

Quantum Entanglement on Fractal Landscapes

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We explore the interplay of fractal geometry and quantum entanglement by analyzing the von Neumann entropy (known as entanglement entropy) and the entanglement contour in the scaling limit. Focusing on free-fermion quantum models known for their simplicity and effectiveness in studying highly entangled quantum systems, we uncover intriguing findings. For gapless ground states exhibiting a finite density of states at the chemical potential, we reveal a super-area law characterized by the presence of a logarithmic divergence in the entanglement entropy. This extends the well-established super-area law observed on translationally invariant Euclidean lattices where the Goev-Klich-Widom conjecture regarding the asymptotic behavior of Toeplitz matrices holds significant influence. Furthermore, we observe the emergence of a self-similar and universal pattern termed an “entanglement fractal” in the entanglement contour data as we approach the scaling limit. Remarkably, this pattern bears resemblance to intricate Chinese paper-cutting designs. We provide general rules to artificially generate this fractal, offering insights into the universal scaling of entanglement entropy. Building upon the insights gained from the entanglement fractal, we explicitly elucidate the origin of the logarithmic divergence on fractals where translation symmetry is broken and the Widom conjecture is inapplicable. For gapped ground states, we observe that the entanglement entropy adheres to a generalized area law, with its dependence on the Hausdorff dimension of the boundary between complementary subsystems.

Introduction—Entanglement provides a quantum-informative perspective for understanding non-local correlations in many-body quantum systems [1–3]. Quantitatively, entanglement entropy (EE), or von Neumann entropy, stands out as one of the most useful quantities for characterizing the strength of quantum entanglement. For quantum many-body systems defined on Euclidean lattice with dimension d_s , it is well-known that the EE of gapped ground states follows a universal scaling feature called the area law¹, given by $S \sim L_A^{d_s-1}$, where L_A is the linear size of the boundary between the two complementary subsystems A and B [2–4]. Interestingly, gapped phases hosting anyons exhibit a universal constant entropy known as topological entanglement entropy. This appears as the subleading term in EE, quantitatively fixed by the total quantum dimension of all anyons [5, 6].

On the other hand, for gapless free-fermion systems defined on translational invariant Euclidean lattice with dimension d_s , when the Fermi surface is of codimension-1, the EE scales as a “super area law”, given by $S \sim L_A^{d_s-1} \log L_A$. This scaling is derived by applying Goev-Klich-Widom (Widom) conjecture of asymptotic behavior of Toeplitz matrices [3, 7–13], where translation symmetry plays a vital role. The scaling also demonstrates how logarithmic divergence is induced by the infinite number of gapless modes on the Fermi surface. However, the situation differs for Fermi surfaces of higher codimension. For example, in 2D systems with Fermi points, such as graphene with Dirac points, the EE still follows the area law. This implies that the vanishing density-of-state of Fermi points cannot provide sufficient long-range quantum correlation to enhance quantum entanglement between the two complementary subsystems.

All the aforementioned investigations into fermion system entanglement are conducted under the assumption of a translationally invariant lattice. However, fermions may also move on a fractal lattice [14] embedded in some Euclidean space, resulting in a scenario where fermions experience fractional dimensionality and translation symmetry is absent. As a type of very exotic geometric patterns, fractals, common in nature, have attracted academic interest since the last century, with statistical models and critical phenomena on fractals gaining substantial research attention [15–21]. Ongoing efforts continue to explore various physical phenomena [22–36], including topological effects, optical and transport properties, and the practical implementation of fractal topological quantum memory. These findings are substantially influenced by the unique features of fractals: fractional dimensions affecting microscopic degrees of freedom and self-similarity with broken translation symmetry. Therefore, considering that the aforementioned super area law depends on translation symmetry, enabling the application of the mathematical Widom conjecture of Toeplitz matrices, it is natural to explore the scaling law of the EE in many-body systems defined on fractal lattices. These systems lack translation symmetry and have non-integer dimensions, where the Widom conjecture is no longer applicable. This line of thinking, exploring the interplay of quantum entanglement and fractal geometry, serves as the motivation for the present work. Ultimately, we aim to gain a deeper understanding of how quantum entanglement is influenced by fractals where macroscopic number of fermionic quantum particles move.

In this paper, we investigate the entanglement properties of many-body systems on the Sierpinski carpet using free-fermion models, with a focus on both the entanglement entropy (EE) and the entanglement contour (EC) [37]. Mathematically, the EE can be decomposed into site-dependent contributions denoted as $s(i)$: $S_A = \sum_{i \in A} s(i)$. Here, $s(i)$ ’s as a whole constitute the EC data. Thus, the EC, serving as a

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¹ Here, the term “area” specifically refers to the boundary area between the two complementary subsystems, denoted as A and B .

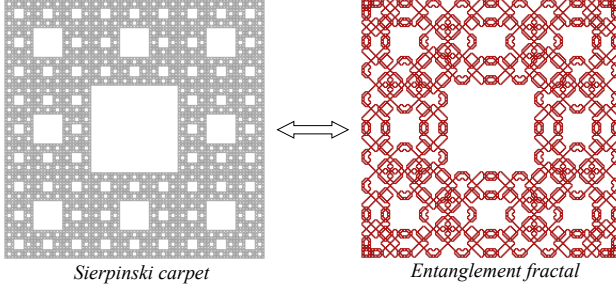


FIG. 1. Pictorial illustration of fractal lattice and entanglement fractal which resembles Chinese papercutting. See main texts for details.

real-space “tomography” of quantum entanglement, provides insight into the fine structure of the real-space distribution of entanglement entropy [38].

Our initial examination considers a gapless system embedded in 2D space with a finite density-of-state (DOS) at the chemical potential within the fractal lattice. To assess entanglement between the two complementary subsystems (labeled as A and B) of a pure state, we employ various bi-partition schemes, even allowing for the fractal structure of the boundary between A and B . For generic free-fermion systems on fractal lattices with a finite DOS at the chemical potential, we observe a super-area scaling of EE: $S_A \sim L_A^{d_s-1} \log L_A$, where $d_s = 2$ denotes the dimension of the embedding space in our numerical setting. This scaling result found in the numerical computation extends generalizes the applicability of the super-area law from translationally invariant Euclidean lattices to fractal lattices.

Next, we turn our attention to the EC data and unveil another intriguing revelation: the emergence of a self-similar and universally present pattern, which we term an “entanglement fractal” (EF) pattern (Fig. 1). Notably, this intricate pattern bears a striking resemblance to the elaborate designs found in Chinese paper-cutting artistry. Surprisingly, despite being discovered through numerical computation, the EF pattern can be reproduced through a set of general rules. This gives us confidence to believe that the EF pattern may encode the origin of the logarithmic divergence in the super-area scaling of the EE. Indeed, upon delving into the EF pattern, we find that the logarithmic divergence can be traced back to the power law decay of $\sum_i' s(i) \sim \frac{1}{\ell}$, where ℓ measures the distance between the site $i \in A$ and the boundary between A and B at the scaling limit. $\sum_i'$ means the summation over all site i with the same ℓ .

For gapped systems, our numerical analysis proposes a universal scaling of EE as $S_A \sim L_A^{d_{\text{bf}}}$, where d_{bf} denotes the Hausdorff dimension of A ’s boundary, thereby extending beyond the conventional area law. Furthermore, in gapped systems, the EC predominantly resides at A ’s boundary and exhibits an exponential decay into the bulk.

