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Conformal perturbations of dirac operators and general Kastler-Kalau-Walze type theorems for even dimensional manifolds with boundary

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Abstract

In this paper, we establish the proof of general Kastler-Kalau-Walze type theorems for conformal perturbations of dirac Operators on even dimensional compact manifolds with (respectively without) boundary.

Keywords: Conformal perturbations of dirac Operators; noncommutative residue; Lichnerowicz type formulas; Kastler-Kalau-Walze type theorem.

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1. Introduction

The noncommutative residue plays a significant role in noncommutative geometry, which has been extensively studied by geometers [1, 2]. Adler discovered the noncommutative residue for one-dimensional manifolds in [3], where he explored the geometric aspects of nonlinear partial differential equations. Wodzicki introduced the noncommutative residue for arbitrary closed compact n-dimensional manifolds in [2] using the theory of zeta functions of elliptic pseudodifferential operators. In [4], Connes used the noncommutative residue to derive a conformal 4-dimensional Polyakov action analogy. Furthermore, Connes claimed the noncommutative residue of the square of the inverse of the Dirac operator was proportioned to the Einstein-Hilbert action in [5]. Kastler provided a brute-force proof of this theorem in [6], while Kalau and Walze proved it in the normal coordinates system simultaneously in [7]. Moreover, Ackermann proved that the Wodzicki residue of the square of the inverse of the Dirac operator $\mathrm{Wres}(D^{-2})$ in turn is essentially the second coefficient of the heat kernel expansion of D^2 in [8].

On the other hand, Fedosov etc. definied a noncommutative residue on Boutet de Monvel's algebra and proved that it was a unique continuous trace in [9]. Schrohe established a relationship between the Dixmier trace and the noncommutative residue for manifolds with boundary in [10]. In [11, 12], Wang computed $\widetilde{\mathrm{Wres}}[\pi^+D^{-1}\circ\pi^+D^{-1}]$ and $\widetilde{\mathrm{Wres}}[\pi^+D^{-2}\circ\pi^+D^{-2}]$, where the two operators are symmetric, in these cases the boundary term vanished. But for $\widetilde{\mathrm{Wres}}[\pi^+D^{-1}\circ\pi^+D^{-3}]$, J. Wang and Y. Wang got a nonvanishing boundary term [13], and give a theoretical explanation for gravitational action on boundary. In others words, Wang provided a kind of method to study the Kastler-Kalau-Walze type theorem for manifolds with boundary.

In [17], Wang established a Kastler-Kalau-Walze type theorem for perturbations of Dirac operators on compact manifolds with (respectively without) boundary. In [16], Wei and Wang establish two Kastler-Kalau-Walze type theorems for conformal perturbations of modified Novikov Operators on 4-dimensional and 6-dimensional compact manifolds with(respectively without) boundary. In [14], J. Wang and Y.

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Wang computed $\widetilde{\mathrm{Wres}}[(\pi^+D^{-2})\circ(\pi^+D^{-n+2})]$ for manifolds with any dimension and boundary, and established a general Kastler-Kalau-Walze type theorem. **The motivation of this paper** is to establish the proof of general Kastler-Kalau-Walze type theorems for conformal perturbations of dirac Operators on even dimensional compact manifolds with (respectively without) boundary. In this paper, the leading symbol of dirac operator is $ic(\xi)$. At the moment, the leading symbol of conformal perturbations of dirac operators is not $ic(\xi)$, which motivates the study of the residue of conformal perturbations of dirac operators. That is, we want to compute $\mathrm{Wres}[\pi^+P_1\circ\pi^+P_2]$, where orders of P_1, P_2 are a_1, a_2 and $-a_1-a_2+2=m$ for even dimensional manifolds with boundary. Motivated by [14, 16], we compute the generalized noncommutative residue $\widetilde{\mathrm{Wres}}\Big[\pi^+(fD^{-1}f^{-1}D^{-1})\circ\pi^+\Big((fD^{-1}f^{-1}D^{-1})^n\Big)\Big]$ and $\widetilde{\mathrm{Wres}}\Big[\pi^+(fD^{-1})\circ\pi^+\Big((f^{-1}D^{-1})\cdot(fD^{-1}\cdot f^{-1}D^{-1})^n\Big)\Big]$ on even dimensional manifolds. Our main theorems are as follows.

Theorem 1.1. *Let M be an $m=2n+4$ dimensional oriented compact spin manifold with boundary ∂M, then we get the following equality:*

$$\begin{aligned}
&\widetilde{\mathrm{Wres}}\Big[\pi^+(fD^{-1}f^{-1}D^{-1})\circ\pi^+\Big((fD^{-1}f^{-1}D^{-1})^n\Big)\Big]\\
=\;&\frac{(2\pi)^{2n+6}}{(2n+4)!}\int_M 2^{2n+6}\Big\{-\frac{1}{12}s-2f^{-1}\Delta(f)-f^{-2}\Big[|\mathrm{grad}_Mf|^2+2\Delta(f)\Big]\Big\}d\mathrm{Vol_M}\\
&+\int_{\partial M}\Big\{\frac{2^nh'(0)\pi ni}{(n+3)!}\mathrm{Vol}(S_{2n+2})Q_0\Big\}d\mathrm{Vol}_{\partial_M},
\end{aligned}\tag{1.1}$$

where Q_0 are defined in (3.42).

Theorem 1.2. *Let M be an $m=2n+4$ dimensional oriented compact spin manifold with boundary ∂M, then we get the following equality:*

$$\begin{aligned}
&\widetilde{\mathrm{Wres}}\Big[\pi^+(fD^{-1})\circ\pi^+\Big((f^{-1}D^{-1})\cdot(fD^{-1}\cdot f^{-1}D^{-1})^n\Big)\Big]\\
=\;&\frac{(2\pi)^{2n+6}}{(2n+4)!}\int_M 2^{2n+6}\Big\{-\frac{1}{12}s-2f^{-1}\Delta(f)-f^{-2}\Big[|\mathrm{grad}_Mf|^2+2\Delta(f)\Big]\Big\}d\mathrm{Vol_M}\\
&+\int_{\partial M}\Big\{\frac{(-1)^nh'(0)\pi}{3\times2^{n+6}(3+n)!}Y_0+\frac{(-1)^n(n+1)f^{-1}\partial_{x_n}(f)\pi}{2^{n+2}}Y_1+\Big[\frac{(i-1)f\cdot\partial_{x_n}(f^{-1})}{2^{n+3}}\\
&-\frac{\partial_{x_n}(f)(1+i)}{2^{n+2}}\Big]Y_2\Big\}d\mathrm{Vol}_{\partial_M},
\end{aligned}\tag{1.2}$$

where $Y_0,\ Y_1,\ Y_2$ are defined in (4.52).

The paper is organized in the following way. In Section 2, we review some basic formulas related to Boutet de Monvel's calculus and the definition of the noncommutative residue for manifolds with boundary. In Section 3, we prove the general Kastler-Kalau-Walze type theorem for $\widetilde{\mathrm{Wres}}\Big[\pi^+(fD^{-1}f^{-1}D^{-1})\circ\pi^+\Big((fD^{-1}f^{-1}D^{-1})^n\Big)\Big]$ on even dimensional manifolds with boundary. In Section 4, we prove the general Kastler-Kalau-Walze type theorem $\widetilde{\mathrm{Wres}}\Big[\pi^+(fD^{-1})\circ\pi^+\Big((f^{-1}D^{-1})\cdot(fD^{-1}\cdot f^{-1}D^{-1})^n\Big)\Big]$ on even dimensional manifolds with boundary.

2. Boutet de Monvel's calculus and the definition of the noncommutative residue

In this section, we recall some basic facts and formulas about Boutet de Monvel's calculus and the definition of the noncommutative residue for manifolds with boundary which will be used in the following. For more details, see Section 2 in [11].

Let M be a 4-dimensional compact oriented manifold with boundary ∂M. We assume that the metric g^{TM} on M has the following form near the boundary,

$$g^M=\frac{1}{h(x_n)}g^{\partial M}+dx_n^2, \tag{2.1}$$

where $g^{\partial M}$ is the metric on ∂M and $h(x_n)\in C^\infty([0,1)):=\{\widehat{h}|_{[0,1)}|\widehat{h}\in C^\infty((-\varepsilon,1))\}$ for some $\varepsilon>0$ and $h(x_n)$ satisfies $h(x_n)>0$, $h(0)=1$ where x_n denotes the normal directional coordinate. Let $U\subset M$ be a collar neighborhood of ∂M which is diffeomorphic with $\partial M\times[0,1)$. By the definition of $h(x_n)\in C^\infty([0,1))$ and $h(x_n)>0$, there exists $\widehat{h}\in C^\infty((-\varepsilon,1))$ such that $\widehat{h}|_{[0,1)}=h$ and $\widehat{h}>0$ for some sufficiently small $\varepsilon>0$. Then there exists a metric g' on $\widetilde{M}=M\bigcup_{\partial M}\partial M\times(-\varepsilon,0]$ which has the form on $U\bigcup_{\partial M}\partial M\times(-\varepsilon,0]$

$$g'=\frac{1}{\widehat{h}(x_n)}g^{\partial M}+dx_n^2, \tag{2.2}$$

such that $g'|_M=g$. We fix a metric g' on the $\widetilde{M}$ such that $g'|_M=g$.

Let the Fourier transformation F' be

$$F':L^2(\mathbf{R}_t)\to L^2(\mathbf{R}_v);\ F'(u)(v)=\int_{\mathbb{R}}e^{-ivt}u(t)dt$$

and let

$$r^+:C^\infty(\mathbf{R})\to C^\infty(\widetilde{\mathbf{R}^+});\ f\to f|\widetilde{\mathbf{R}^+};\ \widetilde{\mathbf{R}^+}=\{x\geq 0;x\in\mathbf{R}\}.$$

We define $H^+=F'(\Phi(\widetilde{\mathbf{R}^+}))$; $H_0^-=F'(\Phi(\widetilde{\mathbf{R}^-}))$ which satisfies $H^+\perp H_0^-$, where $\Phi(\widetilde{\mathbf{R}^+})=r^+\Phi(\mathbf{R})$, $\Phi(\widetilde{\mathbf{R}^-})=r^-\Phi(\mathbf{R})$ and $\Phi(\mathbf{R})$ denotes the Schwartz space. We have the following property: $h\in H^+$ (respectively H_0^-) if and only if $h\in C^\infty(\mathbf{R})$ which has an analytic extension to the lower (respectively upper) complex half-plane $\{\mathrm{Im}\xi<0\}$ (respectively $\{\mathrm{Im}\xi>0\}$) such that for all nonnegative integer l,

$$\frac{d^lh}{d\xi^l}(\xi)\sim\sum_{k=1}^{\infty}\frac{d^l}{d\xi^l}(\frac{c_k}{\xi^k}),$$

as $|\xi|\to+\infty,\mathrm{Im}\xi\leq 0$ (respectively $\mathrm{Im}\xi\geq 0$) and where $c_k\in\mathbb{C}$ are some constants.

Let H' be the space of all polynomials and $H^-=H_0^-\bigoplus H'$; $H=H^+\bigoplus H^-$. Denote by π^+ (respectively π^-) the projection on H^+ (respectively H^-). Let $\widetilde{H}=\{$rational functions having no poles on the real axis$\}$. Then on $\tilde{H}$,

$$\pi^+h(\xi_0)=\frac{1}{2\pi i}\lim_{u\to 0^-}\int_{\Gamma^+}\frac{h(\xi)}{\xi_0+iu-\xi}d\xi, \tag{2.3}$$

where Γ^+ is a Jordan closed curve included $\mathrm{Im}(\xi)>0$ surrounding all the singularities of h in the upper half-plane and $\xi_0\in\mathbf{R}$. In our computations, we only compute π^+h for h in $\widetilde{H}$. Similarly, define π' on $\tilde{H}$,

$$\pi'h=\frac{1}{2\pi}\int_{\Gamma^+}h(\xi)d\xi. \tag{2.4}$$

So $\pi'(H^-)=0$. For $h\in H\bigcap L^1(\mathbf{R})$, $\pi'h=\frac{1}{2\pi}\int_{\mathbf{R}}h(v)dv$ and for $h\in H^+\bigcap L^1(\mathbf{R})$, $\pi'h=0$.

