Unveiling Quasiparticle Berry Curvature Effects in the Spectroscopic Properties of a Chiral p-wave Superconductor

Yunxiang Liao[1,2] and Yi-Ting Hsu[3]

[1]Department of Physics, KTH Royal Institute of Technology, SE-106 91 Stockholm, Sweden
[2]Condensed Matter Theory Center and Joint Quantum Institute,
Department of Physics, University of Maryland, College Park, MD 20742, USA.*
[3]Department of Physics, University of Notre Dame, South Bend, IN 46556 USA†

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The focus of experimental efforts on topological superconductivity (TSC) has predominantly centered around the detection of Majorana boundary modes. On the contrary, the experimental features in the bulk properties of TSC remain relatively unexplored, especially in 2D. Here, we theoretically examine how spectroscopic properties away from the boundaries can be influenced by the Berry curvatures (BCs) in two-dimensional chiral TSC, enabled by spin-orbit couplings (SOC). Specifically, we analyze the semiclassical wavepacket dynamics for quasiparticles in $p+ip$ TSC achieved through Rashba SOC together with proximity to a ferromagnet and conventional superconductor under an external magnetic field $\mathbf{B}$. Beyond the more well-known momentum-space and real-space BC effects, we unveil a new phase-space BC term that emerges exclusively when the SOC and superconductivity coexist, resulting from a Rashba-induced quasiparticle charge dipole moment that couples to $\mathbf{B}$. Importantly, we demonstrate that this BC term grows linearly in $|\mathbf{B}|$, and can have a discernible impact on the energy- or momentum- resolved tunneling spectroscopy by non-uniformly enhancing or suppressing peak intensities. This qualitative effect originates from the BC-modified phase-space density of states, which can be experimentally probed through the non-uniform changes in peak intensities compared to the spectrum at $\mathbf{B} = 0$.

Introduction— Topological superconductors (TSCs) are exotic quantum materials that exhibit non-trivial band topology in the bulk as well as Majorana modes on different dimensional boundaries[1–8]. The experimental detection of a TSC has remained one of the key challenges in the fields of unconventional superconductivity and topological phases, despite intensive studies in the past decade Currently, many TSC experiments have focused on detecting Majorana boundary signatures through local scanning probes[4,9–19], although this approach often faces the challenge of distinguishing signals originating from Majorana modes and from other more mundane sources[18–26]. In contrast, experimental features in the bulk of TSC associated with superconducting band topology remain relatively unexplored.

Bands with non-trivial topology necessarily possess non-trivial band geometry, which characterizes the changes in eigenstates at neighboring momenta $\mathbf{k}$ and could have significant influences on experimental observables in metals, insulators, and superconductors[27–38]. For example, the Berry curvature (BC) $\Omega_{\lambda\lambda'}, \lambda, \lambda' = r, k$, is a geometric quantity expected in two-dimensional (2D) chiral TSC, where the superconducting ground state features a non-zero Chern number $C = \int d\mathbf{k}\Omega_{kk}(\mathbf{k})$ along with quantized thermal Hall conductivity, which results from the quantized integral of the momentum-space BC Ω_{kk}[1]. Besides having non-trivial Ω_{kk}, 2D chiral TSCs can also acquire generalized BCs defined in the real space Ω_{rr} and the phase space Ω_{kr}. The former generally exists in the presence of external perturbations, such as a magnetic field or supercurrent, whereas the latter can exist in superconductors when subjected to external perturbations[35]. Away from the superconducting ground state, these different BCs of Bogoliubov quasiparticles were recently found to influence the density of states and thermal Hall conductivity in spin-singlet superconductors with spin-degenerate normal states[35].

Spin-orbit-coupled (SOC) normal states, on the other hand, have been a crucial ingredient in strategies to achieve TSC materials in 2D since manipulating Fermi surface spin textures can facilitate the formation of effectively spin-triplet Cooper pairs, and thus an odd-parity TSC[10,15,16,39–49]. Candidate materials for 2D chiral TSC that follow this strategy include the superconducting surfaces of proximitized Bi_2Se_3[39], proximitized semiconductors with SOC[42], and hole-doped monolayer transition metal dichalcogenides[48], among other material systems.

Here, we will theoretically show how the quasiparticle BC can impact spectroscopic properties in a $p+ip$ TSC, in particular the tunneling spectrum away from the boundary and the spectral function. The latter can in principle be probed using methods such as momentum- and energy-resolved tunneling spectroscopy (MERTS)[50]. Importantly, while the $p+ip$ pairs here arise from a spin-orbit coupled two-dimensional electron gas (2DEG), we find new BC effects that are *not* inherited directly from the normal state but exist only when the superconducting gap Δ is non-zero. In the following, we will first introduce the minimal model for the $p+ip$ TSC and the quasiparticle wavepacket approach[30,35] we use to examine the BC effects. Then, we will identify the contributions to the equations of motion of wavepackets from the momentum-, real-, and phase-space BCs of the quasiparticles. Finally, we will examine the qualitative features of quasiparticle BC in the tunneling conductance and spectral function, tunable by an external magnetic field.

