Non-linear non-zero-sum Dynkin games with Bermudan strategies

Miryana Grigorova[1*] Marie-Claire Quenez[2 †]

Peng Yuan[3 ‡]

[1,3]University of Warwick
[2]Université Paris Cité

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Abstract: In this paper, we study a non-zero-sum game with two players, where each of the players plays what we call Bermudan strategies and optimizes a general non-linear assessment functional of the pay-off. By using a recursive construction, we show that the game has a Nash equilibrium point.

1 Introduction

Game problems with linear evaluations between a finite number of players are by now classical problems in stochastic control and optimal stopping (cf., e.g., [1], [3], [5], [9], [13], [14], [15], [16], [17], [18], [25], [26] and [28]) with various applications, in particular in economics and finance (cf., e.g., [13], [14], [22] and [25]). In the recent years game problems with *non-linear* evaluation functionals have attracted considerable interest: cf. [2] for the case of non-linear functionals of the form of worst case expectations over a set of possibly singular measures; [6], [7], [8] and [10] for the case of non-linear functionals induced by backward stochastic differential equations (BSDEs). Most of the works dealing with *non-linear* games have focused on the *zero-sum* case (cf.,

*Corresponding Author. Department of Statistics, University of Warwick, E-mail:miryana.grigorova@warwick.ac.uk

†LPSM, University Paris-Cité

‡Department of Statistics, University of Warwick

e.g., [2], [6], [7], [8] and [10]). *Non-zero-sum* games are notoriously more intricate than their zero-sum counterparts even in the case of linear evaluations (cf., e.g., [14], [16], [17], [27], [29] and [31]). Non-zero-sum games with non-linear functionals have been considered in [12] in the discrete-time framework and with non-linear functionals induced by Backward SDEs with Lipschitz driver, in [21] in the continuous time framework and with non-linear functionals of the form of expected exponential utilities.

In the current paper, we address the question of existence of a Nash equilibrium point in a framework with general non-linear evaluations and with a set of stopping strategies which is in between the discrete time and the continuous time stopping strategies. The results of [12] can be seen as a particular case of the current paper.

The paper is organised as follows: In Section 2, we introduce the framework, including the set of optimal stopping strategies of the agents (namely the Bermudan strategies), the pay-off as well as the properties on the risk functionals ρ^1 and ρ^2 of agent 1 and agent 2. In Section 3, we present our main results and show that the non-linear non-zero-sum game with Bermudan strategies has a Nash equilibrium point.

2 The framework

Let $T > 0$ be a **fixed finite** terminal horizon.

Let $(\Omega, \mathcal{F}, P)$ be a (complete) probability space equipped with a right-continuous complete filtration $\mathbb{F} = \{\mathcal{F}_t : t \in [0, T]\}$.

In the sequel, equalities and inequalities between random variables are to be understood in the P-almost sure sense. Equalities between measurable sets are to be understood in the P-almost sure sense.

Let $\mathbb{N}$ be the set of natural numbers, including 0. Let $\mathbb{N}^*$ be the set of natural numbers, excluding 0.

We first define the so-called Bermudan stopping strategies (introduced in [11]).

Let $(\theta_k)_{k\in\mathbb{N}}$ be a sequence of stopping times satisfying the following properties:

(a) The sequence $(\theta_k)_{k\in\mathbb{N}}$ is non-decreasing, i.e. for all $k \in \mathbb{N}$, $\theta_k \leqslant \theta_{k+1}$, a.s.

(b) $\lim_{k\to\infty} \uparrow \theta_k = T$ a.s.

Moreover, we set $\theta_0 = 0$.

We note that the family of σ-algebras $(\mathcal{F}_{\theta_k})_{k\in\mathbb{N}}$ is non- decreasing (as the sequence (θ_k) is non-decreasing). We denote by Θ the set of stopping times τ of the form

$$\tau = \sum_{k=0}^{+\infty} \theta_k \mathbf{1}_{A_k} + T\mathbf{1}_{\bar{A}}, \tag{1}$$

where $\{(A_k)_{k=0}^{+\infty}, \bar{A}\}$ form a partition of Ω such that, for each $k \in \mathbb{N}$, $A_k \in \mathcal{F}_{\theta_k}$, and $\bar{A} \in \mathcal{F}_T$.

The set Θ can also be described as the set of stopping times τ such that for almost all $\omega \in \Omega$, either $\tau(\omega) = T$ or $\tau(\omega) = \theta_k(\omega)$, for some $k = k(\omega) \in \mathbb{N}$.

Note that the set Θ is closed under concatenation, that is, for each $\tau \in \Theta$ and each $A \in \mathcal{F}_\tau$, the stopping time $\tau\mathbf{1}_A + T\mathbf{1}_{A^c} \in \Theta$. More generally, for each $\tau \in \Theta$, $\tau' \in \Theta$ and each $A \in \mathcal{F}_{\tau\wedge\tau'}$, the stopping time $\tau\mathbf{1}_A + \tau'\mathbf{1}_{A^c}$ is in Θ. The set Θ is also closed under pairwise minimization (that is, for each $\tau \in \Theta$ and $\tau' \in \Theta$, we have $\tau \wedge \tau' \in \Theta$) and under pairwise maximization (that is, for each $\tau \in \Theta$ and $\tau' \in \Theta$, we have $\tau \vee \tau' \in \Theta$). Moreover, the set Θ is closed under monotone limit, that is, for each non-decreasing (resp. non-increasing) sequence of stopping times $(\tau_n)_{n\in\mathbb{N}} \in \Theta^{\mathbb{N}}$, we have $\lim_{n\to+\infty} \tau_n \in \Theta$.

We note also that all stopping times in Θ are bounded from above by T.

Remark 1. *We have the following* canonical *writing of the sets in* (1)*:*

$$\begin{aligned} A_0 &= \{\tau = \theta_0\}; \\ A_{n+1} &= \{\tau = \theta_{n+1}, \theta_{n+1} < T\} \backslash (A_n \cup ... \cup A_0); \text{ for all } n \in \mathbb{N}^* \\ \bar{A} &= (\cup_{k=0}^{+\infty} A_k)^c \end{aligned}$$

From this writing, we have: if $\omega \in A_{k+1} \cap \{\theta_k < T\}$, then $\omega \notin \{\tau = \theta_k\}$.

For each $\tau \in \Theta$, we denote by Θ_τ the set of stopping times $\nu \in \Theta$ such that $\nu \geqslant \tau$ a.s. The set Θ_τ satisfies the same properties as the set Θ. We will refer to the set Θ as the set of **Bermudan stopping strategies**, and to the set Θ_τ as the set of Bermudan stopping strategies, greater than or equal to τ (or the set of Bermudan stopping strategies from time τ perspective). For simplicity, the set Θ_{θ_k} will be denoted by Θ_k.

Definition 1. *We say that a family $\phi = (\phi(\tau), \tau \in \Theta)$ is* admissible *if it satisfies the following conditions*

1. *for all $\tau \in \Theta$, $\phi(\tau)$ is a real valued random variable, which is $\mathcal{F}_\tau$-measurable.*
2. *for all $\tau, \tau' \in \Theta$, $\phi(\tau) = \phi(\tau')$ a.s. on $\{\tau = \tau'\}$.*

Moreover, for $p \in [1, +\infty]$ fixed, we say that an admissible family ϕ is p-integrable, if for all $\tau \in \Theta$, $\phi(\tau)$ is in L^p.