METHOD

Consider a many-body ground state $|G\rangle$ with density matrix $\rho = |G\rangle \langle G|$. When partitioning the system into two subsystems A and B , the reduced density matrix can be obtained by tracing over subsystem B :

$$\rho_A = \text{Tr}_B |G\rangle \langle G| = \frac{1}{\mathcal{N}} \exp(-\mathcal{H}^E), \quad (1)$$

where $\mathcal{N}$ is a normalization constant. In the free-fermion limit, owing to the quadratic form of the entanglement Hamiltonian [39, 40]:

$$\mathcal{H}^E = \sum_{i,j} c_i^\dagger h_{i,j}^E c_j \quad (2)$$

with $i, j \in A$, where $c_i^\dagger$ represents a fermionic creation operator at the i -th lattice site. Consequently, entanglement information can be extracted from the entanglement Hamiltonian matrix h^E . Additionally, as discussed in Ref. [39, 40], the spectrum $\{\varepsilon_\ell\}$ of h^E and the spectrum $\{\xi_\ell\}$ of the correlation matrix $C^A(i, j) = \langle G | c_i^\dagger c_j | G \rangle$ with i, j in subsystem A have a one-to-one correspondence with the same eigenvector, expressed as:

$$\varepsilon_\ell = \log[(\xi_\ell)^{-1} - 1]. \quad (3)$$

For convenience in free-fermion systems, the spectrum $\{\xi_\ell\}$ is typically adopted as the entanglement spectrum, and the entanglement entropy (EE) is calculated using the equation:

$$S_A = \text{Tr}[f(C^A)] = - \sum_\ell [\xi_\ell \log \xi_\ell + (1 - \xi_\ell) \log(1 - \xi_\ell)]. \quad (4)$$

Utilizing the eigenvalues and eigenvectors of the correlation matrix C^A , the entanglement contour (EC) in free-fermion systems is defined as:

$$s(i) = \sum_\ell p_i(\ell) S_\ell, \quad (5)$$

where $S_\ell = -[\xi_\ell \log \xi_\ell + (1 - \xi_\ell) \log(1 - \xi_\ell)]$. $p_i(\ell) = |\langle i | \ell \rangle|^2$ represents the probability of the eigenvector $|\ell\rangle$ of the correlation matrix at site i in A , and $\sum_i p_i(\ell) = 1$. These eigenvectors, often referred to as “Schmidt vectors” or “entanglement wavefunctions”, encapsulate information regarding bulk-boundary correspondence in topological gapless systems [41].

EE OF GAPLESS GROUND STATES

At the outset, prior to delving into the examination of entanglement features within fractal systems, it is imperative to address the process of constructing a lattice system exhibiting a fractal structure. Initially, we introduce two variable elements crucial to the fractal lattice—namely, an initial unit cell and a fractal iteration method. These components uniquely

determine a specific type of fractal lattice. Employing the iteration method iteratively n times on the unit cell with lattice sites yields the n th-order approximation of the fractal. For instance, the 5th-order approximation $SC(5, 1)$ of the Sierpinski carpet [14], depicted in the upper part of Fig. 1 (refer to **Supporting Information**, section S1 and Fig. S1 for detailed information; note that “1” in $SC(5, 1)$ represents the width of a unit cell).

Subsequently, we investigate the scaling of Entanglement Entropy (EE) for the gapless system defined on the fractal lattice $SC(n, 1)$ embedded in a 2-dimensional spatial space, as illustrated in Fig. 2A. We consider a spinless tight-binding model described by the Hamiltonian:

$$H_1 = -t \sum_{\langle ij \rangle} c_i^\dagger c_j - \mu \sum_i c_i^\dagger c_i. \quad (6)$$

Here, $c_i^\dagger$ represents a fermionic creation operator at the i th lattice site, $\langle ij \rangle$ denotes nearest-neighbor sites, and μ is the chemical potential. The model H_1 exhibits symmetries due to the properties of the fractal lattice $SC(3, 1)$, such as a four-fold rotation symmetry. To ascertain the bulk gap, it is necessary to compute the Density of States (DOS) of the model H_1 on the fractal lattice $SC(n, 1)$. Given the exponential increase in the number of lattice sites with each iteration of the fractal lattice [14], the numerical exploration of entanglement presents considerable challenges. Specifically, numerical calculations in entanglement for translation-invariant systems hinge on the diagonalization of a non-sparse matrix at a large size limit. The challenge is further amplified when translation symmetry is absent, necessitating a computationally intensive numerical study. In our case, through an analysis of the scaling of the energy gap and the DOS, we demonstrate that the model H_1 on the Sierpinski carpet exhibits a gapless ground state with a finite DOS at the chemical potential (see **Supporting Information**, section S2 and Fig. S2 for technical details), indicative of a metallic ground state on the fractal lattice.

To progress further, we initially establish various bipartition schemes in order to compute entanglement quantities. In Fig. 2A, we present the 3rd-order approximation $SC(3, 1)$ of the Sierpinski carpet as an illustrative example. Four distinct partitioning methods (labeled I-IV) for dividing the original fractal lattice into subsystems A and B are demonstrated, where the area enclosed by dashed lines is designated as the subsystem A for each partition scheme. Specifically, for Partition-I, the subsystem A geometrically corresponds to the 2nd-order approximation $SC(2, 1)$ that preserves all spatial symmetry of the original fractal lattice. In a general sense, for the n th-order approximation $SC(n, 1)$, A represents the $(n - 1)$ th-order approximation $SC(n - 1, 1)$ with linear length $L_A = 3^{n-1}l$ and the number of boundary sites $N_{bA} = 3^{n-1}$, where the lattice constant $l = 1$. By noting $L_A = N_{bA}$, the boundary of A using Partition-I in Fig. 2A forms a regular 1D line (not a fractal line).

Following this, Partition-I is utilized to scrutinize the EE scaling for the model H_1 on $SC(n, 1)$. As outlined in **Supporting Information** section S2 and Fig. S2, the energy gap of the model H_1 on the Sierpinski carpet diminishes at the

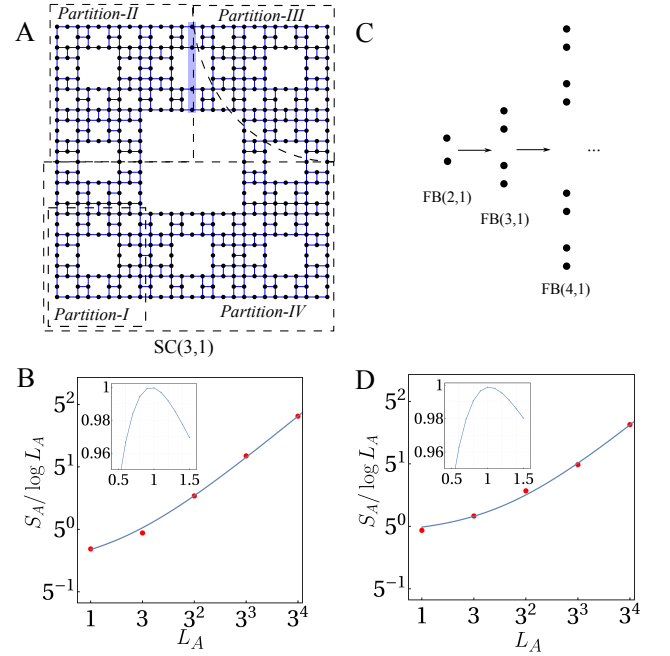


FIG. 2. (A) Four kinds of partitions on the 3rd-order approximation $SC(3, 1)$ of Sierpinski carpet as an example. Partition-I has a normal boundary and Partition-II has a fractal boundary in the blue area. (C) The fractal structure of the boundary for different n . (B) and (D) The EE of the model on the n th-order approximation $SC(n, 1)$ of Sierpinski carpet, using Partition-I and II, respectively. The insets show the coefficient of determination R^2 as a function of α to measure the goodness of fit. Here $t = 1$ and $\mu = 0$.

scaling limit, and the DOS attains a finite value, indicating a metallic ground state on the fractal lattice. Concurrently, it is well-established that the model H_1 on a square lattice respects translation symmetry and displays a one-dimensional Fermi surface with a finite DOS. In accordance with the Widom conjecture of Toeplitz matrices, the EE scales as $S_A = aL_A \log L_A + \dots$, where the coefficient a is determined by the geometric details of the Fermi surface and the specifics of the partitioning method [10].

