

# Stability of Sequential Lateration and of Stress Minimization in the Presence of Noise

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## Abstract

Sequential lateration is a class of methods for multidimensional scaling where a suitable subset of nodes is first embedded by some method, e.g., a clique embedded by classical scaling, and then the remaining nodes are recursively embedded by lateration. A graph is a lateration graph when it can be embedded by such a procedure. We provide a stability result for a particular variant of sequential lateration. We do so in a setting where the dissimilarities represent noisy Euclidean distances between nodes in a geometric lateration graph. We then deduce, as a corollary, a perturbation bound for stress minimization. To argue that our setting applies broadly, we show that a (large) random geometric graph is a lateration graph with high probability under mild conditions, extending a previous result of Aspnes et al (2006).

## 1 Introduction

In multidimensional scaling (MDS), we are provided with some pairwise dissimilarities between a number of items, and the general goal is to embed these items as points in a Euclidean space of given dimension in such a way that the resulting Euclidean distances reproduce, as faithfully as possible, the dissimilarities. MDS is a well-studied problem in psychometrics [14], mathematics and computer science (embedding of metric spaces) [13], in optimization (Euclidean distance matrix completion) [52], and engineering (sensor network localization) [57], and it is an integral part of multivariate statistical analysis [4, 64] and unsupervised machine learning [35]. MDS is closely related to the problem of graph drawing [10, 44].

### 1.1 Setting

More formally, we are given an undirected graph  $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ , with node set  $\mathcal{V} = [n] := \{1, \dots, n\}$  and edge set  $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ , together with non-negative weights on the edges. The weight on  $(i, j) \in \mathcal{E}$  is referred to as the dissimilarity between  $i$  and  $j$ , and denoted  $d_{ij}$ . The (possibly incomplete) matrix  $D = (d_{ij})$  stores these dissimilarities. Based on this information, we seek to embed the nodes into a Euclidean space of given dimension, denoted  $p$ , as accurately as possible. Specifically, we seek a configuration  $y_1, \dots, y_n \in \mathbb{R}^p$  such that  $\|y_i - y_j\| \approx d_{ij}$  for all or most  $(i, j) \in \mathcal{E}$ . A notion of stress, for example, the s-stress of Takane et al. [70] defined as

$$\sum_{(i,j) \in \mathcal{E}} (\|y_i - y_j\|^2 - d_{ij}^2)^2, \quad (1.1)$$

offers a way to quantify the accuracy of the embedding. Throughout, the dimension  $p$  will be assumed given and  $\|\cdot\|$  will denote the Euclidean norm in  $\mathbb{R}^p$ .

We say that the graph is realizable (in dimension  $p$ ) if there is a point set  $y_1, \dots, y_n \in \mathbb{R}^p$  such that  $\|y_i - y_j\| = d_{ij}$  for all  $(i, j) \in \mathcal{E}$ , or in words, if there is a configuration with zero stress. In this paper we are most interested in the noisy realizable situation in which

$$d_{ij}^2 = \|x_i - x_j\|^2 + \varepsilon_{ij}, \quad (i, j) \in \mathcal{E}, \quad (1.2)$$

where  $\{x_1, \dots, x_n\} \in \mathbb{R}^p$  will be referred to as the *latent configuration*<sup>1</sup> and  $\{\varepsilon_{ij} : (i, j) \in \mathcal{E}\}$  represents measurement noise, possibly stochastic. This additive noise model is considered in a number of places, e.g., [3, 41, 54]. It includes, as a special case, the following multiplicative noise model

$$d_{ij} = (1 + \eta_{ij})\|x_i - x_j\|, \quad (i, j) \in \mathcal{E},$$

by simply setting  $\varepsilon_{ij} = 2\eta_{ij}\|x_i - x_j\| + \eta_{ij}^2\|x_i - x_j\|^2$  in (1.2). Although the model (1.2) is in principle completely general, in our results we will bound the error terms. It is possible to study the problem under more general assumptions as recently done in [47, 55], but the model above is most appropriate for our purposes as will become clear below.

## 1.2 Methods

A wide array of approaches have been proposed to tackle this problem, starting with *classical scaling*, the oldest and still the most popular method, proposed by Torgerson [73, 74] and further developed by Gower [32], with roots in a mathematical inquiry by Young and Householder [81] into necessary and sufficient conditions “for a set of numbers to be the mutual distances of a set of real points in Euclidean space” — to quote the abstract of their cornerstone paper. Kruskal [48, 49] formulated the problem as minimizing a notion of stress that he suggested for that purpose — same as (1.1) but without the squares inside the brackets. Many other optimization approaches have been tried, including second order methods [42], as well as other Newton and quasi-Newton variant procedures [30, 43]; augmentation and majorization [22, 36], which include the SMACOF algorithm [23, 24, 56], itself closely related to the fixed point iteration approach of Guttman [34]; incremental and multigrid approaches [15, 19, 78]; divide-and-conquer or patch-stitching algorithms [21, 26, 38, 45, 46, 65, 68, 75, 80, 82]; semidefinite programming (SDP) formulations where the constraint on the embedding dimension is removed [1, 11, 12, 17, 26, 41, 69, 77]; and the completion of the dissimilarity matrix by graph distances before applying a method like classical scaling [50, 60, 62, 66]. See the book by Borg and Groenen [14] and the PhD thesis of Klimentka [44, Ch 2, 3] for partial reviews of the literature.

### Sequential lateration

We place our attention on *sequential lateration*, which is an approach in which a suitable subset of nodes is first embedded by some method — e.g., a clique embedded by classical scaling — and then the remaining nodes are recursively embedded by lateration [8, 9, 27, 29, 33, 43, 53].

*Lateration* is the problem of locating a point based on its (possibly inaccurate) distances to a set of given points often referred to as anchors, beacons or landmarks. The problem is known under

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<sup>1</sup>Note that the latent configuration is only determined up to a rigid transformation, as we do not assume that any anchor is available. However, this duplicity does not cause any trouble.

different names, including ‘trilateration’ or ‘multilateration’, or simply ‘lateration’, in engineering [5, 18, 28, 59, 63, 79], while ‘external unfolding’ is favored in psychometrics [14, 16].