Let $\phi = (\phi(\tau),\, \tau \in \Theta)$ be an admissible family. For a stopping time τ of the form (1), we have

$$\phi(\tau) = \sum_{k=0}^{+\infty} \phi(\theta_k)\mathbf{1}_{A_k} + \phi(T)\mathbf{1}_{\bar{A}} \quad \text{a.s.} \tag{2}$$

Given two admissible families $\phi = (\phi(\tau),\, \tau \in \Theta)$ and $\phi' = (\phi'(\tau),\, \tau \in \Theta)$, we say that ϕ is *equal to* ϕ' and write $\phi = \phi'$ if, for all $\tau \in \Theta$, $\phi(\tau) = \phi'(\tau)$ a.s. We say that ϕ *dominates* ϕ' and write $\phi \geqslant \phi'$ if, for all $\tau \in \Theta$, $\phi(\tau) \geqslant \phi'(\tau)$ a.s.

Let $p \in [1, +\infty]$. We introduce the following properties on the non-linear operators $\rho_{S,\tau}[\cdot]$, which will appear in the sequel.

For $S \in \Theta$, $S' \in \Theta$, $\tau \in \Theta$, for η, η_1 and η_2 in $L^p(\mathcal{F}_\tau)$, for $\xi = (\xi(\tau))$ an admissible p-integrable family:

(i) $\rho_{S,\tau} : L^p(\mathcal{F}_\tau) \longrightarrow L^p(\mathcal{F}_S)$

(ii) *(admissibility)* $\rho_{S,\tau}[\eta] = \rho_{S',\tau}[\eta]$ a.s. on $\{S = S'\}$.

(iii) *(knowledge preservation)* $\rho_{\tau,S}[\eta] = \eta$, for all $\eta \in L^p(\mathcal{F}_S)$, all $\tau \in \Theta_S$.

(iv) *(monotonicity)* $\rho_{S,\tau}[\eta_1] \leqslant \rho_{S,\tau}[\eta_2]$ a.s., if $\eta_1 \leqslant \eta_2$ a.s.

(v) *(consistency)* $\rho_{S,\theta}[\rho_{\theta,\tau}[\eta]] = \rho_{S,\tau}[\eta]$, for all S, θ, τ in Θ such that $S \leqslant \theta \leqslant \tau$ a.s.

(vi) *("generalized zero-one law")* $I_A \rho_{S,\tau}[\xi(\tau)] = I_A \rho_{S,\tau'}[\xi(\tau')]$, for all $A \in \mathcal{F}_S$, $\tau \in \Theta_S$, $\tau' \in \Theta_S$ such that $\tau = \tau'$ on A.

(vii) *(monotone Fatou property with respect to terminal condition)* $\rho_{S,\tau}[\eta] \leqslant \liminf_{n\to+\infty} \rho_{S,\tau}[\eta_n]$, for $(\eta_n), \eta$ such that (η_n) is non-decreasing, $\eta_n \in L^p(\mathcal{F}_\tau)$, $\sup_n \eta_n \in L^p$, and $\lim_{n\to+\infty} \uparrow \eta_n = \eta$ a.s.

(viii) (left-upper-semicontinuity (LUSC) along Bermudan stopping times with respect to the terminal condition and the terminal time), that is,

$$\limsup_{n\to+\infty} \rho_{S,\tau_n}[\phi(\tau_n)] \leqslant \rho_{S,\nu}[\limsup_{n\to+\infty} \phi(\tau_n)],$$

for each non-decreasing sequence $(\tau_n) \in \Theta_S^{\mathbb{N}}$ such that $\lim_{n\to+\infty} \uparrow \tau_n = \nu$ a.s., and for each p-integrable admissible family ϕ such that $\sup_{n\in\mathbb{N}} |\phi(\tau_n)| \in L^p$.

(ix) $\limsup_{n\to+\infty} \rho_{\theta_n,T}[\eta] \leqslant \rho_{T,T}[\eta]$, for all $\eta \in L^p(\mathcal{F}_T)$.

These assumptions on ρ ensure that the one-agent's non-linear optimal stopping problem admits a solution and that the first hitting time (when the value family "hits" the pay-off family) is optimal (cf. [11] for more details).

3 The game problem

We consider two agents, agent 1 and agent 2, whose pay-offs are defined via four admissible families $X^1 = (X^1(\tau))_{\tau\in\Theta}$, $X^2 = (X^2(\tau))_{\tau\in\Theta}$, $Y^1 = (Y^1(\tau))_{\tau\in\Theta}$ and $Y^2 = (Y^2(\tau))_{\tau\in\Theta}$. We assume that X^1, X^2, Y^1 and Y^2 are p-integrable families such that

(A1) $X^1 \leqslant Y^1$, $X^2 \leqslant Y^2$ (that is, for each $\tau \in \Theta$, $X^1(\tau) \leqslant Y^1(\tau)$, and $X^2(\tau) \leqslant Y^2(\tau)$).

(A2) $X^1(T) = Y^1(T)$, $X^2(T) = Y^2(T)$.

(A3) ess $\sup_{\tau\in\Theta} X^1(\tau) \in L^p$, ess $\sup_{\tau\in\Theta} X^2(\tau) \in L^p$,

ess $\sup_{\tau\in\Theta} Y^1(\tau) \in L^p$ and ess $\sup_{\tau\in\Theta} Y^2(\tau) \in L^p$.

(A4) $\limsup_{k\to+\infty} X^1(\theta_k) \leqslant X^1(T)$, $\quad \limsup_{k\to+\infty} X^2(\theta_k) \leqslant X^2(T)$.

The set of stopping strategies of each agent at time 0 is the set Θ of Bermudan stopping times. If the first agent plays $\tau_1 \in \Theta$ and the second agent plays $\tau_2 \in \Theta$, the pay-off of agent 1 (resp. agent 2) at time $\tau_1 \wedge \tau_2$ is given by:

$$I^1(\tau_1, \tau_2) := X^1(\tau_1)\mathbb{1}_{\{\tau_1 \leqslant \tau_2\}} + Y^1(\tau_2)\mathbb{1}_{\{\tau_2 < \tau_1\}}$$

$$(\text{resp. } I^2(\tau_1, \tau_2) := X^2(\tau_2)\mathbb{1}_{\{\tau_2 < \tau_1\}} + Y^2(\tau_1)\mathbb{1}_{\{\tau_1 \leqslant \tau_2\}}),$$

where we have adopted the convention: when $\tau_1 = \tau_2$, it is the first agent who is responsible for stopping the game. The agents evaluate their respective pay-offs via possibly different evaluation functionals. Let $\rho^1 = (\rho_{S,\tau}[\cdot])$ be the family of evaluation operators of agent 1, and let $\rho^2 = (\rho_{S,\tau}[\cdot])$ be the family of evaluation operators of agents 2. If agent 1 plays $\tau_1 \in \Theta$, and agent 2 plays $\tau_2 \in \Theta$, then the assessment (or evaluation) of agent 1 (resp. agent 2) at time 0 of his/her pay-off is given by:

$$J_1(\tau_1, \tau_2) := \rho^1_{0,\tau_1\wedge\tau_2}[X^1(\tau_1)\mathbb{1}_{\{\tau_1 \leqslant \tau_2\}} + Y^1(\tau_2)\mathbb{1}_{\{\tau_2 < \tau_1\}}].$$

$$(\text{resp. } J_2(\tau_1, \tau_2) := \rho^2_{0,\tau_1\wedge\tau_2}[X^2(\tau_2)\mathbb{1}_{\{\tau_2 < \tau_1\}} + Y^2(\tau_1)\mathbb{1}_{\{\tau_1 \leqslant \tau_2\}}]).$$

We assume that both ρ^1 and ρ^2 satisfy the properties (i) - (ix). We will investigate the problem of existence of a Nash equilibrium strategy (τ_1^*, τ_2^*).

