Modeling lower-truncated and right-censored insurance claims with an extension of the MBBEFD class

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Abstract

In general insurance, claims are often lower-truncated and right-censored because insurance contracts may involve deductibles and maximal covers. Most classical statistical models are not (directly) suited to model lower-truncated and right-censored claims. A surprisingly flexible family of distributions that can cope with lower-truncated and right-censored claims is the class of MBBEFD distributions that originally has been introduced by Bernegger (1997) for reinsurance pricing, but which has not gained much attention outside the reinsurance literature. Interestingly, in general insurance, we mainly rely on unimodal skewed densities, whereas the reinsurance literature typically proposes monotonically decreasing densities within the MBBEFD class. We show that this class contains both types of densities, and we extend it to a bigger family of distribution functions suitable for modeling lower-truncated and right-censored claims. In addition, we discuss how changes in the deductible or the maximal cover affect the chosen distributions.

Keywords. General insurance claims, deductible, maximal cover, lower-truncation, right-censoring, MBBEFD distribution, unimodal density, skewed density, normalized loss, exposure curve, Swiss Re exposure curve, Lloyd's exposure curve.

1 Introduction

Insurance contracts in general insurance often involve deductibles $d > 0$ and maximal covers $M > 0$. Deductibles are introduced to reduce the number of small claims which mainly cause administrative expenses but which are not essential in risk mitigation. Maximal covers are introduced to control the maximal loss of an insurer. A maximal cover may, e.g., refer to the property value insured (after subtracting the deductible), or to the maximal insurance coverage warranted to a liability claim. Denote by X the total financial loss. The insurance claim Y after subtracting the deductible $d > 0$ and with a maximal cover of size $M > 0$ is given by

$$Y = \min\{(X-d)_+,\, M\} \mid X > d. \tag{1.1}$$

We say, this financial loss is *lower-truncated* at $d > 0$ and *right-censored* at $M > 0$ (after subtracting the deductible). Statistical modeling of lower-truncated and right-censored claims is a notoriously difficult problem. Most statistical models have an unbounded support, e.g.,

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the supports of the gamma and the log-normal distributions are the entire positive real line $\mathbb{R}_+$. In many cases, this implies that fitting a statistical model to lower-truncated and right-censored data is not a problem that is easily analytically tractable. We give an example. We start from a classical statistical model such as the gamma distribution for the total financial loss $X \sim F_\Gamma$, where F_Γ denotes the gamma distribution with corresponding gamma density f_Γ on $\mathbb{R}_+$. Lower-truncation and right-censoring introduces two difficulties which are illustrated in Figure 1. First, the lower-truncation $(X-d)_+ \mid X > d$ of the total financial loss X implies, in general, that the density of the lower-truncated claim is positive in 0, see Figure 1. In the above mentioned gamma case, this means that the lower-truncation with $d > 0$ leads to a new density given by

$$y \geq 0 \;\mapsto\; \frac{f_\Gamma(d+y)}{\int_d^\infty f_\Gamma(z)\,dz} = \frac{f_\Gamma(d+y)}{1-F_\Gamma(d)} \;>\; 0.$$

Second, right-censoring at $M > 0$ of this lower-truncated claim leads to a point mass in M, resulting in the density f of the lower-truncated and right-censored claim Y

$$y \geq 0 \;\mapsto\; f(y) = \frac{f_\Gamma(d+y)}{1-F_\Gamma(d)}\,\mathbb{1}_{\{y<M\}} + \frac{1-F_\Gamma(d+M)}{1-F_\Gamma(d)}\,\mathbb{1}_{\{y=M\}}, \tag{1.2}$$

where (1.2) is a density w.r.t. the σ-finite measure being the Lebesgue measure on $(0, M)$ and having a point mass in M; this point mass is not illustrated in Figure 1, but only the absolutely continuous part on $(0, M)$; the point mass in M equals one minus the volume of the blue area in Figure 1.

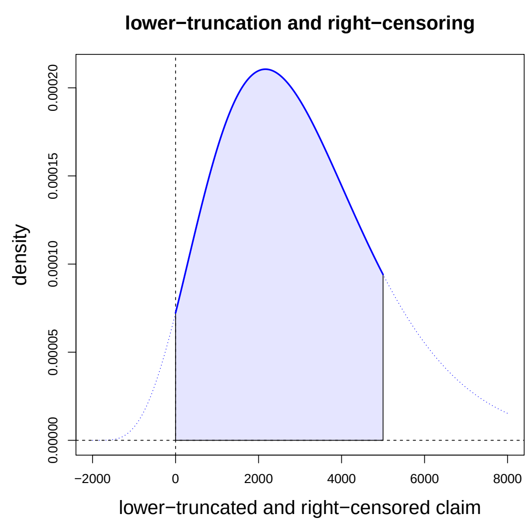

Figure 1: Lower-truncated and right-censored claim with $d = 2000$ and $M = 5000$.

More generally, for maximum likelihood estimation (MLE) based on lower-truncated and right-censored claims $Y \in (0, M]$, $\mathbb{P}$-a.s., we consider the log-likelihood function of an unknown parameter θ given by

$$\theta \;\mapsto\; \ell_Y(\theta) = \log\left(f_\theta(Y)\right)\mathbb{1}_{\{Y<M\}} + \log\left(1 - F_\theta(M_-)\right)\mathbb{1}_{\{Y=M\}}, \tag{1.3}$$

assuming that the response variable Y is absolutely continuous on $(0, M)$ with density $f_\theta(y)$, having a point mass $1 - F_\theta(M_-) = 1 - \lim_{y\uparrow M} F_\theta(y)$ in M, and with (unknown) model parameter θ. Fitting such a model with MLE can be difficult because we need an analytically tractable form for both the density $f_\theta(\cdot)$ and its distribution function $F_\theta(\cdot)$, see (1.3). This is not the case, e.g., in the lower-truncated and right-censored gamma model given in (1.2). Therefore, in such cases, one either needs to rely on numerical integration of the density (which can be computationally demanding, e.g., when performing a regression with fixed covariates) or one uses a version of the Expectation-Maximization (EM) algorithm by interpreting the lower-truncation and right-censoring as a missing information problem; we refer to Verbelen et al. [16], Fung et al. [5] and Sections 6.4.2 and 6.4.3 in Wüthrich–Merz [17]. However, also this EM algorithm approach has its drawbacks as it requires tractability of conditional tail expectations and reasonable dispersion estimates in multi-dimensional parameter settings. These two side constraints lead to further restrictions on the class of solvable models, e.g., these problems can only be solved for a very small number of models within the class of Tweedie's models [15], namely, for the Tweedie's models stated in Theorem 3 of Blæsild–Jensen [3]; we also refer to Landsman–Valdez [6]. In a series of papers, Poudyal [9, 10] and Poudyal–Brazauskas [11] consider trimmed and/or winsorized methods of moments estimators for truncated and/or censored data; in statistics, truncation is also called trimming and censoring winsorizing. In these papers, trimming and winsorizing is also shown to be a useful method of robustifying moment estimation under extreme claims.

We take a different approach in this paper to solve the fitting problem of lower-truncated and right-censored data. In reinsurance, often so-called MBBEFD *exposure curves* are used for exposure rating. Those exposure curves have been introduced by Bernegger [2], and the acronym MBBEFD indicates that this class includes the Maxwell–Boltzmann (MB), the Bose–Einstein (BE) and the Fermi–Dirac (FD) distributions; these are well known distributions in statistical mechanics. These MBBEFD exposure curves are based on the assumption that there is a maximal cover M, and they directly describe right-censored claims up to this maximal cover. Differentiating twice these MBBEFD exposure curves provides us with densities being absolutely continuous on the interval $(0, M)$ and having a point mass in M; we refer to formula (3.7) in Bernegger [2].

The goal of this paper is first to study the properties of these MBBEFD densities and to extend it to a bigger class of models that will be called the *Bernegger class*. Our contribution is to show that the Bernegger class is a rich family of distributions including monotonically decreasing densities, unimodal densities and monotonically increasing densities, and our extension provides new families of lower-truncated and right-censored random variables that allow for skewness in the absolutely continuous part of the distribution. This is of particular interest because unimodal skewed densities are suited for modeling lower-truncated and right-censored claims in general insurance since their empirical density roughly looks like the one given in Figure 1. In particular, the distributions of lower-truncated and right-censored exponential and logistic random variables belong to the Bernegger class.

Surprisingly, the class of MBBEFD densities of Bernegger [2] has only entered the reinsurance literature; see, e.g., Parodi–Watson [8], Abramson [1], Riegel [13], Chapter 21 of Parodi [7], and the R [12] package `mbbefd` of Dutang et al. [4, 14]. Popular examples in reinsurance pricing are the so-called Swiss Re and Lloyd's exposure curves that are special cases of MBBEFD exposure curves; see Bernegger [2]. However, in this reinsurance pricing literature, one mainly focuses on exposure curves and not on the resulting densities nor on their properties. We show that the most popular choices from reinsurance lead to monotonically decreasing densities, whereas

we are mainly interested into the unimodal case, as this is the common situation in general insurance pricing. By fitting a real dataset consisting of private property insurance claims, we introduce a couple of explicit models belonging to the Bernegger class that allow for unimodal and skewed densities.
Finally, we consider the situation where the insurer is interested in understanding how a change in the deductible or the maximal cover affects the expected claim size. For this, we first emphasize that, typically, the insurer only observes the lower-truncated and right-censored claim Y given in (1.1). That is, for statistical modeling, neither are the claims below the deductible d known, nor are the exact claim sizes above the maximal cover M known. Thus, we can only fit a lower-truncated and right-censored density to observations, e.g., of type (1.2). In general, this does not allow us to extrapolate below the lower-truncation point and above the right-censoring point since there are infinitely many candidates for extrapolation. Under some assumptions on the original density and its support, we can at best smoothly extrapolate, e.g., as the dotted lines in Figure 1 suggest, but the true model could also look completely different, as Y does not reveal any information about the claims being outside of its observed support, except for the proportion of claims exceeding the maximal cover. Therefore, we can only perform the opposite operation of either increasing the deductible or decreasing the maximal cover, and we will show that the Bernegger class is closed under these transformations.

Organization. This manuscript is organized as follows. In the next section, we state the necessary properties that any exposure curve has to fulfill in order to describe distribution functions allowing to model lower-truncated and right-censored claims. In Section 3, we start from the MBBEFD class of distributions of Bernegger [2] by stating some of its properties, and then we extend it to a richer family of distributions, the Bernegger class. In Section 4, we introduce a subclass of the Bernegger class that incorporates the logistic distribution as well as the MBBEFD class of distributions, whereas in Section 5, we treat another subclass of distributions that includes the lower-truncated and right-censored exponential distribution. In Section 6, we use a real dataset of lower-truncated and right-censored claims in order to compare the performance of the gamma and the log-normal model to five examples belonging to the Bernegger class, fitting all these models using maximum likelihood estimation (MLE). Finally, in Section 7, we consider the influence of a change in the deductible or the maximal cover on the observed claims distribution. The last section concludes this work. All mathematical proofs and parameters of the fitted models are provided in the appendix.

2 Exposure curves and their resulting densities

2.1 From Exposure curves to distributions

In reinsurance claims modeling, one often works with exposure curves instead of distribution functions. Assume we have a positively supported response variable $Y \sim F_Y$, and assume that the *maximal possible loss* (MPL) is given by $M > 0$, i.e., $0 < Y \leq M$, $\mathbb{P}$-a.s. We define the normalized loss $Z = Y/M$. Denote the distribution function of the normalized loss Z by F_Z, being supported in $(0, 1]$. The *exposure curve* of a normalized loss $Z \sim F_Z$ is defined by

$$z \;\mapsto\; G(z) = \frac{\int_0^z 1 - F_Z(s)\, ds}{\int_0^1 1 - F_Z(s)\, ds} = \frac{\int_0^z 1 - F_Z(s)\, ds}{\mathbb{E}[Z]},$$

for $z \in [0,1]$; see Bernegger [2]. This exposure curve $G : [0,1] \to [0,1]$ is non-decreasing, concave, and satisfies the property $G'(0) > 0$, as well as the normalizations $G(0) = 0$ and $G(1) = 1$; for examples see Figure 2 (lhs), below. In the last integral, a change of variable $s \in [0,1] \mapsto t = sM \in [0,M]$ gives us

$$G(z) = \frac{\int_0^{zM} 1 - F_Z(t/M)\,dt}{M\mathbb{E}[Z]} = \frac{\int_0^{zM} 1 - F_Y(t)\,dt}{\mathbb{E}[Y]} = G_Y(zM),$$

the latter being the exposure curve of Y on $[0,M]$. Thus, we can equally work with the responses Y and Z, but the normalized losses Z will have the advantage that they live on the common unit interval $[0,1]$. Now, let us take the opposite view and characterize the distribution function of a random variable $Z \sim F_Z$ obtained from a function $G : [0,1] \to \mathbb{R}$ satisfying the same properties of an exposure curve. Under the assumptions of the next theorem, this distribution F_Z leads to an absolutely continuous density on $[0,1)$ and a point mass in 1.

Theorem 2.1. *Let $G : [0,1] \to \mathbb{R}$ be a non-decreasing, concave, and twice continuously differentiable function with $G(0) = 0$, $G(1) = 1$, $G'(0) > 0$. The function $F_Z : [0,1] \to \mathbb{R}$ defined by*

$$F_Z(z) = \left(1 - \frac{G'(z)}{G'(0)}\right) \mathbb{1}_{\{z<1\}} + \mathbb{1}_{\{z=1\}} \tag{2.1}$$

is a distribution function on $[0,1]$. Furthermore, this distribution has as density

$$f_Z(z) = -\frac{G''(z)}{G'(0)}, \tag{2.2}$$

for $z \in [0,1)$, and a point mass in 1 given by

$$p = \frac{G'(1)}{G'(0)}. \tag{2.3}$$

Finally, the mean of $Z \sim F_Z$ is equal to $\mathbb{E}[Z] = 1/G'(0)$.

The proofs of all statements are given in the appendix. Due to this last result, functions G satisfying the assumptions of Theorem 2.1 will be called *exposure curves*.

Definition 2.2. *An exposure curve is a function $G : [0,1] \to \mathbb{R}$, which is non-decreasing, concave, and twice continuously differentiable with $G(0) = 0$, $G(1) = 1$, $G'(0) > 0$.*

As seen previously, if we start from any such function G, we can derive a distribution whose density is absolutely continuous and of closed form on $[0,1)$, with a point mass in 1 and a mean that are of closed form too, i.e., we have a class of models that has fully tractable mean, density and point mass, which is suitable to model right-censored claims. Moreover, if $G''(0) < 0$, which implies $f_Z(0) > 0$, it includes lower-truncation in the sense that the density of a lower-truncated random variable Z is positive in zero, see Figure 1. The next result shows that a linear combination of exposure curves allows us to define a mixture of their respective associated distribution functions.