Aspnès et al. [8], developing ideas already present in their prior work [27], introduce<sup>2</sup> the notion of *lateration graph* (in dimension  $p$ ), which they define as a graph with  $n \geq p + 1$  vertices that admits an ordering of its vertices, say  $v_1, \dots, v_n$ , such that the subgraph induced by  $v_1, \dots, v_{p+1}$  is complete and, for each  $j > p + 1$ ,  $v_j$  is connected to at least  $p + 1$  vertices among  $v_1, \dots, v_{j-1}$  — they call this a *laterative ordering* (in dimension  $p$ ). They show in [8, Th 10] that the problem of Section 1.1 is solvable in polynomial time by sequential lateration in the realizable situation (1.2) (with  $\varepsilon_{ij} \equiv 0$ ) when the latent points  $x_1, \dots, x_n$  are in general position and the graph  $(\mathcal{V}, \mathcal{E})$  is a lateration graph. We say that a configuration is in general position if any  $(p + 1)$ -tuple from the configuration spans the entire space.

### 1.3 Contribution and content

Our main contribution is establishing a perturbation bound for sequential lateration. Such a bound helps us understand how the performance of a method degrades with the presence of noise. While, as already mentioned, sequential lateration is exact in the realizable setting when the latent points are in general position and the graph is a lateration graph, our study provides an understanding of how the method behaves in the noisy realizable setting (1.2).

As our second contribution, we use our perturbation bound for sequential lateration to derive a perturbation bound for stress minimization in the same setting of a lateration graph. Although stress minimization is not an algorithm per se, we show that the set of configurations that minimize the stress (1.1) is stable in the presence of noise in the sense that any minimizing configuration is within a distance (up to rigid transformations) to the latent configuration in (1.2) controlled in terms of the amount of noise.

Only a few perturbation bounds exist in the MDS literature. For classical scaling, some partial results were developed early on by Sibson [67] and later revisited by de Silva and Tenenbaum [25], but a true perturbation bound was only established recently in [6], where perturbation bounds for the completion by graph distance method of Kruskal and Seery [50] and the SDP method of Weinberger et al. [77] — in the context of manifold learning in the form of isomap [71] and maximum variance unfolding [76] — were also obtained. Similarly, some perturbative results were derived in [25] for lateration, but a true perturbation bound was only achieved in [6] (to our knowledge). Moore et al. [58], inspired by the earlier work of Eren et al. [27], propose a method for sequential trilateration and carry out a very limited mathematical analysis, restricting themselves to analyzing the probability of a gross error or ‘flip’ in one trilateration step. We are not aware of any other results.

Our perturbation bound for sequential lateration was perhaps anticipated by Anderson et al. [3], who discuss this as an open question in their last section:

“An important problem, linked but separate from the one treated in this paper, is how (numerically) to solve the minimization problem. The corresponding problem in the noiseless case is how to perform localization. For a localization problem to be solvable in polynomial time, it is generally necessary that some special structure holds for the graph; for example, in the case of trilateration graphs, localization can be done in linear time with suitable anchors [2]. We would expect, although we have no formal proof, that

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<sup>2</sup>They use ‘trilateration graph’ and ‘trilaterative ordering’ as they focus on the case of dimension  $p = 2$ .

such geometries will also be important in ensuring that a noisy localization problem is computationally tractable.”

Perturbation bounds and, more generally, a better understanding of the MDS problem under noise, were open problems discussed at length by Mao et al. [57] in their well-cited review paper of the engineering literature on the topic. We leverage our perturbation bound for sequential lateration to obtain another result that contributes to that endeavor: we show, in the same context, that any configuration that minimizes the stress (1.1) is necessarily close to the latent configuration. In doing so, we recover a result of Anderson et al. [3] in the special case of a lateration graph.

Although not all graphs are lateration graphs, the setting covers the main stochastic model used in the literature, that of a random geometric graph. While this was already known to Aspnes et al. [8], Eren et al. [27], as our third contribution, we provide a much more general result, showing that a large random geometric graph is a lateration graph with high probability under very mild assumptions on the underlying sampling distribution.

The remainder of the paper is organized as follows. In Section 2, we derive a perturbation bound for sequential lateration. In Section 3, we obtain as a corollary a perturbation bound for stress minimization. This is placed in the broader context of *rigidity theory*. In Section 4, we provide rather mild conditions under which a large random geometric graph is a lateration graph with high probability. Some numerical experiments meant to illustrate the theory are presented in Section 5. And Section 6 is a discussion section.

## 2 Perturbation bound for sequential lateration

The particular variant of sequential lateration that we work with is based on classical scaling and what we call *classical lateration*, a method for lateration that was originally proposed by Gower [32] and later rediscovered by de Silva and Tenenbaum [25], and is the analog of classical scaling for the lateration problem.

The procedure works as follows. For each  $(p+1)$ -tuple of nodes within  $\mathcal{V} = [n]$ , if it is complete, meaning that the  $(p+1)$ -tuple forms a clique, we embed it by classical scaling; we then recursively embed by classical lateration any node that is neighbor to at least  $p+1$  nodes that have already been embedded. We can think of two main variants: in the ‘first’ variant, we stop at the first full embedding achieved in this manner; in the ‘best’ variant, we go through all full embeddings and select the one with smallest stress (1.1). Both variants run in polynomial time, although the ‘best’ variant is prohibitively expensive to run in practice, having a complexity of order  $\asymp n^{p+1}$  since there are  $(p+1)$ -tuples to go through and, for each of them, running the sequential lateration has complexity  $O(n)$ . Our perturbation bound applies to either variant, and any other variant ‘in between’.

**Theorem 2.1.** *In the context of Section 1.1, consider a noisy realizable situation as in (1.2) in which the network structure  $(\mathcal{V}, \mathcal{E})$  is a lateration graph and the latent configuration is in general position. Then, there is  $\sigma > 0$  and  $C > 0$  such that, if  $\sum_{(i,j) \in \mathcal{E}} \varepsilon_{ij}^2 \leq \sigma^2$ , sequential lateration outputs an embedding  $y_1, \dots, y_n$  satisfying*

$$\min_g \sum_{i \in [n]} \|y_i - g(x_i)\|^2 \leq C \sum_{(i,j) \in \mathcal{E}} \varepsilon_{ij}^2, \quad (2.1)$$

where the minimization is over the rigid group of transformations of  $\mathbb{R}^p$ .

As in [3], the constants  $\sigma$  and  $C$  depend on the graph and the latent configuration.

