

Indicated domination game

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Abstract

Motivated by the success of domination games and by a variation of the coloring game called the indicated coloring game, we introduce a version of domination games called the indicated domination game. It is played on an arbitrary graph G by two players, Dominator and Staller, where Dominator wants to finish the game in as few rounds as possible while Staller wants just the opposite. In each round, Dominator indicates a vertex u of G that has not been dominated by previous selections of Staller, which, by the rules of the game, forces Staller to select a vertex in the closed neighborhood of u . The game is finished when all vertices of G become dominated by the vertices selected by Staller. Assuming that both players are playing optimally according to their goals, the number of selected vertices during the game is the indicated domination number, $\gamma_i(G)$, of G .

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We prove several bounds on the indicated domination number expressed in terms of other graph invariants. In particular, we find a place of the new graph invariant in the well-known domination chain, by showing that $\gamma_i(G) \geq \Gamma(G)$ for all graphs G , and by showing that the indicated domination number is incomparable with the game domination number and also with the upper irredundance number. In connection with the trivial upper bound $\gamma_i(G) \leq n(G) - \delta(G)$, we characterize the class of graphs G attaining the bound provided that $n(G) \geq 2\delta(G) + 2$. We prove that in trees, split graphs and grids the indicated domination number equals the independence number. We also find a formula for the indicated domination number of powers of paths, from which we derive that there exist graphs in which the indicated domination number is arbitrarily larger than the upper irredundance number. We provide some partial results supporting the statement that $\gamma_i(G) = n(G)/2$ if G is a cubic bipartite graph, and leave this as an open question.

Keywords: domination game; indicated coloring; independence number; upper domination number;

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1 Introduction

The coloring game was introduced independently in [15] and [3]. Unlike combinatorial games in which a winner is decided, the result of the coloring game gives a graph invariant, which is based on the assumption that both players are playing optimally according to their goals. A number of variants of the original game have been introduced, see e.g. [2, 5, 22, 25]. The following version was proposed by Grytczuk and defined by Grzesik [17].

The *indicated coloring game* is played on a simple graph G by two players, and a fixed set C of colors. In each round of the game Ann indicates an uncolored vertex, and Ben colors it using a color from C , obeying just the proper coloring rule. The goal of Ann is to achieve a proper coloring of the whole graph, while Ben is trying to prevent this. The minimum cardinality of the set of colors C for which Ann has a winning strategy is called the *indicated chromatic number*, $\chi_i(G)$, of a graph G .

The *domination game*, as introduced in [8], is played on a graph G by two players: Dominator and Staller. They alternate taking moves in which they select a vertex of G . A move is legal if the selected vertex dominates at least one vertex which is not already dominated by previously played vertices. The game ends when there are no legal moves, so when the set of played vertices is a dominating set of G . The goal of Dominator is to finish the game with the minimum number of moves, while the aim of Staller is to maximize the number of moves. If both players play optimally, then

the number of moves played on G is an invariant (see [7, 12]). Therefore, as defined in [8], the *game domination number* $\gamma_g(G)$ is the number of moves on G if Dominator starts the game. Many variants of the domination game have been introduced; see a recent monograph [7] and for example papers [4, 9, 10, 11, 12, 14, 16, 19, 23, 24]. Having in mind the indicated coloring game, we propose the following variant of the domination game.

The *indicated domination game* is played on a graph G by two players, Dominator and Staller, who take turns making a move. In each of his moves, Dominator indicates a vertex v of the graph that has not been dominated in the previous moves, and Staller chooses (or selects) any vertex from the closed neighborhood of v , and adds it to a set D that is being built during the game. The game ends when there is no undominated vertex left, that is, when D is a dominating set. The goal of Dominator is to minimize the size of D , while Staller wants just the opposite. Provided that both players are playing optimally with respect to their goals, the size of the resulting set D is the *indicated domination number* of G , and is denoted by $\gamma_i(G)$.

In the following section, we establish the notation and present basic definitions, while in Section 3 we give some preliminary results. In particular, we prove that $\Gamma(G) \leq \gamma_i(G) \leq \gamma_{gr}(G)$, where $\Gamma(G)$ is the upper domination number and $\gamma_{gr}(G)$ is the Grundy domination number of a graph G . In Section 4, we prove that for a graph G with minimum degree δ and order n , where $n \geq 2\delta + 2$, we have $\gamma_i(G) = n - \delta$ if and only if G contains a spanning subgraph $K_{\delta, n-\delta}$ with an additional property that the part of the bipartition of size $n - \delta$ is an independent set in G . In Section 5, we prove that in several families of graphs (namely trees, split graphs, grids, and connected bipartite cubic graphs with at most 12 vertices) the indicated domination number equals the independence number. On the other hand, the indicated domination number can be arbitrarily larger than the upper irredundance number (and thus also the independence number), which is established in Section 6. This is derived from the formula for the indicated domination number of the k -th power of the path P_n , which is roughly $\gamma_i(P_n^k) = \Theta\left(\frac{\log k}{k} n\right)$ as $n \rightarrow \infty$. In Section 7, we propose several open questions.

2 Notation

Let G be a graph. We denote the number of vertices of G by $n(G)$. If $S \subseteq V(G)$, then the subgraph induced by S is denoted by $G[S]$. For a vertex $v \in V(G)$, the (*open*) *neighborhood* $N(v)$ is the set of neighbors of v , and the *closed neighborhood* is $N[v] = N(v) \cup \{v\}$. If $S \subseteq V(G)$, then $N[S] = \bigcup_{v \in S} N[v]$.