Definition 2. *A pair of Bermudan stopping times $(\tau_1^*, \tau_2^*) \in \Theta \times \Theta$ is called a Nash equilibrium strategy (or a Nash equilibrium point) for the above non-zero-sum non-linear Bermudan Dynkin game if: $J_1(\tau_1^*, \tau_2^*) \geqslant J_1(\tau_1, \tau_2^*)$, for any $\tau_1 \in \Theta$, and $J_2(\tau_1^*, \tau_2^*) \geqslant J_2(\tau_1^*, \tau_2)$, for any $\tau_2 \in \Theta$.*

In other words, any unilateral deviation from the strategy (τ_1^*, τ_2^*) by one of the agent (the strategy of the other remaining fixed) does not render the deviating agent better off.

Theorem 1. *Under assumptions (i) - (ix) on ρ^1 and ρ^2, there exists a Nash equilibrium point (τ_1^*, τ_2^*) for the game described above.*

We will construct a sequence $(\tau_{2n+1}, \tau_{2n})_{n\in\mathbb{N}}$ (by induction), for which we will show that it converges to a Nash equilibrium point.

We set $\tau_1 := T$ and $\tau_2 := T$. We suppose that $\tau_{2n-1} \in \Theta$ and $\tau_{2n} \in \Theta$ have been defined. We set, for each $k \in \mathbb{N}$,

$$\xi^{2n+1}(\theta_k) := X^1(\theta_k)\mathbb{1}_{\{\theta_k<\tau_{2n}\}} + Y^1(\tau_{2n})\mathbb{1}_{\{\tau_{2n}\leqslant\theta_k\}}. \tag{3}$$

Moreover, $\xi^{2n+1}(T) := Y^1(\tau_{2n})$. This definition is "consistent" with the above, as by (3), $\mathbb{1}_{\{\theta_k=T\}}\xi^{2n+1}(\theta_k) = \mathbb{1}_{\{\theta_k=T\}}Y^1(\tau_{2n})$.

For $\tau \in \Theta$ of the form $\tau = \sum_{k\in\mathbb{N}} \theta_k \mathbb{1}_{A_k} + T\mathbb{1}_{\bar{A}}$, where $((A_k), \bar{A})$ is a partition, A_k is $\mathcal{F}_{\theta_k}$-measurable for each $k \in \mathbb{N}$, and $\bar{A}$ is $\mathcal{F}_T$-measurable,

$$\xi^{2n+1}(\tau) := \sum_{k\in\mathbb{N}} \xi^{2n+1}(\theta_k)\mathbb{1}_{A_k} + \xi^{2n+1}(T)\mathbb{1}_{\bar{A}}. \tag{4}$$

We note that $\xi^{2n+1}(\theta_k)$ is the pay-off at $\theta_k \wedge \tau_{2n}$ of agent 1 (up to the equality $\{\theta_k = \tau_{2n}\}$) if agent 1 plays θ_k and agent 2 plays τ_{2n}.

We also note that:

$$\xi^{2n+1}(\tau) = X^1(\tau)\mathbb{1}_{\{\tau<\tau_{2n}\}} + Y^1(\tau_{2n})\mathbb{1}_{\{\tau_{2n}\leqslant\tau\}}.$$

Thus, $\xi^{2n+1}(\tau)$ is the pay-off at $\tau\wedge\tau_{2n}$ of agent 1 (up to the equality $\{\tau = \tau_{2n}\}$) if agent 1 plays τ and agent 2 plays τ_{2n}.

For each $S \in \Theta$, we define

$$\begin{aligned}
&V^{2n+1}(S) := \operatorname{ess\,sup}_{\tau\in\Theta_S}\rho^1_{S,\tau\wedge\tau_{2n}}[\xi^{2n+1}(\tau)]\\
&\tilde{\tau}_{2n+1} := \operatorname{ess\,inf} \tilde{\mathcal{A}}^1, \text{ where } \tilde{\mathcal{A}}^1 := \{\tau \in \Theta : V^{2n+1}(\tau) = \xi^{2n+1}(\tau)\}\\
&\tau_{2n+1} := (\tilde{\tau}_{2n+1} \wedge \tau_{2n-1})\mathbb{1}_{\{\tilde{\tau}_{2n+1}\wedge\tau_{2n-1}<\tau_{2n}\}} + \tau_{2n-1}\mathbb{1}_{\{\tilde{\tau}_{2n+1}\wedge\tau_{2n-1}\geqslant\tau_{2n}\}}.
\end{aligned} \tag{5}$$

Assuming that $\limsup_{k\to+\infty} X^1(\theta_k) \leqslant X^1(T)$ (from (A4)) ensures that $\limsup_{k\to+\infty} \xi^{2n+1}(\theta_k) \leqslant \xi^{2n+1}(T)$. This is a technical condition on the pay-off which we use to apply Theorem 2.3 in [11].

We recall that under the assumptions of Theorem 2.3 in [11], the Bermudan stopping time $\tilde{\tau}_{2n+1}$ is optimal for the optimal stopping problem with value $V^{2n+1}(0)$, that is

$$V^{2n+1}(0) = \rho^1_{0,\tilde{\tau}_{2n+1}\wedge\tau_{2n}}[\xi(\tilde{\tau}_{2n+1})] = \sup_{\tau\in\Theta} \rho^1_{0,\tau\wedge\tau_{2n}}[\xi^{2n+1}(\tau)]. \tag{6}$$

We also recall that $V^{2n+1}(T) = \xi^{2n+1}(T)$, under the assumption of knowledge preservation on ρ.

Remark 2. *i) It is not difficult to show, by induction, that for each $n \in \mathbb{N}$, $(\xi^{2n+1}(\tau))_{\tau\in\Theta}$ is an admissible L^p-integrable family, and τ_{2n+1} is a Bermudan stopping time (for the latter property, we use that Θ has the property of stability by concatenation of two Bermudan stopping times).*

ii) For each $n \in \mathbb{N}$, for each $\tau \in \Theta$, $\xi^{2n+1}(\tau) = \xi^{2n+1}(\tau \wedge \tau_{2n})$.