Lemma 2.3. *Let $(\alpha_i)_{i=1}^n$ be non-negative weights adding up to 1 and let $(G_i)_{i=1}^n$ be exposure curves leading to distributions functions $(F_i)_{i=1}^n$, densities $(f_i)_{i=1}^n$, and point masses in 1 equal to $(p_i)_{i=1}^n$, as in Theorem 2.1. The convex combination*

$$G(z) = \sum_{i=1}^n \alpha_i G_i(z), \tag{2.4}$$

for $z \in [0,1]$*, is again an exposure curve allowing to define the distribution function of a random variable* $Z \sim F_Z$ *given by*

$$F_Z(z) = \sum_{i=1}^{n} w_i F_i(z),$$

for $z \in [0,1]$*, and where* $w_i = \alpha_i G_i'(0) / \sum_{j=1}^{n} \alpha_j G_j'(0)$ *are non-negative weights summing up to 1. In particular, the density of* Z *on* $[0,1)$ *is given by*

$$f_Z(z) = \sum_{i=1}^{n} w_i f_i(z),$$

and the point mass in 1 is equal to

$$p = \sum_{i=1}^{n} w_i p_i.$$

Finally, the mean of $Z \sim F_Z$ *is given by*

$$\mathbb{E}[Z] = \sum_{i=1}^{n} w_i \frac{1}{G_i'(0)}.$$

2.2 Flexibility of the point mass in the right-censoring point

The point mass p in 1 is automatically determined by formula (2.3). Often, one may require more modeling flexibility in the choice of this point mass, while still retaining the tractability of the density and the mean as it was shown in Theorem 2.1. A simple way to do so connects to so-called one-inflated distributions; we refer to Dutang et al. [4]. In the case of Theorem 2.1, this can easily be achieved. The next corollary shows, how we can obtain them by looking at the conditional density of the random variable $Z_0 \overset{(d)}{=} Z|_{\{Z<1\}}$, which corresponds to a lower- and upper-truncated random variable.

Corollary 2.4. *Let* $G : [0,1] \to \mathbb{R}$ *be an exposure curve, we receive an absolutely continuous density on* $[0,1)$

$$f_0(z) = -\frac{1}{1-p} \frac{G''(z)}{G'(0)} = \frac{G''(z)}{G'(1) - G'(0)} \;\geq\; 0,$$

This density f_0 *integrates to 1 and provides the mean for the random variable* $Z_0 \sim f_0$

$$\mathbb{E}[Z_0] = \frac{1}{1-p}(\mathbb{E}[Z] - p) = \frac{1 - G'(1)}{G'(0) - G'(1)}.$$

We can now add a point mass $q \in (0,1)$ in 1 to this density. This adds one more parameter to the model, giving us a mixture distribution between an absolutely continuous part f_0 on $[0,1)$ and a point mass in 1. We have the following corollary.

Corollary 2.5. *Let* $G : [0,1] \to \mathbb{R}$ *be an exposure curve, the random variable* Z *that has an absolutely continuous density on* $[0,1)$ *given by*

$$f_q(z) = (1-q)\, \frac{G''(z)}{G'(1) - G'(0)} \;\geq\; 0,$$

with fixed point mass $q \in (0,1)$ *in 1 has expected value*

$$\mathbb{E}[Z] = (1-q)\, \frac{1 - G'(1)}{G'(0) - G'(1)} + q.$$

Note that the transformation achieved in Corollary 2.5 can also be obtained using Lemma 2.3. Indeed, let us consider the exposure curve

$$\tilde{G}(z) = w\,G(z) + (1-w)z,$$

for $z \in [0,1]$, $w \in [0,1]$, and where G is an exposure curve. Using Theorem 2.1, if we denote by F the distribution function obtained from the exposure curve G, by f the absolutely continuous density on $[0,1)$ and by p the point mass in 1, we can characterize the distribution function $\tilde{F}$, the absolutely continuous density on $[0,1)$ $\tilde{f}$, and the point mass $\tilde{p}$ obtained from the exposure curve $\tilde{G}$, respectively, using

$$\tilde{F}(z) = wF(z) + (1-w)\mathbb{1}_{\{z=1\}},$$

for $z \in [0,1]$,

$$\tilde{f}(z) = wf(z) = -w\frac{G''(z)}{G'(0)},$$

for $z \in [0,1)$, and

$$\tilde{p} = wp + (1-w).$$

Remark 2.6. Note that similarly to Corollary 2.5, one could also add a point mass in 0, thus modeling lower- and right-censored insurance claims with tractable densities, means and point masses.

In what follows, the goal will be to introduce examples of exposure curves that are useful to model lower-truncated and right-censored insurance losses. For this, we start by studying the explicit family of exposure curves introduced by Bernegger [2].

3 The Bernegger class of distributions

3.1 The class of MBBEFD exposure curves and densities

The MBBEFD class of Bernegger [2] selects an explicit family of exposure curves. This family is characterized through two parameters $g \geq 1$ and $b \geq 0$, and it is given as follows for $z \in [0,1]$,

$$G_{b,g}(z) = \begin{cases} z & \text{for } g = 1 \text{ or } b = 0, \\ \frac{\log(1+(g-1)z)}{\log(g)} & \text{for } g > 1 \text{ and } b = 1, \\ \frac{1-b^z}{1-b} & \text{for } g > 1 \text{ and } bg = 1, \\ \frac{\log\left(\frac{(g-1)b+(1-bg)b^z}{(1-b)}\right)}{\log(bg)} & \text{for } g > 1,\ b > 0,\ b \neq 1 \text{ and } bg \neq 1. \end{cases} \tag{3.1}$$

The three cases $bg = 1$, $bg > 1$ and $bg < 1$ give the MB (Maxwell-Boltzmann), the BE (Bose-Einstein) and the FD (Fermi-Dirac) distributions, respectively. Using (2.1), we can calculate the distributions for $z \in [0,1)$, see (3.6) in Bernegger [2],

$$F_{b,g}(z) = \begin{cases} 0 & \text{for } g = 1 \text{ or } b = 0, \\ 1 - (1+(g-1)z)^{-1} & \text{for } g > 1 \text{ and } b = 1, \\ 1 - b^z & \text{for } g > 1 \text{ and } bg = 1, \\ 1 - \frac{1-b}{(g-1)b^{1-z}+(1-bg)} & \text{for } g > 1,\ b > 0,\ b \neq 1 \text{ and } bg \neq 1. \end{cases} \tag{3.2}$$

We observe that the first case is not of interest because it gives a point mass of 1 to $z = 1$. For this reason, we skip this case in the sequel. On $z < 1$, we can calculate the second derivatives of these exposure curves $G_{b,g}$. This gives us the densities for $z \in [0, 1)$, see (3.7) in Bernegger [2],

$$f_{b,g}(z) = \begin{cases} (g-1)\,(1+(g-1)z)^{-2} & \text{for } g>1 \text{ and } b=1, \\ -\log(b)b^z & \text{for } g>1 \text{ and } bg=1, \\ \frac{(g-1)(b-1)\log(b)b^{1-z}}{((g-1)b^{1-z}+(1-bg))^2} & \text{for } g>1,\ b>0,\ b\neq 1 \text{ and } bg\neq 1, \end{cases} \tag{3.3}$$

and we have a point mass in $z = 1$ given by

$$p = \frac{1}{g} \ \in \ (0, 1).$$

Thus, we have an absolutely continuous distribution on $[0, 1)$, with a point mass $p = 1/g$ in 1, and, e.g., in the last case of (3.3), we have a strictly positive density in 0

$$f_{b,g}(0) = \frac{(g-1)\log(b)b}{b-1} \ > \ 0.$$

Such a density may therefore come from a lower-truncated claim. Finally, the mean is given by

$$\mathbb{E}_{b,g}[Z] = \begin{cases} \frac{\log(g)}{g-1} & \text{for } g>1 \text{ and } b=1, \\ \frac{b-1}{\log(b)} & \text{for } g>1 \text{ and } bg=1, \\ \frac{b-1}{\log(b)}\frac{\log(bg)}{bg-1} & \text{for } g>1,\ b>0,\ b\neq 1 \text{ and } bg\neq 1. \end{cases}$$

We give an example of such an exposure curve that is typically used for exposure rating in reinsurance.

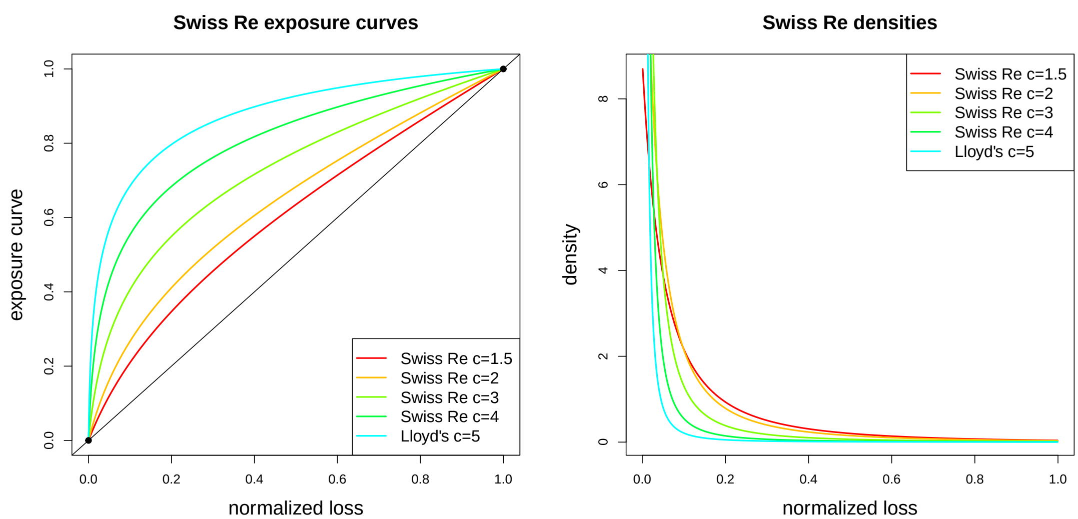

Figure 2: Swiss Re and Lloyd's exposure curves (lhs) and the resulting densities (rhs).

Example 3.1 (Swiss Re and Lloyd's exposure curves)**.** Bernegger [2] provides an explicit parametrization for the MBBEFD class which can be used for reinsurance exposure rating in case of scarce data. Namely, both parameters b and g are parametrized as a function of a single parameter $c > 0$ as follows

$$b = b(c) = \exp\{3.1 - 0.15(1+c)c\} \qquad \text{and} \qquad g = g(c) = \exp\{(0.78 + 0.12c)c\}. \tag{3.4}$$

For $c = 1.5, 2, 3, 4$, one obtains the Swiss Re exposure curves, and for $c = 5$, the Lloyd's exposure curve, these are illustrated in Figure 2 (lhs). The right-hand side of this figure shows the resulting densities $f_{b(c),g(c)}(\cdot)$, and we remark that all of the considered MBBEFD densities are monotonically decreasing on $[0, 1]$. ■

3.2 Properties of the MBBEFD class

Example 3.1 has shown that the exposure curves (3.4) proposed by Bernegger [2] for reinsurance exposure rating lead to monotonically decreasing densities $f_{b,g}$, see Figure 2 (rhs). For general insurance pricing, we are rather interested into unimodal densities similar to Figure 1, because this more commonly reflects the properties of general insurance claims data.

Proposition 3.2. *The density $f_{b,g}$ for $g > 1$ and $b > 0$ given in (3.3) has the following properties:*

- *Case $bg < 1$. The density $f_{b,g}$ is*
  - *monotonically decreasing on $[0, 1)$ for $(1 - bg)/(g - 1) \leq b$;*
  - *unimodal on $[0, 1)$ for $b < (1 - bg)/(g - 1) < 1$, with a maximum in*

$$z^* = 1 - \frac{\log\left((1 - bg)/(g - 1)\right)}{\log(b)} \;\in\; (0, 1); \tag{3.5}$$

  - *monotonically increasing on $[0, 1)$ for $(1 - bg)/(g - 1) \geq 1$.*
- *Case $bg \geq 1$. The density $f_{b,g}$ is monotonically decreasing on $[0, 1)$.*

This proposition shows that in practical applications in general insurance, the FD distributions (with $bg < 1$) are the most interesting ones, as they can be unimodal, or either monotonically decreasing or increasing. This excludes the exposure curves of Example 3.1, as these Swiss Re and Lloyd's exposure curves provide us with $bg > 1$. Next, we show that the MBBEFD distribution $F_{b,g}$ for $bg < 1$ can be derived from the logistic function (distribution)

$$\psi(t) = \frac{e^t}{e^t + 1} \;\in\; (0, 1),$$

for $t \in \mathbb{R}$. The logistic function has first derivative (logistic density)

$$\psi'(t) = \frac{e^t}{(e^t + 1)^2} = \psi(t)\left(1 - \psi(t)\right).$$

This derivative is symmetric around zero, which leads to the following result.

Proposition 3.3. *Let $b > 0$ and $g > 1$. For $bg < 1$, the MBBEFD density has the functional form, for $z \in [0, 1)$,*

$$f_{b,g}(z) = (a + 1)\log(1/b)\,\psi'\left(z\log(1/b) + \log(a)\right),$$

where we set

$$a = \frac{b(g - 1)}{1 - bg} \qquad \text{and} \qquad g = \frac{a + b}{(a + 1)b}, \tag{3.6}$$

respectively. If $b < (1 - bg)/(g - 1) < 1$, i.e. $b < a < 1$, this MBBEFD density is bell shaped around z^* given in (3.5).

For further terminology, we call unimodal symmetric densities, bell shaped densities. Of course, this includes for example the Gaussian density but also the logistic density. We conclude that for $bg < 1$, the MBBEFD density is the logistic density ψ' on the interval

$$[\log(a), \log(a/b)),$$

scaled by a constant factor $(1 + a)\log(1/b) > 0$. It is symmetric around its mode z^*, and it decays slower than the Gaussian density. This now shows why the MBBEFD densities are not sufficient for general insurance claims modeling, because general insurance claims are typically positively skewed, which cannot be captured by the logistic density.

