Existence of global weak solutions to the Navier–Stokes equations in weighted spaces

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Abstract

We obtain a global existence result for the three-dimensional Navier-Stokes equations with a large class of initial data allowing growth at spatial infinity. Our work is a continuation of the results [10], [12] and proves global existence of suitable weak solutions with initial data in different weighted spaces as well as eventual regularity.

1 Introduction

Let (u, p) be a solution the three dimensional Navier-Stokes equations in the sense of distributions:

$$\begin{cases} \partial_t u - \Delta u + (u \cdot \nabla)u + \nabla p = 0, \\ \operatorname{div} u = 0 \end{cases} \quad \text{in } \mathbb{R}^3 \times (0, T) \tag{1.1}$$

For $T > 0$ and initial data $u(x, 0) = u_0(x)$. One of the first results about global in time existence of weak solutions was done by J.Leray in [1] with divergence free initial data $u_0 \in L^2(\mathbb{R}^3)$. Later it was improved by Lemarié-Rieusset in [5] for local initial data $u_0 \in L^2_{uloc}$, the local L^2_{uloc} space is defined as following:

$$\|u_0\|_{L^2_{uloc}} = \sup_{x \in \mathbb{R}^3} \|u_0\|_{L^2(B(x,1))} < \infty. \tag{1.2}$$

The focus of this work is on finding a different class for initial data, for which we can prove global in time existence, for that we need to define local-energy solutions. We will use the following notations, for any cube $Q \in \mathbb{R}^3$ with radius r we will denote Q^* a slightly bigger cube with the same center and radius $5r/4$ and Q^{**} is similarly a small extension of Q^*. Next we introduce the definition of local-energy solutions:

Definition 1.1. *A vector field $u \in L^2_{loc}(\mathbb{R}^3 \times (0, T))$, where $T > 0$ is a local energy solution to (1.1) with initial data $u_0 \in L^2_{loc}(\mathbb{R}^3)$ if the following holds:*

- $u \in \bigcap_{R>0} L^\infty(0, T; L^2(B_R(0)))$, $\nabla u \in L^2_{loc}(\mathbb{R}^3 \times [0, T))$,
- *there is* $p \in L^{\frac{3}{2}}_{loc}(\mathbb{R}^3 \times [0, T))$ *such that* u, p *is a solution to NSE (1.1) in a sense of distributions,*
- *For all compacts* $\Omega \subset \mathbb{R}^3$ *we have* $u(t) \underset{t \to 0+}{\to} u_0$ *in* $L^2(\Omega)$,
- *u is a Caffarelli-Kohn-Nirenberg solution, for all* $\xi \in C_0^\infty(\mathbb{R}^3 \times (0, T))$, $\xi \geq 0$,

$$\begin{aligned} 2 \int\int |\nabla u|^2 \xi \, dx \, dt \leq \int\int |u|^2 (\partial_t \xi + \Delta \xi) \, dx \, dt + \\ \int\int (|u|^2 + 2p)(u \cdot \nabla \xi) \, dx \, dt, \end{aligned} \tag{1.3}$$

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- *the function $t \mapsto \int u(x,t)w(x)\ dx$ is continuous on $[0,T)$ for any $w \in L^2(\mathbb{R}^3)$ with a compact support,*
- *for every cube $Q \subset \mathbb{R}^3$, there exists $p_Q(t) \in L^{\frac{3}{2}}(0,T)$ such that for $x \in Q^*$ and $0 < t, T$,*

$$\begin{aligned} p(x,t) - p_Q(t) &= -\frac{1}{3}|u(x)|^2 + p.v. \int_{y \in Q^{**}} K_{ij}(x-y)(u_i(y,s)u_j(y,s))\ dy \\ &+ \int_{y \notin Q^{**}} (K_{ij}(x-y) - K_{ij}(x_Q - y))(u_i(y,s)u_j(y,s))\ dy, \end{aligned} \tag{1.4}$$

where x_Q is the center of Q and $K_{ij} = \partial_i \partial_j (4\pi|y|)^{-1}$.

This definition was used in [10] and differs from previous definitions in [5, 6, 8, 9, 7] since we only require initial data in L^2_{loc} and not $u_0 \in L^2_{uloc}$. Nevertheless, local energy solutions first were proven to exist locally in time for initial data in $L^2_{uloc}(\Omega)$ for $\Omega = \mathbb{R}^3$ or $\mathbb{R}^3_+$. Most of the results for global existence included a decay assumption on the data, see [5, 6] for $v_0 \in E^2$. One of the examples of such conditions is:

$$\lim_{R \to \infty} \sup_{x_0 \in \mathbb{R}^3} \frac{1}{R^2} \int\limits_{B_R(x_0)} |u_0|^2\ dx = 0. \tag{1.5}$$

Later global existence for non-decaying data in uniform space L^2_{uloc} was done in two dimensional case by Bason [2]. For initial data in $L^2_{loc} \backslash L^2_{uloc}$ Fernandez-Dalgo and Lemarié-Rieusset [11] constructed global solutions with the following condition:

$$\int_{\mathbb{R}^3} \frac{|u_0(x)|^2}{(1+|x|)^2}\ dx < \infty. \tag{1.6}$$

Bradshaw and Kukavica [12] constructed local in time solutions for initial data in the space $\mathring{M}^{p,q}_{\mathcal{C}}$ which is related to our project and will be defined later. In [10] with Tai-Peng Tsai they proved global existence in class $\mathring{M}^{p,q}_{\mathcal{C}}$, this class strictly includes the initial data from [9, 11]. In our paper we want to extend this result to different weighted spaces of initial data F^q_r.

To define spaces $\mathring{M}^{p,q}_{\mathcal{C}}, F^q_r$ we need to construct a certain family of cubes in $\mathbb{R}^3$ that was defined in [10]. For $n \in \mathbb{N}$, we denote $S_0 = \{x : \ |x_i| \leq 2;\ i = 1,2,3\}$ and $R_n = \{x : \ |x_i| \leq 2^n; i = 1,2,3\}$. Let $S_n = R_{n+1} \setminus R_n$ for $n \in \mathbb{N} \setminus \{1\}$ and $S_1 = R_2 \setminus S_0$. Then $|S_n| = 56 \cdot 2^{3n}$. We separate $S_0 = \cup_1^{64} Q_0^{k_0}$ into 64 cubes with side-length 1 and $S_n = \cup_{k_n=1}^{56} Q_n^{k_n}$ into 56 cubes with side-length 2^n, the set of all these cubes we denote as $\mathcal{C} = \cup_{n=0}^{\infty} \cup_{k_n} Q_n^{k_n}$. This set will satisfy the following properties

- Side-length of each cube is proportional to the distance to the origin.
- Adjacent cubes have proportional volume.
- The distance between cubes Q, Q' is proportional to $\min\{|Q|^{\frac{1}{3}}, |Q'|^{\frac{1}{3}}\}$.
- The amount of cubes Q', such that $|Q'| < |Q|$ is bounded above by $|Q|^{\frac{1}{3}}$

Using set $\mathcal{C}$ we reintroduce the definition of $M^{p,q}_{\mathcal{C}}$ and $\mathring{M}^{p,q}_{\mathcal{C}}$ from [10]:

Definition 1.2. *Let $p \in [1,\infty), q \geq 0$. We have $f \in M^{p,q}_{\mathcal{C}}$ if*

$$\|f\|^p_{M^{p,q}_{\mathcal{C}}} := \sup_{Q \in \mathcal{C}} \frac{1}{|Q|^{\frac{q}{3}}} \int_Q |f(x)|^p\ dx < \infty. \tag{1.7}$$

If $f \in M^{p,q}_{\mathcal{C}}$ and satisfies the following condition then $f \in \mathring{M}^{p,q}_{\mathcal{C}}$:

$$\frac{1}{|Q|^{\frac{q}{3}}} \int_Q |f|^p\ dx \to 0,\ \text{as } |Q| \to \infty,\ Q \in \mathcal{C} \tag{1.8}$$

Note that global existence and eventual regularity result [10] considers initial data from $\mathring{M}^{2,2}_{\mathcal{C}}$ and improves the constructions in [9, 12, 11]. The goal of our project is to further improve the result of Bradshaw, Kukavica and Tsai [10] for another set of initial data, using the family of cubes $\mathcal{C}$ let us introduce the following spaces:

Definition 1.3. *Let $q \leq 2$ and $r > 2$, we define spaces F_r^q:*

$$F_r^q = \Big\{ f \in L^2_{loc}(\mathbb{R}^3) \ \Big| \ \sum_{n,k_n} \Big(\frac{1}{|Q_n^{k_n}|^{\frac{q}{3}}} \int_{Q_n^{k_n}} |f|^2 \ dx \Big)^{\frac{r}{2}} < \infty \Big\} \tag{1.9}$$

With the norm $\|f\|^r_{F_r^q} = \sum_{n,k_n}^{\infty} \Big(\frac{1}{|Q_n^{k_n}|^{\frac{q}{3}}} \int_{Q_n^{k_n}} |f|^2 \ dx \Big)^{\frac{r}{2}}$

Both spaces $M_{\mathcal{C}}^{p,q}$ and F_r^q are equivalent to certain Herz spaces. Let $A_k = B_{2^{k+1}} \setminus B_{2^k}$ where B_{2^k} is a ball in $\mathbb{R}^3$ with radius 2^k. Let $n \in \mathbb{N}$, $s \in \mathbb{R}$ and $p, q \in (0, \infty]$, we define the non-homogeneous Herz space $K^s_{p,q}(\mathbb{R}^n)$ as a space of functions $f \in L^p_{loc}(\mathbb{R}^n \setminus \{0\})$ with the following finite norm:

$$\|f\|_{K^s_{p,q}} = \begin{cases} \Big(\sum\limits_{k \in \mathbb{N}_0} 2^{ksq} \|f\|^q_{L^p(A_k)} \Big)^{\frac{1}{q}} \text{ if } q < \infty, \\ \sup\limits_{k \in \mathbb{N}_0} 2^{ks} \|f\|_{L^p(A_k)} \text{ if } q = \infty. \end{cases} \tag{1.10}$$

Then the space $M_{\mathcal{C}}^{p,q}$ is equivalent to Herz space $K^{-q}_{p,\infty}$ and F_r^q is equivalent to $K^{-q}_{2,r}$, $r < +\infty$. Local in time existence of mild solutions with initial data from some Herz spaces was proven by Tsutsui [14] (in case $p > 3$). Global existence of local energy solutions for the case $K^{-2}_{2,\infty}$ follows from [10]. Our goal is to extend the construction of global in time local energy solutions for initial data in Herz spaces $K^{-q}_{2,r}$.

The following is our main theorem on global existence of the solutions with initial data from F_r^q:

Theorem 1.1. *Let $q \leq 2$, $r > 2$ and assume $u_0 \in F_r^q$ is divergence free. Then there exists $u : \mathbb{R}^3 \times (0,\infty) \to \mathbb{R}^3$ and $p : \mathbb{R}^3 \times (0, \infty) \to \mathbb{R}$ so that (u, p) is a local energy solution to the Navier-Stokes equations on $\mathbb{R}^3 \times (0,\infty)$ corresponding to initial data u_0.*

Additionally we will use the following sequence of norms $F_r^{q,m}$ using set of cubes $\mathcal{C}$.

Definition 1.4. *For $m \in \mathbb{N}$ we denote $Q_{0,m} = R_m$ and for any $n \geq 1, k_n = 1...56$, $Q^{k_n}_{n,m} = Q^{k_n}_{m+n}$, where $Q^{k_n}_n$ are cubes of family $\mathcal{C}$. With respect to these sets we can denote norms*

$$\|f\|^r_{F_r^{q,m}} = \sum_{n,k_n} \Big(\frac{1}{|Q^{k_n}_{n,m}|^{\frac{q}{3}}} \int_{Q^{k_n}_{n,m}} |f|^2 \ dx \Big)^{\frac{r}{2}}.$$

We will denote $\mathcal{C}_m$ family of cubes in this definition that corresponds to space $F_r^{q,m}$. The following lemma shows the relation between spaces F_r^q and $F_r^{q,m}$:

Lemma 1.2. *Let $u \in F_r^q$, then we have the following two properties*

- $\|u\|_{F_r^{q,n}} \underset{n\to\infty}{\to} 0.$
- *For any n $u \in F_r^{q,n}$ and there exists constant C independent on n such that*

$$\|u\|_{F_r^{q,n}} \leq C \|u\|_{F_r^q}$$

.

Proof. Let us start with the first property, since the second one will follow from the proof. For the first property we only have to prove the convergence of the first term, corresponding to $Q_{0,n}$, since we know that $u \in F_r^q$ and the rest of the sum converges to 0:

$$\frac{1}{2^{nq}} \int_{B_{2^n}} |u|^2 \ dx \underset{n\to\infty}{\to} 0.$$

We denote $a_{m,k_m} = \left(\frac{1}{2^{mq}}\int_{Q_m^{k_m}} |u|^2\ dx\right)^{\frac{r}{2}}$, for $m \le n$, $k_m \le 64$, here $Q_m^{k_m}$ are cubes in the family $\mathcal{C}$ with radius 2^m, the following estimate holds

$$\begin{aligned} &\frac{1}{2^{nq}}\int_{B_{2^n}} |u|^2\ dx = \frac{1}{2^{nq}}\sum_{m=0}^{n-1}\sum_{k_m}\int_{Q_m^{k_m}} |u|^2\ dx = \frac{1}{2^{nq}}\sum_{m=0}^{n-1}\sum_{k_m} 2^{mq} {a_{m,k_m}}^{\frac{2}{r}} = \\ &= \sum_{m=0}^{n-1}\sum_{k_m}\frac{1}{2^{(n-m)q}}{a_{m,k_m}}^{\frac{2}{r}} \le \sum_{m=0}^{l-1}\sum_{k_m}\frac{1}{2^{(n-m)q}}{a_{m,k_m}}^{\frac{2}{r}} + \sum_{m=l-1}^{n-1}\sum_{k_m}\frac{1}{2^{(n-m)q}}{a_{m,k_m}}^{\frac{2}{r}}. \end{aligned} \tag{1.11}$$

Where $l \le n-1$ is an integer that we will choose later. Next we will estimates last two terms of (1.11) separately

$$\sum_{m=0}^{l-1}\sum_{k_m}\frac{1}{2^{(n-m)q}}{a_{m,k_m}}^{\frac{2}{r}} \le \frac{C}{2^{(n-l)q}}\sup_{n,k_n}|a_{n,k_n}|^{\frac{2}{r}} \tag{1.12}$$

For the last term we will use Holder inequality:

$$\begin{aligned} \sum_{m=l-1}^{n-1}\sum_{k_m}\frac{1}{2^{(n-m)q}}{a_{m,k_m}}^{\frac{2}{r}} &\le \Big(\sum_{m=l-1}^{n-1}\sum_{k_m}\frac{1}{2^{\frac{r(n-k)q}{r-2}}}\Big)^{\frac{r-2}{r}}\Big(\sum_{m=l-1}^{\infty}\sum_{k_m}a_{m,k_m}\Big)^{\frac{2}{r}} \le \\ &\le C(q,r)\Big(\sum_{m=l-1}^{\infty}\sum_{k_m}a_{m,k_m}\Big)^{\frac{2}{r}}. \end{aligned} \tag{1.13}$$

Next we choose n sufficiently large and $l \approx \frac{n}{2}$, with this both terms will be sufficiently small, which implies the convergence:

$$\frac{1}{2^{nq}}\int_{B_{2^n}} |u|^2\ dx \underset{n\to\infty}{\to} 0$$

To check the second statement we again only need to estimate the first term for which we will use (1.11) and apply Holder inequality similarly to (1.13)

$$\begin{aligned} &\frac{1}{2^{nq}}\int_{B_{2^n}} |u|^2\ dx = \sum_{m=0}^{n-1}\sum_{k_m}\frac{1}{2^{(n-m)q}}{a_{m,k_m}}^{\frac{2}{r}} \le \\ &\Big(\sum_{m=1}^{n-1}\sum_{k_m}\frac{1}{2^{\frac{r(n-k)q}{r-2}}}\Big)^{\frac{r-2}{r}}\Big(\sum_{m=1}^{\infty}\sum_{k_m}a_{m,k_m}\Big)^{\frac{2}{r}} \le C(q,r)\Big(\sum_{m=1}^{\infty}\sum_{k_m}a_{m,k_m}\Big)^{\frac{2}{r}} = C\|u\|^2_{F_r^q}. \end{aligned} \tag{1.14}$$

Here $C = C(q,r)$ is independent of n which finishes the proof. □

Using Lemma 1.2 and a priori bounds we can establish the following regularity result:

Theorem 1.3. *(Eventual Regularity) Take $q \le 1$ and $u_0 \in F_r^q$ as divergence-free initial data, and let (u,p) be a local energy solution on $\mathbb{R}^3 \times (0,\infty)$ corresponding to u_0. Assume additionally that:*

$$\|u(\cdot,t)\|^r_{F_r^q} + \sum_{m=1}^{\infty}\Big(\frac{1}{|Q_m|^{\frac{q}{3}}}\int_0^t\int_{Q_m}|\nabla u|^2\ dx\ dt\Big)^{\frac{r}{2}} < \infty, \tag{1.15}$$

for all $t < \infty$. Then for any $\delta > 0$, there exists a time τ depending on u_0, δ so that u is smooth on

$$\{(x,t), t > \max\{\delta|x|^2, \tau\}\} \tag{1.16}$$

The paper organized as follows. In Section 2 we prove a bound for pressure term, using this bound in Section 3 we establish a priori estimate for the solution u in spaces $F_r^{q,m}$. In Section 4 we prove global existence of local energy solutions and in Section 5 we prove Theorem 1.3.

