of h exists almost everywhere on $(0,\infty)$.*
(4) *On (s_0,∞), $h'_+(s)=\mathcal{H}'_+(s)c(s)$, $h'_-(s)=\mathcal{H}'_-(s)c(s)$ exists, and $h'_+(s)\leq h'_-(s)$ for any $s\in(0,\infty)$.*

Proof. Thanks to the convexity of H proven in Theorem 5.3, (2), outside a countable subset Λ of (s_0, ∞), $\mathcal{H}$ is differentiable. On $(s_0, \infty) \setminus \Lambda$, the derivative of $\mathcal{H}$ is decreasing as $\mathcal{H}$ is convex. Now a decreasing function defined on a subset of $\mathbb{R}$ can have only countably many points of discontinuity. So there is a countable subset of (s_0, ∞) outside which h is continuously differentiable.

(2) follows from properties of convex functions.

(3), (4): Recall

$$\mathcal{H}(s, s_0) = h_{M,I,J_\bullet}(s)/c(s) - h_{M,I,J_\bullet}(s_0)/c(s_0) + \int_{s_0}^{s} h_{M,I,J_\bullet}(t)c'(t)/c^2(t)dt.$$

Since in the context of (A) and (B) $h_{M,I,J_\bullet}$ is continuous on $(0, \infty)$, the part of $\mathcal{H}(s, s_0)$ given by the integral is always differentiable. So (3) follows from the analogous properties of $\mathcal{H}(s, s_0)$ in (1) by varying s_0. The formulas in (4) follow from a direct computation. That $h'_+(s) \leq h'_-(s)$ follows from these formulas and (2). □

Remark 5.5. Trivedi asks when the Hilbert-Kunz density function of a graded pair (R, J) is $\dim(R) - 2$ times continuously differentiable; see [Tri23, Question 1]. In general the Hilbert-Kunz density function need not be $\dim(R) - 2$ times continuously differentiable; see [Muk23, Example 8.3.2]. Our work shows that the Hilbert-Kunz density function is always differentiable outside a set of measure zero. Indeed, a convex function on an interval is twice differentiable outside a set of measure zero; see [NP06, Section 1.4]. Thus from Theorem 5.3, it follows that outside a set of measure zero the h function is twice differentiable. Now from Theorem 6.7, we conclude that the Hilbert-Kunz density function of a graded domain of dimension at least two is differentiable outside a set of measure zero.

Remark 5.6. The conclusions of Theorem 5.3 and Theorem 5.4 are deduced in the context of situation (A) or (B), because we prove existence and continuity of $h_{M,I,J_\bullet}$ in those two contexts. So even outside the context of (A) or (B) whenever there is an h-function continuous on $(0, \infty)$, we have a corresponding version of Theorem 5.3 and Theorem 5.4.

We return to the question of existence of $f_{M,I,J_\bullet}(s)$ at a given $s \in \mathbb{R}$. We make comparisons between the limsup and and liminf of the sequence defining $f_{M,I,J_\bullet}(s)$ and the corresponding $h'_+(s)$ and $h'_-(s)$.

Lemma 5.7. *With the notation set in Theorem 5.3, set*

$$D_{n,t} = f_{n,M,I,J_\bullet}(t/p^n) = h_{n,M,I,J_\bullet}((t+1)/p^n) - h_{n,M,I,J_\bullet}(t/p^n).$$

In the context of situation (A) or (B),

(1)

$$h'_+(s) = \lim_{m \to \infty} \lim_{n \to \infty} \frac{\sum\limits_{t=\lceil sp^m p^n \rceil}^{\lceil sp^m p^n \rceil + p^n - 1} D_{m+n,t}}{p^{m(d-1)} p^{nd}}.$$

(2)

$$h'_-(s) = \lim_{m \to \infty} \lim_{n \to \infty} \frac{\sum\limits_{t=\lceil sp^m p^n \rceil - p^n}^{\lceil sp^m p^n \rceil - 1} D_{m+n,t}}{p^{m(d-1)} p^{nd}}.$$

Proof. (1) Note

$$\begin{aligned}\sum_{t=\lceil sp^m p^n\rceil}^{\lceil sp^m p^n\rceil+p^n-1} & D_{m+n,t} \\ &= \sum_{t=\lceil sp^m p^n\rceil}^{\lceil sp^m p^n\rceil+p^n-1} f_{m+n,M}(t/p^m p^n) \\ &= h_{m+n,M}(\lceil sp^m p^n\rceil/p^m p^n + 1/p^m) - h_{m+n,M}(\lceil sp^m p^n\rceil/p^m p^n)\ .\end{aligned}$$

Since in the context of (A) or (B), the h-function exists, the right hand side of the desired equation in (1) is

$$\begin{aligned}\lim_{m\to\infty}\lim_{n\to\infty} & \frac{h_{m+n,M}(\lceil sp^m p^n\rceil/p^m p^n + 1/p^m) - h_{m+n,M}(\lceil sp^m p^n\rceil/p^m p^n)}{p^{m(d-1)}p^{nd}} \\ &= \lim_{m\to\infty}\frac{h_M(s+1/p^m)-h_M(s)}{1/p^m} \\ &= h'_+(s)\ .\end{aligned}$$

(2) Note

$$\begin{aligned}\sum_{t=\lceil sp^m p^n\rceil-p^n}^{\lceil sp^m p^n\rceil-1} & D_{m+n,t} \\ &= \sum_{t=\lceil sp^m p^n\rceil-p^n}^{\lceil sp^m p^n\rceil-1} f_{m+n,M}(t/p^m p^n) \\ &= h_{m+n,M}(\lceil sp^m p^n\rceil/p^m p^n) - h_{m+n,M}(\lceil sp^m p^n\rceil/p^m p^n - 1/p^m)\end{aligned}$$

Thus the right hand side of the desired equation in (1) is

$$\begin{aligned}\lim_{m\to\infty}\lim_{n\to\infty} & \frac{h_{m+n,M}(\lceil sp^m p^n\rceil/p^m p^n) - h_{m+n,M}(\lceil sp^m p^n\rceil/p^m p^n - 1/p^m)}{p^{m(d-1)}p^{n}} \\ &= \lim_{m\to\infty}\frac{h_M(s)-h_M(s-1/p^m)}{1/p^m} \\ &= h'_-(s)\ .\end{aligned}$$

□

Theorem 5.8. *With the same notation as in Theorem 5.3, in the context of situation (A) or (B),*

1. *for any $s>0$,*
$$h'_+(s) \le \underline{\lim}_{n\to\infty} f_{n,M,I,J_\bullet}(s)/p^{n(d-1)} \le \overline{\lim}_{n\to\infty} f_{n,M,I,J_\bullet}(s)/p^{n(d-1)} \le h'_-(s),$$
where $\underline{\lim}$ and $\overline{\lim}$ denote liminf and limsup respectively.
2. *At $s>0$, if h_M is differentiable, then $f_{M,I,J_\bullet}(s)$- the density function of $(M,I,J_\bullet)$ at s exists and is equal to $h'_{M,I,J_\bullet}(s)$. If $h_M(s)$ is a C^1-function, then $f_M(s)$ is continuous.*
3. *There is a countable subset of $(0,\infty)$ outside which $f_{M,I,J_\bullet}(s)$ exists and is equal to $h'_{M,I,J_\bullet}(s)$.*

Proof. (1) In the proof, we also use the notation set in Lemma 5.7, (1). Set

$$\alpha_{\mu,t} = \binom{\mu+t-1}{\mu-1}.$$

Note $D_{n,t} = l((I^t + J_n)M/(I^{t+1} + J_n)M)$. For a fixed n, $D_{n,t}/\alpha_{\mu,t}$ is a decreasing function of t, thanks to Lemma 3.17. So for $\lceil sp^m p^n \rceil \le t \le \lceil sp^m p^n \rceil + p^n - 1$, $D_{m+n,t}/\alpha_{\mu,t} \le D_{m+n,\lceil sp^m p^n \rceil}/\alpha_{\mu,\lceil sp^m p^n \rceil}$, so

$$\begin{aligned} D_{m+n,t} &\le D_{m+n,\lceil sp^m p^n \rceil} \frac{\alpha_{\mu,t}}{\alpha_{\mu,\lceil sp^m p^n \rceil}} \\ &\le D_{m+n,\lceil sp^m p^n \rceil} \frac{\alpha_{\mu,\lceil sp^m p^n \rceil + p^n}}{\alpha_{\mu,\lceil sp^m p^n \rceil}} \ . \end{aligned}$$

Also $\alpha_{\mu,t}$ is a polynomial of degree $\mu - 1$ in t, so

$$\begin{aligned} \lim_{m\to\infty} \lim_{n\to\infty} \frac{\alpha_{\mu,\lceil sp^m p^n \rceil + p^n}}{\alpha_{\mu,\lceil sp^m p^n \rceil}} &= \lim_{m\to\infty} \lim_{n\to\infty} \frac{(\lceil sp^m p^n \rceil + p^n)^{\mu-1}}{\lceil sp^m p^n \rceil^{\mu-1}} \\ &= \lim_{m\to\infty} \frac{(sp^m + 1)^{\mu-1}}{(sp^m)^{\mu-1}} \\ &= 1. \end{aligned}$$

So

$$\begin{aligned} h'_+(s) &= \lim_{m\to\infty} \lim_{n\to\infty} \frac{\sum_{t=\lceil sp^m p^n \rceil}^{\lceil sp^m p^n \rceil + p^n - 1} D_{m+n,t}}{p^{m(d-1)} p^{nd}} \\ &\le \underline{\lim}_{m\to\infty} \underline{\lim}_{n\to\infty} \frac{p^n D_{m+n,\lceil sp^m p^n \rceil}}{p^{m(d-1)} p^{nd}} \frac{\alpha_{\mu,\lceil sp^m p^n \rceil + p^n}}{\alpha_{\mu,\lceil sp^m p^n \rceil}} \\ &= \underline{\lim}_{m\to\infty} \underline{\lim}_{n\to\infty} \frac{p^n D_{m+n,\lceil sp^m p^n \rceil}}{p^{m(d-1)} p^{nd}} \\ &= \underline{\lim}_{m\to\infty} \underline{\lim}_{n\to\infty} \frac{D_{m+n,\lceil sp^m p^n \rceil}}{p^{m(d-1)} p^{n(d-1)}} \ . \end{aligned}$$

For a sequence of real numbers β_n and any m, $\underline{\lim}_{n\to\infty} \beta_{m+n} = \underline{\lim}_{n\to\infty} \beta_n$ is independent of m, so $\underline{\lim}_{m\to\infty} \underline{\lim}_{n\to\infty} \frac{D_{m+n,\lceil sp^m p^n \rceil}}{p^{m(d-1)} p^{n(d-1)}} = \underline{\lim}_{n\to\infty} \frac{D_{n,\lceil sp^n \rceil}}{p^{n(d-1)}}$. Therefore we have

$$h'_+(s) \le \underline{\lim}_{n\to\infty} \frac{D_{n,\lceil sp^n \rceil}}{p^{n(d-1)}} = \underline{\lim}_{n\to\infty} \frac{f_n(s)}{p^{n(d-1)}}.$$

The proof of the last inequality is similar. First we have If $\lceil sp^m p^n \rceil - p^n \le t \le \lceil sp^m p^n \rceil - 1$, then $D_{m+n,t}/\alpha_{\mu,t} \ge D_{m+n,\lceil sp^m p^n \rceil}/\alpha_{\mu,\lceil sp^m p^n \rceil}$, so

$$\begin{aligned} D_{m+n,t} &\ge D_{m+n,\lceil sp^m p^n \rceil} \frac{\alpha_{\mu,t}}{\alpha_{\mu,\lceil sp^m p^n \rceil}} \\ &\ge D_{m+n,\lceil sp^m p^n \rceil} \frac{\alpha_{\mu,\lceil sp^m p^n \rceil - p^n}}{\alpha_{\mu,\lceil sp^m p^n \rceil}} \ . \end{aligned}$$

Also $\alpha_{\mu,t}$ is a polynomial of degree $\mu - 1$ in t, so

$$\begin{aligned} \lim_{m\to\infty} \lim_{n\to\infty} \frac{\alpha_{\mu,\lceil sp^m p^n \rceil - p^n}}{\alpha_{\mu,\lceil sp^m p^n \rceil}} &= \lim_{m\to\infty} \lim_{n\to\infty} \frac{(\lceil sp^m p^n \rceil - p^n)^{\mu-1}}{\lceil sp^m p^n \rceil^{\mu-1}} \\ &= \lim_{m\to\infty} \frac{(sp^m - 1)^{\mu-1}}{(sp^m)^{\mu-1}} \\ &= 1. \end{aligned}$$

So

$$\begin{aligned} h'_-(s) &= \lim_{m\to\infty}\lim_{n\to\infty}\frac{\sum\limits_{t=\lceil sp^mp^n\rceil-p^n}^{\lceil sp^mp^n\rceil-1} D_{m+n,t}}{p^{m(d-1)}p^{nd}} \\ &\geq \overline{\lim}_{m\to\infty}\overline{\lim}_{n\to\infty}\frac{p^nD_{m+n,\lceil sp^mp^n\rceil}}{p^{m(d-1)}p^{nd}}\frac{\alpha_{\mu,\lceil sp^mp^n\rceil-p^n}}{\alpha_{\mu,\lceil sp^mp^n\rceil}} \\ &= \overline{\lim}_{m\to\infty}\overline{\lim}_{n\to\infty}\frac{p^nD_{m+n,\lceil sp^mp^n\rceil}}{p^{m(d-1)}p^{nd}} \\ &= \overline{\lim}_{m\to\infty}\overline{\lim}_{n\to\infty}\frac{D_{m+n,\lceil sp^mp^n\rceil}}{p^{m(d-1)}p^{n(d-1)}}\ . \end{aligned}$$

For a sequence of real numbers β_n and any m, $\overline{\lim}_{n\to\infty}\beta_{m+n}=\overline{\lim}_{n\to\infty}\beta_n$ is independent of m, so $\overline{\lim}_{m\to\infty}\overline{\lim}_{n\to\infty}\frac{D_{m+n,\lceil sp^mp^n\rceil}}{p^{m(d-1)}p^{n(d-1)}}=\overline{\lim}_{n\to\infty}\frac{D_{n,\lceil sp^n\rceil}}{p^{n(d-1)}}$. Therefore we have

$$h'_-(s)\geq\overline{\lim}_{n\to\infty}\frac{D_{n,\lceil sp^n\rceil}}{p^{n(d-1)}}=\overline{\lim}_{n\to\infty}\frac{f_n(s)}{p^{n(d-1)}}.$$

(2) If h_M is differentiable at s, $h'_+(s)=h'_-(s)$. Thus (1) implies that $f_{n,M}(s)/q^{d-1}$ exists and is equal to $h'(s)$, rest of (2) is clear.