For a vertex $x \in S$, every vertex in $N[S] - N[S - \{x\}]$ is called a *private neighbor of x with respect to S* .¹ A set $S \subseteq V(G)$ is an *irredundant set* if every vertex in S has a private neighbor with respect to S . The smallest and largest cardinalities of a maximal irredundant set of G are denoted by $\text{ir}(G)$ and $\text{IR}(G)$, respectively.

A set $S \subseteq V(G)$ is an *independent set* in a graph G if the vertices in S are pairwise nonadjacent. The maximum size of an independent set of G is denoted by $\alpha(G)$. An *edge cover* of G is a set $F \subseteq E(G)$ such that every vertex of G is incident to some edge in F . We denote the minimum size of an edge cover of G by $\rho(G)$. Note that notation $\beta'(G)$ is also used in the literature. Recall that combining König's and Gallai's Theorems gives $\alpha(G) = \rho(G)$ for a bipartite graph G without isolated vertices.

A vertex $v \in V(G)$ *dominates* itself and its neighbors. A subset of vertices $D \subseteq V(G)$ is a *dominating set* of G if it dominates all vertices of G , i.e. $N[D] = V(G)$. This means that every vertex from $V(G) - D$ has a neighbor in D . The minimum cardinality of a dominating set of G is the *domination number*, $\gamma(G)$, of G .

A dominating set D in G is a *minimal dominating set* if no proper subset of D is a dominating set. That is, D is a minimal dominating set if and only if every $x \in D$ has a private neighbor with respect to D . The maximum cardinality of a minimal dominating set is the *upper domination number*, $\Gamma(G)$, of G . We recall the following results.

Theorem 2.1 ([13, Theorem 5]). *If G is a bipartite graph, then $\alpha(G) = \Gamma(G) = \text{IR}(G)$.*

Theorem 2.2 ([20, Theorem 9]). *If G is a chordal graph, then $\alpha(G) = \Gamma(G) = \text{IR}(G)$.*

Given a graph G , a sequence $S = (v_1, \dots, v_k)$ of vertices of G is a *dominating sequence* if for each i

$$N[v_i] - \bigcup_{j=1}^{i-1} N[v_j] \neq \emptyset \quad (1)$$

and the set of vertices from S dominates G . We call the length k of the longest such sequence S the *Grundy domination number*, $\gamma_{\text{gr}}(G)$, of G . Clearly, $\gamma(G) \leq \gamma_{\text{gr}}(G)$.

3 Preliminary results

A well known domination chain [18] was extended with the Grundy domination number in [6]:

$$\text{ir}(G) \leq \gamma(G) \leq i(G) \leq \alpha(G) \leq \Gamma(G) \leq \text{IR}(G) \leq \gamma_{\text{gr}}(G).$$

¹Observe that every vertex isolated in $G[S]$ is viewed as a private neighbor of itself by definition.

The following proposition explains where indicated domination number fits in this chain.

Proposition 3.1. *If G is a graph, then*

$$\Gamma(G) \leq \gamma_i(G) \leq \gamma_{gr}(G).$$

In addition, γ_i is incomparable with both γ_g and IR.

Proof. Note that the definition of the indicated domination game implies that the sequence of vertices selected by Staller is a dominating sequence, so $\gamma_i(G) \leq \gamma_{gr}(G)$.

Let D be a minimal dominating set of cardinality $\Gamma(G)$. We provide a strategy for Staller that will prove that $\Gamma(G) \leq \gamma_i(G)$. The basic ingredient of this strategy is for her to always select a vertex from D . She can do so regardless of which vertex Dominator indicates. Unless all vertices from D have been selected, there exists an undominated vertex (a private neighbor of an unselected vertex from D). This shows that $\Gamma(G) \leq \gamma_i(G)$.

In the rest of the proof we present infinite families of graphs showing that γ_i is incomparable with both γ_g and IR. Stars $K_{1,n}$ show that γ_i can be arbitrarily larger than γ_g , since $\gamma_i(K_{1,n}) = n$ and $\gamma_g(K_{1,n}) = 1$. Second powers of paths (see Corollary 6.3) show that γ_i can be arbitrarily larger than IR.

To see that IR can be arbitrarily larger than γ_i , we consider the following family of graphs. Let H_n be the graph obtained from two disjoint copies of K_n with vertices u_1, \dots, u_n and v_1, \dots, v_n by adding edges $u_i v_i$ for all $i \in \{2, \dots, n\}$. Notice that H_n is isomorphic to the graph $K_n \square K_2$ with one edge removed. It is easy to see that $\text{IR}(H_n) = n - 1$ (it is attained for example by the maximal irredundant set $\{u_2, \dots, u_n\}$). On the other hand, we have $\gamma_i(H_n) \geq \gamma(H_n) = 2$, and if Dominator first indicates u_1 and then v_1 , we get $\gamma_i(H_n) = 2$. Note that this also shows that in general $\gamma_i(G)$ and $\gamma_i(G - e)$ can be arbitrarily far apart since $\gamma_i(K_n \square K_2) \geq \Gamma(K_n \square K_2) = n$.