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2 Pressure estimate

We use a similar representation of pressure as in [10], in this Section we fix some integer N and family of cubes corresponding to the space $F^{q,N}_r$, take Q as one of the cubes in this family:

$$p(x,t) - p_Q(t) = (G^Q_{ij} u_i u_j)(x,t) \ ,\forall x \in Q^*,$$

Where

$$\begin{aligned} G_{ij} f(x) = -\frac{1}{3}\delta_{ij} f(x) + \text{p.v.} \int_{y\in Q^{**}} K_{ij}(x-y) f(y)\ dy + \\ + \int_{y\notin Q^{**}} (K_{ij}(x-y) - K_{ij}(x_Q - y)) f(y)\ dy \end{aligned} \tag{2.1}$$

The following lemma will be the estimate of pressure in spaces $F^{q,N}_r$

Lemma 2.1. *Let $q \le 2$ and (u,p) be a local energy solution and associated pressure on $\mathbb{R}^3 \times [0,T]$. Then for any integer N and cube Q in a family of cubes corresponding to the space $F^{q,N}_r$ we have the following estimate:*

$$\begin{aligned} \frac{1}{|Q|^{\frac{q+1}{3}}} \int_0^t \int_Q |p(x,t) - (p)_{Q(t)}|^{\frac{3}{2}}\ dx\ ds \le \\ \frac{C}{|Q|^{\frac{5-q}{6}}} \int_0^t \Big(\sum_{Q' \subset (Q^{**})^c} \Big(\frac{1}{|Q'|^{\frac{q}{3}}} \int_{Q'} |u|^2\ dy \Big)^{\frac{r}{2}} \Big)^{\frac{3}{r}}\ ds + \frac{C}{|Q|^{\frac{q+1}{3}}} \int_0^t \int_{Q^{**}} |u|^3\ dx\ ds, \end{aligned} \tag{2.2}$$

where Q^{**} is a cube of radius $\frac{5}{4}|Q|^{\frac{1}{3}}$ and the same center as Q.

Proof. We separate G_{ij} from (2.1) into two terms:

$$I_{near} = -\frac{1}{3}\delta_{ij} f(x) + \text{p.v.} \int_{y\in Q^{**}} K_{ij}(x-y) f(y)\ dy,$$

$$I_{far} = \int_{y\notin Q^{**}} (K_{ij}(x-y) - K_{ij}(x_Q - y)) f(y)\ dy.$$

Next we apply Calderon-Zigmund estimates for the term I_{near}:

$$\frac{1}{|Q|^{\frac{1}{3}}} \int_0^t \int_{Q^*} |I_{\text{near}}|^{\frac{3}{2}}\ dx\ ds \le \frac{C}{|Q|^{\frac{1}{3}}} \int_0^t \int_{Q^{**}} |u|^3\ dx\ ds.$$

This term is similar to the one we will have in local energy inequality and will be estimated in the next Section. Now we will estimate the other term I_{far}:

$$\begin{aligned} I_{\text{far}} \le C|Q|^{\frac{1}{3}} \int_{\mathbb{R}^3 \setminus Q^{**}} \frac{1}{|x-y|^4} |u|^2\ dy \le \\ \le C \sum_{Q' \in S_1} |Q|^{\frac{1}{3}} \int_{Q' \cap (Q^{**})^c} \frac{1}{|x-y|^4} |u|^2\ dy + C \sum_{Q' \in S_2} |Q|^{\frac{1}{3}} \int_{Q' \cap (Q^{**})^c} \frac{1}{|x-y|^4} |u|^2\ dy \end{aligned} \tag{2.3}$$

Where S_1 is the subset of cubes with $|Q'| \le 8|Q|$ and S_2 are cubes with $|Q'| > 8|Q|$. There is only finite amount of cubes in S_1 and we can also assume for them that $|Q|^{\frac{1}{3}} \approx |x_Q - x_{Q'}|$ due to construction of our family of cubes $\mathcal{C}$. Moreover, if $x \in Q$ and $y \in Q'$, $Q' \in S_1$ then $|x-y| \approx |x_Q - x_{Q'}| \approx |Q|^{\frac{1}{3}}$. Hence,

$$\begin{aligned} \sum_{Q' \in S_1} |Q|^{\frac{1}{3}} \int_{Q' \cap (Q^{**})^c} \frac{1}{|x-y|^4} |u|^2\ dy \le \sum_{Q' \in S_1} \frac{1}{|Q|} \int_{Q' \cap (Q^{**})^c} |u|^2\ dy \le \\ \le \frac{1}{|Q|} \Big(\sum_{Q' \in S_1} \Big(\frac{1}{|Q'|^{\frac{q}{3}}} \int_{Q'} |u|^2\ dy \Big)^{\frac{r}{2}} \Big)^{\frac{2}{r}} \Big(\sum_{Q' \in S_1} |Q'|^{\frac{qr}{3(r-2)}} \Big)^{\frac{r-2}{r}} \le \\ \le \frac{C}{|Q|^{1-\frac{q}{3}}} \Big(\sum_{Q' \in S_1} \Big(\frac{1}{|Q'|^{\frac{q}{3}}} \int_{Q'} |u|^2\ dy \Big)^{\frac{r}{2}} \Big)^{\frac{2}{r}} \end{aligned} \tag{2.4}$$

On the other hand for $Q' \in S_2$ we have a different relation for $x \in Q, y \in Q' : \ |x-y| \approx |Q'|^{\frac{1}{3}}$

$$\begin{aligned} &\sum_{Q'\in S_2} |Q|^{\frac{1}{3}} \int_{Q'} \frac{1}{|x-y|^4} |u|^2 \, dy \leq C|Q|^{\frac{1}{3}} \sum_{Q'\in S_2} \frac{1}{|Q'|^{\frac{4}{3}}} \int_{Q'} |u|^2 \, dy \leq \\ &\leq C|Q|^{\frac{1}{3}} \Big(\sum_{Q'\in S_2} \Big(\frac{1}{|Q'|^{\frac{q}{3}}} \int_{Q'} |u|^2 \, dy \Big)^{\frac{r}{2}} \Big)^{\frac{2}{r}} \cdot \Big(\sum_{Q'\in S_2} \Big(\frac{1}{|Q'|^{\frac{4-q}{3}}} \Big)^{\frac{r}{r-2}} \Big)^{\frac{r-2}{r}} \end{aligned} \tag{2.5}$$

The last term in the product is finite only for $r > 2$ and $q < 4$, since all of our cubes have radius greater than $|Q|^{\frac{1}{3}}$ we can calculate the multiplier, denote $M = \log_2(|Q|^{\frac{1}{3}})$:

$$\sum_{Q'\in S_2} \Big(\frac{1}{|Q'|^{\frac{4-q}{3}}} \Big)^{\frac{r}{r-2}} = C \sum_{n\geq M} 2^{-\frac{n(4-q)r}{3(r-2)}} = C \frac{2^{-\frac{M(4-q)r}{3(r-2)}}}{1-2^{-\frac{(4-q)r}{3(r-2)}}} = C(q,r)|Q|^{-\frac{(4-q)r}{3(r-2)}} \tag{2.6}$$