Indeed, we have:

$$\begin{aligned}\xi^{2n+1}(\theta_k) &= X^1(\theta_k)\mathbb{1}_{\{\theta_k<\tau_{2n}\}} + Y^1(\tau_{2n})\mathbb{1}_{\{\tau_{2n}\leqslant\theta_k\}}\\ &= X^1(\theta_k\wedge\tau_{2n})\mathbb{1}_{\{\theta_k\wedge\tau_{2n}<\tau_{2n}\}} + Y^1(\tau_{2n})\mathbb{1}_{\{\tau_{2n}\leqslant\theta_k\wedge\tau_{2n}\}} = \xi^{2n+1}(\theta_k\wedge\tau_{2n}).\end{aligned} \tag{7}$$

Now, let $\tau \in \Theta$ be of the form $\tau = \sum_{k\in\mathbb{N}} \theta_k \mathbb{1}_{A_k} + T\mathbb{1}_{\bar{A}}$. By definition of $\xi^{2n+1}(\tau)$, of $\xi^{2n+1}(T)$ and by Eq. (7), we have:

$$\begin{aligned}\xi^{2n+1}(\tau) &= \sum_{k\in\mathbb{N}} \xi^{2n+1}(\theta_k)\mathbb{1}_{A_k} + \xi^{2n+1}(T)\mathbb{1}_{\bar{A}}\\ &= \sum_{k\in\mathbb{N}} \xi^{2n+1}(\theta_k\wedge\tau_{2n})\mathbb{1}_{A_k} + Y^1(\tau_{2n})\mathbb{1}_{\bar{A}} = \xi^{2n+1}(\tau\wedge\tau_{2n}).\end{aligned}$$

Proposition 1. *i) $\xi^{2n+1}(\tau)\mathbb{1}_{\{\tau_{2n}\leqslant\theta_k\}} = Y^1(\tau_{2n})\mathbb{1}_{\{\tau_{2n}\leqslant\theta_k\}}$.*

ii) $V^{2n+1}(\theta_k)\mathbb{1}_{\{\tau_{2n}\leqslant\theta_k\}} = Y^1(\tau_{2n})\mathbb{1}_{\{\tau_{2n}\leqslant\theta_k\}}$.

iii) $V^{2n+1}(\tau)\mathbb{1}_{\{\tau_{2n}\leqslant\tau\}} = Y^1(\tau_{2n})\mathbb{1}_{\{\tau_{2n}\leqslant\tau\}}$.

iv) For each $n \in \mathbb{N}$, $\tilde{\tau}_{2n+1} = \operatorname{ess\,inf}\{\tau \in \Theta : V^{2n+1}(\tau) = X^1(\tau)\} \wedge \tau_{2n}$. In particular, $\tilde{\tau}_{2n+1} \leqslant \tau_{2n}$.

Proof. i) On the set $\{\tau = T\}$, we have $\xi^{2n+1}(\tau) = \xi^{2n+1}(T) = Y^1(\tau_{2n})$. On the set $\{\tau = \theta_k < T\}$, by the second statement in Remark 2, we have $\xi^{2n+1}(\tau) = \xi^{2n+1}(\theta_k) = \xi^{2n+1}(\theta_k \wedge \tau_{2n})$. Hence, on the set $\{\tau = \theta_k < T\} \cap \{\tau_{2n} \leqslant \theta_k\}$, we have $\xi^{2n+1}(\tau) = \xi^{2n+1}(\tau_{2n}) = Y^1(\tau_{2n})$, which proves the desired property.

ii) We have:

$$\begin{aligned}\mathbb{1}_{\{\tau_{2n}\leqslant\theta_k\}}V^{2n+1}(\theta_k) &= \mathbb{1}_{\{\tau_{2n}\leqslant\theta_k\}}\text{ess sup}_{\tau\in\Theta_k}\rho_{\theta_k,\tau\wedge\tau_{2n}}[\xi^{2n+1}(\tau\wedge\tau_{2n})]\\ &= \text{ess sup}_{\tau\in\Theta_k}\mathbb{1}_{\{\tau_{2n}\leqslant\theta_k\}}\rho_{\theta_k,\tau\wedge\tau_{2n}}[\xi^{2n+1}(\tau\wedge\tau_{2n})]\\ &= \text{ess sup}_{\tau\in\Theta_k}\mathbb{1}_{\{\tau_{2n}\leqslant\theta_k\}}\rho_{\theta_k,\tau\wedge\tau_{2n}\wedge\theta_k}[\xi^{2n+1}(\tau\wedge\tau_{2n}\wedge\theta_k)],\end{aligned}$$

where we have used the “genrealized zero-one law” to obtain the last equality.

For any $\tau \in \Theta_k$, $\tau \wedge \tau_{2n} \wedge \theta_k = \tau_{2n} \wedge \theta_k \leqslant \theta_k$. Hence,

$$\begin{aligned}
\mathbb{1}_{\{\tau_{2n} \leqslant \theta_k\}} \rho_{\theta_k, \tau \wedge \tau_{2n} \wedge \theta_k}[\xi^{2n+1}(\tau \wedge \tau_{2n} \wedge \theta_k)] &= \mathbb{1}_{\{\tau_{2n} \leqslant \theta_k\}} \rho_{\theta_k, \tau_{2n} \wedge \theta_k}[\xi^{2n+1}(\tau_{2n} \wedge \theta_k)] \\
&= \mathbb{1}_{\{\tau_{2n} \leqslant \theta_k\}} \xi^{2n+1}(\tau_{2n} \wedge \theta_k),
\end{aligned}$$

where we have used the knowledge-preserving property of ρ to obtain the last equality.

Finally, we get

$$\mathbb{1}_{\{\tau_{2n} \leqslant \theta_k\}} V^{2n+1}(\theta_k) = \mathbb{1}_{\{\tau_{2n} \leqslant \theta_k\}} \xi^{2n+1}(\tau_{2n}) = \mathbb{1}_{\{\tau_{2n} \leqslant \theta_k\}} Y^1(\tau_{2n}).$$

iii) Let $\tau \in \Theta$ be the form $\tau = \sum_{k \in \mathbb{N}} \theta_k \mathbb{1}_{A_k} + T \mathbb{1}_{\bar{A}}$. Then, by admissibility, we have

$$V^{2n+1}(\tau) = \sum_{k \in \mathbb{N}} V^{2n+1}(\theta_k) \mathbb{1}_{A_k} + V^{2n+1}(T) \mathbb{1}_{\bar{A}}.$$

Hence,

$$\begin{aligned}
V^{2n+1}(\tau) \mathbb{1}_{\{\tau_{2n} \leqslant \tau\}} &= \sum_{k \in \mathbb{N}} V^{2n+1}(\theta_k) \mathbb{1}_{A_k \cap \{\tau_{2n} \leqslant \tau\}} + V^{2n+1}(T) \mathbb{1}_{\bar{A} \cap \{\tau_{2n} \leqslant \tau\}} \\
&= \sum_{k \in \mathbb{N}} V^{2n+1}(\theta_k) \mathbb{1}_{A_k \cap \{\tau_{2n} \leqslant \theta_k\}} + V^{2n+1}(T) \mathbb{1}_{\bar{A} \cap \{\tau_{2n} \leqslant T\}} \\
&= \sum_{k \in \mathbb{N}} Y^1(\tau_{2n}) \mathbb{1}_{A_k \cap \{\tau_{2n} \leqslant \theta_k\}} + \xi^{2n+1}(T) \mathbb{1}_{\bar{A} \cap \{\tau_{2n} \leqslant T\}},
\end{aligned}$$

where we have used the previous property (ii) to obtain the last equality. Hence, we get