Last, we present a family of graphs for which γ_g is arbitrarily larger than γ_i . Let D_1 be a graph obtained from a cycle C_9 with vertices x_1, \dots, x_9 and naturally defined edges between them by adding edges $x_1 x_3$, $x_4 x_6$ and $x_7 x_9$. Since Dominator can in turn indicate vertices x_2 , x_5 and x_8 , he has a strategy that ensures $\gamma_i(D_1) \leq 3$. Since $\gamma_i(D_1) \geq \gamma(D_1) = 3$, we have $\gamma_i(D_1) = 3$. On the other hand, it is easy to see that $\gamma_g(D_1) = 4$. Let D_n be a disjoint union of n copies of the graph D_1 . Since $\gamma_i(D_n) \geq \gamma(D_n) = 3n$ and Dominator can in turn indicate vertices of degree two, we have $\gamma_i(D_n) = 3n$. But since Staller's strategy in the domination game on D_n can be to reply in the same copy of D_1 as Dominator played, we get $\gamma_g(D_n) \geq 4n$ (because under that strategy, in each copy the game ends after an even number, four,

of moves). Thus $\gamma_g(D_n) - \gamma_i(D_n) \geq n$. Alternatively, we can consider a connected graph E_n obtained from C_{3n} by adding edges $x_1x_3, x_4x_6, \dots, x_{3n-2}x_{3n}$. In this case, $\gamma_i(E_n) = n$, while $\gamma_g(E_n) \geq n + \frac{n}{10}$ (the last follows by an argument similar as in [21, Theorem 4.1]). \square

The diagram in Figure 1 shows the relations between the parameters from Proposition 3.1.

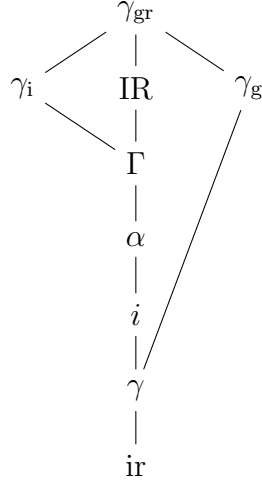


Figure 1: Relations between the domination and independence invariants.

There are several classes of graphs with a transparent structure where the parameters discussed in Proposition 3.1 coincide. For example, the graphs G below all have the property $\Gamma(G) = \gamma_{gr}(G)$ as listed in [1]: hypercubes; complete multipartite graphs, $G = K_{n_1, \dots, n_k}$, such that $n_1 \geq \dots \geq n_k$, $k \geq 2$, and $n_{k-1} \geq 2$; prisms over complete graphs, $G = K_n \square K_2$, for $n \geq 2$; large families of Kneser graphs; the class of (twin-free, connected) cographs. It is also proved in [1] that the join operation preserves that property. By Proposition 3.1 all the graphs from the above families have the indicated domination number equal to their upper domination number. By Theorem 2.1 this in turn implies that the bipartite graphs among the above families have their indicated domination number equal to their independence number.

However, the difference between the indicated domination number and the independence number can be arbitrarily large. For example, $\alpha(K_n \square K_2) = 2$ and $\gamma_i(K_n \square K_2) = n$. Similarly, the difference between the Grundy domination number and the indicated domination number can be arbitrarily large. For example, $\gamma_{gr}(P_n) = n - 1$ and $\gamma_i(P_n) = \lceil \frac{n}{2} \rceil$ as will be proven in Corollary 5.3.

4 Upper bound on γ_i and extremal graphs

Using a bound from Proposition 3.1 combined with [6, Proposition 2.1] we obtain:

$$\gamma_i(G) \leq \gamma_{\text{gr}}(G) \leq n(G) - \delta(G). \quad (2)$$

In the following theorem we characterize the graphs G that attain the bound $\gamma_i(G) = n(G) - \delta(G)$ under the assumption that $n(G) \geq 2\delta(G) + 2$.

Theorem 4.1. *A graph G with minimum degree δ and order n , where $n \geq 2\delta + 2$, has $\gamma_i(G) = n - \delta$ if and only if it contains a spanning subgraph $K_{\delta, n-\delta}$ with an additional property that the part of the bipartition of size $n - \delta$ is an independent set in G .*

Proof. Let G be a graph, which has a spanning subgraph $K_{\delta, n-\delta}$ and the only edges of G that are not in $K_{\delta, n-\delta}$, if any, are between two vertices in the part of size δ . Let the indicated domination game be played on G . Regardless of which vertex is indicated by Dominator in his first move, Staller selects a vertex from the part of size $n - \delta$. In this way, all vertices in the part of size δ become dominated after the first selection of Staller. Therefore, Dominator will in his subsequent moves need to indicate only vertices in the part of size $n - \delta$, and Staller's strategy is simply to select the vertex that Dominator indicated in his move. In this way, Staller ensures that at least $n - \delta$ vertices will be chosen. Thus, $\gamma_i(G) \geq n - \delta$, and combining this with (2) we get $\gamma_i(G) = n - \delta$.

For the converse, let G be a graph on n vertices, where $n \geq 2\delta + 2$, such that $\gamma_i(G) = n - \delta$. Let the indicated domination game be played on G . If after the first selection of Staller more than $\delta + 1$ vertices were dominated, then at most $n - \delta - 1$ vertices will be chosen by Staller when the indicated domination game ends on G , a contradiction. Therefore, Staller needs to select a vertex of degree δ in her first move. For the same reason, since $\gamma_i(G) = n - \delta$, in all of her subsequent moves Staller needs to select a vertex that dominates only one vertex that was not dominated earlier. Let x be the vertex selected in the first move of Staller, and let $N_G(x) = \{u_1, \dots, u_\delta\}$. Let $R = V(G) - N_G[x]$. If there existed a vertex $v \in R$ such that v had no neighbors in $N_G(x)$, then Dominator's next move would be to indicate v , and after Staller selected a vertex that dominates v , at least two new vertices would become dominated, a contradiction. Therefore, every vertex in R has at least one neighbor in $N_G(x)$.