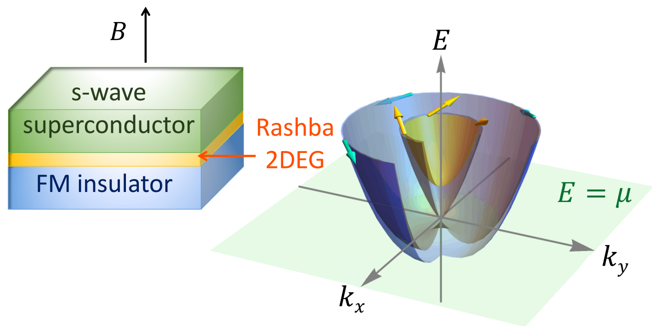

FIG. 1. The left panel sketches the heterostructure we consider under an applied magnetic field $\mathbf{B} = B\hat{z}$. The right panel shows the calculated dispersion of the normal state, i.e. the dispersion of the Rashba 2DEG in the presence of an effective Zeeman field h that results predominantly from the exchange coupling. The green plane represents the chemical potential μ and the blue (yellow) arrows on the paraboloid bands are the calculated spin directions at different momenta for both bands.

Model— The specific model we consider is a Rashba spin-orbit coupled two-dimensional electron gas sandwiched by an s-wave superconductor and a ferromagnetic insulator, proposed by Sau et al.[41,42] for an experimental realization of a 2D $p+ip$ TSC phase in condensed matter systems (see Fig. 1). For this heterostructure, we investigate the quasiparticle dynamics driven by external perturbations, such as an applied out-of-plane magnetic field $\mathbf{B} = B\hat{z}$. The 2DEG layer is described by the following mean-field superconducting model:

$$\begin{aligned} H = &\sum_{\sigma=\uparrow,\downarrow} \int d^2\mathbf{r} c_\sigma^\dagger(\mathbf{r}) \left(\frac{1}{2m} \left(-i\nabla - \mathbf{A}(\mathbf{r})\right)^2 - \mu + h\sigma_z \right) c_\sigma(\mathbf{r}) \\ &+ \sum_{\sigma=\uparrow,\downarrow} \int d^2\mathbf{r} c_\sigma^\dagger(\mathbf{r}) \alpha(\boldsymbol{\sigma} \times (-i\nabla - \mathbf{A}(\mathbf{r}))) \cdot \hat{z} c_\sigma(\mathbf{r}) \\ &+ \int d^2\mathbf{r} \left(\Delta c_\uparrow^\dagger(\mathbf{r}) c_\downarrow^\dagger(\mathbf{r}) + \Delta^* c_\downarrow(\mathbf{r}) c_\uparrow(\mathbf{r}) \right), \end{aligned} \tag{1}$$

where $c_\sigma^\dagger(\mathbf{r})$ creates an electron at position $\mathbf{r}$ with spin $\sigma =\uparrow,\downarrow$ and mass m. μ is the chemical potential, σ_i are the Pauli matrices in spin, and $\mathbf{A}(\mathbf{r}) = \mathbf{B}\times\mathbf{r}/2$ represents the vector potential of the applied magnetic field $\mathbf{B}$. We have set the reduced Planck's constant $\hbar = 1$ and the electric charge $e = 1$. The second and third lines in Eq. 1 describe respectively the Rashba SOC with strength α and the proximity-induced superconducting pairing with pairing potential Δ. For the first line, the chemical potential μ is set to intersect with the lower normal band. We consider the cases where the effective Zeeman splitting h is predominantly induced by the exchange coupling from the ferromagnetic layer with out-of-plane magnetization, and the contribution from the external magnetic field $B < \Delta$ is negligible. This allows the Rashba layer to lie within the TSC regime driven by a relatively large Zeeman splitting $h > \sqrt{|\Delta|^2 + \mu^2}$ without destroying the superconducting layer, which does not directly contact the ferromagnetic layer. Moreover, we approximate h and B as two independent variables assuming that the orbital effect of the ferromagnetic insulator and the Zeeman splitting caused by the weak magnetic field B are negligible. Finally, we do not consider the orbital effect from vortices, which is expected to be insignificant at positions far away from vortices or when the Rashba layer has a large g factor.

Quasiparticle wavepacket— For this superconducting model H, we examine the semiclassical dynamics of Bogoliubov quasiparticles using the wavepacket approach, which has been successfully applied to metals[27,51] (see review Ref.[30] and references therein) and spin-degenerate superconductors[35,52,53]. We consider the case where the external perturbation $\mathbf{A}(\mathbf{r})$ is time independent and slowly varying in the real space $\mathbf{r}$ with a characteristic length scale much larger than the spread of the wavepacket. This allows us to examine the wavepacket dynamics by neglecting the position dependence of the vector potential $\mathbf{A}(\mathbf{r})$ in Eq. 1 and approximating $\mathbf{A}(\mathbf{r})$ by its value at the wavepacket center $\mathbf{r} = \mathbf{r}_c$.