Therefore, we get the estimate for I_{far}

$$\begin{aligned} I_{\text{far}} &\leq C|Q|^{\frac{1}{3}-\frac{(4-q)}{3}} \Big(\sum_{Q'\in S_2} \Big(\frac{1}{|Q'|^{\frac{q}{3}}} \int_{Q'} |u|^2 \, dy \Big)^{\frac{r}{2}} \Big)^{\frac{2}{r}} + \\ &\frac{C}{|Q|^{1-\frac{q}{3}}} \Big(\sum_{Q'\in S_1} \Big(\frac{1}{|Q'|^{\frac{q}{3}}} \int_{Q'} |u|^2 \, dy \Big)^{\frac{r}{2}} \Big)^{\frac{2}{r}} \leq \\ &C|Q|^{\frac{q}{3}-1} \Big(\sum_{Q'\subset (Q^{**})^c} \Big(\frac{1}{|Q'|^{\frac{q}{3}}} \int_{Q'} |u|^2 \, dy \Big)^{\frac{r}{2}} \Big)^{\frac{2}{r}}. \end{aligned} \tag{2.7}$$

Take any cube Q in family of cubes corresponding to space $F_r^{q,N}$, and we get the following estimate for I_{far}

$$\begin{aligned} &\frac{1}{|Q|^{\frac{q+1}{3}}} \int_0^t \int_Q |I_{\text{far}}|^{\frac{3}{2}} \, dx \, ds \leq \\ &\frac{C}{|Q|^{\frac{q+1}{3}+\frac{3-q}{2}-1}} \int_0^t \Big(\sum_{Q'\subset (Q^{**})^c} \Big(\frac{1}{|Q'|^{\frac{q}{3}}} \int_{Q'} |u|^2 \, dy \Big)^{\frac{r}{2}} \Big)^{\frac{3}{r}} \, ds = \\ &\frac{C}{|Q|^{\frac{5-q}{6}}} \int_0^t \Big(\sum_{Q'\subset (Q^{**})^c} \Big(\frac{1}{|Q'|^{\frac{q}{3}}} \int_{Q'} |u|^2 \, dy \Big)^{\frac{r}{2}} \Big)^{\frac{3}{r}} \, ds. \end{aligned} \tag{2.8}$$

Combining the estimates for I_{far} and I_{near} we finish the proof of Lemma (2.1). $\square$

3 Local Energy Inequality

In the Section we will prove a priori bound for solutions of (1.1) in spaces $F_r^{q,N}$.

Lemma 3.1. *Assume $u_0 \in F_r^q$ is divergence free and let (u, p) be a local energy solution with initial data u_0 on $\mathbb{R}^3 \times (0, T)$. Let us assume that for some N we have the following*

$$\|u(\cdot,t)\|^r_{F_r^{q,N}} + \sum_{n=1}^{\infty} \Big(\frac{1}{|Q_n|^{\frac{q}{3}}} \int_0^t \int_{Q_n} |\nabla u|^2 \, dx \, dt \Big)^{\frac{r}{2}} < \infty. \ \forall t < T. \tag{3.1}$$

Then for all $t \in (0, T)$ we have an a priori estimate

$$\begin{aligned} &\|u(\cdot,t)\|^r_{F_r^{q,N}} + \sum_{n=1}^{\infty} \Big(\frac{1}{|Q_n|^{\frac{q}{3}}} \int_0^t \int_{Q_n} |\nabla u|^2 \, dx \, dt \Big)^{\frac{r}{2}} \leq \|u(\cdot,0)\|^r_{F_r^{q,N}} + \\ &\frac{C_0 t^{\frac{r-2}{2}}}{2^{rN}} \int_0^t \|u(\cdot,s)\|^r_{F_r^{q,N}} \, ds + \frac{C_1 t^{\frac{r-2}{2}}}{2^{rN(2-q)}} \int_0^t \|u(\cdot,s)\|^{3r}_{F_r^{q,N}} \, ds \end{aligned} \tag{3.2}$$

Where k_1, k_2, C_0, C_1 are global constants that dont depend on N and Q_n are cubes of the family $\mathcal{C}$ corresponding to the space $F_r^{q,N}$.

Proof. Similarly to previous Section we choose arbitrary integer N and denote Q_n as one of the cubes in the set corresponding to the space $F_r^{q,N}$. The solution satisfies the local energy inequality for any compactly supported positive test function $\phi \le 1$ such that $\phi = 1$ on Q_n, $\operatorname{supp}\{\phi\} \subset Q_n^*$ and $t \le T$:

$$\int_{Q_n} |u|^2(x,t)\,dx + \int_0^t \int_{Q_n} |\nabla u|^2\,dx\,dt \le \int \phi(x,0)|u(x,0)|^2\,dx + \\ \int\int_{Q_n} |u|^2 \Delta\phi\,dx\,dt + \int\int |u|^2 u\cdot\nabla\phi + 2(p-(p)_{Q_n})u\cdot\nabla\phi\,dx\,dt. \tag{3.3}$$

Dividing by $|Q_n|^{\frac{q}{3}}$ and taking to the power $\frac{r}{2}$ we proceed to:

$$\Big(\frac{1}{|Q_n|^{\frac{q}{3}}}\int_{Q_n}|u|^2(x,t)\,dx\Big)^{\frac{r}{2}} + \Big(\frac{1}{|Q_n|^{\frac{q}{3}}}\int_0^t\int_{Q_n}|\nabla u|^2\,dx\,dt\Big)^{\frac{r}{2}} \le \\ C\Big(\frac{1}{|Q_n|^{\frac{q}{3}}}\int\phi(x,0)|u(x,0)|^2\,dx\Big)^{\frac{r}{2}} + C\Big(\frac{1}{|Q_n|^{\frac{q}{3}}}\int_0^t\int|u|^2\Delta\phi\,dx\,dt\Big)^{\frac{r}{2}} + \\ C\Big(\frac{1}{|Q_n|^{\frac{q}{3}}}\int_0^t\int|u|^2u\cdot\nabla\phi + 2(p-(p))u\cdot\nabla\phi\,dx\,dt\Big)^{\frac{r}{2}} \le \\ C\Big(\frac{1}{|Q_n|^{\frac{q}{3}}}\int_{Q_n^*}|u(x,0)|^2\,dx\Big)^{\frac{r}{2}} + C\Big(\frac{1}{|Q_n|^{\frac{q+2}{3}}}\int_0^t\int_{Q_n^*}|u|^2\,dx\,dt\Big)^{\frac{r}{2}} + \\ C\Big(\frac{1}{|Q_n|^{\frac{q+1}{3}}}\int_0^t\int_{Q_n^*}|u|^3\,dx\,dt\Big)^{\frac{r}{2}} + C\Big(\frac{1}{|Q_n|^{\frac{q+1}{3}}}\int_0^t\int|p-(p)_{Q_n}|^{\frac{3}{2}}\,dx\,dt\Big)^{\frac{r}{2}}. \tag{3.4}$$

Here $C = C(r)$ is a uniform constant. For the non-linear term we apply Gagliardo-Nirenberg, Hölder and Young inequalities:

$$\frac{1}{|Q_n|^{\frac{1}{3}}}\int_0^t\int_{Q_n}|u|^3\,dx\,dt \le C(\varepsilon)|Q_n|^{q-\frac{4}{3}}\int_0^t\Big(\frac{1}{|Q_n|^{\frac{q}{3}}}\int_{Q_n}|u|^2\,dx\Big)^3\,dt + \\ \varepsilon\int_0^t\int_{Q_n}|\nabla u|^2\,dx\,ds + C|Q_n|^{\frac{q}{2}-\frac{5}{6}}\int_0^t\Big(\frac{1}{|Q_n|^{\frac{q}{3}}}\int_{Q_n}|u|^2\,dx\Big)^{\frac{3}{2}}\,ds \tag{3.5}$$

