Open boundary conditions of the $D_3^{(2)}$ spin chain and sectors of conformal field theories

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Abstract

We study open boundary conditions for the $D_3^{(2)}$ spin chain, which shares connections with the six-vertex model, under staggering, and also to the antiferromagnetic Potts model. By formulating a suitable transfer matrix, we obtain an integrable, open Hamiltonian, hence allowing for us to classify different regions of the underlying conformal field theory from eigenvalues of the Hamiltonian. [1]

1 Introduction

1.1 Overview

Spin chains have long been objects of study across the fields of quantum physics, high-energy physics, and statistical physics, for connections to computations of finite-size spectra [1], staggered vertex models [2], integrable boundary conditions [3], quantum R-matrices [4], the Bethe ansatz [5], integrability, either through being able to completely solve spin chain models, or through boundary conditions, [7,8], and conformal invariance [9]. To further explore avenues of interest at the intersection of all of these fields, in the following we study the $D_2^{(2)}$, and $D_3^{(2)}$ spin chains. Despite having different rank, each of the two spin chains share similarities, not only from the fact that R-matrices can be constructed which satisfy the Yang Baxter equation, but also from the fact that open boundary conditions can be encoded from K-matrices satisfying variants of the Yang Baxter equation at the leftmost and rightmost endpoints of a finite volume. To determine which sections of the underlying conformal field theory (CFT) are selected depending upon the encoding of open boundary conditions, we introduce the lower, and higher, rank spin chains, from which we distinguish different sectors of the CFT depend on open boundary conditions. From an expansion of the Hamiltonian into a local Hamiltonian, we characterize the ground state with open boundary conditions about the point $(h_1, h_2) \equiv (0, 0)$, and proceed to characterize other sectors of the CFT for $h_1 \equiv 0$, $h_2 \neq 0$, and for $h_1 \neq 0$, $h_2 \equiv 0$.

The root density approach has previous been applied for investigating several aspects of systems in the high-temperature limit, in which one often studies more general aspects of low temperature expansions of probability measures in the presence of different contributions to interactions term of the Hamiltonian. In the presence of different interactions, the root density approach for the Bethe equations implies that there is a change in the density of the roots of the Bethe equations, which not only has connections with conformal field theory, but also with interpretations surrounding excitations to the ground state.

1.2 Spin chain objects

We begin by providing an overview of the higher rank spin chain, and then proceed to describe its relations to the lower rank spin chain. To introduce such a model, define the 36×36 R matrix, with,

$$R(u) \equiv \exp\big(-2u - 6\eta\big) R_J(x) \quad ,$$

as a function of the single parameter u, where $R_J(x)$ denotes the Jimbo matrix [4], and $x \equiv \exp(u)$ and $k \equiv \exp(2\eta)$. The R matrix satisfies the Yang Baxter equation,

$$R_{12}(u-v) R_{13}(u) R_{23}(v) = R_{23}(v) R_{13}(u) R_{12}(u-v) \quad ,$$

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for the anisotropy parameter $\eta \equiv i\gamma$, and another parameter v. Besides the R matrix satisfying the Yang Baxter equation, it also possesses a $U(1)$ symmetry, which is captured by the condition,

$$\left[R(u), \mathbf{h}_j \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{h}_j \right] \equiv 0 \quad ,$$

for $j=1$ and $j=2$, with,

$$\begin{aligned} \mathbf{h}_1 &\equiv \mathcal{M}(1,1) - \mathcal{M}(6,6) \quad , \\ \mathbf{h}_2 &\equiv \mathcal{M}(2,2) - \mathcal{M}(5,5) \quad , \end{aligned}$$

for the matrices $\mathcal{M}(1,1)$, $\mathcal{M}(6,6)$, $\mathcal{M}(2,2)$ and $\mathcal{M}(5,5)$, which are respectively given by the 6×6 matrices with nonzero entries at $(1,1)$, $(6,6)$, $(2,2)$, $(5,5)$, and the identity matrix $\mathbf{I}$. The R matrix also satisfies Parity-Time (PT) symmetry, in which,

$$R_{21}(u) \equiv \mathcal{P}_{12} \mathcal{R}_{12}(u) \mathcal{P}_{12} \equiv R_{12}^{t_1,t_2}(u) \quad ,$$

for the permutation matrix $\mathcal{P}$, for the transposition t. Additional properties, including braiding unitarity, regularity, crossing symmetry, quasi-periodicity, and $\mathbf{Z}_2$ symmetries are also satisfied [1]. From the quantities introduced since the beginning of the section, the transfer matrix of the model takes the form,

$$\mathbf{T}(u) \equiv \mathrm{tr}_0\big(\mathbf{K}_0 \mathbf{T}_0(u)\big) \equiv \mathrm{tr}\Big(\mathbf{K}_0 \prod_{1 \leq j \leq L} \mathbf{R}_{0j}(u)\Big) \quad ,$$

for the twist diagonal matrix,

$$\mathbf{K} \equiv \mathrm{diag}\big(\exp(i\phi_1), \exp(i\phi_2), 1, 1, \exp(-i\phi_2), \exp(-i\phi_1)\big) \quad ,$$

given two angles ϕ_1 and ϕ_2, and product of R matrices for $1 \leq j \leq L$. The angles ϕ_1 and ϕ_2 determine the boundary conditions of the higher rank spin chain, as opposed to the open boundary conditions of the lower rank spin chain that is introduced in the remaining parts of this section.