$$\begin{aligned}
V^{2n+1}(\tau) \mathbb{1}_{\{\tau_{2n} \leqslant \tau\}} &= \sum_{k \in \mathbb{N}} Y^1(\tau_{2n}) \mathbb{1}_{A_k \cap \{\tau_{2n} \leqslant \theta_k\}} + \xi^{2n+1}(T) \mathbb{1}_{\bar{A} \cap \{\tau_{2n} \leqslant T\}} \\
&= \sum_{k \in \mathbb{N}} Y^1(\tau_{2n}) \mathbb{1}_{A_k \cap \{\tau_{2n} \leqslant \theta_k\}} + Y^1(\tau_{2n}) \mathbb{1}_{\bar{A} \cap \{\tau_{2n} \leqslant T\}} = Y^1(\tau_{2n}) \mathbb{1}_{\{\tau_{2n} \leqslant \tau\}}.
\end{aligned}$$

iv) By the previous property (iii), we have, $V^{2n+1}(\tau) = \xi^{2n+1}(\tau)$ if and only if $V^{2n+1}(\tau) \mathbb{1}_{\{\tau < \tau_{2n}\}} = \xi^{2n+1}(\tau) \mathbb{1}_{\{\tau < \tau_{2n}\}}$. Hence,

$$\tilde{\tau}_{2n+1} = \operatorname{ess\,inf}\{\tau \in \Theta : V^{2n+1}(\tau) = X^1(\tau)\} \wedge \tau_{2n}.$$

□

Similarly to (3), (4) and (5), we define:

$$\xi^{2n+2}(\theta_k) := X^2(\theta_k) \mathbb{1}_{\{\theta_k < \tau_{2n+1}\}} + Y^2(\tau_{2n+1}) \mathbb{1}_{\{\tau_{2n+1} \leqslant \theta_k\}}, \text{ and}$$

$$\xi^{2n+2}(T) := Y^2(\tau_{2n+1}).$$

For $\tau \in \Theta$ of the form $\tau = \sum_{k\in\mathbb{N}} \theta_k \mathbb{1}_{A_k} + T\mathbb{1}_{\bar{A}}$, we define

$$\begin{aligned}
&\xi^{2n+2}(\tau) := \sum_{k\in\mathbb{N}} \xi^{2n+2}(\theta_k)\mathbb{1}_{A_k} + \xi^{2n+2}(T)\mathbb{1}_{\bar{A}} \\
&V^{2n+2}(S) := \text{ess sup}_{\tau\in\Theta_S} \rho^2_{S,\tau\wedge\tau_{2n+1}}[\xi^{2n+2}(\tau)] \\
&\tilde{\tau}_{2n+2} := \operatorname{ess\,inf} \tilde{\mathcal{A}}^2, \text{ where } \tilde{\mathcal{A}}^2 := \{\tau \in \Theta : V^{2n+2}(\tau) = \xi^{2n+2}(\tau)\} \\
&\tau_{2n+2} := (\tilde{\tau}_{2n+2} \wedge \tau_{2n})\mathbb{1}_{\{\tilde{\tau}_{2n+2}\wedge\tau_{2n}<\tau_{2n+1}\}} + \tau_{2n}\mathbb{1}_{\{\tilde{\tau}_{2n+2}\wedge\tau_{2n}\geqslant\tau_{2n+1}\}}.
\end{aligned} \tag{8}$$

The random variable $\xi^{2n+2}(\tau)$ is exactly the pay-off at $\tau \wedge \tau_{2n+1}$ of agent 2, if agent 1 plays τ_{2n+1} and agent 2 plays τ. Hence, $V^{2n+2}(S)$ is the optimal value at time S for agent 2, when agent 1's strategy is fixed to τ_{2n+1}. Assuming that $\limsup_{k\to+\infty} X^2(\theta_k) \leqslant X^2(T)$ leads to $\limsup_{k\to+\infty} \xi^{2n+2}(\theta_k) \leqslant \xi^{2n+2}(T)$, which we use in applying Theorem 2.3 in [11]. By Theorem 2.3 in [11], the Bermudan stopping time $\tilde{\tau}_{2n+2}$ is optimal for the problem with value $V^{2n+2}(0)$, that is,

$$V^{2n+2}(0) = \rho^2_{0,\tilde{\tau}_{2n+2}\wedge\tau_{2n+1}}[\xi^{2n+2}(\tilde{\tau}_{2n+2})] = \sup_{\tau\in\Theta} \rho^2_{0,\tau\wedge\tau_{2n+1}}[\xi^{2n+2}(\tau)].$$

Remark 3. *Let us recall that (cf. [11]) $\tilde{\tau}_n = \operatorname{ess\,inf}\{\tau \in \Theta : V^n(\tau) = \xi^n(\tau)\}$ satisfies the property: $V^n(\tilde{\tau}_n) = \xi^n(\tilde{\tau}_n)$. (This is due to the property of stability of Θ by monotone limit and to the right-continuity-along Bermudan stopping strategies of the families $(V^n(\tau))$ and $(\xi^n(\tau))$).*

Remark 4. *By analogy with Remark 2, we have:*

i) $(\xi^{2n+2}(\tau))$ is a admissible L^p-integrable family;

ii) for each $n \in \mathbb{N}$, for each $\tau \in \Theta$, $\xi^{2n+2}(\tau) = \xi^{2n+2}(\tau \wedge \tau_{2n+1})$.

Proposition 2. *We assume that ρ satisfies the usual "zero-one law". Then, for all $m \geqslant 1$, $\tilde{\tau}_{m+2} \leqslant \tau_m$.*

Proof. We suppose, by way of contradiction, that there exists $m \geqslant 1$ such that $P(\tilde{\tau}_{m+2} > \tau_m) > 0$, and we set $n := \min\{m \geqslant 1 : P(\tilde{\tau}_{m+2} > \tau_m) > 0\}$. We have $\tilde{\tau}_{n+1} \leqslant \tau_{n-1}$, by definition of n. This observation, together with the definition of τ_{n+1} and with the inequality of part (iv) of Proposition 1 gives:

$$\begin{aligned}
\tau_{n+1} &= (\tilde{\tau}_{n+1} \wedge \tau_{n-1})\mathbb{1}_{\{\tilde{\tau}_{n+1}\wedge\tau_{n-1}<\tau_n\}} + \tau_{n-1}\mathbb{1}_{\{\tilde{\tau}_{n+1}\wedge\tau_{n-1}\geqslant\tau_n\}} \\
&= \tilde{\tau}_{n+1}\mathbb{1}_{\{\tilde{\tau}_{n+1}<\tau_n\}} + \tau_{n-1}\mathbb{1}_{\{\tilde{\tau}_{n+1}\geqslant\tau_n\}} \\
&= \tilde{\tau}_{n+1}\mathbb{1}_{\{\tilde{\tau}_{n+1}<\tau_n\}} + \tau_{n-1}\mathbb{1}_{\{\tilde{\tau}_{n+1}=\tau_n\}}
\end{aligned} \tag{9}$$

For similar reasons, we have

$$\tau_n = \tilde{\tau}_n \mathbb{1}_{\{\tilde{\tau}_n < \tau_{n-1}\}} + \tau_{n-2} \mathbb{1}_{\{\tilde{\tau}_n = \tau_{n-1}\}}. \tag{10}$$

For the easing of the presentation, we set $\Gamma := \{\tau_n < \tilde{\tau}_{n+2}\}$.