Plugging the above estimate together with pressure estimate in the equation (3.3) we get

$$\Big(\frac{1}{|Q_n|^{\frac{q}{3}}}\int_{Q_n}|u|^2(x,t)\,dx\Big)^{\frac{r}{2}} + \Big(\frac{1}{|Q_n|^{\frac{q}{3}}}\int_0^t\int_{Q_n}|\nabla u|^2\,dx\,dt\Big)^{\frac{r}{2}} \le \\ \le \Big(\frac{1}{|Q_n|^{\frac{q}{3}}}\int_{Q_n^*}|u(x,0)|^2\,dx\Big)^{\frac{r}{2}} + \Big(\frac{1}{|Q_n|^{\frac{q+2}{3}}}\int_0^t\int_{Q_n^*}|u|^2\,dx\,ds\Big)^{\frac{r}{2}} + \\ +\Big(C(\varepsilon)|Q_n|^{\frac{2q-4}{3}}\int_0^t\Big(\frac{1}{|Q_n|^{\frac{q}{3}}}\int_{Q_n^{**}}|u|^2\,dx\Big)^3\,ds\Big)^{\frac{r}{2}} + \\ \varepsilon^{\frac{r}{2}}\Big(\frac{1}{|Q_n|^{\frac{q}{3}}}\int_0^t\int_{Q_n^{**}}|\nabla u|^2\,dx\,ds\Big)^{\frac{r}{2}} + \\ +\Big(|Q_n|^{\frac{q-5}{6}}\int_0^t\Big(\frac{1}{|Q_n|^{\frac{q}{3}}}\int_{Q_n^{**}}|u|^2\,dx\Big)^{\frac{3}{2}}\,ds\Big)^{\frac{r}{2}} + \\ \Big(\frac{C}{|Q_n|^{\frac{5-q}{6}}}\int_0^t\Big(\sum_{Q'\subset(Q^{**})^c}\Big(\frac{1}{|Q'|^{\frac{q}{3}}}\int_{Q'}|u|^2\,dy\Big)^{\frac{r}{2}}\Big)^{\frac{3}{r}}\,ds\Big)^{\frac{r}{2}} \tag{3.6}$$

Next we use Holder inequality in time:

$$\begin{gathered}\Big(\frac{1}{|Q_n|^{\frac{q}{3}}}\int_{Q_n}|u|^2(x,t)\ dx\Big)^{\frac{r}{2}}+\Big(\frac{1}{|Q_n|^{\frac{q}{3}}}\int_0^t\int_{Q_n}|\nabla u|^2\ dx\ dt\Big)^{\frac{r}{2}}\leq\\ \leq\Big(\frac{1}{|Q_n|^{\frac{q}{3}}}\int_{Q_n^*}|u(x,0)|^2\ dx\Big)^{\frac{r}{2}}+t^{\frac{r-2}{2}}\int_0^t\Big(\frac{1}{|Q_n|^{\frac{q+2}{3}}}\int_{Q_n^*}|u|^2\ dx\Big)^{\frac{r}{2}}\ ds+\\ +C(\varepsilon)|Q_n|^{\frac{r(q-2)}{3}}t^{\frac{r-2}{2}}\int_0^t\Big(\frac{1}{|Q_n|^{\frac{q}{3}}}\int_{Q_n^{**}}|u|^2\ dx\Big)^{\frac{3r}{2}}\ ds+\\ \varepsilon^{\frac{r}{2}}\Big(\frac{1}{|Q_n|^{\frac{q}{3}}}\int_0^t\int_{Q_n^{**}}|\nabla u|^2\ dx\ ds\Big)^{\frac{r}{2}}+\\ +|Q_n|^{\frac{r(q-5)}{12}}t^{\frac{r-2}{2}}\int_0^t\Big(\frac{1}{|Q_n|^{\frac{q}{3}}}\int_{Q_n^{**}}|u|^2\ dx\Big)^{\frac{3r}{4}}\ ds+\\ \frac{C}{|Q_n|^{\frac{r(5-q)}{12}}}t^{\frac{r-2}{2}}\int_0^t\Big(\sum_{Q'\in\mathcal{C}}\Big(\frac{1}{|Q'|^{\frac{q}{3}}}\int_{Q'}|u|^2\ dy\Big)^{\frac{r}{2}}\Big)^{\frac{3}{2}}\ ds\end{gathered}\tag{3.7}$$

All of the cubes Q_n have radius at least 2^N and we substitute $|Q_n|$ by 2^{3N} in the following terms:

$$\begin{gathered}\Big(\frac{1}{|Q_n|^{\frac{q+2}{3}}}\int_{Q_n^*}|u|^2\ dx\Big)^{\frac{r}{2}}\leq\frac{C}{2^{rN}}\Big(\frac{1}{|Q_n|^{\frac{q}{3}}}\int_{Q_n^*}|u|^2\ dx\Big)^{\frac{r}{2}},\\ |Q_n|^{\frac{r(q-2)}{3}}t^{\frac{r-2}{2}}\int_0^t\Big(\frac{1}{|Q_n|^{\frac{q}{3}}}\int_{Q_n^{**}}|u|^2\ dx\Big)^{\frac{3r}{2}}\ ds\leq\\ \frac{C}{2^{rN(2-q)}}t^{\frac{r-2}{2}}\int_0^t\Big(\frac{1}{|Q_n|^{\frac{q}{3}}}\int_{Q_n^{**}}|u|^2\ dx\Big)^{\frac{3r}{2}}\ ds,\\ |Q_n|^{\frac{r(q-5)}{12}}t^{\frac{r-2}{2}}\int_0^t\Big(\frac{1}{|Q_n|^{\frac{q}{3}}}\int_{Q_n^{**}}|u|^2\ dx\Big)^{\frac{3r}{4}}\ ds\leq\\ \frac{C}{2^{\frac{rN(5-q)}{4}}}t^{\frac{r-2}{2}}\int_0^t\Big(\frac{1}{|Q_n|^{\frac{q}{3}}}\int_{Q_n^{**}}|u|^2\ dx\Big)^{\frac{3r}{4}}\ ds.\end{gathered}\tag{3.8}$$

Every Q_n^{**} intersects with a fixed amount of cubes from the family $\mathcal{C}_N$, which is independent of n, therefore we can substitute integrals over Q_n^{**} with sum of integrals over Q_n' such that $Q_n'\cap Q_n^{**}\neq\emptyset$. Therefore after taking sum over all Q_n we get the following estimate:

$$\begin{gathered}\|u(\cdot,t)\|^r_{F^q_r}+\sum_{n=1}^{\infty}\Big(\frac{1}{|Q_n|^{\frac{q}{3}}}\int_0^t\int_{Q_n}|\nabla u|^2\ dx\ dt\Big)^{\frac{r}{2}}\leq\|u(\cdot,0)\|^r_{F^q_r}+\\ \frac{C}{2^{rN}}t^{\frac{r-2}{2}}\int_0^t\|u(\cdot,s)\|^r_{F^q_r}\ ds+C(\varepsilon)\frac{C}{2^{rN(2-q)}}t^{\frac{r-2}{2}}\int_0^t\sum_{n=1}^{\infty}\Big(\frac{1}{|Q_n|^{\frac{q}{3}}}\int_{Q_n}|u|^2\ dx\Big)^{\frac{3r}{2}}\ ds+\\ +C\varepsilon^{\frac{r}{2}}\sum_{n=1}^{\infty}\Big(\frac{1}{|Q_n|^{\frac{q}{3}}}\int_0^t\int_{Q_n}|\nabla u|^2\ dx\ ds\Big)^{\frac{r}{2}}+\\ \frac{C}{2^{\frac{rN(5-q)}{4}}}\sum_{n=1}^{\infty}t^{\frac{r-2}{2}}\int_0^t\Big(\frac{1}{|Q_n|^{\frac{q}{3}}}\int_{Q_n^{**}}|u|^2\ dx\Big)^{\frac{3r}{4}}\ ds+\\ \sum_{n=1}^{\infty}\frac{C}{|Q_n|^{\frac{r(5-q)}{12}}}t^{\frac{r-2}{2}}\int_0^t\Big(\sum_{Q'\in\mathcal{C}}\Big(\frac{1}{|Q'|^{\frac{q}{3}}}\int_{Q'}|u|^2\ dy\Big)^{\frac{r}{2}}\Big)^{\frac{3}{2}}\ ds.\end{gathered}\tag{3.9}$$

Next we choose ε sufficiently small to cancel the gradient term and apply the following algebraic inequalities:

$$\sum_{n=1}^{\infty}\Big(\frac{1}{|Q_n|^{\frac{q}{3}}}\int_{Q_n}|u|^2\ dx\Big)^{\frac{\alpha r}{2}}\leq\Big(\sum_{n=1}^{\infty}\Big(\frac{1}{|Q_n|^{\frac{q}{3}}}\int_{Q_n}|u|^2\ dx\Big)^{\frac{r}{2}}\Big)^{\alpha}\tag{3.10}$$