To work towards introducing the higher rank spin chain and open boundary conditions for it, we start with defining the following R matrix, and similar components, for the lower rank spin chain with the following. To construct the R matrix, consider the 6×6 matrix, of the form,

$$\widetilde{R}^{(\mathrm{XXZ})}(u) \equiv \begin{bmatrix} \sinh\big(-\frac{u}{2}+\eta\big) & 0 & 0 & 0 \\ 0 & \sinh\big(\frac{u}{2}\big) & \exp\big(-\frac{u}{2}\big)\sinh(\eta) & 0 \\ 0 & \exp\big(\frac{u}{2}\big)\sinh(\eta) & \sinh\big(\frac{u}{2}\big) & 0 \\ 0 & 0 & 0 & \sinh\big(-\frac{u}{2}+\eta\big) \end{bmatrix} \quad ,$$

from the R matrix for the $A_1^{(1)}$ (XXZ) spin chain, which is related to the R matrix of the lower rank spin chain from the fact that,

$$\widetilde{R}(u) \propto B_{12}B_{34}\mathbf{R}'_{12,34}(u)B_{12}B_{34} \equiv B_{12}B_{34}\Big(R_{14}(u)R_{13}(u)R_{24}(u)R_{23}(u)\Big)B_{12}B_{34} \quad ,$$

and matrices B, which are given by,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\cosh(\frac{\eta}{2})}{\sqrt{\cosh(\eta)}} & -\frac{\sinh(\frac{\eta}{2})}{\sqrt{\cosh(\eta)}} & 0 \\ 0 & -\frac{\sinh(\frac{\eta}{2})}{\sqrt{\cosh(\eta)}} & -\frac{\cosh(\frac{\eta}{2})}{\sqrt{\cosh(\eta)}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad ,$$

satisfying,

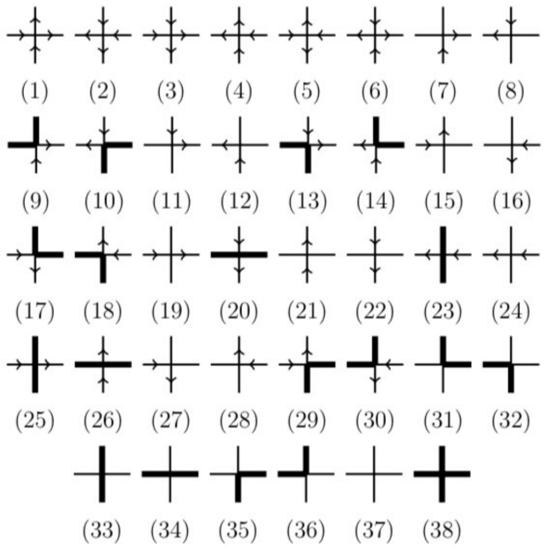

Figure 1: A depiction of the thirty eight possible configuations for the $D_2^{[2]}$ spin chain, reproduced from [8].

$$B^2 = \mathbf{I} \quad ,$$

and R-matrices solving the Yang Baxter equation,

$$R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v) \quad .$$

In contrast to the higher rank case, the R matrix above for the lower rank spin chain satisfies,

$$\mathbf{R}'_{12,34}(u) = R_{43}(-\theta)R_{13}(u)R_{14}(u+\theta)R_{23}(u-\theta)R_{24}(u)R_{34}(\theta) \quad ,$$

which in turn implies,

$$\widetilde{R}(u) \propto B_{12}B_{34}\bigg(R_{14}(u)R_{13}(u)R_{24}(u)R_{23}(u)\bigg)B_{12}B_{34} \equiv B_{12}B_{34}\bigg(R_{43}(-\theta)R_{13}(u)R_{14}(u+\theta)$$
$$\times R_{23}(u-\theta)R_{24}(u)R_{34}(\theta)\bigg)B_{12}B_{34} \quad .$$

To encode open boundary conditions of the lower rank spin chain, we must further describe properties of the $\mathbf{K}$ matrix, which was introduced earlier in the section with the definition of the transfer matrix $\mathbf{T}(u)$ for the higher rank spin chain. In particular, in addition to the R matrices which satisfy the Yang Baxter equation for the lower rank spin chain, open boundary conditions of the chain are enforced from the fact that two other matrices, given by $K_-(u)$ and $K_+(u)$ below, satisfy, [8],

$$R_{12}(u-v)K_{1,-}(u)R_{21}(u+v)K_{2,-}(v) = K_{2,-}(u)R_{12}(u+v)K_{1,-}(u)R_{21}(u-v) \quad ,$$

corresponding to the first, and second, boundary conditions which are reflected through the addition of the terms $K_{1,-}(u)$ and $K_{2,-}(v)$, as well as, [2],

$$R_{1,2}(-u+v)K^{t_1}_{1,+}(u)R_{1,2}(-u-v-2i\gamma)K^{t_2}_{2,+}(v) = K^{t_2}_{2,+}(v)R_{1,2}(-u-v-2i\gamma)K^{t_1}_{1,+}(u)R_{1,2}(-u+v) \quad ,$$

corresponding to the Yang Baxter equation for parameters t_1 and t_2 from the PT symmetric property of the R matrix satisfied by the higher rank spin chain, for the anisotropy parameter γ, where each matrix is respectively given by, [8],

$$K_-(\lambda) \equiv \begin{bmatrix} -\exp(-\lambda)(\exp(2\lambda)+k) & 0 & 0 & 0 \\ 0 & -\frac{1}{2}(1+\exp(2\lambda))\exp(\lambda)(1+k) & \frac{1}{2}(\exp(2\lambda)-1)(1-k)\exp(\lambda) & 0 \\ 0 & \frac{1}{2}(\exp(2\lambda)-1)(1-k)\exp(\lambda) & -\frac{1}{2}(1+\exp(2\lambda))\exp(\lambda)(1+k) & 0 \\ 0 & 0 & 0 & \cdots \end{bmatrix} ,$$

where the last entry along the diagonal is given by,

$$-\exp(3\lambda)(\exp(2\lambda)+k) \quad ,$$

which is equivalent to the matrix with symbols,

$$\begin{bmatrix} Y_1(\lambda) & 0 & 0 & 0 \\ 0 & Y_2(\lambda) & Y_5(\lambda) & 0 \\ 0 & Y_6(\lambda) & Y_3(\lambda) & 0 \\ 0 & 0 & 0 & Y_4(\lambda) \end{bmatrix} ,$$

from the R-matrix basis,

$$\bigg\{ |1\rangle , |2\rangle , |3\rangle , |4\rangle \bigg\} \otimes \bigg\{ |1\rangle , |2\rangle , |3\rangle , |4\rangle \bigg\} \quad .$$