(3) follows from Theorem 5.4, (3). □

Remark 5.9. We prove Theorem 5.8 in the context of situation (A) or (B) defined in Theorem 5.3- which is precisely the contexts where we prove existence of $h_{M,I,J_\bullet}$ in this article. Thus when $(R,\mathfrak{m})$ is a domain, I, $J_\bullet$ satisfy **Condition C**, we get a corresponding density function which is well-defined outside a countable subset of $(0,\infty)$. One particular special case, potentially important for its application to prime characteristic singularity theory, is when $J_\bullet$ is the ideal sequence that defines the F-signature of $(R,\mathfrak{m})$; see Example 3.10.

When $J_n=J^{[q]}$, Theorem 5.8 yields a Hilbert-Kunz density function of (I,J) well defined outside a countable subset of $(0,\infty)$.

The function $h_{M,I,J_\bullet}$ need not be continuous or differentiable at zero. In Theorem 8.12, we prove that for a local domain R, $h_{R,I,J}$ is continuous at zero if and only if $\dim R-\dim R/I\geq 1$ and differentiable at zero if and only if $\dim R-\dim R/I\geq 2$.

The following consequence of Theorem 5.4 and Theorem 5.8 is used in Theorem 8.15. For the notion of integral closure and analytic spread appearing below, we refer to [HS06].

Proposition 5.10. *Let $(R,\mathfrak{m})$ be a noetherian local domain, $I,J_\bullet$ satisfy **Condition C**. Let r be an integer greater than the analytic spread of I. Denote the right and left hand derivatives of $h_{I,J_\bullet}$ at $s>0$ by $h'_+(s)$ and $h'_-(s)$ respectively. Then,*

1. *Both the functions $\frac{h'_+(s)}{s^{r-1}}$ and $\frac{h'_-(s)}{s^{r-1}}$ are decreasing on $(0,\infty)$.*
2. *Let $f_{I,J_\bullet}$ be the corresponding density function. For positive real numbers $s_1<s_2$ such that $f_{I,J_\bullet}(s_1),f_{I,J_\bullet}(s_2)$ exist, we have*

$$\frac{f_{I,J_\bullet}(s_2)}{s_2^{r-1}}\leq\frac{f_{I,J_\bullet}(s_1)}{s_1^{r-1}}.$$

Proof. Once we prove either (1) or (2) for a certain r, the corresponding assertion follows for a larger value of r follows immediately. So we assume r is the analytic spread of I. Without loss of generality we can assume that the residue field of R is infinite. So there exists r elements $x_1,x_2,\dots,x_r$ of I such that the integral closure of $(x_1,x_2,\dots,x_r)$ is I.

We prove (1) now. By Proposition 3.32, we can assume I is generated by r elements. By Theorem 5.3, assertion (2) and Theorem 5.4, assertion (4), both $\frac{h'_+(s)}{s^{r-1}}$ and $\frac{h'_-(s)}{s^{r-1}}$ are right and left hand derivatives of a convex function. Since the right and left hand derivatives of a convex function are always decreasing, we prove (1).

Assertion (2) follows from the following comparisons:

$$\frac{f_{I,J_\bullet}(s_2)}{s_2^{r-1}} \leq \frac{h'_-(s_2)}{s_2^{r-1}} \leq \frac{h'_+(s_1)}{s_1^{r-1}} \leq \frac{f_{I,J_\bullet}(s_1)}{s_1^{r-1}};$$

see Theorem 5.8, assertion (1). □

Example 5.11. We point out that the h-function need not be differentiable on $(0, \infty)$. Our example of a non differentiable h-function comes from [BST13]. Fix a regular local domain $(R, \mathfrak{m})$ of dimension d and a nonzero $f \in R$. For $t \in \mathbb{R}$, [BST13] considers the function $t \to s(R, f^t)$: the F-signature of the pair (R, f^t) which is shown to be the same as

$$s(R, f^t) = \lim_{n\to\infty} \frac{1}{q^d} l\left(\frac{R}{\mathfrak{m}^{[p^n]} : f^{\lceil tp^n \rceil}}\right).$$

With $I = (f)$, $h_{R,I,\mathfrak{m}}(t) = 1 - s(R, f^t)$; see [BST13, section 4]. At $t = 1$, the left hand derivative of h_I is the F-signature of R/f; see [BST13, Theorem 4.6], while the right hand derivative is zero since $h(s) = 1$ for $s \geq 1$. So h is not differentiable at one if and only if the F-signature of R/f is nonzero, precisely when R/f is strongly F-regular. A concrete example comes from the strongly F-regular ring, $\mathbb{F}_p[[x, y, z]]/(x^2 + y^2 + z^2)$ with $p \geq 3$.

Example 5.12. We point out that the limit defining the density function at a particular $s \in \mathbb{R}$, i.e. of $f_{n,M,I,J}(s)/q^{\dim(M)-1}$ may not converge. For example, when $I = 0$, $M = R$, then $f_{n,M,I,J}(0) = l(R/J^{[q]})$; thus $f_{n,M,I,J}(0)/q^{\dim R} = e_{HK}(J, R)$ is a nonzero real number, so $f_{n,M,I,J}(0)/q^{\dim R - 1}$ goes to infinity. This example implies that some assumption is necessary to guarantee the existence of the density function at every point.

Example 5.13. In the definition of the density function if we replace $\lceil sq \rceil$ by $\lfloor sq \rfloor$, then we have more examples where the density function does not exist. We recall Otha's example mentioned in [Kos17, sec 3] which produces such instances. Let R be the power series ring $k[[x_1, \ldots, x_{d+1}]]$, $\alpha_1 \leq \ldots \leq \alpha_{d+1}$ be a sequence of positive integers, $I = (x_1^{\alpha_1} \ldots x_{d+1}^{\alpha_{d+1}})$ be a monomial principal ideal, $J = (x_1, \ldots, x_{d+1})$ be the maximal ideal of R. Assume moreover that $\alpha_d < \alpha_{d+1}$, α_{d+1} does not divide p,and $\epsilon_n \in [0, \alpha_{d+1} - 1]$ is the residue of p^n modulo α_{d+1}. Let $\tilde{f}$ be the density function defined using $\lfloor sq \rfloor$, then $\lim_{n\to\infty} \tilde{f}_{n,R,I,J}(\frac{1}{\alpha_{d+1}})/(p^{nd}\epsilon_n)$ exists and is nonzero. So $\lim_{n\to\infty} \tilde{f}_{n,R,I,J}(\frac{1}{\alpha_{d+1}})/p^{nd}$ exists if and only if ϵ_n is a constant sequence- this is false in general. In general, ϵ_n is a periodic function and its period is the order of $p + \alpha_{n+1}\mathbb{Z}$ in the multiplicative group $(\mathbb{Z}/\alpha_{n+1}\mathbb{Z})^*$.

Example 5.14. We give an example, where the density function exists everywhere although the h-function is not differentiable everywhere. Note that the resulting density function is not continuous in this case; compare with Theorem 6.4. Let $M = R = k[[x]]$ be the power seires ring, $I = J = (x)$. Then $h_n(s) = l(R/I^{\lceil sq \rceil} + J^{[q]}) = min\{\lceil sq \rceil, q\}$. By simple calculation we get $f_n(s) = 1$ when $-1/q < s \leq 1 - 1/q$ and is 0 otherwise. So $f(s) = 1$ when $0 \leq s < 1$ and $f(s) = 0$ otherwise.

Here f_n converges pointwise but not uniformly. Outside an arbitrary neighborhood of 0 and 1 then f_n converges uniformly.

On the other hand, $h(s)$ is 0 when $s \leq 0$, s when $0 \leq s \leq 1$, 1 when $s \geq 1$, and is continuous. We have $f(s) = h'(s)$ when $s \neq 0, 1$; when $s = 0, 1$ $h'(s)$ does not exist and $f(s) = h'_+(s)$. This leads us to guessing that whenever the density function exists at s, it coincides with the right hand derivative $h'_+(s)$.

Remark 5.15. Assume $J_\bullet$ is big and $h_{M,I,J_\bullet}$ is differentiable everywhere. Since $h_{M,I,J_\bullet}$ is eventually constant (Lemma 3.8), the resulting density function $f_{M,I,J_\bullet} = h'_{M,I,J_\bullet}$ is supported on some compact interval $[0,b]$. So the density function has to increase and decrease on $[0,b]$. By Theorem 5.4, $f_{M,I,J_\bullet} = h'(s) = \mathcal{H}'(s)s^{\mu-1}/(\mu-1)!$, where $\mathcal{H}'$ is decreasing since $\mathcal{H}$ is convex; so this gives a natural way to represent $f_{M,I,J_\bullet}$ as a product of a decreasing and an explicit increasing function, namely $c(s)$. This may help analyzing the monotonicity of the density function.

6. Relation among h, density, and Frobenius-Poincaré functions

In Section 4 we developed a notion of Frobenius-Poincaré function in the local setting. Work of Section 5 gives a notion of Hilbert-Kunz density function in the local setting, at least outside a countable subset of $(0,\infty)$. When $(R,\mathfrak{m})$ is graded, we compare these local notions defined using the $\mathfrak{m}$-adic filtration with the classical notion of Frobenius-Poincaré function and Hilbert-Kunz density function defined (see Section 2) using the graded structure of the underlying objects.

Lemma 6.1. *Let $(R,\mathfrak{m})$ be a standard graded ring, M be a finitely generated $\mathbb{Z}$-graded module of dimension d, J be a homogeneous ideal of finite colength. Set*

$$g_{n,M,J,d-1}(s) = \frac{1}{q^{d-1}} l\Big(\frac{M}{J^{[q]}M}\Big)_{\lceil sq\rceil},\ g_{n,M,J}(s) = l\Big(\frac{M}{J^{[q]}M}\Big)_{\lceil sq\rceil}\ .$$

(1) *When M is generated in degree zero, for any graded submodule $N \subseteq M$, $(M/N)_j = \mathfrak{m}^j(M/N)/\mathfrak{m}^{j+1}(M/N)$.*
(2) *When M is generated in degree zero, $g_{n,M,J}(s) = l(\frac{M}{J^{[q]}M})_{\lceil sq\rceil} = f_{n,M,\mathfrak{m},J}(s)$.*

Proof. Let N be any submodule of M, then M/N is also generated in degree 0, so $(M/N)_{\geq j} = \mathfrak{m}^j(M/N)$ and $(M/N)_j = \mathfrak{m}^j(M/N)/\mathfrak{m}^{j+1}(M/N)$ for any j. This implies $g_{n,M,J}(s) = f_{n,M,\mathfrak{m},J}(s)$. $\square$

Lemma 6.2. *We define an equivalence relation $\sim$ on $\mathbb{Z}$-graded modules over a standard graded ring R of positive dimension over a field: we say $M \sim N$ when there is a homogeneous map $\phi : M \to N$ such that* $\dim \mathrm{Ker}\phi, \dim \mathrm{Coker}\phi \leq \dim R - 1$, *and let $\sim$ also denote the minimal equivalence relation generated by such relations. Then M is equivalent to some module generated in degree 0.*

Proof. Since $\dim R > 0$, we can choose an element $c \in R_1$ such that $\dim R/cR \leq \dim R$. First, we find a sufficient large $n > 0$ such that M is generated in degree at most n. Then we truncate at degree n to get $M_{\geq n} := \oplus_{j=n}^{\infty} M_j$, which is generated in degree n. The module $M/M_{\geq n}$ is Artinian. The inclusion $M_{\geq n} \hookrightarrow M$ shows $M_{\geq n} \sim M$. The map $M_{\geq n} \to M_{\geq n}[n]$ given by multiplication by c^n has its kernel and cokernel annihilated by c^n. So the kernel and cokernel have dimension less than $\dim R$. Thus $M \sim M_{\geq n} \sim M_{\geq n}[n]$. Since $M_{\geq n}[n]$ is generated in degree zero, we are done. $\square$

The next result follows directly from the lemma above and Proposition 3.31.

Lemma 6.3. *Let $(R,\mathfrak{m})$ be standard graded, M be a finitely generated $\mathbb{Z}$-graded R-module, I, $J_\bullet$ be homogeneous; assume that the corresponding objects obtained by localizing at $\mathfrak{m}$ satisfy condition (A) or (B) stated in Theorem 5.3. Then there is a finitely generated $\mathbb{N}$-graded R-module M' generated in degree zero such that, $h_{M,I,J_\bullet} = h_{M',I,J_\bullet}$.*

In the context of (A) or (B) stated in Theorem 5.3 there is an h-function and an associated density function defined outside a countable subset of $(0,\infty)$. Although the limit defining the density function may not exist at every point of $(0,\infty)$, we can define the integral of f on any bounded measurable subset Σ of $[0,\infty)$ by integrating the class in $L^1(\Sigma)$ represented by the density function. Fix the maximal subset Λ of $[0,\infty)$ where the density function $f_{M,I,J_\bullet}$ exists. The continuity of f_M at $s \in \Lambda$ refers to the notion of continuity coming from the subspace topology on the domain Λ inherited from $\mathbb{R}$. With this understanding, we have the following theorem.

Theorem 6.4. *Let $(R,\mathfrak{m})$, I, $J_\bullet$, M be as in Theorem 5.3. Then in the context of situation (A) or (B) as stated in Theorem 5.3, we have for any $s > 0$,*

$$h_{M,I,J_\bullet}(s) - \lim_{s_0 \to 0+} h_{M,I,J_\bullet}(s_0) = \int_0^s f_{M,I,J_\bullet}(t)dt.$$

Moreover if the density $f_{M,I,J_\bullet}$ exists and is continuous at $s > 0$, then $h_{M,I,J_\bullet}$ is differentiable at s and $f_M(s) = h'_M(s)$.

Proof. Given $s > 0$, choose $[a,b] \subseteq \mathbb{R}_{>0}$ containing s. For a fixed s_0 in $[a,b]$ and $s > s_0$, we have

$$h_n(s) - h_n(s_0) = \sum_{j=\lceil s_0 q\rceil}^{\lceil sq\rceil - 1} f_n(\frac{j}{q}) \ .$$

Thus

$$\frac{1}{q^d}h_n(s) - \frac{1}{q^d}h_n(s_0) = \int_{s_0-\frac{1}{q}}^{s-\frac{1}{q}} \frac{f_n(t)}{q^{d-1}}dt$$

By Theorem 3.19, we can choose a constant C such that for any $n \in \mathbb{N}$ and $t \in [a,b]$.

$$\frac{1}{q^{d-1}}f_n(t) \leq C.$$

Thus taking limit as n approaches infinity and using dominated convergence, we get

$$h_{M,I,J_\bullet}(s) - h_{M,I,J_\bullet}(s_0) = \int_{s_0}^s f_{M,I,J_\bullet}(t)dt.$$

Taking limit as $s_0 \to 0+$ we get the conclusion involving integrals. Note that $\lim_{s_0 \to 0+}$ exists as h is increasing.