Where α equals to 3 or $\frac{3}{2}$, we can also calculate the multiplier $\sum_{n=1}^{\infty} \frac{C}{|Q_n|^{\frac{r(5-q)}{12}}} \approx \frac{C}{2^{\frac{rN(5-q)}{4}}}$ and plugging it into (3.9) we get:

$$\begin{aligned} \|u(\cdot,t)\|^r_{F^{q,N}_r} + \sum_{n=1}^{\infty} \Big(\frac{1}{|Q_n|^{\frac{q}{3}}} \int_0^t \int_{Q_n} |\nabla u|^2 \, dx \, dt\Big)^{\frac{r}{2}} \le \\ \|u(\cdot,0)\|^r_{F^{q,N}_r} + \frac{C}{2^{rN}} t^{\frac{r-2}{2}} \int_0^t \|u(\cdot,s)\|^r_{F^{q,N}_r} \, ds + \\ Ct^{\frac{r-2}{2}} \int_0^t \frac{1}{2^{rN(2-q)}} \|u(\cdot,s)\|^{3r}_{F^{q,N}_r} + \frac{1}{2^{\frac{rN(5-q)}{4}}} \|u(\cdot,s)\|^{\frac{3r}{2}}_{F^q_r} \, ds \le \|u(\cdot,0)\|^r_{F^{q,N}_r} + \\ \frac{C_0 t^{\frac{r-2}{2}}}{2^{rN}} \int_0^t \|u(\cdot,s)\|^r_{F^{q,N}_r} \, ds + \frac{C_1 t^{\frac{r-2}{2}}}{2^{rN(2-q)}} \int_0^t \|u(\cdot,s)\|^{3r}_{F^{q,N}_r} \, ds. \end{aligned} \tag{3.11}$$

In the last line we also used Young inequality and C_0, C_1 are global constants. □

4 Global Existence

The idea of the proof relies on getting a-priori estimates with help of a variation of Gronwall which we will apply to the norm of u in spaces $F^{q,m}_r$. We use the same lemma as in [10]:

Lemma 4.1. *Suppose $f(t) \in L^\infty([0,T];[0,\infty))$ satisfies, for some $m \ge 1$,*

$$f(t) \le a + \int_0^t (b_1 f(s) + b_2 f(s)^m) \, ds \quad \text{for } 0 < t < T, \tag{4.1}$$

where $a, b_1, b_2 \ge 0$ then for $T_0 = \min(T, T_1)$, with

$$T_1 = \frac{a}{2ab_1 + (2a)^m b_2}$$

we have $f(t) \le 2a$ for $t \in (0, T_0)$.

Remark 4.2. In our case we will apply this lemma for arbitrary large T,$m = 3$ and $f(t) = \|u(\cdot,t)\|^r_{F^{q,N}_r}$, $a = f(0)$, $b_1 = \frac{C_0 T^{\frac{r-2}{2}}}{2^{rN}}$, $b_2 = \frac{C_1 T^{\frac{r-2}{2}}}{2^{rN(2-q)}}$, which will guarantee the uniform apriori bound up to the time $\min\{T_1, T\}$, where

$$T_1 = \frac{1}{2\frac{C_0 T^{\frac{r-2}{2}}}{2^{rN}} + (2\|u_0\|^r_{F^{q,N}_r})^2 \frac{C_1 T^{\frac{r-2}{2}}}{2^{rN(2-q)}}}$$

We can take $T = n$ for some large integer n and find N such that $T_1 > n$, as will be shown in the next lemma. For eventual regularity Theorem 1.3 we will take $T = 2^{2N}$ and therefore

$$T_1 = \frac{1}{C_0 2^{-2N} + C_1 2^{rN(q-1)-2N} (2\|u_0\|^r_{F^{q,N}_r})^2} \approx 2^{2N}, \text{ if } q \le 1$$

Lemma 4.3. *Assume $u_0 \in F^q_r$ is divergence-free, and let (u,p) be a local energy solution with initial data u_0 on $\mathbb{R}^3 \times (0,\infty)$, we also assume that (u,p) satisfies the following for all N:*

$$\|u(\cdot,t)\|^r_{F^{q,N}_r} + \sum_{m=1}^{\infty} \Big(\frac{1}{|Q_m|^{\frac{q}{3}}} \int_0^t \int_{Q_m} |\nabla u|^2 \, dx \, dt\Big)^{\frac{r}{2}} \le \infty \quad \forall t < \infty \tag{4.2}$$

Where Q_m are cubes in $\mathcal{C}_N$. Then for any $n \in \mathbb{N}$ there exists N and such that

$$\|u(\cdot,t)\|^r_{F^{q,N}_r} + \sum_{m=1}^{\infty} \Big(\frac{1}{|Q_m|^{\frac{q}{3}}} \int_0^t \int_{Q_m} |\nabla u|^2 \, dx \, dt\Big)^{\frac{r}{2}} \le c_0 \|u_0\|^r_{F^{q,N}_r} \quad \forall t < n \tag{4.3}$$

Here c_0 is a global constant that does not depend on n.

Proof. We will combine Lemma 3.1 and Lemma 4.1, take some integer n and $T = n$ in Lemma 3.1. From local energy estimate for all N we have the following

$$\|u(\cdot,t)\|^r_{F^{q,N}_r} + \sum_{n=1}^{\infty}\Big(\frac{1}{|Q_n|^{\frac{q}{3}}}\int_0^t\int_{Q_n}|\nabla u|^2\,dx\,dt\Big)^{\frac{r}{2}} \le \|u(\cdot,0)\|^r_{F^{q,N}_r} + \frac{C_0 n^{\frac{r-2}{2}}}{2^{rN}}\int_0^t \|u(\cdot,s)\|^r_{F^{q,N}_r}\,ds + \frac{C_1 n^{\frac{r-2}{2}}}{2^{rN(2-q)}}\int_0^t \|u(\cdot,s)\|^{3r}_{F^{q,N}_r}\,ds \tag{4.4}$$

For all $t \le n$, therefore we can apply Gronwall lemma with $f = \|u(\cdot,t)\|_{F^{q,N}_r}$. Since we know that $u_0 \in F^q_r$ and $\lim_{N\to\infty}\|u_0\|_{F^{q,N}_r} = 0$ we can find N such that

$$T_1 = \frac{1}{2\frac{C_0 n^{\frac{r-2}{2}}}{2^{rN}} + (2\|u_0\|^r_{F^{q,N}_r})^2\frac{C_1 n^{\frac{r-2}{2}}}{2^{rN(2-q)}}} > n$$

Here T_1 is given to us by Lemma 4.1 and therefore we have a uniform bound (4.3) $\square$

Lemma 4.4. *Assume $f \in F^q_r$ is divergence free then for any $\varepsilon > 0$ there exists a divergence free $g \in L^2$ such that $\|f - g\|_{F^q_r} \le \varepsilon$ and $\|f - g\|_{F^{q,m}_r} \le \varepsilon$ for any $m \in \mathbb{N}$*

Proof. Let $\varepsilon > 0$ and $f \in F^q_r$ is divergence free, we choose $R > 0$ such that $\|f\chi_{\mathbb{R}^3\setminus B_R}\|_{F^q_r} \le \varepsilon$. Then we can find smooth cut-off radial function $\phi \le 1$ such that $\operatorname{supp}\phi \subset B_{2R}$ and $\phi = 1$ on B_R. Then we can use Bogovskii map [3] to construct function h such that

$$\operatorname{div} h = -f\nabla\phi,\ \ \operatorname{supp} h \subset B_{2R}\setminus B_R,\ \ \|h\|_{W^{1,2}} \le \|f\nabla\phi\|_{2,B_{2R}\setminus B_R}. \tag{4.5}$$

Moreover, we can estimate the following norm:

$$\int_{B_{2R}\setminus B_R}|h|^2\,dx \le C_0R^2\int_{B_{2R}\setminus B_R}|f|^2|\nabla\phi|^2\,dx \le C_0\int_{B_{2R}\setminus B_R}|f|^2\,dx \tag{4.6}$$

For some universe constant C_0. Next we choose $g = f\phi + h$, from (4.5)we get that $\operatorname{div} g = 0$. Now we only need to check that $\|f - g\|_{F^q_r}$ is small, indeed

$$\|f-g\|_{F^q_r} \le \|f\chi_{\mathbb{R}^3\setminus B_R}\|_{F^q_r} + \|h\|_{F^q_r} \le \|f\chi_{\mathbb{R}^3\setminus B_R}\|_{F^q_r} + \Big(\sum_{Q\in\mathcal{C}|\ Q\cap B_{2R}\setminus B_R\neq\emptyset}\Big(\frac{1}{|Q|^{q/3}}\int_Q|h|^2\,dx\Big)^{r/2}\Big)^{\frac{1}{r}}. \tag{4.7}$$