The effective Hamiltonian H which governs the wavepacket dynamics can then be diagonalized by introducing the Bogoliubov-de Gennes (BdG) quasiparticle annihilation operator $\gamma_s(\mathbf{k})$ in the momentum space $\mathbf{k}$:

$$\begin{aligned} \gamma_s(\mathbf{k}) &= \Phi_s^\dagger(\mathbf{k})\Psi(\mathbf{k}), \qquad s = \pm, \\ \Psi(\mathbf{k}) &= \left[c_\uparrow(\mathbf{k}) \;\; c_\downarrow(\mathbf{k}) \;\; c_\downarrow^\dagger(-\mathbf{k}) \;\; -c_\uparrow^\dagger(-\mathbf{k}) \right]^T, \\ \Phi_s(\mathbf{k}) &= \left[u_{\uparrow s}(\mathbf{k}) \;\; u_{\downarrow s}(\mathbf{k}) \;\; v_{\downarrow s}(\mathbf{k}) \;\; v_{\uparrow s}(\mathbf{k}) \right]^T, \end{aligned} \tag{2}$$

where $c_\sigma(\mathbf{k}) = \int d^2\mathbf{r} e^{-i(\mathbf{k}+\mathbf{A}(\mathbf{r}_c))\cdot\mathbf{r}} c_\sigma(\mathbf{r})$ for spin $\sigma =\uparrow,\downarrow$, and the helicity $s = \pm$ labels the upper and lower BdG bands. The explicit expression for the BdG eigenfunction $\Phi_s(\mathbf{k})$ can be found in the Supplementary Material (SM)[54]. After the BdG transformation, the Hamiltonian acquires the form

$$H = \sum_{s=\pm1} \int_\mathbf{k} E_s(k) \gamma_s^\dagger(\mathbf{k}) \gamma_s(\mathbf{k}) + \text{const.}. \tag{3}$$

Here $\int_\mathbf{k}$ is short for $\int d^2\mathbf{k}/(2\pi)^2$, and the BdG quasiparticle energy $E_s(k)$ is given by

$$E_s(k) = \sqrt{b_k^2 + |\Delta|^2 + h^2 + \xi_k^2 + 2s\sqrt{\xi_k^2\left(b_k^2 + h^2\right) + |\Delta|^2 h^2}}, \tag{4}$$

with $\xi_k = \frac{k^2}{2m} - \mu$ and $b_k = \alpha k$.

To investigate the BC effects in the TSC phase, we construct a wavepacket for the quasiparticles from the lower BdG band with $s = -$, which originates from the lower normal band on which the chemical potential μ intersects,

$$|W_s\rangle = \int_\mathbf{k} w_s(\mathbf{k}, t) \gamma_s^\dagger(\mathbf{k}) |G\rangle. \tag{5}$$

Here $|G\rangle$ denotes the superconducting ground state. The envelope function $w_s(\mathbf{k},t)$ obeys the normalization condition $\int_{\mathbf{k}} |w_s(\mathbf{k},t)|^2 = 1$, and is sharply peaked around the center $\mathbf{k}_c$. The coordinates of the wavepacket centers in the momentum and real spaces $\mathbf{k}_c$ and $\mathbf{r}_c$ are given respectively by[53]

$$\begin{aligned}\mathbf{k}_c &= \int_{\mathbf{k}} |w_s(\mathbf{k},t)|^2 \mathbf{k},\\ \mathbf{r}_c &= i\int_{\mathbf{k}} \left(w_s(\mathbf{k},t)\Phi_s(\mathbf{k})\right)^\dagger \partial_{\mathbf{k}} \left(w_s(\mathbf{k},t)\Phi_s(\mathbf{k})\right).\end{aligned} \tag{6}$$

For cases where the perturbation is strong, a treatment involving both $s=\pm$ bands is required[29,30,55], which falls beyond the scope of our current study.

Wavepacket dynamics and quasiparticle BC— Next, we investigate the semiclassical dynamics of this quasiparticle wavepacket $|W_s\rangle$ by studying the Lagrangian of the wavepacket L, which is defined as the wavepacket average[56] of the operator $\hat{L} = i\frac{d}{dt} - \hat{H}$[30,35]. It can be expressed in terms of the momentum-space and real-space Berry connections $\mathcal{A}_{ks}$ and $\tilde{\mathcal{A}}_{rs}$:

$$L = -E_s(k_c) - \dot{\mathbf{k}}_c\cdot\mathbf{r}_c + \dot{\mathbf{k}}_c\cdot\mathcal{A}_{ks} + \dot{\mathbf{r}}_c\cdot\tilde{\mathcal{A}}_{rs}. \tag{7}$$

The momentum-space Berry connection $\mathcal{A}_{ks}$ is defined in terms of the BdG wavefunction $\Phi_s(\mathbf{k})$ as $\mathcal{A}_{ks} = i\Phi_s^\dagger(\mathbf{k}_c)\partial_{\mathbf{k}_c}\Phi_s(\mathbf{k}_c)$, and can be further simplified to [54]

$$\mathcal{A}_{ks} = -\frac{1}{2}\rho_{0s}(\mathbf{k}_c)\partial_{\mathbf{k}_c}\chi + \frac{1}{2}\rho_{1s}(\mathbf{k}_c)\partial_{\mathbf{k}_c}\phi_{\mathbf{k}_c}. \tag{8}$$