By comparing these two scenarios, we are motivated to test whether or not the EE when the model is placed on the Sierpinski carpet would also scale as $S_A = aL_A^\alpha \log L_A + \dots$, where the parameter α might encode the fractal information of the lattice, and a remains a nonuniversal constant determined by certain details of the model. To substantiate this proposal, we numerically compute S_A for different n of $SC(n, 1)$ to increase the fractal lattice size, as depicted in Fig. 2B. By fitting the numerical data with $\alpha = 1$, we obtain $S_A / \log L_A = 0.22372L_A + 0.37143$. Additionally, we utilize the *coefficient of determination* denoted as R^2 , which is a function of α , to measure the goodness of fit, as shown in the inset of Fig. 2B. We observe that $\alpha \approx 1$ yields the best fit, with R^2 closest to 1. Based on this fitting expression in 2D where the fractal lattice is embedded, we propose that the EE scaling of the model H_1 on the fractal lattice is given by:

$$S_A = aL_A^{d_s-1} \log L_A + \dots \quad (7)$$

Here, d_s is the integer-valued spatial dimension of the Euclidean space where the fractal lattice is embedded. However, as only one specific partition scheme is applied during the above analysis, one may wonder if this scaling is universal enough.

To assess the universality of (7) across different partition schemes, we incorporate Partition-II alongside Partition-I as depicted in Fig. 2A. In the case of Partition-II, A explicitly breaks the symmetry and fractal structure of the original Sierpinski carpet. Notably, the linear length of A 's boundary is $L_A = 3^{n-1}l$ in $SC(n, 1)$ and the number of sites on the boundary of the subsystem A is $N_{bA} = 2^{n-1}$, illustrated in Fig. 2C for varying n . Since $L_A > N_{bA}$, the boundary “ $FB(n, 1)$ ” of A exhibits a distinct fractal structure, contrasting sharply with Partition-I where the boundary is a regular 1D line. For convenience, we define a boundary Hausdorff dimension $d_{bf} = \log_{L_A} N_{bA} = \log_3 2$ for $FB(n, 1)$, akin to the n th-order approximation of the Cantor set [14].

With this foundation, we numerically determine the EE on $SC(n, 1)$ as depicted in Fig. 2D. We anticipate that the EE scaling in this case remains $S_A = aL_A^\alpha \log L_A + \dots$. Fitting the data with $\alpha = 1$ yields $S_A / \log L_A = 0.15858L_A + 0.82249$. The optimal fit is obtained with $\alpha \approx 1$, with the coefficient of determination R^2 closest to 1, as shown in the inset of Fig. 2D. Based on these findings, the EE scaling of the model H_1 on the fractal lattice is consistent with (7). While it is impractical to exhaustively consider an infinite array of partition schemes, the two partitions discussed, i.e., Partition-I and Partition-II, which respectively preserve and break the symmetry of the original Sierpinski carpet, serve as representative cases. Thus, we conclude that (7) universally governs the scaling of EE for gapless free-fermion systems with finite DOS on a fractal lattice embedded in d_s dimensional Euclidean space.

ENTANGLEMENT FRACTAL AND ITS ARTIFICIAL GENERATING

To deeply probe into the entanglement characteristics of fractal geometry and the intricate details of EE, we embark on an investigation of a measure known as the entanglement contour (EC) [37], denoted as $s(i)$ where i represents any lattice site within the subsystem A . EC provides a real-space “tomography” of EE and is defined in (5) (for more details, refer to **Methods**). By utilizing EC, we can reconstruct EE as:

$$S_A = \sum_{i \in A} s(i). \quad (8)$$

Initially, we employ Partition-II to examine the distribution of EC $s(i)$ for the model H_1 on $SC(5, 1)$. As illustrated in Fig. 4A, our numerical findings indicate that the predominant $s(i)$ in the scaling limit, whose color tends to be red, exhibits a distinct pattern within the bulk of the subsystem A , reminiscent of Chinese papercutting. Additionally, we observe self-similarity in the EC, as indicated by the black rhombuses in Fig. 4A. To advance our exploration, we employ Partition-I,

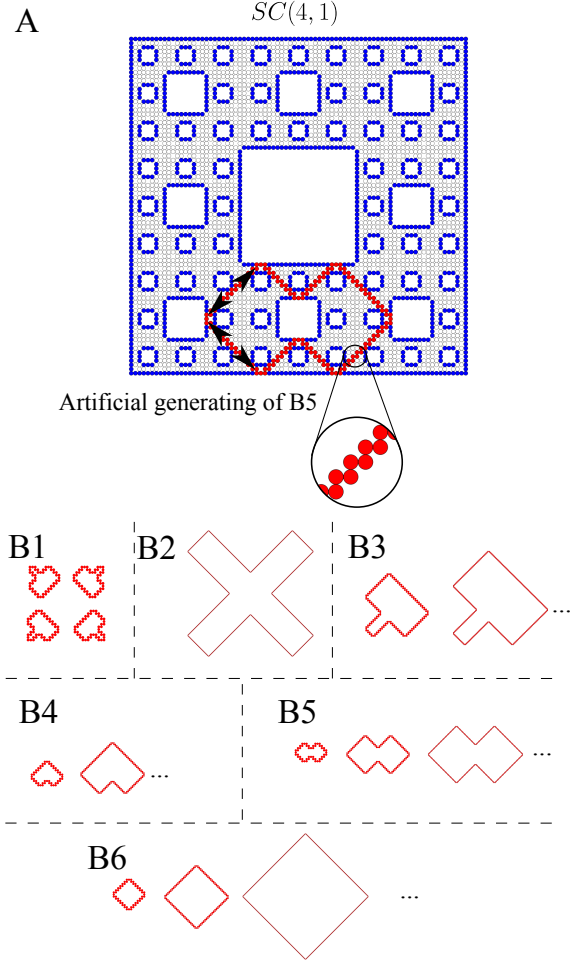


FIG. 3. (A) Illustration of the artificial generating rules for *entanglement fractal* on the 4th-order approximation $SC(4, 1)$. The red closed loop shows the artificial generating structure of (B5) in the gapless fractal systems and the blue sites are the boundaries of $SC(4, 1)$. (B1-B6) All possible structures by the artificial generating rules.

III, and IV to examine the EC of the model H_1 on $SC(5, 1)$. All numerical results are presented in Fig. 4A, B, D, and E, where the predominant EC data (formed by significantly large value of $s(i)$ whose color tends to be red) display a clearly self-similar structure.

Up to this point, we haven't yet discussed the pattern in Fig. 4C. This pattern is referred to as “Entanglement fractals” (EF), which is not obtained through numerical computations but is instead artificially generated using a set of rules. We will discuss EF below and discover that EF effectively represents the real-space distribution of the predominant EC data observed in Fig. 4A, B, D, and E (as indicated by the four arrows from C to A, B, D, and E).