Whenever $f_M(t)$ exists at s and is continuous at s, the differentiability of h_M at s and that $h'_M(s) = f_M(s)$ follows from the second fundamental theorem of Calculus. ☐

Proposition 6.5. *Continue with the same notation as in Lemma 6.1 but M not necessarily generated in degree zero. Set*

$$\tilde{g}_{n,M,J,d-1}(s) = l(M/J^{[q]}M)_{\lfloor sq \rfloor}/q^{d-1}.$$

If additionally $d = \dim(M) \geq 2$, the two limits below exist for all $s \in \mathbb{R}$:

$$\tilde{g}_{M,J}(s) = \lim_{n\to\infty} \tilde{g}_{n,M,J,d-1}(s),\ g_{M,J}(s) = \lim_{n\to\infty} g_{n,M,J,d-1}(s).$$

Moreover $\tilde{g}_{M,J}(s) = g_{M,J}(s)$.

Proof. By [Tri18], $\tilde{g}_{n,M,J,d-1}(s)$ converges for all $s \in \mathbb{R}$. For $s \in \mathbb{Z}[1/p]$, $g_{n,M,J,d-1}(s) = \tilde{g}_{n,M,J,d-1}(s)$ for q large; so we conclude convergence of $g_{n,M,J,d-1}(s)$. When s is not in $\mathbb{Z}[1/p]$,

$$g_{n,M,J,d-1}(s) = \tilde{g}_{n,M,J,d-1}(s + \frac{1}{q}).$$

Now for $d \geq 2$, the uniform convergence of the sequence of functions $\tilde{g}_{n,M,J,d-1}$ and continuity of $\tilde{g}_{M,J}$ imply that the sequence $\tilde{g}_{n,M,J,d-1}(s+\frac{1}{q})$ converges to $\tilde{g}_{M,J}(s)$. ☐

Theorem 6.6. *Let $(R, \mathfrak{m})$ be standard graded, J be a homogeneous $\mathfrak{m}$-primary ideal, M be an R-module of dimension $d \geq 2$. Then*

(1) *$h_{M,\mathfrak{m},J}$ is differentiable on $\mathbb{R}$. The density function $f_{M,\mathfrak{m},J}(s)$ exists everywhere on $\mathbb{R}$ and is the same as $h'_{M,\mathfrak{m},J}(s)$.*
(2) *Moreover $f_{M,\mathfrak{m},J}$ is the same as Trivedi's Hilbert-Kunz density function $\tilde{g}_{M,J}(s)$; see Section 2.*

Proof. (1) It follows from [Tay18, Lemma 3.3], that for $s \leq 1$, $h_M(s) = e(\mathfrak{m}, M)s^d/d!$. So h_M is differentiable at zero and the derivative is zero. A direct computation shows that the density function at zero exists and is zero. So we can restrict to $(0, \infty)$. Thanks to Theorem 5.8, (2), it is enough to show that h_M is differentiable on $(0, \infty)$. By using Lemma 6.3, we can assume that M is generated in degree zero. Thus by Lemma 6.1

$$f_{n,M,\mathfrak{m},J}(s) = g_{n,M,J}(s) := l([\frac{M}{J^{[p^n]}M}]_{\lceil sq \rceil}) \text{ for all } s \in \mathbb{R}.$$

As $d \geq 2$, by Proposition 6.5, $g_{n,M,J}(s)/q^{d-1}$ converges to Trivedi's density function $\tilde{g}_{M,J}(s)$ for all s. Since $\tilde{g}_{M,J}(s)$ is continuous, $f_{M,\mathfrak{m},J}(s)$ is also continuous. Now by Theorem 6.4, (2), $h_{M,I,J}$ is differentiable on $(0, \infty)$.

(2) Fix an M' which is generated in degree zero and equivalent to M in the sense of Lemma 6.2. Thanks to Lemma 6.3 and part (1)

$$h_M = h_{M'}\,, f_M = f_{M'}.$$

The associativity formula for Trivedi's density function implies $\tilde{g}_{M,J} = \tilde{g}_{M',J}$; see [Tri18, Prop 2.14]. Since M' is generated in degree zero and has dimension at least two, by Lemma 6.1 and Proposition 6.5, $\tilde{g}_{M',J} = f_{M',\mathfrak{m},J}$. Putting together we conclude that $f_{M,\mathfrak{m},J} = \tilde{g}_{M,J}$. □

We further strengthen the above theorem by proving it for any homogeneous J which not necessarily has finite colength,

Theorem 6.7. *Let $(R, \mathfrak{m})$ be standard graded, J be a homogeneous ideal, $s \in \mathbb{R}$, M be a finitely generated $\mathbb{Z}$-graded module of dimension d. Assume $d \geq 2$. Set $\tilde{g}_{n,M,J,d-1}(s) = l(M/J^{[q]}M)_{\lfloor sq \rfloor}/q^{d-1}$. Then:*

(1) *The sequence $(\tilde{g}_{n,M,J,d-1}(s))_n$ converges uniformly on every compact subset of $\mathbb{R}$. The limiting function is continuous.*
(2) *$h_{M,\mathfrak{m},J}$ is differentiable and*

$$h'_{M,\mathfrak{m},J}(s) = f_{M,\mathfrak{m},J}(s) = \lim_{n \to \infty} \tilde{g}_{n,M,J,d-1}(s).$$

Proof. (1) For a positive integer N, set $J' = J + \mathfrak{m}^{N+1}$. Then on $[0, N]$, $\tilde{g}_{n,M,J,d-1} = \tilde{g}_{n,M,J',d-1}$. Since J' is $\mathfrak{m}$-primary, by [Tri18], $\tilde{g}_{n,M,J',d-1}$ converges uniformly to a continuous function. Thus on $[0, N]$, $\tilde{g}_{n,M,J,d-1}$ converges uniformly to a continuous function.

(2) Fix a compact interval $[a, b] \subseteq \mathbb{R}$. By Theorem 3.13, (1), we can choose t_0 such that for all $t \geq t_0$, $h_{M,\mathfrak{m},J} = h_{M,\mathfrak{m},J+\mathfrak{m}^t}$ on $[a, b]$. Using the ideas from the argument in part(1), fix an integer $t \geq t_0$, ensure $\tilde{g}_{n,M,J,d-1} = \tilde{g}_{n,M,J+\mathfrak{m}^t,d-1}$ on $[a, b]$ for all n. By Theorem 6.6, $h_{M,\mathfrak{m},J+\mathfrak{m}^t}$ is differentiable on $\mathbb{R}$ with derivative $\tilde{g}_{M,J+\mathfrak{m}^t}$. Thus on (a, b), $h_{M,\mathfrak{m},J}$ is differentiable with derivative being the continuous function $\tilde{g}_{M,J}$. Since by Theorem 5.8 $h'_M = f_M$ on (a, b), we are done. □

We point out below that in the graded context the Frobenius-Poincaré function defined using the underlying grading and the maximal ideal adic filtration coincide. Recall that by Ω, we denote the open lower half complex plane. Let $(R, \mathfrak{m})$ be standard graded, M be an $\mathbb{N}$-graded R-module, J be a homogeneous ideal. For $y \in \Omega$,

Proposition 6.8. *Let $(R, \mathfrak{m})$ be standard graded, M a finitely generated $\mathbb{Z}$-graded R-module of dimension d, J be a homogeneous ideal. Consider the sequence of functions on the open lower half plane*

$$G_{n,M,J}(y) = \sum_{j=0}^{\infty} l([\frac{M}{J^{[q]}M}]_j)e^{-iyj/q}$$

(1) *$\frac{1}{q^d}G_{n,M,J}(y)$ defines a holomorphic function on Ω for every n.*
(2) *Recall that $F_{M,\mathfrak{m},J}$ denotes the Frobenius-Poincaré function defined in Definition 4.5. The sequence*

$$\lim_{n\to\infty} \frac{1}{q^d} G_{n,M,J}(y)$$

converges to $F_{M,\mathfrak{m},J}(y)$.
(3) *When J is $\mathfrak{m}$-primary, $G_{n,M,J}(y)/q^d$ converges to $F_{M,\mathfrak{m},J}(y)$ on $\mathbb{C}$.*

Proof. Fix an $\mathbb{N}$-graded module M' generated in degree zero and equivalent to M in the sense of Lemma 6.2.

(3) Since J is $\mathfrak{m}$-primary, G_n is a sum of finitely many entire functions. So G_n is entire. Fix a compact subset K of $\mathbb{C}$. By [Muk23, Lemma 3.2.5], we can find a constant D such that

$$|\frac{1}{q^d}G_{n,M,J}(y) - \frac{1}{q^d}G_{n,M',J}(y)| \leq \frac{D}{q} \text{ for all } n \text{ and } y \in K.$$

Since M' is generated in degree zero, $F_{n,M',\mathfrak{m},J} = G_{n,M',J}$. Since $F_{n,M',\mathfrak{m},J}/q^d$ uniformly converges to $F_{M',\mathfrak{m},J}$ on K, the last inequality implies that $\frac{1}{q^d}G_{n,M,J}$ converges uniformly to $F_{M',\mathfrak{m},J}$ on K; see Theorem 4.3. Thanks to Lemma 6.3 and Theorem 4.3, $F_{M',\mathfrak{m},J} = F_{M,\mathfrak{m},J}$ on $\mathbb{C}$.

(1) There is a polynomial P of degree d with non-negative coefficients such that

$$l([\frac{M}{J^{[q]}M}]_j) \leq l(M_j) \leq P(j).$$

Fix a compact subset $K \subseteq \Omega$. Choose $\epsilon > 0$ such that $\Im y < -\epsilon$ for every $y \in K$. Since

$$\sum_{j=0}^{\infty} \frac{1}{q^d}|P(j)|e^{-j\epsilon/q}$$

is convergent, we conclude that the sequence of holomorphic functions

$$(\frac{1}{q^d}\sum_{j=0}^{N} l([\frac{M}{J^{[q]}M}]_j)e^{-iyj/q})_N$$

converges uniformly to $\frac{1}{q^d}G_{n,M,J}(y)$ on K. This proves the holomorphicity of $\frac{1}{q^d}G_{n,M,J}$ on Ω.

(2) When $d = 0$, the conclusion follows from a direct computation. Assume $d \geq 1$. Since

$$l([\frac{M}{J^{[q]}M}]_j) = l([\frac{M}{J^{[q]}M}]_{\leq j}) - l([\frac{M}{J^{[q]}M}]_{\leq j-1}),$$

a direct computation using the equation above shows that,

$$\sum_{j=0}^{\infty} l([\frac{M}{J^{[q]}M}]_j)e^{-iyj/p^n} = \sum_{j=0}^{\infty} l([\frac{M}{J^{[q]}M}]_{\leq j})e^{-iyj/p^n}(1-e^{-iy/p^n}) \ . \tag{6.1}$$

Since

$$l(\frac{(\mathfrak{m}^j + J^{[q]})M}{(\mathfrak{m}^{j+1} + J^{[q]})M}) = l([\frac{M}{(\mathfrak{m}^{j+1} + J^{[q]})M}]) - l([\frac{M}{(\mathfrak{m}^{j} + J^{[q]})M}]),$$

a direct computation shows that,

$$\sum_{j=0}^{\infty} l(\frac{(\mathfrak{m}^j + J^{[q]})M}{(\mathfrak{m}^{j+1} + J^{[q]})M})e^{-iyj/p^n} = \sum_{j=0}^{\infty} l(\frac{M}{(\mathfrak{m}^{j+1} + J^{[q]})M})e^{-iyj/p^n}(1-e^{-iy/p^n}) \tag{6.2}$$

Choose a such that as an R-module M is generated by homogeneous elements of degree at most a. Therefore

$$\mathfrak{m}^j M \subseteq M_{\geq j} \subseteq \mathfrak{m}^{j-a}M.$$

So,

$$\begin{aligned}
l(\frac{M}{(\mathfrak{m}^{j+1} + J^{[q]})M}) - l([\frac{M}{J^{[q]}M}]_{\leq j}) &= l(\frac{M_{\geq j+1} + J^{[q]}M}{\mathfrak{m}^{j+1}M + J^{[q]}M}) \\
&\leq l(\frac{\mathfrak{m}^{j+1-a}M + J^{[q]}M}{\mathfrak{m}^{j+1}M + J^{[q]}M}) \\
&\leq l(\frac{\mathfrak{m}^{j+1-a}M}{\mathfrak{m}^{j+1}M}) \\
&\leq Cj^{d-1},
\end{aligned}$$

for some C, which is independent of q and j. Using Equation (6.1), Equation (6.2) and the comparison above, we get that for any $y \in \Omega$,

$$\begin{aligned}
|\frac{1}{q^d}G_{n,M,J}(y) - \frac{1}{q^d}F_{n,\mathfrak{m},J}(y)| &\leq \sum_{j=0}^{\infty} C\frac{1}{q}(\frac{j}{q})^{d-1}e^{-\Im yj/q}|1-e^{-iy/q}| \\
&= C|1-e^{-iy/q}|\int_0^{\infty} \lfloor s \rfloor^{d-1} e^{-\Im y \lfloor s \rfloor} ds \\
&\leq C|1-e^{-iy/q}|\int_0^{\infty} s^{d-1}e^{-\Im y(s-1)}ds.
\end{aligned}$$

Since $\Im y < 0$ for $y \in \Omega$, the last integral is convergent. It follows from the last chain of inequalities that on a compact subset of Ω,

$$|\frac{1}{q^d}G_{n,M,J}(y) - \frac{1}{q^d}F_{n,\mathfrak{m},J}(y)|$$

uniformly converges to zero. This finishes the proof of (2). □

7. Arithmetic properties

In this section, we record some arithmetic properties of the function we have constructed in the previous sections.

7.1. **$\mathfrak{m}$-adic continuity.** We have proven that the h-function is continuous with respect to the $\mathfrak{m}$-adic topology on the set of ideals in R.