An operator of order $m\in\mathbf{Z}$ and type d is a matrix

$$\widetilde{A}=\begin{pmatrix}\pi^+P+G & K\\ T & \widetilde{S}\end{pmatrix}:\quad \begin{matrix}C^\infty(M,E_1)\\ \bigoplus\\ C^\infty(\partial M,F_1)\end{matrix}\quad\longrightarrow\quad\begin{matrix}C^\infty(M,E_2)\\ \bigoplus\\ C^\infty(\partial M,F_2)\end{matrix},$$

where M is a manifold with boundary ∂M and E_1, E_2 (respectively F_1, F_2) are vector bundles over M (respectively ∂M). Here, $P : C_0^{\infty}(\Omega, \overline{E_1}) \to C^{\infty}(\Omega, \overline{E_2})$ is a classical pseudodifferential operator of order m on Ω, where Ω is a collar neighborhood of M and $\overline{E_i}|M = E_i$ $(i = 1, 2)$. P has an extension: $\mathcal{E}'(\Omega, \overline{E_1}) \to \mathcal{D}'(\Omega, \overline{E_2})$, where $\mathcal{E}'(\Omega, \overline{E_1})$ $(\mathcal{D}'(\Omega, \overline{E_2}))$ is the dual space of $C^{\infty}(\Omega, \overline{E_1})$ $(C_0^{\infty}(\Omega, \overline{E_2}))$. Let $e^+ : C^{\infty}(M, E_1) \to \mathcal{E}'(\Omega, \overline{E_1})$ denote extension by zero from M to Ω and $r^+ : \mathcal{D}'(\Omega, \overline{E_2}) \to \mathcal{D}'(\Omega, E_2)$ denote the restriction from Ω to X, then define

$$\pi^+ P = r^+ P e^+ : C^{\infty}(M, E_1) \to \mathcal{D}'(\Omega, E_2).$$

In addition, P is supposed to have the transmission property; this means that, for all j, k, α, the homogeneous component p_j of order j in the asymptotic expansion of the symbol p of P in local coordinates near the boundary satisfies:

$$\partial_{x_n}^k \partial_{\xi'}^{\alpha} p_j(x', 0, 0, +1) = (-1)^{j-|\alpha|} \partial_{x_n}^k \partial_{\xi'}^{\alpha} p_j(x', 0, 0, -1),$$

then $\pi^+ P : C^{\infty}(M, E_1) \to C^{\infty}(M, E_2)$. Let G, T be respectively the singular Green operator and the trace operator of order m and type d. Let K be a potential operator and S be a classical pseudodifferential operator of order m along the boundary. Denote by $B^{m,d}$ the collection of all operators of order m and type d, and $\mathcal{B}$ is the union over all m and d.

Recall that $B^{m,d}$ is a Fréchet space. The composition of the above operator matrices yields a continuous map: $B^{m,d} \times B^{m',d'} \to B^{m+m', \max\{m'+d, d'\}}$. Write

$$\widetilde{A} = \begin{pmatrix} \pi^+ P + G & K \\ T & \widetilde{S} \end{pmatrix} \in B^{m,d}, \widetilde{A}' = \begin{pmatrix} \pi^+ P' + G' & K' \\ T' & \widetilde{S}' \end{pmatrix} \in B^{m',d'}.$$

The composition $\widetilde{A}\widetilde{A}'$ is obtained by multiplication of the matrices (For more details see [6]). For example $\pi^+ P \circ G'$ and $G \circ G'$ are singular Green operators of type d' and

$$\pi^+ P \circ \pi^+ P' = \pi^+(PP') + L(P, P').$$

Here PP' is the usual composition of pseudodifferential operators and $L(P, P')$ called leftover term is a singular Green operator of type $m' + d$. For our case, P, P' are classical pseudo differential operators, in other words $\pi^+ P \in \mathcal{B}^{\infty}$ and $\pi^+ P' \in \mathcal{B}^{\infty}$.

Let M be a n-dimensional compact oriented manifold with boundary ∂M. Denote by $\mathcal{B}$ the Boutet de Monvel's algebra. We recall that the main theorem in [9, 11].

Theorem 2.1. [9]**(Fedosov-Golse-Leichtnam-Schrohe)** *Let M and ∂M be connected,* $\dim M = n \geq 3$, *and let $\widetilde{S}$ (respectively $\widetilde{S}'$) be the unit sphere about ξ (respectively ξ') and $\sigma(\xi)$ (respectively $\sigma(\xi')$) be the corresponding canonical $n-1$ (respectively $(n-2)$) volume form. Set $\widetilde{A} = \begin{pmatrix} \pi^+ P + G & K \\ T & \widetilde{S} \end{pmatrix} \in \mathcal{B}$, and denote by p, b and s the local symbols of P, G and $\widetilde{S}$ respectively. Define:*

$$\begin{aligned} \widetilde{\mathrm{Wres}}(\widetilde{A}) = & \int_X \int_{\mathbf{\widetilde{S}}} \mathrm{trace}_E \left[p_{-n}(x, \xi)\right] \sigma(\xi) dx \\ & + 2\pi \int_{\partial X} \int_{\mathbf{\widetilde{S}}'} \left\{\mathrm{trace}_E \left[(\mathrm{tr} b_{-n})(x', \xi')\right] + \mathrm{trace}_F \left[s_{1-n}(x', \xi')\right]\right\} \sigma(\xi') dx', \end{aligned} \tag{2.5}$$

where $\widetilde{\mathrm{Wres}}$ denotes the noncommutative residue of an operator in the Boutet de Monvel's algebra.
Then a) $\widetilde{\mathrm{Wres}}([\widetilde{A}, B]) = 0$, for any $\widetilde{A}, B \in \mathcal{B}$; b) It is the unique continuous trace on $\mathcal{B}/\mathcal{B}^{-\infty}$.

Theorem 2.2. [18] *For even m-dimensional compact spin manifolds without boundary, the following equality holds:*

$$\mathrm{Wres}\left[fDf^{-1}D^{-1}\right]^{\frac{m-2}{2}} = \frac{(2\pi)^{\frac{m}{2}}}{(\frac{m}{2}-2)!} \int_M \mathrm{trace}\left\{-\frac{1}{12}s - 2f^{-1}\Delta(f) - f^{-2}[|\mathrm{grad}_M f|^2 + 2\Delta(f)]\right\} dvol_M, \tag{2.6}$$

where s is the scaler curvature.

3. The noncommutative residue $\widetilde{\mathbf{Wres}}\Big[\boldsymbol{\pi^+(fD^{-1}f^{-1}D^{-1})\circ\pi^+\Big((fD^{-1}f^{-1}D^{-1})^n\Big)}\Big]$ on even dimensional manifolds with boundary

Firstly, we recall the definition of the Dirac operator. Let M be an $m=2n+4$ dimensional oriented compact spin Riemannian manifold with a Riemannian metric g^M and let ∇^L be the Levi-Civita connection about g^M.

Set $\widetilde{e}_m=\frac{\partial}{\partial x_m}$, $\widetilde{e}_j=\sqrt{h(x_m)}e_j\ \ (1\leq j\leq m-1)$, where $\{e_1,\cdots,e_{m-1}\}$ are orthonormal basis of $T\partial_M$. In the local coordinates $\{x_i;1\leq i\leq m\}$ and the fixed orthonormal frame $\{\widetilde{e}_1,\cdots,\widetilde{e}_m\}$, the connection matrix $(\omega_{s,t})$ is defined by

$$\nabla^L(\widetilde{e}_1,\cdots,\widetilde{e}_m)=(\widetilde{e}_1,\cdots,\widetilde{e}_m)(\omega_{s,t}).\tag{3.1}$$

Let $c(\widetilde{e}_i)$ denotes the Clifford action, which satisfies

$$c(\widetilde{e}_i)c(\widetilde{e}_j)+c(\widetilde{e}_j)c(\widetilde{e}_i)=-2g^M(\widetilde{e}_i,\widetilde{e}_j).\tag{3.2}$$

In [19], the Dirac operator is given

$$D=\sum_{i=1}^m c(\widetilde{e}_i)\Big[\widetilde{e}_i-\frac{1}{4}\sum_{s,t}\omega_{s,t}(\widetilde{e}_i)c(\widetilde{e}_s)c(\widetilde{e}_t)\Big].\tag{3.3}$$

Set a Clifford action $c(X)$ on M and $X=\sum_{\alpha=1}^m a_\alpha\widetilde{e}_\alpha=X^T+X_m\partial_{x_m}=\sum_{j=1}^m X_j\partial_j$ is a vector field. We define $\nabla_X^{S(TM)}:=X+\frac{1}{4}\sum_{ij}\langle\nabla_X^L\widetilde{e}_i,\widetilde{e}_j\rangle c(\widetilde{e}_i)c(\widetilde{e}_j)$, which is a spin connection, where $L(X)=\frac{1}{4}\sum_{ij}\langle\nabla_X^L\widetilde{e}_i,\widetilde{e}_j\rangle c(\widetilde{e}_i)c(\widetilde{e}_j)$. And let $g^{ij}=g(dx_i,dx_j)$, $\xi=\sum_k\xi_j dx_j$ and $\nabla^L_{\partial_i}\partial_j=\sum_k\Gamma_{ij}^k\partial_k$, we denote that

$$\sigma_i=-\frac{1}{4}\sum_{s,t}\omega_{s,t}(\widetilde{e}_i)c(\widetilde{e}_i)c(\widetilde{e}_s)c(\widetilde{e}_t);\quad \xi^j=g^{ij}\xi_i;\quad \Gamma^k=g^{ij}\Gamma_{ij}^k;\quad \sigma^j=g^{ij}\sigma_i.\tag{3.4}$$

Then by [11] and $\sigma(\partial_{x_j})=i\xi_j$, we have the following lemmas.

Lemma 3.1. *The following identities hold:*

$$\begin{aligned}
&\sigma_1(D)=ic(\xi);\\
&\sigma_0(D)=-\frac{1}{4}\sum_{i,s,t}\omega_{s,t}(\widetilde{e}_i)c(\widetilde{e}_i)c(\widetilde{e}_s)c(\widetilde{e}_t)\\
&\sigma_0(\nabla_X^{S(TM)})=L(X);\\
&\sigma_1(\nabla_X^{S(TM)})=i\sum_{j=1}^n X_j\xi_j.
\end{aligned}$$

By the composition formula of pseudodifferential operators, we have

Lemma 3.2. *The following identities hold:*

$$\begin{aligned}
&\sigma_{-1}(D^{-1})=\frac{\sqrt{-1}c(\xi)}{|\xi|^2};\\
&\sigma_{-2}(fD^{-1}f^{-1}D^{-1})=\sigma_{-2}(D^{-2})=|\xi|^{-2};\\
&\sigma_{-2n}\Big[(fD^{-1}f^{-1}D^{-1})^n\Big]=\sigma_{-2n}(D^{-2n})=|\xi|^{-2n};\\
&\sigma_{-2n-1}(D^{-2n-1})=\sqrt{-1}c(\xi)|\xi|^{-2n-2};\\
&\sigma_{-2}(D^{-1})=\frac{c(\xi)\sigma_0(D)c(\xi)}{|\xi|^4}+\frac{c(\xi)}{|\xi|^6}\sum_j c(dx_j)\Big[\partial_{x_j}(c(\xi))|\xi|^2-c(\xi)\partial_{x_j}(|\xi|^2)\Big]\\
&\sigma_{-3}(D^{-2})=-\sqrt{-1}|\xi|^{-4}\xi_k(\Gamma^k-2\sigma^k)-\sqrt{-1}|\xi|^{-6}2\xi^j\xi_\alpha\xi_\beta\partial_jg^{\alpha\beta};\\
&\sigma_{-2n-1}\Bigg((fD^{-1}f^{-1}D^{-1})^n\Bigg)=n\cdot\sigma_2^{(1-n)}\sigma_{-3}(fD^{-1}f^{-1}D^{-1})-i\cdot\sum_{k=0}^{n-2}\partial_{\xi_\mu}\sigma_2^{(1-n+k)}\partial_{x_\mu}\sigma_2^{-1}\big(\sigma_2^{-1}\big)^k, \qquad (3.5)
\end{aligned}$$

where $\sigma_2=(1+\xi_m^2)^2$.