More concretely, to comprehend the numerical outcomes, we delineate a set of generative rules to artificially construct predominant EC on the fractal lattice $SC(n, 1)$ illustrated in Fig. 4A, B, D, and E. As an illustrative instance, we depict a red closed loop within $SC(4, 1)$ in Fig. 3A, where the blue

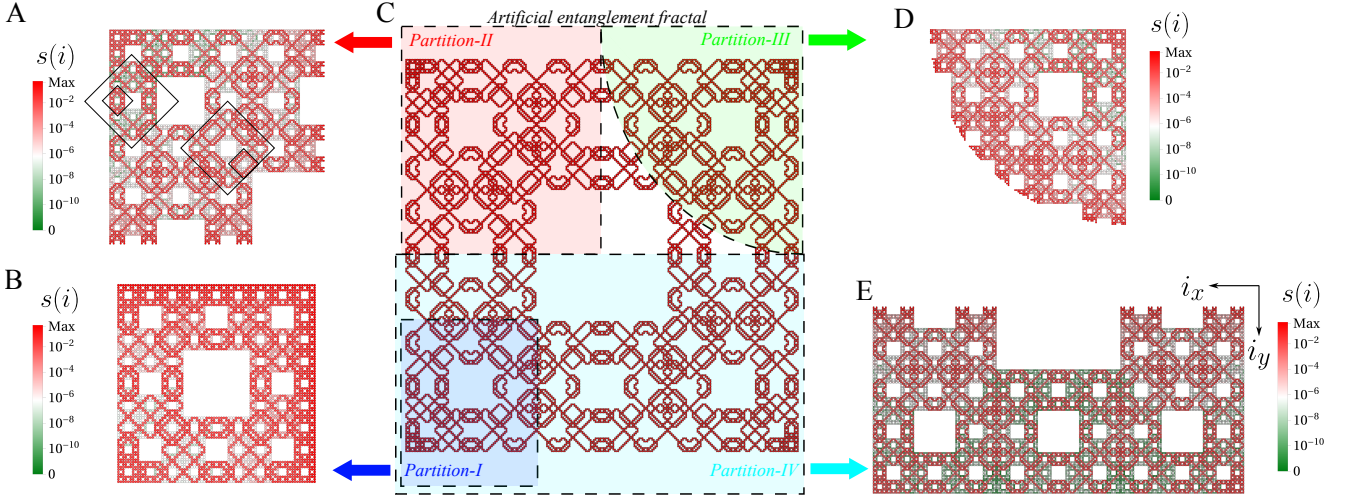


FIG. 4. (A), (B), (D) and (E) are EC of the model H_1 using Partition-II, I, III and IV, respectively. (C) The artificially generated structure, called *entanglement fractal* (EF), and its four partitions. The predominant EC data (formed by significantly large value of $s(i)$ whose color tends to be red) of (A), (B), (D), and (E) match the artificially generated EF pattern in (C).

sites form the boundaries of $SC(4, 1)$. The fundamental rules encompass:

- i. The predominant EC is constituted by closed loops;
- ii. Each loop line consists of two sites arranged in a zig-zag pattern, as depicted in the magnified view in Fig. 3A.
- iii. Each loop line extends along one of the four diagonal directions: 45° , 135° , 225° , or 315° , as indicated by the black arrows in Fig. 3A;
- iv. Each loop line originates from and reflects specularly at the boundaries of $SC(n, 1)$.

By adhering to these rules, we artificially generate all conceivable structures on the fractal lattice $SC(n, 1)$ as presented in Fig. 3B1-B6. Notably, the patterns in Fig. 3B3-B6 exhibit self-similarity, while the pattern set in Fig. 3B3 does not manifest in the EC of H_1 on $SC(n, 1)$. The prohibited EC may be contingent on the fractal structure or the Hamiltonian model (see **Supporting Information**, section S3 and Fig. S3 and Fig. S4 for more details).

By applying the patterns in Fig. 3B1-B2 and B4-B6 to $SC(5, 1)$, we successfully generate a fractal structure on $SC(5, 1)$, as depicted in Fig. 4C. Subsequently, by employing Partition-I-IV to Fig. 4C and contrasting the resulting patterns with the numerical outcomes in Fig. 4A, B, D, and E, we observe that the distribution of predominant EC data in the bulk of A closely resembles the artificially generated pattern shown in Fig. 4C. This similarity indicates that the chosen rules have been successful. The distinctive fractal pattern, artificially generated by a set of rules, is referred to as the EF, which can be considered as the entanglement “fingerprint” of the fractal geometry of the original lattice.

Although the EF pattern may appear aesthetically pleasing, its physical significance and relevance to physics may be

questioned. In the next section, we will demonstrate that the EF pattern leads to the logarithmic divergence observed in the scaling law of the EE in (7), which plays a vital role in generalizing the super-area law from translationally invariant lattice to fractal lattice (see **Supporting Information**, section S4 for more details).

RECONSTRUCTING EE VIA ENTANGLEMENT FRACTAL

As shown in (8), the EE can be decomposed into the real-space distribution within the subsystem A . Thus, we infer that the universal scaling of the entanglement entropy originates from the aforementioned EF pattern. Let us elaborate on this point in more detail below.

Since EF only captures the distribution of predominant EC data, the EC data in the rest of the subsystem A , where EC data are relatively weaker, are entirely neglected (see Fig. 4). To explore the distinct behaviors of $s(i)$ on EF and the remaining part of subsystem A , as illustrated in Fig. 4E where Partition-IV is applied, we introduce two quantities defined as:

$$s_{A_s(\bar{A}_s)}(i_y) = \frac{\sum_{i_x \in A_s(\bar{A}_s)} s(i_x, i_y)}{\mathcal{N}(i_y)}, \quad (9)$$

where A_s and $\bar{A}_s$ represent the lattice sites covered by EF and the rest sites in the subsystem A , respectively. Each lattice site i is uniquely labeled by two integers: (i_x, i_y) in Fig. 4E. Here $\mathcal{N}(i_y) = \sum_{i_x \in A_s} 1$ is the number of lattice sites with the same i_y in Fig. 4E. The coordinate i_y measures the distance of lattice site i from the boundary between the two subsystems A and B . From (9), $s_{A_s(\bar{A}_s)}(i_y)$ is simply the average value of EC data for a given i_y for all sites $i_x \in A_s(\bar{A}_s)$. As depicted in Fig. 5A as i_y increases, we numerically find that:

$$s_{A_s}(i_y) \sim 1/(i_y)^{1.12}, \quad s_{\bar{A}_s}(i_y) \sim 1/(i_y)^{3.69} \quad (10)$$

which motivates us to propose the following equation for calculating the EE from the EC:

$$S_A = \sum_{i \in A_s} s(i) + \dots \quad (11)$$

Here, the ellipsis represents the subleading term $\sum_{i \in \bar{A}_s} s(i)$ of the EE. Only the first term of (11) significantly contributes to the EE at the scaling limit. In terms of the quantity $s_{A_s}(i_y)$, we have

$$S_A \approx \sum_{i \in A_s} s(i) = \sum_{i_y} \mathcal{N}(i_y) s_{A_s}(i_y). \quad (12)$$

Since numerically we find that the denominator $\mathcal{N}(i_y)$ in (9) is always less than L_A , we can rewrite the denominator as $\mathcal{N}(i_y) = L_A \cdot p(i_y)$ with $p(i_y) \equiv \mathcal{N}(i_y)/L_A$ and $0 < p(i_y) < 1$. Then we obtain:

$$S_A \approx L_A \sum_{i_y} p(i_y) s_{A_s}(i_y). \quad (13)$$

Here the series $\sum_{i_y} p(i_y) s_{A_s}(i_y)$ can be regarded as a random series whose convergence depends on the distribution of $p(i_y)$ and the series $\sum_{i_y} s_{A_s}(i_y)$ [42].

