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On the Construction of Relativistic Quantum Wave Equation and General Solution of the Second Order Differential Equation

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Abstract

Using the elementary axioms of special relativity and quantum mechanics we construct a wave equation which generalizes the Schrodinger equation. We also solve the general second order differential equation ($y''(x) = V(x)y(x)$).

1 Introduction

Einstein belived that the complete equivalence of mass and energy is given from the equation

$$E = m_0 c^2, \tag{1}$$

where m is the mass of a moving particle and E the total energy (c is the velocity of light in vacum). This formula is not completely right. It has to do with error terms of series representing the mass of a moving particle (see relation (5) below) and restrictions using Newton formula to write

$$m = m_0\gamma(v) = m_0 + \frac{1}{c^2}\left(\frac{1}{2}m_0 v^2\right) + \dots \tag{2}$$

Taking the first two terms of (2) the energy of a moved particle $E = mc^2$ can divided into $E_0 = m_0c^2$ (which m_0 is the rest mass) and $\frac{1}{2}m_0v^2$ is the kinetic energy.

In 1927 Klein and Gordon manage to constuct a relativistic quantum mechanical equation for moving particles using the following equation

$$E^2 = p^2c^2 + m_0^2c^4. \tag{3}$$

Here p is the momentum of a moving particle and m_0 is the rest mass. By attaching where E and p the quantum operators $+i\hbar\partial_t$ and $-i\hbar\partial_x$ respectively (energy and momentum), they arived to an equation which bear their names (Klein-Gordon equation):

$$\partial_{xx}^2 Y - \frac{1}{c^2}\partial_{tt}^2 Y - \frac{m_0^2c^2}{\hbar^2}Y = 0. \tag{4}$$

In some cases the above equation holds good. But in general is not satisfactory. However the folowing formula for the mass of a traveling particle is reliable

$$m = m_0\gamma(v) = \frac{m_0}{\sqrt{1-\left(\frac{v}{c}\right)^2}} \tag{5}$$

In this article, under the same axioms used by Klein and Gordon and not using energy cuts and avoiding the square in energy we arive to a generalized equation similar to that of Schrodinger. Below we state the two axioms we need:

Axiom 1.
Relation (5) holds for every moving particle.

Axiom 2.
i) In the energy E corespond the quantum operator $\widehat{E} = i\hbar\partial_t$
ii) In the momentum p corspond the quantum operator $\widehat{p} = -i\hbar\partial_x$.

2 Main Results

The form of the equation we looking for must read as

$$\widehat{E}Y = \left(\frac{(\widehat{p})^2}{2m_0} + E_0\widehat{I}\right)Y, \tag{6}$$

where $E_0 = m_0c^2$, with m_0, c the rest mass and the velocity of light in vacum respectively.
We shall work with simple single-valued calculus and not operators to arive to an equation formed by (5) (Axiom 1), then we will use the quantum corespodence (Axiom 2) to reduce the desired diferential equation.
Starting right away we have:

$$E = \frac{p^2}{2m} + m_0c^2 = \frac{m^2v^2}{2m} + m_0c^2 = \frac{1}{2}mv^2 + m_0c^2 = \frac{1}{2}m_0\gamma(v)v^2 + m_0c^2 =$$

$$= \frac{m_0c^2}{2\sqrt{1-\left(\frac{v}{c}\right)^2}}\left(\frac{v}{c}\right)^2 + m_0c^2. \tag{7}$$

But

$$\frac{v}{c} = \frac{p}{mc} = \frac{p}{m_0c}\sqrt{1-\left(\frac{v}{c}\right)^2} \tag{8}$$

We interested to expand E in a form of $E = f(m_0, p)$ and then use Axiom (2). As someone can see this is quite dificult because of the nature of the relation $v = p/m$ in which m depends by v (relation (5)) and v by p/m again. In this point we mention that we can solve (8) with respect to $\frac{v}{c}$ to get

$$\xi = \frac{v}{c} = \frac{p}{m_0c}\sqrt{1-\xi^2}. \tag{9}$$

Hence

$$\xi = \frac{v}{c} = \pm \frac{p}{m_0 c\sqrt{1+\frac{p^2}{m_0^2c^2}}}. \tag{10}$$

Seting this in (7) we get

$$E = \frac{m_0c^2}{2}\frac{\xi^2}{\sqrt{1-\xi^2}} + m_0c^2 = \frac{E_0}{2}\frac{p^2}{m_0^2c^2\sqrt{1+\frac{p^2}{m_0^2c^2}}} + E_0. \tag{11}$$

Hence from the expansion

$$\frac{x^2}{\sqrt{1+x^2}} = \sum_{n=0}^{\infty}\begin{pmatrix} -1/2 \\ n \end{pmatrix} x^{2n+2}$$

we get

$$E = E_0 + \frac{E_0}{2}\sum_{n=0}^{\infty}\begin{pmatrix} -1/2 \\ n \end{pmatrix}\frac{p^{2n+2}}{(m_0c)^{2n+2}}.$$

Replacing the arithmetical quntities of p and E with operators (Axiom 2), we get

Theorem 1.

$$i\hbar\frac{\partial Y}{\partial t} = E_0Y + \frac{E_0}{2}\sum_{n=0}^{\infty}(-1)^{n+1}\begin{pmatrix} -1/2 \\ n \end{pmatrix}\left(\frac{\hbar}{m_0c}\right)^{2n+2}\frac{\partial^{2n+2}Y}{\partial x^{2n+2}}. \tag{12}$$

For solving (12), we use the folowing Lemma's found in [3]:

Lemma 1.
If

$$f(x) = \sum_{n=1}^{\infty} A_n e^{-nx},$$

then

$$f(x) = \sum_{n=1}^{\infty}\frac{1}{e^{nx}-1}\sum_{d|n} A_d\mu(n/d)$$

and the oposite.

Lemma 2.
If $x > 0$

$$f(x) = \sum_{n=1}^{\infty}\frac{1}{e^{nx}-1}\sum_{d|n} A_d\mu(n/d),$$

then

$$f^{(\nu)}(x) = \sum_{n=1}^{\infty}\frac{1}{e^{nx}-1}\sum_{d|n} A_d(-d)^{\nu}\mu(n/d).$$

Set now

$$Y(x,t)=\sum_{k=1}^{\infty}\frac{\sum_{d|k}A_d(t)\mu(k/d)}{e^{kx}-1}.$$

Then

$$i\hbar\sum_{k=1}^{\infty}\frac{1}{e^{kx}-1}\sum_{d|k}A'_d(t)\mu(k/d)=E_0\sum_{k=1}^{\infty}\frac{\sum_{d|k}A_d(t)\mu(k/d)}{e^{kx}-1}+$$

$$+\frac{E_0}{2}\sum_{n=0}^{\infty}(-1)^{n+1}C_{-1/2,n}l_0^{2n+2}\sum_{k=1}^{\infty}\frac{1}{e^{kx}-1}\sum_{d|k}A_d(t)d^{2n+2}\mu(k/d). \qquad (13)$$

Rearaging the double series in the above equation and summing with respect to n we get

$$i\hbar\sum_{k=1}^{\infty}\frac{1}{e^{kx}-1}\sum_{d|k}A'_d(t)\mu(k/d)=$$

$$=E_0\sum_{k=1}^{\infty}\frac{1}{e^{kx}-1}\sum_{d|k}A_d(t)\mu(k/d)\left(1-\frac{l_0^2d^2}{2\sqrt{1-l_0^2d^2}}\right). \qquad (14)$$

Hence it must be

$$\frac{i\hbar}{m_0c^2}A'_d(t)=\left(1-\frac{l_0^2d^2}{2\sqrt{1-l_0^2d^2}}\right)A_d(t),$$

or

$$A_n(t)=f_n\exp\left(-\frac{iE_0t}{\hbar}+\frac{iE_0l_0^2n^2t}{2\hbar\sqrt{1-l_0^2n^2}}\right),\text{ for } n=1,2,\ldots$$

Hence

$$Y(x,t)=\sum_{n=1}^{\infty}\frac{1}{e^{nx}-1}\sum_{d|n}f_d\exp\left(-\frac{iE_0t}{\hbar}+\frac{iE_0l_0^2d^2t}{2\hbar\sqrt{1-l_0^2d^2}}\right)\mu(n/d).$$

Returning to Taylor series and assuming that $A_n=\chi_{n\le 1/l_0}A_n^*$, where $\chi_{n\le 1/l_0}$ is the characteristic function (1, if $1\le n\le[1/l_0]$ and 0 otherwise), we arive to

Theorem 2.

$$Y(x,t)=\exp\left(-\frac{iE_0t}{\hbar}\right)\sum_{n=1}^{1/l_0}f_n\exp\left(i\frac{E_0tl_0^2n^2}{2\hbar\sqrt{|l_0^2n^2-1|}}\right)e^{-nx}, \qquad (15)$$

where it is $Y(x,0)=\sum_{n=1}^{1/l_0}f_ne^{-nx}=f(x)$ and $l_0=\frac{\hbar}{m_0c}$ is the Compton's wavelength of the particle.

Notes.
i) The Plank's lenght l_0 has the following meaning: If $l_0 \geq 1$, then we can not interchage the double series in (13), as also the series becomes no convergent and equation (12) is no loger valid. For particles such as the neutrino's we have Compton length $l_0 = \frac{\hbar}{m_\nu c} \approx 0.000164$ cm . Electron have $l_0 = 3,86 \times 10^{-13}$ cm . This in literature is called Compton wavelength of electron.
ii) Taking the first term of the sum in (12) we arive easily to Schrodinger's equation

$$i\hbar \frac{\partial Y}{\partial t} = E_0 Y - \frac{\hbar^2}{2m_0} \frac{\partial^2 Y}{\partial x^2}. \tag{16}$$

Set now

$$C_{m,n} := \left(\begin{array}{c} m \\ n \end{array} \right). \tag{17}$$

$$Y(t;x,y,x) = \sum_{k,l,m=1}^{\infty} \frac{\sum_{d_1|k} \sum_{d_2|l} \sum_{d_3|m} A_{d_1 d_2 d_3}(t) \mu\left(k/d_1\right) \mu(l/d_2) \mu(m/d_3)}{(e^{kx}-1)(e^{ly}-1)(e^{mz}-1)}.$$

Hence

$$\partial_x^{2\nu+2} Y(t;x,y,x) =$$
$$\sum_{k,l,m=1}^{\infty} \frac{\sum_{d_1|k} \sum_{d_2|l} \sum_{d_3|m} A_{d_1 d_2 d_3}(t) d_1^{2\nu+2} \mu\left(k/d_1\right) \mu(l/d_2) \mu(m/d_3)}{(e^{kx}-1)(e^{ly}-1)(e^{mz}-1)},$$

$$\partial_y^{2\kappa+2} Y(t;x,y,x) =$$
$$\sum_{k,l,m=1}^{\infty} \frac{\sum_{d_1|k} \sum_{d_2|l} \sum_{d_3|m} A_{d_1 d_2 d_3}(t) d_2^{2\kappa+2} \mu\left(k/d_1\right) \mu(l/d_2) \mu(m/d_3)}{(e^{kx}-1)(e^{ly}-1)(e^{mz}-1)},$$

$$\partial_z^{\rho} Y(t;x,y,x) =$$
$$\sum_{k,l,m=1}^{\infty} \frac{\sum_{d_1|k} \sum_{d_2|l} \sum_{d_3|m} A_{d_1 d_2 d_3}(t) d_3^{2\rho+2} \mu\left(k/d_1\right) \mu(l/d_2) \mu(m/d_3)}{(e^{kx}-1)(e^{ly}-1)(e^{mz}-1)}.$$