Since Θ is a global form on ∂M, so for any fixed point $x_0\in\partial M$, we can choose the normal coordinates U of x_0 in ∂M(not in M) and compute $\Theta(x_0)$ in the coordinates $\widetilde{U}=U\times[0,1)$ and the metric $\frac{1}{h(x_m)}g^{\partial M}+dx_m^2$. The dual metric of g^M on $\widetilde{U}$ is $h(x_m)g^{\partial M}+dx_m^2$. Write $g_{ij}^M=g^M(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j})$; $g^{ij}_M=g^M(dx_i,dx_j)$, then

$$[g^M_{i,j}]=\begin{bmatrix}\frac{1}{h(x_m)}[g^{\partial M}_{i,j}] & 0\\ 0 & 1\end{bmatrix};\quad [g_M^{i,j}]=\begin{bmatrix}h(x_m)[g^{i,j}_{\partial M}] & 0\\ 0 & 1\end{bmatrix},$$

and

$$\partial_{x_s}g_{ij}^{\partial M}(x_0)=0,\quad 1\leq i,j\leq m-1;\quad g^M_{i,j}(x_0)=\delta_{ij}.$$

Let $\{e_1,\cdots,e_{m-1}\}$ be an orthonormal frame field in U about $g^{\partial M}$ which is parallel along geodesics and $e_i=\frac{\partial}{\partial x_i}(x_0)$, then $\{\widetilde{e_1}=\sqrt{h(x_m)}e_1,\cdots,\widetilde{e_{m-1}}=\sqrt{h(x_n)}e_{m-1},\widetilde{e_m}=dx_m\}$ is the orthonormal frame field in $\widetilde{U}$ about g^M. Locally $S(TM)|\widetilde{U}\cong\widetilde{U}\times\wedge^*_C(\frac{m}{2})$. Let $\{f_1,\cdots,f_m\}$ be the orthonormal basis of $\wedge^*_C(\frac{m}{2})$. Take a spin frame field $\sigma:\widetilde{U}\to Spin(M)$ such that $\pi\sigma=\{\widetilde{e_1},\cdots,\widetilde{e_m}\}$ where $\pi:Spin(M)\to O(M)$ is a double covering, then $\{[\sigma,f_i],1\leq i\leq m\}$ is an orthonormal frame of $S(TM)|_{\widetilde{U}}$. In the following, since the global form Θ is independent of the choice of the local frame, so we can compute $\mathtt{tr}_{S(TM)}$ in the frame $\{[\sigma,f_i],1\leq i\leq m\}$. Let $\{\hat{e}_1,\cdots,\hat{e}_m\}$ be the canonical basis of $\mathbb{R}^m$ and $c(\hat{e}_i)\in cl_C(m)\cong Hom(\wedge^*_C(\frac{m}{2}),\wedge^*_C(\frac{m}{2}))$ be the Clifford action. Then

$$c(\widetilde{e_i})=[(\sigma,c(\hat{e}_i))];\quad c(\widetilde{e_i})[(\sigma,f_i)]=[\sigma,(c(\hat{e}_i))f_i];\quad \frac{\partial}{\partial x_i}=[(\sigma,\frac{\partial}{\partial x_i})],$$

then we have $\frac{\partial}{\partial x_i}c(\widetilde{e_i})=0$ in the above frame. By Lemma 2.2 in [11], we have

Lemma 3.3. *With the metric g^M on M near the boundary*

$$\partial_{x_j}(|\xi|^2_{g^M})(x_0)=\begin{cases}0, & \text{if } j<m;\\ h'(0)|\xi'|^2_{g^{\partial M}}, & \text{if } j=m.\end{cases}\qquad (3.6)$$

$$\partial_{x_j}[c(\xi)](x_0)=\begin{cases}0, & \text{if } j<m;\\ \partial x_n(c(\xi'))(x_0), & \text{if } j=m,\end{cases}\qquad (3.7)$$

where $\xi=\xi'+\xi_m dx_m$.

In the following, we will compute the residue $\widetilde{\mathrm{Wres}}\Big[\pi^+(fD^{-1}f^{-1}D^{-1})\circ\pi^+\Big((fD^{-1}f^{-1}D^{-1})^n\Big)\Big]$ for nonzero smooth functions $f,\ f^{-1}$ on even dimensional oriented compact spin manifolds with boundary and get a general Kastler-Kalau-Walze type theorem in this case. By Theorem 2.1, we have

$$\begin{aligned}&\widetilde{\mathrm{Wres}}\Big[\pi^+(fD^{-1}f^{-1}D^{-1})\circ\pi^+\Big((fD^{-1}f^{-1}D^{-1})^n\Big)\Big]\\ =&\int_M\int_{|\xi|=1}\mathrm{trace}_{S(TM)}\big[\sigma_{-n}\big((fD^{-1}f^{-1}D^{-1})^{n+1}\big)\big]\sigma(\xi)dx+\int_{\partial M}\Phi,\end{aligned}\tag{3.8}$$

where

$$\begin{aligned}\Phi=&\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{j,k=0}^{\infty}\sum\frac{(-i)^{|\alpha|+j+k+\ell}}{\alpha!(j+k+1)!}\mathrm{trace}_{S(TM)}\Big[\partial_{x_m}^j\partial_{\xi'}^\alpha\partial_{\xi_m}^k\sigma_r^+(fD^{-1}f^{-1}D^{-1})(x',0,\xi',\xi_m)\\&\times\partial_{x_m}^\alpha\partial_{\xi_m}^{j+1}\partial_{x_m}^k\sigma_l\Big((fD^{-1}f^{-1}D^{-1})^n\Big)(x',0,\xi',\xi_m)\Big]d\xi_m\sigma(\xi')dx',\end{aligned}\tag{3.9}$$

and the sum is taken over $r-k+|\alpha|+\ell-j-1=-(2n+4),r\leq-2,\ell\leq-2n$.

Then, by Theorem 2.2 and direct computations, we have the following theorem.

Theorem 3.4. *If M is a $2n{+}4$-dimensional compact oriented manifolds without boundary, then the following equality holds:*

$$\begin{aligned}&\mathrm{Wres}\Big[\Big(fD^{-1}f^{-1}D^{-1}\Big)^{n+1}\Big]\\ =&\frac{(2\pi)^{2n+6}}{(2n+4)!}\int_M 2^{2n+6}\mathrm{trace}\Big\{-\frac{1}{12}s-2f^{-1}\Delta(f)-f^{-2}[|\mathrm{grad}_Mf|^2+2\Delta(f)]\Big\}d\mathrm{Vol_M},\end{aligned}\tag{3.10}$$

where s is the scalar curvature.

Locally we can use Theorem 3.4 to compute the interior term of (3.8), then

$$\begin{aligned}&\int_M\int_{|\xi|=1}\mathrm{trace}_{S(TM)}\Big[\sigma_{-n}\Big((fD^{-1}f^{-1}D^{-1})^{n+1}\Big)\Big]\sigma(\xi)dx\\ =&\frac{(2\pi)^{2n+6}}{(2n+4)!}\int_M 2^{2n+6}\Big\{-\frac{1}{12}s-2f^{-1}\Delta(f)-f^{-2}\Big[|\mathrm{grad}_Mf|^2+2\Delta(f)\Big]\Big\}d\mathrm{Vol_M},\end{aligned}\tag{3.11}$$

so we only need to compute $\int_{\partial M}\Phi$.

When $m=2n+4$ is even, then $\mathrm{trace}_{S(TM)}[\mathrm{id}]=2^{\frac{m}{2}}$, the sum is taken over $r-k+|\alpha|+\ell-j=-2n-3,r\leq-2,\ell\leq-2n$, then we have the $\int_{\partial M}\Phi$ is the sum of the following five cases:

case (a) (I) $r=-2,\ l=-2n,\ k=j=0,\ |\alpha|=1$

By (3.9), we get

$$\begin{aligned}&\text{case (a) (I)}\\ =&-\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{|\alpha|=1}\mathrm{trace}\Big[\partial_{\xi'}^\alpha\pi_{\xi_m}^+\sigma_{-2}(fD^{-1}f^{-1}D^{-1})\times\partial_{x'}^\alpha\partial_{\xi_m}\sigma_{-2n}\Big((fD^{-1}f^{-1}D^{-1})^n\Big)\Big](x_0)\\&\times d\xi_m\sigma(\xi')dx',\end{aligned}\tag{3.12}$$

By Lemma 3.3, for $i<m$, then

$$\partial_{x_i}\Big(\sigma_{-2n}\Big((fD^{-1}f^{-1}D^{-1})^n\Big)\Big)(x_0)=\partial_{x_i}\Big(|\xi|^{-2n}\Big)(x_0)=(-n)|\xi|^{-2n-2}\partial_{x_i}(|\xi|^2)(x_0)=0,\tag{3.13}$$

so **case (a) (I)** vanishes.

case (a) (II) $r=-2,\ l=-2n,\ k=|\alpha|=0,\ j=1$

By (3.9), we get

$$\begin{aligned}
&\text{case (a) (II)}\\
=&\ -\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}\Big[\partial_{x_m}\pi^+_{\xi_m}\sigma_{-2}(fD^{-1}f^{-1}D^{-1})\times\partial^2_{\xi_m}\sigma_{-2n}\Big((fD^{-1}f^{-1}D^{-1})^n\Big)\Big](x_0)\\
&\times d\xi_m\sigma(\xi')dx'. \qquad (3.14)
\end{aligned}$$

By Lemma 3.3, we have

$$\partial_{x_m}\sigma_{-2}(fD^{-1}f^{-1}D^{-1})(x_0)|_{|\xi'|=1}=-\frac{h'(0)}{(1+\xi_m^2)^2}. \qquad (3.15)$$

By the Cauchy integral formula, then

$$\begin{aligned}
\pi^+_{\xi_m}\partial_{x_m}\sigma_{-2}(fD^{-1}f^{-1}D^{-1})(x_0)|_{|\xi'|=1} &= -h'(0)\frac{1}{2\pi i}\lim_{u\to 0^-}\int_{\Gamma^+}\frac{\frac{1}{(\eta_m+i)^2(\xi_m+iu-\eta_m)}}{(\eta_m-i)^2}d\eta_m\\
&= \frac{h'(0)(i\xi_m+2)}{4(\xi_m-i)^2}. \qquad (3.16)
\end{aligned}$$

From Lemma 3.2, we have

$$\begin{aligned}
&\partial^2_{\xi_m}\sigma_{-2n}\Big((fD^{-1}f^{-1}D^{-1})^n\Big)(x_0)=\partial^2_{\xi_m}\big(|\xi|^{-2n}\big)(x_0)\\
=&\ n(n+1)(|\xi|^2)^{-n-2}\big(\partial_{\xi_m}|\xi|^2\big)^2(x_0)-n(|\xi|^2)^{-n-1}\partial^2_{\xi_m}\big(|\xi|^2(x_0)\big)\\
=&\ \Big((4n+2)\xi_m^2-2\Big)n(1+\xi_m^2)^{(-n-2)}. \qquad (3.17)
\end{aligned}$$

We note that

$$\begin{aligned}
&\int_{-\infty}^{\infty}\left[\frac{h'(0)(i\xi_m+2)}{4(\xi_m-i)^2}\times\Big((4n+2)\xi_m^2-2\Big)n(1+\xi_m^2)^{(-n-2)}\right]d\xi_m\\
=&\ \frac{h'(0)\cdot n}{4}\int_{\Gamma^+}\frac{(4n+2)i\xi_m^3+(4+8n)\xi_m^2-2i\xi_m-4}{(\xi_m-i)^{(n+4)}(\xi_m+i)^{(n+2)}}d\xi_m\\
=&\ \frac{h'(0)\cdot n}{4}\frac{2\pi i}{(n+3)!}\left[\frac{(4n+2)i\xi_m^3+(4+8n)\xi_m^2-2i\xi_m-4}{(\xi_m+i)^{(n+2)}}\right]^{(n+3)}|_{\xi_m=i}. \qquad (3.18)
\end{aligned}$$

Since $m=2n+4$ is even, $\mathrm{trace}_{S(TM)}[\mathrm{id}]=\dim(\wedge^*(\frac{2n+4}{2}))=2^{n+2}$. Then we obtain

$$\textbf{case (a) (II)}=\frac{-2^nh'(0)\cdot n\pi i}{(n+3)!}\mathrm{Vol}(S_{2n+2})\left[\frac{(4n+2)i\xi_m^3+(4+8n)\xi_m^2-2i\xi_m-4}{(\xi_m+i)^{(n+2)}}\right]^{(n+3)}|_{\xi_m=i}dx', \qquad (3.19)$$

where $\mathrm{Vol}(S_{2n+2})$ is the canonical volume of S_{2n+2}.