Now, suppose that there exists a vertex in R , say w_0 , that has a neighbor in R , and let w_1, \dots, w_k be the neighbors of w_0 from R . For every vertex w_i , where $i \in \{0, \dots, k\}$, there exists a vertex $u_{j_i} \in N_G(x)$ such that w_i is the only neighbor of u_{j_i} in R . Indeed, if for some w_i such a vertex u_{j_i} would not exist, then Dominator

could indicate w_i in his second move, and regardless of which vertex Staller would select in her second move to dominate w_i at least two new vertices would become dominated in that move, which is a contradiction. Since $\deg(w_0) \geq \delta$, we infer that w_0 has (at least) $\delta - k$ neighbors in $N_G(x)$. Clearly, all of the neighbors of w_0 in $N_G(x)$ are distinct from vertices u_{j_i} , where $i \in [k]$, which implies that w_0 has exactly $\delta - k$ neighbors in $N_G(x)$. In addition, $k \leq \delta - 1$, since all vertices u_{j_i} , where $i \in \{0, 1, \dots, k\}$, are pairwise distinct. Therefore, since $n > 2\delta + 1 \geq |N_G[x]| + k + 1$, we infer that there exists a vertex $z \in R - \{w_0, \dots, w_k\}$. If z has a neighbor in R , then, by the same reason as earlier, there exists a vertex u' in $N_G(x)$ such that z is the only neighbor of u' from R . However, every vertex in $N_G(x)$ has a neighbor in $\{w_0, \dots, w_k\}$, therefore z cannot be adjacent to any vertex in R . Since $\deg(z) \geq \delta$, we infer $N_G(z) = N_G(x)$, which is again a contradiction, since each $u_{j_i} \in N_G(x)$ has only one neighbor in R . The assumption that there exists an edge between two vertices in R led us to a contradiction, therefore $R \cup \{x\}$ is independent, and $K_{\delta, n-\delta}$ is a spanning subgraph of G with the part of the bipartition of size $n - \delta$ being independent. \square

Note that a graph G with minimum degree δ and order n , where $n \geq 2\delta + 2$ and $\gamma_i(G) = n - \delta$, can equivalently be described as being the join of an independent set of size $n - \delta$ and any graph of order δ .

5 Graphs with $\gamma_i = \alpha$

In this section, we present several classes of graphs whose indicated domination number equals their independence number. Naturally, due to Proposition 3.1, the independence number of any of these graphs is equal to the upper domination number.

Recall that a graph is *split* if its vertex set can be partitioned into a clique and an independent set. A *split partition* of a split graph G is a pair (K, I) such that K is a clique, I is an independent set, $K \cup I = V(G)$ and $K \cap I = \emptyset$. It was proven in [6] that if G is a split graph with split partition (K, I) , then

$$\gamma_{gr}(G) = \begin{cases} \alpha(G), & \text{if every two vertices in } K \text{ have a common neighbor in } I; \\ \alpha(G) + 1, & \text{otherwise.} \end{cases}$$

We next show that the second line in the above equation never appears if γ_{gr} is replaced with γ_i .

Proposition 5.1. *If G is a split graph, then $\gamma_i(G) = \alpha(G)$.*

Proof. Let (K, I) be the split partition of G . We can assume that I is a maximum independent set, and thus every vertex in K has a neighbor in I . Dominator's strategy in each of his moves is to indicate a vertex x from I that has not been dominated by the vertices previously chosen by Staller. After all vertices from I have been dominated by Staller's selected vertices, the vertices in K have also been dominated since K is a clique and each vertex from K has a neighbor in I . Hence at most $|I| = \alpha(G)$ vertices were chosen by Staller, and we infer that $\gamma_i(G) \leq \alpha(G)$. The reverse inequality follows from Proposition 3.1. \square

To see that trees enjoy the equality considered in this section is a bit less straightforward.

Theorem 5.2. *If T is a tree, then $\gamma_i(T) = \alpha(T)$.*

Proof. Let F be a minimum edge cover of the tree T . Since T is bipartite, $|F| = \rho(G) = \alpha(T)$. Choose a leaf r of T and consider T as a rooted tree with r as its root, by which the notions of parent and child can be used in T . Let an indicated domination game be played in T . We will present a strategy of Dominator, which ensures that at most $\alpha(T)$ vertices will be selected during the game.

In the first move, Dominator indicates vertex r , which forces Staller to dominate both r and its child s . Note that $rs \in F$, and that after the first move of Staller both endvertices of this edge from F become dominated. We will say that an edge $f \in F$ is *saturated* if both of its endvertices are dominated. We claim that Dominator can ensure that after every move of Staller, a new edge from F becomes saturated. In other words, after the i^{th} move of Staller, Dominator can ensure that at least i edges from F are saturated. This is clearly true after the first move of Staller. In the subsequent moves, as long as there are still some undominated vertices left, Dominator considers the vertex of T which is the closest to r among all undominated vertices. Let this vertex be denoted by u , and let $uv \in F$ for a neighbor v of u . Now, we consider several possibilities that need to be reflected in Dominator's strategy. With this, Dominator will achieve that a new edge of F becomes saturated after every move of Staller, and, in addition, that the set of dominated vertices induces a connected subgraph of T (there is one exception case to this property, in which case Dominator can achieve that after the subsequent move of Staller the set of dominated vertices again induces a connected subgraph of T).