Here, $\chi \equiv \arg(\Delta)$ denotes the phase of the order parameter[57], $\phi_{\mathbf{k}} \equiv \arctan(k_y/k_x)$ denotes the intrinsic angle for the Rashba SOC. $\rho_{as}(\mathbf{k})$ for $a=0,1$ is given by

$$\rho_{as}(\mathbf{k}) = \sum_\sigma (\zeta_\sigma)^a \left(|u_{\sigma s}(\mathbf{k})|^2 - |v_{\sigma s}(\mathbf{k})|^2\right), \tag{9}$$

with $\zeta_{\uparrow/\downarrow} = \pm 1$. Physically speaking, these two factors $\rho_{0s}(\mathbf{k})$ and $\rho_{1s}(\mathbf{k})$ at the wavepacket center $\mathbf{k}_c$ correspond to the wavepacket average[56] of the total charge $\hat{Q} = \sum_\sigma \int_{\mathbf{r}} \hat{c}_\sigma^\dagger(\mathbf{r})\hat{c}_\sigma(\mathbf{r})$ and two times the spin $\hat{S} = \sum_\sigma \int_{\mathbf{r}} \zeta_\sigma \hat{c}_\sigma^\dagger(\mathbf{r})\hat{c}_\sigma(\mathbf{r})/2$, respectively.

The real-space Berry connection $\tilde{\mathcal{A}}_{rs}$ in Eq. 7 consists of three parts:

$$\tilde{\mathcal{A}}_{rs} = \mathcal{A}_{rs} + \rho_{0s}(\mathbf{k}_c)\mathbf{A}(\mathbf{r}_c) + \frac{1}{2}\mathbf{B}\times\mathbf{d}. \tag{10}$$

The first term $\mathcal{A}_{rs}$ differs from the momentum-space counterpart $\mathcal{A}_{ks}$ in Eq. 8 by only the replacement $\partial_{\mathbf{k}_c} \to \partial_{\mathbf{r}_c}$. The second term describes the coupling between total charge $\rho_{0s}(\mathbf{k}_c)$ and the external vector potential $\mathbf{A}(\mathbf{r}_c)$. The last term describes the coupling between the external magnetic field $\mathbf{B}$ and the wavepacket average[56] $\mathbf{d}$ of the charge dipole moment operator $\hat{\mathbf{d}} = \sum_\sigma \int d^2\mathbf{r}\, \hat{c}_\sigma^\dagger(\mathbf{r})\hat{c}_\sigma(\mathbf{r})(\mathbf{r}-\mathbf{r}_c)$. Note that the second and last terms, which respectively originate from the total charge and the dipole of the wavepacket, enter the Lagrangian Eq. 7 in the same way as the well-known Lagrangian of a charged object with a dipole moment in a magnetic field[58].

We find that the wavepacket average $\mathbf{d}$ consists of two terms $\mathbf{d} = \mathbf{d}_1 + \mathbf{d}_2$:

$$\begin{aligned}\mathbf{d}_1 &= \frac{1}{2}\left(\rho_{0s}^2(\mathbf{k}_c) - 1\right)\partial_{\mathbf{k}_c}\chi,\\ \mathbf{d}_2 &= \frac{1}{2}\left(\rho_{2s}(\mathbf{k}_c) - \rho_{0s}(\mathbf{k}_c)\rho_{1s}(\mathbf{k}_c)\right)\partial_{\mathbf{k}_c}\phi_{\mathbf{k}_c},\end{aligned} \tag{11}$$

where $\rho_{2s}(\mathbf{k}_c) = \sum_\sigma \zeta_\sigma \left(|u_{\sigma s}(\mathbf{k}_c)|^2 + |v_{\sigma s}(\mathbf{k}_c)|^2\right)$[54]. Here the first term $\mathbf{d}_1$ vanishes for our case of an s-wave order parameter with $\partial_{\mathbf{k}}\chi = 0$. Importantly, the second term $\mathbf{d}_2$ arises from the winding of the Rashba angle $\phi_{\mathbf{k}_c}$ around the Fermi surface, but is *not* inherited from the normal state. Instead, $\mathbf{d}_2$ originates from the momentum-dependent mixtures of electron and hole components in the quasiparticle eigenstates, which therefore exists only in the superconducting but not the normal state. To be explicit, both $\mathbf{d}_1$ and $\mathbf{d}_2$ become vanishing in a normal state because the superconducting phase χ does not exist and the prefactor $\rho_{2s} - \rho_{0s}\rho_{1s} = 0$. $\mathbf{d}_1$ becomes nonzero for nontrivial superconducting pairing as discussed in Ref.[35]. The new component $\mathbf{d}_2$ in the wavepacket charge dipole moment can be viewed as a new feature of spin-orbit coupled superconductors that is reported for the first time to our best knowledge.