3.3 Extension of Bernegger's idea using non-bell shaped densities

We extend the class of bell-shaped MBBEFD densities to a more general class of exposure curves, which allows, in particular, for skewness in their corresponding densities. We call this extended family the Bernegger class. In Section 2.1, we have started from a generic exposure curve $G : [0, 1] \to [0, 1]$ which is a non-decreasing, concave and twice continuously differentiable function with the normalizations $G(0) = 0$, $G(1) = 1$ and with $G'(0) > 0$. The MBBEFD exposure curve (3.1) can be reparametrized. Indeed, by using (3.6), we obtain for $z \in [0, 1]$

$$G_{b,a}(z) = \frac{\log(a + b^z) - \log(a + 1)}{\log(a + b) - \log(a + 1)}, \tag{3.7}$$

for parameters $b \geq 0$ and $a > -\min(1, b)$ chosen such that $G_{b,a}$ is an exposure curve; we refer to Section 3.1 of Bernegger [2]. This structure can be used to design exposure curve forms that do not have the bell-shape property of Proposition 3.3. We modify the modeling set-up (3.7) as follows. Choose a function $B : [0, 1] \to \mathbb{R}$ that satisfies

$$B(z) = h(b(z)), \tag{3.8}$$

for some functions h and b, in order to define an exposure curve (under further assumptions on h and b)

$$z \mapsto G(z) = \frac{B(z) - B(0)}{B(1) - B(0)}, \tag{3.9}$$

which ensures that the normalization property $G(0) = 0$ and $G(1) = 1$ is satisfied. The function h will be denoted as the *link function*, whereas the function b will be named the *inner function*, and we notice that Bernegger's original choice was $b(z) = a + b^z$ and $h(x) = \log(x)$, meaning that he used a *logarithmic linked exposure curve*. We will first explore some examples using the same link function and then introduce the *exponentially linked exposure curves*, which use the link function $h(x) = \exp(x)$. We call the class of distributions induced by exposure curves of the form (3.9) the Bernegger class.

4 Logarithmic linked exposure family

We start by considering logarithmic linked examples of the Bernegger class, which are obtained by choosing $h(x) = \log(x)$ in (3.8).

Proposition 4.1. *Choose a function $b : [0,1] \to (0,\infty)$ with $b(0) \neq b(1)$ that is twice continuously differentiable and define for $z \in [0,1]$ the function*

$$G(z) = \frac{\log(b(z)) - \log(b(0))}{\log(b(1)) - \log(b(0))}. \tag{4.1}$$

The function G is an exposure curve if and only if one of the following two holds:

$$b'(0) > 0,\ b'(z) \geq 0 \text{ and } b''(z)b(z) - b'(z)^2 \leq 0 \text{ for all } z \in [0,1], \tag{4.2}$$

or

$$b'(0) < 0,\ b'(z) \leq 0 \text{ and } b''(z)b(z) - b'(z)^2 \geq 0 \text{ for all } z \in [0,1]. \tag{4.3}$$

Using Theorem 2.1, one can then derive the distribution function of a random variable leading to an absolutely continuous density on $[0,1)$ and a point mass in 1.

Corollary 4.2. *Assume that a twice continuously differentiable function $b : [0,1] \to (0,\infty)$ with $b(0) \neq b(1)$ fulfills condition* (4.2) *or* (4.3)*. The exposure curve G defined in* (4.1) *provides the distribution of a random variable $Z \sim F_Z$*

$$F_Z(z) = \left(1 - \frac{b'(z)}{b'(0)}\frac{b(0)}{b(z)}\right)\mathbb{1}_{\{z<1\}} + \mathbb{1}_{\{z=1\}}, \tag{4.4}$$

for $z \in [0,1]$, with density for $z \in [0,1)$

$$f_Z(z) = \frac{b(0)}{-b'(0)}\frac{b''(z)b(z) - b'(z)^2}{b(z)^2},$$

and with point mass in $z = 1$ equal to

$$p = \frac{b'(1)}{b'(0)}\frac{b(0)}{b(1)}.$$

Moreover, the mean of $Z \sim F_Z$ is equal to

$$\mathbb{E}[Z] = \frac{b(0)}{-b'(0)}\log\left(\frac{b(0)}{b(1)}\right).$$

For general insurance pricing, we are interested into unimodal densities and we can thus derive the first derivative of f_Z in order to characterize the maximum of the density

$$f_Z'(z) = \frac{b(0)}{-b'(0)}\frac{b'''(z)b(z)^2 - 3b''(z)b'(z)b(z) + 2b'(z)^3}{b(z)^3}.$$

Note that this derivative only exists if the third derivative of b exists. The next result shows an explicit member of the Bernegger class that belongs to the logarithmic linked exposure family.

Example 4.3 (Two-parameter logistic distribution)**.** Consider a random variable X following a two-parameter logistic distribution with density

$$f_X(z) = \frac{e^{(z-\mu)/\sigma}}{\sigma\left(1 + e^{(z-\mu)/\sigma}\right)^2}, \quad \text{for } -\infty < z < \infty,\ -\infty < \mu < \infty,\ \sigma > 0,$$

and distribution

$$F_X(z) = \frac{e^{(z-\mu)/\sigma}}{1+e^{(z-\mu)/\sigma}}, \quad \text{for } -\infty < z < \infty,\ -\infty < \mu < \infty,\ \sigma > 0.$$

Let $d \in \mathbb{R}$ and $M > 0$ in order to define the scaled lower-truncated and right-censored random variable

$$Z = \frac{1}{M} \min\{(X-d)_+, M\} \,|\, X > d. \tag{4.5}$$

The distribution of Z is given by

$$F_Z(z) = \frac{F_X(d+zM) - F_X(d)}{1 - F_X(d)} \mathbb{1}_{\{z\in[0,1)\}} + \mathbb{1}_{\{z=1\}}, \tag{4.6}$$

which implies

$$F_Z(z) = \frac{e^{(d+zM-\mu)/\sigma} - e^{(d-\mu)/\sigma}}{1+e^{(d+zM-\mu)/\sigma}} \mathbb{1}_{\{z\in[0,1)\}} + \mathbb{1}_{\{z=1\}},$$

for $z \in [0,1]$. Furthermore, we have

$$\int_0^z 1 - F_Z(s)\,ds = \frac{-\sigma}{M}\left(1+e^{(d-\mu)/\sigma}\right)\left[\log\left(1+e^{-(d+zM-\mu)/\sigma}\right) - \log\left(1+e^{-(d-\mu)/\sigma}\right)\right],$$

for $z \in [0,1]$. This implies that the exposure curve of Z is given by

$$\begin{aligned} G(z) &= \frac{\int_0^z 1 - F_Z(s)\,ds}{\int_0^1 1 - F_Z(s)\,ds} \\ &= \frac{\log\left(1+e^{-(d+zM-\mu)/\sigma}\right) - \log\left(1+e^{-(d-\mu)/\sigma}\right)}{\log\left(1+e^{-(d+M-\mu)/\sigma}\right) - \log\left(1+e^{-(d-\mu)/\sigma}\right)}, \end{aligned}$$

for $z \in [0,1]$, which shows that the distribution of a (scaled) lower-truncated and right-censored two-parameter logistic random variable belongs to the Bernegger class with a logarithmic link function $h(x) = \log(x)$ and inner function $b(z) = 1 + e^{-(d+zM-\mu)/\sigma}$. ■

5 Exponentially linked exposure family

Next we introduce the exponentially linked exposure family by setting $h(x) = \exp(x)$ in (3.8).

Proposition 5.1. *Choose a function $b : [0,1] \to \mathbb{R}$ with $b(0) \neq b(1)$ that is twice continuously differentiable and define for $z \in [0,1]$ the function*

$$G(z) = \frac{e^{b(z)} - e^{b(0)}}{e^{b(1)} - e^{b(0)}}. \tag{5.1}$$

The function G is an exposure curve if and only if one of the following two holds:

$$b'(0) > 0,\ b'(z) \geq 0 \text{ and } b''(z) + b'(z)^2 \leq 0 \text{ for all } z \in [0,1], \tag{5.2}$$

or

$$b'(0) < 0,\ b'(z) \leq 0 \text{ and } b''(z) + b'(z)^2 \geq 0 \text{ for all } z \in [0,1]. \tag{5.3}$$

As for the logarithmic linked exposure curves, one can then derive, using Theorem 2.1, a distribution function leading to an absolutely continuous density on $[0,1)$ and to a point mass in 1.

Corollary 5.2. *Assume that a twice continuously differentiable function $b : [0,1] \to \mathbb{R}$ with $b(0) \neq b(1)$ fulfills condition* (5.2) *or* (5.3)*. The exposure curve G defined in* (5.1) *provides the distribution of a random variable $Z \sim F_Z$*

$$F_Z(z) = \left(1 - \frac{e^{b(z)}}{e^{b(0)}} \frac{b'(z)}{b'(0)}\right) \mathbb{1}_{\{z<1\}} + \mathbb{1}_{\{z=1\}}, \tag{5.4}$$

for $z \in [0,1]$, with density for $z \in [0,1)$

$$f_Z(z) = -\frac{e^{b(z)}}{e^{b(0)}} \frac{b'(z)^2 + b''(z)}{b'(0)}, \tag{5.5}$$

and with point mass in $z = 1$ equal to

$$p = \frac{e^{b(1)}}{e^{b(0)}} \frac{b'(1)}{b'(0)}.$$

The mean of $Z \sim F_Z$ is equal to

$$\mathbb{E}[Z] = \frac{e^{b(1)-b(0)} - 1}{b'(0)}.$$

The first derivative of the density f_Z given in (5.5) allows us to characterize its extrema and is given by

$$f_Z'(z) = -\frac{e^{b(z)}}{e^{b(0)}} \frac{b'(z)^3 + 3b'(z)b''(z) + b'''(z)}{b'(0)}. \tag{5.6}$$

Note that this derivative only exists if the third derivative of b exists. The next example shows that lower-truncated and right-censored exponential random variables belong to the exponentially linked exposure family.

Example 5.3 (Exponential distribution)**.** Consider a total financial loss $X \sim \mathrm{Exp}(\lambda)$ as well as a deductible $d > 0$ and a maximal cover $M > 0$. Then the scaled lower-truncated and right-censored insurance claim is given by

$$Z = \frac{1}{M} \min\{(X-d)_+,\, M\} \mid X > d,$$

and its distribution reads as

$$\begin{aligned} F_Z(z) &= \frac{F_X(d+zM) - F_X(d)}{1 - F_X(d)} \mathbb{1}_{\{z\in[0,1)\}} + \mathbb{1}_{\{z=1\}} \\ &= \left(1 - e^{-\lambda z M}\right) \mathbb{1}_{\{z\in[0,1)\}} + \mathbb{1}_{\{z=1\}}, \end{aligned}$$

for $z \in [0,1]$. This implies that the exposure curve of the random variable Z is given by

$$G(z) = \frac{\int_0^z 1 - F_Z(s)\,ds}{\int_0^1 1 - F_Z(s)\,ds} = \frac{e^{-\lambda z M} - 1}{e^{-\lambda M} - 1},$$

for $z \in [0,1]$, which shows that the distribution of a (scaled) lower-truncated and right-censored exponential random variable belongs to the Bernegger class with an exponential link function $h(x) = \exp(x)$ and a linear inner function $b(z) = -\lambda z M$, where M stands for the maximal cover. ■

We point out that the exponentially linked exposure family is equal to the logarithmic linked exposure family as stated in the next proposition. Thus, the choice of using one family over the other is mainly motivated by having simpler forms in the inner function b.

Proposition 5.4. *The logarithmic linked exposure family and the exponentially linked exposure family coincide.*

6 Real dataset example

In this section, the goal is to exploit some examples belonging to the logarithmic and the exponentially linked exposure family of the Bernegger class. These examples will be used to fit general insurance claims data using the tractability of the models described in Section 2.1. In other words, MLE can directly be used since the distribution functions as well as their associated densities are of closed form.
For this, we will vary the choice of the inner function $b(z)$. In the following, $\theta \in \Theta$ will denote the set of parameters appearing in the inner function. Using a dataset for Z taking values in $(0, 1]$, we will fit the models with MLE, maximizing the log-likelihood function

$$\theta \mapsto \ell_Z(\theta) = \log(f^{(\theta)}(Z))\mathbb{1}_{\{Z<1\}} + \log(p^{(\theta)})\mathbb{1}_{\{Z=1\}}, \tag{6.1}$$

where the absolutely continuous density $f^{(\theta)}$ on $[0, 1)$ and the point mass $p^{(\theta)}$ are obtained as in Section 2.1. We also study the model of Corollary 2.5, which extends the previous model by adding a flexible point mass q in 1. Its log-likelihood function is

$$\begin{aligned}(\theta, q) \mapsto \ell_Z(\theta, q) &= \log((1-q)f_0^{(\theta)}(Z))\mathbb{1}_{\{Z<1\}} + \log(q)\mathbb{1}_{\{Z=1\}} \\ &= \log(f_0^{(\theta)}(Z))\mathbb{1}_{\{Z<1\}} + \log(1-q)\mathbb{1}_{\{Z<1\}} + \log(q)\mathbb{1}_{\{Z=1\}}.\end{aligned} \tag{6.2}$$

The maximization problem in (6.1) will be denoted as the *MLE of the standard problem*, whereas the maximization problem in (6.2) will be called the *MLE of the extended problem.*
The dataset used in this section consists of claims observations from private property insurance. Private property usually includes deductibles to reduce the number of small claims, and hence, administrative expenses, e.g. , a sufficiently high deductible implies that not every lost umbrella gets reported to the insurance company as stolen. Secondly, private property includes maximal covers that may depend on the underlying peril. We have $n = 126\,026$ claims Y_i above the deductible d and we scale them by the maximal cover providing us with normalized lower-truncated and right-censored claims Z_i for $i = 1, \dots, n$. We assume that these normalized claims Z_i are i.i.d. and follow a distribution belonging to the Bernegger class.

	Min.	Q1	Median	Q3	Max.	Mean
Normalized claims Z_i	0.00001	0.160	0.280	0.457	1	0.339

Table 1: Summary statistics of the dataset containing 126 026 normalized lower-truncated and right-censored claims Z_i.