The next question is: can we really reconstruct the super-area law of the EE in (7) ($d_s = 2$) by using the asymptotic behavior $s_{A_s}(i_y) \sim 1/(i_y)^{1.12}$? To achieve this goal, let us consider the linear size of A is L_A along both horizontal and vertical directions, i.e., $i_y \in (0, L_A)$, such that the subsystem is properly scaled to a thermodynamical two-dimensional region. Mathematically, the series $\sum_{n=1}^{\infty} 1/n^\beta$ converges to a constant called the Riemann zeta function $\zeta(\beta)$ if $\beta > 1$. Consequently, when $s_{A_s}(i_y) \sim 1/(i_y)^{1.12}$ is utilized, we can deduce the series $\sum_{i_y} p(i_y) s_{A_s}(i_y) < \sum_{i_y} s_{A_s}(i_y)$ is convergent by noting that $0 < p(i_y) < 1$. As a result, based on (13), we can deduce that the EE $S_A \lesssim L_A$, which corresponds to, at most, the area law but not the super-area law at large L_A limit.

Therefore, to recover the logarithmic divergence $\log L_A$ appeared in the super-area law in (7), firstly, we assume the following asymptotic behavior:

$$s_{A_s}(i_y) \sim 1/i_y. \quad (14)$$

We observe that the exponent “1.12” in (10) is very close to 1; the deviation here can be considered a consequence of the finite size effect of the numerical calculation. Secondly, we assume that $p(i_y)$ should have a non-zero lower bound $p(i_y) > \epsilon$. As shown in Fig. 5B, we plot $p(i_y)$ and numerically confirm that the nonzero lower bound of $p(i_y)$ indeed exists. Based on these two assumptions, and combining (13) and (14), we can infer the EE should adhere to $\epsilon L_A \log L_A \lesssim S_A \lesssim L_A \log L_A$, i.e., $S_A \sim L_A \log L_A$ at large L_A limit, which aligns with the super-area law in (7) ($d_s = 2$).

In conclusion, we have shown how to reconstruct the super-area law of the EE through the asymptotic behavior of the EC data distributed on the EF pattern, especially providing

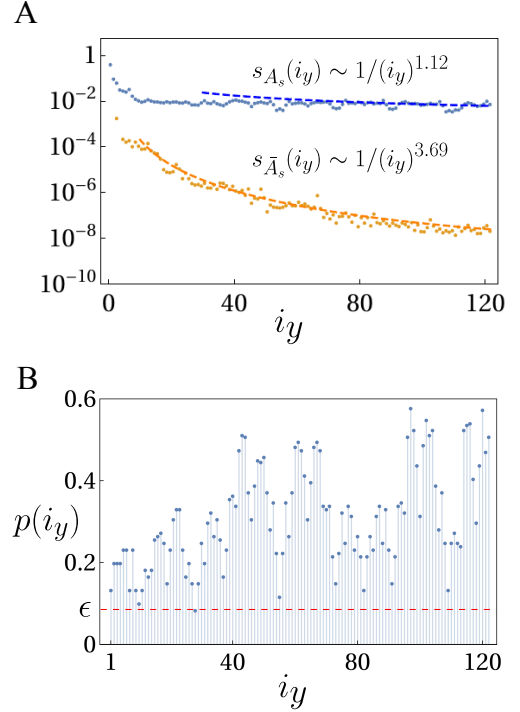


FIG. 5. (A) The behavior of EC $s(i)$ on the self-similar structure A_s and the rest of the subsystem $\bar{A}_s$ by adopting Partition-IV in Fig. 4(E), where the blue and yellow lines represent $s_{A_s}(i_y) \sim 1/(i_y)^{1.12}$ and $s_{\bar{A}_s}(i_y) \sim 1/(i_y)^{3.69}$, respectively. (B) The distribution of $p(i_y)$ with i_y in the EC of Partition-IV, where ϵ is the lower bound of $p(i_y)$.

insight into the origin of the logarithmic divergence in the EE formula. In comparison, as aforementioned, the logarithmic divergence for the case of translationally invariant Euclidean lattices arises from the application of the Widom conjecture of the asymptotic behavior of Toeplitz matrices.

Finally, noting that the EE of free-fermion systems can be formally expressed as $S_A = \text{Tr}[f(C^A)]$ as shown in **Methods** [39, 40], (11) provides information about the asymptotic behavior of the spectrum of the correlation matrix C^A . This may assist in analytically generalizing the Widom conjecture of Toeplitz matrices to matrices with self-similarity. Mathematically, Brownian motion on infinitely ramified self-similar fractals (see Refs. [43, 44] and references therein), such as the Sierpinski carpet, presents a challenging problem concerning the asymptotic behaviors of the Laplacian on the fractals [45, 46]. Formulating a new ‘conjecture’ for the asymptotic behaviors of matrices with self-similarity would be instrumental in understanding Brownian motion on fractals, representing an important avenue for future study.

EE & EC OF GAPPED GROUND STATES

Turning to the entanglement of gapped systems on fractals, we consider a tight-binding model on the fractal lattice $SC(n, 1)$ embedded in a two-dimensional space, as shown in

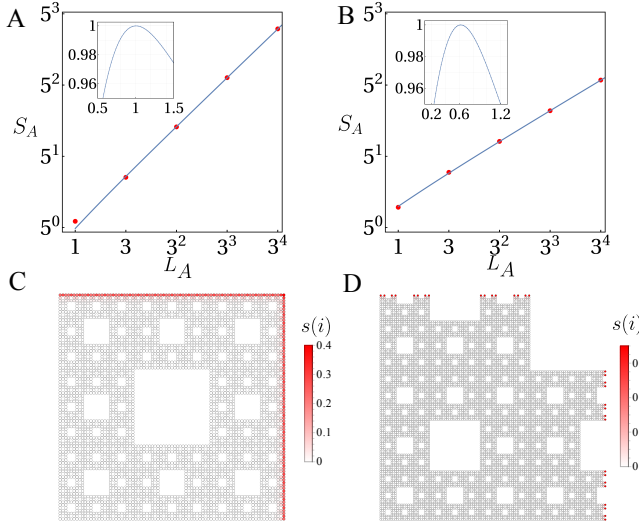


FIG. 6. (A) and (B) are the EE of the model H_2 as a function of the number of iteration n by using Partition-I and II, respectively. The insets show the coefficient of determination R^2 as a function of α to measure the goodness of fit. (C) and (D) are the distribution of EC of the model H_2 in the subsystem A of $SC(5, 1)$ by using Partition-I and II, respectively. Here $t_1 = 0.5$ and $t = 1$.

Fig. 2A. The model is given by

$$H_2 = \sum_i (c_{s,i}^\dagger c_{p,i}^\dagger) t_1 \sigma_x \begin{pmatrix} c_{s,i} \\ c_{p,i} \end{pmatrix} + \sum_{\langle i,j \rangle} (c_{s,i}^\dagger c_{p,i}^\dagger) t \sigma_z \begin{pmatrix} c_{s,j} \\ c_{p,j} \end{pmatrix}, \quad (15)$$

where $c_{s(p),i(j)}^\dagger$ is a fermionic creation operator of $s(p)$ orbital at the $i(j)$ th lattice site and $\sigma_{x,y}$ are Pauli matrices. As discussed in **Supporting Information** section S2 and Fig. S2, through analyzing the scaling of the energy gap and the DOS, we find that the ground state of the model is gapped.