The proof of Theorem 2.1 occupies the rest of this section. It uses two perturbation bounds from [6], one for classical scaling (Lemma 2.2) and one for classical lateration (Lemma 2.3). We start with a stability result for classical scaling.

**Lemma 2.2** (Corollary 2 in [6]). *Consider a configuration  $x_1, \dots, x_m \in \mathbb{R}^p$  that affinely spans the entire space, and a complete set of dissimilarities  $(d_{ij})$ , and define*

$$\eta^4 = \sum_{1 \leq i < j \leq m} (d_{ij}^2 - \|x_i - x_j\|^2)^2.$$

*Then there are constants  $\eta_0 > 0$  and  $A > 0$  depending on the configuration such that, if  $\eta \leq \eta_0$ , then classical scaling with input dissimilarities  $(d_{ij})$  (and given dimension  $p$ ) returns a configuration  $y_1, \dots, y_m$  satisfying*

$$\min_g \sum_{i \in [m]} \|y_i - g(x_i)\|^2 \leq A\eta^4,$$

*where the minimization is over the rigid group of transformations.*

Next is a stability result for classical lateration — the lateration method that we consider — where stability is considered with respect to noise both at the level of the dissimilarities and at the level of the landmarks.

**Lemma 2.3** (Corollary 3 in [6]). *Consider a configuration  $x_1, \dots, x_m \in \mathbb{R}^p$  that affinely spans the entire space, and an arbitrary point  $x \in \mathbb{R}^p$ . Let  $y_1, \dots, y_m \in \mathbb{R}^p$  be another configuration and let  $d_1, \dots, d_m$  be set of dissimilarities, and define*

$$\nu^2 = \sum_{i \in [m]} \|y_i - x_i\|^2, \quad \text{and} \quad \eta^4 = \sum_{i \in [m]} (d_i^2 - \|x - x_i\|^2)^2.$$

*Then there are constants  $\nu_0 > 0$  and  $B > 0$  depending on the configuration  $x_1, \dots, x_m$  such that, if  $\nu \leq \nu_0$ , classical lateration with inputs  $y_1, \dots, y_m$  and  $d_1, \dots, d_m$  outputs an embedding  $y$  satisfying*

$$\|y - x\|^2 \leq B(\nu^2 + \eta^4).$$

In the statement,  $x_1, \dots, x_m$  play the role of landmarks and  $x$  is the unknown point to be recovered;  $y_1, \dots, y_m$  should be seen as noisy versions of  $x_1, \dots, x_m$ , and  $d_1, \dots, d_m$  should be seen as noisy versions of  $\|x - x_1\|, \dots, \|x - x_m\|$ .

*Proof of Theorem 2.1.* Assume without loss of generality that  $(1, \dots, n)$  is already a laterative ordering. We first apply classical scaling to  $(d_{ij})_{1 \leq i < j \leq p+1}$ , which we can do since these dissimilarities are available because of the assumption that  $(1, \dots, n)$  is a laterative ordering. We let  $y_1, \dots, y_{p+1}$  be the output and use Lemma 2.2 to quantify its accuracy. By assumption,

$$\eta^4 := \sum_{1 \leq i < j \leq p+1} \varepsilon_{ij}^2 \leq \sum_{(i,j) \in \mathcal{E}} \varepsilon_{ij}^2 \leq \sigma^2.$$

Let  $\eta_0 > 0$  and  $A > 0$  denote the constants of Lemma 2.2, which depend on the configuration  $x_1, \dots, x_{p+1}$ , and therefore, on the entire configuration  $x_1, \dots, x_n$ . Assuming  $\sigma$  is small enough that  $\sigma^2 \leq \eta_0^4$ , so that  $\eta \leq \eta_0$ , that same lemma gives that

$$\sum_{1 \leq i \leq p+1} \|y_i - g(x_i)\|^2 \leq A \sum_{1 \leq i < j \leq p+1} \varepsilon_{ij}^2,$$

for some rigid transformation  $g$ . Henceforth, we assume that  $g$  is the identity transformation, which we can do without loss of generality.

Define

$$\mathcal{E}_k := \{(i, j) \in \mathcal{E} : i, j \in [k]\} = \mathcal{E} \cap [k]^2,$$

and, for  $k \geq p + 1$ , assume that we have already embedded  $y_1, \dots, y_k$  in such a way that, for some constant  $C_k$  depending on the configuration,

$$\sum_{1 \leq i \leq k} \|y_i - x_i\|^2 \leq C_k \sum_{(i,j) \in \mathcal{E}_k} \varepsilon_{ij}^2. \quad (2.2)$$

We are now poised to obtain  $y_{k+1}$  by lateration based on  $p + 1$  points in  $\{y_1, \dots, y_k\}$  and the corresponding dissimilarities. Assume these points to be  $y_{i_1}, \dots, y_{i_{p+1}}$ , so that the corresponding dissimilarities are  $d_{k+1, i_1}, \dots, d_{k+1, i_{p+1}}$ , and  $y_{k+1}$  is the output of classical lateration based on these inputs. We have

$$\begin{aligned} \nu_{k+1}^2 &:= \sum_{j=1}^{p+1} \|y_{i_j} - x_{i_j}\|^2 \leq \sum_{1 \leq i \leq k+1} \|y_i - x_i\|^2 \\ &\leq C_k \sum_{(i,j) \in \mathcal{E}_k} \varepsilon_{ij}^2 \leq C_k \sum_{(i,j) \in \mathcal{E}} \varepsilon_{ij}^2 \leq C_k \sigma^2, \end{aligned}$$

using our induction hypothesis (2.2) and by assumption. And we also have

$$\eta_{k+1}^4 := \sum_{j=1}^{p+1} (d_{k+1, i_j}^2 - \|x_{k+1} - x_{i_j}\|^2)^2 = \sum_{j=1}^{p+1} \varepsilon_{k+1, i_j}^2 \leq \sum_{i=1}^k \varepsilon_{k+1, i}^2.$$