On the set Γ, we have:

1) $\tau_n < \tilde{\tau}_{n+2} \leqslant \tau_{n+1}$, the last inequality being due to property (iv) of Proposition 1.

2) $\tau_{n+1} = \tau_{n-1}$. This is due to (1), together with Eq. (9).

3) $\xi^{n+2} = \xi^n$. This is a consequence of (2) and the definitions of ξ^{n+2} and ξ^n.

4) $\tau_n = \tilde{\tau}_n$.

We prove that $\{\tilde{\tau}_n = \tau_{n-1}\} \cap \Gamma = \varnothing$, which together with Eq. (10), gives the desired statement.

Due to Eq. (10), we have $\{\tilde{\tau}_n = \tau_{n-1}\} = \{\tilde{\tau}_n = \tau_{n-1}\} \cap \{\tau_n = \tau_{n-2}\}$. Thus, we have

$$\{\tilde{\tau}_n = \tau_{n-1}\} \cap \Gamma = \{\tilde{\tau}_n = \tau_{n-1}, \tau_n = \tau_{n-2} < \tilde{\tau}_{n+2}\}.$$

Now, we have $\tilde{\tau}_n \leqslant \tau_{n-2}$ (due to the definition of n). Hence,

$$\{\tilde{\tau}_n = \tau_{n-1}\} \cap \Gamma = \{\tilde{\tau}_n = \tau_{n-1} \leqslant \tau_{n-2} = \tau_n < \tilde{\tau}_{n+2}\} = \varnothing,$$

where the equality with $\varnothing$ is due to $\tilde{\tau}_{n+2} \leqslant \tau_{n-1}$.

We note that combining properties (1) and (4) gives $\tilde{\tau}_n < \tilde{\tau}_{n+2}$ on Γ. We will obtain a contradiction with this property. To this end, we will show that:

$$\mathbb{1}_\Gamma V^{n+2}(\tilde{\tau}_n) = \mathbb{1}_\Gamma \xi^{n+2}(\tilde{\tau}_n). \tag{11}$$

By definition of $\tilde{\tau}_n$ and by Remark 3, we have:

$$V^n(\tilde{\tau}_n) = \xi^n(\tilde{\tau}_n).$$

This property, together with property (3) on Γ, gives $V^n(\tilde{\tau}_n) = \xi^n(\tilde{\tau}_n) = \xi^{n+2}(\tilde{\tau}_n)$ on Γ. In order to show Eq. (11), it suffices to show

$$\mathbb{1}_\Gamma V^{n+2}(\tilde{\tau}_n) = \mathbb{1}_\Gamma V^n(\tilde{\tau}_n).$$

By property (4) on Γ and Proposition 4 (applied with $A = \Gamma \in \mathcal{F}_{\tau_n}$ and $\tau = \tau_n$), we have

$$\mathbb{1}_\Gamma V^{n+2}(\tilde{\tau}_n) = \mathbb{1}_\Gamma V^{n+2}(\tau_n) = \mathbb{1}_\Gamma V_\Gamma^{n+2}(\tau_n).$$

Due to property (3) on Γ, V_Γ^{n+2} and V_Γ^n have the same pay-off, and by applying again Proposition 4 and property (4) on Γ, we have

$$\mathbb{1}_\Gamma V_\Gamma^{n+2}(\tau_n) = \mathbb{1}_\Gamma V_\Gamma^n(\tau_n) = \mathbb{1}_\Gamma V^n(\tau_n) = \mathbb{1}_\Gamma V^n(\tilde{\tau}_n).$$

We have shown that $\mathbb{1}_\Gamma V^{n+2}(\tilde{\tau}_n) = \mathbb{1}_\Gamma V^n(\tilde{\tau}_n)$, which is the desired equality. Hence, we get $\tilde{\tau}_{n+2} \leqslant \tilde{\tau}_n$ on Γ (as by definition $\tilde{\tau}_{n+2} = \operatorname{ess\,inf}\{\tau \in \Theta : V^{n+2}(\tau) = \xi^{n+2}(\tau)\}$). However, this is in contradiction with the property $\tilde{\tau}_{n+2} > \tilde{\tau}_n$ on Γ. The proof is complete.

$\square$

Lemma 1. *i) For all* $n \geqslant 2$, $\tau_{n+1} = \tilde{\tau}_{n+1}\mathbb{1}_{\{\tilde{\tau}_{n+1} < \tau_n\}} + \tau_{n-1}\mathbb{1}_{\{\tilde{\tau}_{n+1} = \tau_n\}}$.

ii) For all $n \geqslant 2$, $\tilde{\tau}_{n+1} = \tau_{n+1} \wedge \tau_n$.

iii) On $\{\tau_n = \tau_{n-1}\}$, $\tau_m = T$, *for all* $m \in \{1, ..., n\}$.

Proof. i) This property follows from the definition of τ_{n+1}, together with Proposition 2, and with property (iv) of Proposition 1.

ii) By using (i), we get

$$\begin{aligned}
\tau_{n+1} \wedge \tau_n &= (\tilde{\tau}_{n+1} \wedge \tau_n)\mathbb{1}_{\{\tilde{\tau}_{n+1} < \tau_n\}} + (\tau_{n-1} \wedge \tau_n)\mathbb{1}_{\{\tilde{\tau}_{n+1} = \tau_n\}} \\
&= \tilde{\tau}_{n+1}\mathbb{1}_{\{\tilde{\tau}_{n+1} < \tau_n\}} + (\tau_{n-1} \wedge \tilde{\tau}_{n+1})\mathbb{1}_{\{\tilde{\tau}_{n+1} = \tau_n\}} \\
&= \tilde{\tau}_{n+1}\mathbb{1}_{\{\tilde{\tau}_{n+1} < \tau_n\}} + \tilde{\tau}_{n+1}\mathbb{1}_{\{\tilde{\tau}_{n+1} = \tau_n\}},
\end{aligned}$$

where we have used Proposition 2 for the last equality.

Finally, by using property (iv) of Proposition 1, we get $\tau_{n+1} \wedge \tau_n = \tilde{\tau}_{n+1}$.

iii) To prove this property, we proceed by induction. The property is true for $n = 2$. We suppose that the property is true at rank $n-1$ (where $n \geqslant 3$), that is on $\{\tau_{n-1} = \tau_{n-2}\}$, $\tau_m = T$, for all $m \in \{1, ..., n-1\}$.

From the expression for τ_n from statement (i), we get

$$\tau_n = \tilde{\tau}_n\mathbb{1}_{\{\tilde{\tau}_n < \tau_{n-1}\}} + \tau_{n-2}\mathbb{1}_{\{\tilde{\tau}_n = \tau_{n-1}\}}.$$

Hence, $\tau_n = \tau_{n-2}$ on the set $\{\tau_n = \tau_{n-1}\}$. We conclude by the induction hypothesis. $\square$

Lemma 2. *The following inequalities hold true:*

i) $J_1(\tau, \tau_{2n}) \leqslant J_1(\tau_{2n+1}, \tau_{2n})$, *for all* $\tau \in \Theta$.

ii) $J_2(\tau_{2n+1}, \tau) \leqslant J_2(\tau_{2n+1}, \tau_{2n+2})$, *for all* $\tau \in \Theta$.