With such an encoding of the boundary conditions of the spin chain with $K_-(u)$ and $K_+(u)$, the transfer matrix takes on a similar form, in which,

$$\begin{aligned} \mathbf{T}_{D_2^{(2)}}(u) &\equiv \mathrm{tr}_a\big(K_{+,a}(u)R_{a1}(u)\cdots\times R_{aL}(u)K_{-,a}(u)R_{1a}(u)\cdots\times R_{La}(u)\big) \\ &\equiv \mathrm{tr}_a\bigg(K_{+,a}(u)\prod_{1\leq j\leq L} R_{aj}(u)K_{-,a}(u)\prod_{1\leq j'\leq L} R_{j'a}(u)\bigg) \quad . \end{aligned}$$

With $\mathbf{T}_{D_2^{(2)}}(u)$, which satisfies the condition $\big[\mathbf{T}_{D_2^{(2)}}(u), \mathbf{T}_{D_2^{(2)}}(v)\big] = 0$, we also stipulate, in order to properly construct open boundary conditions for the lower rank spin chain, that,

$$K_{+,a}(\lambda) = K^{-t}(-\rho-\lambda)M \quad ,$$

where t denotes the transposition of the matrix, and $\rho \equiv -\log(k)$, and $M \equiv \mathrm{diag}(k,1,1,\frac{1}{k})$. Explicitly, the entries of K_+, from K_- and the parameters ρ and M, is given by,

$$\begin{bmatrix} Y_1(\lambda)(-\rho-\lambda) & 0 & 0 & 0 \\ 0 & Y_2(\lambda)(-\rho-\lambda) & Y_6(\lambda)(-\rho-\lambda) & 0 \\ 0 & Y_5(\lambda)(-\rho-\lambda) & Y_3(\lambda)(-\rho-\lambda) & 0 \\ 0 & 0 & 0 & Y_4(\lambda)(-\rho-\lambda) \end{bmatrix} \begin{bmatrix} k & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{k} \end{bmatrix}$$

1.3 Paper overview

Equipped with the overview in *1.1* and definitions of lower, and higher, rank spin chains in *1.2*, in the remaining sections of the paper we apply the open boundary framework to the higher rank spin chain, in an effort to determine how the boundary conditions determine the CFT sector. From information on how open boundary conditions are encoded in the Yang Baxter equation, and transfer matrix, for the lower rank spin chain, we incorporate open boundary conditions in the higher rank spin chain. In the higher rank case, we obtain an expansion for the local Hamiltonian, and provide a formulation of the Bethe equations whose roots are analyzed to study the ground state.

2 Encoding open boundary conditions in the higher rank spin chain

2.1 Obtaining the higher rank spin chain Hamiltonian from an expansion of the derivative of the transfer matrix about $u \equiv 0$

In comparison to twisted boundary conditions encoded with the angles ϕ_1 and ϕ_2, open boundary conditions for the higher rank spin chain can be encoded by introducting a K matrix for the 36×36 R matrix, from the basis,

$$\bigg\{ |1\rangle , |2\rangle , |3\rangle , |4\rangle , |5\rangle , |6\rangle \bigg\} \otimes \bigg\{ |1\rangle , |2\rangle , |3\rangle , |4\rangle , |5\rangle , |6\rangle \bigg\} \quad ,$$

in which the trace would then take the form,

$$\mathbf{T}^{\mathrm{open}}(u) \equiv \mathrm{tr}_0\big(\mathbf{K}_{+,0}(u)\mathbf{T}_{+,0}(u)\mathbf{K}_{-,0}(u)\mathbf{T}_{-,0}(u)\big) \equiv \mathrm{tr}_0\bigg(\mathbf{K}_{+,0}(u) \prod_{1 \leq j \leq L} \mathbf{R}_{+,0j}(u)\mathbf{K}_{-,0}(u) \prod_{1 \leq j' \leq L} \mathbf{R}_{-,j'0}(u)\bigg)$$
$$\equiv \mathrm{tr}_0\bigg(\mathbf{K}^{\mathrm{open}}_{+,0}(u) \prod_{1 \leq j \leq L} \mathbf{R}_{+,a0}(u)\mathbf{K}^{\mathrm{open}}_{-,0}(u) \prod_{1 \leq j' \leq L} \mathbf{R}_{-,j'0}(u)\bigg) \quad ,$$

for the higher rank spin chain transfer matrix,

$$\mathbf{T}^{\mathrm{open}}_{D^{(2)}_3}(u) \equiv \mathbf{T}^{\mathrm{open}}(u) \quad ,$$

with open boundary conditions enforced through the K matrix,

$$\mathbf{K}^{\mathrm{open}}_{-}(u) \equiv \mathbf{K}_{-}(u) \equiv \begin{bmatrix} k_0(u) & 0 & 0 & 0 & 0 & 0 \\ 0 & k_0(u) & 0 & 0 & 0 & 0 \\ 0 & 0 & k_1(u) & k_2(u) & 0 & 0 \\ 0 & 0 & k_3(u) & k_4(u) & 0 & 0 \\ 0 & 0 & 0 & 0 & k_5(u) & 0 \\ 0 & 0 & 0 & 0 & 0 & k_5(u) \end{bmatrix} \quad ,$$

from the fact that the K matrix is a special $n \equiv 2$ case of the matrix, [6],

$$\begin{bmatrix} k_0(u)\mathbf{I}_{n\times n} & & & \\ & k_1(u) & k_2(u) & \\ & k_3(u) & k_4(u) & \\ & & & k_5(u)\mathbf{I}_{n\times n} \end{bmatrix} \quad ,$$

which amounts to the matrix,

$$\begin{bmatrix} k_0(u)\mathbf{I}_{2\times 2} & & & \\ & k_1(u) & k_2(u) & \\ & k_3(u) & k_4(u) & \\ & & & k_5(u)\mathbf{I}_{2\times 2} \end{bmatrix} \quad ,$$