Hence as in the one dimensional case we have

$$i\hbar \sum_{k,l,m=1}^{\infty} \frac{1}{(e^{kx}-1)(e^{ly}-1)(e^{mz}-1)} \sum_{d_1|k, d_2|l, d_3|m} A'_{d_1 d_2 d_3}(t) \mu(k/d_1) \mu(l/d_2) \mu(m/d_3) =$$

$$= E_0 \sum_{k,l,m=1}^{\infty} \frac{1}{(e^{kx}-1)(e^{ly}-1)(e^{mz}-1)} \sum_{d_1|k, d_2|l, d_3|m} A_{d_1 d_2 d_3}(t) \mu(k/d_1) \mu(l/d_2) \mu(m/d_3) \times$$
$$\times \left(1 - \frac{l_0^2 d_1^2}{2\sqrt{1 - l_0^2 d_1^2}} - \frac{l_0^2 d_2^2}{2\sqrt{1 - l_0^2 d_2^2}} - \frac{l_0^2 d_3^2}{2\sqrt{1 - l_0^2 d_3^2}} \right).$$

Hence it must be

$$\frac{i\hbar}{E_0}A'_{d_1d_2d_3}(t)=\left(1-\frac{l_0^2d_1^2}{2\sqrt{1-l_0^2d_1^2}}-\frac{l_0^2d_2^2}{2\sqrt{1-l_0^2d_2^2}}-\frac{l_0^2d_3^2}{2\sqrt{1-l_0^2d_3^2}}\right)A_{d_1d_2d_3}(t),$$

or

$$A_{d_1d_2d_3}(t)=$$

$$f_{d_1d_2d_3}\exp\left[\left(-i\frac{E_0t}{\hbar}+i\frac{E_0l_0^2d_1^2t}{2\hbar\sqrt{1-l_0^2d_1^2}}+i\frac{E_0l_0^2d_2^2t}{2\hbar\sqrt{1-l_0^2d_2^2}}+i\frac{E_0l_0^2d_3^2t}{2\hbar\sqrt{1-l_0^2d_3^2}}\right)\right].$$

Hence the equation

$$i\hbar\frac{\partial Y}{\partial t}=E_0Y+\frac{E_0}{2}\sum_{n=0}^{\infty}(-1)^{n+1}C_{-1/2,n}\left(\frac{\hbar}{m_0c}\right)^{2n+2}\left(\frac{\partial^{2n+2}Y}{\partial x^{2n+2}}+\frac{\partial^{2n+2}Y}{\partial y^{2n+2}}+\frac{\partial^{2n+2}Y}{\partial z^{2n+2}}\right),$$

admits solution

$$Y(t;x,y,z)=\exp\left(-\frac{iE_0t}{\hbar}\right)\times$$

$$\times\sum_{k,l,m=1}^{1/l_0}f_{klm}\exp\left[\frac{iE_0l_0^2t}{2\hbar}\left(\frac{k^2}{\sqrt{|l_0^2k^2-1|}}+\frac{l^2}{\sqrt{|l_0^2l^2-1|}}+\frac{m^2}{\sqrt{|l_0^2m^2-1|}}\right)\right]e^{-kx-ly-mz}.$$

QED

Theorem 3.
Assume the equation

$$i\hbar\frac{\partial Y}{\partial t}=\frac{E_0}{2}\sum_{n=0}^{\infty}(-1)^{n+1}C_{-1/2,n}\left(\frac{\hbar}{m_0c}\right)^{2n+2}\frac{\partial^{2n+2}Y}{\partial x^{2n+2}}+E_0Y+V(x)Y. \quad (17.1)$$

If the potential $V(x)$ is zero, then

$$Y(x,t)=\frac{\exp\left(-\frac{iE_0t}{\hbar}\right)}{2\pi}\int_{-1/l_0}^{1/l_0}\widehat{f}(\gamma)\exp\left(-\frac{iE_0t}{2\hbar}\frac{l_0^2\gamma^2}{\sqrt{1+l_0^2\gamma^2}}\right)e^{i\gamma x}d\gamma. \quad (18)$$

If the potential $V(x)$ is a function of x ($2\pi-$periodic) i.e.

$$V(x)=\sum_{k=0}^{\infty}a_kx^k=\sum_{k=1}^{\infty}c_ke^{-ikx},$$

then

$$i\hbar\frac{\partial}{\partial t}\widehat{Y}(\gamma,t)=\left(E_0+\frac{E_0}{2}\frac{l_0^2\gamma^2}{\sqrt{1+l_0^2\gamma^2}}+D\right)\widehat{Y}(\gamma,t), \quad (19)$$

where

$$D(.)=\sum_{k=0}^{\infty}a_ki^k\left(\frac{\partial}{\partial\gamma}\right)^k(.)=V\left(i\frac{\partial}{\partial\gamma}\right)(.) \quad (20)$$

and

$$\widehat{Y}(\gamma,t)=\sum_{m=-\infty}^{\infty}\Lambda_m\exp\left(\frac{-iC_mt}{\hbar}\right)\sum_{n=1}^{\infty}\{c_k-g_m(\log_2\gamma)\delta_{k,1}\}_2^{(-1)}(n)e^{in2^{\gamma}}. \tag{20.1}$$

For the symbol $\{c_n\}_2^{(-1)}(n)$ see relations (24.01),(24.1),(25) below.

Proof.
Assuming the equation

$$i\hbar\frac{\partial Y}{\partial t}=\frac{E_0}{2}\sum_{n=0}^{\infty}(-1)^{n+1}C_{-1/2,n}\left(\frac{\hbar}{m_0c}\right)^{2n+2}\frac{\partial^{2n+2}Y}{\partial x^{2n+2}}+E_0Y+V(x)Y, \tag{21}$$

where $Y=Y(x,t)$ and $V(x)=\sum_{k=0}^{\infty}a_kx^k$. Then taking the Fourier transform in both sides of (21), we get

$$i\hbar\frac{\partial\widehat{Y}(\gamma,t)}{\partial t}=\frac{E_0}{2}\sum_{n=0}^{\infty}(-1)^{n+1}C_{-1/2,n}l_0^{2n+2}(i\gamma)^{2n+2}\widehat{Y}(\gamma,t)+E_0\widehat{Y}(\gamma,t)+$$

$$+\sum_{n=0}^{\infty}a_ni^n\int_{-\infty}^{\infty}Y(x,t)(-ix)^ne^{-ix\gamma}dx,\ |\gamma|\leq 1/l_0$$

and $\widehat{Y}(\gamma,t)=\int_{-\infty}^{+\infty}Y(x,t)e^{-ix\gamma}dx$. Hence

$$i\hbar\frac{\partial\widehat{Y}(\gamma,t)}{\partial t}=\left(E_0+\frac{E_0}{2}\frac{l_0^2\gamma^2}{\sqrt{1+l_0^2\gamma^2}}\right)\widehat{Y}(\gamma,t)+\sum_{n=0}^{\infty}a_ni^n\frac{\partial^n\widehat{Y}(\gamma,t)}{\partial\gamma^n}. \tag{21.1}$$

Hence if we denote the differential operator

$$D(.)=\sum_{n=0}^{\infty}a_ni^n\left(\frac{\partial}{\partial\gamma}\right)^n(.)=V\left(i\frac{\partial}{\partial\gamma}\right)(.), \tag{22}$$

we get the result.
For to solve (21) we write $\widehat{Y}(\gamma,t)=A_n(\gamma)B_n(t)$. Then

$$i\hbar A_n(\gamma)B_n'(t)=\left(E_0+\frac{E_0}{2}\frac{l_0^2\gamma^2}{\sqrt{1+l_0^2\gamma^2}}\right)A_n(\gamma)B_n(t)+B_n(t)DA_n(\gamma).$$

Hence

$$i\hbar\frac{B_n'(t)}{B_n(t)}=C_n\text{ and }DA_n(\gamma)=\left(C_n-E_0-\frac{E_0}{2}\frac{l_0^2\gamma^2}{\sqrt{1+l_0^2\gamma^2}}\right)A_n(\gamma).$$

Hence

$$B_n(t)=\Lambda_n\exp\left(\frac{-iC_n}{\hbar}t\right) \tag{22.1}$$

and

$$\sum_{k=0}^{\infty} a_k i^k \left(\frac{\partial}{\partial\gamma}\right)^k A_n(\gamma) = \left(C_n - E_0 - \frac{E_0}{2}\frac{l_0^2\gamma^2}{\sqrt{1+l_0^2\gamma^2}}\right) A_n(\gamma) \Leftrightarrow$$

$$V\left(i\frac{\partial}{\partial\gamma}\right) A_n(\gamma) = A_n(\gamma) g_n(\gamma), \qquad (eq)$$

where

$$g_n(\gamma) = C_n - E_0 - \frac{E_0}{2}\frac{l_0^2\gamma^2}{\sqrt{1+l_0^2\gamma^2}}. \qquad (22.2)$$

This equation (eq) is hold for general potential $V(x)$. Assuming that

$$V(x) = \sum_{k=1}^{\infty} c_k e^{-ikx},$$

we have

$$V\left(i\frac{\partial}{\partial\gamma}\right) A_n(\gamma) = \sum_{k=1}^{\infty} c_k \exp\left(k\frac{\partial}{\partial\gamma}\right) A_n(\gamma) = \sum_{k=1}^{\infty} c_k A_n(\gamma+k).$$

Hence it is intersesting to solve the equation

$$\sum_{k=1}^{\infty} c_k A_m(\gamma+k) = g_m(\gamma) A_m(\gamma),\ m \in \mathbf{Z}. \qquad (23)$$

We solve the equation:

$$\sum_{k=1}^{\infty} c_k A_m(\gamma+k) = g_m(\gamma) A_m(\gamma) + 1. \qquad (24)$$

Setting $A_m^*(\gamma) = \sum_{l=1}^{\infty} b_{m,l} e^{il\gamma}$ and $A_m(\gamma) = A_m^*\left(2^\gamma\right)$, we get

$$\sum_{k=1}^{\infty} c_k A_m\left(\log_2(\gamma)+k\right) = \sum_{k=1}^{\infty} c_k A_m^*\left(\gamma 2^k\right) =$$

$$= \sum_{k,l=1}^{\infty} c_k b_{m,l} e^{i2^k l\gamma} = \sum_{n=1}^{\infty} e^{in\gamma} \sum_{2^k|n} c_k b_{m,n/2^k}.$$

Hence

$$\sum_{n=1}^{\infty} e^{in\gamma} \sum_{2^k|n} c_k b_{m,n/2^k} = g_m(\log_2\gamma) \sum_{n=1}^{\infty} b_{m,n} e^{in\gamma} + 1.$$