If v is the parent of u , then it is clear, by how u is defined, that v has been dominated in some of the previous moves. In this case, the strategy of Dominator is to indicate u . In this way, u will become dominated after the following move of Staller, and so the edge uv becomes saturated, by which the number of saturated edges from F increases by one. Additionally, the set of dominated vertices still

induces a connected subgraph. Next, assume that u is the parent of v . It may happen that v is already dominated. (This is the case when the set of dominated vertices does not induce a tree, and how this happens will be explained in the next case.) If this is indeed so, then Dominator indicates u , and in the following move of Staller, u will become dominated and $uv \in F$ will thus become saturated. The remaining possibility is that v is not dominated. By the strategy of Dominator, we can argue that all children of v are undominated at this point. Dominator's strategy in this case is to indicate v . If v or u is selected by Staller in the next move, then $uv \in F$ becomes saturated and the set of dominated vertices is connected, as desired. Otherwise, a child x of v is selected by Staller, and let xy be the edge of F , which covers x . Note that either $y = v$ or y is a child of x . In either case, the edge $xy \in F$ becomes saturated. In this case (that is, when a child x of v is selected by Staller), we are in the situation when the set of dominated vertices is not connected, but, as explained earlier, Dominator's next move is to indicate u . In this way, u becomes dominated, and since v was dominated in the preceding move, the edge $uv \in F$ becomes saturated. Thus, after these two consecutive moves, the set of vertices in D that are dominated induces a connected subgraph again.

Since after each move of Staller, the number of saturated edges from F increased, we infer that at most $|F| = \alpha(T)$ vertices are chosen by Staller when the game ends. Since $\gamma_i(T) \geq \alpha(T)$ by Proposition 3.1, the stated equality is proved. \square

Corollary 5.3. *If $n \geq 1$, then $\gamma_i(P_n) = \lceil \frac{n}{2} \rceil$.*

In the next result, we prove for yet another family of graphs, namely the grids, that their indicated domination number equals the independence number.

Theorem 5.4. *If $m, n \geq 1$, then $\gamma_i(P_m \square P_n) = \alpha(P_m \square P_n) = \lceil \frac{mn}{2} \rceil$.*

Proof. Let $V(P_m) = \{x_1, \dots, x_m\}$ and $V(P_n) = \{y_1, \dots, y_n\}$. We divide the proof into two cases, depending on whether both m and n are odd or not.

First, assume that at least one of m and n is an even integer, and, by symmetry, we may assume with no loss of generality that m is even. Consider the partition of $V(P_m \square P_n)$ into pairs of adjacent columns. Namely, let $C_i = \{x_{2i-1}, x_{2i}\} \times V(P_n)$, for all $i \in [m/2]$. The strategy of Dominator, to achieve his goal that at most $mn/2$ vertices will be selected during the indicated domination game on $P_m \square P_n$, is divided into two phases. In the first phase, he deals with each of the sets C_i , proceeding in their natural order, while in the process some of the vertices of C_i , for all $i \in [m/2]$, may be left undominated. In the second phase, he indicates the vertices that were left undominated in the first phase one by one.

Let us first present the strategy of Dominator in C_1 . Whenever Dominator indicates a vertex of C_1 , that vertex is in the column $\{x_2\} \times V(P_n)$. Dominator can

start by indicating an arbitrary vertex of that column, say (x_2, y_1) , and then proceeds by indicating vertices of that column by obeying only one rule, which we explain next. At a given point in the game we say that the pair $\{(x_{2i-1}, y_j), (x_{2i}, y_j)\}$, where $i \in [m/2]$ and $j \in [n]$, is *empty* if neither vertex in the pair has been dominated up to that point in the game. The strategy of Dominator while dealing with the set C_1 in the first phase is to find an arbitrary empty pair $\{(x_1, y_j), (x_2, y_j)\}$, if it exists, and indicate (x_2, y_j) , as long as there exists an empty pair in C_1 . Thus, in her next move, Staller has to dominate (x_2, y_j) . She can do this by choosing either (x_2, y_j) or one of its neighbors. In either case, the pair $\{(x_1, y_j), (x_2, y_j)\}$ is no longer empty after her move. Clearly, if the move of Staller is to choose (x_1, y_j) , (x_2, y_{j-1}) , (x_2, y_j) or (x_2, y_{j+1}) , then it may happen that additional pairs of C_1 become non-empty. In this case, regardless of which of the vertices (x_1, y_j) , (x_2, y_{j-1}) , (x_2, y_j) or (x_2, y_{j+1}) is chosen by Staller, the chosen vertex is the first vertex of its corresponding pair that was chosen during the game, and since its neighboring pairs have become non-empty (if they were not already non-empty before that move), the other vertex of the pair in which the chosen vertex lies will never be selected by Staller during the first phase of the game. It is also possible, that Staller chooses (x_3, y_j) to dominate (x_2, y_j) . In this case, (x_2, y_j) becomes dominated and the pair $\{(x_1, y_j), (x_2, y_j)\}$ becomes non-empty, yet the move of Staller choosing (x_3, y_j) will be considered when dealing with C_2 . The first step of the game in the first phase, dealing with C_1 , ends when all pairs of C_1 have become non-empty. Note that by the strategy of Dominator, in each pair $\{(x_1, y_j), (x_2, y_j)\}$ at most one of the vertices has been chosen by Staller, and since all pairs in C_1 have become non-empty, at most one of the vertices in each pair remains undominated. Clearly, if a vertex from a pair was chosen, then in the corresponding pair both vertices have been dominated. We summarize these observations as follows: after dealing with C_1 in the first phase, ℓ_1 vertices of C_1 have been chosen, and at most $n - \ell_1$ vertices from C_1 have been left undominated.