Some summary statistics of the claims Z_i are provided in Table 1 and the empirical (observed) density of the claims that are strictly smaller than 1 is shown in Figure 3, we have an empirical

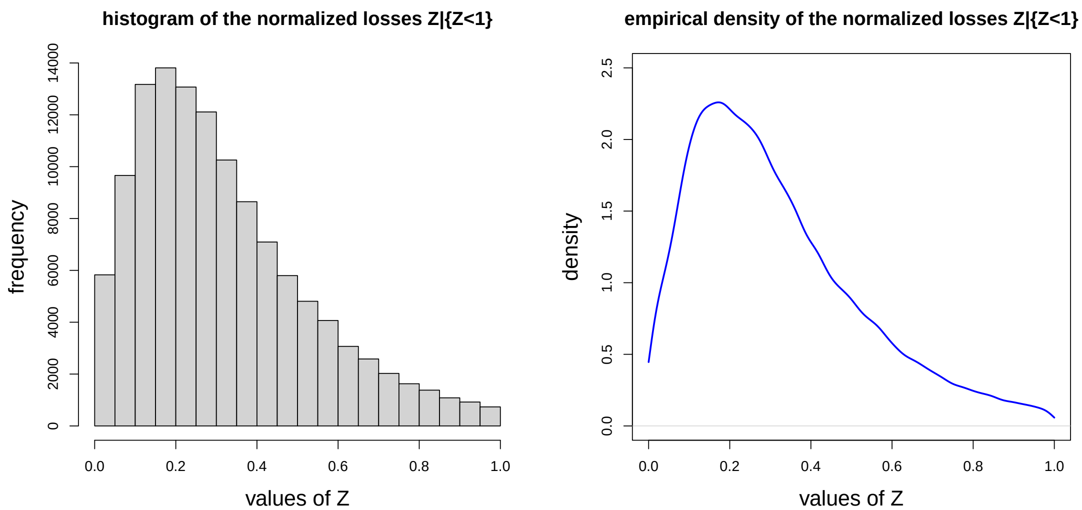

Figure 3: Histogram (lhs) and empirical density (rhs) of the claims Z_i, only showing the claims strictly smaller than 1, and the point mass in 1 is 3.4%.

mean of 0.339 and the observed point mass in 1 is 0.034, i.e., the insurance company pays the maximal cover on 3.4% of the claims.
These normalized losses Z_i, along with different models from the Bernegger class, will be used to solve the above MLE maximization problems in order to produce the results of this section. For this, we use the R function *optim* in order to minimize the sum of the negative of the log-likelihoods evaluated at Z_i, after having possibly transformed our parameters in a way to ensure that they lie in a suitable open domain, this is described below. The results of the best model from the Bernegger class will then be compared to lower-truncated and right-censored log-normal and gamma models at the end of this section. The first fitted model is the classical MBBEFD model of Bernegger [2].

6.1 The MBBEFD example

We have seen in Section 3.3 that the MBBEFD example belongs to the logarithmic linked exposure family and is obtained by choosing the inner function

$$b(z) = a + b^z,$$

for parameters $b \geq 0$ and $a > -\min(1, b)$ chosen in a way to obtain a well-defined exposure curve. Using the parametrization in (3.6), it is possible to give the conditions under which this class of distributions leads to unimodal densities on $[0, 1)$. Indeed, according to Proposition 3.2, such unimodal densities are obtained if and only if

$$bg < 1 \qquad \text{and} \qquad b < \frac{1 - bg}{g - 1} < 1,$$

for parameters $g > 1$ and $b > 0$. Therefore, we use the domain

$$\Theta = \left\{ g > 1, \max\left(0, \frac{2-g}{g}\right) < b < \frac{1}{2g-1} \right\},$$

in order to find unimodal MLE solutions of the standard and extended maximization problems described in (6.1) and (6.2). In Figure 4, the empirical density of the data points strictly smaller than 1 is plotted in blue color. Since this corresponds to a true density integrating to one, we further show the conditional density of $Z|_{\{Z<1\}}$ of the fitted models, using the parameters obtained by solving the standard problem (6.1) (in green) and the extended problem (6.2) (in red). As for all the fitted models in this section, these parameters are provided in the appendix.. The point mass and the mean are shown in Table 2, as well as the log-likelihoods of the random variables $Z|_{\{Z<1\}}$ and Z, and the AIC scores that are computed using ℓ_Z. We see that, as expected, the MLE solution of the extended problem gives better results, even if the densities obtained are not close to the empirical density. Proposition 3.3 helps us to understand why the fit is not accurate since the empirical density is skewed to the right, whereas the MBBEFD class only allows for symmetric densities described by the derivative of the logistic function.

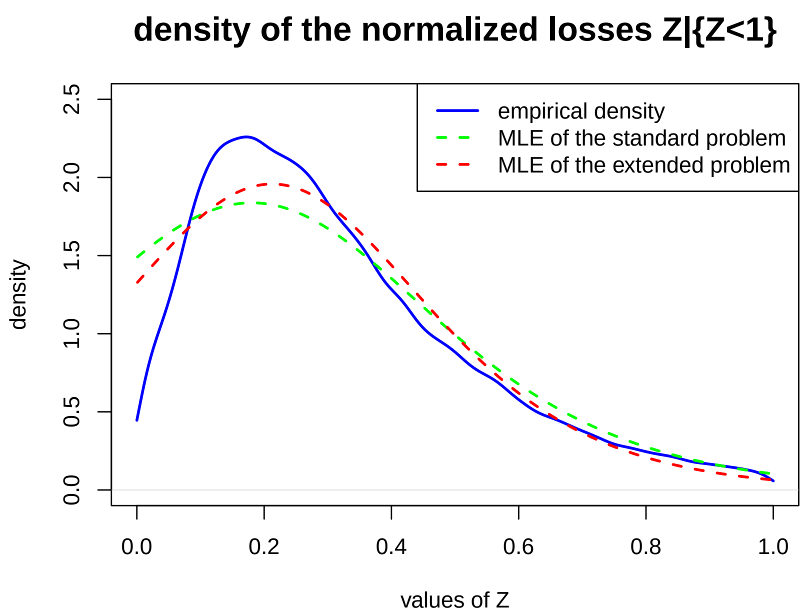

Figure 4: The MBBEFD example: densities of the random variable $Z|_{\{Z<1\}}$.

	Point mass	Mean	$\ell_{Z\|_{\{Z<1\}}}$	ℓ_Z	AIC
Empirical density (Blue)	0.034	0.339	-	-	-
MLE of the standard problem (Green)	0.020	0.337	32 682	13 475	-26 947
MLE of the extended problem (Red)	0.034	0.336	33 257	14 587	-29 168

Table 2: The MBBEFD example: results.

6.2 Power logarithmic linked exposure example

Another example belonging to the logarithmic linked exposure family is obtained by choosing the power function

$$b(z) = \left(1 - \frac{z}{\alpha}\right)^{\delta} + a,$$

for $\alpha > 1$, $\delta > 1$ and a sufficiently large parameter $a > 0$. This is a smooth and strictly convex curve on $[0, 1]$ with $b(1) = (1 - 1/\alpha)^{\delta} + a < 1 + a = b(0)$. The first and second derivatives for $z \in [0, 1]$ are given by

$$b'(z) = -\frac{\delta}{\alpha}\left(1 - \frac{z}{\alpha}\right)^{\delta-1} \; < \; 0 \qquad \text{and} \qquad b''(z) = \frac{\delta(\delta-1)}{\alpha^2}\left(1 - \frac{z}{\alpha}\right)^{\delta-2} \; > \; 0.$$

In order to achieve $b''(z)b(z) - b'(z)^2 \geq 0$ for all $z \in [0, 1]$, which is a necessary condition due to Proposition 4.1, the parameter a has to satisfy

$$a > \max_{z\in[0,1]} \left(\frac{b'(z)^2}{b''(z)} - \left(1 - \frac{z}{\alpha}\right)^{\delta}\right). \tag{6.3}$$

This provides in this example

$$a > \max_{z\in[0,1]} \left(\frac{1}{\delta - 1}\left(1 - \frac{z}{\alpha}\right)^{\delta}\right) \; = \; \frac{1}{\delta - 1}. \tag{6.4}$$

With Corollary 4.2, we then obtain as density for $z \in [0, 1)$

$$f_Z(z) = \frac{a+1}{\alpha} \frac{(1 - z/\alpha)^{\delta-2}}{(a + (1 - z/\alpha)^{\delta})^2} \left(a(\delta - 1) - (1 - z/\alpha)^{\delta}\right), \tag{6.5}$$

with point mass in $z = 1$ equal to

$$p = \left(1 - \frac{1}{\alpha}\right)^{\delta-1} \frac{a+1}{a + (1 - 1/\alpha)^{\delta}},$$

and mean

$$\mathbb{E}[Z] = \frac{\alpha(a+1)}{\delta} \log\left(\frac{a+1}{a + (1 - 1/\alpha)^{\delta}}\right).$$

The derivative of the density f_Z is given by

$$f'_Z(z) = -\frac{a+1}{\alpha^2} \frac{\left[a^2(\delta-1)(\delta-2) - a(\delta-1)(\delta+4)(1 - z/\alpha)^{\delta} + 2(1 - z/\alpha)^{2\delta}\right]}{(a + (1 - z/\alpha)^{\delta})^3\,(1 - z/\alpha)^{-\delta+3}}. \tag{6.6}$$

Lemma 6.1. *The power logarithmic linked exposure example with the above parameters leads to a well-defined distribution. The density f_Z of this power logarithmic linked exposure example given in* (6.5) *can only be unimodal on* $[0, 1)$ *if* $\delta > 2$.

Therefore, we restrict to the domain

$$\Theta = \left\{\alpha > 1, \delta > 2, a > \frac{1}{\delta - 1}\right\},$$

in order to find unimodal solutions to the MLE of the standard and extended maximization problems described in (6.1) and (6.2). The results displayed in Figure 5 are very similar to the ones obtained for the MBBEFD example. This can be confirmed by comparing Table 2 and Table 3, where most of the values coincide, although they are actually different if we look at digits after the decimal point. We conclude that this power logarithmic linked example does not improve the fit provided by the MBBEFD example, although this model allows for skewness.

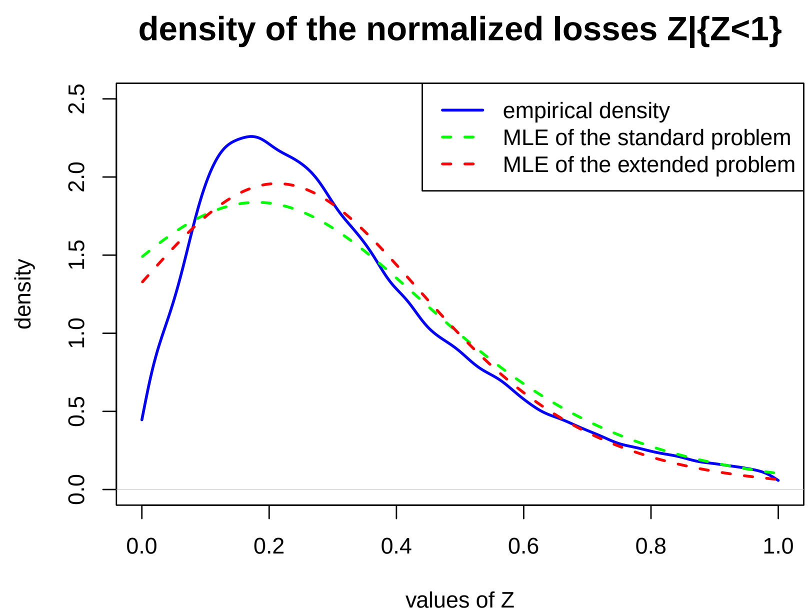

Figure 5: Power logarithmic linked exposure example: densities of the random variable $Z|_{\{Z<1\}}$.

	Point mass	Mean	$\ell_{Z\|_{\{Z<1\}}}$	ℓ_Z	AIC
Empirical density (Blue)	0.034	0.339	-	-	-
MLE of the standard problem (Green)	0.020	0.337	32 680	13 475	-26 945
MLE of the extended problem (Red)	0.034	0.336	33 257	14 587	-29 166

Table 3: Power logarithmic linked exposure example: results.

6.3 Sine logarithmic linked exposure example

A third example belonging to the logarithmic linked exposure family is given by the sine function

$$b(z) = \sin(\alpha z + \beta) + a,$$

for $\beta \in \left(-\frac{\pi}{2}, 0\right), \alpha \in \left(0, \frac{\pi}{2} - \beta\right)$ and $-\sin(\beta) < a < -1/\sin(\beta)$. This is a smooth curve on $[0, 1]$ with $b(0) = \sin(\beta) + a < \sin(\alpha + \beta) + a = b(1)$. The first and second derivatives for $z \in [0, 1]$ are given by

$$b'(z) = \alpha \cos(\alpha z + \beta) \;>\; 0 \qquad \text{and} \qquad b''(z) = -\alpha^2 \sin(\alpha z + \beta).$$

Note that in this case, the function b is in general neither concave, nor convex on the entire interval $[0, 1]$. We claim that $b''(z)b(z) - b'(z)^2 \geq 0$ holds for all $z \in [0, 1]$, which is a necessary condition in order to obtain a distribution function due to Proposition 4.1. Moreover, the density for $z \in [0, 1)$ reads as

$$f_Z(z) = \frac{\sin(\beta) + a}{\cos(\beta)} \frac{\alpha \left[1 + a \sin(\alpha z + \beta)\right]}{\left[\sin(\alpha z + \beta) + a\right]^2}, \tag{6.7}$$

with point mass in $z = 1$

$$p = \frac{\cos(\alpha + \beta)}{\cos(\beta)} \frac{\sin(\beta) + a}{\sin(\alpha + \beta) + a},$$

and mean

$$\mathbb{E}[Z] = -\frac{\sin(\beta)+a}{\alpha\cos(\beta)} \log\left(\frac{\sin(\beta)+a}{\sin(\alpha+\beta)+a}\right).$$

The derivative of the density f_Z is given by

$$f_Z'(z) = -\frac{\sin(\beta)+a}{\cos(\beta)} \frac{\alpha^2\cos(\alpha z+\beta)\left[-a^2+a\sin(\alpha z+\beta)+2\right]}{\left[\sin(\alpha z+\beta)+a\right]^3}. \tag{6.8}$$

Lemma 6.2. *The sine logarithmic linked exposure example with the above parameters leads to a well-defined distribution. Moreover, the density f_Z given in* (6.7) *can only be unimodal on* $[0,1)$ *if* $1 \leq a \leq 2$.