Proof. Let us first prove statement i):

By (A1), we have $X^1 \leqslant Y^1$; it follows

$$X^1(\tau)\mathbb{1}_{\{\tau \leqslant \tau_{2n}\}} + Y^1(\tau_{2n})\mathbb{1}_{\{\tau_{2n} < \tau\}} \leqslant \xi^{2n+1}(\tau).$$

Hence, by monotonicity, and by definition of $V^{2n+1}(0)$, we have

$$J_1(\tau, \tau_{2n}) \leqslant V^{2n+1}(0) \tag{12}$$

We will now show that $V^{2n+1}(0) = J_1(\tau_{2n+1}, \tau_{2n})$, which will complete the proof of statement i).

We have

$$\begin{aligned} J_1(\tau_{2n+1}, \tau_{2n}) &= \rho^1_{0,\tau_{2n+1}\wedge\tau_{2n}}(X^1(\tau_{2n+1})\mathbb{1}_{\{\tau_{2n+1} \leqslant \tau_{2n}\}} + Y^1(\tau_{2n})\mathbb{1}_{\{\tau_{2n} < \tau_{2n+1}\}}) \\ &= \rho^1_{0,\tau_{2n+1}\wedge\tau_{2n}}(\xi^{2n+1}(\tau_{2n+1})), \end{aligned}$$

where we have used iii) from Lemma 1, and $X^1(T) = Y^1(T)$ from (A2) to show the last equality.

On the other hand, by ii) from Remark 2 and ii) from Lemma 1,

$$\rho^1_{0,\tau_{2n+1}\wedge\tau_{2n}}(\xi^{2n+1}(\tau_{2n+1})) = \rho^1_{0,\tau_{2n+1}\wedge\tau_{2n}}(\xi^{2n+1}(\tau_{2n+1}\wedge\tau_{2n})) = \rho^1_{0,\tilde{\tau}_{2n+1}\wedge\tau_{2n}}(\xi^{2n+1}(\tilde{\tau}_{2n+1})).$$

By optimality of $\tilde{\tau}_{2n+1}$ for $V^{2n+1}(0)$ (cf. Eq. (6)), we get

$$\rho^1_{0,\tilde{\tau}_{2n+1}\wedge\tau_{2n}}(\xi^{2n+1}(\tilde{\tau}_{2n+1})) = V^{2n+1}(0).$$

Hence, we conclude

$$J_1(\tau_{2n+1}, \tau_{2n}) = \rho^1_{0,\tau_{2n+1}\wedge\tau_{2n}}(\xi^{2n+1}(\tau_{2n+1})) = \rho^1_{0,\tilde{\tau}_{2n+1}\wedge\tau_{2n}}(\xi^{2n+1}(\tilde{\tau}_{2n+1})) = V^{2n+1}(0). \tag{13}$$

From Eq. (13) and Eq. (12), we get

$$J_1(\tau, \tau_{2n}) \leqslant J_1(\tau_{2n+1}, \tau_{2n}).$$

Let us now prove statement ii):

We have

$$J_2(\tau_{2n+1}, \tau) \leqslant V^{2n+2}(0), \tag{14}$$

by definition of $V^{2n+2}(0)$ (cf. Eq. (8)).

We will now show that $J_2(\tau_{2n+1}, \tau_{2n+2}) = V^{2n+2}(0)$, which will complete the proof.

By definition of $\xi^{2n+2}(\tau_{2n})$, by ii) from Remark 4, and by ii) from Lemma 1, we have

$$
\begin{aligned}
J_2(\tau_{2n+1}, \tau_{2n+2}) &= \rho^2_{0,\tau_{2n+1}\wedge\tau_{2n+2}}(X^2(\tau_{2n+2})\mathbb{1}_{\{\tau_{2n+2}<\tau_{2n+1}\}} + Y^2(\tau_{2n+1})\mathbb{1}_{\{\tau_{2n+1}\leqslant\tau_{2n+2}\}})\\
&= \rho^2_{0,\tau_{2n+1}\wedge\tau_{2n+2}}(\xi^{2n+2}(\tau_{2n+2}))\\
&= \rho^2_{0,\tau_{2n+1}\wedge\tau_{2n+2}}(\xi^{2n+2}(\tau_{2n+2}\wedge\tau_{2n+1}))\\
&= \rho^2_{0,\tilde{\tau}_{2n+2}\wedge\tau_{2n+1}}(\xi^{2n+2}(\tilde{\tau}_{2n+2})).
\end{aligned}
$$

By Eq. (8), we have

$$\rho^2_{0,\tilde{\tau}_{2n+2}\wedge\tau_{2n+1}}(\xi^{2n+2}(\tilde{\tau}_{2n+2})) = V^{2n+2}(0).$$

Hence, we conclude

$$J_2(\tau_{2n+1}, \tau_{2n+2}) = \rho^2_{0,\tau_{2n+1}\wedge\tau_{2n+2}}(\xi^{2n+2}(\tau_{2n+2})) = \rho^2_{0,\tilde{\tau}_{2n+2}\wedge\tau_{2n+1}}(\xi^{2n+2}(\tilde{\tau}_{2n+2})) = V^{2n+2}(0). \tag{15}$$

From Eq. (15) and Eq. (14), we get

$$J_2(\tau_{2n+1}, \tau) \leqslant J_2(\tau_{2n+1}, \tau_{2n+2}).$$

$\square$

Remark 5. *As a by-product of the previous proof, we find that τ_{2n+1} is optimal for the problem with value $V^{2n+1}(0)$, and τ_{2n+2} is optimal for the problem with value $V^{2n+2}(0)$.*

Definition 3. *We define $\tau_1^* = \lim_{n\to+\infty}\tau_{2n+1}$, and $\tau_2^* = \lim_{n\to+\infty}\tau_{2n}$.*

Proposition 3. *We assume that ρ^1 and ρ^2 satisfy properties $(i)-(vii)$, and the following additional property: for $i \in \{1, 2\}$,*

$$\limsup_{n\to+\infty} \rho^i_{0,\nu_n}[\xi(\nu_n)] = \rho^i_{0,\nu}[\xi(\nu)], \tag{16}$$

for any sequence $(\nu_n) \subset \Theta^{\mathbb{N}}$, $\nu \in \Theta$, such that $\nu_n \downarrow \nu$. We have:

i) For all $\tau \in \Theta$, $\lim_{n\to+\infty} J_1(\tau, \tau_{2n}) = J_1(\tau, \tau_2^*)$.

ii) For all $\tau \in \Theta$, $\lim_{n\to+\infty} J_2(\tau_{2n+1}, \tau) = J_2(\tau_1^*, \tau)$.

iii) For all $\tau \in \Theta$, $\lim_{n\to+\infty} J_1(\tau_{2n+1}, \tau_{2n+2}) = J_1(\tau_1^*, \tau_2^*)$.

iv) For all $\tau \in \Theta$, $\lim_{n\to+\infty} J_2(\tau_{2n+1}, \tau_{2n+2}) = J_2(\tau_1^*, \tau_2^*)$.