Let  $\nu_0 > 0$  and  $B > 0$  denote the constants of Lemma 2.3, which depend on the configuration  $x_{i_1}, \dots, x_{i_{p+1}}$ , and therefore, on the entire configuration  $x_1, \dots, x_n$ . Assuming that  $\sigma$  is small enough that  $C_k \sigma^2 \leq \nu_0^2$ , so that  $\nu_{k+1} \leq \nu_0$ , the same lemma then gives us the error bound

$$\begin{aligned} \|y_{k+1} - x_{k+1}\|^2 &\leq B\nu_{k+1}^2 + B\eta_{k+1}^4 \\ &\leq BC_k \sum_{(i,j) \in \mathcal{E}_k} \varepsilon_{ij}^2 + B \sum_{i=1}^k \varepsilon_{k+1, i}^2 \\ &\leq B(C_k + 1) \sum_{(i,j) \in \mathcal{E}_{k+1}} \varepsilon_{ij}^2, \end{aligned}$$

thus enabling us to continue the induction. At the end of the induction, when all the  $y_i$  have been embedded, we obtain the announced bound (2.1).  $\square$

### 3 Perturbation bound for stress minimization

In the noisy realizable setting (1.2), the stress clearly functions as a proxy for the *noiseless stress*, defined as

$$\sum_{(i,j) \in \mathcal{E}} (\|y_i - y_j\|^2 - \|x_i - x_j\|^2)^2. \quad (3.1)$$

In turn, the noiseless stress functions as a proxy for the *complete noiseless stress*, defined as

$$\sum_{1 \leq i < j \leq n} (\|y_i - y_j\|^2 - \|x_i - x_j\|^2)^2. \quad (3.2)$$

We establish below that, in some actionable sense, the stress tracks the complete noiseless stress in the context of a lateration graph.

### 3.1 Rigidity theory

To investigate this, we turn to *rigidity theory*, which examines the question of uniqueness (up to a rigid transformation) when realizing a weighted graph in a given Euclidean space [7, 51, 72]. We introduce some vocabulary from that literature (in particular, from [20]). As we have already seen, a configuration is a set of  $n$  points in  $\mathbb{R}^p$  indexed by  $[n] = \{1, \dots, n\}$ . A configuration is *generic* if the set of its coordinates do not satisfy any nonzero polynomial equation with integer coefficients. We say that two configurations  $\mathbf{y} = \{y_1, \dots, y_n\}$  and  $\mathbf{z} = \{z_1, \dots, z_n\}$  are *congruent* if there is a rigid transformation  $f : \mathbb{R}^p \rightarrow \mathbb{R}^p$  such that  $z_i = f(y_i)$  for all  $i \in [n]$ . A configuration  $\mathbf{y} = \{y_1, \dots, y_n\}$  and a graph  $\mathcal{G} = (\mathcal{V} = [n], \mathcal{E})$ , together, form a *framework*, denoted  $\mathcal{G}(\mathbf{y})$ . We say that two frameworks,  $\mathcal{G}(\mathbf{y})$  and  $\mathcal{G}(\mathbf{z})$  are *equivalent* if

$$\|y_i - y_j\| = \|z_i - z_j\|, \quad \forall (i, j) \in \mathcal{E}.$$

The framework  $\mathcal{G}(\mathbf{y})$  is said to be *globally rigid* if, whenever  $\mathcal{G}(\mathbf{y})$  and  $\mathcal{G}(\mathbf{z})$  are equivalent, then necessarily  $\mathbf{y}$  and  $\mathbf{z}$  are congruent. The graph  $\mathcal{G}$  is said to be *generically globally rigid* if  $\mathcal{G}(\mathbf{y})$  is globally rigid whenever  $\mathbf{y}$  is generic.

The complete noiseless stress (3.2) is exactly zero when  $\|y_i - y_j\| = \|x_i - x_j\|$  for all  $i < j$ , and we know this to be equivalent to  $\mathbf{y} = \{y_1, \dots, y_n\}$  and  $\mathbf{x} = \{x_1, \dots, x_n\}$  being congruent. For the noiseless stress (3.1), the same is true if  $\mathcal{G}(\mathbf{x})$  is globally rigid. This is by mere definition, and we would like to know when this happens. Also by definition, it happens when  $\mathbf{x}$  is a generic configuration and  $\mathcal{G}$  is *generically globally rigid*.

Generic configurations are ‘common’ in the sense that those configurations that are not generic have zero Lebesgue measure (in  $\mathbb{R}^{np}$ ). This is simply because there are countably many polynomials with integer coefficients and each one of these defines a surface (its null set) of zero Lebesgue measure. In particular, if a configuration is drawn iid at random from a density, then the configuration is generic with probability one.

The question of whether a graph is generically globally rigid or not, is a delicate question. In the very special but useful case of dimension  $p = 2$ , Jackson and Jordán [39] have shown that if the graph is 6-vertex connected, meaning that it remains connected even after the removal of any 5 vertices, then the graph is generically globally rigid. The situation in dimension  $p \geq 3$  is more complex, although some useful results exist; see, e.g., [2, 37]. A necessary and sufficient condition exists in terms of the existence of an *equilibrium stress matrix*, which for a framework  $\mathcal{G}(\mathbf{x})$  is defined as a matrix  $\omega = (\omega_{ij})$  satisfying  $\sum_{j:(i,j) \in \mathcal{E}} \omega_{ij} (x_i - x_j) = 0$  for all  $i \in [n]$ . To a stress matrix  $\omega$ , we associate another matrix  $\Omega = (\Omega_{ij})$  with  $\Omega_{ij} = -\omega_{ij}$  when  $i \neq j$ , and  $\Omega_{ii} = \sum_j \omega_{ij}$ . (If we see  $\omega$  as the weight matrix of a graph, then  $\Omega$  is the corresponding Laplacian.) Connelly [20] and Gortler et al. [31], together, have shown that if  $\mathcal{G}$  has  $n \geq p + 2$  nodes and is not the complete graph, and if  $\mathbf{x}$  is a generic configuration, then  $\mathcal{G}(\mathbf{x})$  is globally rigid if and only if there is a equilibrium stress matrix  $\omega$  with  $\text{rank } \Omega = n - p - 1$ .

### 3.2 Rigidity theory in the presence of noise

What we have learned so far is that, if the graph  $\mathcal{G} = (\mathcal{V}, \mathcal{E})$  given in the embedding problem is generically globally rigid, and we are in a realizable situation with an underlying configuration  $x_1, \dots, x_n$  that is generic, then the noiseless stress (3.1) is minimized exactly where the complete noiseless stress (3.2) is minimized, that is, at all the rigid transformations of the configuration. These conditions are fulfilled with high probability by a random geometric graph under additional mild conditions (Section 4). But all this does not imply much about the noisy stress (1.1).