Hence

$$\sum_{2^k|n} b_{m,n/2^k}\left(c_k - g_m(\log_2\gamma)\delta_{k,0}\right) = \delta_{n,1}, \qquad (24.01)$$

where $\delta_{n,1}$ is 1 when $n = 1$ and 0 otherwise and $\delta_{n,0}$ is 1 when $n = 0$ and 0 otherwise. Hence using the deffinition of the inverse of an arithmetical function, in the sense

$$\sum_{2^k|n} a_k b_{n/2^k} = \delta_{n,1} \Leftrightarrow a_n = \{b_k\}_2^{(-1)}(n), \tag{24.1}$$

we get

$$b_{m,n} = b_{m,n}(\gamma) = \{c_k - g_m(\log_2 \gamma)\,\delta_{k,0}\}_2^{(-1)}(n). \tag{25}$$

Setting $A_m(\gamma + k) = B_m(\gamma + k) - h_k$, with $h_0 = 0$ in (24) we get

$$\sum_{k=1}^{\infty} c_k\,(B_m(\gamma + k) - h_k) = g_m(\gamma)\,(B_m(\gamma) - h_0) + 1 \Leftrightarrow$$

$$\sum_{k=1}^{\infty} c_k B_m(\gamma + k) - \sum_{k=1}^{\infty} c_k h_k = g_m(\gamma) B_m(\gamma) + 1$$

If we choose h_k such that

$$\sum_{k=1}^{\infty} c_k h_k = -1, \tag{26}$$

we get the equivalence of (23) and (24). Hence

$$A_m(\gamma) = \sum_{n=1}^{\infty} \{c_k - g_m(\log_2 \gamma)\delta_{k,0}\}_2^{(-1)}(n) e^{in2^\gamma}. \tag{27}$$

Then then solution $\widehat{Y}(\gamma, t)$ of (21.1), is

$$\widehat{Y}(\gamma, t) = \sum_{m=-\infty}^{\infty} \Lambda_m \exp\left(\frac{-iC_m t}{\hbar}\right) \sum_{n=1}^{\infty} \{c_k - g_m(\log_2 \gamma)\delta_{k,0}\}_2^{(-1)}(n) e^{in2^\gamma}. \tag{28}$$

QED

Hence we also get the next

Theorem 4.1
Assuming the equation

$$y''(x) = V(x)y(x), \tag{29}$$

where $V(x) = \sum_{k=1}^{\infty} c_k e^{-ikx}$, we have

$$\widehat{y}(\gamma) = \sum_{n=1}^{\infty} \{c_k + \log_2(\gamma)^2 \delta_{k,0}\}_2^{(-1)}(n) e^{in2^\gamma}, \tag{30}$$

where $\{b_k\}_2^{(-1)}(n)$ is the arthmetic inverse of b_k in the sense of (24.1).

Proof.
We have

$$\sum_{k=0}^{\infty} b_k y^{(k)}(x) = V(x)y(x) \Leftrightarrow$$

$$\sum_{k=0}^{\infty} b_k \int_{-\infty}^{\infty} y^{(k)}(x) e^{-ix\gamma} dx = \int_{-\infty}^{\infty} y(x) \left(\sum_{n=0}^{\infty} a_n x^n \right) e^{-ix\gamma} dx \Leftrightarrow$$

$$\sum_{k=0}^{\infty} b_k (i\gamma)^k \int_{-\infty}^{\infty} y(x) e^{-ix\gamma} dx = \sum_{n=0}^{\infty} a_n \int_{-\infty}^{\infty} y(x) x^n e^{-ix\gamma} dx \Leftrightarrow$$

$$B(i\gamma)\widehat{y}(\gamma) = \sum_{n=0}^{\infty} a_n i^n \frac{d^n}{d\gamma^n} \left(\int_{-\infty}^{\infty} y(x) e^{-ix\gamma} dx \right) \Leftrightarrow$$

$$B(i\gamma)\widehat{y}(\gamma) = \sum_{n=0}^{\infty} a_n i^n \frac{d^n}{d\gamma^n} \widehat{y}(\gamma) \Leftrightarrow$$

$$B(i\gamma)\widehat{y}(\gamma) = V\left(i\frac{d}{d\gamma} \right) \widehat{y}(\gamma). \tag{30.01}$$

Assume now that $V(x) = \sum_{n=-\infty}^{\infty} c_n e^{-inx}$.

$$B(i\gamma)\widehat{y}(\gamma) = \sum_{n=-\infty}^{\infty} c_n \exp\left(n\frac{d}{d\gamma} \right) \widehat{y}(\gamma). \tag{30.02}$$

Or equivalent

$$B(i\gamma)\widehat{y}(\gamma) = \sum_{n=-\infty}^{\infty} c_n \widehat{y}(\gamma + n). \tag{30.03}$$

Hence working as above we get for $B(x) = x^2$ and $c_n = 0$, if $n = 0, -1, -2, \ldots$.

$$\widehat{y}(\gamma) = \sum_{n=1}^{\infty} \{c_k + \log_2(\gamma)^2 \delta_{k,0}\}_2^{(-1)}(n) e^{in2^\gamma}.$$

QED.

The equation

$$\sum_{n=0}^{\infty} g_n (-i)^n y^{(n)}(x) = V(x)y(x), \tag{30.1}$$

where $g_n = \frac{g^{(n)}(0)}{n!}$, can be written in the form

$$\sum_{n=0}^{\infty} g_n \gamma^n \widehat{y}(\gamma) = V\left(i\frac{d}{d\gamma} \right) \widehat{y}(\gamma).$$

Hence

$$V\left(i\frac{d}{d\gamma}\right)\widehat{y}(\gamma) = g(\gamma)\widehat{y}(\gamma).$$

which have solution

$$\sum_{n=-\infty}^{\infty} c_n \widehat{y}(\gamma+n) = g(\gamma)\widehat{y}(\gamma).$$

Hence one solution is

$$\widehat{y}(\gamma) = \frac{1}{g(\gamma)} \sum_{n_1} \frac{c_{n_1}}{g(\gamma+n_1)} \sum_{n_2} \frac{c_{n_2}}{g(\gamma+n_1+n_2)} \sum_{n_3} \frac{c_{n_3}}{g(\gamma+n_1+n_2+n_3)} \cdots.$$

Hence holds the next

Theorem 4.2
The equation

$$\sum_{n=0}^{\infty} b_n \frac{d^n}{dx^n} y(x) = V(x)y(x),$$

where

$$V(x) = \sum_{n=-\infty}^{\infty} c_n e^{-inx}.$$

Have solution

$$\widehat{y}(\gamma) = \frac{1}{B(i\gamma)} \sum_{n_1} \frac{c_{n_1}}{B(i\gamma+in_1)} \sum_{n_2} \frac{c_{n_2}}{B(i\gamma+in_1+in_2)} \sum_{n_3} \frac{c_{n_3}}{B(i\gamma+in_1+in_2+in_3)} \cdots,$$

where

$$B(x) = \sum_{n=0}^{\infty} b_n x^n$$

and

$$\widehat{y}(\gamma) = \int_{-\infty}^{\infty} y(t) e^{-it\gamma} dt,$$

is the Fourier transform of $y(x)$.

Set $A_n(z)$ and $\theta_l(z)$ such that

$$\theta_l(z) = \sum_{n=-\infty}^{\infty} A_n(z)\theta_l(z+n) \qquad (eq1)$$

and

$$\sum_{k=-\infty}^{\infty} c_k A_{n-k}(z+k) = \delta_{n,0} g(z), \qquad (eq2)$$

we have

$$\sum_{k=-\infty}^{\infty} c_k \theta_l(z+k) = \sum_{k=-\infty}^{\infty} c_k \sum_{n=-\infty}^{\infty} A_n(z+k)\theta_l(z+n+k) =$$

$$= \sum_{k=-\infty}^{\infty} c_k \sum_{n=-\infty}^{\infty} A_{n-k}(z+k)\theta_l(z+n) =$$

$$= \sum_{n=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} c_k A_{n-k}(z+k) \right) \theta_l(z+n) =$$

$$= \sum_{n=-\infty}^{\infty} \delta_{n,0} g(z)\theta_l(z+n) = g(z)\theta_l(z).$$

Hence we search for functions $\theta_l(z)$ and $A_n(z)$ such that $(eq1)$ and $(eq2)$ hold. But equation $(eq2)$ can be written as

$$\sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} c_k A_{n-k}(z+k) e^{inx} = g(z) \sum_{n=-\infty}^{\infty} \delta_{n,0} e^{inx}. \tag{31}$$

Hence

$$\sum_{n,k=-\infty}^{\infty} c_k A_n(z+k) e^{inx} e^{ikx} = g(z).$$

Taking in both sides the Fourier transform we find

$$\sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} c_k \widehat{A}_n(\gamma) e^{i\gamma k} e^{inx} e^{ikx} = \widehat{g}(\gamma) \Leftrightarrow$$

$$\overline{V}(x+\gamma) \sum_{n=-\infty}^{\infty} \widehat{A}_n(\gamma) e^{inx} = \widehat{g}(\gamma) \Leftrightarrow$$

$$\sum_{n=-\infty}^{\infty} \widehat{A}_n(\gamma) e^{inx} = \frac{\widehat{g}(\gamma)}{\overline{V}(x+\gamma)}.$$

Hence

$$\widehat{A}_n(\gamma) = \widehat{g}(\gamma) \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{-itn}}{\overline{V}(t+\gamma)} dt \Rightarrow$$

$$A_n(x) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \widehat{g}(\gamma) \left(\int_0^{2\pi} \frac{e^{-itn}}{\overline{V}(t+\gamma)} dt \right) e^{ix\gamma} d\gamma.$$

Hence knowing $A_n(x)$, then from $(eq1)$ we can find $\theta_l(z)$ by iteration i.e.

$$\theta_l(z) = \sum_{n_1} A_{n_1}(z) \sum_{n_2} A_{n_2}(z+n_1) \sum_{n_3} A_{n_3}(z+n_1+n_2) \dots$$

and we have the next

Theorem 5.
I. Assume that

$$V(x) = \sum_{n=-\infty}^{\infty} c_n e^{-inx} \tag{32}$$

and $g(x)$ is a given function. Then equation

$$V\left(i\frac{d}{dw}\right)\widehat{y}(w) = \sum_{n=-\infty}^{\infty} c_n \widehat{y}(w+n) = g(w)\widehat{y}(w), \tag{33}$$

have solution

$$\widehat{y}(w) = \sum_{n_1} A_{n_1}(w) \sum_{n_2} A_{n_2}(w+n_1) \sum_{n_3} A_{n_3}(w+n_1+n_2)\ldots, \tag{34}$$

where $A_n(x)$ is such that

$$A_n(x) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty} \widehat{g}(\gamma)\left(\int_0^{2\pi} \frac{e^{-itn}}{\overline{V}(t+\gamma)}dt\right)e^{ix\gamma}d\gamma. \tag{35}$$

II. When

$$V(x) = \sum_{n=-\infty}^{\infty} c_n e^{-inx} \tag{36}$$

and $g(x) = -x^2$, then equation

$$y''(x) = V(x)y(x), \tag{37}$$

have solution $y(x)$ such that its Fourier transform is

$$\widehat{y}(w) = \sum_{n_1} A_{n_1}(w) \sum_{n_2} A_{n_2}(w+n_1) \sum_{n_3} A_{n_3}(w+n_1+n_2)\ldots, \tag{38}$$

where

$$A_n(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta''(\gamma)\left(\int_0^{2\pi} \frac{e^{-itn}}{\overline{V}(t+\gamma)}dt\right)e^{ix\gamma}d\gamma. \tag{39}$$