for arbitrary boundary parameter ξ_-, and functions,

$$k_0(u) \equiv \big(\exp(2u) + \exp(2n\eta)\big)\Big(\xi^2_- \exp(u + 2n\eta) - \exp(-u)\Big) \quad ,$$
$$k_1(u) \equiv \frac{1}{2}\big(\exp(2u)+1\big)\Big(2\xi_- \exp(2n\eta)\big(\exp(2u)-1\big) - \exp(u)\big(1-\xi^2_-\exp(2n\eta)\big)\big(1+\exp(2n\eta)\big)\Big) \quad ,$$
$$k_2(u) \equiv k_3(u) \equiv \frac{1}{2}\exp(u)\big(\exp(2u)-1\big)\big(1+\xi^2_-\exp(2n\eta)\big)\big(1-\exp(2n\eta)\big) \quad ,$$
$$k_4(u) \equiv \frac{1}{2}\big(\exp(2u)+1\big)\Big(-2\xi_-\exp(2n\eta)\big(\exp(2u)-1\big) - \exp(u)\big(1-\xi^2_-\exp(2n\eta)\big(1+\exp(2n\eta)\big)\Big) \quad ,$$
$$k_5(u) \equiv \big(\exp(2u)+\exp(2n\eta)\big)\big(\xi^2_-\exp(u+2n\eta) - \exp(3u)\big) \quad ,$$

for a real parameter η. The trace of the product of matrices enforcing open boundary conditions, and the R matrices, is obtained by setting $a \equiv 0$ from,

$$\mathrm{tr}_a\bigg(\mathbf{K}^{\mathrm{open}}_{+,a}\big(u\big)\prod_{1\leq j\leq L}\mathbf{R}_{+,aj}\big(u\big)\mathbf{K}^{\mathrm{open}}_{-,a}\big(u\big)\prod_{1\leq j'\leq L}\mathbf{R}_{-,j'a}\big(u\big)\bigg)\quad .$$

From the transfer matrix with open boundary conditions, one can introduce an open integrable Hamiltonian, which can be obtained from rearranging the expression above. To obtain the desired expression for the open, integrable Hamiltonian, we analyze the derivative of the transfer matrix above, upon set $u \equiv 0$,

$$\bigg(\mathbf{T}^{\mathrm{open}}\big(0\big)\bigg)' \equiv \bigg(\mathrm{tr}_0\bigg(\mathbf{K}_{+,0}\big(0\big)\prod_{1\leq j\leq L}\mathbf{R}_{+,j0}\big(0\big)\mathbf{K}_{-,0}\big(0\big)\prod_{1\leq j'\leq L}\mathbf{R}_{-,j'0}\big(0\big)\bigg)\bigg)'\quad ,$$

from solutions to the Bethe equations, which can be formulated by observing that the transfer matrix, with open boundary conditions for the higher rank spin chain, satisfies, along the lines of arguments presented in [1],

$$\begin{aligned}\mathbf{T}\big(u\big)\left|\Lambda\right\rangle &= \Lambda\big(u\big)\left|\Lambda\right\rangle \quad ,\\ \mathbf{h}_j\left|\Lambda\right\rangle &= h_j\left|\Lambda\right\rangle \quad ,\end{aligned}$$

for $1 \leq j \leq 2$, where $|\Lambda\rangle$ denotes the normalized eigenstate of $\mathbf{T}\big(u\big)$. From the two relations provided above, in the presence of twisted boundary conditions parameterized by the angles ϕ_1 and ϕ_2, the eigenvalues take the form, [1],

$$\begin{aligned}\Lambda\big(u\big) \equiv \big[4\mathrm{sinh}\big(u-2i\gamma\big)\mathrm{sinh}\big(u-4i\gamma\big)\big]^L\mathrm{exp}\big(i\phi_1\big)A\big(u\big)\\ +\big[\big(4\mathrm{sinh}\big(u-4i\gamma\big)\mathrm{sinh}\big(u\big)\big]^L\bigg(\mathrm{exp}\big(i\phi_2\big)B_1\big(u\big)+B_2\big(u\big)+B_3\big(u\big)+\mathrm{exp}\big(-i\phi_2\big)B_4\big(u\big)\bigg)\\ +\big[4\mathrm{sinh}\big(u-2i\gamma\big)\mathrm{sinh}\big(u\big)\big]^L\mathrm{exp}\big(-i\phi_1\big)C\big(u\big)\quad ,\end{aligned}$$

for quantities exhibiting the dependencies,

$$\begin{aligned}A\big(u,u^{[1]}_j,\gamma\big) &\equiv A\big(u\big)\quad ,\\ B_1\big(u,u^{[1]}_j,\gamma\big) &\equiv B_1\big(u\big)\quad ,\\ B_2\big(u,u^{[2]}_j,\gamma\big) &\equiv B_2\big(u\big)\quad ,\\ B_3\bigg(B_2\big(u\big),u,u^{[2]}_j,\gamma\bigg) &\equiv B_3\big(u\big)\quad ,\\ B_4\big(u,u^{[1]}_j,u^{[2]}_j,\gamma\big) &\equiv B_4\big(u\big)\quad ,\\ C\bigg(A\big(u\big),u,u^{[1]}_j,\gamma\bigg) &\equiv C\big(u\big)\quad ,\end{aligned}$$

for the parameter $\gamma \in \big(0,\frac{\pi}{4}\big)$, and Bethe roots of the first, and second types, $u^{[1]}_j$, and $u^{[2]}_j$, respectively. In the presence of twisted boundary conditions, the Bethe equations are,

$$\bigg[\frac{\mathrm{sinh}\big(u^{[1]}_j-i\gamma\big)}{\mathrm{sinh}\big(u^{[1]}_j+i\gamma\big)}\bigg]^L = \prod_{k\neq j}^{m_1}\prod_{k=1}^{m_2}\bigg[\bigg[\frac{\mathrm{sinh}\big(u^{[1]}_j-u^{[1]}_k-2i\gamma\big)}{\mathrm{sinh}\big(u^{[1]}_j-u^{[1]}_k+2i\gamma\big)}\bigg]\bigg[\frac{\mathrm{sinh}\big(u^{[1]}_j-u^{[2]}_k+i\gamma\big)}{\mathrm{sinh}\big(u^{[1]}_j-u^{[2]}_k-i\gamma\big)}\bigg]\bigg]\quad .$$

For the higher rank spin chain of the same length L with open boundary conditions, the normalized eigenstates, $|\Lambda^{\mathrm{open}}\rangle$ would satisfy,

$$\mathbf{T}^{\text{open}}(u)\,|\Lambda^{\text{open}}\rangle = \Lambda^{\text{open}}(u)\,|\Lambda^{\text{open}}\rangle \quad ,$$
$$\mathbf{h}_j\,|\Lambda^{\text{open}}\rangle = h_j\,|\Lambda^{\text{open}}\rangle \quad .$$