Theorem 7.1. *Let $t \in \mathbb{N}$, I_t, J_t be two sequences of ideals such that $I_t + J_t \subset \mathfrak{m}^t$. Then for any s, $\lim_{t\to\infty} h_{M,I+I_t,J+J_t}(s) = h_{M,I,J}(s)$. This convergence is uniform with respect to s on any compact set in $(0,\infty)$.*

Proof. If $s \leq 0$ then both sides are 0, so there is nothing to prove. Fix $0 < s_1 < s_2 < \infty$ and it suffices to prove the uniform convergence on $[s_1, s_2]$, which is true by Theorem 3.13 and Theorem 3.20. □

The Frobenius-Poincaré function also satisfies a similar property:

Proposition 7.2. *Let $t \in \mathbb{N}$, I_t, J_t be two sequences of ideals such that $I_t + J_t \subset \mathfrak{m}^t$. Then for any $y \in \Omega$: the open lower half complex plane, $\lim_{t\to\infty} F_{M,I+I_t,J+J_t}(y) = F_{M,I,J}(y)$. If J is $\mathfrak{m}$-primary, then the above holds for $y \in \mathbb{C}$. In either case, the convergence is uniform on a compact subset of Ω or $\mathbb{C}$.*

Proof. Fix a compact subset K of Ω. Choose $\epsilon > 0$ such that $\Im y < -\epsilon$ for all $y \in K$. Recall from Theorem 3.16, that there is a polynomial $P \in \mathbb{R}[t]$ such that $h_{n,M,I,J}(s) \leq P(s)$ for all $s \in \mathbb{R}$ and all n; so $h_{M,I+I_t,J+J_t}(s) \leq P(s)$ for all s. Notice $|P(s)e^{-\epsilon s}|$ is integrable on $\mathbb{R}_{\geq 0}$ and the sequence $h_{M,I+I_t,J+J_t}$ converges to $h_{M,I,J}$; the convergence is uniform on every compact subset of $(0,\infty)$; see Theorem 3.13. Say the absolute values of elements of K is bounded above by D. Given $\delta > 0$, the observations above allows us to choose an interval $[a,b] \subseteq (0,\infty)$ and $t_0 \in \mathbb{N}$ such that,

(a) $2\int_0^a |P(s)|e^{-\epsilon s} ds + 2\int_b^\infty |P(s)|e^{-\epsilon s} ds \leq \frac{\delta}{2D}$.

(b) $|h_{M,I+I_t,J+J_t}(x) - h_{M,I,J}(x)| \leq \frac{\delta}{2D\int_a^b e^{-\epsilon s}ds}$ for all $t \geq t_0$ and all $s \in [a,b]$.

Therefore by using Theorem 4.3, for $y \in K$ and all $t \geq t_0$

$$\begin{aligned}|F_{M,I+I_t,J+J_t}(y) - F_{M,I,J}(y)| &\leq \int_0^\infty |y||h_{M,I+I_t,J+J_t}(s) - h_{M,I,J}(s)|e^{-\epsilon s} ds\\ &\leq D[2\int_0^a |P(s)|e^{-\epsilon s} ds + 2\int_b^\infty |P(s)|e^{-\epsilon s} ds\\ &+ \int_a^b |h_{M,I+I_t,J+J_t}(s) - h_{M,I,J}(s)|e^{-\epsilon s} ds]\\ &\leq \delta\ .\end{aligned}$$

This proves uniform convergence of $(F_{M,I+I_t,J+J_t}(y))_t$ to $F_{M,I,J}(y)$ on every compact subset of Ω. The assertion for $\mathfrak{m}$-primary J follows from a similar argument. □

7.2. **Basic properties.** Let R be a local ring, t be an indeterminate, I, J be $\mathfrak{m}$-primary ideals, M be a finitely generated R-module.

Theorem 7.3. *[Tay18, Proposition 2.6] Assume I, J are two $\mathfrak{m}$-primary ideals. Then*
1. $\dim M < d$*, then* $h_{M,I,J,d}(s) = 0$.
2. $h_{M,I,J}$ *is increasing.*
3. $h_{M,I,J}(s) \leq e(I,M)s^d/d!$.
4. $h_{M,I,J}(s) \leq e_{HK}(J,M)$.

Theorem 7.4. *The above (1) and (2) is still true if only $I + J$ is $\mathfrak{m}$-primary. (3) remains valid when I is $\mathfrak{m}$-primary and (4) remains valid when J is $\mathfrak{m}$-primary.*

Proof. By $\mathfrak{m}$-adic continuity $\lim_{t\to\infty} h_{M,I+\mathfrak{m}^t,J+\mathfrak{m}^t}(s) = h_{M,I,J}(s)$ and $I+\mathfrak{m}^t$, $J+\mathfrak{m}^t$ are $\mathfrak{m}$-primary. We have:

(1) $\dim M < d$, then $h_{M,I+\mathfrak{m}^t,J+\mathfrak{m}^t,d}(s) = 0$. Let $t\to\infty$, $h_{M,I,J,d}(s) = 0$.
(2) For $s_1 < s_2$, $h_{M,I+\mathfrak{m}^t,J+\mathfrak{m}^t}(s_1) \le h_{M,I+\mathfrak{m}^t,J+\mathfrak{m}^t}(s_2)$. Let $t\to\infty$, $h_{M,I,J}(s_1) \le h_{M,I,J}(s_2)$.
(3) $h_{M,I,J+\mathfrak{m}^t}(s) \le e(I,M)s^d/d!$. Let $t\to\infty$, we have $h_{M,I,J}(s)\le e(I,M)s^d/d!$.
(4) $h_{M,I+\mathfrak{m}^t,J}(s) \le e_{HK}(J,M)$. Let $t\to\infty$, we have $h_{M,I,J}(s)\le e_{HK}(J,M)$.

□

Proposition 7.5 (Additivity)**.** *Let $0\to M'\to M\to M''\to 0$ be an exact sequence of modules of dimension at most d. Let I,J be ideals such that $I+J$ is $\mathfrak{m}$-primary. Recall that the Kronecker delta notation $\delta_{a,b}$ represents zero if $a\neq b$ and 1 if $a=b$.*

(1) $\mathcal{F}_{M,I,J} = \delta_{\dim(M),\dim(M')}\mathcal{F}_{M',I,J} + \delta_{\dim(M),\dim(M'')}\mathcal{F}_{M'',I,J}$ *for* $\mathcal{F}=h,F$.
(2)
$$f_M(s) = \delta_{\dim(M),\dim(M')}f_{M'}(s) + \delta_{\dim(M),\dim(M'')}f_{M''}(s),$$
whenever $h_{M,I,J}$, $\delta_{\dim(M),\dim(M')}h_{M',I,J}$, $\delta_{\dim(M),\dim(M'')}h_{M'',I,J}$ are all differentiable at s.

Proof. (1)When $\mathcal{F}=h$, this is true by Proposition 3.31. Then Theorem 4.3 implies the statement for $\mathcal{F}=F_M$.
(2) follows from Theorem 5.8. □

Corollary 7.6 (Associativity formula)**.** The h-function, density function and Frobenius-Poincaré function satisfy the associativity formula. To be precise,

(1) let $\mathcal{F}\in\{h,F\}$, then
$$\mathcal{F}_{M,d}(s) = \sum_{P\in\mathrm{Spec}(R),\dim R/P=\dim R} \lambda_{R_P}(M_P)\mathcal{F}_{R/P}(s),$$
for all $s\in\mathbb{R}$.
(2) At a point s where $h_{R/P}$ is differentiable for all $P\in\mathrm{Assh}(R)$, the same associativity formula holds for the density function (i.e. $\mathcal{F}=f$) at s.

Theorem 7.7. *Let $(R,\mathfrak{m},k)$ be a noetherian local ring of dimension d, M be a finitely generated module of dimension d, I,I',J,J' be R-ideals such that $I'\subset I$, $J'\subset J$, $I'+J'$ is $\mathfrak{m}$-primary. Then $h_{M,I',J'}(s)\ge h_{M,I,J}(s)$ and equality holds if $I\subset \bar{I'}$ and $J\subset J'^*$.*

Proof. The inequality is clear. For the rest, note both sides of the inequality are additive on M. So by the associativity formula, we can replace M with R/P where $\dim R/P = d$. The containment hypotheses on the ideals also hold for their images in R/P for any prime ideal P. So we may assume $M=R$ and R is a domain. By definition of the integral closure and tight closure we can choose a nonzero $c\in R$ such that $cI^n\subset I'^n$ and $cJ^{[q]}\subset J'^{[q]}$, thus $I^{\lceil sq\rceil}+J^{[q]}/I'^{\lceil sq\rceil}+J'^{[q]}$ is annihilated by c. So
$$l\left(\frac{I^{\lceil sq\rceil}+J^{[q]}}{I'^{\lceil sq\rceil}+J'^{[q]}}\right) \le l\left(0:_{\frac{R}{I'^{\lceil sq\rceil}+J'^{[q]}}} c\right) = l\left(\frac{R}{cR+I'^{\lceil sq\rceil}+J'^{[q]}}\right) \le Cq^{d-1}.$$
The last equation is true because $\dim R/cR < \dim R$. This means
$$0\le h_{n,M,I',J'}(s) - h_{n,M,I,J}(s) \le Cq^{d-1}.$$
Dividing by q^d and take the limit when $q\to\infty$, we get $h_{M,I',J'}(s) = h_{M,I,J}(s)$. □

Theorem 7.8. *Let $n_0 \in \mathbb{N}$, then*

$$h_{M,I^{n_0},J}(s) = h_{M,I,J}(sn_0), h_{M,I,J^{[p^{n_0}]}}(s) = p^{n_0 d} h_{M,I,J}(s/p^{n_0}).$$

Proof. If $s \leq 0$ then both sides of the equation are 0 and the equality holds. Now we assume $s > 0$. By definition $h_{n,M,I^{n_0},J}(s) = l(M/I^{n_0\lceil sq\rceil} + J^{[q]}M)$. Since $\lceil sqn_0\rceil \leq n_0\lceil sq\rceil \leq \lceil sqn_0\rceil + n_0 - 1$, $h_{n,M,I,J}(sn_0) \leq h_{n,M,I^{n_0},J}(s) \leq h_{n,M,I,J}(sn_0 + (n_0-1)/q)$. We have $\lim_{n\to\infty}(h_{n,M,I,J}(sn_0 + (n_0-1)/q) - h_{n,M,I,J}(sn_0))/q^d = 0$ by Theorem 3.20. So

$$\lim_{n\to\infty} h_{n,M,I^{n_0},J}(s)/q^d = \lim_{n\to\infty} h_{n,M,I,J}(sn_0)/q^d,$$

which means $h_{M,I^{n_0},J}(s) = h_{M,I,J}(sn_0)$. We have $h_{n,M,I,J^{[p^{n_0}]}}(s) = l(M/I^{\lceil sq\rceil} + J^{[qp_0^n]}M) = l(M/I^{\lceil s/p^{n_0}\cdot qp^{n_0}\rceil} + J^{[qp_0^n]}M)$. So

$$\begin{aligned} &\lim_{n\to\infty} \frac{h_{n,M,I,J^{[p^{n_0}]}}(s)}{q^d} \\ = p^{n_0 d} &\lim_{n\to\infty} \frac{h_{n+n_0,M,I,J}(s/p^{n_0})}{q^d p^{n_0 d}} \\ &= p^{n_0 d} h_{M,I,J}(s/p^{n_0}). \end{aligned}$$

□

7.3. **Integration and h-function.** Let R be a local ring of characteristic p, $R[[t]]$ be a power series ring with indeterminate t. Let M be a finitely generated R-module, I, J be two R-ideals such that $I + J$ is $\mathfrak{m}$-primary. Let $M[[t]] = M \otimes_R R[[t]]$. We want to express $h_{M[[t]],R[[t]],(I,t^\alpha),(J,t^\beta)}$ in terms of $h_{M,R,I,J}$.

Theorem 7.9. *(1)$h_{M[[t]],R[[t]],(I,t^\alpha),(J,t^\beta)}(s) = \alpha\int_{s-\beta/\alpha}^{s} h_{M,R,I,J}(x)dx$*
(2)$h_{M[[t]],R[[t]],(I,t^\alpha),J}(s) = \alpha\int_0^s h_{M,R,I,J}(x)dx$
(3)$h_{M[[t]],R[[t]],I,(J,t^\beta)}(s) = \beta h_{M,R,I,J}(s)$.

Proof. We will use the convention $I^s = R$ when $s \leq 0$. To prove the equality we may assume $s = s_0/q_0 \in \mathbb{Z}[1/p]$ because the functions on both sides are continuous when $s > 0$. Then for $q \geq q_0$, sq is an integer.

$$h_{n,M[[t]],R[[t]],(I,t^\alpha),(J,t^\beta)} = l(\frac{M[[t]]}{((I,t^\alpha)^{sq} + (J^{[q]}, t^{\beta q}))M[[t]]})$$

The above length is also equal to

$$l(\frac{M[[t]]}{(\sum_{0\leq j\leq sq} I^{sq-j}t^{\alpha j} + (J^{[q]}, t^{\beta q}))M[[t]]})$$

But by the convention, it is also

$$l(M[[t]]/\sum_{0\leq j\leq\infty} I^{sq-j}t^{\alpha j} + (J^{[q]}, t^{\beta q})M[[t]])$$

and because the existence of the $t^{\beta q}$-term, it is also equal to

$$l(M[[t]]/(\sum_{0\leq j\leq\lfloor\beta q/\alpha\rfloor} I^{sq-j}t^{\alpha j} + (J^{[q]}, t^{\beta q}))M[[t]])$$

Note that the module inside is nonzero only in t-degree at most $\beta q - 1$. So summing up over the lengths in different t-degrees, the above length is also equal to the following sum:

$$L = \sum_{0\leq x\leq\beta q-1} l(M/(J^{[q]} + I^{sq-\lfloor x/\alpha\rfloor})M)$$

Let $y = \lfloor x/\alpha \rfloor$ and

$$L_1 = \sum_{\alpha\lfloor \beta q/\alpha \rfloor \le x \le \beta q - 1} l(M/(J^{[q]} + I^{sq - \lfloor x/\alpha \rfloor})M),$$

$$L_2 = \sum_{0 \le x \le \alpha\lfloor \beta q/\alpha \rfloor - 1} l(M/(J^{[q]} + I^{sq - \lfloor x/\alpha \rfloor})M).$$

Here we denote $L_1 = 0$ if $\beta q/\alpha \in \mathbb{Z}$. Then $L = L_1 + L_2$, and L_1 has at most α terms and each term is of order $O(q^d)$, so L_1 has order $O(q^d)$. Now

$$\begin{aligned} L_2 &= \alpha \sum_{0 \le y \le \lfloor \beta q/\alpha \rfloor - 1} l(M/J^{[q]} + I^{sq-y}M) \\ &= \alpha \sum_{0 \le y \le \lfloor \beta q/\alpha \rfloor - 1} h_{n,M,I,J}(s - y/q) \\ &= \alpha q \int_{s - \lfloor \beta q/\alpha \rfloor / q}^{s} h_{n,M,I,J}(x)dx \end{aligned}$$

Now $\lim_{q\to\infty} L_1/q^{d+1} = 0$, so

$$\lim_{q\to\infty} L/q^{d+1} = \lim_{q\to\infty} L_2/q^{d+1} = \alpha \int_{s-\beta/\alpha}^{s} h_{M,I,J}(x)dx.$$

Since the equation

$$h_{M[[t]],R[[t]],(I,t^\alpha),(J,t^\beta)} = \alpha \int_{s-\beta/\alpha}^{s} h_{M,R,I,J}(x)dx$$

is true on $\mathbb{Z}[1/p]$ and both sides are continuous with respect to s, they are equal on all of $\mathbb{R}$. The rest of the two equations can be obtained by taking limit as α or β goes to infinity and using the $\mathfrak{m}$-adic continuity proven in Theorem 3.13. $\square$

7.4. **Ring extension.**

Proposition 7.10. *Let $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a local map such that $\mathfrak{m}S$ is $\mathfrak{n}$-primary and $\dim R = \dim S$. Then*

$$h_{M\otimes_R S,S,IS,JS}(s) \le l_S(S/\mathfrak{m}S) h_{M,R,I,J}(s).$$

The equality holds when S is flat over R.