Due to the existence of a finite gap, we propose that the EE of this model H_2 on the Sierpinski carpet would scale as $S_A = aL_A^\alpha + \dots$, where α is a universal parameter to be determined. To verify this proposal, we first adopt Partition-I in Fig. 2A to study the scaling of EE of the model H_2 on the n th-order approximation $SC(n, 1)$ of Sierpinski carpet. As shown in Fig. 6A, by fitting the data with $\alpha = 1$, the numerical data of EE is fit to $S_A = 1.10033L_A - 0.120981$. The best fit is $\alpha \approx 1$ with the coefficient of determination R^2 closest to 1, as shown in the inset of Fig. 6A. Furthermore, we consider Partition-II in Fig. 2A to divide $SC(n, 1)$ into A and B , where the boundary $FB(n, 1)$ in the blue area of Fig. 2A has fractal structure. Through numerical calculation, the EE of the model H_2 in this case is demonstrated in Fig. 6B. By fitting the numerical data, the EE scales to $S_A = 1.84893L_A^{0.62} - 0.234274$, where $\alpha \approx 0.62$ is the best fit with the coefficient of determination R^2 closest to 1. Note that for Sierpinski carpet, its spatial dimension $d_s = 2$ and Hausdorff dimension $d_f = \log_3 8 \approx 1.8928$, while the boundary $FB(n-1)$ of the subsystem A has the Hausdorff dimension $d_{bf} = \log_3 2 \approx 0.6309$. Combining the numerical results of EE with Partition-I and II, we observe that the EE of the

gapped systems can reflect the fractal feature of the subsystem's boundaries. Then, for gapped systems on fractal, we obtain a generalized area law written as

$$S_A = aL_A^{d_{bf}} + \dots, \quad (16)$$

where d_{bf} is the boundary Hausdorff dimension. Finally, considering the EC $s(i)$ in this case, as demonstrated in Fig. 6C and D, the distribution of $s(i)$ is localized at the boundaries of the subsystem A .

DISCUSSIONS AND OUTLOOK

In this study, we have investigated the interaction between fractal geometry and quantum entanglement of free fermions using two entanglement measures: entanglement entropy (EE) and entanglement contour (EC). Several intriguing questions arise, prompting further exploration. For instance, we have introduced several guidelines for artificially generating entanglement fractals (EF) as depicted in Fig. 4C, which emerge from the predominant EC data at the scaling limit. However, the foundational theory of EF remains to be established, representing a critical next step in comprehending the two assumptions outlined in the main text. To formulate this theory, one approach could involve continuously adjusting the hopping energies of a tight-binding model. This adjustment would lead to a gradual evolution of the hopping energy distribution from a regular translationally invariant lattice to a fractal lattice. By examining the changes in EE and EC throughout this evolution, we suggest that the topology of the Fermi surface [47] on the regular translationally invariant lattice could significantly influence the EF structure on the fractal lattice.

Furthermore, drawing inspiration from the Widom conjecture and analytical findings [48, 49] concerning Toeplitz matrices in translation-invariant systems for computing EE, it becomes imperative to investigate the asymptotic behavior of the correlation matrix C^A on fractals. Such exploration could potentially aid in determining the analytical expression of the non-universal coefficient in the EE scaling formulas. Additionally, similar analyses can be conducted on hyperbolic lattices, where insights from non-Abelian hyperbolic band theory may prove beneficial [50]. Furthermore, for exactly solvable models of interacting many-body systems exhibiting exotic scaling of EE [51], exploring the scaling of their EC presents an intriguing avenue, potentially revealing universal behaviors extending beyond free-fermion systems.

The correlation matrix typically retains symmetries inherited from the Hamiltonian of lattice systems, including properties like translation invariance and self-similarity. By delving into conjectures regarding the asymptotic behavior of the correlation matrix on fractals, we stand to glean valuable insights into the asymptotic tendencies of a particular matrix class exhibiting self-similarity. Moreover, such endeavors pave the way for examining Brownian motion on fractals, establishing connections with the spectral characteristics of the Laplacian on fractals [45, 46].

With advancements in experimental techniques, it has become feasible to experimentally realize lattice systems with

fractal structures in physics and chemistry [22–26]. This development opens avenues for studying many-body systems with fractal geometries. Furthermore, in the realm of phononic platforms, it is now possible to simulate and measure the entanglement of many-body systems using pumping-probe responses in fractal phononic lattices [13, 33, 52]. For free-fermion systems, the entanglement contour can be de-

composed into the summation of particle-number cumulants' densities, providing a method to measure the entanglement fractal in the transport of quantum point contacts [38].

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Supporting information

S1. FRACTAL LATTICE AND THE DEFINITION OF ENTANGLEMENT QUANTITY

Here, we discuss the construction of a lattice system with a fractal structure. Initially, we consider an initial unit cell U and a fractal iteration method F to generate the fractal lattice. As illustrated in Fig. S1, employing the method F on the cell U iteratively n times allows us to obtain the n th-order approximation $SC(n, s)$ of the Sierpinski carpet [14]. Here, n represents the n th iteration, and the number of lattice sites in a unit cell is s^2 . Without loss of generality, when $n \rightarrow \infty$ and setting the lattice constant $l = 1$, the Hausdorff fractal dimension of the Sierpinski carpet is defined as

$$d_f = \lim_{n \rightarrow \infty} \log_{\mathcal{L}} \mathcal{N} = \log_3 8, \quad (17)$$

where the number of lattice sites $\mathcal{N} = s^2 \times 8^n$ in the n th-order approximation $SC(n, s)$ of the Sierpinski carpet, and the width of $SC(n, s)$ is $\mathcal{L} = s \times 3^n$. The Hausdorff fractal dimension d_f does not depend on the number s^2 of lattice sites in a unit cell U when $n \rightarrow \infty$. Then, for convenience, we set $s = 1$ in this work.

S2. SCALING OF ENERGY GAP AND DOS OF THE MODEL H_1 AND H_2 ON FRACTAL $SC(n, 1)$

In this section, we delve into the properties of the energy spectrum for the models H_1 and H_2 . For the n th-order approximation $SC(n, 1)$ with a finite number of lattice sites, its energy spectrum exhibits a finite number of gaps E_j . We then examine the scaling behavior of the maximum energy gap $\max(E_j)$ as the number n of iterations increases. As illustrated in Fig. S2(a) and (b), we observe that the scaling of $\max(E_j)$ for model H_1 decreases exponentially, while $\max(E_j)$ for model H_2 remains invariant.

Furthermore, we consider the density of states (DOS) of the models H_1 and H_2 on $SC(n, 1)$ to provide more information about their energy spectrum. Specifically, when model H_1 is defined on the square lattice with translation invariance, lattice constant $l = 1$, and periodic boundary conditions, we can utilize Bloch band theory to clearly determine its energy spectrum. Moreover, its density of states in the thermodynamic limit can be obtained from the energy dispersion:

$$E(\mathbf{k}) = -2t(\cos k_x + \cos k_y) - \mu, \quad (18)$$

where the DOS in Fig. S2(c) is depicted with $t = 1$ and $\mu = 0$. From the continuity of the DOS in Fig. S2(c), we observe that the energy spectrum of the model on the square lattice is gapless, and the maximum point of the DOS indicates the presence of a nesting Fermi surface.