Irrespective of an explicit form of the eigenstates from the first equality above, asymptotically the Hamiltonian takes the form,

$$\frac{\mathrm{d}}{\mathrm{d}u}\bigg(\log\big(\mathbf{T}^{\text{open}}(u)\big)\bigg)\bigg|_{u\equiv 0} \quad ,$$

from the logarithmic derivative of the higher rank spin chain transfer matrix with open boundary conditions, [6],

$$\mathcal{H}(k,\kappa,\mathbf{K}_-,\mathbf{K}_+) \equiv \mathcal{H} \sim \sum_{1\leq k\leq N-1} h_{k,k+1} + \frac{1}{2\kappa}\bigg(\mathbf{K}_1^-(0)\bigg)' + \frac{1}{\operatorname{tr}\big(\mathbf{K}_+(0)\big)}\operatorname{tr}_0\mathbf{K}_{0,+}(0)h_{N0} \quad ,$$

for the two-site Hamiltonian appearing in the first term,

$$h_{k,k+1} = \frac{1}{\xi(0)}\mathcal{P}_{k,k+1}\bigg(\mathbf{R}_{k,k+1}(0)\bigg)' \quad ,$$

and another Hamiltonian term appearing in the third term,

$$h_{N0} \equiv \frac{1}{\xi(0)}\mathcal{P}_{N,0}\bigg(\mathbf{R}_{N,0}(0)\bigg)' \quad ,$$

for some parameter κ and a permutation $\mathcal{P}$, given by,

$$\mathcal{P} \equiv \sum_{1\leq a,\beta\leq d} e_{\alpha\beta}\otimes e_{\beta\alpha} \quad ,$$

over the basis for the tensor product of d-dimensional vector spaces, $\mathcal{V}\otimes\mathcal{V}$, and the function,

$$\xi(u) \equiv 4\sinh(u+2\eta)\sinh(u+4\eta) \quad .$$

2.2 The open, integrable Hamiltonian

Equipped with the transfer matrix under open boundary conditions, and the corresponding Hamiltonian, we identify eigenvectors of the Hamiltonian obtained in the previous section. To do this, observe,

$$E \propto \big(\Lambda^{\text{open}}(0)\big)'' \quad ,$$

from which we write, [1],

$$E \equiv -\sum_{1\leq k\leq m_1}\frac{2\,\sinh^2(2i\gamma)}{\cosh(2u_k^{[1]}) - \cosh(2i\gamma)} \quad ,$$

corresponding to the energy of the eigenvalues, termed the eigenenergies. To establish the connection between the transfer matrix, integrable Hamiltonian, and boundary CFT, the summation for E above can be expressed as, [8],

$$E = f_0 L + f_s - \frac{\pi v_F\big(\frac{c}{24}-h\big)}{L} + \mathcal{O}(L^{-2}) \quad ,$$

for the length L of the chain, which coincides with the system size, the central charge, conformal weight of the field, the bulk energy density f_0, surface energy f_s, and Fermi velocity,

$$v_F \equiv \frac{2\pi \sin\big(\pi - 2\gamma\big)}{\pi - 2\gamma} \quad .$$

Furthermore, observe from the equation,

$$\mathbf{h}_j \left|\Lambda^{\mathrm{open}}\right\rangle = h_j \left|\Lambda^{\mathrm{open}}\right\rangle \quad ,$$

that the eigenvalues are given by,

$$\begin{aligned} h_1^{\mathrm{open}} &\equiv h_1 \equiv L - m_1 \quad , \\ h_2^{\mathrm{open}} &\equiv h_2 \equiv m_1 - m_2 \quad , \end{aligned}$$

for parameters $m_1 \geq m_2 \geq 0$. The expression for the summation over k given for E above is obtained from the leading term of an expansion of the transfer matrix,

$$\mathbf{T}^{\mathrm{open}}\big(0\big) \approx \big[4\sinh\big(2i\gamma\big)\sinh\big(4i\gamma\big)\big]^L \exp\big(i\mathbf{P}\big) \quad ,$$

where the term multiplying the L th power is given by,

$$\prod_{1 \leq i \leq L} \delta_{a_i}^{b_{a+i}} \quad ,$$

the translation operator under open boundary conditions, under the equivalence, for some $j > 0$,

$$\begin{aligned} \big(a_{L+i+j}\big) \mathrm{mod}\ L &\equiv a_{i+j} \quad , \\ \big(b_{L+i+j}\big) \mathrm{mod}\ L &\equiv b_{i+j} \quad , \\ a_L &\equiv b_1 \quad . \end{aligned}$$

In turn, substituting the leading order term for the natural logarithm of $\mathbf{T}^{\mathrm{open}}\big(0\big)$ into the expansion for the Hamiltonian,

$$\mathbf{H}^{\mathrm{open}} \approx -\sinh\big(2i\gamma\big) \bigg[\frac{\mathrm{d}}{\mathrm{d}u} \bigg(\log\big(\mathbf{T}^{\mathrm{open}}\big(u\big)\big) \bigg) \bigg|_{u \equiv 0} \bigg] + L \sinh\big(2i\gamma\big) \big[\coth\big(2i\gamma\big) + \coth\big(4i\gamma\big)\big] \mathbf{I}^{\otimes L} \quad ,$$

yields an expression for a local Hamiltonian,

$$-\sinh\big(2i\gamma\big) \bigg[\frac{\mathrm{d}}{\mathrm{d}u} \bigg(\log \bigg[\mathrm{tr}_0 \bigg(\mathbf{K}_{+,0}^{\mathrm{open}}\big(u\big) \prod_{1 \leq j \leq L} \mathbf{R}_{+,a0}\big(u\big) \mathbf{K}_{-,0}^{\mathrm{open}}\big(u\big) \prod_{1 \leq j' \leq L} \mathbf{R}_{-,j'0}\big(u\big) \bigg) \bigg] L \sinh\big(2i\gamma\big) \big[\coth\big(2i\gamma\big) + \coth\big(4i\gamma\big)\big] \mathbf{I}^{\otimes L}$$