Proof. For any $\mathfrak{m}$-primary ideal $\mathfrak{a}$, we have

$$l_S(M \otimes_R S/(\mathfrak{a}S)M \otimes_R S) \le l_R(M/\mathfrak{a}M) l_S(S/\mathfrak{m}S).$$

This means $h_{n,M\otimes_R S,S,IS,JS}(s) \le l(S/\mathfrak{m}S) h_{n,M,R,I,J}(s)$. All these equalities will hold if S is flat over R. $\square$

8. h-function and density function near boundaries

In this section, we discuss the behaviour of $h(s)$ near zero and s large enough. The regions near zero and away from zero where the h-function often shows interesting behaviour are marked by two other already known invariants, namely F-limbus and F-threshold. Recall that F-threshold is a well-known numerical invariant in characteristic p which compares the ordinary power and Frobenius power; it was defined as a limsup in [Hun+08a] and [MTW05], and is shown to be a limit in [DNP18]. The F-limbus is less known, which is defined in [Tay18].

Definition 8.1. Let R be a ring of characteristic $p>0$ which is not necessarily local, and let I, J be ideals of R. Define

$$c_I^J(n) = \sup\{t \in \mathbb{N} : I^t \not\subseteq J^{[p^n]}\},$$
$$c^J(I) = \lim_{n\to\infty} \frac{c_I^J(n)}{p^n},$$
$$b_I^J(n) = \inf\{t \in \mathbb{N} : J^{[p^n]} \not\subseteq I^t\},$$
$$b^J(I) = \lim_{n\to\infty} \frac{b_I^J(n)}{p^n}.$$

The number $c^J(I)$ is called the F-threshold of I with respect to J and the number $b^J(I)$ is called the F-limbus of I with respect to J. The following properties are proven in [Tay18, Lemma 3.2].

Lemma 8.2. *Let R be a ring of characteristic $p>0$, and let I, J be proper ideals of R.*

(1) *For any I, J, any limit above either exists or goes to infinity.*
(2) *If I is contained in the Jacobson radical of R, $I \not\subseteq nil(R)$, then $b^J(I) \le c^J(I)$.*
(3) *If $I \not\subseteq \sqrt{J}$ then $c^J(I) = \infty$.*
(4) *If $I \subset \sqrt{J}$ then $0 \le c^J(I) < \infty$.*
(5) *If $J \not\subseteq \sqrt{I}$ then $b^J(I) = 0$.*
(6) *If $J \subset \sqrt{I}$ then $0 < b^J(I) \le \infty$.*
(7) *If $I \subset Rad(R)$, $I \not\subseteq nil(R)$, $I \subset \sqrt{J}$, $J \subset \sqrt{I}$, then $0 < b^J(I) \le c^J(I) < \infty$.*

Lemma 8.3. *Let $(R, \mathfrak{m})$ be a local ring of dimension d and characteristic p, let I, J be two proper ideals of R, and let M be a finitely generated R-module.*

(1) *If I is $\mathfrak{m}$-primary, then $b^J(I) > 0$ and for $s \le b^J(I)$, $h_M(s) = \frac{s^d}{d!}e(I, M)$.*
(2) *If J is $\mathfrak{m}$-primary, then $c^J(I) < \infty$ and for $s \ge c^J(I)$, $h_M(s) = e_{HK}(J, M)$.*

Proof. The above Lemma is a generalization of Lemma 3.3 of [Tay18]. The proof is identically the same since it only uses the containment relation, which does not depend on whether I, J are $\mathfrak{m}$-primary or not. If I is $\mathfrak{m}$-primary then $J \subset \sqrt{I}$, so $b^J(I) > 0$; if J is $\mathfrak{m}$-primary then $I \subset \sqrt{J}$, so $c^J(I) < \infty$. ☐

8.1. **Tail: F-threshold, minimal stable point and maximal support.** Let $(R, \mathfrak{m})$ be a local ring of characteristic $p>0$, I, J are R-ideals. Assume J is $\mathfrak{m}$-primary. By Lemma 8.3, (2), when J is $\mathfrak{m}$-primary, the $h_{M,I,J}(s)$-becomes the constant $e_{HK}(J, M)$ for large enough s. Since h is increasing and $h_M(s) \le e_{HK}(J, M)$ for any s, and there is a smallest point after which $h_{M,I,J}(s)$ becomes a constant. We relate this smallest point to another seemingly unrelated invariant of (I, J) which we call the F-threshold upto tight closure; see Definition 8.5. The next lemma guarantees the existence of this invariant.

Lemma 8.4. *$(R, \mathfrak{m}, k)$ be a local ring of characteristic $p>0$, I, J be two R-ideal, $I \subset \sqrt{J}$. Let*

$$r_I^J(n) = \max\{t \in \mathbb{N} | \, I^t \not\subseteq (J^{[p^n]})^*\},$$

Then $(r_I^J(n)/p^n)_n$ is a non-decreasing sequence converging to a real number.

Proof. Given a natural number n, pick $x \in I^{r_I^J(n)} \setminus (J^{[q]})^*$. Note that x^p cannot be in $(J^{[pq]})^*$. Indeed, in contrary say $x^p \in (J^{[pq]})^*$. Then there is a $c \in R$ not in any minimal primes of R such that $cx^{p^{m+1}} \in (J^{[q]})^{[p^{m+1}]}$ for any large m. This implies $x \in (J^{[q]})^*$. So we conclude

$$r_I^J(n+1) \ge p r_I^J(n),$$

whence the desired non-decreasingness follows. By Lemma 3.6, $(r_I^J(n)/p^n)_n$ is bounded and hence converges to a real number. $\square$

Definition 8.5. Let $(r_I^J(n))_n$ be the same as in Lemma 8.4. The limit of $(r_I^J(n)/p^n)_n$, which exists by Lemma 8.4 is called the *F-threshold up to tight closure* for the ideal pair I, J and is denoted by $r_{R,I,J}$.

Lemma 8.6. *Let $(R, \mathfrak{m}, k)$ be a d-dimensional reduced local ring of characteristic $p > 0$, J be an $\mathfrak{m}$-primary R-ideal. Then $e_{HK}(J, R) = \lim_{n\to\infty} l(R/(J^{[q]})^*)/q^d$.*

Proof. It suffices to show $\lim_{n\to\infty} l((J^{[q]})^*/J^{[q]})/q^d = 0$. There is a test element $c \in R$, which is in not contained in any minimal primes of R such that, $c(J^{[q]})^* \subseteq J^{[q]}$ for all q; see [Hun96]. So we have $l((J^{[q]})^*/J^{[q]}) \leq l(0 :_{R/J^{[q]}} c) = l(R/cR + J^{[q]}) \leq Cq^{d-1}$ for some constant C, so

$$\lim_{n\to\infty} l((J^{[q]})^*/J^{[q]})/q^d = 0.$$

$\square$

Theorem 8.7. *Let $(R, \mathfrak{m}, k)$ be a of characteristic $p > 0$, I be an R-ideal, J be an $\mathfrak{m}$-primary R-ideal, M be a finitely generated R-module. Define*

$$\alpha_{M,I,J} = \sup\{s|\, h_{M,I,J}(s) \neq e_{HK}(J, M)\} = \min\{s|\, h_{M,I,J}(s) = e_{HK}(J, M)\}.$$

Then

$$\alpha_{R,I,J} = r_{R,I,J}.$$

Proof. For simplicity, first assume R is a complete local domain. It suffices to prove:

(1) For $x \in \mathbb{Z}[1/p]$, if $x > r_{R,I,J}$, then $x \geq \alpha_{R,I,J}$;
(2) For $x \in \mathbb{Z}[1/p]$, if $x < r_{R,I,J}$, then $x \leq \alpha_{R,I,J}$.

(1): If $x > r_{R,I,J}$, then there is an infinite sequence n_i, such that $xp^{n_i} > r_I^J(n_i)$ and xp^{n_i} is an integer for all i. By definition of r_n, $I^{xp^{n_i}} \subset (J^{[p^{n_i}]})^*$. So

$$h_{R,I,J}(x) = \lim_{i\to\infty} l(R/I^{\lceil xp^{n_i}\rceil} + (J^{[p^{n_i}]})^*)/q^d = \lim_{i\to\infty} l(R/(J^{[p^{n_i}]})^*)/q^d = e_{HK}(J, R).$$

The last equality in the above chain follows from Lemma 8.6. So $x \geq \alpha_{R,I,J}$.

(2): If $x < r_{R,I,J}$, then there is an integer n_0, such that $xp^{n_0} \leq r_I^J(n_0)$ and xp^{n_0} is an integer. Let $q_0 = p^{n_0}$. By definition of $r_I^J(n)$, $I^{xq_0} \nsubseteq (J^{[q_0]})^*$. Choose $f \in I^{xq_0}\backslash(J^{[q_0]})^*$. Let $\tilde{J} = J^{[q_0]} + fR$; then $e_{HK}(\tilde{J}, R) < e_{HK}(J^{[q_0]}, R)$; see [Hun13, Theorem 5.5], [HH90, Theorem 8.17]. Now fix an $s < xq_0$, then for any $q = p^n$, $sq < xqq_0$. Since $f \in I^{xq_0}$, $f^q \in I^{xqq_0} \subseteq I^{\lceil sq\rceil}$. Therefore,

$$I^{\lceil sq\rceil} + (J^{[q_0]} + fR)^{[q]} = I^{\lceil sq\rceil} + (J^{[q_0]})^{[q]}.$$

This means $h_{R,I,\tilde{J}}(s) = h_{R,I,J^{[q_0]}}(s)$. So for $s < xq_0$, $h_{R,I,J^{[q_0]}}(s) = h_{R,I,\tilde{J}}(s) \leq e_{HK}(\tilde{J}, R) < e_{HK}(J^{[q_0]}, R)$. This means $\alpha_{R,I,J^{[q_0]}} \geq xq_0$. By Theorem 7.8, $h_{R,I,J^{[q_0]}}(s) = q_0^d h_{R,I,J}(s/q_0)$, $\alpha_{R,I,J} = \frac{\alpha_{R,I,J^{[q_0]}}}{q_0} \geq x$.

Now we argue that without loss of generality R can be taken to be a complete domain. Note,

$$\alpha_{R,I,J} = \max_{Q \text{ minimal prime of } R}\{\alpha_{\frac{R}{Q}, I\frac{R}{Q}, J\frac{R}{Q}}\}, \quad r_{R,I,J} = \max_{Q \text{ minimal prime of } R}\{r_{\frac{R}{Q}, I\frac{R}{Q}, J\frac{R}{Q}}\}. \tag{8.1}$$

The above description of $\alpha_{R,I,J}$ follows from Proposition 3.31. The above description of $r_{R,I,J}$ follows from [Hun96, Thm 1.3]. Thanks to Equation (8.1), it suffices to prove the present theorem when R is a domain. Assume R is a domain. Since J is $\mathfrak{m}$-primary, $r_{R,I,J}$ coincides with $r_{\hat{R},I\hat{R},J\hat{R}}$. Indeed as $J^{[q]}$ is $\mathfrak{m}$-primary for all q and R is a domain,

$(J^{[q]}\hat{R})^* = (J^{[q]})^*\hat{R}$; see [HH94, Thm 7.16, (a)]. On the other hand, $h_{R,I,J} = h_{\hat{R},I\hat{R},J\hat{R}}$; so $\alpha_{R,I,J} = \alpha_{\hat{R},I\hat{R},J\hat{R}}$. So without loss of generality R can be taken to be $\mathfrak{m}$-adically complete. We can pass to the complete domain case using Equation (8.1). □

Since $h_M(s)$ is the integration of $f_M(s)$, we see the minimal stable point of h_M is the maximal support of f_M. Precisely,

Corollary 8.8. Let $(R, \mathfrak{m}, k)$ be a local ring of characteristic $p > 0$, I be an R-ideal, J be an $\mathfrak{m}$-primary R-ideal. Then $\alpha_{R,I,J} = \sup\{s| \, f_{R,I,J}(s) \text{ exists and is nonzero}\}$. Moreover for $s > \alpha_{R,I,J}$, $f_{R,I,J}(s)$ is zero, and for $0 < s < \alpha_{R,I,J}$, it is nonzero whenever it is well-defined.

Proof. For $s > \alpha_{R,I,J}$, $h_{I,J}(s)$ is constant. So by Theorem 5.8, $f_{I,J}$ exists and is zero. Since $h_{I,J}$ is the integral of the density function by Theorem 6.4 and h is a non-constant increasing function on $(a, \alpha_{R,I,J})$ for any $0 < a < \alpha_{R,I,J}$, $f_{I,J}$ has to be nonzero on a subset of $(a, \alpha_{R,I,J})$ of nonzero measure. So when $f_{I,J}(a)$ exists, it is nonzero by Proposition 5.10. □

Remark 8.9. Recall from Theorem 6.7 that for standard graded $(R, \mathfrak{m})$ of Krull dimension at least two and a finite colength homogeneous ideal J, Trivedi's density function $\tilde{g}_{R,J}$ coincides with $f_{R,\mathfrak{m},J}$ and both are continuous. So Theorem 8.7 gives a precise description of the support of $\tilde{g}_{R,J}$. Thus Theorem 8.7 and the theorem below extends [TW22, Theorem 4.9], where $\alpha_{R,J}$ is shown to coincide with the F-threshold $c^J(\mathfrak{m})$ under suitable hypothesis.