Proof. Let us first show statement i):

Let us recall the following notation:

for a fixed $\tau \in \Theta$,

$$I^1(\tau,\nu) := X^1(\tau)\mathbb{1}_{\{\tau\leqslant\nu\}} + Y^1(\nu)\mathbb{1}_{\{\nu<\tau\}},$$

$$I^1(\tau,\tau_2^*) := X^1(\tau)\mathbb{1}_{\{\tau\leqslant\tau_2^*\}} + Y^1(\tau_2^*)\mathbb{1}_{\{\tau_2^*<\tau\}}.$$

With this notation, we have

$$J_1(\tau,\tau_{2n}) = \rho^1_{0,\tau\wedge\tau_{2n}}[I(\tau,\tau_{2n})], \quad \text{and} \quad J_1(\tau,\tau_2^*) = \rho^1_{0,\tau\wedge\tau_2^*}[I(\tau,\tau_2^*)].$$

We note that the sequence (τ_{2n}) and $(\tau\wedge\tau_{2n})$ converges from above to τ_2^* and $\tau\wedge\tau_2^*$, respectively. Moreover, for each $\tau\in\Theta$, the family $(I^1(\tau,\nu))_{\nu\in\Theta}$ is admissible. Indeed, for each $\nu\in\Theta$, $I^1(\tau,\nu)$ is $\mathcal{F}_\nu$-measurable. Moreover, if $\{\nu=\nu'\}$, $I(\tau,\nu)=I(\tau,\nu')$ a.s. Hence, as any admissible family in our framework is right-continuous along Bermudan stopping strategies (cf. Remark 2.10 in [11]), we get

$$\lim_{n\to+\infty} I(\tau,\tau_{2n}) = I(\tau,\tau_2^*).$$

Hence, by property (16) on ρ^1, we get

$$\limsup_{n\to+\infty} \rho^1_{0,\tau\wedge\tau_{2n}}[I(\tau,\tau_{2n})] = \rho^1_{0,\tau\wedge\tau_2^*}[I(\tau,\tau_2^*)].$$

Now, let us prove statement ii):

For $\tau\in\Theta$, we recall the following notation:

$$I^2(\nu,\tau) := X^2(\tau)\mathbb{1}_{\{\tau<\nu\}} + Y^2(\nu)\mathbb{1}_{\{\nu\leqslant\tau\}},$$

The family $(I^2(\nu,\tau)_{\nu\in\Theta}$ is admissible. Indeed, for each $\nu\in\Theta$, $I^2(\nu,\tau)$ is $\mathcal{F}_\nu$-measurable. Moreover, on $\{\nu_1=\nu_2\}$, $I^2(\nu_1,\tau)=I^2(\nu_2,\tau)$ a.s.

As (τ_{2n+1}) converges from above to τ_1^*, and as $(I^2(\nu,\tau)_{\nu\in\Theta})$ is right-continuous along Bermudan stopping strategies (cf. Remark 2.10 in [11]), we get

$$\lim_{n\to+\infty} I^2(\tau_{2n+1},\tau) = I^2(\tau_1^*,\tau).$$

By property (16) on ρ^2, we get

$$\limsup_{n\to+\infty} \rho^2_{0,\tau_{2n+1}\wedge\tau}[I^2(\tau_{2n+1},\tau)] = \rho^2_{0,\tau_1^*\wedge\tau}[I^2(\tau_1^*,\tau)].$$

We now prove statement iii).

The proof relies again on the Bermudan structure on Θ. For any sequence $(\tau_n)\in\Theta^{\mathbb{N}}$ converging from above to $\tau\in\Theta$, we have: for almost each $\omega\in\Omega$, there exists $n_0=n_0(\omega)$ such that for all $n\geqslant n_0$, $\tau_n(\omega)=\tau(\omega)$ (cf. Remark

10 in [11]).

Hence, for almost each $\omega \in \Omega$, there exists $n_0 = n_0(\omega)$ such that for $n \geqslant n_0$, $\tau_{2n+1}(\omega) = \tau_1^*(\omega)$, $\tau_{2n+2}(\omega) = \tau_2^*(\omega)$ and

$$I(\tau_{2n+1}, \tau_{2n+2})(\omega) = X^1(\tau_1^*)(\omega)\mathbb{1}_{\{\tau_1^* \leqslant \tau_2^*\}}(\omega) + Y^1(\tau_2^*)(\omega)\mathbb{1}_{\{\tau_2^* < \tau_1^*\}}(\omega).$$

By property (16) on ρ^1, we get

$$\lim_{n \to +\infty} J_1(\tau_{2n+1}, \tau_{2n+2}) = J_1(\tau_1^*, \tau_2^*).$$

The proof of iv) is based on the same arguments. $\square$

We are now ready to complete the proof of Theorem 1.

Proof of Theorem 1. By combining Lemma 2 and Proposition 3, we get:

$$J_1(\tau, \tau_2^*) \leqslant J_1(\tau_1^*, \tau_2^*), \text{ for all } \tau \in \Theta.$$

$$J_2(\tau_1^*, \tau) \leqslant J_2(\tau_1^*, \tau_2^*), \text{ for all } \tau \in \Theta.$$

Hence, (τ_1^*, τ_2^*) is a Nash equilibrium point. $\square$

We have thus shown that the non-linear non-zero-sum Dynkin game with Bermudan strategies admits a Nash equilibrium point.

4 Appendix

Proposition 4. *(Localisation property) Let $(\xi(\tau))_{\tau\in\Theta}$ be a given admissible p-integrable family. Let $(V(\tau))_{\tau\in\Theta}$ be the value family of the optimal stopping problem: for $S\in\Theta$,*

$$V(S) = ess\ sup_{\tau\in\Theta_S}\rho_{S,\tau}[\xi(\tau)].$$

Let $S\in\Theta$, and let A be in $\mathcal{F}_S$. We consider the pay-off family $(\xi(\tau)\mathbb{1}_A)_{\tau\in\Theta_S}$, and we denote by $(V_A(\tau))_{\tau\in\Theta_S}$ the corresponding value family, defined by:

$$V_A(\tau) = ess\ sup_{\nu\in\Theta_\tau}\rho_{\tau,\nu}[\xi(\nu)\mathbb{1}_A].$$

If ρ satisfies the usual "zero-one law" (that is $\mathbb{1}_A\rho_{S,\tau}[\eta] = \mathbb{1}_A\rho_{S,\tau}[\mathbb{1}_A\eta]$ for all $A\in\mathcal{F}_S$, for all $\eta\in L^p(\mathcal{F}_\tau)$), then for each $\tau\in\Theta_S$,

$$\mathbb{1}_A V_A(\tau) = \mathbb{1}_A V(\tau).$$

Proof. By the definition of $V(\tau)$ and the usual "zero-one law", we have

$$\begin{aligned}\mathbb{1}_A V(\tau) &= \mathbb{1}_A \text{ess sup}_{\nu\in\Theta_\tau}\rho_{\tau,\nu}[\xi(\nu)] = \text{ess sup}_{\nu\in\Theta_\tau}\mathbb{1}_A\rho_{\tau,\nu}[\xi(\nu)]\\ &= \text{ess sup}_{\nu\in\Theta_\tau}\mathbb{1}_A\rho_{\tau,\nu}[\mathbb{1}_A\xi(\nu)] = \mathbb{1}_A\text{ess sup}_{\nu\in\Theta_\tau}\rho_{\tau,\nu}[\mathbb{1}_A\xi(\nu)] = \mathbb{1}_A V_A(\tau).\end{aligned}$$

□

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