From the Lagrangian L in Eq. 7, we obtain the equations of motion for the wavepacket center:

$$\begin{bmatrix}\Omega_{rrs}^{ij} & \Omega_{rks}^{ij} - \delta_{ij}\\ \Omega_{krs}^{ij} + \delta_{ij} & \Omega_{kks}^{ij}\end{bmatrix}\begin{bmatrix}\dot{\mathbf{r}}_c^j\\ \dot{\mathbf{k}}_c^j\end{bmatrix} = \begin{bmatrix}\partial_{r_c^i}E_s(k_c)\\ \partial_{k_c^i}E_s(k_c)\end{bmatrix}, \tag{12}$$

where the sum of repeated Cartesian indices is implied. The momentum-, real-, and phase-space BCs are defined as the derivatives of the Berry connections $\tilde{\mathcal{A}}_{rs}$ and $\mathcal{A}_{ks}$:

$$\begin{aligned}\Omega_{kks}^{ij} &\equiv \partial_{k_c^i}\mathcal{A}_{ks}^j - \partial_{k_c^j}\mathcal{A}_{ks}^i, \qquad \Omega_{rrs}^{ij} \equiv \partial_{r_c^i}\tilde{\mathcal{A}}_{rs}^j - \partial_{r_c^j}\tilde{\mathcal{A}}_{rs}^i,\\ \Omega_{rks}^{ij} &\equiv \partial_{r_c^i}\mathcal{A}_{ks}^j - \partial_{k_c^j}\tilde{\mathcal{A}}_{rs}^i.\end{aligned} \tag{13}$$

Note that Eq. 12 is the generic equations of motion for both the superconducting quasiparticles and Bloch electrons under time-independent perturbations[27,30,35]. However, the explicit expressions of BCs are model dependent. For the $p+ip$ TSC in Eq 1, we find that up to the leading order in the real-space gradient expansion, the real-, momentum-, and phase-space BCs are given by (see the full expression in SM[54])

$$\begin{aligned}\mathbf{\Omega}_{ks} &= -\frac{1}{2}\partial_{\mathbf{k}_c}\rho_{0s}(\mathbf{k}_c)\times\partial_{\mathbf{k}_c}\chi + \frac{1}{2}\partial_{\mathbf{k}_c}\rho_{1s}(\mathbf{k}_c)\times\partial_{\mathbf{k}_c}\phi_{\mathbf{k}_c},\\ \mathbf{\Omega}_{rs} &= \rho_{0s}(\mathbf{k}_c)\mathbf{B},\\ \Omega_{rks}^{ij} &= (\partial_{k_c^j}\rho_{0s}(\mathbf{k}_c))p_s^i - \frac{1}{2}\partial_{k_c^j}(\mathbf{B}\times\mathbf{d})^i,\end{aligned} \tag{14}$$

where $\mathbf{\Omega}_{ks}$ ($\mathbf{\Omega}_{rs}$) denotes the vector form of Ω_{kks} (Ω_{rrs}) and $\mathbf{p}_s = -\mathbf{A}(\mathbf{r}_c)$ is the supercurrent for an s-wave order parameter. For our case of an s-wave order parameter with $\partial_{\mathbf{k}}\chi = 0$ and an absence of supercurrent $\mathbf{p}_s = 0$, we find a non-zero momentum-space BC $\mathbf{\Omega}_{ks}$ inherited from the spin-orbit coupled normal metal and a real-space BC $\mathbf{\Omega}_{rs}$ due to the applied magnetic field $\mathbf{B}$. Importantly, the phase-space BC Ω_{krs} receives a charge dipole moment $\mathbf{d}$-dependent contribution under $\mathbf{B}$. Since the quasiparticle charge dipole moment $\mathbf{d}$ in Eq. 11 only exists when superconductivity and SOC coexist (when $\partial_{\mathbf{k}}\chi = 0$), we expect that this dipole-induced BC can induce detectable features of Rashba superconductors in superconducting properties.

BC-induced violation of Liouville theorem— The presence of BC in the equations of motion Eq. 12 leads to a breakdown of Liouville theorem for the conservation of the phase-space volume $\Delta V = \Delta k \Delta r$. This is because the change of volume in time δt is given by $\Delta V(t+\delta t) - \Delta V(t) = (\nabla_{\mathbf{r}}\dot{\mathbf{r}} + \nabla_{\mathbf{k}}\dot{\mathbf{k}})\Delta V(t)\delta t$. From the equation of motion in Eq. 12, one can see that this volume change vanishes when $\Omega_{\lambda\lambda' s} = 0$, $\lambda, \lambda' = r, k$, but acquires BC-induced terms when $\Omega_{\lambda\lambda' s} \neq 0$[28,30]. Therefore, to preserve the total number of states in volume element ΔV, i.e. $D(\mathbf{r},\mathbf{k})\Delta V$, a correction to the phase-space density of states (DOS) $D(\mathbf{r},\mathbf{k})$ is required [28,30]. It is known that the modified phase-space DOS D associated with the equation of motion Eq. 12 up to an inessential overall coefficient, is given by [30,35,59]

$$D(\mathbf{r},\mathbf{k}) = \sqrt{\det\begin{bmatrix}\Omega_{rrs} & \Omega_{rks} - I \\ \Omega_{krs} + I & \Omega_{kks}\end{bmatrix}} \approx 1 + \mathrm{Tr}\,\Omega_{krs} - \mathbf{\Omega}_{rs}\cdot\mathbf{\Omega}_{ks}, \tag{15}$$

where a spatial gradient expansion is applied in the second line for slowly varying external fields.