Theorem 8.10. *Let $(R, \mathfrak{m}, k)$ be a local ring of characteristic $p > 0$, I be an R-ideal, J be an $\mathfrak{m}$-primary R-ideal. Then $c^J(I) = r_{R,I,J}$ is true under either of the assumptions below:*

1. *There exists a sequence of positive numbers r'_n such that $I^{r'_n} \subset J^{[q]} : (J^{[q]})^*$ for infinitely many $q \gg 0$ and $\lim_n r'_n/p^n \to 0$.*
2. *There exists a constant n_0 such that $I^{n_0} \subset J^{[q]} : (J^{[q]})^*$ for infinitely many $q \gg 0$.*
3. *R is F-rational[4], i.e. the tight closure of every parameter ideal coincides with the ideal and J is a parameter ideal.*
4. *$I \subset \sqrt{\tau(R)}$, where $\tau(R) = \cap_{\mathfrak{a} \subset R}\mathfrak{a} : \mathfrak{a}^*$ is the test ideal of R. See [HH90, Definition 8.22, Proposition 8.23] for details on the test ideal.*
5. *(Theorem 4.9, [TW21])R is strongly F-regular on the punctured spectrum.*

Proof.
1. By definition $r_I^J(n) \le c_I^J(n)$, and the condition implies $c_I^J(n) \le r_I^J(n) + r_n$, so $\lim_n(c_I^J(n) - r_I^J(n))/p^n = 0$ and $c^J(I) = r^J(I)$.
2. By (1) and the fact that $\lim_n n_0/n = 0$.
3. If J is a parameter ideal, so is $J^{[q]}$. Since R is F-rational, $J^{[q]} : (J^{[q]})^* = R$ for any q, so $n_0 = 1$ satisfies the assumption of (2).
4. There exist an n_0 such that $I^{n_0} \subset \tau(R) \subset \cap_q J^{[q]} : (J^{[q]})^*$, and this n_0 satisfies the assumption of (2).
5. In this case $\tau(R)$ is either $\mathfrak{m}$-primary or is the unit ideal, so $I \subset \sqrt{\tau(R)}$ always holds.

□

8.2. **Head: Order of vanishing at 0 and Hilbert-Kunz multiplicity of quotient rings.** So far we have proven continuity of the h-function on $\mathbb{R}_{>0}$; see Theorem 3.20 and Theorem 3.30. In this section we determine when $h_{M,I,J}$ is continuous at $s = 0$; see Theorem 8.14. In Theorem 8.12, we determine the order of vanishing of h-functions near the origin and show that the asymptotic behaviour of $h_{I,J}$ near the origin captures other numerical invariants of (R, I, J). A major intermediate step involved in proving

[4] see [FW89], [Smi97]

Theorem 8.12 is Theorem 8.11, which boils down to proving commutation of the order of a double limit. We lay the groundwork for that.

Let $(R, \mathfrak{m}, k)$ be a local of characteristic $p > 0$, I, J be two R-ideals such that $I + J$ is $\mathfrak{m}$-primary. Let $d = \dim R$, $d' = \dim R/I$. For a positive integer s_0, consider the sequence of real numbers:

$$\Gamma_{s_0,m,n} = \frac{l(R/I^{s_0p^n} + J^{[p^np^m]})}{p^{nd}p^{md'}s_0^{d-d'}}.$$

$$\lim_{n\to\infty} \Gamma_{s_0,m,n} = \frac{h_R(s_0/p^m)}{(s_0/p^m)^{d-d'}}. \tag{8.2}$$

$$\begin{aligned}\lim_{m\to\infty} \Gamma_{s_0,m,n} &= \frac{e_{HK}(J^{[p^n]}, R/I^{s_0p^n})}{p^{nd}s_0^{d-d'}}\\ &= \frac{e_{HK}(J, R/I^{s_0p^n})}{(s_0p^n)^{d-d'}}\\ &= \frac{1}{(s_0p^n)^{d-d'}} \sum_{P\in \mathrm{Assh}(R/I)} e_{HK}(J, R/P) l_{R_P}(R_P/I^{s_0p^n}R_P) \ ,\end{aligned}$$

where $\mathrm{Assh}(R/I)$ is the set of associated primes of R/I in R of dimension $\dim(R/I)$; see Definition 3.23.

For $P \in \mathrm{Assh}(R/I)$, we have $\mathrm{ht}(P) \leq \dim R - \dim R/P = \dim R - \dim R/I = d - d'$. So

$$\begin{aligned}\lim_{n\to\infty}\lim_{m\to\infty} \Gamma_{s_0,m,n} &= \lim_{n\to\infty} \frac{1}{(s_0p^n)^{d-d'}} \sum_{P\in \mathrm{Assh}(R/I)} e_{HK}(J, R/P) l_{R_P}(R_P/I^{s_0p^n}R_P)\\ &= \frac{1}{(d-d')!} \sum_{P\in \mathrm{Assh}(R/I), \mathrm{ht}P = d-d'} e_{HK}(J, R/P) e(I, R_P) \ .\end{aligned}$$

When R is an F-finite domain and hence an excellent domain (see [Kun76]), for all $P \in \mathrm{Assh}(R/I)$, $\mathrm{ht}(P) = d - d'$. So the above quantity is

$$\frac{1}{(d-d')!} \sum_{P\in \mathrm{Assh}(R/I)} e_{HK}(J, R/P) e(IR_P, R_P).$$

When R is a Cohen-Macaulay domain and I is part of a system of parameters, the above quantity recovers the Hilbert-Kunz multiplicity $e_{HK}(J, R/I)$ as,

$$\begin{aligned}&\sum_{P\in \mathrm{Assh}(R/I)} e_{HK}(J, R/P) e(IR_P, R_P)\\ &= \sum_{P\in \mathrm{Assh}(R/I)} e_{HK}(J, R/P) l(R_P/IR_P)\\ &= e_{HK}(J, R/I) \ .\end{aligned}$$

Theorem 8.11. *Assume R is a domain and $I \neq 0$ and J be such that $I + J$ is $\mathfrak{m}$-primary. Fix a positive integer s_0. Set $\dim(R/I) = d'$. Then*

$$\lim_{m\to\infty} \frac{h(s_0/p^m)}{(s_0/p^m)^{d-d'}} = \frac{1}{(d-d')!} \sum_{P\in \mathrm{Assh}(R/I)} e_{HK}(J, R/P) e(I, R_P) \ ,$$

where $\mathrm{Assh}(R/I)$ is the set of associated primes of R/I in R of dimension $\dim(R/I)$.

Proof. We use the notation set above in this subsection. It follows from Equation (8.2) and above that we need to show

$$\lim_{m\to\infty}\lim_{n\to\infty}\Gamma_{s_0,m,n}=\lim_{n\to\infty}\lim_{m\to\infty}\Gamma_{s_0,m,n}.$$

We already see that $\lim_{n\to\infty}\Gamma_{s_0,m,n}$ and $\lim_{n\to\infty}\lim_{m\to\infty}\Gamma_{s_0,m,n}$ exist. We claim that that the sequence $n\to\Gamma_{s_0,m,n}$ is uniformly convergent in terms of m; then, by argument of analysis, we get $\lim_{m\to\infty}\lim_{n\to\infty}\Gamma_{s_0,m,n}$ exists, and is equal to $\lim_{n\to\infty}\lim_{m\to\infty}\Gamma_{s_0,m,n}$.

To this end, we prove that there exist a constant C such that $|\Gamma_{s_0,m,n+1}-\Gamma_{s_0,m,n}|\leq C/p^n$ for all m, which implies that $|\lim_{n\to\infty}\Gamma_{s_0,m,n}-\Gamma_{s_0,m,n}|\leq 2C/p^n$ for all m. We can prove it in two steps: we first prove there is a constant C_1 such that $\Gamma_{s_0,m,n+1}-\Gamma_{s_0,m,n}\leq C_1/p^n$, then we prove there is a constant C_2 such that $\Gamma_{s_0,m,n}-\Gamma_{s_0,m,n+1}\leq C_2/p^n$, then $C=\max\{|C_1|,|C_2|\}$ satisfies the statement of the claim. Without loss of generality we assume $R/\mathfrak{m}$ is a perfect field; see Remark 3.15.

Choice of C_1: since $\dim R=d$, there is an exact sequence

$$0\to R^{\oplus p^d}\to F_*R\to N\to 0$$

where N is an R-module with $\dim N<d$. Then we have

$$(R/I^{s_0p^n}+J^{[p^np^m]})^{\oplus p^d}\to F_*R/(I^{s_0p^n}+J^{[p^np^m]})F_*R\to N/(I^{s_0p^n}+J^{[p^np^m]})N\to 0.$$

This means

$$\begin{aligned}l(\frac{R}{I^{s_0p^{n+1}}+J^{[p^{n+1}p^m]}})&\leq l(\frac{R}{I^{s_0p^n[p]}+J^{[p^{n+1}p^m]}})\\&\leq p^dl(\frac{R}{I^{s_0p^n}+J^{[p^np^m]}})+l(\frac{N}{(I^{s_0p^n}+J^{[p^np^m]})N}).\end{aligned}$$

So dividing $p^{(n+1)d}p^{md'}s_0^{d-d'}$, we get

$$\Gamma_{s_0,m,n+1}\leq\Gamma_{s_0,m,n}+l(N/(I^{s_0p^n}+J^{[p^np^m]})N)/p^{(n+1)d}p^{md'}s_0^{d-d'}.$$

Now we claim that there is a constant $C_1>0$ that depends on N,I,J and s_0 but is independent of m,n such that $l(N/I^{s_0p^n}+J^{[p^np^m]}N)/p^{n(d-1)+d}p^{md'}s_0^{d-d'}\leq C_1$. We have

$$\begin{aligned}l(N/(I^{s_0p^n}+J^{[p^np^m]})N)&\leq l(N/(I^{s_0[p^n]}+J^{[p^np^m]})N)\\&=l(F^n_*N/(I^{s_0}+J^{[p^m]})F^n_*N)\\&\leq\mu_R(F^n_*N)l(R/I^{s_0}+J^{[p^m]}).\end{aligned}$$

Since $\dim N\leq d-1$ and $\dim R/I=d'$, $\mu_R(F^n_*N)/p^{n(d-1)}$ and $l(R/I^{s_0}+J^{[p^m]})/p^{md'}$ are both bounded. And $p^{-d}s_0^{d-d'}$ is independent of m,n. This means there is a constant $C_1>0$ that depends on N,I,J and s_0 but is independent of m,n such that $l(N/I^{s_0p^n}+J^{[p^np^m]}N)/p^{n(d-1)+d}p^{md'}s_0^{d-d'}\leq C_1$. Thus we have

$$\Gamma_{s_0,m,n+1}\leq\Gamma_{s_0,m,n}+C_1/p^n.$$

Choice of C_2: since $\dim R=d$, there is an injection $F_*R\xrightarrow{\phi}R^{\oplus p^d}$ where $\dim\mathrm{Coker}\phi<\dim R$. Let μ be the minimal number of generators of I. Choose $0\neq c\in I$ and let $\psi=c^\mu\phi$. Since R is a domain, ψ is still an injection, and we have a short exact sequence

$$0\to F_*R\xrightarrow{\psi}R^{\oplus p^d}\to N'\to 0$$

and we have $\dim N'<\dim R$. Therefore, we get an exact sequence:

$$F_*R/(I^{s_0p^n}+J^{[p^np^m]})F_*R\xrightarrow{\bar{\phi}}(R/I^{s_0p^n}+J^{[p^np^m]})^{\oplus p^d}\to N'/(I^{s_0p^n}+J^{[p^np^m]})N'\to 0.$$

We claim that $\bar{\phi}$ induces an R-linear map $\Phi : F_*(R/(I^{s_0p^{n+1}} + J^{[p^{n+1}p^m]})) \xrightarrow{\bar{\phi}} (R/I^{s_0p^n} + J^{[p^np^m]})^{\oplus p^d}$. It suffices to show $\psi(F_*(I^{s_0p^{n+1}} + J^{[p^{n+1}p^m]})) \in (I^{s_0p^n} + J^{[p^np^m]})^{\oplus p^d}$. We have $I^{s_0p^{n+1}} = I^{s_0p^np} \subset I^{(s_0p^n-\mu)[p]}$. So

$$\begin{aligned}
&\psi(F_*(I^{s_0p^{n+1}} + J^{[p^{n+1}p^m]}))\\
&\subset \psi(F_*(I^{(s_0p^n-\mu)[p]} + J^{[p^{n+1}p^m]}))\\
&\subset (I^{(s_0p^n-\mu)} + J^{[p^np^m]})\psi(F_*R)\\
&\subset c^\mu(I^{(s_0p^n-\mu)} + J^{[p^np^m]})\phi(F_*R)\\
&\subset (I^{(s_0p^n)} + J^{[p^np^m]})\phi(F_*R)\\
&\subset (I^{(s_0p^n)} + J^{[p^np^m]})^{\oplus p^d}.
\end{aligned}$$

This induces an exact sequence

$$F_*(R/(I^{s_0p^{n+1}} + J^{[p^{n+1}p^m]})) \to (R/I^{s_0p^n} + J^{[p^np^m]})^{\oplus p^d} \to N'/(I^{s_0p^n} + J^{[p^np^m]})N' \to 0$$

Therefore,

$$p^d l(R/I^{s_0p^n} + J^{[p^np^m]}) \leq l(R/I^{s_0p^{n+1}} + J^{[p^{n+1}p^m]}) + l(N'/(I^{s_0p^n} + J^{[p^np^m]})N')$$

So dividing $p^{(n+1)d}p^{md'}s_0^{d-d'}$, we get

$$\Gamma_{s_0,m,n+1} \leq \Gamma_{s_0,m,n} + l(N'/(I^{s_0p^n} + J^{[p^np^m]})N')/p^{(n+1)d}p^{md'}s_0^{d-d'}$$

Since $\dim N' < \dim R$, we can use the same proof in the previous step to show that there is a constant $C_2 > 0$ that depends on N', I, J and s_0 but independent of m, n such that $l(N'/(I^{s_0p^n} + J^{[p^np^m]})N')/p^{n(d-1)+d}p^{md'}s_0^{d-d'} \leq C_2$, so

$$\Gamma_{s_0,m,n} \leq \Gamma_{s_0,m,n+1} + C_2/p^n.$$

$\square$

Theorem 8.12. *Let $(R, \mathfrak{m}, k)$ be a local domain, I, J be two R-ideals, $I \neq 0$, $I + J$ is $\mathfrak{m}$-primary. Let $d = \dim R$, $d' = \dim R/I$. Then:*

(1) $\lim_{s\to 0+} h(s)/s^{d-d'} = \frac{1}{(d-d')!}\sum_{P\in \mathrm{Assh}(R/I)} e_{HK}(J, R/P)e(I, R_P)$.
(2) *The order of vanishing of $h(s)$ at $s = 0$ is exactly $d - d'$.*
(3) *$h(s)$ is continuous at* 0.