Dominator proceeds by dealing with C_2 , and the only difference from the initial case when dealing with C_1 is that some vertices (x_3, y_j) may have already been chosen by Staller. More generally, when Dominator starts to deal with C_i , where $2 \leq i \leq m/2$, some vertices (x_{2i-1}, y_j) may have already been chosen by Staller while dealing with C_{i-1} . This also implies that the corresponding pairs $\{(x_{2i-1}, y_j), (x_{2i}, y_j)\}$, and their eventual neighboring pairs, $\{(x_{2i-1}, y_{j-1}), (x_{2i}, y_{j-1})\}$ and $\{(x_{2i-1}, y_{j+1}), (x_{2i}, y_{j+1})\}$, are already non-empty when Dominator starts to deal with C_i . The strategy of Dominator is the same as when dealing with C_1 . Notably, Dominator finds an arbitrary empty pair $\{(x_{2i-1}, y_j), (x_{2i}, y_j)\}$, if it exists, and indicates (x_{2i}, y_j) , as long as there exists an empty pair in C_i . By using the same arguments as in the previous paragraph, Dominator can ensure that after dealing with C_i in the first phase, ℓ_i vertices of C_i have been chosen, and at most $n - \ell_i$ vertices from C_i have been left undom-

inated. The first phase is over after Dominator deals with $C_{m/2}$. By that time, Staller has chosen $\sum_{i=1}^{m/2} \ell_i$ vertices, while at most

$$\sum_{i=1}^{m/2} (n - \ell_i) = \frac{mn}{2} - \sum_{i=1}^{m/2} \ell_i$$

vertices of $P_m \square P_n$ have been left undominated. Now, the second phase begins, in which Dominator indicates the remaining undominated vertices one by one, in any order. Hence, Staller will additionally select at most $\frac{mn}{2} - \sum_{i=1}^{m/2} \ell_i$ vertices, which together with vertices selected in the first phase contributes to at most $\frac{mn}{2}$ chosen vertices during the entire game. Hence, $\gamma_i(P_m \square P_n) \leq \frac{mn}{2} = \alpha(P_m \square P_n)$, which combined with Proposition 3.1 implies the stated result.

Second, consider the case when both m and n are odd. In this case, partition $V(P_m \square P_n)$ into $\frac{m+1}{2}$ sets, namely $C_0 = \{x_1\} \times V(P_n)$ and $C_i = \{x_{2i}, x_{2i+1}\} \times V(P_n)$, where $1 \leq i \leq \frac{m-1}{2}$. The proof is similar to the proof of the previous case in which m was even; the only difference is in the first step, where Dominator deals with the first column C_0 , which we explain next.

Dominator uses the following strategy when dealing with C_0 , which lasts as long as there is an undominated vertex left in C_0 . The strategy consists of two phases. In the first phase, Dominator starts by indicating (x_1, y_1) , and then, in every further step, he indicates the vertex (x_1, y_{i+2}) if i is the largest index such that (x_1, y_i) is dominated and $i \leq n - 2$. By the choices of Staller, a vertex of C_0 becomes dominated by itself or one of its neighbors (possibly it is dominated by a neighbor from C_1 , which will then be considered when dealing with C_1). If Staller chose (x_1, y_{i+3}) in one of her moves, then after the first phase is over, vertex (x_1, y_{i+1}) remains undominated. Now, note that vertices chosen by Staller together with vertices that are not yet dominated after the first phase form an independent set of the path induced by the first column. In the second phase, Dominator selects vertices of C_0 that have not been dominated in the first phase (if any) one by one, and in this way all vertices of C_0 become dominated. By the previous observation, Staller has chosen at most $\alpha(P_n) = \frac{n+1}{2}$ vertices in C_0 .

The rest of the strategy of Dominator is exactly the same as in the previous case. Hence we conclude that Dominator can ensure that at most $\frac{(m-1)n}{2}$ vertices will be selected to dominate the vertices in $C_1 \cup \dots \cup C_{\frac{m-1}{2}}$. Together with the bound on the number of vertices from C_0 selected in the first step of the game, we infer that at most

$$\frac{n+1}{2} + \frac{(m-1)n}{2} = \frac{mn+1}{2} = \left\lceil \frac{mn}{2} \right\rceil$$

vertices will be selected during the game on $P_m \square P_n$. We readily infer that $\gamma_i(P_m \square P_n) = \left\lceil \frac{mn}{2} \right\rceil$. \square

The next family of graphs G for which we suspect that $\gamma_i(G) = \alpha(G)$ are cubic bipartite graphs. We checked by computer that the equality holds for all (cubic) bipartite graphs G with $n(G) \leq 10$. Next, we extend this result to $n(G) \leq 12$.

Proposition 5.5. *If G is a connected, cubic bipartite graph of order at most 12, then $\gamma_i(G) = \frac{n(G)}{2} = \alpha(G)$.*

Proof. By the observations preceding the proposition, it remains to consider the case when $n(G) = 12$. Let $A = \{a_1, \dots, a_6\}$ and $B = \{b_1, \dots, b_6\}$ be the partite sets of G . For $j \geq 1$, we let s_j denote the vertex chosen by Staller on her j^{th} move and let $S_j = \{s_1, \dots, s_j\}$. In addition, d_i will denote the vertex indicated by Dominator in his i^{th} move, for $i \geq 2$. We provide a strategy for Dominator that will ensure at most 6 vertices are chosen by Staller. Without loss of generality $s_1 = a_1$ and $N(a_1) = \{b_1, b_2, b_3\}$. Dominator indicates $d_2 = b_4$. If $s_2 = b_4$, then $|N[S_2]| = 8$, which leaves four undominated vertices. At most four more vertices will be chosen by Staller. Otherwise, $s_2 \in N(b_4) \cap (A - \{a_1\})$. Reindexing if necessary we assume $s_2 = a_2$. If $\{b_5, b_6\} \subseteq N(a_2)$, then S_2 dominates eight vertices and thus at most six vertices will be chosen by Staller. If a_2 is adjacent to only one, say b_5 , of b_5 or b_6 , then Dominator points to b_6 . Now, regardless of which vertex Staller chooses from $\{b_6, a_3, a_4, a_5, a_6\}$ it is easy to see that $|N[S_3]| \geq 9$, which implies that Staller will choose at most 6 vertices. Therefore, we may assume that $N(s_2) \cap \{b_4, b_5, b_6\} = \{b_4\}$, and this implies that $|N(a_2) \cap \{b_1, b_2, b_3\}| = 2$. Note that we now have $(N(b_5) \cup N(b_6)) \subseteq \{a_3, a_4, a_5, a_6\}$. Dominator then points to $d_3 = b_5$. If $s_3 = b_5$, then $|N[S_3]| = 10$ which implies that Staller can choose at most five vertices when the game has ended. Thus, we may assume that Staller chooses $s_3 \in N(b_5) \cap \{a_3, a_4, a_5, a_6\}$. By reindexing if necessary we assume that $s_3 = a_3$. Similar to the above argument, if $a_3 b_6 \in E(G)$, then Staller can choose at most six vertices when the game has ended. Hence, we may assume that $b_6 \notin N(a_3)$. Dominator now indicates $d_4 = b_6$. Staller must choose $s_4 \in \{b_6, a_4, a_5, a_6\}$, and it follows that $|V(G) - N[S_4]| \leq 2$. Therefore, $\gamma_i(G) \leq 6$. The reverse inequality follows by Proposition 3.1. \square