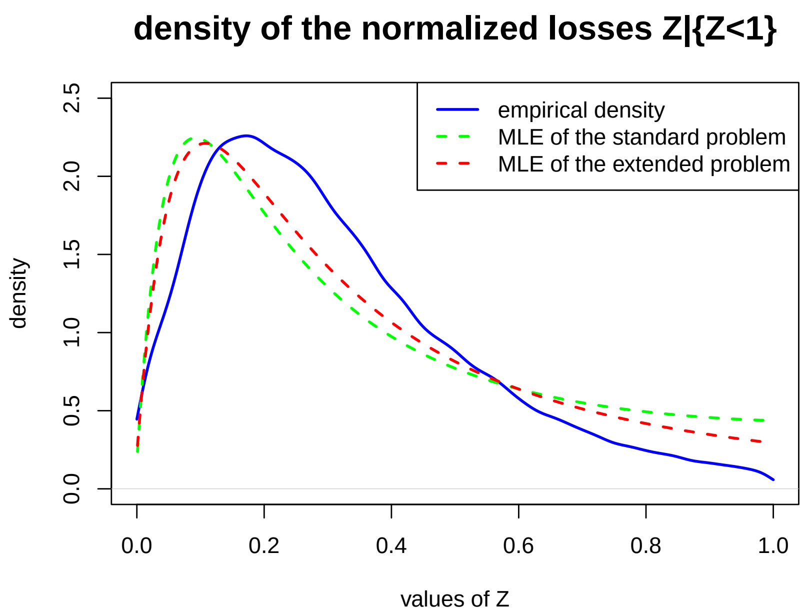

Figure 6: Sine logarithmic linked exposure example: densities of the random variable $Z|_{\{Z<1\}}$.

	Point mass	Mean	$\ell_{Z\|_{\{Z<1\}}}$	ℓ_Z	AIC
Empirical density (Blue)	0.034	0.339	-	-	-
MLE of the standard problem (Green)	0.040	0.380	26 907	8 164	-16 323
MLE of the extended problem (Red)	0.034	0.362	30 453	11 783	-23 558

Table 4: Sine logarithmic linked exposure example: results.

Thus, we use the domain

$$\Theta = \left\{\beta \in \left(-\frac{\pi}{2}, 0\right), \alpha \in \left(0, \frac{\pi}{2} - \beta\right), 1 < a < \min\left(-\frac{1}{\sin(\beta)}, 2\right)\right\},$$

in order to find unimodal solutions to the standard and extended maximization problems described in (6.1) and (6.2). Similarly as for the previous examples, we obtain the results displayed in Figure 6 and Table 4. Although this example manages to produce rather skewed densities, see Figure 6, the fit does not seem to be accurate for this data, i.e., it is less accurate than the first examples, and we clearly prefer the previous models.

6.4 Quadratic exponentially linked exposure example

Let us now treat an example belonging to the exponentially linked exposure family, and given by the quadratic inner function

$$b(z) = \alpha z^2 + \beta z,$$

for $\alpha < 0$ and $\beta < -\sqrt{-2\alpha}$. This is a smooth curve on $[0,1]$ with $b(0) = 0 > \alpha + \beta = b(1)$. The first and second derivatives for $z \in [0,1]$ are given by

$$b'(z) = 2\alpha z + \beta \ < \ 0 \qquad \text{and} \qquad b''(z) = 2\alpha \ < \ 0.$$

This function b is smooth and strictly concave on $[0,1]$. We claim that $b''(z) + b'(z)^2 \geq 0$ holds for all $z \in [0,1]$, which is a necessary condition in order to obtain a distribution function due to Proposition 5.1. Moreover, the density for $z \in [0,1)$ is given by

$$f_Z(z) = -\frac{e^{\alpha z^2+\beta z}\left[4\alpha^2 z^2 + 4\alpha\beta z + \beta^2 + 2\alpha\right]}{\beta}, \tag{6.9}$$

with point mass in $z = 1$

$$p = \frac{e^{\alpha+\beta}\left[2\alpha + \beta\right]}{\beta},$$

and mean

$$\mathbb{E}[Z] = \frac{e^{\alpha+\beta} - 1}{\beta}.$$

The derivative of the density f_Z is given by

$$f_Z'(z) = -\frac{e^{\alpha z^2+\beta z}\left(2\alpha z + \beta\right)\left[(2\alpha z + \beta)^2 + 6\alpha\right]}{\beta}. \tag{6.10}$$

Lemma 6.3. *The quadratic exponentially linked exposure example with the above parameters leads to a well-defined distribution. Moreover, the density f_Z given in* (6.9) *is unimodal on* $[0,1)$ *if and only if* $-\sqrt{-6\alpha} < \beta < -2\alpha - \sqrt{-6\alpha}$.

Thus, we restrict to the domain

$$\Theta = \Big\{\alpha < 0, \beta \in \left(-\sqrt{-6\alpha}\,, \min\left(-2\alpha - \sqrt{-6\alpha}, -\sqrt{-2\alpha}\right)\right)\Big\},$$

in order to find unimodal solutions to the standard and extended maximization problems described in (6.1) and (6.2). Similarly as for the previous examples, we obtain the results shown in Figure 7 and Table 5. Although a new family of exposure curves is used here, the results are close to the ones obtained with the power logarithmic linked exposure example or the MBBEFD example. In fact, in the extended problem, we obtain a slightly better model than in the MBBEFD class according to AIC. However, looking at Figure 7, this new model is again not satisfactory for this data.

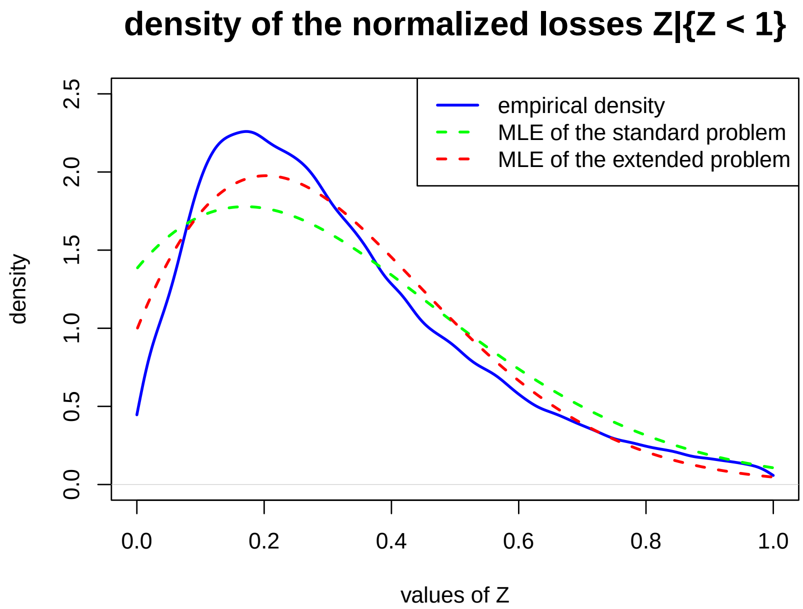

Figure 7: Quadratic exponentially linked exposure example: densities of the random variable $Z|_{\{Z<1\}}$.

	Point mass	Mean	$\ell_{Z\|_{\{Z<1\}}}$	ℓ_Z	AIC
Empirical density (Blue)	0.034	0.339	-	-	-
MLE of the standard problem (Green)	0.016	0.345	32 101	12 425	-24 846
MLE of the extended problem (Red)	0.034	0.341	33 397	14 726	-29 447

Table 5: Quadratic exponentially linked exposure example: results.

6.5 Power exponentially linked exposure example

We consider a final example belonging to the exponentially linked exposure family. This example is a bit more difficult in handling, but it provides the best results for our dataset. Choose the power function

$$b(z) = \epsilon(z+\delta)^\alpha - \beta z,$$

for $\alpha \in (1,2), \delta > 0, \epsilon < 0$ and $\beta > \epsilon\alpha\delta^{\alpha-1} + \sqrt{-\epsilon\alpha(\alpha-1)\delta^{\alpha-2}}$. This is a smooth curve on $[0,1]$ with $b(0) = \epsilon\delta^\alpha > \epsilon(1+\delta)^\alpha - \beta = b(1)$. The first and second derivatives for $z \in [0,1]$ are given by

$$b'(z) = \alpha\epsilon(z+\delta)^{\alpha-1} - \beta \ < \ 0 \qquad \text{and} \qquad b''(z) = \alpha(\alpha-1)\epsilon(z+\delta)^{\alpha-2} \ < \ 0.$$

This function b is smooth and strictly concave on $[0,1]$. We claim that $b''(z) + b'(z)^2 \geq 0$ holds for all $z \in [0,1]$, which is a necessary condition in order to obtain a distribution function due to Proposition 5.1. Moreover, the density for $z \in [0,1)$ reads as

$$f_Z(z) = -\frac{e^{\epsilon(z+\delta)^\alpha - \beta z}\left[(\alpha\epsilon(z+\delta)^{\alpha-1} - \beta)^2 + \alpha(\alpha-1)\epsilon(z+\delta)^{\alpha-2}\right]}{e^{\epsilon\delta^\alpha}\left[\alpha\epsilon\delta^{\alpha-1} - \beta\right]},$$

with point mass in $z = 1$

$$p = \frac{e^{\epsilon(1+\delta)^\alpha - \beta}\left[\alpha\epsilon(1+\delta)^{\alpha-1} - \beta\right]}{e^{\epsilon\delta^\alpha}\left[\alpha\epsilon\delta^{\alpha-1} - \beta\right]},$$

and mean

$$\mathbb{E}[Z] = \frac{e^{\epsilon[(1+\delta)^\alpha - \delta^\alpha] - \beta} - 1}{\alpha\epsilon\delta^{\alpha-1} - \beta}.$$

The derivative of the density f_Z can be obtained using (5.6). It does however not allow to explicitly characterize the extrema of the density in this example. Nevertheless, we observe in Figure 8 that this example contains unimodal densities.

Lemma 6.4. *The power exponentially linked exposure example with the above parameters leads to a well-defined distribution.*

Thus, we restrict to the domain

$$\Theta = \left\{\alpha \in (1,2), \delta > 0, \epsilon < 0, \beta > \epsilon\alpha\delta^{\alpha-1} + \sqrt{-\epsilon\alpha(\alpha-1)\delta^{\alpha-2}}\right\},$$

in order to find solutions to the standard and extended maximization problems described in (6.1) and (6.2). Similarly as for the previous examples, we obtain the results shown in Figure 8 and Table 6. This time, the fit seems much better. This is especially the case for the red curve, which represents the solution of the extended maximization problem. The tail behavior on the right and left ends seems adequate in contrast to the plots presented in the previous examples. This observation is confirmed by the AIC scores attained here, which are lower than the AIC scores of all other examples, i.e., we give clear preference to this last example for this data.

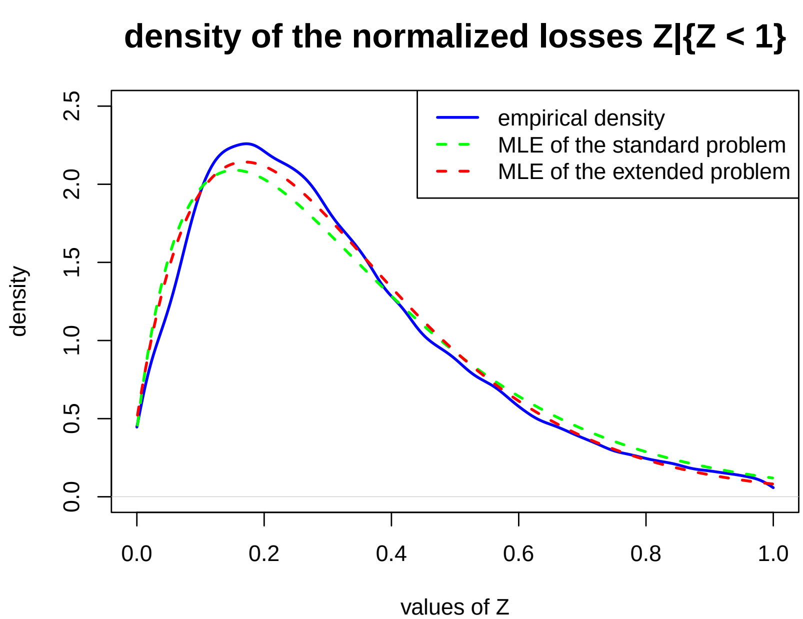

Figure 8: Power exponentially linked exposure example: densities of the random variable $Z|_{\{Z<1\}}$.

	Point mass	Mean	$\ell_{Z\|_{\{Z<1\}}}$	ℓ_Z	AIC
Empirical density (Blue)	0.034	0.339	-	-	-
Standard MLE density (Green)	0.025	0.341	34 068	15 199	-30 390
Flexible MLE density (Red)	0.034	0.339	34 402	15 731	-31 453

Table 6: Power exponentially linked exposure example: results.

6.6 Comparison with the log-normal and gamma distribution

For this real dataset example, we performed an MLE estimation without taking into account any covariates. In that case, performing MLE using lower-truncated and right-censored log-normal and gamma models is feasible. Our aim is to compare the extended fit of the power exponentially linked exposure example to the fit of these classical distributions. For this, we first consider the normalized claim Z and write

$$Z = \frac{1}{M}\min\left\{(X-d)_+, M\right\} \mid X > d = \min\left\{\left(\frac{X}{M} - \frac{d}{M}\right)_+, 1\right\} \Bigg| \frac{X}{M} > \frac{d}{M},$$

where X is assumed to have a log-normal and gamma distribution, respectively. For an arbitrary $M > 0$, both distributions share the following scaling property

$$X \sim \mathrm{LN}(\mu, \sigma^2) \implies \frac{X}{M} \sim \mathrm{LN}(\mu - \log(M), \sigma^2) \quad \text{and} \quad X \sim \Gamma(\gamma, c) \implies \frac{X}{M} \sim \Gamma(\gamma, cM),$$

where $\mu \in \mathbb{R}$ and $\sigma^2 > 0$ are the parameters of a log-normal distribution, and $\gamma > 0$ and $c > 0$ are the shape and scale parameters of a gamma distribution. Let us denote by θ the parameters of the log-normal, respectively, gamma distribution. Due to the above scaling property and by setting $\tilde{d} = d/M$, we can maximize without loss of generality the log-likelihood function

$$(\theta, \tilde{d}) \mapsto \ell_Z(\theta, \tilde{d}) = \log(f^{(\theta,\tilde{d})}(Z))\mathbb{1}_{\{Z<1\}} + \log(p^{(\theta,\tilde{d})})\mathbb{1}_{\{Z=1\}}, \tag{6.11}$$

where the absolutely continuous density $f^{(\theta,\tilde{d})}$ is given by

$$f^{(\theta,\tilde{d})}(y) = \frac{f_X(\tilde{d} + y)}{1 - F_X(\tilde{d})},$$

for $y \in [0, 1)$, $\tilde{d} \in (0, \infty)$, and where the point mass $p^{(\theta,\tilde{d})}$ is given by

$$p^{(\theta,\tilde{d})} = \frac{1 - F_X(\tilde{d} + 1)}{1 - F_X(\tilde{d})},$$

see (1.2) and (1.3). For both distributions, the number of parameters is thus equal to three and the results of these MLE fits are given in Figure 9 and Table 7. Compared to the extended MLE of the power exponentially linked example, the fit on the absolutely continuous density seems slightly worse for the gamma model, whereas the fit using the log-normal model seems to be accurate for this dataset when looking at Figure 9. These observations are confirmed in Table 7. On the one hand, the gamma model fails to obtain a suitable value for the point mass, and on the other hand, the values obtained for the point mass and the mean are close to the empirical values for the log-normal model, which also achieves the lowest AIC score. Therefore we give preference to the log-normal model over the other models for this dataset.