Notice that there are only finite amount of cubes in $\mathcal{C}$ intersecting $B_{2R}\setminus B_R$ and the amount does not depend on R due to properties of C moreover all such cubes have comparable radius to R, therefore for some universe constant C we have the following

$$\sum_{Q\in\mathcal{C}|\ Q\cap B_{2R}\setminus B_R\neq\emptyset}\Big(\frac{1}{|Q|^{q/3}}\int_Q|h|^2\,dx\Big)^{r/2} \le \Big(\frac{C}{|B_R|^{q/3}}\int_{B_{2R}\setminus B_R}|h|^2\,dx\Big)^{r/2} \le \Big(\frac{C}{|B_R|^{q/3}}\int_{B_{2R}\setminus B_R}|f|^2\,dx\Big)^{r/2} \le C\|f\chi_{\mathbb{R}^3\setminus B_R}\|_{F^q_r} \le C\varepsilon. \tag{4.8}$$

Here we used (4.6). Therefore g satisfies all the conditions of the lemma. $\square$

Next we will prove the main theorem:

Proof. Take $n \in \mathbb{N}$, from Lemma 4.4 there exists $u^n_0 \in L^2(\mathbb{R}^3)$ such that $\|u_0 - u^n_0\|_{F^q_r} \le \frac{1}{n}$ and $\|u_0 - u^n_0\|_{F^{q,m}_r} \le \frac{1}{n}$ for any $m \in \mathbb{N}$. Let $(u^n, \bar{p}^n)$ be a global Leray solution for initial data u^n_0. By Lemma 4.3 there exists a sequence N_n such that $N_n > n$ and u^n is bounded uniformly on $B_{N_n}\times[0,n]$. Hence, there exists a sub-sequence $u^{1,k}$ that converges on $B_{N_1}\times(0,1)$ in the following sense:

$$\begin{aligned} u^{1,k} &\overset{*}{\rightharpoonup} u_1 \text{ in } L^\infty(0,1;L^2(B_{N_1})),\\ u^{1,k} &\rightharpoonup u_1 \text{ in } L^\infty(0,1;H^1(B_{N_1})),\\ u^{1,k} &\to u_1 \text{ in } L^3(0,1;L^3(B_{N_1})). \end{aligned} \tag{4.9}$$

By Lemma 4.3 the sequence $u^{1,k}$ is also uniformly bounded on $B_{N_n} \times [0,n]$ for any $n \in \mathbb{N}$. Therefore by induction we construct a subsequence $\{u^{n,k}\}_{k\in\mathbb{N}}$ from $\{u^{n-1,k}\}_{k\in\mathbb{N}}$ which converges to a vector field u_n on $B_{N_n} \times (0,n)$ as $k \to \infty$ in the following sense:

$$\begin{aligned} u^{n,k} &\overset{*}{\rightharpoonup} u_n \text{ in } L^\infty(0,n; L^2(B_{N_n})), \\ u^{n,k} &\rightharpoonup u_n \text{ in } L^\infty(0,n; H^1(B_{N_n})), \\ u^{n,k} &\to u_n \text{ in } L^3(0,n; L^3(B_{N_n})). \end{aligned} \tag{4.10}$$

Denote $\tilde{u_n}$ as a 0 extension of u_n to $\mathbb{R}^3 \times (0,\infty)$. From our construction $\tilde{u_n} = \tilde{u_{n-1}}$ on $B_{N_{n-1}} \times (0, n-1)$. Let $u = \lim_{n\to\infty} \tilde{u_n}$. Then, $u = u_n$ on $B_{N_n} \times (0,n)$ for every $n \in \mathbb{N}$.

Let $u^{(k)} = u^{k,k}$ on $B_{N_k} \times (0,k)$ and 0 elsewhere. Then for any $n \in \mathbb{N}$ we have the following as $k \to \infty$:

$$\begin{aligned} u^{(k)} &\overset{*}{\rightharpoonup} u \text{ in } L^\infty(0,n; L^2(B_{N_n})), \\ u^{(k)} &\rightharpoonup u \text{ in } L^\infty(0,n; H^1(B_{N_n})), \\ u^{(k)} &\to u \text{ in } L^3(0,n; L^3(B_{N_n})). \end{aligned} \tag{4.11}$$

Using Lemma 4.3 we have the following bound for u:

$$\sup_{0<t<n} \|u(\cdot,t)\|^r_{F^{q,N}_r} + \sum_{m=1}^{\infty} \Big(\frac{1}{|Q_m|^{\frac{q}{3}}} \int_0^n \int_{Q_m} |\nabla u|^2 \, dx \, dt \Big)^{\frac{r}{2}} \leq c_0 \|u_0\|^r_{F^{q,N}_r} \tag{4.12}$$

The pressure is dealt similarly to [12], indeed we know that there is weak limit $p^{(k)} \rightharpoonup p$ in $L^{3/2}((0,n)\times Q_{N_n})$, we will check the pressure decomposition formula and strong convergence. Take $Q \in \mathcal{C},\ T > 0$, then for any k $(u^{(k)}, p^{(k)})$ are global Leray energy solutions and there exists $p^{(k)}_Q$ such that for all $(x,t) \in Q \times [0,T]$:

$$\begin{aligned} p^{(k)}(x,t) - p^{(k)}_Q = G^Q_{ij}(u^{(k)}_i u^{(k)}_j) = -\frac{1}{3}\delta_{ij} u^{(k)}_i u^{(k)}_j + \\ \text{p.v.} \int_{y\in Q^{**}} K_{ij}(x-y)(u^{(k)}_i u^{(k)}_j)(y)\, dy + \\ +\text{p.v.} \int_{y\notin Q^{**}} (K_{ij}(x-y) - K_{ij}(x_Q - y))(u^{(k)}_i u^{(k)}_j)(y)\, dy \end{aligned} \tag{4.13}$$

Choose $m > T$ so that $Q^{**} \subset Q_m = Q(0, 2^m)$ and separate $G^Q_{ij}(u_i u_j)$ into two terms

$$\begin{aligned} p^{(k)}_1(x,t) = -\frac{1}{3}\delta_{ij} u^{(k)}_i u^{(k)}_j + \text{p.v.} \int_{y\in Q^{**}} K_{ij}(x-y)(u^{(k)}_i u^{(k)}_j)(y)\, dy + \\ +\text{p.v.} \int_{Q_m \setminus Q^{**}} (K_{ij}(x-y) - K_{ij}(x_Q - y))(u^{(k)}_i u^{(k)}_j)(y)\, dy \\ p^{(k)}_2(x,t) = \text{p.v.} \int_{y\notin Q_m} (K_{ij}(x-y) - K_{ij}(x_Q - y))(u^{(k)}_i u^{(k)}_j)(y)\, dy \end{aligned} \tag{4.14}$$

We denote p_1, p_2 as similar terms but where $u^{(k)}$ is substituted by u. To prove convergence we choose $\varepsilon > 0$ arbitrary small and estimate the difference

$$\begin{aligned} |p_2 - p^{(k)}_2| \leq \int_{y\notin Q_m} (K_{ij}(x-y) - K_{ij}(x_Q - y))|u^{(k)}_i u^{(k)}_j - u_i u_j|(y)\, dy \leq \\ \sum_{\tilde{Q}\in\mathcal{C}\ |\tilde{Q}|^{1/3}\geq 2^{m-1}} \int_{\tilde{Q}} (K_{ij}(x-y) - K_{ij}(x_Q - y))|u^{(k)}_i u^{(k)}_j - u_i u_j|(y)\, dy \leq \\ \sum_{\tilde{Q}\in\mathcal{C}\ |\tilde{Q}|^{1/3}\geq 2^{m-1}} \int_{\tilde{Q}} \frac{|Q|^{1/3}}{|\tilde{Q}|^{4/3}} |u^{(k)}_i u^{(k)}_j - u_i u_j|(y)\, dy. \end{aligned} \tag{4.15}$$