Next, we consider model H_1 defined on the n th-order approximation of the Sierpinski carpet. To practically determine the energy gap of the model on $SC(n, 1)$, we adopt the method outlined in Ref. [53] to calculate its DOS. This

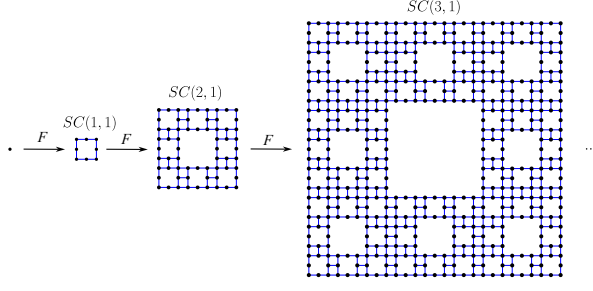


FIG. S1. Illustration of using iteration method F to generate the n th-order approximation $SC(n, 1)$ of Sierpinski carpet with $N = 8^n$ lattice sites.

method proves to be very useful and efficient for large lattice systems without translation invariance. Utilizing this approach, we demonstrate the DOS of the model on the 6th-order approximation $SC(6, 1)$ in Fig. S2(e) as an example, where $SC(6, 1)$ comprises 8^6 lattice sites. Upon comparison with the DOS in Fig. S2(c), we observe that although the fractal structure induces fluctuations in the DOS of the model, it remains continuous, with its maximum point still located at $E = 0$. Furthermore, with an increase in the number n of iterations, the maximum energy gap $\max(E_j)$ of the model on $SC(n, 1)$ becomes progressively smaller. Therefore, we propose that the model on the Sierpinski carpet, in the thermodynamic limit, can be regarded as a gapless system.

Here, we consider the DOS of model H_2 to determine the nature of its energy spectrum. Firstly, when considering model H_2 on the square lattice with periodic boundary conditions, we can determine its DOS in the thermodynamic limit using Bloch band theory, which is determined by the energy dispersion:

$$E(\mathbf{k}) = \pm \sqrt{4t^2(\cos k_x + \cos k_y)^2 + t_1^2}, \quad (19)$$

where the DOS is illustrated in Fig. S2(d) with $t = 1$ and $t_1 = 0.5$. We observe that the energy spectrum of the model in this case exhibits an energy gap. Next, considering the model defined on the approximation $SC(6, 1)$ of the Sierpinski carpet as an example, by employing the efficient method outlined in Ref. [53], we find that the energy gap of the model still persists, as shown in Fig. S2(f). Additionally, compared with the results in Fig. S2(d), the fractal structure of $SC(6, 1)$ induces fluctuations in the DOS and increases the DOS at the maximum point, as depicted in Fig. S2(f). Furthermore, by increasing the number n of iterations, the maximum energy gap $\max(E_j)$ of model H_2 on $SC(n, 1)$ remains invariant, as shown in Fig. S2(b). Therefore, we propose that the model on the Sierpinski carpet in the thermodynamic limit is still a gapped system.

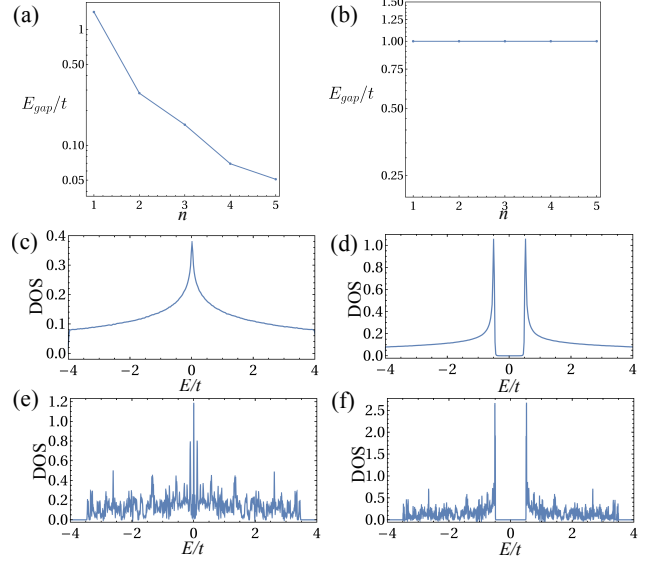


FIG. S2. Scaling of maximum energy gap $\max(E_j)$ for the model and in (a) and (b). (c) and (d) are the DOS of the model and on a square lattice with periodic boundary condition, respectively. (e) and (f) are the DOS of the model and on the approximation $SC(6, 1)$ of Sierpinski carpet, respectively. Here $t = 1$ and $\mu = 0$ in the model, $t = 1$, $t_1 = 0.5$ in the model.

S3. MORE NUMERICAL RESULTS FOR ENTANGLEMENT CONTOUR

For fractal systems, the fractal dimension is determined by the iteration method F and the unit cell U . As the number of iterations n approaches infinity, the fractal dimension solely depends on the iteration method F . Therefore, it is imperative to study the entanglement contour (EC) of model H_1 on a fractal lattice with identical iteration method but different unit cells to elucidate the entanglement fingerprint of fractal geometry. Specifically, in Fig. S3, we present numerical results of the EC of model H_1 in three types of approximations $SC(n, s)$ of the Sierpinski carpet, where the unit cell U comprises s^2 lattice sites. Remarkably, we observe that for model H_1 , its EC exhibits a universal pattern across the three types of approximations $SC(n, s)$, indicating that it is not influenced by the unit cell U . These numerical findings suggest that the structure of EC may solely depend on the iteration method.

In the following, we delve into the effect of the iteration method F on the EC of model H_1 . Firstly, we provide a detailed discussion of the generalized Sierpinski carpet. As depicted in Fig. S4, considering a unit cell with one lattice site, we employ an iteration method $F(m, m_f)$ to act on the unit cell once. Then, the system comprises $m^2 - m_f^2$ lattice sites with a width of m unit cells. Here $m > m_f$ and both are positive integer. If $m_f = 0$, we obtain the normal lattice system $NL(n, 1)$ with a trivial fractal structure and fractal dimension $d_f = 2$, as shown in Fig. S4 (a), where n is the number of iterations. When $m_f \neq 0$, we obtain the n th-order approximation $GSC_{m_f}(n, 1)$ of the generalized Sierpinski carpet in

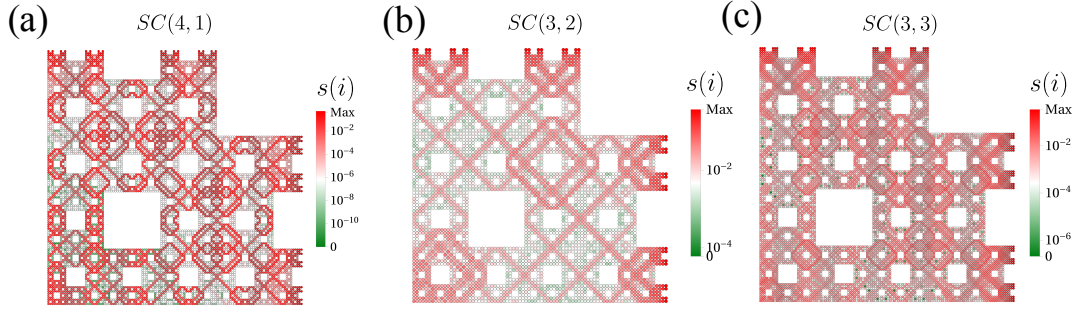


FIG. S3. (a-c) is the EC of the model H_1 on the n th-order approximation $SC(n, s)$ with different unit cell. Here $t = 1$ and $\mu = 0$.