case (a) (III) $r=-2,\ l=-2n,\ j=|\alpha|=0,\ k=1$

By (3.9), we get

$$\begin{aligned}
&\text{case (a) (III)}\\
=\ & -\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}\Big[\partial_{\xi_m}\pi^+_{\xi_m}\sigma_{-2}(fD^{-1}f^{-1}D^{-1})\times\partial_{\xi_m}\partial_{x_m}\sigma_{-2n}\Big((fD^{-1}f^{-1}D^{-1})^n\Big)\Big](x_0)\\
&\times d\xi_m\sigma(\xi')dx'\\
=\ & \frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}\Big[\partial^2_{\xi_m}\pi^+_{\xi_m}\sigma_{-2}(fD^{-1}f^{-1}D^{-1})\times\partial_{x_m}\sigma_{-2n}\Big((fD^{-1}f^{-1}D^{-1})^n\Big)\Big](x_0)\\
&\times d\xi_m\sigma(\xi')dx'.
\end{aligned}\tag{3.20}$$

By Lemma 3.2, we have

$$\partial^2_{\xi_m}\pi^+_{\xi_m}\sigma_{-2}(fD^{-1}f^{-1}D^{-1})(x_0)|_{|\xi'|=1}=\frac{-i}{(\xi_m-i)^3},\tag{3.21}$$

and

$$\partial_{x_m}\bigg(\sigma_{-2n}\Big((fD^{-1}f^{-1}D^{-1})^n\Big)\bigg)(x_0)=\partial_{x_m}\bigg((|\xi|^2)^{-n}\bigg)(x_0)=h'(0)(-n)(1+\xi_m^2)^{-n-1}.\tag{3.22}$$

Then

$$\begin{aligned}
&\int_{-\infty}^{\infty}\mathrm{trace}\Big[\frac{-i}{(\xi_m-i)^3}\times h'(0)(-n)(1+\xi_m^2)^{-n-1}\Big]d\xi_m\\
=\ & i\cdot n\cdot h'(0)\cdot 2^{n+2}\int_{\Gamma^+}\frac{1}{(\xi_m-i)^{(n+4)}(\xi_m+i)^{(n+1)}}d\xi_m\\
=\ & -n\cdot h'(0)\cdot 2^{n+3}\cdot\frac{\pi}{(n+3)!}\left[\frac{1}{(\xi_m+i)^{n+1}}\right]^{(n+3)}|_{\xi_m=i}.
\end{aligned}\tag{3.23}$$

Then

$$\textbf{case (a) (III)}=-\frac{2^{n+2}\pi nh'(0)}{(n+3)!}\mathrm{Vol}(S_{2n+2})\left[\frac{1}{(\xi_m+i)^{n+1}}\right]^{(n+3)}|_{\xi_m=i}dx'.\tag{3.24}$$

case (b) $r=-2,\ l=-2n-1,\ k=j=|\alpha|=0$

By (3.9) and an integration by parts,, we get

$$\begin{aligned}
&\text{case (b)}\\
=\ & -i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}\Big[\pi^+_{\xi_m}\sigma_{-2}(fD^{-1}f^{-1}D^{-1})\times\partial_{\xi_m}\sigma_{-2n-1}\Big((fD^{-1}f^{-1}D^{-1})^n\Big)\Big](x_0)\\
&\times d\xi_m\sigma(\xi')dx'\\
=\ & i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}\Big[\partial_{\xi_m}\pi^+_{\xi_m}\sigma_{-2}(fD^{-1}f^{-1}D^{-1})\times\sigma_{-2n-1}\Big((fD^{-1}f^{-1}D^{-1})^n\Big)\Big](x_0)\\
&\times d\xi_m\sigma(\xi')dx'.
\end{aligned}\tag{3.25}$$

By Lemma 3.2, we have

$$\partial_{\xi_m}\pi^+_{\xi_m}\sigma_{-2}(fD^{-1}f^{-1}D^{-1})(x_0)|_{|\xi'|=1}=\frac{i}{2(\xi_m-i)^2}.\tag{3.26}$$

Using the recursion formula (4.20) in [7], we get

$$\sigma_{3-n}(D^{-n+4})(x,\xi)=\sigma_{5-n}(D^{-n+6})\sigma_2^{-1}+\sigma_2^{(-\frac{n}{2}+3)}\sigma_{-3}(D^{-2})-\sqrt{-1}\partial_{\xi_\mu}\sigma_2^{(-\frac{n}{2}+3)}\partial_{x_\mu}\sigma_2^{-1}. \tag{3.27}$$

Then we obtain by induction

$$\begin{aligned}
&\sigma_{-2n-1}\Big((fD^{-1}f^{-1}D^{-1})^n\Big)(x,\xi)\\
=\;&\sigma_{-2n+1}\Big((fD^{-1}f^{-1}D^{-1})^{n-1}\Big)\sigma_2^{-1}+\sigma_2^{(1-n)}\sigma_{-3}(fD^{-1}f^{-1}D^{-1})-i\cdot\partial_{\xi_\mu}\sigma_2^{(1-n)}\partial_{x_\mu}\sigma_2^{-1}\\
=\;&n\cdot\sigma_2^{(1-n)}\sigma_{-3}(fD^{-1}f^{-1}D^{-1})-i\cdot\sum_{k=0}^{n-2}\partial_{\xi_\mu}\sigma_2^{(1-n+k)}\partial_{x_\mu}\sigma_2^{-1}\big(\sigma_2^{-1}\big)^k.
\end{aligned}\tag{3.28}$$

In the normal coordinate, $g^{ij}(x_0)=\delta_i^j$ and $\partial_{x_j}(g^{\alpha\beta})(x_0)=0$, if $j<m$; $\partial_{x_j}(g^{\alpha\beta})(x_0)=h'(0)\delta^\alpha_\beta$, if $j=m$. So by Lemma A.2 in [11], we have $\Gamma^m(x_0)=\frac{2n+3}{2}h'(0)$ and $\Gamma^k(x_0)=0$ for $k<m$. By the definition of δ^k and Lemma 3.2, we have $\delta^m(x_0)=0$ and $\delta^k=\frac{1}{4}h'(0)c(\widetilde{e_k})c(\widetilde{e_m})$ for $k<m$. So

$$\begin{aligned}
&\sigma_{-3}(fD^{-1}f^{-1}D^{-1})(x_0)|_{|\xi'|=1}\\
=\;&-i|\xi|^{-4}\xi_k(\Gamma^k-2\delta^k)(x_0)|_{|\xi'|=1}-i|\xi|^{-6}2\xi^j\xi_\alpha\xi_\beta\partial_jg^{\alpha\beta}(x_0)|_{|\xi'|=1}-i|\xi|^{-4}\xi_k\big[c(\partial^j)\cdot f\cdot c(df^{-1})\big]\\
=\;&\frac{i}{(1+\xi_m^2)^2}\Big(\frac{1}{2}h'(0)\sum_{k<m}\xi_kc(\widetilde{e_k})c(\widetilde{e_n})-\frac{2n+3}{2}h'(0)\xi_m\Big)-\frac{2ih'(0)\xi_m}{(1+\xi_m^2)^3}-\frac{i}{(1+\xi_m^2)^2}\sum_{k=1}^m\xi_k\big[c(\widetilde{e_k})\\
&\times f\cdot c(df^{-1})\big].
\end{aligned}\tag{3.29}$$

We note that $\int_{|\xi'|=1}\xi_1\cdots\xi_{2q+1}\sigma(\xi')=0$, so the first term and the fourth term in (3.29) has no contribution for computing **case (b)**.

On the other hand, we have

$$\sigma_2^{(1-n)}(x_0)=(1+\xi_m^2)^{(1-n)}, \tag{3.30}$$

and

$$\partial_{x_j}(|\xi|^{-2})(x_0)=0,\quad j<m. \tag{3.31}$$

Then

$$\begin{aligned}
&-i\sum_{k=0}^{n-2}\partial_{\xi_\mu}\sigma_2^{(1-n+k)}\partial_{x_\mu}\sigma_2^{-1}\big(\sigma_2^{-1}\big)^k(x_0)\\
=\;&-i\sum_{k=0}^{n-2}\partial_{\xi_n}\Big[(|\xi|^2)^{(1-n+k)}\Big]\partial_{x_n}(|\xi|^2)^{-1}(|\xi|)^{-2k}=i\sum_{k=0}^{n-2}(|\xi|^2)^{(-n+k)}(1-n+k)2\xi_n|\xi|^{-4}h'(0)(1+\xi_m^2)^{-k}\\
=\;&i\sum_{k=0}^{n-2}(1+\xi_m^2)^{(-n)}(1+\xi_m^2)^{-2}h'(0)\xi_n(-2n+2k+2)=i\sum_{k=0}^{n-2}h'(0)(-2n+2k+2)\xi_n(1+\xi_m^2)^{(-n)}\\
=\;&ih'(0)(-n^2+n)\xi_n(1+\xi_m^2)^{(-n-2)}.
\end{aligned}\tag{3.32}$$

In conclusion, we obtain

$$\begin{aligned}
&\sigma_{-2n-1}\Big((fD^{-1}f^{-1}D^{-1})^n\Big)(x,\xi)\\
=\;&n(1+\xi_m^2)^{1-n}\Big(\frac{-i(2n+3)h'(0)\xi_m}{2(1+\xi_m^2)^2}-\frac{2ih'(0)\xi_m}{(1+\xi_m^2)^3}\Big)-ih'(0)(n^2-n)\xi_n(1+\xi_m^2)^{(-n-2)}.
\end{aligned}\tag{3.33}$$

From (3.26) and (3.33), we obtain

$$\begin{aligned}
\textbf{case (b)} &= i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}\bigg\{\frac{i}{2(\xi_m-i)^2}\times\bigg[n(1+\xi_m^2)^{1-n}\Big(\frac{-i}{(1+\xi_m^2)^2}\times\frac{2n+3}{2}h'(0)\xi_m \\
&\quad -\frac{2ih'(0)\xi_m}{(1+\xi_m^2)^3}\Big)-ih'(0)(n^2-n)\xi_n(1+\xi_m^2)^{(-n-2)}\bigg]\bigg\}d\xi_m\sigma(\xi')dx' \\
&= -\frac{ih'(0)}{8}\mathrm{Vol}(S_{2n+2})\int_{\Gamma^+}\frac{-2n(2n+3)\xi_n^3+(-8n^2-10n-4)\xi_n}{(\xi_m-i)^{n+4}(\xi_m+i)^{n+2}}d\xi_n dx' \\
&= -\frac{ih'(0)}{8}\mathrm{Vol}(S_{2n+2})2^{n+2}\frac{2\pi i}{(n+3)!}\left[\frac{-2n(2n+3)\xi_n^3-(8n^2+10n+4)\xi_n}{(\xi_m+i)^{n+2}}\right]^{n+3}|_{\xi_m=i}dx' \\
&= \frac{2^n\pi h'(0)}{(n+3)!}\mathrm{Vol}(S_{2n+2})\left[\frac{-2n(2n+3)\xi_n^3-(8n^2+10n+4)\xi_n}{(\xi_m+i)^{n+2}}\right]^{(n+3)}|_{\xi_m=i}dx'. \qquad (3.34)
\end{aligned}$$

case (c) $r=-3,\ l=-2n,\ k=j=|\alpha|=0$

By (3.9), we get

$$\begin{aligned}
\text{case (c)} &= -i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}\Big[\pi^+_{\xi_m}\sigma_{-3}(fD^{-1}f^{-1}D^{-1})\times\partial_{\xi_m}\sigma_{-2n}\Big((fD^{-1}f^{-1}D^{-1})^n\Big)\Big](x_0) \\
&\quad\times d\xi_m\sigma(\xi')dx' \qquad (3.35)
\end{aligned}$$

By Lemma 3.2, we have

$$\partial_{\xi_m}\Big[\sigma_{-2n}\Big((fD^{-1}f^{-1}D^{-1})^n\Big)\Big](x_0)=\partial_{\xi_m}\Big((|\xi|^2)^{-n}\Big)(x_0)=-2n\xi_m(1+\xi_m^2)^{-n-1}. \qquad (3.36)$$

By the Cauchy integral formula, we obtain

$$\begin{aligned}
\pi^+_{\xi_m}\Big(\frac{\xi_m}{(1+\xi_m^2)^2}\Big) &= \frac{1}{2\pi i}\int_{\Gamma^+}\frac{\eta_m}{(\xi_m-\eta_m)(1+\eta_m^2)^2}d\eta_m \\
&= \left[\frac{\eta_m}{(\xi_m-\eta_m)(\eta_m^2+i)^2}\right]^1\Big|_{\eta_n=i} \\
&= \frac{-i}{4(\xi_m-i)^2}, \qquad (3.37)
\end{aligned}$$

and

$$\pi^+_{\xi_m}\Big(\frac{\xi_m}{(1+\xi_m^2)^3}\Big)=\frac{-i}{16(\xi_m-i)^2}-\frac{1}{8(\xi_m-i)^3}. \qquad (3.38)$$