While most of the literature on rigidity theory focuses on the noiseless setting, Anderson et al. [3] consider the question of stability in the presence of noise. They do so in the realizable setting in dimension  $p = 2$ , and in the setting where *anchors* are given. (Anchors are points whose position is known.) The graph is generically globally rigid with an underlying generic configuration. With anchors, the configuration is effectively unique, not just up to a rigid transformation. In this context, they show that the distance between the minimizer of the stress (1.1) constrained by the anchors and the underlying configuration is bounded by a constant multiple of the noise amplitude. Their analysis is based on the results of Connolly [20] and Gortler et al. [31] mentioned above.

We prove an analogous result in the present anchor-free setting for an arbitrary embedding dimension. We do so for lateration graphs, which in addition to including important models (Section 4), allows for a completely different proof based on the perturbation bound just established in Theorem 2.1.

**Theorem 3.1.** *In the context of Section 1.1, consider a noisy realizable situation as in (1.2) in which the network structure  $(\mathcal{V}, \mathcal{E})$  is a lateration graph and the latent configuration is in general position. Then, there is  $\sigma > 0$  and  $C > 0$  such that, if  $\sum_{(i,j) \in \mathcal{E}} \varepsilon_{ij}^2 \leq \sigma^2$ , any minimizer  $y_1^*, \dots, y_n^*$  of the stress (1.1) satisfies*

$$\min_g \sum_{i \in [n]} \|y_i^* - g(x_i)\|^2 \leq C \sum_{(i,j) \in \mathcal{E}} \varepsilon_{ij}^2,$$

where the minimization is over the rigid group of transformations.

Once again, and as is the case in [3], the constants  $\sigma$  and  $C$  depend on the graph and the latent configuration, namely, on the framework  $\mathcal{G}(\mathbf{x})$ . (As it turns out, the proof below shows that we can use the same  $\sigma$  and a small multiple of the constant  $C$  of Theorem 2.1.)

*Proof.* We first bound the minimum value of the stress. Let  $y_1, \dots, y_n$  be the embedding given by sequential lateration. Let  $\sigma > 0$  and  $C_0 > 0$  denote the constants of Theorem 2.1. Assuming that  $\sum_{(i,j) \in \mathcal{E}} \varepsilon_{ij}^2 \leq \sigma^2$ , the theorem gives that

$$\sum_{i \in [n]} \|y_i - g_0(x_i)\|^2 \leq C_0 \sum_{(i,j) \in \mathcal{E}} \varepsilon_{ij}^2, \quad (3.3)$$

for some rigid transformation  $g_0$ . Let  $y_1^*, \dots, y_n^*$  be a stress minimizer. Since  $x_1, \dots, x_n$  is feasible, it must be the case that the stress achieved by  $y_1^*, \dots, y_n^*$  is not larger than the stress achieved by  $x_1, \dots, x_n$ , so that

$$\sum_{(i,j) \in \mathcal{E}} (\|y_i^* - y_j^*\|^2 - d_{ij}^2)^2 \leq \sum_{(i,j) \in \mathcal{E}} (\|x_i - x_j\|^2 - d_{ij}^2)^2 = \sum_{(i,j) \in \mathcal{E}} \varepsilon_{ij}^2. \quad (3.4)$$

Therefore, if we define  $\xi_{ij} = d_{ij}^2 - \|y_i^* - y_j^*\|^2$ , we have that

$$\sum_{(i,j) \in \mathcal{E}} \xi_{ij}^2 \leq \sum_{(i,j) \in \mathcal{E}} \varepsilon_{ij}^2 \leq \sigma^2. \quad (3.5)$$

Applying Theorem 2.1 with the same dissimilarities  $(d_{ij})$  and same graph structure but the configuration  $y_1^*, \dots, y_n^*$  instead of the configuration  $x_1, \dots, x_n$ , which we can do because of (3.4), we obtain that

$$\sum_{i \in [n]} \|y_i - g_1(y_i^*)\|^2 \leq C_0 \sum_{(i,j) \in \mathcal{E}} \xi_{ij}^2, \quad (3.6)$$

for some rigid transformation  $g_1$ . Combining (3.3) and (3.6), together with (3.5), and using the triangle inequality, yields

$$\begin{aligned} \sum_{i \in [n]} \|g_1(y_i^*) - g_0(x_i)\|^2 &\leq 2 \sum_{i \in [n]} \|g_1(y_i^*) - y_i\|^2 + 2 \sum_{i \in [n]} \|y_i - g_0(x_i)\|^2 \\ &\leq 2C_0 \sum_{i \in [n]} \xi_{ij}^2 + 2C_0 \sum_{(i,j) \in \mathcal{E}} \varepsilon_{ij}^2 \leq 4C_0 \sum_{(i,j) \in \mathcal{E}} \varepsilon_{ij}^2. \end{aligned}$$

We then conclude the proof by observing that  $\|g_1(y_i^*) - g_0(x_i)\| = \|y_i^* - g_2(x_i)\|$  with  $g_2 := g_1^{-1} \circ g_0$  being a rigid transformation.  $\square$

## 4 Random geometric graphs

In the literature, the main stochastic model is a *random geometric graph* [61]. Such a graph has node set representing points that are drawn iid from some distribution on  $\mathbb{R}^p$  and edges between any two of these points within distance  $r$ . For example, Aspnes et al. [8] show that, for the uniform distribution on  $[0, 1]^2$ , as the size of the configuration increases, if the connectivity radius is not too small, the probability that the resulting graph is generically globally rigid, and that the corresponding framework is globally rigid, tends to one. We generalize their result.

**Theorem 4.1.** *Suppose a configuration of cardinality  $n$  is drawn iid from a density supported on  $\bar{\Omega} \subset \mathbb{R}^p$ , where  $\Omega$  is bounded, open, and connected. Considering the asymptotic regime  $n \rightarrow \infty$ , there is  $r_n \rightarrow 0$  such that a graph built on this configuration with a connectivity radius  $r \geq r_n$  is a lateration graph with probability tending to one.*

The conditions on the support of the distribution generating the locations of the sensors are very mild. We could even relax the condition that  $\Omega$  is connected as long as the connectivity radius  $r$  exceeds the maximum separation between its connected components.