Proof. (1) Let $\frac{1}{(d-d')!}\sum_{P\in \mathrm{Assh}(R/I)} e_{HK}(J, R/P)e(I, R_P) = c = c_{I,J}$, which is a constant that only depends on I, J. The last theorem implies for any fixed s_0,

$$\lim_{m\to\infty} h(s_0/p^m)/(s_0/p^m)^{d-d'} = c$$

Choose a sequence $\{s_i\}_i \subset (0, \infty)$ such that $\lim_{i\to\infty} s_i = 0$ and $\lim_{i\to\infty} h(s_i)/s_i^{d-d'}$ exists. Below we argue that $\lim_{i\to\infty} h(s_i)/s_i^{d-d'} = c$; then (1) follows. Fix any $n_0 \in \mathbb{N}$. There exists an integer α_i for each s_i such that $s_ip^{\alpha_i} \in (p^{n_0-1}, p^{n_0}]$. Since $h(s)$ is an increasing function,

$$\frac{h(\lfloor s_iq^{\alpha_i}\rfloor/q^{\alpha_i})}{((\lfloor s_iq^{\alpha_i}\rfloor+1)/q^{\alpha_i})^{d-d'}}\leq\frac{h(s_i)}{s_i^{d-d'}}\leq\frac{h(\lceil s_iq^{\alpha_i}\rceil/q^{\alpha_i})}{((\lceil s_iq^{\alpha_i}\rceil-1)/q^{\alpha_i})^{d-d'}}$$
$$\Longrightarrow(\frac{\lfloor s_iq^{\alpha_i}\rfloor}{\lfloor s_iq^{\alpha_i}\rfloor+1})^{d-d'}\frac{h(\lfloor s_iq^{\alpha_i}\rfloor/q^{\alpha_i})}{(\lfloor s_iq^{\alpha_i}\rfloor/q^{\alpha_i})^{d-d'}}\leq\frac{h(s_i)}{s_i^{d-d'}}\leq(\frac{\lceil s_iq^{\alpha_i}\rceil}{\lceil s_iq^{\alpha_i}\rceil-1})^{d-d'}\frac{h(\lceil s_iq^{\alpha_i}\rceil/q^{\alpha_i})}{(\lceil s_iq^{\alpha_i}\rceil/q^{\alpha_i})^{d-d'}}$$
$$\Longrightarrow(\frac{p^{n_0-1}}{p^{n_0-1}+1})^{d-d'}\frac{h(\lfloor s_iq^{\alpha_i}\rfloor/q^{\alpha_i})}{(\lfloor s_iq^{\alpha_i}\rfloor/q^{\alpha_i})^{d-d'}}\leq\frac{h(s_i)}{s_i^{d-d'}}\leq(\frac{p^{n_0-1}}{p^{n_0-1}-1})^{d-d'}\frac{h(\lceil s_iq^{\alpha_i}\rceil/q^{\alpha_i})}{(\lceil s_iq^{\alpha_i}\rceil/q^{\alpha_i})^{d-d'}}.$$

Let $i\to\infty$, then $s_i\to 0$, $\alpha_i\to\infty$. Since $\lfloor s_iq^{\alpha_i}\rfloor,\lceil s_iq^{\alpha_i}\rceil$ lies in $[p^{n_0-1},p^{n_0}]$, so there are only finitely many possible values of $\lfloor s_iq^{\alpha_i}\rfloor,\lceil s_iq^{\alpha_i}\rceil$. So by Theorem 8.11,

$$\lim_{i\to\infty}\frac{h(\lfloor s_iq^{\alpha_i}\rfloor/q^{\alpha_i})}{(\lfloor s_iq^{\alpha_i}\rfloor/q^{\alpha_i})^{d-d'}}=\lim_{i\to\infty}\frac{h(\lceil s_iq^{\alpha_i}\rceil/q^{\alpha_i})}{(\lceil s_iq^{\alpha_i}\rceil/q^{\alpha_i})^{d-d'}}=c.$$

This means

$$(\frac{p^{n_0-1}}{p^{n_0-1}+1})^{d-d'}c\leq\lim_{i\to\infty}h(s_i)/s_i^{d-d'}\leq(\frac{p^{n_0-1}}{p^{n_0-1}-1})^{d-d'}c.$$

Since this is true for arbitrary n_0, we get

$$\lim_{i\to\infty}h(s_i)/s_i^{d-d'}=c.$$

This finishes the proof of (1).

(2) follows from (1).

(3) Since R is a domain and $I\neq 0$, $d'=\dim R/I<\dim R=d$, $d-d'\geq 1$. So the order of $h(s)$ at 0 is at least 1; in particular, $\lim_{s\to 0+}h(s)=0=h(0)$. □

Lemma 8.13. *Let $(R,\mathfrak{m})$ be a noetherian local domain, I,J be two R-ideal such that $I+J$ is $\mathfrak{m}$-primary. Then $h_{R,I,J}(s)$ is continuous at 0 if and only if $I\neq 0$.*

Proof. If $I\neq 0$ then by previous theorem it is continuous at 0. If $I=0$, then $h_R(s)=e_{HK}(J,R)\neq 0=h_R(0)$ for $s>0$, so it is discontinuous at 0. □

Theorem 8.14. *Let $(R,\mathfrak{m})$ be a noetherian local ring, I,J be two R-ideals such that $I+J$ is $\mathfrak{m}$-primary, M be a finitely generated R-module. Then $h_{M,I,J}(s)$ is continuous at 0 if and only if $I\nsubseteq P$ for any $P\in\operatorname{Supp}(M)$ with $\dim R/P=\dim M$. In particular, $h_{R,I,J}(s)$ is continuous at 0 if and only if $\dim R>\dim R/I$. If h_M is discontinuous at 0 then we have*

$$\lim_{s\to 0+}h_M(s)=\sum_{P\in\operatorname{Supp}(M),I\subset P,\dim R/P=\dim M}l_{R_P}(M_P)e_{HK}(J,R/P).$$

Proof. By the associativity formula for h-function in Corollary 7.6,

$$h_M(s)=\sum_{P\in\operatorname{Supp}(M),\dim R/P=\dim M}l_{R_P}(M_P)h_{R/P}(s).$$

For any $P\in\operatorname{Supp}(M)$, $\lim\limits_{s\to 0+}h_{R/P,I,J}(s)$ is always non-negative; the limit is positive if and only if $I\subseteq P$, in which case the limit is $e_{HK}(J,R/P)$; see Lemma 8.13. Thus taking limit as s approaches zero from the right, we get the expression of the right hand limit of h_M. Since h_M is continuous at 0 if and only if $\lim_{s\to 0^+}h_{R/P}(s)=0$ for any $P\in\operatorname{Supp}(M)$ with $\dim R/P=\dim M$, the continuity of h_M at zero is equivalent to asking $I\nsubseteq P$ for any $P\in\operatorname{Supp}(M)$ with $\dim R/P=\dim M$. If $M=R$, then this means $I\nsubseteq P$ for any $P\in\operatorname{Assh}(R)$ which means $\dim R>\dim R/I$. □

Now we analyse the behaviour of the density function $f_{R,I,J}$ near the origin. Our argument below uses the monotonicity property proven in Proposition 5.10. This forces us to make a simplifying assumption about I.

Theorem 8.15. *Let (R,m) be a noetherian local domain. Let $x_1, x_2, \dots, x_r$ be part of system of parameters of R, where $r \geq 1$. Set I to be the integral closure of $(x_1, \dots, x_r)$. Fix an ideal J such that $I + J$ is $\mathfrak{m}$-primary. Denote the left and right hand derivatives of h at s by h'_+ and h'_- respectively. Then,*

(1) *Then both $\lim_{s \to 0+} \frac{h'_+(s)}{s^{r-1}}$ and $\lim_{s\to 0+} \frac{h'_-(s)}{s^{r-1}}$ exist and coincide with*

$$r \lim_{s\to 0+} \frac{h_{R,I,J}(s)}{s^r} = r\frac{1}{r!} \sum_{P \text{ is minimal over} I} e_{HK}(J\frac{R}{P}, \frac{R}{P}) e(IR_P, R_P).$$

(2) *Let $f_{R,I,J}(s)$ be the density function associated to the pair (I, J) at s, when it exists. Then*

$$\lim_{s\to 0+} \frac{f_{R,I,J}(s)}{s^{r-1}} = r \lim_{s\to 0+} \frac{h_{R,I,J}(s)}{s^r} = r\frac{1}{r!} \sum_{P \text{ is minimal over} I} e_{HK}(J\frac{R}{P}, \frac{R}{P}) e(IR_P, R_P).$$

Proof. (1) First consider the case of $\lim_{s\to 0+} \frac{h'_+(s)}{s^{r-1}}$. By Proposition 5.10, $h'_+(s)/s^{r-1}$ is a decreasing function on the positive real line. So the limit exists. For positive real numbers $s_0 \leq t \leq s$, the decreasingness above implies

$$h'_+(s_0)\frac{t^{r-1}}{s_0^{r-1}} \geq h'_+(t) \geq h'_+(s)\frac{t^{r-1}}{s^{r-1}}. \tag{8.3}$$

Recall that outside a countable subset of $(0, \infty)$ the derivative of $h_{R,I,J}$ is h'_+. Thus taking integration on $[s_0, s]$, the above inequality implies

$$\frac{h'_+(s_0)}{s_0^{r-1}} \geq r\frac{h_{R,I,J}(s) - h_{R,I,J}(s_0)}{s^r - s_0^r} \geq \frac{h'_+(s)}{s^{r-1}}.$$

Taking limits the above chain of inequality gives,

$$\lim_{s_0\to 0+} \frac{h'_+(s_0)}{s_0^{r-1}} \geq \lim_{s\to 0+} \lim_{s_0\to 0+} r\frac{h_{R,I,J}(s) - h_{R,I,J}(s_0)}{s^r - s_0^r} = \lim_{s\to 0+} r\frac{h_{R,I,J}(s)}{s^r} \geq \lim_{s\to 0+} \frac{h'_+(s)}{s^{r-1}}.$$

The claimed equality in the last chain follows as $h(t)$ approaches zero as t approaches zero from right since we assume $r \geq 1$; see Theorem 8.12.

The case of $\lim_{s\to 0+} \frac{h'_-(s)}{s^{r-1}}$ follows by a similar argument once we use the decreasingness of $\frac{h'_-(s)}{s^{r-1}}$ on $(0, \infty)$ and that outside a countable set the derivative of $h_{R,I,J}$ is h'_-.

(2) Whenever $f_{R,I,J}(s)$ exists at some positive s, we have

$$h'_+(s) \leq f_{R,I,J}(s) \leq h'_-(s); \quad \text{see Theorem 5.8.}$$

Rest follows from these comparisons and part (1). ☐

Remark 8.16. (1) Note that Theorem 8.15 includes the case when I is $\mathfrak{m}$-primary. Indeed we can always assume that the residue field is infinite without loss of generality. So we can assume that I is integral over an ideal generated by system of parameters; see [HS06, Ch. 8].

(2) When $(R, \mathfrak{m})$ is not necessarily a domain, one can obtain analogues of Theorem 8.12 and Theorem 8.15 using the associativity formula Proposition 3.31.

9. Applications

9.1. **Comparison between Hilbert-Kunz and Hilbert-Samuel multiplicity.** Let $(R, \mathfrak{m})$ be an F-finite local ring of dimension d. It is well known that for an $\mathfrak{m}$-primary ideal I,

$$e(I) \geq e_{HK}(I, R) \geq \frac{e(I)}{d!}.$$

When d is at least 2, Watanabe and Yoshida asked whether the right most inequality is always strict; see [Kei00, Question 2.9][5]. Watanabe-Yoshida's question was affirmatively answered by Hanes by approximating an appropriate length function; see [Han03, Thm 2.2, 2.4]. We show that Watanabe-Yoshida's question is equivalent to a question of containment of ideals, which is a priori much weaker. The translation of the question regarding multiplicities to containment of ideals is facilitated by the appropriate h-function.

Proposition 9.1. *Let $(R, \mathfrak{m})$ be an F-finite noetherian ring of dimension d. For an $\mathfrak{m}$-primary ideal I, the following statements are equivalent.*

(1) $e_{HK}(I, R) > \frac{e(I)}{d!}$.
(2) *The minimal stable point $\alpha_{R,I,I}$ of $h_{R,I,I}$ is strictly larger than 1; see Theorem 8.7.*
(3) *There exists a $q = p^e$ such that I^{q+1} is not contained in $(I^{[q]})^*$.*

Remark 9.2. The fact that $e_{HK}(I, R)$ is strictly greater than $e(I)/d!$ implies that I^{q+1} can be contained in $(I^{[q]})^*$ only for finitely many q's. Indeed, otherwise $l(R/I^{q+1})/q^d$ is at least $l(R/(I^{[q]})^*)/q^d$ for infinitely many q's. Taking limit as q approaches infinity, this implies $e(I)/d! \geq e_{HK}(I, R)$; see Lemma 8.6. The point of the previous proposition is that assertion (3), which is much weaker, implies assertion (1).

Proof. The value of $h_{R,I,I}(s)$ at 1 and $\alpha_{R,I,I}$ are $e(I)/d!$ and $e_{HK}(I, R)$ respectively; see Lemma 8.3, Theorem 8.7. If (1) holds, $h_{R,I,I}(s)$ cannot be a constant on $[1, \alpha_{R,I,I}]$, so (2) follows. Now (2) implies that $h_{R,I,I}$ is a non-constant increasing function on $[1, \alpha_{R,I,I}]$; see Theorem 8.7. So (1) follows.

Now we argue that (2) and (3) are equivalent. Let $r_I^I(n)$ be as in Lemma 8.4. Then $(r_I^I(n)/p^n)_n$ is an nondecreasing sequence converging to $\alpha_{R,I,I}$; see Lemma 8.4, Theorem 8.7. If (2) holds $r_I^I(e)$ is strictly greater than p^e for some e, so (3) follows. Conversely if (3) holds, $r_I^I(e)$ must be strictly greater than p^e. So $\alpha_{R,I,I}$ must be strictly greater than 1. □

The line of argument in the above proposition shows that:

Corollary 9.3. Suppose $J \subset I$ are two $\mathfrak{m}$-primary ideals in a local ring $(R, \mathfrak{m})$. Suppose there exists some $q = p^e$ such that $I^{q+1} \not\subseteq (J^{[q]})^*$. Then

$$e_{HK}(J, R) > \frac{e(I)}{\dim(R)!}.$$

Remark 9.4. We do not know what motivated Watanabe-Yoshida to formulate the question mentioned above. But from the point of view of h-functions, this inequality seems probable. Indeed assume additionally that the density function $f_{R,I,I}$ is continuous at 1. Since the value of the density function at 1 is $\frac{1}{(\dim R-1)!}e(I) > 0$, $f_{R,I,I}$ remains positive in a neighborhood of 1. This implies $\alpha_{R,I,I} > 1$ and hence the inequality sought for by Watanabe-Yoshida follows; see Proposition 9.1. Although we do not know whether $f_{R,I,I}$ is continuous when $\mathrm{ht}(I)$ is at least 2, we expect that to be the case; see Question 10.2.