We could not extend the above reasoning to graphs of larger order. Thus it remains open whether $\gamma_i(G) = \frac{n(G)}{2}$ holds for all cubic bipartite graph G .

6 Graphs with γ_i larger than Γ

In this section, we present a class of graphs whose indicated domination numbers exceed their upper irredundance numbers (and therefore also the independence number) by an arbitrarily large amount.

Recall that the k^{th} power G^k of a graph G has $V(G^k) = V(G)$, and $uv \in E(G^k)$ if and only if $d_G(u, v) \leq k$, where d_G is the standard (shortest paths) distance in G .

Theorem 6.1. *For every $n \geq k \geq 2$, the k^{th} powers P_n^k of paths satisfy*

$$\gamma_i(P_n^k) = \Theta\left(\frac{\log k}{k} n\right)$$

as $n \rightarrow \infty$. More explicitly,

$$(\lceil \log(k+1) \rceil + 1) \left\lfloor \frac{n}{4k} \right\rfloor \leq \gamma_i(P_n^k) \leq \frac{\lceil \log(k+1) \rceil + 2}{2k+2} n + \lceil \log(k+1) \rceil + 2$$

where \log means logarithm of base 2.

Proof. For the lower bound we may assume without loss of generality that n is a multiple of $4k$. We divide P_n^k into $\frac{n}{4k}$ sections of $4k$ consecutive vertices. Staller's strategy is to consider each section separately. Denote the vertices in one such section S by v_1, \dots, v_{4k} . Note that the “middle” subsection S^- formed by the vertices v_{k+1}, \dots, v_{3k} can only be dominated by vertices within S . Let us also use the notation S' for the “left” half v_1, \dots, v_{2k} of S , and write S'' for its “right” half v_{2k+1}, \dots, v_{4k} .

The first time Dominator indicates a vertex in S , with no loss of generality it is one of the vertices from S' . Staller then selects v_k . This selection dominates the entire $S^- \cap S'$, but leaves the k vertices of $S^- \cap S''$ undominated. More generally, no matter what happened during the game before this move, the set U of undominated vertices inside S is a set $\{v_i, v_{i+1}, \dots, v_j\}$ of consecutive vertices (or just a singleton $\{v_i\}$), because only some vertices at the right end of S'' may possibly be dominated by a selection from the successor section of S . Here $j > 3k$ may occur, but Staller's strategy will handle this situation as if $j \leq 3k$ held. So, attention will be restricted to $U' = \{v_i, v_{i+1}, \dots, v_{j'}\}$ where $j' = \min(j, 3k)$; hence $2k+1 \leq i \leq j' \leq 3k$. The crucial point is:

- (*) If $|U'| \geq 2^s$, for an integer $s \geq 0$, then Staller can achieve that at least $s+1$ steps are needed to make the entire U' dominated during the rest of the game.

We prove this by induction on s , the case of $s = 0$ being trivial. Once (*) is proved, it follows that the S^- subsection inside each of the $\frac{n}{4k}$ sections S requires at least $\lceil \log(k+1) \rceil + 1$ moves under a properly chosen strategy of Staller, hence implying the claimed lower bound.

In a general move of the game when at least one selection has already been made in S , assume that the set of undominated vertices inside S is $U \subseteq S''$ (also allowing $U \not\subseteq S^-$, but $U' \subset S^-$ will hold by definition), and let Dominator indicate vertex v_ℓ from U .

- (a) If $\ell > 3k$ (i.e., $v_\ell \in U \setminus U'$), then Staller imagines as if Dominator has indicated $\ell = 3k$ and proceeds as in case (c) below.
- (b) If $\ell \leq 3k$ and $\ell - i < j - \ell$, then Staller selects $v_{\ell-k}$.
- (c) If $\ell \leq 3k$ and $\ell - i \geq j - \ell$, then Staller selects $v_{\ell+k}$.

In either case, Staller's selection is inside S , hence it has no influence on the middle subsection of any section other than S . Moreover, in case (a) the selected vertex is v_{4k} , therefore it dominates v_ℓ . Finally, the number of vertices in U' that become dominated by Staller's selection is at most $\lceil \frac{1}{2}|U'| \rceil$. Consequently, if $|U'| \geq 2^s$, then at least 2^{s-1} vertices remain undominated inside S^- after this move. This implies (*) by induction, and completes the proof of the lower bound.

As a preparation to the proof of the upper bound, we observe the following.

- (**) If $U = \{v_i, v_{i+1}, \dots, v_j\}$ is a set of consecutive undominated vertices, and $j - i + 1 \leq 2k + 1$, then Dominator can achieve in at most $\lceil \log(j - i + 2) \rceil$ moves that the entire U becomes dominated.