*Proof.* Let  $\mathcal{G}_r(\mathbf{x})$  be the neighborhood graph with connectivity radius  $r$  built on the point set  $\mathbf{x} = \{x_1, \dots, x_n\}$ . It is obvious that the property of being a lateration graph is monotonic in  $r$  in the sense that if  $\mathcal{G}_r(\mathbf{x})$  is a lateration graph then so is  $\mathcal{G}_s(\mathbf{x})$  for any  $s > r$ . It therefore suffices to find  $r_n \rightarrow 0$  such that  $\mathcal{G}_{r_n}(x_1, \dots, x_n)$  is lateration graph with probability tending to 1. (All limits are as  $n \rightarrow \infty$  unless otherwise specified.)

$\Omega$  being bounded, for any  $m \geq 1$  integer, it can be covered with finitely many, say  $N_m$ , open balls of radius  $1/2m$  centered on points belonging to  $\Omega$ . (We even know that the minimum number  $N_m$  satisfies  $N_m \leq C_0 m^p$ , where  $C_0$  depends on  $\text{diam}(\Omega)$  and  $p$ , although this will not play a role in what follows.) We consider such a covering, with balls denoted  $B_1^m, \dots, B_{N_m}^m$ . Let  $A_j := B_j \cap \Omega \neq \emptyset$  for all  $j$ . Form the following graph: the node set is  $A_1^m, \dots, A_{N_m}^m$ , and  $A_j^m$  and  $A_k^m$  are connected if they intersect. We call this the *cover graph*. Because  $\Omega$  is connected, the cover graph must also be connected, and may therefore be traversed by, say, depth-first search, which starting at any  $A_{j_0}^m$  results in a (finite) path in the cover graph that passes through the entire graph, meaning, a

sequence  $(A_{j_s}^m : s = 0, \dots, S_m)$  with  $I_s^m := A_{j_{s-1}}^m \cap A_{j_s}^m \neq \emptyset$  for all  $s$ , with the property that, for any  $j$ , there is  $s$  such that  $A_{j_s}^m = A_j^m$ . Note that, by construction, each  $I_s^m$  is a nonempty open subset of  $\Omega$  of diameter  $< 1/m$ ; together, these subsets cover  $\Omega$ . (Note that some of these sets might coincide, but this is unimportant.)

Now, let  $f$  be a density with support  $\bar{\Omega}$ , and let  $x_1, \dots, x_n$  denote an iid sample from  $f$ . For a Borel set  $A$ , let  $P(A) = \int_A f$ . Consider the event  $E_s^{n,m}$  that  $I_s^m$  contains at least  $p+1$  sample points, and define  $E^{n,m} = \bigcap_s E_s^{n,m}$ , which is the event that each one of the subsets  $I_1^m, \dots, I_{S_m}^m$  contains at least  $p+1$  sample points. Let  $a(n, m) = 1 - \mathbb{P}(E^{n,m})$ , which is the probability that  $E^{n,m}$  fails to happen. Note that, essentially by definition,  $a(n, m)$  is decreasing in  $n$ . In addition to that, we also have  $\lim_{n \rightarrow \infty} a(n, m) = 0$ . To see this, we derive, by the union bound and the fact that the number of points falling in a Borel set  $A$  is binomial with parameters  $n$  and  $P(A)$ ,

$$\begin{aligned} a(n, m) &\leq \sum_{s=0}^{S_m} (1 - \mathbb{P}(E_s^{n,m})) \\ &\leq \sum_{s=0}^{S_m} \sum_{k=0}^p \binom{n}{k} P(I_s^m)^k (1 - P(I_s^m))^{n-k} \\ &\leq (S_m + 1)(p+1)n^p (1 - b_m)^{n-p}, \quad b_m := \min_{s=0, \dots, S_m} P(I_s^m). \end{aligned}$$

Since each  $I_s^m$  is a nonempty open subset of  $\Omega$ , we have that  $b_m > 0$ , and so  $a(n, m) \rightarrow 0$  as  $n \rightarrow \infty$  when  $m$  remains fixed. (The convergence is exponentially fast, although this will not play a role.) The fact that  $a(n, m)$  is decreasing in  $n$  and  $\lim_{n \rightarrow \infty} a(n, m) = 0$  implies, via elementary arguments, that there is sequence  $m_n \rightarrow \infty$  such that  $\lim_{n \rightarrow \infty} a(n, m_n) = 0$ , or equivalently,  $\mathbb{P}(E^{n, m_n}) \rightarrow 1$  as  $n \rightarrow \infty$ .

We now prove that, under  $E^{n,m}$ , the neighborhood graph built on the sample points  $x_1, \dots, x_n$  with connectivity radius  $r = 1/m$  is a lateration graph. Thus, we work under the situation where each  $I_s^m$  contains at least  $p+1$  sample points. First, consider  $p+1$  such points in  $I_1^m$ , and label them  $v_1, \dots, v_{p+1}$  in any order. Since  $\text{diam}(I_1^m) < r$ , the subgraph that these points induce is complete. Recall that  $I_1^m \subset A_{j_0}^m$ . Label the remaining points in  $A_{j_0}^m$  as  $v_{p+1}, \dots, v_{n_0}$  and note that, since  $\text{diam}(A_{j_0}^m) < r$ , each of these points is connected to all the points  $v_1, \dots, v_{p+1}$ . Let  $\mathcal{V}_0$  denote  $\{v_1, \dots, v_{n_0}\}$ . Similarly, recall that  $I_1^m \subset A_{j_1}^m$ ; label the remaining points in  $A_{j_1}^m$  as  $v_{n_0+1}, \dots, v_{n_1}$ , and note that, since  $\text{diam}(A_{j_1}^m) < r$ , each of these points is connected to all the points  $v_1, \dots, v_{p+1}$ , and therefore to at least  $p+1$  points inside  $\mathcal{V}_0$ ; let  $\mathcal{V}_1 := \{v_1, \dots, v_{n_1}\}$ . Suppose that we are at a stage where we have built an ordering  $\mathcal{V}_{s-1} = \{v_1, \dots, v_{n_{s-1}}\}$  of the sample points in  $A_{j_0}^m, \dots, A_{j_{s-1}}^m$  such that, for each  $p+1 < j \leq j_{s-1}$ ,  $v_j$  is connected to at least  $p+1$  points among  $v_1, \dots, v_{j-1}$ . In particular, this includes all the points in  $I_s^m$  since  $I_s^m \subset A_{j_{s-1}}^m$ . Now,  $I_s^m \subset A_{j_s}^m$  also; label the remaining points in  $A_{j_s}^m$  as  $v_{n_{s-1}+1}, \dots, v_{n_s}$ , and since  $\text{diam}(A_{j_s}^m) < r$ , each of these points is connected to all the points  $I_s^m$ . Since  $I_s^m$  contains at least  $p+1$  points (because  $E^{n,m}$  holds), we may continue the recursion by letting  $\mathcal{V}_s = \{v_1, \dots, v_{n_s}\}$ . Doing so until all the sample points have been processed provides a laterative ordering of the entire neighborhood graph  $\mathcal{G}_r(x_1, \dots, x_n)$ .  $\square$