In conclusion, we obtain

$$\begin{aligned}
\pi^+_{\xi_m}\Big(\sigma_{-3}(fD^{-1}f^{-1}D^{-1})(x_0)|_{|\xi'|=1}\Big) &= -ih'(0)\pi^+_{\xi_m}\Big(\frac{(2n+3)\xi_m}{2(1+\xi_m^2)^2}+\frac{2\xi_m}{(1+\xi_m^2)^3}\Big) \\
&= -ih'(0)\Big[(2n+3)\pi^+_{\xi_m}\Big(\frac{\xi_m}{(1+\xi_m^2)^2}\Big)+2\pi^+_{\xi_m}\Big(\frac{\xi_m}{(1+\xi_m^2)^3}\Big)\Big] \\
&= ih'(0)\Big[\frac{i(2n+2)}{8(\xi_m-i)^2}+\frac{1}{4(\xi_m-i)^3}\Big]. \qquad (3.39)
\end{aligned}$$

Therefore, by (3.36) and (3.39), we have

$$\begin{aligned}
\textbf{case (c)} &= -i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}\Big[\frac{-2n\xi_m}{(1+\xi_m^2)^{n+1}}\times ih'(0)\Big[\frac{i(2n+2)}{8(\xi_m-i)^2}+\frac{1}{4(\xi_m-i)^3}\Big]\Big]d\xi_m\sigma(\xi')dx' \\
&= (-n)2^n\mathrm{Vol}(S_{2n+2})h'(0)\int_{\Gamma^+}\frac{\big(2i(n+1)\xi_m+2n+4\big)\xi_n}{(\xi_m+i)^{n+1}(\xi_m-i)^{n+4}}d\xi_m dx' \\
&= (-n)2^n\mathrm{Vol}(S_{2n+2})h'(0)\frac{2\pi i}{(n+3)!}\Bigg[\frac{\big(2i(n+1)\xi_n+2n+4\big)\xi_m}{(\xi_m+i)^{n+1}}\Bigg]^{(n+3)}|_{\xi_m=i}dx'. \qquad (3.40)
\end{aligned}$$

Since Φ is the sum of the **case (a)**, **case (b)** and **case (c)**, so

$$\begin{aligned}
\Phi &= \frac{2^n h'(0)\pi ni}{(n+3)!}\mathrm{Vol}(S_{2n+2})dx'\cdot\Bigg[\frac{-4ni\xi_m^3-4(2n+2)\xi_m^2+(4ni+8i)\xi_m}{(\xi_m+i)^{n+2}}\Bigg]^{(n+3)}|_{\xi_m=i} \\
&:= \frac{2^n h'(0)\pi ni}{(n+3)!}\mathrm{Vol}(S_{2n+2})dx'\cdot Q_0, \qquad (3.41)
\end{aligned}$$

where

$$\begin{aligned}
Q_0 &= \Bigg[\frac{-4ni\xi_m^3-4(2n+2)\xi_m^2+(4ni+8i)\xi_m}{(\xi_m+i)^{n+2}}\Bigg]^{(n+3)}|_{\xi_m=i} \\
&= (-1)^n(n+3)!(1+i)4^{-(n+1)}\Bigg[(2+2i)nC_{-n-2}^{n}+(1+i)(n-2)C_{-n-2}^{n+1}-(3+i)C_{-n-2}^{n+2} \\
&\quad -C_{-n-2}^{n+3}\Bigg]. \qquad (3.42)
\end{aligned}$$

Combining (3.11) and (3.41), we obtain Theorem 1.1.

4. The noncommutative residue $\widetilde{\mathrm{Wres}}\Big[\pi^+(fD^{-1})\circ\pi^+\Big((f^{-1}D^{-1})\cdot(fD^{-1}f^{-1}D^{-1})^n\Big)\Big]$ on even dimensional manifolds with boundary

In the following, we will compute the residue $\widetilde{\mathrm{Wres}}\Big[\pi^+(fD^{-1})\circ\pi^+\Big((f^{-1}D^{-1})\cdot(fD^{-1}f^{-1}D^{-1})^n\Big)\Big]$ for nonzero smooth functions $f,\ f^{-1}$ on even dimensional oriented compact spin manifolds with boundary and get a general Kastler-Kalau-Walze type theorem in this case. By Theorem 2.1, we have

$$\begin{aligned}
&\widetilde{\mathrm{Wres}}\Big[\pi^+(fD^{-1})\circ\pi^+\Big((f^{-1}D^{-1})\cdot(fD^{-1}f^{-1}D^{-1})^n\Big)\Big] \\
&= \int_M\int_{|\xi|=1}\mathrm{trace}_{S(TM)}\big[\sigma_{-n}\big((fD^{-1}f^{-1}D^{-1})^{n+1}\big)\big]\sigma(\xi)dx+\int_{\partial M}\Psi, \qquad (4.1)
\end{aligned}$$

where

$$\begin{aligned}
\Psi &= \int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{j,k=0}^{\infty}\sum\frac{(-i)^{|\alpha|+j+k+\ell}}{\alpha!(j+k+1)!}\mathrm{trace}_{S(TM)}\Big[\partial_{x_m}^j\partial_{\xi'}^\alpha\partial_{\xi_m}^k\sigma_r^+(fD^{-1})(x',0,\xi',\xi_m) \\
&\quad \times\partial_{x_m}^\alpha\partial_{\xi_m}^{j+1}\partial_{x_m}^k\sigma_l\Big((f^{-1}D^{-1})\cdot(fD^{-1}f^{-1}D^{-1})^n\Big)(x',0,\xi',\xi_m)\Big]d\xi_m\sigma(\xi')dx', \qquad (4.2)
\end{aligned}$$

and the sum is taken over $r-k+|\alpha|+\ell-j-1=-(2n+4), r\leq -1, \ell\leq -2n-1$.

Locally we can use Theorem 3.4 to compute the interior term of (3.8), then

$$
\begin{aligned}
&\int_M\int_{|\xi|=1}\mathrm{trace}_{S(TM)}\Big[\sigma_{-n}\Big((fD^{-1}f^{-1}D^{-1})^{n+1}\Big)\Big]\sigma(\xi)dx\\
=\;&\frac{(2\pi)^{2n+6}}{(2n+4)!}\int_M 2^{2n+6}\Big\{-\frac{1}{12}s-2f^{-1}\Delta(f)-f^{-2}\Big[|\mathrm{grad}_M f|^2+2\Delta(f)\Big]\Big\}d\mathrm{Vol_M}, \qquad (4.3)
\end{aligned}
$$

so we only need to compute $\int_{\partial M}\Psi$.

When $m=2n+4$ is even, then $\mathrm{trace}_{S(TM)}[\mathrm{id}]=2^{\frac{m}{2}}$, the sum is taken over $r-k+|\alpha|+\ell-j=-2n-3, r\leq -1, \ell\leq -2n-1$, then we have the $\int_{\partial M}\Psi$ is the sum of the following five cases:

case (1) $r=-1, l=-2n-1, j=k=0, |\alpha|=1$.

By (4.2), we get

$$
\begin{aligned}
&\mathrm{case}\ (1)\\
=\;&-\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{|\alpha|=1}\mathrm{trace}\Big[\partial^\alpha_{\xi'}\pi^+_{\xi_m}\sigma_{-1}(fD^{-1})\times\partial^\alpha_{x'}\partial_{\xi_m}\sigma_{-2n-1}\Big((f^{-1}D^{-1})\cdot(fD^{-1}\cdot f^{-1}D^{-1})^n\Big)\Big](x_0)\\
&\times d\xi_m\sigma(\xi')dx'\\
=\;&-\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{|\alpha|=1}\mathrm{trace}\Big[\partial^\alpha_{\xi'}\pi^+_{\xi_m}\big(f\sigma_{-1}(D^{-1})\big)\times\partial^\alpha_{x'}\partial_{\xi_m}\Big(f^{-1}\sigma_{-2n-1}(D^{-2n-1})\Big)\Big](x_0)d\xi_m\sigma(\xi')dx'\\
=\;&-\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{|\alpha|=1}\mathrm{trace}\Big\{f\cdot\partial^\alpha_{\xi'}\pi^+_{\xi_m}\big(\sigma_{-1}(D^{-1})\big)\times\Big[\big(\partial^\alpha_{x'}(f^{-1})\big)\partial_{\xi_m}\Big(\sigma_{-2n-1}(D^{-2n-1})\Big)+f^{-1}\\
&\times\partial^\alpha_{x'}\partial_{\xi_m}\Big(\sigma_{-2n-1}(D^{-2n-1})\Big)\Big]\Big\}(x_0)d\xi_m\sigma(\xi')dx'\\
=\;&-\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{|\alpha|=1}\mathrm{trace}\Big[\partial^\alpha_{\xi'}\pi^+_{\xi_m}\big(\sigma_{-1}(D^{-1})\big)\times\partial^\alpha_{x'}\partial_{\xi_m}\Big(\sigma_{-2n-1}(D^{-2n-1})\Big)\Big](x_0)d\xi_m\sigma(\xi')dx'\\
&-f\cdot\sum_{j<1}\partial_{x_j}(f^{-1})\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{|\alpha|=1}\mathrm{trace}\Big[\partial^\alpha_{\xi'}\pi^+_{\xi_m}\big(\sigma_{-1}(D^{-1})\big)\times\partial_{\xi_m}\Big(\sigma_{-2n-1}(D^{-2n-1})\Big)\Big](x_0)\\
&\times d\xi_m\sigma(\xi')dx'. \qquad (4.4)
\end{aligned}
$$

By Lemma 2.2 in [11] and (3.12) in [14], we have for $j<m$

$$
\begin{aligned}
&\partial_{x_j}\sigma_{-2n-1}(D^{-2n-1})(x_0)=\partial_{x_j}\big[\sqrt{-1}c(\xi)|\xi|^{-2n-2}\big]\\
=\;&\sqrt{-1}\big[\partial_{x_j}c(\xi)\big](x_0)|\xi|^{-2n-2}+\sqrt{-1}c(\xi)\partial_{x_j}(|\xi|^{-2n-2})(x_0)=0, \qquad (4.5)
\end{aligned}
$$

so

$$
-\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{|\alpha|=1}\mathrm{trace}\Big[\partial^\alpha_{\xi'}\pi^+_{\xi_m}\big(\sigma_{-1}(D^{-1})\big)\times\partial^\alpha_{x'}\partial_{\xi_m}\Big(\sigma_{-2n-1}(D^{-2n-1})\Big)\Big](x_0)d\xi_m\sigma(\xi')dx'=0. \qquad (4.6)
$$

By Lemma 3.2 and direct calculations, for $i<m$, we obtain

$$
\begin{aligned}
&\partial^\alpha_{\xi'}\pi^+_{\xi_m}\sigma_{-1}(D^{-1})(x_0)|_{|\xi'|=1}=\partial_{\xi_i}\pi^+_{\xi_m}\sigma_{-1}(D^{-1})(x_0)|_{|\xi'|=1}\\
=\;&\frac{c(dx_i)}{2(\xi_m-i)}-\frac{\xi_i(\xi_m-2i)c(\xi')+\xi_ic(dx_m)}{2(\xi_m-i)^2}, \qquad (4.7)
\end{aligned}
$$

and we get

$$\partial_{\xi_m}\left(\sigma_{-2n-1}(D^{-2n-1})\right)=\frac{\sqrt{-1}c(dx_m)}{|\xi|^{2n+2}}-\frac{(2n+2)\sqrt{-1}\Big[\xi_n c(\xi')+\xi_n^2c(dx_m)\Big]}{|\xi|^{2n+4}}. \tag{4.8}$$

Then for $i<m$, we have

$$\begin{aligned}&\mathrm{trace}\left[\partial^{\alpha}_{\xi'}\pi^+_{\xi_m}\sigma_{-1}(D^{-1})\times\partial_{\xi_m}\left(\sigma_{-2n-1}(D^{-2n-1})\right)\right](x_0)\\ =\ &-\xi_i\mathrm{trace}\Big[\frac{c(dx_m)^2}{2(\xi_m-i)^2|\xi|^{2n+2}}\Big]-4i\xi_m\xi_i\mathrm{trace}\Big[\frac{c(dx_i)^2}{2(\xi_m-i)|\xi|^{2n+4}}\Big]+4i\xi_m\xi_i(\xi_m-2i)\\ &\times\mathrm{trace}\Big[\frac{c(\xi')^2}{2(\xi_m-i)^2|\xi|^{2n+4}}\Big]+4i\xi_m^2\xi_i\mathrm{trace}\Big[\frac{c(dx_m)^2}{2(\xi_m-i)^2|\xi|^{2n+4}}\Big].\end{aligned} \tag{4.9}$$