Next, we investigate the BC effects in experimental observables that arise from the BC-induced phase-space DOS $\delta D = D - 1$. To this end, we divide $\delta D = \delta D_1 + \delta D_2$ into two portions, depending on whether the BC is driven by an external magnetic field $\mathbf{B}$ or a supercurrent $\mathbf{p}_s$:

$$\begin{aligned}\delta D_1 &= -\frac{1}{2}\mathbf{B}\cdot(\nabla_{\mathbf{k}}\times\mathbf{d}_2) - \frac{1}{2}(\rho_{0s}\nabla_{\mathbf{k}}\rho_{1s}\times\nabla_{\mathbf{k}}\phi_{\mathbf{k}})\cdot\mathbf{B} \\ &= -\frac{1}{4}\mathbf{B}\cdot\left[(\nabla_{\mathbf{k}}\rho_{2s} + \rho_{0s}\nabla_{\mathbf{k}}\rho_{1s} - \rho_{1s}\nabla_{\mathbf{k}}\rho_{0s})\times\nabla_{\mathbf{k}}\phi_{\mathbf{k}}\right],\end{aligned} \tag{16a}$$

$$\delta D_2 = -\nabla_{\mathbf{k}}\rho_{0s}\cdot\mathbf{p}_s, \tag{16b}$$

where we apply the expressions of BC in Eq. 14 and keep only the leading terms in spatial gradient $\partial_{\mathbf{r}}$. Note that the BC effect δD_2 driven by a supercurrent is found to be similar to that in spin-degenerate superconductors[35] and is not the focus of this work. On the contrary, the BC effects δD_1 driven by a magnetic field $\mathbf{B}$ is proportional to the winding of the Rashba angle $\phi_{\mathbf{k}}$, and therefore only exists under SOC. More specifically, the first term in the first equality of the expression for δD_1 originates from the coupling between the $\mathbf{B}$ field and the new component in the quasiparticle charge dipole moment $\mathbf{d}_2$ which only exists in the presence of a spin-orbit coupled superconductor, while the second term persists in a spin-orbit coupled normal state. We note that the expression for δD_1 (Eq. 16a) remains valid even if $\partial_{\mathbf{k}}\chi \neq 0$. In the following, we will focus on how the BC effects arising from the $\mathbf{B}$ field-induced phase-space DOS δD_1 influence experimental observables.

BC effects in spectroscopic properties— Specifically, the BC-modified phase-space DOS $D = 1 + \delta D_1$ influence the normal metal-to-superconductor tunneling conductance[60] $G(\omega)$ and the spectral function $A(\mathbf{k},\omega)$ in the following way:

$$\begin{aligned}G(\mathbf{r},\omega) = \sum_\sigma \int_{\mathbf{k}} D(\mathbf{r},\mathbf{k})\,&\left[|u_{\sigma s}(\mathbf{k})|^2\delta\left(\omega - E_s(k)\right)\right.\\ &\left.+|v_{\sigma s}(\mathbf{k})|^2\delta\left(\omega + E_s(k)\right)\right],\end{aligned} \tag{17a}$$

$$\begin{aligned}A(\mathbf{k},\omega) = \sum_\sigma \int_{\mathbf{r}} D(\mathbf{r},\mathbf{k})\,&\left[|u_{\sigma s}(\mathbf{k})|^2\delta\left(\omega - E_s(k)\right)\right.\\ &\left.+|v_{\sigma s}(\mathbf{k})|^2\delta\left(\omega + E_s(k)\right)\right].\end{aligned} \tag{17b}$$

Note that we focus on the lower BdG band $s = -$ and an s-wave pairing potential Δ with $\partial_{\mathbf{k}}\chi = 0$, and we neglect the small spatial variation in $D(\mathbf{r},\mathbf{k})$ given the slowly varying perturbation. The tunneling conductance can be measured by STM or tunneling junctions, whereas the spectral function can in principle be probed by methods such as momentum- and energy-resolved tunneling spectroscopy (MERTS)[50] even in the presence of a magnetic field. For STM measurements to assess open surfaces, besides Fig. 1, one can also consider bilayer structures consisting of a Rashba superconductor with a ferromagnet[16].

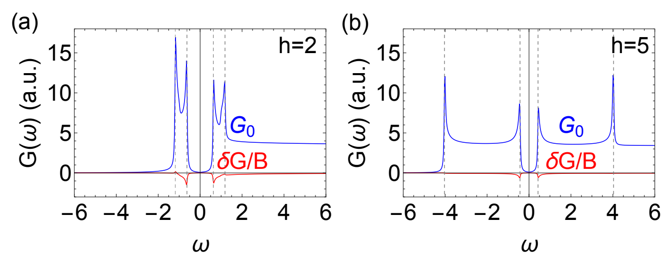

FIG. 2. BC effects on the tunneling conductance $G(\omega)$ at an effective Zeeman splitting of (a) $h = 2$ and (b) $h = 5$. The blue curves G_0 are the bare tunneling conductance calculated from Eq. 17 with $D(\mathbf{k}) = 1$. G_0 can be obtained by the tunneling conductance at $B = 0$, assuming that the Zeeman splitting h is predominantly given by the exchange coupling. The red curves $\delta G/B$ are the corrections per unit magnetic field B to G_0 from the BC-induced change in the phase-space DOS, calculated from Eq. 17 with $D(\mathbf{k})$ replaced by $\delta D_1(\mathbf{k})$. The chosen model parameters are (μ, m, α)=(0.2,0.5,1) in the units of $\Delta = 1$, where the chemical potential μ intersects with the lower normal band. This figure and Fig. 3 are generated by replacing the Dirac delta function in Eq. 17 with a Lorentzian $L(\omega) = \frac{1}{\pi}\frac{\eta/2}{\omega^2+(\eta/2)^2}$ of width $\eta = 0.05$.