## 5 Numerical experiments

We probe the accuracy of the stability bound in Theorem 3.1 in the following numerical experiments. We begin by noting that the constants  $\sigma, C > 0$  in Theorem 3.1 depend on the graph  $\mathcal{G}$  and the latent configuration  $x_1, x_2, \dots, x_n$ . In particular, for a fixed graph  $\mathcal{G}$  the constant  $C$  depends on the

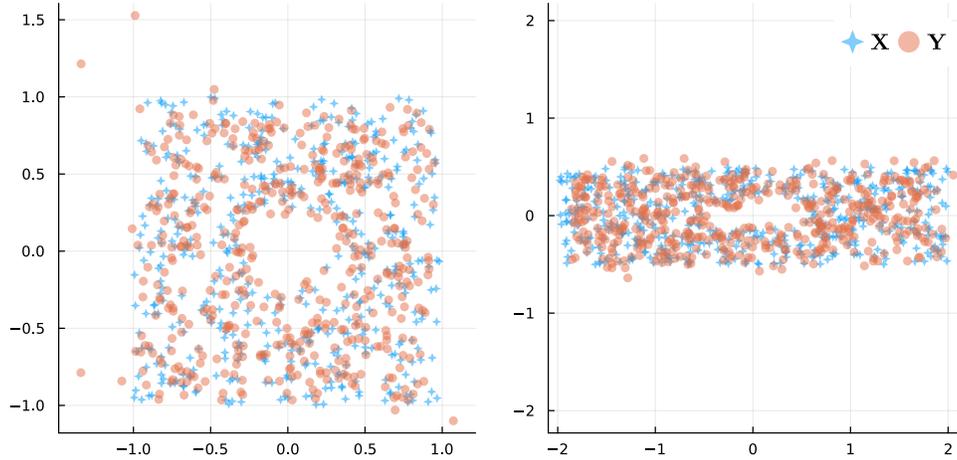


Figure 4.1: Examples of latent configurations  $x_1, x_2, \dots, x_n$  and the embedding  $y_1, y_2, \dots, y_n$  obtained from sequential lation when (left)  $h = 0.5$  and  $\kappa = 1$ , and when (right)  $h = 0.5$  and  $\kappa = 2$ . The model is (1.2), with  $\varepsilon_{ij} \sim N(0, \zeta^2)$  for  $\zeta^2 = 0.1$ .

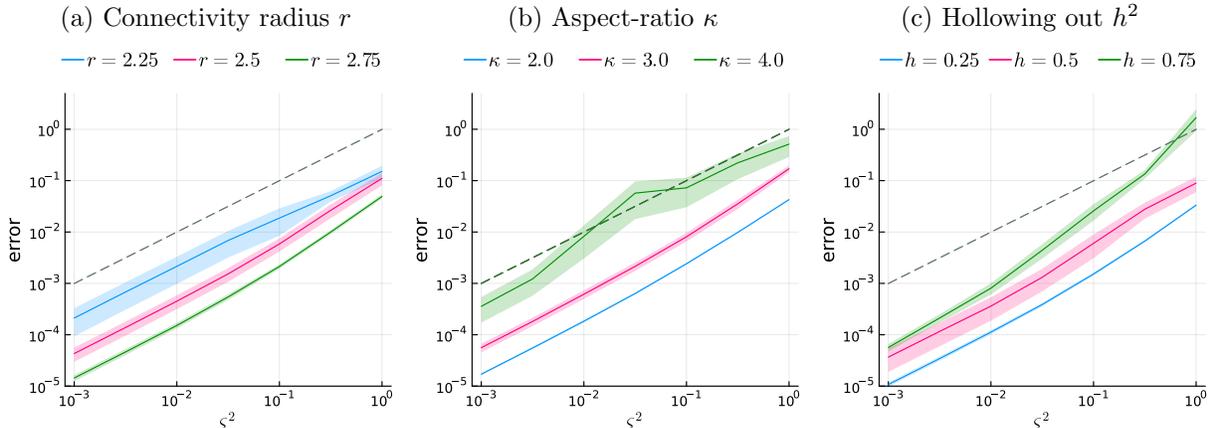


Figure 4.2: Results of the numerical experiments. The vertical axis in all plots is the embedding error and the horizontal axis is the variance of the noise,  $\zeta^2$ . The results are shown on a log-log scale. The dashed line in (a), (b) and (c) is the 45° line corresponding to the mean perturbation  $s(\varepsilon)^2$  defined in (5.1).

aspect-ratio, i.e., the ratio of the largest to the smallest eigenvalue of the the latent configuration [6, Section 3]. On the flipside, for a fixed latent configuration, the constant  $\sigma$  depends on the number of edges,  $|\mathcal{E}|$ , in the graph  $\mathcal{G}$ .

Therefore, in order to investigate the stability bound, we consider the setting where the latent configuration  $x_1, x_2, \dots, x_n$  is drawn *iid* from a uniform distribution on the domain  $\Omega(h, \kappa)$ , where for  $h \in (0, 1)$  and scale  $\kappa > 0$ ,

$$\Omega(h, \kappa) := [-\kappa, \kappa] \times [-\kappa^{-1}, \kappa^{-1}] \setminus [-h\kappa, h\kappa] \times [-h\kappa^{-1}, h\kappa^{-1}],$$

is a rectangle with aspect ratio  $\kappa^2 \in (0, 1)$  and a fraction  $h^2 \in (0, 1)$  of its area hollowed out from the center. The parameters  $h$  and  $\kappa$  together account for the complexity of the latent configuration.