We note that $i<m$, $\int_{|\xi'|=1}\xi_i\sigma(\xi')=0$, so

$$\begin{aligned}&-f\sum_{j<m}\partial_{x_j}(f^{-1})\int_{|\xi'|=1}\int^{+\infty}_{-\infty}\sum_{|\alpha|=1}\mathrm{trace}\left[\partial^{\alpha}_{\xi'}\pi^+_{\xi_m}\sigma_{-1}(D^{-1})\times\partial_{\xi_m}\left(\sigma_{-2n-1}(D^{-2n-1})\right)\right](x_0)\\ &\times d\xi_m\sigma(\xi')dx'\\ =\ &0.\end{aligned} \tag{4.10}$$

Then we have **case** $(\mathbf{1})=0$.

case $\;(\mathbf{2})$ $r=-1,l=-2n-1,|\alpha|=k=0,j=1.$

By (4.2), we have

$$\begin{aligned}&\text{case (2)}\\ =\ &-\frac{1}{2}\int_{|\xi'|=1}\int^{+\infty}_{-\infty}\mathrm{trace}\left[\partial_{x_m}\pi^+_{\xi_m}\sigma_{-1}(fD^{-1})\times\partial^2_{\xi_m}\sigma_{-2n-1}\left((f^{-1}D^{-1})\cdot(fD^{-1}\cdot f^{-1}D^{-1})^n\right)\right](x_0)\\ &\times d\xi_m\sigma(\xi')dx'\\ =\ &-\frac{1}{2}\int_{|\xi'|=1}\int^{+\infty}_{-\infty}\mathrm{trace}\left[\partial_{x_m}\pi^+_{\xi_m}\sigma_{-1}(D^{-1})\times\partial^2_{\xi_m}\sigma_{-2n-1}\left(D^{-2n-1}\right)\right](x_0)d\xi_m\sigma(\xi')dx'\\ &-\frac{1}{2}f^{-1}\partial_{x_m}(f)\int_{|\xi'|=1}\int^{+\infty}_{-\infty}\mathrm{trace}\left[\pi^+_{\xi_m}\sigma_{-1}(D^{-1})\times\partial^2_{\xi_m}\sigma_{-2n-1}\left(D^{-2n-1}\right)\right](x_0)d\xi_m\sigma(\xi')dx'\\ =\ &-\frac{1}{2}\int_{|\xi'|=1}\int^{+\infty}_{-\infty}\mathrm{trace}\left[\partial^2_{\xi_m}\partial_{x_m}\pi^+_{\xi_m}\sigma_{-1}(D^{-1})\times\sigma_{-2n-1}\left(D^{-2n-1}\right)\right](x_0)d\xi_m\sigma(\xi')dx'-\frac{1}{2}\\ &\times f^{-1}\partial_{x_m}(f)\int_{|\xi'|=1}\int^{+\infty}_{-\infty}\mathrm{trace}\left[\pi^+_{\xi_m}\sigma_{-1}(D^{-1})\times\partial^2_{\xi_m}\sigma_{-2n-1}\left(D^{-2n-1}\right)\right](x_0)d\xi_m\sigma(\xi')dx'.\end{aligned} \tag{4.11}$$

By (2.2.23) in [11], we have

$$\begin{aligned}&\pi^+_{\xi_m}\partial_{x_m}\sigma_{-1}(D^{-1})(x_0)|_{|\xi'|=1}\\ =\ &\frac{\partial_{x_m}\big[c(\xi')\big](x_0)}{2(\xi_m-i)}+\sqrt{-1}h'(0)\left[\frac{ic(\xi')}{4(\xi_m-i)}+\frac{c(\xi')+ic(dx_m)}{4(\xi_m-i)^2}\right].\end{aligned} \tag{4.12}$$

So

$$\begin{aligned}
&\partial^2_{\xi_m}\pi^+_{\xi_m}\partial_{x_m}\sigma_{-1}(D^{-1})(x_0)|_{|\xi'|=1}\\
=&\frac{\partial_{x_m}\big[c(\xi')\big](x_0)}{4(\xi_m-i)^3}+\sqrt{-1}h'(0)\left[\frac{ic(\xi')}{8(\xi_m-i)^3}+\frac{c(\xi')+ic(dx_m)}{24(\xi_m-i)^4}\right].
\end{aligned}\tag{4.13}$$

We know that

$$\sigma_{-2n-1}(D^{-2n-1})=\frac{\sqrt{-1}[c(\xi')+\xi_nc(dx_m)]}{(1+\xi_m^2)^{n+1}},\tag{4.14}$$

By the relation of the Clifford action and trace$AB=$ traceBA, then we have the equalities:

$$\begin{aligned}
&\mathrm{trace}\big[c(\xi')c(dx_m)\big]=0;\quad \mathrm{trace}\big[c(dx_m)^2\big]=-2^{n+2};\quad \mathrm{trace}\big[c(\xi')^2\big](x_0)|_{|\xi'|=1}=-2^{n+2};\\
&\mathrm{trace}\big[\partial_{x_m}c(\xi')c(dx_m)\big]=0;\quad \mathrm{trace}\big[\partial_{x_m}c(\xi')c(\xi')\big](x_0)|_{|\xi'|=1}=-2^{n+1}h'(0).
\end{aligned}$$

By (4.11), (4.13) and (4.14), we have

$$\begin{aligned}
&\mathrm{trace}\Big[\partial^2_{\xi_m}\partial_{x_m}\pi^+_{\xi_m}\sigma_{-1}(D^{-1})\times\sigma_{-2n-1}(D^{-2n-1})\Big](x_0)|_{|\xi'|=1}\\
=&\mathrm{trace}\bigg\{\Big[\frac{\partial_{x_m}[c(\xi')](x_0)}{4(\xi_m-i)^3}+ih'(0)\Big[\frac{ic(\xi')}{8(\xi_m-i)^3}+\frac{c(\xi')+ic(dx_m)}{24(\xi_m-i)^4}\Big]\Big]\times\frac{i[c(\xi')+\xi_mc(dx_m)]}{(1+\xi_m^2)^{n+1}}\bigg\}\\
=&\mathrm{trace}\bigg\{\Big[\frac{\partial_{x_m}[c(\xi')](x_0)}{4(\xi_m-i)^3}+\frac{(4i-3\xi_m)h'(0)}{24(\xi_m-i)^4}c(\xi')-\frac{h'(0)c(dx_m)}{24(\xi_m-i)^4}\Big]\times\frac{\sqrt{-1}[c(\xi')+\xi_mc(dx_m)]}{(1+\xi_m^2)^{n+1}}\bigg\}\\
=&\frac{2^{n+1}h'(0)i}{12(\xi_m+i)^{n+1}(\xi_m-i)^{n+4}}.
\end{aligned}\tag{4.15}$$

Thus, we have

$$\begin{aligned}
&-\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}\left[\partial^2_{\xi_m}\partial_{x_m}\pi^+_{\xi_m}\sigma_{-1}(D^{-1})\times\sigma_{-2n-1}\Big(D^{-2n-1}\Big)\right](x_0)d\xi_m\sigma(\xi')dx'\\
=&\mathrm{Vol}(S_{2n+2})\frac{2^{n+1}h'(0)\pi}{12(n+3)!}\Big[(\xi_m+i)^{-n-1}\Big]^{(n+3)}|_{\xi_m=i}dx',
\end{aligned}\tag{4.16}$$

where $\mathrm{Vol}(S_{2n+2})$ is the canonical volume of S_{2n+2} and denote the p-th derivative of $f(\xi_m)$ by $[f(\xi_m)]^{(p)}$. By (4.14) and direct calculations, we have

$$\partial_{\xi_m}\sigma_{-2n-1}(D^{-2n-1})=\frac{-2(n+1)i\xi_m[c(\xi')+\xi_nc(dx_m)]}{(1+\xi_m^2)^{n+2}}\tag{4.17}$$

and

$$\begin{aligned}
&\partial^2_{\xi_m}\sigma_{-2n-1}(D^{-2n-1})\\
=&i\Big[\frac{4\xi_m^2(n+1)(n+2)\big(c(\xi')+\xi_mc(dx_m)\big)}{(1+\xi_m^2)^{n+3}}-\frac{6\xi_mc(dx_m)(n+1)+2(n+1)c(\xi')}{(1+\xi_m^2)^{n+2}}\Big].
\end{aligned}\tag{4.18}$$

On the other hand, by calculations, we have

$$\pi^+_{\xi_m}\sigma_{-1}(D^{-1})(x_0)|_{|\xi'|=1}=-\frac{c(\xi')+\sqrt{-1}c(dx_m)}{2(\xi_m-\sqrt{-1})}.\tag{4.19}$$

By (4.11), (4.18) and (4.19), we get

$$\begin{aligned}
& \operatorname{trace}\Big[\pi^+_{\xi_m}\sigma_{-1}(D^{-1})\times\partial^2_{\xi_m}\sigma_{-2n-1}(D^{-2n-1})\Big](x_0)\\
= & \frac{2^{n+2}(n+1)\Big[2i\xi_m^2(n+2)(1+\xi_m)-(2\xi_m-i-i\xi_m)(1+\xi_m^2)\Big]}{(\xi_m-i)^{n+4}(\xi_m+i)^{n+3}},
\end{aligned}\tag{4.20}$$

then we have

$$\begin{aligned}
& -\frac{1}{2}f^{-1}\partial_{x_m}(f)\int_{|\xi'|=1}\int^{+\infty}_{-\infty}\operatorname{trace}\Big[\pi^+_{\xi_m}\sigma_{-1}(D^{-1})\times\partial^2_{\xi_m}\sigma_{-2n-1}\Big(D^{-2n-1}\Big)\Big](x_0)d\xi_m\sigma(\xi')dx'\\
= & \frac{-\pi i f^{-1}\partial_{x_m}(f)}{(n+3)!}\mathrm{Vol}(S_{2n+2})G_0dx',
\end{aligned}\tag{4.21}$$

where

$$G_0=\left[\frac{2^{n+2}(n+1)\big[(2ni+3i+2)\xi_m^3+(2n+3)i\xi_m^2+(2-i)\xi_m-i\big]}{(\xi_m+1)^{n+3}}\right]^{(n+3)}|_{\xi_m=i}.$$

Combining (4.11), (4.16) and (4.21), we obtain

$$\textbf{case (2)} = \left[\frac{2^{n+1}h'(0)\pi}{12(n+3)!}\Big(\frac{1}{(\xi_m+i)^{n+1}}\Big)^{(n+3)}|_{\xi_m=i}-\frac{\pi i f^{-1}\partial_{x_m}(f)}{(n+3)!}G_0\right]\mathrm{Vol}(S_{2n+2})dx'.$$

case (3) $r=-1, l=-2n-1, |\alpha|=j=0, k=1.$

By (4.2), we have

$$\begin{aligned}
& \text{case (a) (3)}\\
= & -\frac{1}{2}\int_{|\xi'|=1}\int^{+\infty}_{-\infty}\operatorname{trace}\Big[\partial_{\xi_m}\pi^+_{\xi_m}\sigma_{-1}(fD^{-1})\times\partial_{\xi_m}\partial_{x_m}\sigma_{-2n-1}\Big((f^{-1}D^{-1})\cdot(fD^{-1}\cdot f^{-1}D^{-1})^n\Big)\Big](x_0)\\
& \times d\xi_m\sigma(\xi')dx'\\
= & -\frac{1}{2}\int_{|\xi'|=1}\int^{+\infty}_{-\infty}\operatorname{trace}\Big[\partial_{\xi_m}\pi^+_{\xi_m}\sigma_{-1}(D^{-1})\times\partial_{\xi_m}\partial_{x_m}\sigma_{-2n-1}\Big(D^{-2n-1}\Big)\Big](x_0)d\xi_m\sigma(\xi')dx'\\
& -\frac{1}{2}f\cdot\partial_{x_m}(f^{-1})\int_{|\xi'|=1}\int^{+\infty}_{-\infty}\operatorname{trace}\Big[\partial_{\xi_m}\pi^+_{\xi_m}\sigma_{-1}(D^{-1})\times\partial_{\xi_m}\sigma_{-2n-1}\Big(D^{-2n-1}\Big)\Big](x_0)d\xi_m\sigma(\xi')dx'\\
= & \frac{1}{2}\int_{|\xi'|=1}\int^{+\infty}_{-\infty}\operatorname{trace}\Big[\partial^2_{\xi_m}\pi^+_{\xi_m}\sigma_{-1}(D^{-1})\times\partial_{x_m}\sigma_{-2n-1}\Big(D^{-2n-1}\Big)\Big](x_0)d\xi_m\sigma(\xi')dx'\\
& -\frac{1}{2}f\cdot\partial_{x_m}(f^{-1})\int_{|\xi'|=1}\int^{+\infty}_{-\infty}\operatorname{trace}\Big[\partial_{\xi_m}\pi^+_{\xi_m}\sigma_{-1}(D^{-1})\times\partial_{\xi_m}\sigma_{-2n-1}\Big(D^{-2n-1}\Big)\Big](x_0)\\
& \times d\xi_m\sigma(\xi')dx'.
\end{aligned}\tag{4.22}$$