Remarks.
1) The Fourier transform of $f(x)$ is defined as

$$\widehat{f}(x) = \int_{-\infty}^{\infty} f(t)e^{-itx}dt \Leftrightarrow f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(\gamma)e^{i\gamma x}d\gamma. \tag{40}$$

2) The function $\delta(x)$ is the Dirac delta function, while $\delta_{k,\nu}$ is the Kronecker's delta symbol, i.e. $\delta_{k,\nu} = 1$ if $k = \nu$ and $\delta_{k,\nu} = 0$ if $k \neq \nu$.
3) From

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \delta''(w)F(w)e^{iwx}dw = \frac{1}{2\pi}\left(-x^2F_n(0) + 2ixF_n'(0) + F_n''(0)\right),$$

we get

$$A_n(x) = \frac{1}{2\pi}\int_0^{2\pi}\left(-\frac{x^2}{\overline{V}(t)} - \frac{2ix\overline{V}'(t)}{\overline{V}(t)^2} + \frac{2\overline{V}'(t)^2}{\overline{V}(t)^3} - \frac{\overline{V}''(t)}{\overline{V}(t)^2}\right)e^{-itn}dt.$$

Example 1.
Assume $V(x) = 1$, then $c_n = \delta_{n,0}$. Hence from (39) we get

$$A_n(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\delta''(\gamma)2\pi\delta_{n,0}e^{i\gamma x}d\gamma = -x^2\delta_{n,0}. \qquad (a)$$

Hence $\theta(z) = -z^2\theta(z)$. Hence one leads to $\theta(z) = \delta(z-i)$. Hence $\widehat{y}(z) = \delta(z-i)$, which gives $y(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\delta(z-i)e^{-izx}dz = \frac{e^x}{2\pi}$: (a). Hence indeed one solution of $y''(x) = V(x)y(x)$, with $V(x) = 1$ is $y(x) = \frac{e^x}{2\pi}$. However one observes that working with functions of $L_2(\mathbf{R})$ is better than working with distributions. In this example the Fourier integral (a) is over the real line and the distribution is $\delta(x-i)$. Hence $x-i$ is never 0 at the real line. However many books use this Fourier pair (actually we need here a generalization of the Fourier transform or the notion of distribution functions).

Example 2.
Assume $V(x) = e^{-ix}$. Hence $c_n = \delta_{n,1}$. In this case $\overline{V}(x) = e^{ix}$ and

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{-itn}}{e^{i(t+\gamma)}}dt = 2\pi e^{-i\gamma}\delta_{n,-1}.$$

Hence

$$A_n(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\delta''(\gamma)2\pi e^{-i\gamma}\delta_{n,-1}e^{i\gamma x}d\gamma = -(x-1)^2\delta_{n,-1}.$$

Hence from

$$\sum_{n=-\infty}^{\infty}c_n\theta(z+n) = -z^2\theta(z) \Leftrightarrow \theta(z+1) = -z^2\theta(z),$$

we get easily

$$\theta(z) = e^{-i\pi z}\Gamma(z)^2.$$

Therefore, there exists a solution of $y''(x) = e^{-ix}y(x)$ such that its Fourier transform is

$$\widehat{y}(w) = e^{-i\pi w}\Gamma(w)^2.$$

However this evaluation is not in $L_2(\mathbf{R})$.

Remarks.

If we assume that $y(it) = (Mg)(t)$, where $(Mg)(t)$ is the Mellin transform of $g(t)$, then the Fourier transform of a function is

$$\widehat{y}(x) = \int_{-\infty}^{+\infty} y(t)e^{-itx}dt = \int_{-\infty/i}^{+\infty/i} y(it)e^{-i(it)x}idt =$$

$$= -i\int_{-i\infty}^{+i\infty} y(it)e^{tx}dt = -i2\pi i\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} Mg(t)\left(e^{-x}\right)^{-t}dt =$$

$$= 2\pi g\left(e^{-x}\right).$$

Hence if $\widehat{y}\,(x) = 2\pi g\,(e^{-x})$, then $y\,(ix) = (Mg)(x)$.

Example 3.
Assume that

$$g(x) = \frac{1}{x^2+1},$$

then

$$(Mg)(s) = \frac{\pi}{2}\csc\left(\frac{\pi s}{2}\right),\ 0 < Re(s) < 2.$$

Thus

$$\widehat{y}(\gamma) = \frac{2\pi}{1+e^{-2\gamma}}$$

and

$$y(x) = \frac{\pi}{2}\csc\left(\frac{-i\pi x}{2}\right).$$

3 Some notes on $y''(x) = V(x)y(x)$.

Assume now that

$$\frac{1}{V(x)} = \sum_{n=-\infty}^{\infty} c_n^* e^{-inx}. \tag{41}$$

Then

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{-itn}}{\overline{V}(t+\gamma)}dt = \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{k=-\infty}^{\infty} c_k^* e^{-itn}e^{ik(t+\gamma)}\right)dt =$$

$$= \sum_{k=-\infty}^{\infty} c_k^*\left(\frac{1}{2\pi}\int_0^{2\pi} e^{it(k-n)}dt\right)e^{ik\gamma} = \sum_{k=-\infty}^{\infty} c_k^*\delta_{k,n}e^{ik\gamma} = c_n^* e^{in\gamma}$$

and

$$A_n(x) = \int_{-\infty}^{\infty} \delta''(\gamma)c_n^* e^{in\gamma}e^{ix\gamma}d\gamma = -(n+x)^2 c_n^*.$$

Theorem 5.1
If we denote c_n^* such that

$$\frac{1}{V(x)} = \sum_{n=-\infty}^{\infty} c_n^* e^{-inx}, \tag{41.1}$$

then
I. If $g(x) = -x^2$, then

$$A_n(x) = -(n+x)^2 c_n^*. \tag{41.2}$$

II. If $g(x)$ is any analytic function, then

$$A_n(x) = g(n+x) c_n^* \tag{41.3}$$

and the solution of

$$\sum_n c_n \widehat{y}(z+n) = g(z)\widehat{y}(z), \tag{41.4}$$

is

$$\widehat{y}(z) = \lim_{N\to\infty} \sum_{n_1}\sum_{n_2}\dots\sum_{n_N} g(n_1+z)g(n_1+n_2+z)\dots$$
$$\dots g(n_1+n_2+\dots+n_N+z)c_{n_1}^* c_{n_2}^* \dots c_{n_N}^*. \tag{41.5}$$

Using (38) we get

$$\widehat{y}(w) = \lim_{N\to\infty} \sum_{n_1}\sum_{n_2}\sum_{n_3}\dots\sum_{n_N} (-1)^N (n_1+w)^2 (n_2+n_1+w)^2 (n_3+n_2+n_1+w)^2 \dots$$
$$\dots (n_N+n_{N-1}+\dots+n_2+n_1+w)^2 c_{n_1}^* c_{n_2}^* \dots c_{n_N}^* =$$
$$= \lim_{N\to\infty} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty} \delta''(t_1)\delta''(t_2)\dots\delta''(t_N)\times$$
$$\times \frac{1}{V(t_1+t_2+\dots+t_N)} \frac{1}{V(t_2+t_3+\dots+t_N)} \dots \frac{1}{V(t_{N-1}+t_N)} \frac{1}{V(t_N)} \times$$
$$\times e^{-it_1 w} e^{-it_2 w} \dots e^{-it_N w} dt_1 dt_2 \dots dt_N \Rightarrow$$
$$\widehat{y}(w) = \lim_{N\to\infty} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty} \delta''(t_1)\delta''(t_2)\dots\delta''(t_N)\times$$
$$\times \frac{1}{V(t_1)} \frac{1}{V(t_1+t_2)} \dots \frac{1}{V(t_1+t_2+\dots+t_N)} \times$$
$$\times e^{-it_1 w} e^{-it_2 w} \dots e^{-it_N w} dt_1 dt_2 \dots dt_N =$$
$$= \lim_{N\to\infty} \frac{\partial^2}{\partial h_1^2} \frac{\partial^2}{\partial h_2^2} \dots \frac{\partial^2}{\partial h_N^2} \left(f_N(h_1, h_2, \dots, h_N) e^{-ih_1 w - ih_2 w - \dots - ih_N w} \right)_{h_1=h_2=\dots=h_N=0},$$

where

$$f_N(t_1,t_2,\ldots,t_N)=\frac{1}{V(t_1)V(t_1+t_2)\ldots V(t_1+t_2+\ldots+t_N)}.$$

Also if we denote

$$\epsilon_{n_1,n_2,\ldots,n_N}:=2^{n_1 mod 2+n_2 mod 2+\ldots+n_N mod 2}.$$

and

$$f_N^{(n_1,n_2,\ldots,n_N)}=\left(\frac{\partial^{n_1+n_2+\ldots+n_N}}{\partial t_1^{n_1}\partial t_2^{n_2}\ldots\partial t_N^{n_N}}f_N(t_1,t_2,\ldots,t_N)\right)_{t_1=t_2=\ldots=t_N=0},$$

then

$$\widehat{y}(w)=\lim_{N\to\infty}\sum_{n_1,n_2,\ldots,n_N=0}^{2}(-1)^{n_1+n_2+\ldots+n_N}\epsilon_{n_1,n_2,\ldots,n_N}f_N^{(n_1,n_2,\ldots,n_N)}\cdot(iw)^{2N-n_1-n_2-\ldots-n_N}.$$

Hence

$$\widehat{y}\,(w)=\lim_{N\to\infty}(iw)^{2N}\sum_{n_1,n_2,\ldots,n_N=0}^{2}\epsilon_{n_1,n_2,\ldots,n_N}f_N^{(n_1,n_2,\ldots,n_N)}\cdot(-iw)^{-n_1-n_2-\ldots-n_N}.$$

$$\widehat{y}(w)=\lim_{N\to\infty}\sum_{k=0}^{2N}\left(\sum_{\substack{0\le n_1,n_2,\ldots,n_N\le 2\\ n_1+n_2+\ldots+n_N=k}}\epsilon_{n_1,n_2,\ldots,n_N}f_N^{(n_1,n_2,\ldots,n_N)}\right)(-iw)^{2N-k}.$$

$$\widehat{y}(w)=\lim_{N\to\infty}\sum_{k=0}^{2N}\left(\sum_{\substack{0\le n_1,n_2,\ldots,n_N\le 2\\ n_1+n_2+\ldots+n_N=2N-k}}\epsilon_{n_1,n_2,\ldots,n_N}f_N^{(n_1,n_2,\ldots,n_N)}\right)(-iw)^{k}. \tag{42}$$

But

$$\left(\frac{\partial^{n_1+n_2+\ldots+n_N}}{\partial x_1^{n_1}\partial x_2^{n_2}\ldots\partial x_N^{n_N}}f_N(P)\right)_{P=(0,0,\ldots,0)}=\frac{1}{(2\pi)^N}\int_{\mathbf{R}}\int_{\mathbf{R}}\ldots\int_{\mathbf{R}}\widehat{f}_N(\gamma_1,\gamma_2,\ldots,\gamma_N)\times$$

$$\times(i\gamma_1)^{n_1}(i\gamma_2)^{n_2}\ldots(i\gamma_N)^{n_N}d\gamma_1 d\gamma_2\ldots d\gamma_N. \tag{43}$$