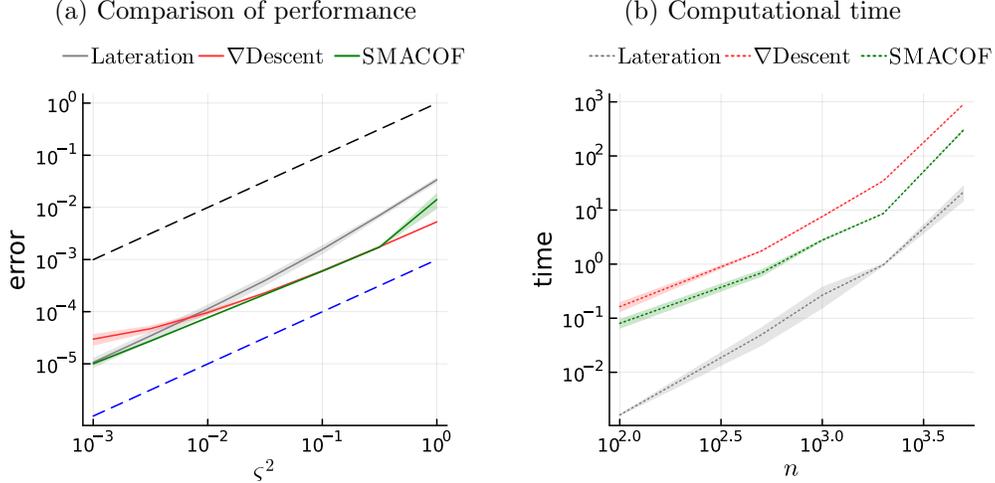


Figure 4.3: (a) Comparison of the embedding error for SMACOF and Gradient Descent shown on a log-log scale. The black dashed line corresponds the mean perturbation,  $s(\varepsilon)^2$ , and the blue dashed line is a plot of  $s(\varepsilon)^2/10^3$  which provides evidence of a lower bound for the embedding error. (b) Computational time for sequential lateration, SMACOF and Gradient Descent for varying sample sizes  $n$ .

We consider the setting where the dissimilarities are corrupted by additive noise  $\varepsilon_{ij}$ , i.e.,  $d_{ij}^2 = \max\{\|x_i - x_j\|^2 + \varepsilon_{ij}, 0\}$ , where  $\varepsilon_{ij}$  are drawn *iid* from  $N(0, \zeta^2)$ . See Figure 4.1 for an illustration. Given the graph  $\mathcal{G}$  with dissimilarities  $d_{ij}$ , we obtain the embedding  $y_1, y_2, \dots, y_n$  using the sequential lateration method described earlier in the paper and then compute the embedding error

$$\frac{1}{n} \sum_{i \in [n]} \|y_i - \hat{g}(x_i)\|^2,$$

where the rigid transformation  $\hat{g} : \mathbb{R}^p \rightarrow \mathbb{R}^p$  is obtained via Procrustes alignment [6]. In all experiments we compare the embedding error of  $y_1, y_2, \dots, y_n$  to the mean perturbation,

$$s(\varepsilon)^2 := \frac{1}{|\mathcal{E}|} \sum_{(i,j) \in \mathcal{E}} \varepsilon_{ij}^2, \quad (5.1)$$

which is the normalized bound on the right hand side of (2.1) in Theorem 2.1. Note that, when  $|\mathcal{E}|$  is large,  $s(\varepsilon)^2 \approx \mathbb{E}(\varepsilon_{ij}^2) = \zeta^2$  by the law of large numbers.

The results are summarized in Figure 4.2. For fixed  $n = 500$ ,  $h = 0.2$  and  $\kappa = 1$ , Figure 4.2a shows the effect of the effect of the connectivity radius ( $r \in \{2.25, 2.5, 2.75\}$ ) of the random geometric graph on the accuracy of the bound. Figure 4.2b illustrates the effect of the aspect ratio ( $\kappa \in \{2, 3, 4\}$ ) at a fixed connectivity radius of  $r = 0.3$ . Lastly, for fixed  $\kappa = 1$  and  $r = 0.3$ , Figure 4.2c shows the effect of the hollowing out ( $h \in \{0.25, 0.5, 0.75\}$ ) of the domain of the latent configuration. In all cases, the results corroborate the bound established in Theorem 2.1. Furthermore, as seen in the plots, the constants are likely larger for more complex latent configurations, i.e., when  $r$  is small,  $\kappa$  is small, or when  $h$  is large.

Figure 4.3 investigates the accuracy of the bound established in Theorem 3.1. We compare the embedding error of the sequential lateration method to the embedding error from stress-minimizers,

$y_1^*, y_2^*, \dots, y_n^*$ , obtained using (i) gradient descent, and (ii) the SMACOF algorithm [24]. Figure 4.3a shows the embedding error of the three methods compared to the mean perturbation  $s(\varepsilon)^2$  (the black dashed line), and confirms the bound in Theorem 3.1. While the embedding error of  $y_1^*, y_2^*, \dots, y_n^*$  marginally improves on the embedding error of sequential lateration, the advantage of the sequential lateration procedure is the reduced computational time which, as shown in Figure 4.3b, is between one to two orders of magnitude faster than SMACOF and gradient descent.

## 6 Discussion

Our main contribution in this paper is a perturbation bound for sequential lateration. This provides a way to understand and, to some extent, quantify the stability of sequential lateration in the presence of noise. As a corollary, we obtained a perturbation bound for stress minimization in the setting of a lateration graph. As we mentioned earlier, this addresses the issue of noise in multidimensional scaling / network localization discussed and formulated as a set of open questions by Mao et al. [57] in their well-known review paper.

A related but distinct issue is the presence of outliers, by which we mean gross errors (i.e., some of the error terms  $\varepsilon_{ij}$  in (1.2) could be quite large). There are robust methods<sup>3</sup> for MDS, e.g., [17, 36], but their robustness properties are not well-understood. Converting the available (metric) data into ordinal data, by replacing  $d_{ij}$  by its rank among all dissimilarities  $(d_{kl})_{(k,l) \in \mathcal{E}}$ , and then applying a method for ordinal MDS is likely to yield a robust method, but the robustness of such methods are also poorly understood. For some effort in this direction, see [40].

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<sup>3</sup>We use the term ‘robust’ in the way it is used in statistics. In the broad MDS literature, this term is sometimes used to mean what we refer to here as stability to noise.

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