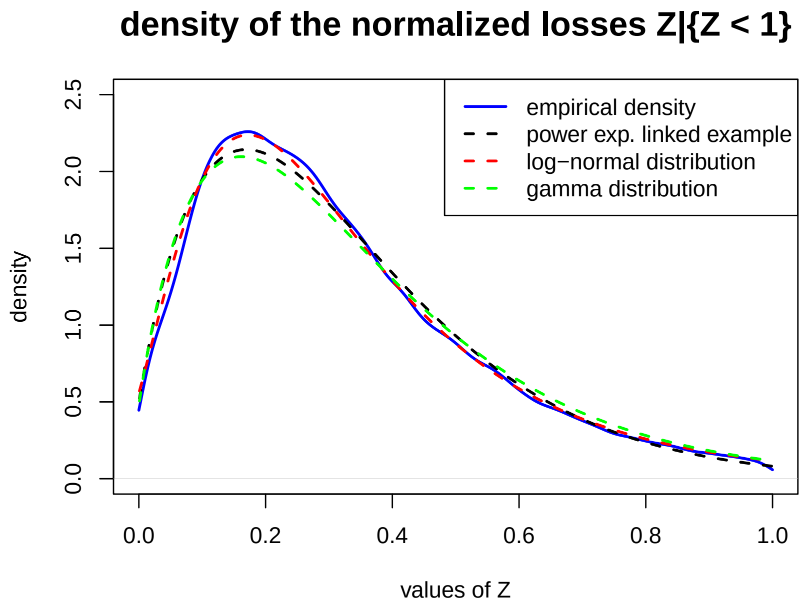

Figure 9: Comparison with the gamma and the log-normal distribution: densities of the random variable $Z|_{\{Z<1\}}$.

	Point mass	Mean	$\ell_{Z\|_{\{Z<1\}}}$	ℓ_Z	AIC
Empirical density (Blue)	0.034	0.339	-	-	-
Flexible MLE density of the power exponentially linked example (Black)	0.034	0.339	34 402	15 731	-31 453
MLE density of the log-normal distribution (Red)	0.031	0.340	34 640	15 956	-31 907
MLE density of the gamma distribution (Green)	0.024	0.341	34 236	15 355	-30 704

Table 7: Comparison with the gamma and the log-normal distribution: results.

Thus, it seems that the log-normal distribution has a better performance than our examples from the Bernegger class. However, we would like to point out that for fitting the gamma and the log-normal models, we do not work with the original deductible d and maximal cover M of the product since we optimize over $\tilde{d} = d/M$, see (6.11). This means that we do not assume the log-normal and the gamma distribution to fit the total financial losses, but we allow for arbitrary scaling. We also remark that fitting the gamma and log-normal models under the original deductible d and maximal cover M does not lead to competitive models for this dataset. Of course, it is then unclear how one could derive the distribution of the total financial loss from the distribution of the observed insurance claims. We show in the next section that such a derivation is in general not unique and implies to make possibly wrong assumptions on the distribution of the total financial loss.

7 Changes in the deductible or the maximal cover

The primary objective of modeling lower-truncated and right-censored insurance claims is to fit the size of observed claims, or equivalently, the size of financial losses falling in the interval $(d, d+M)$. However, the insurer might also be interested in understanding how a change in the deductible and/or in the maximal cover influences the expected claim size. Let us denote by $\tilde{d}$ the new deductible and by $\tilde{M}$ the new maximal cover. Two cases may arise.
If the deductible increases or the maximal cover decreases such that $(\tilde{d}, \tilde{d}+\tilde{M}) \subseteq (d, d+M)$, the insurer can use his current claims observations in order to derive a new model for the new range of interest. However, in the case where $(\tilde{d}, \tilde{d}+\tilde{M}) \not\subseteq (d, d+M)$, e.g., when the deductible decreases and the maximal cover increases, an extrapolation based on the distribution of the observed total financial losses being in $(d, d+M)$ has to be made in order to obtain a candidate for the distribution on the new range of interest. We discuss in this section how the insurer can evaluate the new expected claim size after a change in the deductible or the maximal cover by treating separately the two above cases.

7.1 Increase in the deductible or decrease in the maximal cover

As in the previous sections, we denote the total financial loss by X, whereas the insurance claim is denoted by Y, see (1.1). Furthermore, we define the normalized insurance claim by

$$Z = \frac{1}{M} \min\{(X-d)_+, M\} \mid Z > d.$$

After an increase in the deductible or a decrease in the maximal cover such that the new range of interest satisfies $(\tilde{d}, \tilde{d}+\tilde{M}) \subseteq (d, d+M)$, the new normalized insurance claim becomes

$$\begin{aligned}\tilde{Z} &= \frac{1}{\tilde{M}} \min\left\{(X-\tilde{d})_+, \tilde{M}\right\} \mid X > \tilde{d} \\ &= \frac{M}{\tilde{M}} \min\left\{\left(Z - \frac{\tilde{d}-d}{M}\right)_+, \frac{\tilde{M}}{M}\right\} \Bigg| \, Z > \frac{\tilde{d}-d}{M}.\end{aligned}$$

That is, the distribution of $\tilde{Z}$ is equal to a scaled lower-truncated and right-censored distribution of Z, where the lower-truncation point is given by $\bar{d} = (\tilde{d}-d)/M$ and the right-censoring point is $\bar{M} = \tilde{M}/M$ (after subtracting the lower-truncation). The next result shows that if Z belongs to the Bernegger class, then $\tilde{Z}$ also belongs to the Bernegger class. In other words, the Bernegger class is closed under lower-truncation and right-censoring.

Proposition 7.1. *Let Z be a member of the Bernegger class with exposure curve G and let $0 \leq \bar{d}, \bar{M} \leq 1$ such that $\bar{d} + \bar{M} \leq 1$. Moreover, define the scaled lower-truncated and right-censored random variable*

$$\tilde{Z} = \frac{1}{\bar{M}} \min\{(Z-\bar{d})_+, \bar{M}\} \mid Z > \bar{d}.$$

This random variable belongs again to the Bernegger class and its exposure curve is given by

$$\tilde{G} : [0,1] \to [0,1], \; z \mapsto \frac{G(\bar{d}+z\bar{M})}{G(\bar{d}+\bar{M})}.$$

We point out that in the case where an increase in the deductible and/or a decrease in the maximal cover leads to a new interval satisfying $(\tilde{d}, \tilde{d} + \tilde{M}) \not\subseteq (d, d + M)$, this last result does not apply since either the lower-truncation point $\bar{d} = (\tilde{d} - d)/M$ is negative or the sum $\bar{d} + \bar{M} = (\tilde{d} - d + \tilde{M})/M$ exceeds 1. This situation has then to be handled by performing some extrapolation on the observed part of the total financial loss distribution as in the case where the deductible decreases or the maximal cover increases, see next section.

7.2 Decrease in the deductible or increase in the maximal cover

The goal of this last section is to treat the case where the new deductible and the new maximal cover satisfy $(\tilde{d}, \tilde{d}+\tilde{M}) \not\subseteq (d, d+M)$. This typically happens when the deductible d decreases or when the maximal cover M increases, but also in some cases when d increases and M decreases, as discussed above. Since the insurer only has full knowledge on the distribution of the total final loss on the interval $(d, d + M)$, he will have to perform an extrapolation of the observed density in order to obtain a candidate for the distribution on the new range of interest.

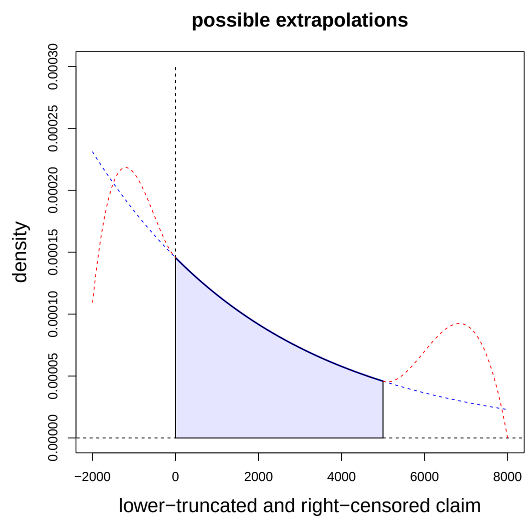

Figure 10: Two possible extrapolations of the density of a lower-truncated and right-censored exponential random variable with $d = 2000$ and $M = 5000$.

In Example 5.3, we showed that a scaled lower-truncated and right-censored exponential random variable belongs to the Bernegger class. Using this example, we show in Figure 10 that extrapolating a density on a given interval to a larger interval leads to infinitely many possible candidates. For this, we first plot in blue the density of an exponential random variable $X \sim \mathrm{Exp}(\lambda)$ for $\lambda = \log(2)/3000$. Then, we set the deductible to $d = 2000$ and the maximal cover $M = 5000$. The blue area in Figure 10 corresponds to the observable part of the density of the total financial loss of the insurer. Let us denote by p_- the probability of X being smaller than 2000 and by p_+ the probability of X being larger than 7000. We plot in red another possible extrapolation of the observed density such that the density of the total financial loss under this new extrapolation is continuously differentiable and such that the area under the red curve at the left and right ends is equal to p_- and p_+, respectively. Using the observations falling in the interval $(d, d + M)$ as well as the point mass in $d + M$ indicating the proportion exceeding this threshold, the insurer may choose either one of the two above extrapolations and, actually, the number of possible candidates is in general infinite. Therefore, we point out that in the case where $(\tilde{d}, \tilde{d}+\tilde{M}) \not\subseteq (d, d+M)$, the insurer cannot derive the distribution of the total financial loss on this new range without making additional assumptions on the random variable X. Using the two-parameter logistic distribution of Example 4.3, the next example shows a possible extrapolation.

Example 7.2 (Lower-truncated and right-censored two-parameter logistic distribution)**.** Let us assume that X follows a two-parameter logistic distribution as in Example 4.3. Moreover, let $d \in \mathbb{R}$ and $M > 0$. By Corollary 4.2, the normalized lower-truncated and right-censored random variable Z in (4.5) has as density

$$f_Z(z) = \frac{M}{\sigma}\left(1+e^{(d-\mu)/\sigma}\right)\frac{e^{(zM+d-\mu)/\sigma}}{\left(1+e^{(zM+d-\mu)/\sigma}\right)^2}, \tag{7.1}$$

for $z \in [0,1)$, with a point mass in 1 equal to

$$p = \frac{1+e^{(d-\mu)/\sigma}}{1+e^{(M+d-\mu)/\sigma}}.$$

Suppose now that we know the values of the d, M and p and that we want to retrieve the unknown distribution of the random variable X. This distribution is related to the distribution of Z by

$$F_Z(z) = \frac{F_X(d+zM)-F_X(d)}{1-F_X(d)}\,\mathbb{1}_{\{z\in[0,1)\}} + \mathbb{1}_{\{z=1\}}. \tag{7.2}$$

At this point, we have to make the following (possibly wrong) assumptions. We first assume that the random variable X is absolutely continuous and that the support of its density is the whole real line. Moreover, we assume that

$$f_X(z) = C f_Z\left(\frac{z-d}{M}\right), \tag{7.3}$$

for $z \in \mathbb{R}$ and where C is a normalizing constant such that f_X becomes a density on $\mathbb{R}$. Note that in this last assumption, f_Z is now seen as a function defined on the whole real line. In general, it might not be possible to obtain a well-defined function by extending the domain of f_Z, which is a priori only defined for $z \in [0,1)$, see (7.1). In this case, however, extending the domain of the density f_Z to the whole real line leads to a well-defined function. Note that assumption (7.3) might seem natural in view of (7.2). Our above assumptions give us again the original density in this case, namely,

$$f_X(z) = \frac{e^{(z-\mu)/\sigma}}{\sigma\left(1+e^{(z-\mu)/\sigma}\right)^2}, \quad \text{for } -\infty < z < \infty,\ -\infty < \mu < \infty,\ \sigma > 0.$$

■

This example shows how we can start from a member of the Bernegger class in order to obtain a candidate for the original distribution. Under some assumptions, we managed to retrieve the original distribution in this case. If we assumed however that the support of the density f_X is the positive real line or that X is not an absolutely continuous random variable but rather has some point masses at selected locations, we would have ended with a wrong candidate for the original distribution. Of course, the point mass in 1 can help us in order to verify whether the made assumptions are plausible but even with this information, the number of possible candidates for the original distribution is in general infinite. That is, we point out again that it is in general impossible to retrieve the original distribution by performing some extrapolation based on the lower-truncated and right-censored distribution.

8 Conclusion

Most classical statistical models are not directly suited to model lower-truncated and right-censored claims in general insurance since they lead to problems that are not easily analytically tractable. Bernegger introduced in [2] the MBBEFD class of distributions that can model such claims using a distribution function, an absolutely continuous density, and a point mass that are all of closed form. This class was introduced in the reinsurance literature, where densities are typically monotonically decreasing.

In general insurance, however, we are mainly interested in unimodal skewed densities. Therefore, we extended the MBBEFD class to a much bigger family of distributions that we called the Bernegger class. By starting from the properties of an exposure curve, we introduced two subfamilies, namely the logarithmic and exponentially linked exposure families. Through various examples, we used the full tractability and flexibility of the Bernegger class in order to fit parameters to general insurance claims using maximum likelihood estimations. It turned out that this large class of distributions contains models allowing to obtain a suitable approximation for the distribution of the used dataset, and in general, we have a rich family of unimodal and skewed densities within the Bernegger class. This class of distribution allows to model lower-truncated and right-censored random variables without making any assumption on the original unobserved distribution. We showed further that it is in general impossible to obtain a unique extrapolation from the observable lower-truncated and right-censored distribution in order to derive the full distribution of the original random variable. That is, from an actuarial perspective, it is not possible to know the distribution of the total financial loss from the insurance claims observations.