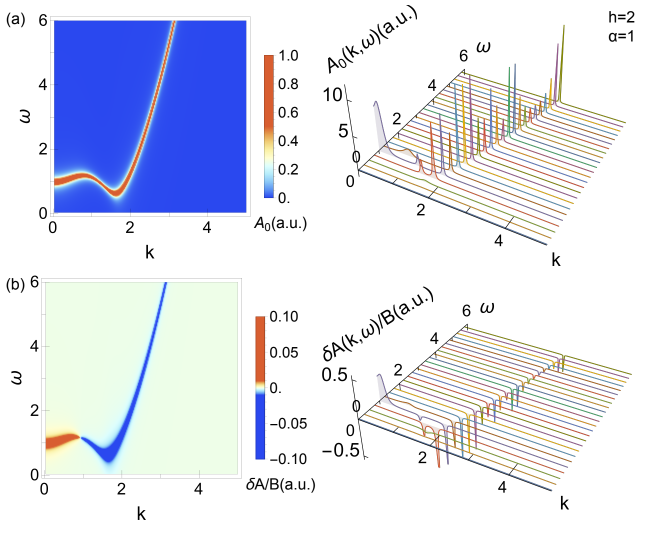

FIG. 3. BC effect on the the spectral function $A(\mathbf{k},\omega)$. (a) The bare energy- and momentum-resolved spectral function $A_0(\mathbf{k},\omega)$, calculated using Eq. 17b with $D(\mathbf{k})=1$, and (b) the correction per unit magnetic field $\delta A(\mathbf{k},\omega)/B$ to the bare spectral function $A_0(\mathbf{k},\omega)$ from the BC-induced change in the phase-space DOS, calculated with $D(\mathbf{k})$ replaced with $\delta D_1(\mathbf{k})$. The bare spectral function $A_0(\mathbf{k},\omega)$ and the BC correction $\delta A(\mathbf{k},\omega)/B$ are shown in an intensity plot (left) and momentum distribution curves (right). The chosen model parameters are (μ, m, α, h)=(0.2,0.5,1,2) in the units of Δ.

We now demonstrate the BC effects in the tunneling conductance $G(\omega)=G_0(\omega)+\delta G(\omega)$ in Fig. 2, where the bare conductance G_0 and the BC correction δG are plotted using Eq. 17a by taking the phase-space DOS $D(\mathbf{k})$ to be 1 and $\delta D_1(\mathbf{k})$, respectively. The bare conductance G_0 shows four coherence peaks at energies ω labeled by the dotted lines in Fig. 2(a). These peaks originate from the band extrema of the lower BdG band. The number of peaks is in principle determined by the number of extrema in the BdG band. Nonetheless, the number of practically visible coherence peaks in experiments is further determined by the actual material band structure, thermal and impurity-caused smearing, as well as the focused frequency range. Moreover, due to the effective p-wave pairing, we do not find the bare tunneling conductance $G_0(\omega)$ to be even in frequency ω, which is only required for s-wave pairing[60].

We now move onto the BC contribution δG that arises solely from the BC-modified phase-space DOS δD_1, driven by an external magnetic field $\mathbf{B}$. Under an intermediate Zeeman field $h=2|\Delta|$, we find that there is a BC-induced considerable suppression δG to the strengths of the two lower-energy coherence peaks but not the higher-energy peaks (see Fig. 2(a)). As we vary the strengths of the Zeeman splitting h or SOC α, we find non-uniform suppression and/or enhancement across different coherence peaks (see Fig. 2(b) and SM[54]). Such BC-induced alteration to the peak intensities can be linearly magnified by the magnetic field strength $|\mathbf{B}|$ since

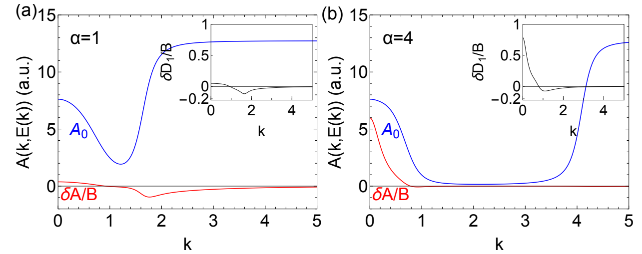

FIG. 4. The bare spectral function $A_0(\mathbf{k},\omega)$ (blue) and the BC contribution per unit magnetic field strength $\delta A(\mathbf{k},\omega)/B$ (red) as functions of momentum k along the superconducting band $\omega=E_s(k)$. The inset shows the ratio $\delta A/(BA_0)$, identical to the Berry curvature correction to the phase-space DOS $\delta D_1/B$. The SOC strength is set to (a) $\alpha=1$ and (b) $\alpha=4$. The remaining model parameters are (μ, m, h)=(0.2,0.5,2). Note that the BC effect $\delta A/A_0$ at $k=0$ increases significantly with α.