By (2.2.29) in [11], we have

$$\partial^2_{\xi_m}\pi^+_{\xi_m}\sigma_{-1}(D^{-1})(x_0)|_{|\xi'|=1}=\frac{c(\xi')+ic(dx_m)}{(\xi_m-i)^3}.\tag{4.23}$$

By Lemma (3.2), direct computations show that

$$\partial_{x_m}\sigma_{-1-2n}(D^{-1-2n})(x_0)|_{|\xi'|=1}=\frac{\sqrt{-1}\partial_{x_m}[c(\xi')](x_0)}{(1+\xi_m^2)^{n+1}}-\frac{\sqrt{-1}(n+1)h'(0)c(\xi)}{(1+\xi_m^2)^{n+2}}.\tag{4.24}$$

According to the above three formulas and the Cauchy integral formula, we have

$$\begin{aligned}
&\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}\left[\partial^2_{\xi_m}\pi^+_{\xi_m}\sigma_{-1}(D^{-1})\times\partial_{x_m}\sigma_{-2n-1}\Big(D^{-2n-1}\Big)\right](x_0)d\xi_m\sigma(\xi')dx'\\
=\;&\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}\left\{\frac{c(\xi')+ic(dx_m)}{(\xi_m-i)^3}\times\left[\frac{i\partial_{x_m}[c(\xi')](x_0)}{(1+\xi_m^2)^{n+1}}-\frac{i(n+1)h'(0)c(\xi)}{(1+\xi_m^2)^{n+2}}\right]\right\}(x_0)d\xi_m\sigma(\xi')dx'\\
=\;&\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}2^{n+1}h'(0)\times\frac{-i\xi_m^2-2(n+1)\xi_m+2ni}{(\xi_m+i)^{n+2}(\xi_m-i)^{n+5}}d\xi_m\sigma(\xi')dx'\\
=\;&\frac{\pi i h'(0)2^{n+1}\mathrm{Vol}(S_{2n+2})dx'}{(n+4)!}G_1.
\end{aligned}\tag{4.25}$$

where

$$G_1:=\left[\frac{-i\xi_m^2-2(n+1)\xi_m+2ni}{(\xi_m+i)^{n+2}}\right]^{(n+4)}|_{\xi_m=i}.$$

By (2.2.29) in [11], we have

$$\partial_{\xi_m}\pi^+_{\xi_m}\sigma_{-1}(D^{-1})(x_0)|_{|\xi'|=1}=-\frac{c(\xi')+ic(dx_m)}{2(\xi_m-i)^2}.\tag{4.26}$$

Combining (4.17) and (4.26), we have

$$\mathrm{trace}\Big[\partial_{\xi_m}\pi^+_{\xi_m}\sigma_{-1}(D^{-1})\times\partial_{\xi_m}\big(\sigma_{-2n-1}(D^{-2n-1})\big)\Big](x_0)|_{|\xi'|=1}=-\frac{2^{n+2}(n+1)i\xi_m(1+\xi_m)}{(\xi_m-i)^{n+4}(\xi_m+i)^{n+2}},\tag{4.27}$$

then we obtain

$$\begin{aligned}
&-\frac{1}{2}f\cdot\partial_{x_m}(f^{-1})\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}\left[\partial_{\xi_m}\pi^+_{\xi_m}\sigma_{-1}(D^{-1})\times\partial_{\xi_m}\sigma_{-2n-1}\Big(D^{-2n-1}\Big)\right](x_0)d\xi_m\sigma(\xi')dx'\\
=\;&\frac{f\cdot\partial_{x_m}(f^{-1})\pi i}{(n+3)!}\cdot\mathrm{Vol}(S_{2n+2})dx'\cdot G_2,
\end{aligned}\tag{4.28}$$

where

$$G_2:=\left[\frac{2^{n+2}(n+1)i\xi_m(1+\xi_m)}{(\xi_m+i)^{n+2}}\right]^{(n+3)}|_{\xi_m=i}.$$

Then

$$\textbf{case (3)}\quad=\quad\left[\frac{h'(0)2^{n+1}}{(n+4)!}G_1+\frac{f\cdot\partial_{x_m}(f^{-1})}{(n+3)!}G_2\right]\cdot\pi i\mathrm{Vol}(S_{2n+2})dx'.$$

case (4) $r=-1, l=-2n-2, |\alpha|=j=k=0.$

By (4.2), we have

$$\begin{aligned}
&\text{case (4)}\\
=&\ -i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}\Big[\pi^+_{\xi_m}\sigma_{-1}(fD^{-1})\times\partial_{\xi_m}\sigma_{-2n-2}\Big((f^{-1}D^{-1})\cdot(fD^{-1}\cdot f^{-1}D^{-1})^n\Big)\Big](x_0)\\
&\times d\xi_m\sigma(\xi')dx'\\
=&\ i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}\Big[\partial_{\xi_m}\pi^+_{\xi_m}\sigma_{-1}(fD^{-1})\times\sigma_{-2n-2}\Big((f^{-1}D^{-1})\cdot(fD^{-1}\cdot f^{-1}D^{-1})^n\Big)\Big](x_0)\\
&\times d\xi_m\sigma(\xi')dx'\\
=&\ i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}\Big\{\partial_{\xi_m}\pi^+_{\xi_m}\big(f\sigma_{-1}(D^{-1})\big)\times\Big[(f^{-1}\sigma_{-2n-2}(D^{-2n-1})+\sum_{j=1}^{m}\partial_{\xi_j}(|\xi|^{-2n-2})\\
&\times\Big(\sigma_1(D)\partial_{x_j}(f)\Big)\Big]\Big\}(x_0)d\xi_m\sigma(\xi')dx'\\
=&\ i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}\Big[\partial_{\xi_m}\pi^+_{\xi_m}\Big(\sigma_{-1}(D^{-1})\Big)\times\sigma_{-2n-2}(D^{-2n-1})\Big](x_0)d\xi_m\sigma(\xi')dx'\\
&+i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}\Big[\partial_{\xi_m}\pi^+_{\xi_m}\big(\sigma_{-1}(D^{-1})\big)\times\sum_{j=1}^{m}\partial_{\xi_j}(|\xi|^{-2n-2})\Big[\sigma_1(D)\partial_{x_j}(f)\Big]\Big](x_0)\\
&\times d\xi_m\sigma(\xi')dx'.
\end{aligned}\tag{4.29}$$

By (3.8) in [14], we have:

$$\sigma_{-2n-1}(D^{-2n})=n\sigma_2(D^2)^{(-n+1)}\sigma_{-3}(D^{-2})-i\sum_{k=0}^{n-2}\sum_{\mu=1}^{2n+2}\partial_{\xi_\mu}\sigma_2^{-n+k+1}(D^2)\partial_{x_\mu}\sigma_2^{-1}(D^2)\big(\sigma_2(D^2)\big)^{-k}.\tag{4.30}$$

By Lemma 2.2 in [11], we have

$$\sum_{j=1}^{m}\partial_{\xi_j}(|\xi|^{-2n-2})\partial_{x_j}(c(\xi))(x_0)|_{|\xi'|=1}=-2(n+1)\xi_m(1+\xi_m^2)^{-n-2}\partial_{x_m}[c(\xi')](x_0),\tag{4.31}$$

and

$$\Big[-i\sum_{k=0}^{n-1}\sum_{\mu=1}^{m}\partial_{\xi_\mu}\sigma_2^{-n+k}(D^2)\partial_{x_\mu}\sigma_2^{-1}(D^2)(\sigma_2(D^2))^{-k}\Big]ic(\xi)(x_0)|_{|\xi'|=1}=\frac{c(\xi)h'(0)\xi_m(n+1)n}{(1+\xi_m^2)^{n+3}}.\tag{4.32}$$

By (3.26) in [14], we have

$$\begin{aligned}
\sigma_{-3}(D^{-2})(x_0)|_{|\xi'|=1}&=\frac{i}{(1+\xi_m^2)^2}\Big(\frac{1}{2}h'(0)\sum_{k<m}\xi_kc(\widetilde{e_k})c(\widetilde{e_m})-\frac{n-1}{2}h'(0)\xi_m\Big)-\frac{2ih'(0)\xi_m}{(1+\xi_m^2)^3}\\
&=\frac{i}{(1+\xi_m^2)^2}\Big(\frac{1}{2}h'(0)c(\xi')c(dx_m)-\frac{n-1}{2}h'(0)\xi_m\Big)-\frac{2ih'(0)\xi_m}{(1+\xi_m^2)^3}.
\end{aligned}\tag{4.33}$$

So by (4.30), we have

$$
\begin{aligned}
&\sigma_{-2n-2}(D^{-2n-1})(x_0)|_{|\xi'|=1} = \sigma_{-2n-2}(D^{-2n-2}\cdot D)\\
= &\left\{\sum_{|\alpha|=0}^{+\infty}(-i)^{|\alpha|}\frac{1}{\alpha!}\partial_\xi^\alpha[\sigma(D^{-2n-2})]\partial_x^\alpha[\sigma(D)]\right\}_{-2n-2}\\
= &\sigma_{-2n-2}(D^{-2n-2})\sigma_0(D)+\sigma_{-2n-3}(D^{-2n-2})\sigma_1(D)+\sum_{|\alpha|=1}(-i)\partial_\xi^\alpha[\sigma_{-2n-2}(D^{-2n-2})]\partial_x^\alpha[\sigma_1(D)]\\
= &|\xi|^{-2n-2}\sigma_0(D)+\sum_{j=1}^{2n+4}\partial_{\xi_j}(|\xi|^{-2n-2})\partial_{x_j}c(\xi)+\Bigg[(n+1)\sigma_2(D^2)^{(-n)}\sigma_{-3}(D^{-2})-i\sum_{k=0}^{n-1}\sum_{\mu=1}^{2n+4}\partial_{\xi_\mu}\\
&\times\sigma_2^{-n+k}(D^2)\partial_{x_\mu}\sigma_2^{-1}(D^2)(\sigma_2(D^2))^{-k}\Bigg]\sqrt{-1}c(\xi)\\
= &\frac{(-2n-3)h'(0)c(dx_m)}{4(1+\xi_m^2)^{n+1}}-2(n+1)\xi_m(1+\xi_m^2)^{-n-2}\partial_{x_m}[c(\xi')](x_0)+(n+1)i(1+\xi_m^2)^{-n}[c(\xi')+\xi_m\\
&\times c(dx_m)]\times\left[\frac{-ih'(0)c(\xi')c(dx_m)}{2(1+\xi_m^2)^2}-\frac{(2n+3)h'(0)i\xi_m}{2(1+\xi_m^2)^2}-\frac{2ih'(0)\xi_m}{(1+\xi_m^2)^3}\right]+[c(\xi')+\xi_mc(dx_m)]h'(0)\xi_m\\
&\times(n^2+n)(1+\xi_m^2)^{-n-3}.
\end{aligned}\tag{4.34}
$$

By (4.26) and (4.34), we have

$$
\begin{aligned}
&\mathrm{trace}[\partial_{\xi_m}\pi^+_{\xi_m}\sigma_{-1}(D^{-1})\times\sigma_{-2n-2}(D^{-2n-1})](x_0)|_{|\xi'|=1}\\
= &\frac{2^{n+1}h'(0)}{4(\xi_m-i)^{n+4}(\xi_m+i)^{n+3}}\times\Big[(2n+3)(2n+1)i\xi_m^3+(2\pi-2n-1)\xi_m^2+\big(8n^2+16n+7\big)i\xi_m\\
&+(2n+3+2\pi)\Big]^{(n+3)}|_{\xi_n=i}
\end{aligned}\tag{4.35}
$$