Here we used that $Q^{**} \subset Q_m$ and therefore $|K_{ij}(x-y) - K_{ij}(x_Q - y)| \leq C\frac{|Q|^{1/3}}{|\tilde{Q}|^{4/3}}$. Next we will apply Holder inequality

$$
\begin{aligned}
\sum_{\tilde{Q}\in\mathcal{C}\ |\tilde{Q}|^{1/3}\geq 2^{m-1}} \int_{\tilde{Q}} \frac{|Q|^{1/3}}{|\tilde{Q}|^{4/3}} |u_i^{(k)} u_j^{(k)} - u_i u_j|(y)\ dy \leq \\
|Q|^{1/3}\Big(\sum_{\tilde{Q}\in\mathcal{C}\ |\tilde{Q}|^{1/3}\geq 2^{m-1}} |\tilde{Q}|^{\frac{(q-4)r}{3(r-2)}} \Big)^{\frac{r-2}{r}} \\
\Big(\sum_{\tilde{Q}\in\mathcal{C}\ |\tilde{Q}|^{1/3}\geq 2^{m-1}} \Big(\frac{1}{|\tilde{Q}|^{q/3}} \int_{\tilde{Q}} |u_i^{(k)} u_j^{(k)} - u_i u_j|(y)\ dy \Big)^{\frac{r}{2}} \Big)^{2/r} \leq \\
|Q|^{1/3}\Big(\sum_{\tilde{Q}\in\mathcal{C}\ |\tilde{Q}|^{1/3}\geq 2^{m-1}} |\tilde{Q}|^{\frac{(q-4)r}{3(r-2)}} \Big)^{\frac{r-2}{r}} (\|u\|^2_{F_r^{q,m}} + \|u^{(k)}\|^2_{F_r^{q,m}}) \leq \\
C|Q|^{1/3} 2^{m(q-4)} (\|u_0\|^2_{F_r^{q,m}} + \|u_0^{(k)}\|^2_{F_r^{q,m}}).
\end{aligned}
\tag{4.16}
$$

Here we used (4.3), since we have uniform bound for $u, u^{(k)}$ up to time T with our choice of m. Notice that $q < 4$ and $u_0^{(k)}$ converges to u_0, therefore we choose m, k large enough so that $\|u_0^{(k)}\|^2_{F_r^{q,m}} \leq \varepsilon + \|u_0\|^2_{F_r^{q,m}}$ and

$$
C|Q|^{1/3} 2^{m(q-4)} (\|u_0\|^2_{F_r^{q,m}} + \|u_0^{(k)}\|^2_{F_r^{q,m}}) \leq 2\varepsilon \tag{4.17}
$$

Next we apply (4.11) to get large enough k so that

$$
\|p_1^{(k)} - p^1\|_{L^{3/2}(Q\times[0,T])} \leq \varepsilon \tag{4.18}
$$

This proves that $G^Q_{ij}(u_i^{(k)} u_j^{(k)}) \to G^Q_{ij}(u_i u_j)$ in $L^{3/2}(Q \times [0,T])$. Lastly, in any fixed domain $Q \times [0,T]$ pair $(u, G^Q_{ij}(u_i u_j)$ solves (1.1). Therefore, $\nabla p = \nabla G^Q_{ij}(u_i u_j)$ in $\mathcal{D}'(\mathbb{R}^3)$ at every time t and so there exists a function of time $p_Q(t)$ such that

$$
p(x,t) = G^Q_{ij}(u_i u_j)(x) + p_Q(t), \text{ for } (x,t) \in Q \times [0,T]
$$

Also from this identity we get $p_Q(t) \in L^{\frac{3}{2}}(0,T)$. The rest of the proof follows from the similar argument to [12], because we have the same convergence of u^k, p^k on any $Q \times T_0$ for any fixed $T_0 > 0$.

□

5 Eventual regularity

In this Section we will prove Theorem 1.3 , we will use one of Caffarelli-Kohn-Nirenberg epsilon-regularity criteria variations, see [4]

Lemma 5.1. *For any $\sigma \in (0,1)$, there exists a universal constant $\varepsilon_*(\sigma) > 0$ such that, if a pair (u,p) is a suitable weak solution of (1.1) in $Q_r = B_r(x_0) \times (t_0 - r^2, t_0)$, and satisfies the bound*

$$
\varepsilon^3 = \frac{1}{r^2}\int_{Q_r} (|u|^3 + |p|^{\frac{3}{2}})\ dx\ dt < \varepsilon_*, \tag{5.1}
$$

Then $u \in L^\infty(Q_{\sigma r})$. Moreover, there is L^∞ estimate

$$
\|\nabla^k u\|_{L^\infty(Q_{\sigma r})} \leq C_k \varepsilon r^{-k-1},\ \forall k \in \mathbb{Z}_+ \tag{5.2}
$$

for universal constants $C_k = C_k(\sigma)$.

Next we will prove Theorem (1.3)

Proof. Choose sufficiently large T, N_1 such that $\|u_0\|_{F_r^{q,m}} < 1$ for all $m > N_1$ and Lemma 4.3 holds for $N > N_1$ and $t < T$. We take some $N > N_1$ and denote the following:

$$Q = (-2^N, 2^N)^3. \tag{5.3}$$

Note that $Q \in \mathcal{C}_N$. From Lemmas 2.1,4.3 we have that

$$\begin{aligned} J = \frac{1}{|Q|^{\frac{2}{3}}} \int_0^t \int_Q (|u|^3 + |p - (p)_Q(s)|^{\frac{3}{2}} \, dx \, dx \leq \\ \leq \frac{C}{|Q|^{\frac{1-q}{2}}} \int_0^t \Big(\sum_{Q' \cap (Q^{**})^c \neq \emptyset} \Big(\frac{1}{|Q'|^{\frac{q}{3}}} \int_{Q'} |u|^2 \, dy \Big)^{\frac{r}{2}} \Big)^{\frac{3}{r}} \, ds + \frac{C}{|Q|^{\frac{2}{3}}} \int_0^t \int_{Q^{**}} |u|^3 \, dx \, ds, \end{aligned} \tag{5.4}$$

here cubes Q' are part of family $\mathcal{C}_N$. Next we apply (3.5):

$$\begin{aligned} \frac{1}{|Q|^{\frac{2}{3}}} \int_0^t \int_Q |u|^3 \, dx \, dt \leq C(\varepsilon) |Q|^{q-\frac{5}{3}} \int_0^t \Big(\frac{1}{|Q|^{\frac{q}{3}}} \int_Q |u|^2 \, dx \Big)^3 \, dt + \\ \frac{\varepsilon}{|Q|^{\frac{1}{3}}} \int_0^t \int_Q |\nabla u|^2 \, dx \, ds + C |Q|^{\frac{q}{2}-\frac{7}{6}} \int_0^t \Big(\frac{1}{|Q|^{\frac{q}{3}}} \int_Q |u|^2 \, dx \Big)^{\frac{3}{2}} \, ds \end{aligned} \tag{5.5}$$

Since $q \leq 1$ we can apply Lemma 4.3 to get the following:

$$J = \frac{1}{|Q|^{\frac{2}{3}}} \int_0^t \int_Q (|u|^3 + |p - (p)_Q(s)|^{\frac{3}{2}} \, dx \, dx \leq \|u_0\|_{F_N^{1,r}}, \ \forall t < T. \tag{5.6}$$

Take $\sigma < 1$ from Lemma 5.1 arbitrary small. From Lemma 1.2 we can choose m large enough so that $\|u_0\|_{F_m^{1,r}} \leq \varepsilon(\sigma)$. Then by Remark 4.2 we can take $T = c_* 2^m$ with $c_* < 1$ and get that

$$\frac{1}{c_* 2^{2m}} \int_0^{c_* 2^{2m}} \int_{B_{2^m}(0)} (|u|^3 + |p - (p)_Q|^{\frac{3}{2}}) \, dx \, dt \leq \varepsilon(\sigma) \tag{5.7}$$

By epsilon-regularity Lemma 5.1 we get that u is regular in

$$Z_m = B_{\sigma\sqrt{c_*}2^m} \times [(1-\sigma^2)c_* 2^{2m}, c_* 2^{2m}], \tag{5.8}$$

and $\|u\|_{L^\infty(Z_m)} \leq C 2^{-m}$. Take $\delta < 1$ and choose c_*, σ so that Z_m contains

$$P_m = \{(x,t) \in \mathbb{R}^4_+ : \ \delta|x|^2 \leq t, \ \ (1-\sigma^2)c_* 2^{2m} \leq t \leq 4(1-\sigma^2)c_* 2^{2m}\}. \tag{5.9}$$

After taking the union by $m \geq N_1$ we prove the statement of (1.3).

□

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