Hence

$$\widehat{y}(w)=\lim_{N\to\infty}\sum_{k=0}^{2N}\left(\sum_{\substack{0\le n_1,n_2,\ldots,n_N\le 2\\ n_1+n_2+\ldots+n_N=2N-k}}\epsilon_{n_1,n_2,\ldots,n_N}f_N^{(n_1,n_2,\ldots,n_N)}\right)(-iw)^{k}=$$

$$= \lim_{N\to\infty} \sum_{k=0}^{2N} \left(\sum_{\substack{0 \le n_1, n_2, \ldots, n_N \le 2 \\ n_1 + n_2 + \ldots + n_N = k}} \epsilon_{n_1, n_2, \ldots, n_N} f_N^{(2-n_1, 2-n_2, \ldots, 2-n_N)} \right) (-iw)^k =$$

$$= \lim_{N\to\infty} \frac{1}{(2\pi)^N} \int_{\mathbf{R}^N} \widehat{f}_N(\gamma_1, \gamma_2, \ldots, \gamma_N) \times$$

$$\times \sum_{k=0}^{2N} \left(\sum_{\substack{0 \le n_1, n_2, \ldots, n_N \le 2 \\ n_1 + n_2 + \ldots + n_N = 2N-k}} \epsilon_{n_1, n_2, \ldots, n_N} (i\gamma_1)^{n_1} (i\gamma_2)^{n_2} \ldots (i\gamma_N)^{n_N} \right) (-iw)^{2N-k} \times$$

$$\times d\gamma_1 d\gamma_2 \ldots d\gamma_N.$$

Hence we get the next

Theorem 6.
Assume the equation

$$y''(x) = V(x)y(x). \tag{44}$$

Then

$$\widehat{y}(w) = \lim_{N\to\infty} \left(\frac{\partial^2}{\partial h_1^2} \frac{\partial^2}{\partial h_2^2} \cdots \frac{\partial^2}{\partial h_N^2} \frac{e^{-ih_1 w - ih_2 w - \ldots - ih_N w}}{V(h_1)V(h_1+h_2)\ldots V(h_1+h_2+\ldots+h_N)} \right)_{h_1=h_2=\ldots=h_N=0}. \tag{45}$$

Also if we set

$$\widehat{y}_N(w) := \frac{1}{(2\pi)^N} \int_{\mathbf{R}^N} \widehat{f}_N(\gamma_1, \gamma_2, \ldots, \gamma_N)(\gamma_1 - w)^2 (\gamma_2 - w)^2 \ldots (\gamma_N - w)^2 d\gamma_1 d\gamma_2 \ldots d\gamma_N, \tag{46}$$

then

$$\widehat{y}(w) = \lim_{N\to\infty} \widehat{y}_N(w) \tag{47}$$

and

$$\widehat{y}(w) = \lim_{N\to\infty} \sum_{k=0}^{2N} \left(\sum_{\substack{0 \le n_1, n_2, \ldots, n_N \le 2 \\ n_1 + n_2 + \ldots + n_N = k}} \epsilon_{n_1, n_2, \ldots, n_N} f_N^{(2-n_1, 2-n_2, \ldots, 2-n_N)} \right) (-iw)^k. \tag{47.1}$$

However

$$\frac{d^{2N}}{dw^{2N}} \widehat{y}_N(w) = \text{costant}.$$

Hence exists costants $C_{N,k}$, such that

$$\widehat{y}_N(w) = \sum_{k=0}^{2N} C_{N,k} w^k.$$

Set

$$Y(x) := \frac{1}{V(x)}.$$

We have

$$\widehat{f}_N(\gamma_1, \gamma_2, \ldots, \gamma_N) = \int_{\mathbf{R}^N} Y(t_1)Y(t_1+t_2)\ldots Y(t_1+t_2+\ldots+t_N)e^{-it_1\gamma_1-it_2\gamma_2-it_N\gamma_N}dt_1dt_2\ldots dt_N. \tag{47.2}$$

Also

$$\widehat{Y}(x) = \int_{\mathbf{R}} Y(t)e^{-itx}dt.$$

Then from (46) we get

$$\widehat{y}_N(w) = \frac{1}{(2\pi)^N}\int_{\mathbf{R}^N} \widehat{Y}(t_1-t_2)\widehat{Y}(t_2-t_3)\ldots\widehat{Y}(t_{N-1}-t_N)\widehat{Y}(t_N+w)t_1^2t_2^2\ldots t_N^2dt_1dt_2\ldots dt_N.$$

Hence

$$\widehat{y}_N(w) = \frac{1}{(2\pi)^N}\int_{\mathbf{R}^N} \widehat{Y}(t_1+w)\widehat{Y}(t_2-t_1)\widehat{Y}(t_3-t_2)\ldots\widehat{Y}(t_{N-1}-t_{N-2})\widehat{Y}(t_N-t_{N-1})\times$$

$$\times t_1^2t_2^2\ldots t_N^2dt_1dt_2\ldots dt_N =$$

$$= \frac{1}{(2\pi)^N}\int_{\mathbf{R}^N} \widehat{Y}(t_1+w)\widehat{Y}(t_2-t_1)\ldots\widehat{Y}(t_{N-1}-t_{N-2})\widehat{Y}(t_N-t_{N-1})t_1^2t_2^2\ldots t_N^2dt_1dt_2\ldots dt_N.$$

Hence

$$\frac{1}{2\pi}\int_{\mathbf{R}} \widehat{y}_N(w)e^{iww'}dw = \frac{1}{(2\pi)^N}\int_{\mathbf{R}^N} \widehat{Y}(t_1-t_2)\widehat{Y}(t_2-t_3)\ldots\widehat{Y}(t_{N-1}-t_N)\times$$

$$\times\left(\frac{1}{2\pi}\int_{\mathbf{R}} \widehat{Y}(t_N+w)e^{iww'}dw\right)t_1^2t_2^2\ldots t_N^2dt_1dt_2\ldots dt_N.$$

Hence

$$y_N(w') = \frac{1}{(2\pi)^N}\int_{\mathbf{R}^N} \widehat{Y}(t_1-t_2)\widehat{Y}(t_2-t_3)\ldots\widehat{Y}(t_{N-1}-t_N)\times$$

$$\times\left(\frac{1}{2\pi}\int_{\mathbf{R}} \widehat{Y}(w)e^{i(w-t_N)w'}dw\right)t_1^2t_2^2\ldots t_N^2dt_1dt_2\ldots dt_N.$$

Hence

$$y_N(w') = \frac{1}{(2\pi)^N}\int_{\mathbf{R}^N} \widehat{Y}(t_1-t_2)\widehat{Y}(t_2-t_3)\ldots\widehat{Y}(t_{N-1}-t_N)\times$$

$$\times Y(w')e^{-it_Nw'}t_1^2t_2^2\ldots t_N^2dt_1dt_2\ldots dt_N \Rightarrow$$

$$y_N(w') = Y(w')\frac{1}{(2\pi)^N}\int_{\mathbf{R}^N} \widehat{Y}(t_1-t_2)\widehat{Y}(t_2-t_3)\ldots\widehat{Y}(t_{N-1}-t_N)e^{-it_Nw'}t_1^2t_2^2\ldots t_N^2dt_1dt_2\ldots dt_N.$$

Hence

$$\frac{y_N(w)}{Y(w)} = \frac{1}{(2\pi)^N}\int_{\mathbf{R}^N} \widehat{Y}(t_1-t_2)\widehat{Y}(t_2-t_3)\dots\widehat{Y}(t_{N-1}-t_N)e^{-it_N w}t_1^2t_2^2\dots t_N^2 dt_1dt_2\dots dt_N.$$

Hence

$$\frac{y_N(w)}{Y(w)} = \frac{1}{(2\pi)^N}\int_{\mathbf{R}^N} \widehat{Y}(t_2-t_1)\widehat{Y}(t_3-t_2)\dots\widehat{Y}(t_N-t_{N-1})e^{-it_1 w}t_1^2t_2^2\dots t_N^2 dt_1dt_2\dots dt_N =$$

$$= \frac{1}{(2\pi)^N}\int_{\mathbf{R}}\int_{\mathbf{R}}\dots\int_{\mathbf{R}} e^{i(t_2-t_1)w}\widehat{Y}(t_2-t_1)t_1^2dt_1e^{i(t_3-t_2)w}\widehat{Y}(t_3-t_2)t_2^2dt_2\dots\times$$
$$\times e^{i(t_N-t_{N-1})w}\widehat{Y}(t_N-t_{N-1})t_{N-1}^2dt_{N-1}e^{-it_N w}t_N^2dt_N.$$

Hence if we define the operator $F_n(x,w)$ such that

$$F_{n+1}(x,w) = \frac{1}{2\pi}\int_{\mathbf{R}} F_n(t,w)e^{(x-t)iw}\widehat{Y}(x-t)t^2dt, \tag{47.2}$$

then

$$\widehat{F}_{N+1}(\gamma,w) = -\frac{1}{2\pi}e^{i\gamma w}\widehat{Y}(\gamma)\frac{d^2}{d\gamma^2}\widehat{F}_N(\gamma,w)$$

and

$$\frac{y_N(w)}{Y(w)} = \left[\int_{\mathbf{R}} F_N(t,w)e^{-it\gamma}t^2dt\right]_{\gamma=w} = -\left[\frac{d^2}{d\gamma^2}\widehat{F}_N(\gamma,w)\right]_{\gamma=w}.$$

Hence

$$\frac{y_\infty(w)}{Y(w)} = \frac{y(w)}{Y(w)} = -\frac{d^2}{d\gamma^2}\left[-\frac{1}{2\pi}e^{i\gamma w}\widehat{Y}(\gamma)\frac{d^2}{d\gamma^2}\left[-\frac{1}{2\pi}e^{i\gamma w}\widehat{Y}(\gamma)\frac{d^2}{d\gamma^2}[\dots]\right]\right]_{\gamma=w}$$

and we get the next

Theorem 7.
Assume the equation

$$y''(x) = V(x)y(x), \tag{48}$$

where the potential $V(x)$ is of the form

$$V(x) = \sum_{n=-\infty}^{\infty} c_n e^{-inx}. \tag{49}$$

If

$$Y(x) := \frac{1}{V(x)}, \tag{50}$$

then it holds

$$y(x) = -Y(x)\frac{d^2}{d\gamma^2}\left[-\frac{1}{2\pi}e^{i\gamma x}\widehat{Y}(\gamma)\frac{d^2}{d\gamma^2}\left[-\frac{1}{2\pi}e^{i\gamma x}\widehat{Y}(\gamma)\frac{d^2}{d\gamma^2}[\dots]\right]\right]_{\gamma=x}. \tag{51}$$

Also and obviously one can see that immediately from (48):

$$y(x) = \frac{1}{V(x)} \frac{d^2}{dx^2} \left[\frac{1}{V(x)} \frac{d^2}{dx^2} \left[\frac{1}{V(x)} \frac{d^2}{dx^2} \left[\frac{1}{V(x)} \dots \right] \right] \right]. \tag{52}$$

It is of very interest to find the Fourier transform $\widehat{f}_N(\gamma_1, \gamma_2, \dots, \gamma_N)$. From relation (47.2) and discusion below of it, we find (setting $t_1 = t'_N$, $t_2 = t'_{N-1}$, $t_k = t'_{N-k+1}$ and then $t'_1 = h_1 - h_2$, $t'_2 = h_2 - h_3$, ..., $t'_{N-2} = h_{N-2} - h_{N-1}$, $t'_{N-1} = h_{N-1} - h_N$, $t'_N = h_N$).