which is equivalent to, after collecting like terms,

$$-\sinh\big(2i\gamma\big) \bigg[\frac{\mathrm{d}}{\mathrm{d}u} \bigg(\log \bigg[\mathrm{tr}_0 \bigg(\mathbf{K}_{+,0}^{\mathrm{open}}\big(u\big) \prod_{1 \leq j \leq L} \mathbf{R}_{+,a0}\big(u\big) \mathbf{K}_{-,0}^{\mathrm{open}}\big(u\big) \prod_{1 \leq j' \leq L} \mathbf{R}_{-,j'0}\big(u\big) \bigg) \bigg] + L \Big[\coth\big(2i\gamma\big) + \coth\big(4i\gamma\big) \big] \mathbf{I}^{\otimes L} \Big] \quad ,$$

in terms of the site translation operator. Computing the derivative of the natural logarithm of the transfer matrix for the lower rank spin chain under open boundary conditions, and evaluating at $u \equiv 0$, yields approximately to first order,

$$\left(\mathrm{tr}_0\left(\mathbf{K}^{\mathrm{open}}_{+,0}(u)\prod_{1\leq j\leq L}\mathbf{R}_{+,a0}(u)\mathbf{K}^{\mathrm{open}}_{-,0}(u)\prod_{1\leq j'\leq L}\mathbf{R}_{-,j'0}(u)\right)\right)^{-1}\left(\left[4\mathrm{sinh}(2i\gamma)\mathrm{sinh}(4i\gamma)\right]^L\exp(i\mathbf{P})\right) \quad .$$

implies the approximation,

$$-\mathrm{sinh}(2i\gamma)\Bigg[\left(\mathrm{tr}_0\left(\mathbf{K}^{\mathrm{open}}_{+,0}(u)\prod_{1\leq j\leq L}\mathbf{R}_{+,a0}(u)\mathbf{K}^{\mathrm{open}}_{-,0}(u)\prod_{1\leq j'\leq L}\mathbf{R}_{-,j'0}(u)\right)\right)^{-1}\left(\left[4\mathrm{sinh}(2i\gamma)\mathrm{sinh}(4i\gamma)\right]^L\exp(i\mathbf{P})\right)$$
$$+L\left[\coth(2i\gamma)+\coth(4i\gamma)\right]\mathbf{I}^{\otimes L}\Bigg] \quad .$$

for the open boundary Hamiltonian.

2.3 Statement of the Bethe equations for anisotropy parameters approaching 0, and the root density

For anisotropy parameters that are very close to 0, the Bethe equations,

$$\left[\frac{\sinh(u^{[1]}_j-i\gamma)}{\sinh(u^{[1]}_j+i\gamma)}\right]^L=\prod_{k\neq j}^{m_1}\prod_{k=1}^{m_2}\left[\left[\frac{\sinh(u^{[1]}_j-u^{[1]}_k-2i\gamma)}{\sinh(u^{[1]}_j-u^{[1]}_k+2i\gamma)}\right]\left[\frac{\sinh(u^{[1]}_j-u^{[2]}_k+i\gamma)}{\sinh(u^{[1]}_j-u^{[2]}_k-i\gamma)}\right]\right] \quad ,$$

can be approximated with the relations,

$$\left[\frac{u^{[1]}_j-i}{u^{[1]}_j+i}\right]^L=\prod_{k\neq j}^{m_1}\prod_{k=1}^{m_2}\left[\left[\frac{u^{[1]}_j-u^{[k]}_k-2i}{u^{[1]}_j-u^{[2]}_k+2i}\right]\left[\frac{u^{[1]}_j-u^{[2]}_k+i}{u^{[1]}_j-u^{[2]}_k-i}\right]\right] \quad .$$

From the fact that the spin-chain has rank two, there exists a mapping between pairs $\{\lambda_j,-\lambda_j\}$, and two possible solutions to the Bethe equations,

$$\begin{aligned}\lambda_j &\Longleftrightarrow u^{[1]}_j \quad ,\\ -\lambda_j &\Longleftrightarrow -u^{[1]}_j \quad ,\\ \lambda_k &\Longleftrightarrow u^{[1]}_k \quad ,\\ -\lambda_k &\Longleftrightarrow -u^{[1]}_k \quad ,\end{aligned}$$

take the form,

$$\left[\frac{\lambda_j-i}{\lambda_j+i}\right]^L\approx\prod_{k\neq j}^{m_1}\prod_{k=1}^{m_2}\left[\left[\frac{\lambda_j-\lambda_k-2i}{\lambda_j-\lambda_k+2i}\right]\left[\frac{\lambda_j-\lambda_k+i}{\lambda_j-\lambda_k-i}\right]\right] \quad ,$$

under the assumption that,

$$\begin{aligned}\sin(u^{[1]}_j-i\gamma)&\approx u^{[1]}_j-i \quad ,\\ \sin(u^{[1]}_j+i\gamma)&\approx u^{[1]}_j+i \quad ,\end{aligned}$$

for $\gamma\approx 0$. Under the identification, [1],

$$\begin{aligned}u^{[1]}_j &\longrightarrow x_j+\delta^{[1]}_j+i\frac{\pi}{2}-i(\gamma-\epsilon^{[1]}_j) \quad ,\\ u^{[2]}_j &\longrightarrow x_j+\delta^{[2]}_j+i(\frac{\pi}{2}+\epsilon^{[2]}_j) \quad ,\end{aligned}$$

of the first and second roots of the Bethe equation, for sufficiently small parameters,

$$\delta_j^{[1]}, \delta_j^{[2]}, \epsilon_j^{[1]}, \epsilon_j^{[2]} \in \mathbf{R} \quad ,$$

whose complex conjugates satisfy,

$$\overline{u_j^{[1]}} \longrightarrow x_j + \delta_j^{[1]} - i\frac{\pi}{2} + i\big(\gamma - \epsilon_j^{[1]}\big) \quad ,$$
$$\overline{u_j^{[2]}} \longrightarrow x_j + \delta_j^{[2]} - i\big(\frac{\pi}{2} + \epsilon_j^{[2]}\big) \quad ,$$

one can substitute these expressions for the first and second root types appearing in the Bethe equation, with the following rearrangements.