The strategy is simple:

1. Indicate v_ℓ , where $\ell = \lfloor \frac{1}{2}(i + j) \rfloor$.
2. Depending on Staller's selection, update U to its part that remains undominated.
3. Return to step 1 as long as U is non-empty.

To verify that this strategy proves (**), it suffices to observe that the undominated vertices of U remain consecutive after each move—this is because $j - i + 1 \leq 2k + 1$ has been assumed—and that their number gets halved in each move.

Now we provide a strategy for Dominator, who will proceed from left to right in P_n^k . The situation before each move can be described with an alternating sequence of sections that we call “gaps” U_i , consisting of consecutive undominated vertices, and “intervals” D_i , consisting of consecutive dominated vertices. At the beginning we have no intervals, and just one gap $U_1 = \{v_1, \dots, v_n\}$. Let Dominator play according to the following rules.

1. While v_n is undominated and the last gap has at least $2k + 2$ vertices:
 - (a) Identify the first vertex v_g of the gap that contains v_n .

- (b) Indicate the vertex v_ℓ , where $\ell = g + 2k + 1$.
- (c) Depending on Staller's selection, update the alternating sequence $U_1, D_1, U_2, D_2, \dots$ of gaps and intervals.

2. In each of the gaps U_1, U_2, \dots , apply the strategy described above for (**).

After the first phase, for the sake of a more transparent computation, we split the gap-interval vertex partition into blocks $B_1 = (U_1, D_1)$, $B_2 = (U_2, D_2)$, \dots , $B_m = (U_m, D_m)$. If v_n remains undominated before the second phase, i.e. $v_n \in U_m$, we artificially define D_m as the emptyset. Observe that except for the last block B_m , each gap U_i has at least 1 and at most $2k + 1$ vertices, and each interval D_i has exactly $2k + 1$ vertices. In particular, also the first block begins with a gap, as v_1 is undominated.

Consider any B_i with $i < m$. If $2^s \leq |U_i| < 2^{s+1}$, (**) guarantees that Dominator can achieve that U_i becomes dominated within $s + 1 \leq \lceil \log(2k + 2) \rceil$ moves (and, according to (*), that is the best he can do). Moreover, D_i was dominated in just one step. Since $|U_i \cup D_i| \geq 2k + 1 + 2^s \geq 2k + 2$ whenever $i < m$, the average number of moves required to dominate one vertex in B_i is

$$\frac{s + 2}{|U_i| + |D_i|} \leq \frac{\lceil \log(2k + 2) \rceil + 1}{2k + 2}$$

because s is an integer and $2^s \leq |U_i| \leq 2k + 1$. This upper bound is valid for all vertices in the whole $B_1 \cup \dots \cup B_{m-1}$.

To complete the proof of the theorem, it suffices to note that U_m becomes dominated in at most $\lceil \log(2k + 2) \rceil = \lceil \log(k + 1) \rceil + 1$ moves and D_m in just one move (or none if it is empty). \square

Let us state the particular case $k = 2$ of the general lower bound separately, as it already has important consequences.

Corollary 6.2. *If $n \geq 2$, then $\gamma_i(P_n^2) \geq 3\lfloor \frac{n}{8} \rfloor$*

Since all P_n^2 are chordal graphs, we have $\text{IR}(P_n^2) = \Gamma(P_n^2) = \alpha(P_n^2) = \lceil \frac{n}{3} \rceil \leq \frac{n+2}{3}$ by Theorem 2.2. Together with Corollary 6.2, noting also that $3\lfloor \frac{n}{8} \rfloor \geq \frac{3}{8}n - \frac{21}{8}$, we get $\gamma_i(P_n^2) - \text{IR}(P_n^2) \geq \frac{n-79}{24}$, which can be arbitrarily large when n grows.

Corollary 6.3. *There exist graphs G such that $\gamma_i(G) - \text{IR}(G)$ is arbitrarily large.*

The above corollary also implies that there exist graphs G such that $\gamma_i(G) - \Gamma(G)$ is arbitrarily large.

7 Concluding remarks

In Section 4 we characterized the graphs G with minimum degree δ and order $n \geq 2\delta + 2$, that satisfy the extremal value $\gamma_i(G) = n - \delta$. It would be interesting to extend the characterization to graphs of smaller order relative to δ .

Question 7.1. *For which graphs G of order n and minimum degree δ such that $n < 2\delta + 2$ does $\gamma_i(G) = n - \delta$ hold?*

In Section 5, we proved for two important families of bipartite graphs (namely trees and grids) that the indicated domination number equals their independence number. In addition, computer check confirmed this to hold for all (bipartite) graphs of order at most 10. We could not find any bipartite graph that would not have this property, so we pose this as an open problem:

Question 7.2. *Is it true that for every bipartite graph G we have $\gamma_i(G) = \alpha(G)$?*

In the same section we also proved that connected, cubic bipartite graphs G of order at most 12 satisfy $\gamma_i(G) = \frac{n(G)}{2} = \alpha(G)$. We think this result could be generalized to all connected, cubic bipartite graphs, but we were unable to prove it. Therefore we pose the following question.

Question 7.3. *Is it true that $\gamma_i(G) = \frac{n(G)}{2}$ for any cubic bipartite graph G ?*

It would also be interesting to find whether there is an upper bound on the indicated domination number in the class of all connected cubic graphs. In particular, is there a constant $C < 1$ such that $\gamma_i(G) \leq C \cdot n$ holds for all connected cubic graphs G ? Note that $C \geq \frac{1}{2}$ due to the 3-cube and the Petersen graph, as checked by computer.

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