$$\widehat{f}_N(\gamma_1, \gamma_2, \dots, \gamma_N) = \int_{\mathbf{R}^N} Y(t'_N) Y(t'_N + t'_{N-1}) \dots \times$$

$$\times \dots Y(t'_N + t'_{N-1} + \dots + t'_2) Y(t'_N + t'_{N-1} + \dots + t'_2 + t'_1) \times \dots$$

$$\times \exp\left(-it'_N \gamma_1 - it'_{N-1} \gamma_2 - \dots - it'_2 \gamma_{N-1} - it'_1 \gamma_N\right) dt'_1 dt'_2 \dots dt'_N =$$

$$= \int_{\mathbf{R}^N} Y(h_N) Y(h_{N-1}) \dots Y(h_2) Y(h_1) \times$$

$$\times \exp\left(-i\gamma_1 h_N - i(h_{N-1} - h_N)\gamma_2 - \dots - i(h_2 - h_1)\gamma_{N-1} - i(h_1 - h_2)\gamma_N\right) dh_1 dh_2 \dots dh_N =$$

$$= \int_{\mathbf{R}^N} Y(h_N) Y(h_{N-1}) \dots Y(h_2) Y(h_1) \times$$

$$\times \exp[-ih_N(\gamma_1 - \gamma_2) - ih_{N-1}(\gamma_2 - \gamma_3) - \dots \times$$

$$\times \dots - ih_3(\gamma_{N-2} - \gamma_{N-1}) - ih_2(\gamma_{N-1} - \gamma_N) - ih_1 \gamma_N] dh_1 dh_2 \dots dh_N =$$

$$= \widehat{Y}(\gamma_1 - \gamma_2) \widehat{Y}(\gamma_2 - \gamma_3) \dots \widehat{Y}(\gamma_{N-2} - \gamma_{N-1}) \widehat{Y}(\gamma_{N-1} - \gamma_N) \widehat{Y}(\gamma_N).$$

Hence we get the next

Theorem 8.
Assume

$$f_N(x_1, x_2, \dots, x_N) = Y(x_1) Y(x_1 + x_2) \dots Y(x_1 + x_2 + \dots + x_N). \tag{52.1}$$

Then

$$\widehat{f}_N(\gamma_1, \gamma_2, \dots, \gamma_N) = \widehat{Y}(\gamma_1 - \gamma_2) \widehat{Y}(\gamma_2 - \gamma_3) \dots \widehat{Y}(\gamma_{N-2} - \gamma_{N-1}) \widehat{Y}(\gamma_{N-1} - \gamma_N) \widehat{Y}(\gamma_N), \tag{52.2}$$

where

$$Y(x) = \frac{1}{V(x)}.$$

4 Some interesting solutions

We know that equation

$$\sum_{n=0}^{\infty} g_n(-i)^n \frac{d^n}{dx^n} y(x) = V(x)y(x),$$

can be written in the form

$$\sum_{n=0}^{\infty} c_n \widehat{y}(z+n) = g(z)\widehat{y}(z),$$

where $g(x) = \sum_{n=0}^{\infty} g_n x^n$, when

$$V(x) = \sum_{n=0}^{\infty} c_n e^{-inx}.$$

Also if

$$Y(x) = \frac{1}{V(x)} = \sum_{n=-\infty}^{\infty} c_n^* e^{-inx},$$

then

$$\widehat{y}(z) = \lim_{N\to\infty} \sum_{n_1\geq 0} \sum_{n_2\geq 0} \sum_{n_3\geq 0} \ldots \times$$

$$\times \ldots \sum_{n_N\geq 0} A_{n_1}(z) A_{n_2}(z+n_1) A_{n_3}(z+n_1+n_2) \ldots A_{n_N}(z+n_1+n_2+\ldots+n_N), \tag{52.3}$$

where

$$A_n(x) = g(n+x)c_n^*.$$

Also

$$g(z) = \sum_k c_k \prod_{j=0}^{k-1} f_k(z+j)$$

Hence $\theta(z)$ is solution of (30.1) and consequently

$$\widehat{y}(z) = \theta(z),$$

where

$$\frac{\theta(z+k)}{\theta(z)} = \prod_{j=0}^{k-1} f_k(z+j).$$

Hence if $f(z) = z$, then we get $\theta(z) = \Gamma(z)$. If $f(z) = z^2$, then $\theta(z) = \Gamma(z)^2$. If $f(z) = \frac{z^2}{z-1}$, then $\theta(z) = (z-1)\Gamma(z)$. If $f(z) = 2z+1$, then $\theta(z) = 2^z\Gamma(z+\frac{1}{2})$. If $f(z) = 2z^2+z$, then $\theta(z) = 2^z\Gamma(z+\frac{1}{2})\Gamma(z)$. If $f(z) = 2z^2+z+1$, then $\theta(z) = 2^z\Gamma(z-\rho_1)\Gamma(z-\rho_2)$, where $\rho_1 = \frac{-1-i\sqrt{7}}{4}$, $\rho_2 = \frac{-1+i\sqrt{7}}{4}$. By this way we

get the next

Theorem 9.
Suppose $P(z) = A\prod_{j=1}^{\nu}(z-\rho_j)$, $Q(z) = B\prod_{j=1}^{\mu}(z-\sigma_j)$ and $R(z) = \sum_{j=0}^{\tau} a_j z^j$, $a_\tau > 0$, are three polynomials of degree ν, μ, τ resp. and P, Q, have no common roots and $H(z) = Cz^m e^{-R(z)}P(z)/Q(z)$. Then if

$$g(z) = \sum_{k=0}^{\infty} c_k \left(\frac{A}{B}\right)^k \prod_{j=0}^{k-1} H(z+j) = \sum_{k=0}^{\infty} \left(\frac{A}{B}\right)^k \prod_{j=0}^{k-1} c_j^* H(z+j), \tag{53}$$

where $\prod_{j=0}^{k-1} c_j^* = c_k$, the equation

$$\sum_{n=0}^{\infty} g_n(-i)^n y^{(n)}(x) = V(x)y(x),\ V(x) = \sum_{n=0}^{\infty} c_n e^{-inx}, \tag{54}$$

can be set in the form

$$\sum_{n=0}^{\infty} c_n \widehat{y}(z+n) = g(z)\widehat{y}(z) \tag{55}$$

and a solution of (55) is

$$\widehat{y}(z) = \left(\frac{A}{B}\right)^z C^z \Gamma(z)^m e^{-R_1(z)} \frac{\prod_{j=1}^{\nu}\Gamma(z-\rho_j)}{\prod_{j=1}^{\mu}\Gamma(z-\sigma_j)}. \tag{56}$$

The function $R_1(z)$ is a polynomial of degree $\tau+1$.
Notes.
I. Actualy if $R(z) = z$, then $R_1(z) = \frac{z^2}{2} - \frac{z}{2}$. If $R(z) = z^2$, then $R_1(z) = \frac{z^3}{3} - \frac{z^2}{2} + \frac{z}{6}$. If $R(z) = z^3$, then $R_1(z) = \frac{z^4}{4} - \frac{z^3}{2} + \frac{z^2}{4}$, ... etc. In general if $R(z) = z^\tau$ and $p_\tau(n) = \sum_{j=0}^{n-1} j^\tau = \sum_{k=0}^{\tau+1} h_k n^k$, then $R(z) = \sum_{k=0}^{\tau+1} h_k z^k$. Obviously this procedure is linear i.e. if for example $R(z) = z^2 - 2z$, then $R_1(z) = \frac{z^3}{3} - \frac{z^2}{2} + \frac{z}{6} - 2\left(\frac{z^2}{2} - \frac{z}{2}\right) = \frac{z^3}{3} - \frac{3z^2}{2} + \frac{7z}{6}$.
II. From Weierstrass factorization theorem we have that if $H(z)$ is meromorphic, with $\{\rho_j\}_{j\in\mathbf{Z}}$ roots and $\{\sigma_j\}_{j\in\mathbf{Z}}$ poles, then exists polynomial $R(z)$ such that (provided that the products converge):

$$H(z) = Cz^m e^{-R(z)} \frac{\prod_{j=1}^{\infty}\left(1 - \frac{z}{\rho_j}\right)}{\prod_{j=1}^{\infty}\left(1 - \frac{z}{\sigma_j}\right)}$$

and if

$$g(z) = \sum_k c_k \prod_{j=0}^{k-1} H(z+j), \tag{56.00}$$

then a solution of

$$\sum_k g_k(-i)^k y^{(k)}(x) = V(x)y(x),\ V(x) = \sum_{n=0}^{\infty} c_n e^{-inx},$$

is

$$\widehat{y}(z) = \left(\frac{A}{B}\right)^z C^z \Gamma(z)^m e^{-R_1(z)} \frac{\prod_{j=1}^{\infty}\Gamma(z-\rho_j)}{\prod_{j=1}^{\infty}\Gamma(z-\sigma_j)}.$$

Also it holds

$$\frac{\widehat{y}(z+1)}{\widehat{y}(z)} = H(z) \Leftrightarrow \widehat{y}(z) = \prod_{n=0}^{\infty} \frac{1}{H(z+n)}. \tag{56.01}$$

III. Note that if $H(z)$ is $1-$periodic, then

$$g(z) = \sum_k c_k (H(z))^k.$$

Hence given the c_k of the potential $V(z)$ and g_n of $g(z)$ we get: $g(z)$ is $1-$periodic. For example if $g(z) = \cos(2\pi z)$ and $V(z) = \sum_{n=1}^{\infty} \frac{e^{-inz}}{n}$, then $\cos(2\pi z) = \sum_{n=1}^{\infty} \frac{H(z)^n}{n} = -\log(1-H(z))$. Hence $H(z) = 1 - e^{-\cos(2\pi z)}$. Hence

$$\frac{\widehat{y}(z+1)}{\widehat{y}(z)} = 1 - e^{-\cos(2\pi z)}. \tag{56.1}$$

However we can not solve with iteration this equation because it is $1-$periodic and

$$\widehat{y}(z) = \prod_{n=0}^{\infty} \frac{1}{1-e^{-\cos(2\pi(z+n))}} = \prod_{n=0}^{\infty} \frac{1}{1-e^{-\cos(2\pi z)}}.$$

IV. If we assume the equation $y''(x) = V(x)y(x)$, where $V(x) = \sum_{n=0}^{N} c_n e^{-inx}$, is a trigonometric polynomial, then under certain conditions we have

$$\sum_{n=0}^{N} c_n \widehat{y}(z+n) = -z^2 \widehat{y}(z).$$

This equation is a simple special functions functional equation and we can assume is taking known solutions. For example, with $N = 2$ the equation

$$c_0 f(x) + c_1 f(x+1) + c_2 f(x+2) = -x^2 f(x)$$

gives a solution $f(x)$ and hence the equation

$$y''(x) = \left(\sum_{n=0}^{2} c_n e^{-inx}\right) y(x),$$

have solution $\widehat{y}(x) = f(x) \Leftrightarrow y(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{itx}dt$.