Given the two possible root types for solutions to the Bethe equation, for even L,

$$\log\bigg[\, \bigg|\frac{\sinh\big(u_j^{[1]} - i\big)}{\sinh\big(u_j^{[1]} + i\big)}\bigg|^L \,\bigg] \approx \log\bigg[\, \bigg|\frac{u_j^{[1]} - i}{u_j^{[1]} + i}\bigg|^L \,\bigg] = \log\bigg[\, \bigg|\frac{x_j + \delta_j^{[1]} + i\frac{\pi}{2} - i\big(\gamma - \epsilon_j^{[1]}\big)}{x_j + \delta_j^{[2]} + i\big(\frac{\pi}{2} + \epsilon_j^{[2]}\big)}\bigg|^L \,\bigg]$$
$$= L\bigg[\, \log\big[\big|x_j + \delta_j^{[1]} + i\frac{\pi}{2} - i\big(\gamma - \epsilon_j^{[1]}\big)\big|\big] - \log\big[\big|x_j + \delta_j^{[2]} + i\big(\frac{\pi}{2} + \epsilon_j^{[2]}\big)\big|\big] \,\bigg]$$
$$= L\bigg[\, \log\big[x_j + \delta_j^{[1]} + i\frac{\pi}{2} - i\big(\gamma - \epsilon_j^{[1]}\big)\big] - \log\big[x_j + \delta_j^{[2]} + i\big(\frac{\pi}{2} + \epsilon_j^{[2]}\big)\big] \,\bigg] \quad ,$$

corresponding to terms on LHS of the Bethe equations, and,

$$\log\bigg[\, \prod_{k\neq j}^{m_1} \prod_{k=1}^{m_2} \bigg[\, \bigg[\frac{\sinh\big(u_j^{[1]} - u_k^{[k]} - 2i\big)}{\sinh\big(u_j^{[1]} - u_k^{[2]} + 2i\big)}\bigg] \bigg[\frac{\sinh\big(u_j^{[1]} - u_k^{[2]} + i\big)}{\sinh\big(u_j^{[1]} - u_k^{[2]} - i\big)}\bigg] \,\bigg] \,\bigg] = \log\bigg[\, \prod_{k\neq j}^{m_1} \prod_{k=1}^{m_2} \bigg[\frac{\sinh\big(u_j^{[1]} - u_k^{[k]} - 2i\big)}{\sinh\big(u_j^{[1]} - u_k^{[2]} + 2i\big)}\bigg] \,\bigg]$$
$$+\log\bigg[\, \prod_{k\neq j}^{m_1} \prod_{k=1}^{m_2} \bigg[\frac{\sinh\big(u_j^{[1]} - u_k^{[2]} + i\big)}{\sinh\big(u_j^{[1]} - u_k^{[2]} - i\big)}\bigg] \,\bigg]$$
$$\approx \log\bigg[\prod_{k\neq j}^{m_1} \prod_{k=1}^{m_2} \bigg[\frac{u_j^{[1]} - u_k^{[k]} - 2i}{u_j^{[1]} - u_k^{[2]} + 2i}\bigg] \,\bigg] + \log\bigg[\prod_{k\neq j}^{m_1} \prod_{k=1}^{m_2} \bigg[\frac{u_j^{[1]} - u_k^{[2]} + i}{u_j^{[1]} - u_k^{[2]} - i}\bigg] \,\bigg] \quad ,$$

corresponding to terms on the RHS of the Bethe equations, which can be expressed as,

$$\log\bigg[\prod_{k\neq j}^{m_1} \prod_{k=1}^{m_2} \bigg[\frac{x_j - x_k + \delta_j^{[1]} - \delta_k^{[2]} + i\big(1 + \frac{\pi}{2}\big) - i\big(\gamma - \epsilon_j^{[1]} - \frac{\pi}{2} - \epsilon_k^{[2]}\big)}{x_j - x_k + \delta_j^{[1]} - \delta_k^{[2]} + i\big(-1 + \frac{\pi}{2}\big) - i\big(\gamma - \epsilon_j^{[1]} - \frac{\pi}{2} - \epsilon_k^{[2]}\big)}\bigg]\bigg]$$
$$+\log\bigg[\prod_{k\neq j}^{m_1} \prod_{k=1}^{m_2} \bigg[\frac{x_j - x_k + \delta_j^{[1]} - \delta_k^{[2]} + i\big(-2 + \frac{\pi}{2}\big) - i\big(\gamma - \epsilon_j^{[1]} - \frac{\pi}{2} - \epsilon_k^{[2]}\big)}{x_j - x_k + \delta_j^{[1]} - \delta_k^{[2]} + i\big(2 + \frac{\pi}{2}\big) - i\big(\gamma - \epsilon_j^{[1]} - \frac{\pi}{2} - \epsilon_k^{[2]}\big)}\bigg]\bigg] \quad .$$

Hence,

$$L\bigg[\, \log\big[x_j + \delta_j^{[1]} + i\frac{\pi}{2} - i\big(\gamma - \epsilon_j^{[1]}\big)\big] - \log\big[x_j + \delta_j^{[2]} + i\big(\frac{\pi}{2} + \epsilon_j^{[2]}\big)\big] \,\bigg]$$
$$\approx \log\bigg[\prod_{k\neq j}^{m_1} \prod_{k=1}^{m_2} \bigg[\frac{x_j - x_k + \delta_j^{[1]} - \delta_k^{[2]} + i\big(1 + \frac{\pi}{2}\big) - i\big(\gamma - \epsilon_j^{[1]} - \frac{\pi}{2} - \epsilon_k^{[2]}\big)}{x_j - x_k + \delta_j^{[1]} - \delta_k^{[2]} + i\big(-1 + \frac{\pi}{2}\big) - i\big(\gamma - \epsilon_j^{[1]} - \frac{\pi}{2} - \epsilon_k^{[2]}\big)}\bigg]\bigg]$$
$$+\log\bigg[\prod_{k\neq j}^{m_1} \prod_{k=1}^{m_2} \bigg[\frac{x_j - x_k + \delta_j^{[1]} - \delta_k^{[2]} + i\big(-2 + \frac{\pi}{2}\big) - i\big(\gamma - \epsilon_j^{[1]} - \frac{\pi}{2} - \epsilon_k^{[2]}\big)}{x_j - x_k + \delta_j^{[1]} - \delta_k^{[2]} + i\big(2 + \frac{\pi}{2}\big) - i\big(\gamma - \epsilon_j^{[1]} - \frac{\pi}{2} - \epsilon_k^{[2]}\big)}\bigg]\bigg] \quad .$$