Going forward, it will be interesting to further characterize and classify the members of the Bernegger class based on different properties. Of course, this might involve exploring other link functions. Another next step is to lift these models to regression models allowing for integrating fixed effects described by covariates. In this last setup, we point out that relying on numerical approximation for the distribution function in order to perform MLE is not possible for a large number of covariates.

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References

[1] Abramson, D. (2022). A nonproportional premium rating method for construction risks. *North American Actuarial Journal* **26/4**, 626-645.

[2] Bernegger, S. (1997). The Swiss Re exposure curves and the MBBEFD distribution class. *ASTIN Bulletin* **27/1**, 99-111.

[3] Blæsild, P., Jensen, J.L. (1985). Saddlepoint formulas for reproductive exponential models. *Scandinavian Journal of Statistics* **12/3**, 193-202.

[4] Dutang, C., Gesmann, M., Spedicato, G. (2021). Exposure rating, destruction rate models and the `mbbefd` package. R *Vignettes*.

[5] Fung, T.C., Badescu, A.L., Lin, X.S. (2022). Fitting censored and truncated regression data using the mixture of experts models. *North American Actuarial Journal*, **26/4**, 496-520.

[6] Landsman, Z., Valdez, E.A. (2005). Tail conditional expectation for exponential dispersion models. *ASTIN Bulletin* **35/1**, 189-209.

[7] Parodi, P. (2014). *Pricing in General Insurance*. CRC Press.

[8] Parodi, P., Watson, P. (2019). Property graphs - a statistical model for fire and explosion losses based on graph theory. *ASTIN Bulletin* **49/2**, 263-297.

[9] Poudyal, C. (2021). Robust estimation of loss models for lognormal insurance payment severity data. *ASTIN Bulletin* **51/2**, 475-507.

[10] Poudyal, C. (2021). Truncated, censored, and actuarial payment-type moments for robust fitting of a single-parameter Pareto distribution. *Journal of Computational and Applied Mathematics* **388**, 113310.

[11] Poudyal, C., Brazauskas, V. (2022). Robust estimation of loss models for truncated and censored severity data. *Variance* **15/2**.

[12] R Core Team (2018). R: A language and environment for statistical computing. R *Foundation for Statistical Computing*, Vienna, Austria. `https://www.R-project.org/`

[13] Riegel, U. (2010). On fire exposure rating and the impact of the risk profile type. *ASTIN Bulletin* **40/2**, 727-777.

[14] Spedicato, G.A., Dutang, C. (2015). The `mbbefd` package: a package for handling destruction rates in R including MBBEFD distributions. R *Vignettes*.

[15] Tweedie, M.C.K. (1984). An index which distinguishes between some important exponential families. In: *Statistics: Applications and New Directions*. Ghosh, J.K., Roy, J. (Eds.). Proceeding of the Indian Statistical Golden Jubilee International Conference, Indian Statistical Institute, Calcutta, 579-604.

[16] Verbelen, R., Gong, L., Antonio, K., Badescu, A., Lin, S. (2015). Fitting mixtures of Erlangs to censored and truncated data using the EM algorithm. *ASTIN Bulletin* **45/3**, 729-758.

[17] Wüthrich, M.V., Merz, M. (2023). *Statistical Foundations of Actuarial Learning and its Applications.* Springer Actuarial. `https://link.springer.com/book/10.1007/978-3-031-12409-9`

A Proofs

We prove all statements in this appendix.

Proof of Theorem 2.1. The function F_Z, as defined in (2.1), is continuously differentiable on the interval $(0,1)$ by our assumptions on G. This means that a derivative exists, and is equal to

$$f_Z(z) = F_Z'(z) = -\frac{G''(z)}{G'(0)} \geq 0,$$

for $z \in [0,1)$. Since G was assumed to be twice continuously differentiable, non-decreasing and concave, we obtain that $0 \leq G'(z) \leq G'(0)$ and $f_Z(z) \geq 0$ for all $z \in [0,1]$. This implies

$$0 = F_Z(0) \leq F_Z(z) \leq F_Z(1) = 1, \quad \text{for all } z \in [0,1].$$

Since G' is continuous by assumption, we conclude from (2.1) that F_Z is right-continuous, and hence, a distribution function on $[0,1]$. The point mass in 1 is then given by

$$p = 1 - F_Z(1_-) = \frac{G'(1)}{G'(0)} \in [0,1],$$

and the mean of $Z \sim F_Z$ is equal to

$$\mathbb{E}[Z] = \int_0^1 1 - F_Z(s)\, ds = \int_0^1 \frac{G'(s)}{G'(0)}\, ds = \frac{G(1)-G(0)}{G'(0)} = \frac{1}{G'(0)}.$$

This proves the theorem. □

Proof of Lemma 2.3. First, it is clear that the function $G : [0,1] \to \mathbb{R}$ defined in (2.4) is an exposure curve. Moreover, we have by (2.1) that the distribution function F_Z generated by G satisfies

$$\begin{aligned}
F_Z(z) &= \left(1 - \frac{G'(z)}{G'(0)}\right) \mathbb{1}_{\{z<1\}} + \mathbb{1}_{\{z=1\}} \\
&= \left(1 - \frac{\sum_{i=1}^n \alpha_i G_i'(z)}{\sum_{j=1}^n \alpha_j G_j'(0)}\right) \mathbb{1}_{\{z<1\}} + \mathbb{1}_{\{z=1\}} \\
&= \left(1 - \sum_{i=1}^n \frac{\alpha_i G_i'(0)}{\sum_{j=1}^n \alpha_j G_j'(0)} \frac{G_i'(z)}{G_i'(0)}\right) \mathbb{1}_{\{z<1\}} + \mathbb{1}_{\{z=1\}} \\
&= \left(1 - \sum_{i=1}^n w_i \frac{G_i'(z)}{G_i'(0)}\right) \mathbb{1}_{\{z<1\}} + \mathbb{1}_{\{z=1\}} \\
&= \sum_{i=1}^n w_i \left(\left(1 - \frac{G_i'(z)}{G_i'(0)}\right) \mathbb{1}_{\{z<1\}} + \mathbb{1}_{\{z=1\}}\right) = \sum_{i=1}^n w_i F_i(z),
\end{aligned}$$

for $z \in [0,1]$, and where the elements

$$w_i = \frac{\alpha_i G_i'(0)}{\sum_{j=1}^n \alpha_j G_j'(0)} \;\; \geq \;\; 0$$

are weights summing up to 1. The proof then follows similarly to Theorem 2.1. □

Proof of Proposition 3.2. We calculate the first derivative for $g > 1$, $b > 0$, $b \neq 1$ and $bg \neq 1$

$$\begin{aligned}
f_{b,g}'(z) &= -\frac{(g-1)(b-1)(\log(b))^2 b^{1-z}}{((g-1)b^{1-z} + (1-bg))^2} + 2\frac{(g-1)^2(b-1)(\log(b))^2 b^{2-2z}}{((g-1)b^{1-z} + (1-bg))^3} \\
&= \frac{(g-1)(b-1)(\log(b))^2 b^{1-z}}{((g-1)b^{1-z} + (1-bg))^3} \left[(g-1)b^{1-z} - (1-bg)\right] \\
&= (g-1)(b-1)\left(\log(b)\right)^2 b^{1-z}\, \frac{(g-1)b^{1-z} - (1-bg)}{((g-1)b^{1-z} + (1-bg))^3}. \qquad \text{(A.1)}
\end{aligned}$$

This derivative can be zero (for $g > 1$, $b > 0$, $b \neq 1$) if and only if the last ratio is zero.

(a) Case $bg < 1$. This implies $1 - bg > 0$ and $b < 1$, and the term in front of the ratio in (A.1) is negative and the denominator in the ratio is positive. In this case, the derivative is thus positive if $0 < b^{1-z} < (1-bg)/(g-1)$ and negative if $0 < (1-bg)/(g-1) < b^{1-z}$. Since b^{1-z} is increasing in z, we have a monotonically increasing density on $[0, 1)$ if $(1-bg)/(g-1) \geq 1$ and we have a monotonically decreasing density on $[0, 1)$ if $(1-bg)/(g-1) \leq b$. For $b < (1-bg)/(g-1) < 1$, we have a unimodal density with critical point

$$z^* = 1 - \frac{\log\left((1-bg)/(g-1)\right)}{\log(b)} \in (0, 1).$$

(b) Case $bg = 1$. This implies $b < 1$. The density $f_{b,g}(z) = -\log(b)b^z$ is monotonically decreasing in z.

(c) Case $bg > 1$. This implies $bg - 1 > 0$. which means that the numerator of the last ratio in (A.1) is thus always positive. We therefore need to analyze for $b \neq 1$ the ratio

$$\frac{b-1}{(g-1)b^{1-z} + (1-bg)}, \tag{A.2}$$

in order to determine the sign of the derivative $f'_{b,g}$.

(c1) Consider the first case $b < 1$. In this case, we have a negative numerator in (A.2) and b^{1-z} is increasing in $z \in [0, 1]$. If $0 < b^{1-z} < (bg-1)/(g-1)$, we have a positive derivative, and for $0 < (bg-1)/(g-1) < b^{1-z}$, we have a negative derivative. Since in this case, $(bg-1)/(g-1) < b < b^{1-z}$ holds for all $z \in (0, 1]$, we have a monotonically decreasing density.

(c2) Consider the second case $b > 1$. In this case, we have a positive numerator in (A.2) and b^{1-z} is decreasing in $z \in [0, 1]$. If $0 < b^{1-z} < (bg-1)/(g-1)$, we have a negative derivative, and for $0 < (bg-1)/(g-1) < b^{1-z}$, we have a positive derivative. Since in this case, we have $(bg-1)/(g-1) > b > b^{1-z}$ for all $z \in (0, 1]$, the density is decreasing.

Finally, the case $b = 1$ also immediately follows. This completes the proof. □

Proof of Proposition 3.3. For $bg < 1$, we have the MBBEFD density

$$\begin{aligned} f_{b,g}(z) &= \frac{(g-1)(b-1)\log(b)b^{1-z}}{\left((g-1)b^{1-z} + (1-bg)\right)^2} = \frac{\frac{b(g-1)(b-1)\log(b)}{(1-bg)^2} b^{-z}}{\left(\frac{b(g-1)}{1-bg} b^{-z} + 1\right)^2} \\ &= \frac{(b-1)\log(b)}{1-bg} \frac{\exp\left\{z\log(1/b) + \log\left(\frac{b(g-1)}{1-bg}\right)\right\}}{\left(\exp\left\{z\log(1/b) + \log\left(\frac{b(g-1)}{1-bg}\right)\right\} + 1\right)^2}. \end{aligned}$$

This proves the first claim. For the second claim, we remark that the function $t \mapsto \psi'(t)$ is symmetric around the origin $t = 0$. In view of our claim, there is $z = z^* \in (0, 1)$ such that $z\log(1/b) + \log(b(g-1)/(1-bg)) = 0$ if and only if $b < (1-bg)/(g-1) < 1$. Using reparametrization (3.6) completes the proof. □

Proof of Proposition 4.1. First the function $G(z)$ is well-defined for all $z \in [0, 1]$ due to the assumption that the function b maps to the interval $(0, \infty)$ and that $b(0) \neq b(1)$. We compute the first and second derivative of G and obtain

$$\begin{aligned} G'(0) &= \frac{b'(0)}{b(0)} \frac{1}{\log(b(1)) - \log(b(0))} > 0, \\ G'(z) &= \frac{b'(z)}{b(z)} \frac{1}{\log(b(1)) - \log(b(0))} \geq 0, \\ G''(z) &= \frac{b''(z)b(z) - b'(z)^2}{b(z)^2} \frac{1}{\log(b(1)) - \log(b(0))} \leq 0, \end{aligned}$$

where the inequalities hold if and only if $b'(0) > 0$, $b'(z) \geq 0$ and $b''(z)b(z) - b'(z)^2 \leq 0$ for all $z \in [0, 1]$, or $b'(0) < 0$, $b'(z) \leq 0$ and $b''(z)b(z) - b'(z)^2 \geq 0$ for all $z \in [0, 1]$. □

Proof of Corollary 4.2. According to Theorem 2.1, the function F_Z defined in (4.4) satisfies for $z \in [0, 1]$

$$\begin{aligned} F_Z(z) &= \left(1 - \frac{G'(z)}{G'(0)}\right) \mathbb{1}_{\{z<1\}} + \mathbb{1}_{\{z=1\}} \\ &= \left(1 - \frac{b'(z)}{b'(0)} \frac{b(0)}{b(z)}\right) \mathbb{1}_{\{z<1\}} + \mathbb{1}_{\{z=1\}}. \end{aligned}$$

The remaining statements then follow. $\square$

Proof of Proposition 5.1. First the function $G(z)$ is well-defined for all $z \in [0,1]$ due to the assumption that $b(0) \neq b(1)$. We compute now the first and second derivative of G and obtain

$$\begin{aligned} G'(0) &= \frac{e^{b(0)}}{e^{b(1)} - e^{b(0)}} \, b'(0) \; > \; 0, \\ G'(z) &= \frac{e^{b(z)}}{e^{b(1)} - e^{b(0)}} \, b'(z) \; \geq \; 0, \\ G''(z) &= \frac{e^{b(z)}}{e^{b(1)} - e^{b(0)}} [b'(z)^2 + b''(z)] \; \leq \; 0, \end{aligned}$$

where the inequalities hold if and only if $b'(0) > 0$, $b'(z) \geq 0$ and $b'(z)^2 + b''(z) \leq 0$ for all $z \in [0,1]$, or $b'(0) < 0$, $b'(z) \leq 0$ and $b'(z)^2 + b''(z) \geq 0$ for all $z \in [0,1]$. $\square$

Proof of Corollary 5.2. According to Theorem 2.1, the function F_Z defined in (5.4) satisfies for $z \in [0,1]$

$$\begin{aligned} F_Z(z) &= \left(1 - \frac{G'(z)}{G'(0)}\right) \mathbb{1}_{\{z<1\}} + \mathbb{1}_{\{z=1\}} \\ &= \left(1 - \frac{e^{b(z)}}{e^{b(0)}} \frac{b'(z)}{b'(0)}\right) \mathbb{1}_{\{z<1\}} + \mathbb{1}_{\{z=1\}}. \end{aligned}$$