Theorem 9.1
Assume that $e_n(z)$ is a base in $\mathbf{A}$ sunbset of $\mathbf{R}$ and $g_n^* = \langle g(z), e_n(z)\rangle$, where $g(z) = \sum_{n=0}^{\infty} g_n^* e_n(x)$. Also

$$V(z) = \sum_{n=0}^{\infty} c_n e^{-inz}$$

and

$$\sum_{n=0}^{\infty} g_n (-i)^n \frac{d^n}{dx^n} y(z) = V(x)y(x).$$

For to solve the above equation, set $J_{n,k}(a_{(.)})$ to be an operator such that:

$$\exp\left(\sum_{k=0}^{\infty} a_k B_{k,n}(z)\right) = \sum_{k=0}^{\infty} J_{n,k}(a_{(.)}) B_{k,n}(z),$$

where

$$B_{k,n}(z) = \sum_{l=0}^{k} C_{k,l} p_{k-l}(n) z^l,\ p_\tau(n) = \sum_{j=0}^{n-1} j^\tau.$$

If $e_n(z)$ is a certain base, we write

$$B_{k,n}(z) = \sum_{l=1}^{\infty} S_{k,n,l} e_l(z).$$

and $S_{k,n,l}^*$ is the matrix inverse of $S_{k,n,l}$, in the sense

$$\sum_{l=1}^{\infty} S_{k,n,l} S_{k',n,l}^* = R_n \delta_{k',k},$$

where $\delta_{k',k} = 1$, if $k = k'$ and $\delta_{k',k} = 0$, if $k' \neq k$. Also we define $a'_{n,k}$ to be the arithmetic inverse of $\sum_{d|n} c_d \mu(n/d)$, in the sense

$$\sum_{d_1|A} \left(\sum_{d|d_1} c_d \mu(d_1/d)\right) a'_{A/d_1,B} = \delta_{A,1}\delta_{B,1}.$$

Then with a suitable choise of R_n, we get

$$y(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \prod_{n=0}^{\infty} \exp\left(-\sum_{k=0}^{\infty} J_{n,k}^{(-1)} \left(\sum_{l=1}^{\infty} g_l^* \sum_{d_1|n, d_2|l} S_{k,d_1,d_2}^* a'_{n/d_1,l/d_2}\right)(z+n)^k\right) e^{izx} dz.$$

Proof.
We write

$$g(z) = \sum_{n=0}^{\infty} c_n \prod_{j=0}^{n-1} H(z+j),$$

where

$$H(z) = \exp\left(\sum_{k=0}^{\infty} a_k z^k\right)$$

Hence if $C_{n,k}$ denotes the binomial function i.e. $C_{n,k} = \frac{n!}{(n-k)!k!}$, then

$$\prod_{j=0}^{n-1} H(z+j) = \exp\left(\sum_{k=0}^{\infty} a_k \sum_{j=0}^{n-1} (z+j)^k\right) =$$

$$= \exp\left(\sum_{k=0}^{\infty} a_k \sum_{j=0}^{n-1} \sum_{l=0}^{k} C_{k,l} j^{k-l} z^l\right) =$$

$$= \exp\left(\sum_{k=0}^{\infty} a_k \sum_{l=0}^{k} C_{k,l} \sum_{j=0}^{n-1} j^{k-l} z^l\right) =$$

$$= \exp\left(\sum_{k=0}^{\infty} a_k \sum_{l=0}^{k} C_{k,l} p_{k-l}(n) z^l\right).$$

Hence if we denote

$$B_{k,n}(z) := \sum_{l=0}^{k} C_{k,l} p_{k-l}(n) z^l,$$

then

$$g(z) = \sum_{n=0}^{\infty} c_n \exp\left(\sum_{k=0}^{\infty} a_k B_{k,n}(z)\right).$$

Now we assume the transformation J such that

$$\exp\left(\sum_{k=0}^{\infty} a_k B_{k,n}(z)\right) = \sum_{k=0}^{\infty} J_{n,k}(a_{(.)}) B_{k,n}(z).$$

This equivalently give (if we denote $h_{n,k} = J_{n,k}(a_{(.)})$):

$$\exp\left(\sum_{j=0}^{n-1} \sum_{k=0}^{\infty} a_k (z+j)^k\right) = \sum_{j=0}^{n-1} \sum_{k=0}^{\infty} h_{n,k} (z+j)^k \Leftrightarrow$$

$$\prod_{j=0}^{n-1} \exp\left(\sum_{k=0}^{\infty} a_k (z+j)^k\right) = \sum_{j=0}^{n-1} \log\left(\exp\left(\sum_{k=0}^{\infty} h_{n,k} (z+j)^k\right)\right) \Leftrightarrow$$

$$\prod_{j=0}^{n-1} \exp\left(\sum_{k=0}^{\infty} a_k (z+j)^k\right) = \log\left(\prod_{j=0}^{n-1} \exp\left(\sum_{k=0}^{\infty} h_{n,k} (z+j)^k\right)\right) \Leftrightarrow$$

$$\exp\left(\prod_{j=0}^{n-1}\exp\left(\sum_{k=0}^{\infty}a_k(z+j)^k\right)\right)=\prod_{j=0}^{n-1}\exp\left(\sum_{k=0}^{\infty}h_{n,k}(z+j)^k\right).$$

Hence the problem reduces to find if for any function $X(z)$ exists function $Y_n(z)$ such that

$$\prod_{j=0}^{n-1}X(z+j)=\sum_{j=0}^{n-1}Y_n(z+j),\ \forall n\in\mathbf{N}.$$

However

$$g(z)=\sum_{n=1}^{\infty}c_n\sum_{k=1}^{\infty}J_{n,k}(a_{(.)})B_{k,n}(z)=\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}c_nJ_{n,k}(a_{(.)})B_{k,n}(z). \qquad (a)$$

Assume now $e_n(z)$ is any base function in a set $\mathbf{A}$ subset of $\mathbf{R}$, then

$$B_{k,n}(z)=\sum_{l=1}^{\infty}S_{k,n,l}e_l(z)$$

and relation (a) becomes

$$g(z)=\sum_{l=1}^{\infty}g_l^*e_l(z)=\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}c_nJ_{n,k}(a_{(.)})\sum_{l=1}^{\infty}S_{k,n,l}e_l(z)=$$

$$=\sum_{l=1}^{\infty}\left(\sum_{n,k=1}^{\infty}c_nJ_{n,k}(a_{(.)})S_{k,n,l}\right)e_l(z).$$

Hence

$$g_l^*=\sum_{n,k=1}^{\infty}c_nJ_{n,k}(a_{(.)})S_{k,n,l}.$$

If

$$\sum_{n=1}^{\infty}I_{n,l}c_n=g_l^*,$$

where

$$I_{n,l}=\sum_{k=1}^{\infty}J_{n,k}(a_{(.)})S_{k,n,l}$$

and

$$\sum_{l=1}^{\infty}S_{k,n,l}S^*_{k',n,l}=R_n\delta_{k,k'},$$

then

$$\sum_{l=1}^{\infty}I_{n,l}S^*_{k',n,l}=\sum_{k=1}^{\infty}J_{n,k}(a_{(.)})\sum_{l=1}^{\infty}S_{k,n,l}S^*_{k',n,l}=$$

$$= \sum_{k=1}^{\infty} J_n(a_k) R_n \delta_{k,k'} = J_{n,k'}(a_{(.)}) R_n.$$

Assume

$$\sum_{n_1,l_1=1}^{\infty} I_{nn_1,ll_1} b_{n_1,l_1} = \phi(n,l)$$

Then

$$\sum_{n_1,l_1=1}^{\infty} I_{nn_1n_2,ll_1l_2} b_{n_1,l_1} = \phi(nn_2, ll_2)$$

$$\sum_{n_1,l_1=1}^{\infty} I_{nn_1n_2,ll_1l_2} b_{n_1,l_1} a'_{n_2,l_2} = \phi(nn_2, ll_2) a'_{n_2,l_2}$$

$$\sum_{n_2,l_2=1}^{\infty} \sum_{n_1,l_1=1}^{\infty} I_{nn_1n_2,ll_1l_2} b_{n_1,l_1} a'_{n_2,l_2} = \sum_{n_2,l_2=1}^{\infty} \phi(nn_2, ll_2) a'_{n_2,l_2}$$

$$\sum_{A,B=1}^{\infty} I_{nA,lB} \sum_{d_1|A, d_2|B} b_{d_1,d_2} a'_{A/d_1,B/d_2} = \sum_{n_2,l_2=1}^{\infty} \phi(nn_2, ll_2) a'_{n_2,l_2}$$

Assume

$$\sum_{d_1|A, d_2|B} b_{d_1,d_2} a'_{A/d_1,B/d_2} = \delta_{A,1}\delta_{B,1}. \qquad (b)$$

Then

$$I_{n,l} = \sum_{n_2,l_2=1}^{\infty} \phi(nn_2, ll_2) a'_{n_2,l_2}$$

Assume now that $b_{n_1,l_1} = \delta_{l_1,1} c_{n_1}$, then

$$\sum_{n=1}^{\infty} \sum_{n_1,l_1=1}^{\infty} I_{nn_1,ll_1} b_{n_1,l_1} = \sum_{n=1}^{\infty} \phi(n,l) \Rightarrow$$

$$\sum_{n,l_1=1}^{\infty} I_{n,ll_1} \sum_{d|n} b_{d,l_1} = \sum_{n=1}^{\infty} \phi(n,l) = g_l^*.$$

Hence if

$$\sum_{d|n} b_{d,l_1} = \delta_{l_1,1} c_n \Leftrightarrow b_{n,l} = \delta_{l,1} \sum_{d|n} c_d \mu(n/d),$$

then

$$\sum_{n=1}^{\infty} I_{n,l} c_n = g_l^*.$$

Also $a'_{n,l}$ is the inverse arithmetic function of $b_{n,l}$ in the sense (b), then

$$\sum_{d_1|A, d_2|B} \delta_{d_2,1} \left(\sum_{d|d_1} c_d \mu(d_1/d) \right) a'_{A/d_1,B/d_2} = \delta_{A,1}\delta_{B,1} \Leftrightarrow$$

$$\sum_{d_1|A}\left(\sum_{d|d_1} c_d\mu(d_1/d)\right) a'_{A/d_1,B} = \delta_{A,1}\delta_{B,1}. \qquad (c)$$

From equation (c) we get the $a'_{n,l}$. Finaly

$$B_{k,n}(z) = \sum_{l=1}^{\infty} S_{k,n,l} e_l(z)$$

$$\sum_{l=1}^{\infty} S_{k,n,l} S^*_{k',n,l} = R_n \delta_{k,k'}$$

$$J_{n,k}(a_{(.)}) = \frac{1}{R_n}\sum_{l=1}^{\infty} I_{n,l} S^*_{k',n,l}$$

and

$$I_{n,l} = \sum_{n_2,l_2=1}^{\infty} \phi(nn_2, ll_2) a'_{n_2,l_2},$$

$$\sum_{n=1}^{\infty} \phi(n,l) = g^*_l.$$

Assuming further that $\phi(n,l) = \phi_1(n)\phi_2(l)$, we get

$$\sum_{n=1}^{\infty} \phi_1(n)\phi_2(l) = g^*_l.$$

Hence

$$\phi_2(l) = \frac{g^*_l}{G},\ G = \sum_{n=1}^{\infty} \phi_1(n)$$

Hence there exists constant G such that

$$\phi(n,l) = \phi_1(n)\frac{g^*_l}{G}$$

and

$$I_{n,l} = \frac{1}{G}\sum_{n_2,l_2=1}^{\infty} \phi_1(nn_2) g^*_{ll_2} a'_{n_2,l_2}$$