In terms of λ_j and λ_k, the approximate relation for the Bethe equations for anisotropy parameters that are approximately 0 reads,

$$L\ \log\Big[\frac{\lambda_j - i}{\lambda_j + i}\Big] \approx \log\Big[\prod_{k\neq j}^{m_1}\prod_{k=1}^{m_2}\Big[\ \Big[\frac{\lambda_j-\lambda_k-2i}{\lambda_j-\lambda_k+2i}\Big]\ \Big[\frac{\lambda_j-\lambda_k+i}{\lambda_j-\lambda_k-i}\Big]\ \Big]\ \Big]\quad ,$$

under the identification,

$$\lambda_j \longrightarrow \hat{x_j} + \hat{\delta_j^{[1]}} + i\frac{\pi}{2} - i\big(\gamma - \hat{\epsilon_j^{[1]}}\big)\quad ,$$
$$\lambda_k \longrightarrow \hat{x_j} + \hat{\delta_j^{[2]}} + i\big(\frac{\pi}{2} + \hat{\epsilon_j^{[2]}}\big)\quad ,$$

for sufficiently small parameters,

$$\hat{x_j}, \hat{\delta_j^{[1]}}, \hat{\epsilon_j^{[1]}}, \hat{x_j}, \hat{\delta_j^{[2]}}, \hat{\epsilon_j^{[2]}} \in \mathbf{R}\quad .$$

Under invariance of solutions to the Bethe equations, in which solutions come in pairs $\{\lambda_j, -\lambda_j\}$ and $\{\lambda_k, -\lambda_k\}$, the Bethe equations also take the form,

$$L\Big[\ \log\Big[-\Big(x_j + \delta_j^{[1]} + i\frac{\pi}{2} - i\big(\gamma - \epsilon_j^{[1]}\big)\Big)\Big] - \log\Big[-\Big(x_j + \delta_j^{[2]} + i\big(\frac{\pi}{2} + \epsilon_j^{[2]}\big)\Big)\Big]\ \Big]$$
$$\approx \log\Big[\prod_{k\neq j}^{m_1}\prod_{k=1}^{m_2}\Big[\frac{-\Big(x_j - x_k + \delta_j^{[1]} - \delta_k^{[2]} + i\big(1+\frac{\pi}{2}\big) - i\big(\gamma - \epsilon_j^{[1]} - \frac{\pi}{2} - \epsilon_k^{[2]}\big)\Big)}{-\Big(x_j - x_k + \delta_j^{[1]} - \delta_k^{[2]} + i\big(-1+\frac{\pi}{2}\big) - i\big(\gamma - \epsilon_j^{[1]} - \frac{\pi}{2} - \epsilon_k^{[2]}\big)\Big)}\Big]\Big]$$
$$+\log\Big[\prod_{k\neq j}^{m_1}\prod_{k=1}^{m_2}\Big[\frac{-\Big(x_j - x_k + \delta_j^{[1]} - \delta_k^{[2]} + i\big(-2+\frac{\pi}{2}\big) - i\big(\gamma - \epsilon_j^{[1]} - \frac{\pi}{2} - \epsilon_k^{[2]}\big)\Big)}{-\Big(x_j - x_k + \delta_j^{[1]} - \delta_k^{[2]} + i\big(2+\frac{\pi}{2}\big) - i\big(\gamma - \epsilon_j^{[1]} - \frac{\pi}{2} - \epsilon_k^{[2]}\big)\Big)}\Big]\Big]\quad ,$$

from the fact that the identification from [1] also takes the form,

$$-u_j^{[1]} \longrightarrow -\Big(x_j + \delta_j^{[1]} + i\frac{\pi}{2} - i\big(\gamma - \epsilon_j^{[1]}\big)\Big)\quad ,$$
$$-u_j^{[2]} \longrightarrow -\Big(x_j + \delta_j^{[2]} + i\big(\frac{\pi}{2} + \epsilon_j^{[2]}\big)\Big)\quad ,$$

for pairs of solutions, $\{u_j^{[1]}, -u_j^{[1]}\}$ and $\{u_k^{[1]}, -u_k^{[1]}\}$. From the set of relations above for the Bethe equations after taking natural logarithms, one obtains the density for roots of the Bethe equations which can be used to study the ground state, [1],

$$\rho^x(x) \equiv \frac{1}{2(\pi - 4\gamma)}\Big(\cosh\big(\frac{\pi x}{\pi - 4\gamma}\big)\Big)^{-1}\quad ,$$

from the expression for the centers of the counting function, [1], from the root density approach,

$$z^x(x) \equiv \frac{1}{2\pi}\Big(\psi(x, 2\gamma) + \frac{1}{L}\sum_{1\leq k\leq\frac{L}{2}}\chi(x - x_k, 4\gamma)\Big)\quad ,$$

for roots of the Bethe equation. For the counting function above, the two functions are given by,

$$\chi(x,y) \equiv 2\ \arctan\Big(\tanh(x)\cot(y)\Big)\quad ,$$
$$\psi(x,y) \equiv 2\ \arctan\Big(\tanh(x)\tan(y)\Big)\quad .$$

Altogether, the density approximation of the roots to the Bethe equation for anisotropy parameters which almost vanishes falls into the following characterization:

- Ground state: $h_1 \equiv h_2 \equiv 0$,

- Type I excitation to the ground state: $h_1 > 0$, $h_2 \equiv 0$,

- Type II excitation to the ground state: $h_1 \equiv 0$, $h_2 > 0$,

- Type III excitation to the ground state: $h_1, h_2 > 0$.

Under each set of possible choices for h_1 and h_2 provided above, one can characterize solutions to the Bethe equations from the density provided earlier with , similar to the arrangement of roots provided in *Figure 1*, *Figure 2*, *Figure 3*, *Figure 4*, and *Figure 5* of [1].

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