[5]The original question is restricted to the case $I = \mathfrak{m}$.

9.2. **F-threshold and multiplicity.** Comparisons among Hilbert-Samuel multiplicity, Hilbert-Kunz multiplicity, F-threshold are abound in the literature. We show that general properties of the h function combined with a very coarse approximation of it recover some of these.

Motivated by the comparison between Hilbert-Samuel multiplicity and log canonical threshold in [FEM04] and the analogy between F-threshold and log canonical threshold (see [MTW04, Thm 3.3, 3.4]) the following was conjectured:

Conjecture 9.5. *(see [Hun+08b, Conj 5.1]) Let $(R, \mathfrak{m})$ be a noetherian local ring of dimension d containing a positive charactersitic field. Let J be an ideal generated by a full system of parameters and I be an $\mathfrak{m}$-primary ideal. Then*

$$e(I) \geq \frac{d^d}{c^J(I)^d} e(J).$$

Here $e(-)$ denotes the Hilbert-Samuel multiplicity of the corresponding ideal.

Remark 9.6. We can assume I is generated by a system of parameters without loss of generality, in Conjecture 9.5. Indeed in Conjecture 9.5 one can first assume that the residue field is infinite by making standard constructions. Recall $e(I) = e(\overline{I})$ and $c^J(I) = c^J(\overline{I})$, where $\overline{I}$ is the integral closure of I (see [Hun+08b, Prop 2.2, (2)]). When the residue field is infinite, we can choose a system of parameters $f_1, \ldots, f_d$ such that $I = \overline{(f_1, \ldots, f_d)}$.

The above conjecture is settled when $(R, \mathfrak{m})$ is graded; see [HTW11], [Hun+08b]. Drawing motivations from [TW04, Prop 4.5] which confirms a special case of the above conjecture the following conjecture was made:

Conjecture 9.7. *(see [NS20, Conj 1.1]) Let $f_1, f_2, \ldots, f_r$ be part of a system of parameters of a noetherian local ring $(R, \mathfrak{m})$ of prime characteristic. Let J be an $\mathfrak{m}$-primary ideal. Set $I = (f_1, \ldots, f_r)R$. Then*

$$e_{HK}(J, R) \leq \left(\frac{c^J(I)}{r}\right)^r e_{HK}\left(J\frac{R}{I}, \frac{R}{I}\right).$$

We next point out that Conjecture 9.7, as stated, is false even when R is regular.

Proposition 9.8. *Take $(R, \mathfrak{m})$ to be the localization of a polynomial ring in d variables over a prime characteristic field. Take $J = \mathfrak{m}^t$ and $I = \mathfrak{m}$. Then for large enough t,*

$$e_{HK}(J, R) > \left(\frac{c^J(I)}{d}\right)^d e_{HK}\left(J\frac{R}{I}, \frac{R}{I}\right).$$

Thus for large t, Conjecture 9.7 fails.

Proof. Since R is regular, $e_{HK}(J, R)$ is the same as $l(R/J)$ which is just $\binom{d+t}{d}$. The F-threshold is $t + d - 1$; see [Hun+08b, Example 2.7, (iii)]. Since for large t,

$$\binom{t+d}{d} > \left(\frac{t+d-1}{d}\right)^d,$$

we are done. □

We now relate Conjecture 9.5 to Conjecture 9.7.

Proposition 9.9. *Let $(R, \mathfrak{m})$ be a noetherian local ring of prime characteristics. The following are equivalent:*

(1) *For every pair of $\mathfrak{m}$-primary ideals I, J generated by system of parameters*
$$e(I) \geq \frac{d^d}{c^J(I)^d} e(J).$$
(2) *For every pair of $\mathfrak{m}$-primary ideals I, J generated by system of parameters*
$$e_{HK}(J, R) \leq (\frac{c^J(I)}{d})^d e_{HK}(J\frac{R}{I}, \frac{R}{I}).$$
That is the restricted case of Conjecture 9.7, where both I, J are generated by system of parameters, is equivalent to Conjecture 9.5.

Proof. Assume (1). For an ideal generated by system of parameters the Hilbert-Kunz and Hilbert-Samuel multiplicities are the same; see [Lec57, Thm 2]. Since I is generated by a system of parameters
$$e_{HK}(J\frac{R}{I}, \frac{R}{I}) = l(\frac{R}{I}) \geq e(I).$$
This implies (2).

Now assume (2). Choose a system of parameters $f_1, f_2, \ldots, f_d$ so that $I = (f_1, f_2, \ldots, f_d)$. For any positive integer n, (2) yields,
$$e_{HK}(J, R) \leq (\frac{c^J((f_1^n, f_2^n, \ldots, f_d^n))}{d})^d l(\frac{R}{(f_1^n, f_2^n, \ldots, f_d^n)}).$$
Since I^n is in the integral closure of $(f_1^n, f_2^n, \ldots, f_d^n)$, $c^J((f_1^n, f_2^n, \ldots, f_d^n)) = c^J(I^n)$. Moreover $c^J(I^n) = c^J(I)/n$; see [Hun+08b, Prop 2.2, (3)]. So the last inequality gives
$$e_{HK}(J, R) \leq (\frac{c^J(I)}{d})^d \frac{l(\frac{R}{(f_1^n, f_2^n, \ldots, f_d^n)})}{n^d},$$
for all n. Taking limit as n approaches infinity in the last inequality, we obtain (1). □

In view of the above proposition, we believe that the corrected version of Conjecture 9.7 should be as follows:

Conjecture 9.10. *Let $f_1, f_2, \ldots, f_r$ be part of a system of parameters of a noetherian local ring $(R, \mathfrak{m})$ of prime characteristic. Let J be an ideal generated by a (full) system of parameters of R. Set $I = (f_1, \ldots, f_r)R$. Then*
$$e_{HK}(J, R) \leq (\frac{c^J(I)}{r})^r e_{HK}(J\frac{R}{I}, \frac{R}{I}).$$

In [NS20, Prop 2.1] establishes a comparison between Hilbert-Kunz multiplicity and the F-threshold, which proves Conjecture 9.7 when $r = 1$. This comparison appears as assertion (2) of the next theorem. We strengthen their result in Theorem 9.12 by using the property of h-function proven below.

Theorem 9.11. *Let $(R, \mathfrak{m})$ be a noetherian local domain. Let $x_1, x_2, \ldots, x_r$ be part of a system of parameters of R, where $r \geq 1$. Let I be the integral closure of $(x_1, \ldots, x_r)$. Let $J_\bullet$ be a family of ideals such that $I, J_\bullet$ satisfy **Condition C**. Then $h_{R,I,J_\bullet}(s)/s^r$ is a decreasing function on $(0, \infty)$*

Proof. For a positive s, let $h'_+(s)$ be the right hand derivative of $h_{R,I,J_\bullet}$ at s, which exists by Theorem 5.4, assertion (1). By Proposition 5.10, $h'_+(s)/s^{r-1}$ is decreasing on the positive real line. By Theorem 5.4, outside a countable subset of $(0, \infty)$, $h'_+(s)$ is the derivative of $h_{R,I,J_\bullet}$ at s. Now given positive real numbers $s_1 < s_2$ and any $s_0 \in (0, s_1)$, integrating $h'_+(s)$, we get

$$\frac{h_{R,I,J_\bullet}(s_1) - h_{R,I,J_\bullet}(s_0)}{s_1^r - s_0^r} \geq r\frac{h'_+(s_1)}{s_1^{r-1}} \geq \frac{h_{R,I,J_\bullet}(s_2) - h_{R,I,J_\bullet}(s_1)}{s_2^r - s_1^r}.$$

Taking limit as s_0 approaches zero from right in the above chain of inequalities and using Theorem 8.12, we get

$$\frac{h_{R,I,J_\bullet}(s_1)}{s_1^r} \geq \frac{h_{R,I,J_\bullet}(s_2) - h_{R,I,J_\bullet}(s_1)}{s_2^r - s_1^r}.$$

The last inequality implies $h_{R,I,J_\bullet}(s_1)/s_1^r \geq h_{R,I,J_\bullet}(s_2)/s_2^r$. $\square$

Theorem 9.12. *Let $f_1, f_2, \ldots, f_r$ be part of a system of parameters of a noetherian local domain $(R, \mathfrak{m})$ of prime characteristics with r positive; set $I = (f_1, \ldots, f_r)$. Let J be an $\mathfrak{m}$-primary ideal. Let $\alpha_{R,I,J}$ be the minimal stable point of $h_{R,I,J}$ as defined in Section 8.1. Then*

(1)

$$e_{HK}(J, R) \leq \frac{\alpha_{R,I,J}^r}{r!} \sum_{P \text{ is a minimal over prime } I} e_{HK}(J\frac{R}{P}, \frac{R}{P})e(IR_P, R_P).$$

(2)

$$e_{HK}(J, R) \leq \frac{c^J(I)^r}{r!} e_{HK}(J\frac{R}{I}, \frac{R}{I}).$$

Proof. First we point out (1) implies (2). We know $\alpha_{R,I,J} \leq c^J(I)$ (see Lemma 8.3, assertion (2)) and $e(IR_P, R_P) \leq l_{R_p}(R_P/IR_P)$. The last inequality holds as IR_P is generated by a system of parameters. Using these two comparisons in (1), we get

$$e_{HK}(J, R) \leq \frac{c^J(I)^r}{r!} \sum_{P \text{ minimal prime over} I} e_{HK}(J\frac{R}{P}, \frac{R}{P}) l_{R_p}(\frac{R_P}{IR_P}).$$

The right hand side of the above inequality is $e_{HK}(J\frac{R}{I}, \frac{R}{I})$; see [Hun13, Thm 3.14]. So (2) follows.

For (1), note, since $h_{R,I,J}(s)/s^r$ is decreasing on $(0, \infty)$ by Theorem 9.11, we have

$$\lim_{s \to 0+} \frac{h_{R,I,J}(s)}{s^r} \geq \frac{h_{R,I,J}(\alpha_{R,I,J})}{\alpha_{R,I,J}^r} = \frac{e_{HK}(J, R)}{\alpha_{R,I,J}^r}.$$

The above one sided limit is

$$\frac{1}{r!} \sum_{P \text{ minimal prime over } I} e_{HK}(J\frac{R}{P}, \frac{R}{P})e(IR_P, R_P),$$

by Theorem 8.12. So assertion (1) follows.

$\square$

Now we point out an equivalent formulation of Conjecture 9.5 phrased in terms of h functions.

Proposition 9.13. *Let $(R,)$ be a local domain of Krull dimension $d \geq 1$, J be an ideal generated by system of parameters, I be an $\mathfrak{m}$-primary ideal. The following are equivalent.*

(1)

$$e(I) \geq \frac{d^d}{c^J(I)^d} e(J).$$

(2) *There exists* $x_0 \in [0, c^J(I)]$ *such that*

$$\frac{e(I)}{d^d} \geq \frac{h_{R,I,J}(x_0)}{x_0^d}.$$

Proof. Assume (1) holds. One can extend $h_{R,I,J}(s)/s^d$ to a continuous function on $[0, \infty)$ whose value at zero is $e(I)/d!$; see Theorem 8.12, Theorem 8.14. Since $h_{R,I,J}(s)/s^d$ evaluates to $e_{HK}(J, R)/c^J(I)^d$ at $s = c^J(I)$ and

$$\frac{e(I)}{d!} \geq \frac{e(I)}{d^d} \geq \frac{e_{HK}(J, R)}{c^J(I)^d} = \frac{e(J)}{c^J(I)^d},$$

using the decreasingness of $h_{R,I,J}(s)/s^d$, we get (2).

Now assume (2) holds. Since $h_{R,I,J}(s)/s^d$ is decreasing by Theorem 9.11,

$$\frac{h_{R,I,J}(x_0)}{x_0^d} \geq \frac{h_{R,I,J}(c^J(I))}{c^J(I)^d} = \frac{e(J)}{c^J(I)^d}.$$

So (1) follows from (2). □

10. Questions

Question 10.1. Let $(R, \mathfrak{m})$ be an F-finite ring; J be an $\mathfrak{m}$-primary ideal, I be any ideal. Is the minimal stable point $\alpha_{R,I,J}$ of $h_{R,I,J}$ the same as the F-threshold $c^J(I)$?

In view of Theorem 8.7, the above question is a question about asymptotic comparisons of $J^{[q]}$ and $(J^{[q]})^*$. Moreover, in view of Theorem 9.12, one may hope to replace $c^J(I)$ by potentially the smaller number $\alpha_{R,I,J}$ in Conjecture 9.10 or Conjecture 9.5. So this question tests the veracity of this naive hope.

Question 10.2. Let $(R, \mathfrak{m})$ be an F-finite ring; I, J be ideals such that $I + J$ is $\mathfrak{m}$-primary.

(1) Find conditions on (R, I, J) such that the limit defining the Hilbert-Kunz density function $f_{R,I,J}(s)$:

$$\lim_{q\to\infty} \frac{1}{q^{d-1}} l\left(\frac{I^{\lceil sq\rceil} + J^{[q]}}{I^{\lceil sq\rceil+1} + J^{[q]}}\right),$$

exists at all $s \in \mathbb{R}$.

(2) Find conditions on (R, I, J) such the Hilbert-Kunz density function $f_{R,I,J}(s)$ is continuous on $(0, \infty)$.

Recall that continuity of the density function is equivalent to the corresponding h-function being continuously differentiable; see Theorem 6.4. Our result suggests that a larger value of $\mathrm{ht}(I)$ may imply a better smoothness property of the h-function; see Theorem 8.12. So we wonder whether both the questions above have affirmative answers when $\mathrm{ht}(I)$ is at least 2; see Remark 9.4 for a consequence of affirmative answers. Recall that when R is standard graded of dimension at least 2 and $I = \mathfrak{m}$, for any homogeneous ideal J, the answer to both the questions are affirmative; see Theorem 6.7.

Inspired by Trivedi's question [Tri23, Question 2], we ask

Question 10.3. Let I, J be $\mathfrak{m}$-primary ideals of a noetherian local ring R of dimension at least two. Is $h_{R,I,J}$ a piecewise polynomial? In other words, does there exists a countable subset S of $\mathbb{R}$ and a partition $\mathbb{R} \setminus S = \coprod_{n\in\mathbb{N}} (a_n, b_n)$ such that on each (a_n, b_n), $h_{R,I,J}$ is given by a polynomial function?

We point out that, in the context of the question, $h_{R,I,J}(s)$ is $e_{HK}(J, R)$ for large s, $e(I, R)s^{\dim(R)}/\dim(R)!$ on some interval $(0, a]$ and zero for s nonpositive.

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