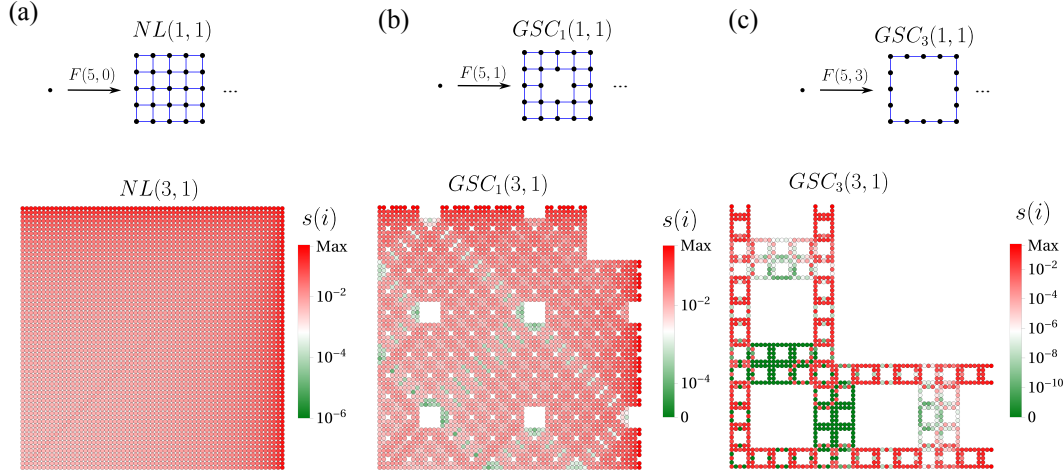


FIG. S4. The upper of (a-c) show the generating process of normal lattice $NL(n, 1)$ and the approximation $GSC_{1(3)}(n, 1)$ generalized Sierpinski carpet. The bottom of (a-c) is the EC of the model H_1 on the normal lattice and the approximation $GSC_{1(3)}(3, 1)$ of generalized Sierpinski carpet. Here $t = 1$ and $\mu = 0$.

Fig. S4(b) and (d), with the fractal dimension represented as:

$$d_f = \log_m(m^2 - m_f^2). \quad (20)$$

Next, we investigate the EC of model H_1 on the approximation $GSC_{m_f}(n, 1)$ of the generalized Sierpinski carpet. To eliminate irrelevant effects on the EC, we set $n = 3$ to ensure that three lattice systems have identical widths, as shown in Fig. S4. From the numerical results of the EC depicted in Fig. S4(a-c), we observe that for the normal lattice $NL(n, 1)$, the EC does not exhibit the special pattern consistent with the results in Ref. [37]. However, for the generalized Sierpinski carpet, the EC is influenced by the iteration method F , as shown in Fig. S4(b-c). Based on these numerical findings, we conjecture that the EC of gapless systems with fractal geometry may exhibit a correspondence with the iteration method.

S4. THE SUPER-AREA LAW OF FREE-FERMION SYSTEMS ON TRANSLATIONAL INVARIANT LATTICE

In this section, we delve into the scaling behavior of the EE in free-fermion systems with translational invariance. The correlation matrix C^A , defined in the main text, typically manifests as a Toeplitz matrix, making it challenging to discern its asymptotic spectrum. Consequently, even in translationally invariant lattices, analytically studying the scaling of EE poses significant difficulties. Ref. [49] provides an alternative definition of the correlation matrix C^A for the subsystem A . By employing a projection operator $\hat{R} = \sum_{x \in \Omega} |x\rangle \langle x|$ onto the lattice set Ω , C^A can be succinctly defined as:

$$C^A = \hat{R} \hat{P} \hat{R}, \quad (21)$$

where $\hat{P}$ is also a projection operator that projects onto the occupied states via $\hat{P} = \theta(-\mathcal{H}) = \sum_k \theta(-\varepsilon_F) |\psi_k\rangle \langle \psi_k|$. Here, $|\psi_k\rangle$ and ε_k denote the eigenvectors and eigenvalues of the single-particle Hamiltonian matrix $\mathcal{H}$ with the momentum k , and θ represents the step function, indicating occupied

single-particle states with energy $\varepsilon_k \leq \varepsilon_F$. Based on the expression of EE S_A in free-fermion systems in Eq.(15) of the main text, we can rewrite EE as follows:

$$\begin{aligned}
S_A &= \text{Tr}(\rho_A \ln \rho_A) \\
&= - \sum_i [\xi_i \ln \xi_i + (1 - \xi_i) \ln(1 - \xi_i)] \\
&= \sum_i f(\xi_i) \\
&= \text{Tr} [f(\hat{R}\hat{P}\hat{R})],
\end{aligned} \tag{22}$$

where the function $f(t) = -t \ln t - (1-t) \ln(1-t)$. Ref. [10] discusses that the sum of function f acting on the spectrum of the Hermitian operator $\hat{R}\hat{P}\hat{R}$ equals the trace of the function f acting on the Hermitian operator $\hat{R}\hat{P}\hat{R}$. Meanwhile, the kernel of the operator $\hat{R}\hat{P}\hat{R}$ is given by

$$\langle \psi_k | \hat{R}\hat{P}\hat{R} | \psi_{k'} \rangle = \chi_\Gamma(k) \chi_\Gamma(k') \frac{V}{(2\pi)^d} \int_\Omega e^{i(k-k')x} dx, \tag{23}$$

where $\chi_\Gamma(k)$ is defined for Γ as $\chi_\Gamma(k) = 1$ if $k \in \Gamma$ and $\chi_\Gamma(k) = 0$ otherwise. In a d -dimensional system with a $(d-1)$ -dimensional Fermi surface, the Fermi sea region of the system is represented as $\Gamma = \{k | \varepsilon_k \leq \varepsilon_F\}$ in momentum space. Ref. [10] discusses that when $L \rightarrow \infty$, the scaling of the system is determined by the Widom conjecture.

Next, we delve into employing the Widom conjecture to derive the super-area law for d -dimensional systems with $(d-1)$ -dimensional Fermi surfaces. In Ref. [10], considering the operator $\hat{R}\hat{P}\hat{R}$ defined in two sets Ω and Γ , with the kernel in (23), and a general class of functions f , the asymptotic formula of the trace for $\hat{R}\hat{P}\hat{R}$ is derived as follows:

$$\begin{aligned}
\text{Tr} f(\hat{R}\hat{P}\hat{R}) &= \left(\frac{L_A}{2\pi} \right)^d f(1) \int_\Omega \int_\Gamma dx dk \\
&+ \left(\frac{L_A}{2\pi} \right)^{d-1} \frac{\ln 2 \ln L_A}{4\pi^2} U(f) \int_{\partial\Omega} \int_{\partial\Gamma} |\mathbf{n}_x \cdot \mathbf{n}_k| dS_x dS_k \\
&+ o(L_A^{d-1} \ln L_A),
\end{aligned} \tag{24}$$

where $\partial\Gamma$ and $\partial\Omega$ represent the boundaries of the Fermi sea and the subsystem A respectively. $\mathbf{n}_x$ and $\mathbf{n}_k$ denote unit normal vectors to $\partial\Gamma$ and $\partial\Omega$ respectively, and $U(f) = \int_0^1 \frac{f(t) - t f(1)}{t(1-t)} dt$. Utilizing this formula, we readily obtain the scaling of EE in d -dimensional systems with $(d-1)$ -dimensional Fermi surfaces. From (24), the first term vanishes due to $\lim_{x \rightarrow 1^-} f(x) = 0$. The second term of (24) determines the scaling of EE and is proportional to $L_A^{d-1} \log L_A$. It is essential to note that this formula holds true only when $\hat{R}\hat{P}\hat{R}$ is Hermitian. In conclusion, by leveraging the Widom conjecture, we establish the logarithmic divergence of the scaling of EE on translationally invariant lattice.