Also then

$$J_n(a_k) = \frac{1}{GR_n}\sum_{l,n_2,l_2=1}^{\infty} \phi_1(nn_2) g^*_{ll_2} a'_{n_2,l_2} S^*_{k,n,l}$$

$$g(z) = \frac{1}{G}\sum_{n,k=1}^{\infty} \frac{c_n}{R_n} \sum_{l,n_2,l_2=1}^{\infty} \phi_1(nn_2) g^*_{ll_2} a'_{n_2,l_2} S^*_{k,n,l} B_{k,n}(z).$$

$$g(z)=\frac{1}{G}\sum_{n,k=1}^{\infty}\frac{c_n}{R_n}\sum_{l=1}^{\infty}\left(\sum_{n_2,l_2=1}^{\infty}\phi_1(nn_2)g^*_{ll_2}a'_{n_2,l_2}\right)S^*_{k,n,l}B_{k,n}(z).$$

Assume that $\psi(n)$ is such that

$$\sum_{n=1}^{\infty}J_{n,k}(a_{(.)})\psi(n)=\sum_{n=1}^{\infty}\frac{\psi(n)}{GR_n}\sum_{l,n_2,l_2=1}^{\infty}\phi_1(nn_2)g^*_{ll_2}a'_{n_2,l_2}S^*_{k,n,l}.$$

Hence if $\psi(n)=GR_n$, we get

$$\sum_{n=1}^{\infty}J_{n,k}(a_{(.)})\psi(n)=\sum_{n=1}^{\infty}\sum_{l,n_2,l_2=1}^{\infty}\phi_1(nn_2)g^*_{ll_2}a'_{n_2,l_2}S^*_{k,n,l}\Rightarrow$$

$$\sum_{n=1}^{\infty}J_{n,k}(a_{(.)})\psi(n)=\sum_{n,l=1}^{\infty}\phi_1(n)g^*_l\sum_{d_1|n,d_2|l}S^*_{k,d_1,d_2}a'_{n/d_1,l/d_2}=$$

$$=\sum_{n=1}^{\infty}\phi_1(n)\sum_{l=1}^{\infty}g^*_l\sum_{d_1|n,d_2|l}S^*_{k,d_1,d_2}a'_{n/d_1,l/d_2}.$$

If we set now $\phi_1(n)=\psi(n)=z^n$, then

$$J_{n,k}(a_{(.)})=\sum_{l=1}^{\infty}g^*_l\sum_{d_1|n,d_2|l}S^*_{k,d_1,d_2}a'_{n/d_1,l/d_2},$$

where the $a'_{n,m}$ is given from (c). Using now

$$H(z)=\exp\left(\sum_{k=0}^{\infty}a_kz^k\right),$$

we get

$$y(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\prod_{n=0}^{\infty}\exp\left(-\sum_{k=0}^{\infty}J^{(-1)}_{n,k}\left(\sum_{l=1}^{\infty}g^*_l\sum_{d_1|n,d_2|l}S^*_{k,d_1,d_2}a'_{n/d_1,l/d_2}\right)(z+n)^k\right)e^{izx}dz.$$

Theorem 10.
Given a potential

$$V(z;\tau)=\sum_{n=0}^{\infty}c_n(\tau)e^{-i\tau nz}$$

and

$$B(z;\tau)=\sum_{n=0}^{\infty}b_n(\tau)z^n=g(-iz;\tau)=\sum_{n=0}^{\infty}(-i)^ng_n(\tau)z^n,$$

we form the equation

$$\sum_{n=0}^{\infty} b_n(\tau)\frac{d^n}{dz^n}y(z;\tau) = V(z;\tau)y(z;\tau),$$

which may be formed as

$$V\left(i\frac{d}{dz}\right)\widehat{y}(z;\tau) = g(z;\tau)\widehat{y}(z;\tau),$$

where

$$\widehat{y}(z;\tau) = \int_{-\infty}^{\infty} y(x;\tau)e^{-ixz}dx.$$

Also this may be written as

$$\sum_{n=0}^{\infty} c_n(\tau)\widehat{y}(z+\tau n;\tau) = g(z;\tau)\widehat{y}(z;\tau). \tag{56.1a}$$

I. If we assume that

$$\widehat{y}(z;\tau) = \sum_{l=1}^{\infty} C_l(z;\tau), \tag{a}$$

where

$$C_l(z+n\tau;\tau) = A_n(z;\tau)C_l(z;\tau),\ n \in \mathbf{Z},$$

then it must be

$$g(z;\tau) = \sum_{n=0}^{\infty} c_n(\tau)A_n(z;\tau). \tag{b}$$

From where we find $A_n(z;\tau)$. In this case also

$$\widehat{y}(z+n\tau;\tau) = A_n(z;\tau)\widehat{y}(z;\tau). \tag{c}$$

II. If $H(z;\tau)$ is such that

$$\frac{\widehat{y}(z+\tau;\tau)}{\widehat{y}(z;\tau)} = H(z;\tau),\ Im(\tau), Im(z) > 0 \tag{d}$$

and $H(z+\tau;\tau) = A^*(z;\tau)H(z;\tau)$, then

$$H(z+j\tau;\tau) = \left(\prod_{l=0}^{j-1} A^*(z+l\tau;\tau)\right)H(z;\tau)$$

and from (d):

$$\widehat{y}(z+n\tau;\tau) = \left(\prod_{j=0}^{n-1} H(z+j\tau;\tau)\right)\widehat{y}(z;\tau). \tag{e}$$

Hence from (c):

$$A_n(z;\tau) = \prod_{j=0}^{n-1} H(z + j\tau;\tau). \tag{56.2}$$

Hence

III.

$$A_1(z;\tau) = H(z;\tau)$$

and from (b) we get

$$\widehat{y}(z;\tau) = \prod_{n=0}^{\infty} \frac{1}{A_1(z + n\tau;\tau)}. \tag{f}$$

Proof.

For the proof of (II), we have

$$g(z;\tau) = \sum_{n=0}^{\infty} c_n(\tau) \prod_{j=0}^{n-1} H(z + j\tau;\tau). \tag{56.4}$$

Hence from (56.2), we get

$$g(z;\tau) = \sum_{n=0}^{\infty} c_n(\tau) \left(\prod_{j=0}^{n-1} \prod_{l=0}^{j-1} A^*(z + l\tau;\tau) \right) H(z;\tau)^n.$$

Hence given the $(-i)^n g_n(\tau) = b_n(\tau)$ of $B(z;\tau)$ and $c_n(\tau)$ of the potential, we get $H(z;\tau)$ from (56.2) and the solution of (eq) is

$$\widehat{y}(z;\tau) = \prod_{n=0}^{\infty} \frac{1}{H(z + n\tau;\tau)}.$$

We have

$$\sum_{k=0}^{\infty} b_k(\tau) y^{(k)}(x;\tau) = V(x;\tau) y(x;\tau) \Leftrightarrow$$

$$\sum_{k=0}^{\infty} b_k(\tau) \int_{-\infty}^{\infty} y^{(k)}(x;\tau) e^{-ix\gamma} dx = \int_{-\infty}^{\infty} y(x;\tau) \left(\sum_{n=0}^{\infty} c_n(\tau) x^n \right) e^{-ix\gamma} dx \Leftrightarrow$$

$$\sum_{k=0}^{\infty} b_k(\tau) (i\gamma)^k \int_{-\infty}^{\infty} y(x;\tau) e^{-ix\gamma} dx = \sum_{n=0}^{\infty} c_n(\tau) \int_{-\infty}^{\infty} y(x;\tau) x^n e^{-ix\gamma} dx \Leftrightarrow$$

$$B(i\gamma;\tau)\widehat{y}(\gamma) = \sum_{n=0}^{\infty} c_n(\tau) i^n \frac{d^n}{d\gamma^n} \left(\int_{-\infty}^{\infty} y(x;\tau) e^{-ix\gamma} dx \right) \Leftrightarrow$$

$$B(i\gamma;\tau)\widehat{y}(\gamma;\tau) = \sum_{n=0}^{\infty} c_n(\tau) i^n \frac{d^n}{d\gamma^n} \widehat{y}(\gamma;\tau) \Leftrightarrow$$

$$B(i\gamma;\tau)\widehat{y}(\gamma;\tau) = V\left(i\frac{d}{d\gamma}\right)\widehat{y}(\gamma;\tau). \tag{57}$$

Assume now that $V(x;\tau) = \sum_{n=0}^{\infty} c_n(\tau)e^{-in\tau x}$. Hence

$$B(i\gamma;\tau)\widehat{y}(\gamma;\tau) = \sum_{n=0}^{\infty} c_n(\tau)\exp\left(\tau n\frac{d}{d\gamma}\right)\widehat{y}(\gamma;\tau). \tag{58}$$

Or equivalent

$$B(i\gamma;\tau)\widehat{y}(\gamma;\tau) = \sum_{n=-\infty}^{\infty} c_n(\tau)\widehat{y}(\gamma+n\tau;\tau). \tag{59}$$

Assume that $Im(z)$, $Im(\tau) > 0$ and

$$\widehat{y}(z;\tau) = \sum_{l=1}^{\infty} C_l(z;\tau). \tag{60}$$

Then if n is integer, we have

$$C_l(z+n\tau;\tau) = A_n(z;\tau)C_l(z;\tau).$$

Hence for almost all $a_n(\tau)$ we have

$$\sum_{n=0}^{\infty} a_n(\tau)\widehat{y}(z+n\tau) = \sum_{n=0}^{\infty} a_n(\tau)\sum_{l=1}^{\infty} C_l(z+n\tau;\tau) =$$

$$= \sum_{n=0}^{\infty}\sum_{l=1}^{\infty} a_n(\tau)A_n(z;\tau)C_l(z;\tau) = \sum_{n=0}^{\infty} a_n(\tau)A_n(z;\tau)\sum_{l=1}^{\infty} C_l(z;\tau) =$$

$$= \left(\sum_{n=0}^{\infty} a_n(\tau)A_n(z;\tau)\right)\widehat{y}(z;\tau).$$

Hence we can write

$$\sum_{n=0}^{\infty} a_n(\tau)\left(\widehat{y}(z+n\tau;\tau) - A_n(z;\tau)\widehat{y}(z;\tau)\right) = 0$$

and it must be

$$\widehat{y}(z+n\tau;\tau) = A_n(z;\tau)\widehat{y}(z;\tau). \tag{61}$$

Hence if $a_n(\tau) = c_n(\tau)$ and

$$g(z;\tau) = B(iz;\tau) = \sum_{n=0}^{\infty} b_n(\tau)(iz)^n = \sum_{n=0}^{\infty} c_n(\tau)A_n(z;\tau),$$

then from (d) $\widehat{y}(z)$ is given from (f) and satisfies (56.1a) and $y(x;\tau)$ is solution of

$$\sum_{n=0}^{\infty} b_n(\tau)y^{(n)}(x;\tau) = V(x;\tau)y(x;\tau), \tag{62}$$

where

$$V(x) = \sum_{n=0}^{\infty} c_n(\tau) e^{-in\tau x}.$$

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