The remaining statements then follow. $\square$

Proof of Proposition 5.4. Let G be an exposure curve belonging to the logarithmic linked exposure family and write

$$G(z) = \frac{\log(b(z)) - \log(b(0))}{\log(b(1)) - \log(b(0))},$$

for some twice differentiable function $b : [0,1] \to (0,\infty)$ fulfilling condition (4.2) or (4.3). Then, by defining $m = \min_{z\in[0,1]} \log(b(z)) - 1$ and $\tilde{b}(z) = \log(\log(b(z)) - m)$, we obtain for $z \in [0,1]$,

$$\begin{aligned} G(z) &= \frac{\log(b(z)) - \log(b(0))}{\log(b(1)) - \log(b(0))} \\ &= \frac{(\log(b(z)) - m) - (\log(b(0)) - m)}{(\log(b(1)) - m) - (\log(b(0)) - m)} \\ &= \frac{\exp\left(\tilde{b}(z)\right) - \exp\left(\tilde{b}(0)\right)}{\exp\left(\tilde{b}(1)\right) - \exp\left(\tilde{b}(0)\right)}. \end{aligned}$$

This means that if the function $\tilde{b}$ fulfills condition (5.2) or (5.3), then the exposure curve G also belongs to the exponentially linked exposure family. The latter holds since for $z \in [0,1]$,

$$\begin{aligned} \tilde{b}'(z) &= \frac{1}{\log(b(z)) - m} \frac{b'(z)}{b(z)}, \\ \tilde{b}''(z) &= \frac{\log(b(z)) - m}{[b(z)\,(\log(b(z)) - m)]^2} \left(b''(z)b(z) - b'(z)^2\right) - \frac{b'(z)^2}{[b(z)\,(\log(b(z)) - m)]^2}, \end{aligned}$$

which implies

$$\tilde{b}'(z)^2 + \tilde{b}''(z) = \frac{\log(b(z)) - m}{[b(z)\,(\log(b(z)) - m)]^2} \left(b''(z)b(z) - b'(z)^2\right).$$

This means that for any $z \in [0,1]$,

$$\begin{aligned} \tilde{b}'(z)^2 + \tilde{b}''(z) \leq 0 \quad &\Longleftrightarrow \quad b''(z)b(z) - b'(z)^2 \leq 0, \\ \tilde{b}'(z)^2 + \tilde{b}''(z) \geq 0 \quad &\Longleftrightarrow \quad b''(z)b(z) - b'(z)^2 \geq 0. \end{aligned}$$

This shows that any exposure curve belonging to the logarithmic linked exposure family belongs to the exponentially linked exposure family but either b or $\tilde{b}$ may take a more complicated form. The other direction follows by using a similar argument. $\square$

Proof of Lemma 6.1. The power logarithmic linked exposure example leads to a well-defined distribution due to (6.3) and Propositon 4.1. Set $y = (1 - z/\alpha)^\delta \in ((1 - 1/\alpha)^\delta, 1]$ for $z \in [0, 1)$. The derivative (6.6) is zero only if

$$a^2(\delta - 1)(\delta - 2) - a(\delta - 1)(\delta + 4)y + 2y^2 = 0.$$

If there exist real-valued solutions, they are given by

$$y_\pm \quad = \quad a\, \frac{(\delta - 1)(\delta + 4) \pm \sqrt{(\delta - 1)^2(\delta + 4)^2 - 8(\delta - 1)(\delta - 2)}}{4}. \tag{A.3}$$

We start with the case $\delta \in (1, 2]$. In that case, we have

$$(\delta - 1)(\delta + 4) \leq \sqrt{(\delta - 1)^2(\delta + 4)^2 - 8(\delta - 1)(\delta - 2)}.$$

This implies for the smaller solution $y_- \leq 0$, thus, this solution provides $y_- \notin ((1 - 1/\alpha)^\delta, 1]$. For the bigger solution, we have, using (6.4) in the second step,

$$y_+ \geq a\, \frac{(\delta - 1)(\delta + 4)}{4} > \frac{\delta + 4}{4} > 1. \tag{A.4}$$

Thus, also this second solution provides $y_+ \notin ((1 - 1/\alpha)^\delta, 1]$, therefore the density is monotone for $\delta \in (1, 2]$. Next, we analyze the case $\delta > 2$. The term under the square root in (A.3) is given by

$$\begin{aligned} (\delta - 1)^2(\delta + 4)^2 - 8(\delta - 1)(\delta - 2) \quad &= \quad (\delta - 1) \left[(\delta - 1)(\delta + 4)^2 - 8(\delta - 2)\right] \\ &= \quad (\delta - 1) \left[(\delta - 1)(\delta^2 + 8\delta + 8) + 8\right] \;\; > \;\; 0. \end{aligned}$$

Thus, there are two real-valued solutions to (A.3). The bigger solution y_+ also provides (A.4), and henceforth, $y_+ \notin ((1 - 1/\alpha)^\delta, 1]$. Therefore, we can focus on the smaller solution y_-. It is given by

$$y_- \quad = \quad a(\delta - 1) \left[(\delta/4 + 1) - \sqrt{(\delta/4 + 1)^2 - \frac{\delta - 2}{2\delta - 2}}\right].$$

The square bracket is in $[0, 0.1]$, therefore there are $\alpha > 1$ and $a > 1/(\delta - 1)$ such that $y_- \in ((1 - 1/\alpha)^\delta, 1]$. In particular, there is a critical point $z^* \in (0, 1]$ with $f_Z(z^*) = 0$ in these cases and we easily see from (6.6) that z^* is a maximum. This does however not hold for any $\alpha > 1$ and $a > 1/(\delta - 1)$. □

Proof of Lemma 6.2. In order to prove that the sine logarithmic linked exposure example leads to a well-defined distribution function, we need to show that the function h defined through $h(z) = b''(z)b(z) - b'(z)^2$ satisfies $h(z) \leq 0$ for all $z \in [0, 1]$ according to Proposition 4.1. For this, it suffices to show that $h(0) \leq 0$ and $h'(z) \leq 0$ for any $0 \leq z \leq 1$. This indeed holds since

$$\begin{aligned} h(0) &= b''(0)b(0) - b'(0)^2 \\ &= -\alpha^2 \sin(\beta) \Big[\sin(\beta) + a \Big] - \Big[\alpha \cos(\beta) \Big]^2 \\ &= -\alpha^2 [1 + a \sin(\beta)] \;\; < \;\; 0, \end{aligned}$$

where the last inequality is due to $a < -\frac{1}{\sin(\beta)}$. Furthermore, let $z \in [0, 1]$, then

$$\begin{aligned} h'(z) &= b'''(z)b(z) - b'(z)b''(z) \\ &= -\alpha^3 \cos(\alpha z + \beta) \Big[\sin(\alpha z + \beta) + a \Big] - \alpha \cos(\alpha z + \beta)(-\alpha^2 \sin(\alpha z + \beta)) \\ &= -a\alpha^3 \cos(\alpha z + \beta) \;\; < \;\; 0, \end{aligned}$$

since $a > 0$, $\alpha > 0$ and $\alpha z + \beta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ for any $z \in [0, 1]$. From (6.8), we see that this derivative has either no root or a unique root $z^* \in [0, 1]$ satisfying

$$z^* = \frac{1}{\alpha} \left[\sin^{-1} \left(\frac{a^2 - 2}{a} \right) - \beta \right], \tag{A.5}$$

where the inverse function $\sin^{-1}(\cdot)$ is chosen to map to the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$. This root can only exist if $-2 \leq a \leq -1$ or $1 \leq a \leq 2$. Since a was assumed to be positive, this means that the density f_Z can only be unimodal when $1 \leq a \leq 2$. Note that this condition is not sufficient since we do not a priori obtain that there is a root z^* lying in the interval $(0,1)$. However, in the case where this root $z^* \in (0,1)$ exists, it is clear from (6.8) that it corresponds to a maximum of the density.

$\square$

Proof of Lemma 6.3. The function b of this example is strictly decreasing, which means that the necessary condition in order to obtain a well-defined distribution reads $b''(z) + b'(z)^2 \geq 0$ for all $z \in [0,1]$ according to Proposition 5.1. This condition holds due to the assumption $\beta < -\sqrt{-2\alpha}$. Furthermore, the roots of the derivative of the density in (6.10) are given by solving

$$2\alpha z + \beta = 0 \quad \text{or} \quad (2\alpha z + \beta)^2 + 6\alpha = 0.$$

Note that by the conditions imposed on the parameters, we have $2\alpha z + \beta < 0$ for all $z \in [0,1]$. This means that the only extrema in the interval $[0,1]$ can lie at

$$z_- = \frac{-\beta - \sqrt{-6\alpha}}{2\alpha} \quad \text{and} \quad z_+ = \frac{-\beta + \sqrt{-6\alpha}}{2\alpha}.$$

Furthermore, since $\alpha < 0$ and $\beta < 0$, we obtain $z_+ < 0$. This means that the density can have at most one extremum z^* lying between 0 and 1, and we have that $z^* \in (0,1) \iff 0 > -\beta - \sqrt{-6\alpha} > 2\alpha$, which is equivalent to $-\sqrt{-6\alpha} < \beta < -2\alpha - \sqrt{-6\alpha}$. Note finally that this extremum $z^* \in (0,1)$ corresponds to the maximum of the density due to (6.10).

$\square$

Proof of Lemma 6.4. Since the function b of this example is strictly decreasing, the necessary condition in order to obtain a well-defined distribution reads $b''(z) + b'(z)^2 \geq 0$ for all $z \in [0,1]$ according to Proposition 5.1. This condition holds due to the assumptions made on the different parameters. Indeed, these assumptions imply that $b'(z)$ is negative and decreasing, while $b''(z)$ is increasing for all $z \in [0,1]$. This means in particular that $b''(z) + b'(z)^2$ is increasing for all $z \in [0,1]$. In order to prove the above necessary condition, it thus suffices to show that

$$b''(0) + b'(0)^2 > 0.$$

Given that $\alpha > 0$, $\delta > 0$ and $\epsilon < 0$, this is equivalent to

$$\alpha(\alpha-1)\epsilon\delta^{\alpha-2} + (\alpha\epsilon\delta^{\alpha-1} - \beta)^2 > 0 \iff \left|\alpha\epsilon\delta^{\alpha-1} - \beta\right| > \sqrt{-\alpha(\alpha-1)\epsilon\delta^{\alpha-2}}.$$

By assumption, we have that $\beta > \alpha\epsilon\delta^{\alpha-1}$. This implies that the interior of the absolute value is negative, we can thus write

$$\begin{aligned} b''(0) + b'(0)^2 > 0 &\iff \beta - \alpha\epsilon\delta^{\alpha-1} > \sqrt{-\alpha(\alpha-1)\epsilon\delta^{\alpha-2}} \\ &\iff \beta > \alpha\epsilon\delta^{\alpha-1} + \sqrt{-\alpha(\alpha-1)\epsilon\delta^{\alpha-2}}. \end{aligned}$$

The parameter β needs thus to satisfy precisely this last condition in order to obtain a well-defined distribution function.

$\square$

Proof of Proposition 7.1. Using (2.1), the distribution of $\tilde{Z}$ can be written as

$$\begin{aligned} F_{\tilde{Z}}(z) &= \frac{F_Z(\bar{d} + z\bar{M}) - F_Z(\bar{d})}{1 - F_Z(\bar{d})} \mathbb{1}_{\{z\in[0,1)\}} + \mathbb{1}_{\{z=1\}} \\ &= \frac{G'(\bar{d}) - G'(\bar{d} + z\bar{M})}{G'(\bar{d})} \mathbb{1}_{\{z\in[0,1)\}} + \mathbb{1}_{\{z=1\}}, \end{aligned}$$

for $z \in [0,1]$. By defining a new exposure curve $\tilde{G} : [0,1] \to [0,1]$, $z \mapsto G(\bar{d} + z\bar{M})/G(\bar{d} + \bar{M})$ and using Theorem 2.1, we can derive the distribution function induced by $\tilde{G}$, which reads as

$$\begin{aligned} \tilde{F}(z) &= \left(1 - \frac{\tilde{G}'(z)}{\tilde{G}'(0)}\right) \mathbb{1}_{\{z\in[0,1)\}} + \mathbb{1}_{\{z=1\}} \\ &= \left(1 - \frac{G'(\bar{d} + z\bar{M})}{G'(\bar{d})}\right) \mathbb{1}_{\{z\in[0,1)\}} + \mathbb{1}_{\{z=1\}}. \end{aligned}$$

This distribution is equal to the distribution of $\tilde{Z}$, which shows that $\tilde{Z}$ is a member of the Bernegger class with exposure curve $\tilde{G}$. $\square$

B MLE parameters of the models of Section 6

We provide in this appendix the values of all MLE parameters obtained when fitting the real dataset of Section 6 in the following tables.

MLE parameters	b	g	q
Standard MLE density (Green)	$5.66 \cdot 10^{-3}$	50.5	-
Flexible MLE density (Red)	$2.29 \cdot 10^{-3}$	94.9	0.0339

Table 8: MLE parameters of the MBBEFD example rounded to three significant digits (Section 6.1).

MLE parameters	α	δ	a	q
Standard MLE density (Green)	$3.76 \cdot 10^4$	$1.95 \cdot 10^5$	0.393	-
Flexible MLE density (Red)	$9.32 \cdot 10^3$	$5.67 \cdot 10^4$	0.275	0.0339

Table 9: MLE parameters of the power logarithmic linked exposure example rounded to three significant digits (Section 6.2).

MLE parameters	α	β	a	q
Standard MLE density (Green)	2.57	-1.25	1.05	-
Flexible MLE density (Red)	$1.94 \cdot 10^{-3}$	-1.57	1.00	0.0339

Table 10: MLE parameters of the sine logarithmic linked exposure example rounded to three significant digits (Section 6.3).

MLE parameters	α	β	q
Standard MLE density (Green)	-2.19	-2.88	-
Flexible MLE density (Red)	-3.28	-3.10	0.0339

Table 11: MLE parameters of the quadratic exponentially linked exposure example rounded to three significant digits (Section 6.4).

MLE parameters	α	β	δ	ϵ	q
Standard MLE density (Green)	1.00	-191	0.130	-195	-
Flexible MLE density (Red)	1.00	-3000	0.235	-3000	0.0339

Table 12: MLE parameters of the power exponentially linked exposure example rounded to three significant digits (Section 6.5).

MLE parameters	μ	σ	$\tilde{d}$
MLE density	-0.947	0.563	0.110

Table 13: MLE parameters of the lower-truncated and right-censored log-normal distribution rounded to three significant digits (Section 6.6).

MLE parameters	γ	c	$\tilde{d}$
MLE density	1.97	5.46	0.0154

Table 14: MLE parameters of the lower-truncated and right-censored gamma distribution rounded to three significant digits (Section 6.6).