Then by the Cauchy integral formula, we get

$$
\begin{aligned}
&i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}\Big[\partial_{\xi_m}\pi^+_{\xi_m}\big(\sigma_{-1}(D^{-1})\big)\times\sigma_{-2n-2}(D^{-2n-1})\Big](x_0)d\xi_m\sigma(\xi')dx'\\
= &\frac{-\pi 2^n h'(0)\mathrm{Vol}(S_{2n+2})dx'}{(n+3)!}G_3.
\end{aligned}\tag{4.36}
$$

where

$$
G_3:=\left\{\frac{1}{(\xi_m+i)^{n+3}}\Big[(2n+3)(2n+1)i\xi_m^3+(2\pi-2n-1)\xi_m^2+\big(8n^2+16n+7\big)i\xi_m+(2n+3+2\pi)\Big]\right\}^{(n+3)}|_{\xi_m=i}.
$$

And we have

$$
\sum_{j=1}^m\partial_{\xi_j}(|\xi|^{-2n-2})\Big(\sigma_1(D)\partial_{x_j}(f)\Big)(x_0)|_{|\xi'|=1}=-\sum_{j=1}^m\Big(\xi_j\partial_{x_j}(f)\Big)\cdot(2n+2)i\cdot(\xi_m^2+1)^{-n-2}\cdot c(\xi).\tag{4.37}
$$

We note that $i<m$, $\int_{|\xi'|=1}\xi_i\sigma(\xi')=0$, so $\sum_j\xi_j\partial_{x_j}(f)\mathrm{trace}[\mathrm{id}]$ have no contribution for computing **case**

(4). Then we obtain

$$\begin{aligned}
& i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}\Big[\partial_{\xi_m}\pi^+_{\xi_m}\big(\sigma_{-1}(D^{-1})\big)\times\sum_{j=1}^{m}\partial_{\xi_j}(|\xi|^{-2n-2})\big(\sigma_1(D)\partial_{x_j}(f)\big)\Big](x_0)d\xi_m\sigma(\xi')dx' \\
= & \ i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\Big\{-\frac{c(\xi')+ic(dx_m)}{2(\xi_m-i)^2}\cdot\Big[-\Big(\xi_m\partial_{x_m}(f)\Big)\cdot(2n+2)i\cdot(\xi_m^2+1)^{-n-2}\cdot c(\xi)\Big]\Big\}(x_0) \\
& \times d\xi_m\sigma(\xi')dx' \\
= & \ \frac{-2^{n+4}\partial_{x_m}(f)i\pi(n+1)}{(n+3)!}\mathrm{Vol}(S_{2n+2})G_4dx'. \qquad (4.38)
\end{aligned}$$

where

$$G_4 := \left[\frac{(1+\xi_m)\xi_m}{2(\xi_m+i)^{n+2}}\right]^{(n+3)}|_{\xi_m=i}.$$

Then

$$\textbf{case (4)} = \left[\frac{-\pi 2^n h'(0)}{(n+3)!}G_3-\frac{2^{n+4}\partial_{x_m}(f)i\pi(n+1)}{(n+3)!}G_4\right]\mathrm{Vol}(S_{2n+2})dx'.$$

case (5) $r=-2,\ l=-2n-1,\ k=j=|\alpha|=0.$

By (4.2), we get

$$\begin{aligned}
\text{case (5)} \ = & \ -i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}\Big[\pi^+_{\xi_m}\sigma_{-2}(fD^{-1})\times\partial_{\xi_m}\sigma_{-2n-1}\Big((f^{-1}D^{-1})\cdot(fD^{-1}\cdot f^{-1}D^{-1})^n\Big)\Big](x_0) \\
& \times d\xi_m\sigma(\xi')dx' \\
= & \ -i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}\Big[\pi^+_{\xi_m}\sigma_{-2}(D^{-1})\times\partial_{\xi_m}\sigma_{-2n-1}(D^{-2n-1})\Big](x_0)d\xi_m\sigma(\xi')dx'. \qquad (4.39)
\end{aligned}$$

By (2.2.34)-(2.2.37) in [11], we have

$$\pi^+_{\xi_n}\sigma_{-2}(D^{-1})(x_0)|_{|\xi'|=1}=J_1-J_2, \qquad (4.40)$$

where

$$J_1=-\frac{H_1}{4(\xi_n-i)}-\frac{H_2}{4(\xi_n-i)^2}, \qquad (4.41)$$

and

$$H_1=ic(\xi')\sigma_0(D)c(\xi')+ic(dx_n)\sigma_0(D)c(dx_n)+ic(\xi')c(dx_n)\partial_{x_n}[c(\xi')]; \qquad (4.42)$$

$$H_2=[c(\xi')+ic(dx_n)]\sigma_0(D)[c(\xi')+ic(dx_n)]+c(\xi')c(dx_n)\partial_{x_n}c(\xi')-i\partial_{x_n}[c(\xi')]; \qquad (4.43)$$

$$J_2=\frac{h'(0)}{2}\left[\frac{c(dx_n)}{4i(\xi_n-i)}+\frac{c(dx_n)-ic(\xi')}{8(\xi_n-i)^2}+\frac{3\xi_n-7i}{8(\xi_n-i)^3}[ic(\xi')-c(dx_n)]\right]. \qquad (4.44)$$

Similar to (2.2.38) in [11], we have

$$\partial_{\xi_m}\sigma_{-2n-1}(D^{-2n-1})(x_0)|_{|\xi'|=1}=\sqrt{-1}\left[\frac{c(dx_m)}{(1+\xi_m^2)^{n+1}}-(n+1)\times\frac{2\xi_mc(\xi')+2\xi_m^2c(dx_m)}{(1+\xi_m^2)^{n+2}}\right]. \qquad (4.45)$$

By (4.39), (4.44) and (4.45), we have

$$\begin{aligned}
&\mathrm{tr}[J_2\times\partial_{\xi_m}\sigma_{-1-2n}(D^{-1-2n})(x_0)]|_{|\xi'|=1}\\
=&\ \frac{\sqrt{-1}}{2}h'(0)\mathrm{trace}\Bigg\{\Bigg[\Big(\frac{1}{4i(\xi_m-i)}+\frac{1}{8(\xi_m-i)^2}-\frac{3\xi_m-7i}{8(\xi_m-i)^3}\Big)c(dx_m)+\Big(\frac{-1}{8(\xi_m-i)^2}+\frac{3\xi_m-7i}{8(\xi_m-i)^3}\Big)\\
&\times ic(\xi')\Bigg]\times\Bigg[\Big(\frac{1}{(1+\xi_m^2)^{1+n}}-\frac{2(n+1)\xi_m^2}{(1+\xi_m^2)^{n+2}}\Big)c(dx_m)-\frac{2(n+1)\xi_m}{(1+\xi_m^2)^{n+2}}c(\xi')\Bigg]\Bigg\}\\
=&\ h'(0)2^{n-1}\times\frac{(2n+1)\xi_m^3-2i(2n+1)\xi_m^2-(6n+5)\xi_m+4i}{(\xi_m-i)^2(1+\xi_m^2)^{n+2}}. \qquad (4.46)
\end{aligned}$$

By (2.2.40) in [11], we have

$$\begin{aligned}
J_1 =&\ \frac{-1}{4(\xi_m-i)^2}[(2+i\xi_m)c(\xi')\sigma_0(D)c(\xi')+i\xi_mc(dx_m)\sigma_0(D)c(dx_m)+(2+i\xi_m)c(\xi')c(dx_m)\\
&\times\partial_{x_m}c(\xi')+ic(dx_m)\sigma_0(D)c(\xi')+ic(\xi')\sigma_0(D)c(dx_m)-i\partial_{x_m}c(\xi')]. \qquad (4.47)
\end{aligned}$$

Similar to Lemma 2.4 in [11], we have

$$\sigma_0(D)(x_0)=c_0c(dx_m),\quad \text{where } c_0=\frac{-m}{4}h'(0). \qquad (4.48)$$

By the relation of the Clifford action and trace$AB=$ traceBA, then we have the equalities:

$$\mathrm{trace}[c(\xi')\sigma_0(D)c(\xi')c(dx_m)]=-c_02^{n+2};\quad \mathrm{trace}[c(dx_m)\sigma_0(D)c(dx_m)^2]=c_02^{n+2};$$

$$\mathrm{trace}[c(\xi')c(dx_m)\partial_{x_m}c(\xi')c(dx_m)](x_0)|_{|\xi'|=1}=-2^{n+1}h'(0);\ \mathrm{trace}[c(dx_m)\sigma_0(D)c(\xi')^2]=c_02^{n+2}.$$

By (4.47) and (4.48), considering for $i<m$, $\int_{|\xi'|=1}\{$odd number product of $\xi_i\}\sigma(\xi')=0$, then

$$\begin{aligned}
&\mathrm{tr}[J_1\times\partial_{\xi_m}\sigma_{-1-2n}(D^{-1-2n})(x_0)]|_{|\xi'|=1}\\
=&\ \frac{2^{n+1}ih'(0)}{4(\xi_m-i)^2(1+\xi_m^2)^{n+2}}\cdot\Big\{(m-1)[(2n+1)\xi_m^2-2i(n+1)\xi_m-1]+[-(1+2n)i\xi_m^3-2(1+2n)\xi_m^2\\
&+(2n+3)i\xi_m+2]\Big\}. \qquad (4.49)
\end{aligned}$$

By combining (4.46), (4.49) and the Cauchy integral formula, we have

$$\begin{aligned}
&\text{case (5)}\\
=&\ -i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}[(J_1-J_2)\times\partial_{\xi_m}\sigma_{-1-2n}(D^{-1-2n})](x_0)d\xi_m\sigma(\xi')dx'\\
=&\ 2^{n+1}h'(0)\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\frac{[4n^2+8n+3]\xi_m^2-[4n^2+14n+8]i\xi_m-(m+1)}{4(\xi_m-i)^{n+4}(\xi_m+i)^{n+2}}d\xi_m\sigma(\xi')dx'\\
=&\ \frac{2^{n+2}h'(0)\mathrm{Vol}(S_{2n+2})\pi i dx'}{(n+3)!}G_5, \qquad (4.50)
\end{aligned}$$

where

$$G_5:=\left[\frac{[4n^2+8n+3]\xi_m^2-[4n^2+14n+8]i\xi_m-(m+1)}{4(\xi_m+i)^{n+2}}\right]^{(n+3)}|_{\xi_m=i}.$$

Since Ψ is the sum of the **case (1)-case (5)**, so

$$\begin{aligned}
\Psi =&\ \Bigg\{\frac{(-1)^nh'(0)\pi}{3\times2^{n+6}(3+n)!}Y_0+\frac{(-1)^n(n+1)f^{-1}\partial_{x_n}(f)\pi}{2^{n+2}}Y_1+\Bigg[\frac{(i-1)f\cdot\partial_{x_n}(f^{-1})}{2^{n+3}}\\
&-\frac{\partial_{x_n}(f)(1+i)}{2^{n+2}}\Bigg]Y_2\Bigg\}\mathrm{Vol}(S_{2n+2})dx', \qquad (4.51)
\end{aligned}$$

where

$$\begin{aligned}
Y_0 &= -24(1+i)\Bigg[(4n^2+8n+3)C^{n}_{-n-3}+(6n^2+13n+5-\pi)C^{n+1}_{-n-3}+(n^2+3n-\pi+1)C^{n+2}_{-n-3}\\
&\quad -2n(1+i)(1+n)C^{n+3}_{-n-3}+24C^{n+2}_{-n-2}-24C^{n+3}_{-n-2}-6C^{n+4}_{-n-2}\Bigg]\times(3+n)!+A^{3+n}_{-1-n};\\
Y_1 &= (4n+4+6i)C^{n}_{-n-3}+(9i+8n+9)C^{n+1}_{-n-3}+(i+n+1)\Bigg[5C^{n+2}_{-n-3}+C^{n+3}_{-n-3}\Bigg]\\
Y_2 &= (n+1)(-1)^n\Bigg[2(1+i)C^{n+1}_{-n-2}+(3+i)C^{n+2}_{-n-2}+C^{n+3}_{-n-2}\Bigg]. \qquad (4.52)
\end{aligned}$$

Combining (3.11) and (4.51), we obtain Theorem 1.2.

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