$\delta D_1\propto|\mathbf{B}|$ (see Eq. 16a), although the field strength $|\mathbf{B}|$ is capped by the local approximation that treats $A(\mathbf{r}_c)$ as a constant as well as the influence of vortices.

Experimentally, we expect that the quasiparticle BC in a Rashba-driven $p+ip$ TSC can be detected by measuring tunneling spectra in the following way: The uncorrected tunneling spectrum G_0 is in fact the tunneling conductance in the absence of the magnetic field B, assuming that the Zeeman splitting h is predominantly from the exchange coupling. When the magnetic field is applied $B\neq 0$, we propose that the linear-in-B changes from G_0 that are non-uniform across different coherence peaks are signatures of the BC-modified phase-space DOS $D=1+\delta D_1$[61].

We now turn to the BC effects in the spectral function $A=A_0+\delta A$. The uncorrected spectral function $A_0(\mathbf{k},\omega)$ is defined with a bare phase-space DOS $D(\mathbf{k})=1$, while the BC contribution $\delta A(\mathbf{k},\omega)$ is obtained by replacing $D(\mathbf{k})$ with $\delta D_1(\mathbf{k})$ in Eq. 17b. The magnitude of A_0 reaches its peaks at frequencies along the lower superconducting band $\omega=E_{s=-1}(\mathbf{k})$ (see Figs. 3 (a)). Instead of shifting the positions of these peaks, we find that the BC contribution δA modulates the peak intensities in A_0 in a momentum-dependent way (see Fig. 3(b)). Specifically, the peak intensities are substantially enhanced at small momenta, but suppressed at large momenta for the parameter space we investigate. As in the case of tunneling conductance, since $\delta D_1\propto B$, this BC effect δA grows linearly with the external magnetic field B until the local approximation breaks down.

Furthermore, we find that this qualitative BC effect $\delta A(\mathbf{k},\omega)$ in the spectral function is sensitive to the Rashba SOC strength α in a momentum-dependent way, where the α-dependence implicitly enters δD_1 through the coherence factors in ρ_i's in Eq. 16a. Specifically, we find that at $\mathbf{k}=0$ the BC correction to the phase space DOS $\delta D_1\propto\alpha^2$. Such an enhancement in the magnitude of δD_1 is evident in Fig. 4, where we show the BC contribution per unit magnetic field strength $\delta A(\mathbf{k},\omega)/B$ and the bare spectral function $A_0(\mathbf{k},\omega)$ along the lower superconducting band $\omega = E_{s=-}(k)$ at different α's. Their ratio $\delta A/A_0$ is exactly given by the phase-space DOS correction δD_1 (see the insets in Fig. 4), which can be shown using Eq. 17b. This explains the drastic enhancement in the BC effect $\delta A/A_0 \propto \alpha^2$ at zero momentum as we increase α. We thus expect that this consequence of BC in the spectral function could be tuned to be large enough for experimental detection by the magnetic field strength B and the SOC strength α.

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* liao2@kth.se
† yhsu2@nd.edu

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Unveiling Quasiparticle Berry Curvature Effects in the Spectroscopic Properties of a Chiral p-wave Superconductor
Supplemental Material

Yunxiang Liao[1,2] and Yi-Ting Hsu[3]

[1] *Department of Physics, KTH Royal Institute of Technology, SE-106 91 Stockholm, Sweden*
[2] *Condensed Matter Theory Center and Joint Quantum Institute, Department of Physics, University of Maryland, College Park, MD 20742, USA.**
[3] *Department of Physics, University of Notre Dame, South Bend, IN 46556 USA*†

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I. MODEL

We study a Rashba spin-orbit coupled two-dimensional electron gas (2DEG) sandwiched by an s-wave superconductor and a ferromagnetic insulator [S1, S2], in the presence of an applied out-of-plane magnetic field $\mathbf{B} = B\hat{z}$. It is governed by the Hamiltonian:

$$\begin{aligned} H = &\sum_{\sigma=\uparrow,\downarrow} \int d^2\mathbf{r} c^\dagger_\sigma(\mathbf{r}) \left[\frac{1}{2m} \left(-i\nabla - \mathbf{A}(\mathbf{r})\right)^2 - \mu + \alpha \boldsymbol{\sigma} \times \left(-i\nabla - \mathbf{A}(\mathbf{r})\right) \cdot \hat{z} + h\sigma_z \right] c_\sigma(\mathbf{r}) \\ &+ \int d^2\mathbf{r} \Delta c^\dagger_\uparrow(\mathbf{r}) c^\dagger_\downarrow(\mathbf{r}) + \int d^2\mathbf{r} \Delta^* c_\downarrow(\mathbf{r}) c_\uparrow(\mathbf{r}). \end{aligned} \tag{S1}$$

Here $c_\sigma(\mathbf{r})$ indicates the annihilation operator for an electron at position $\mathbf{r}$ with spin $\sigma =\uparrow,\downarrow$ and mass m. μ denotes the chemical potential, and α indicates the Rashba spin-orbit coupling strength. Pauli matrices σ_i act in the spin space. $\mathbf{A}(\mathbf{r}) = \mathbf{B} \times \mathbf{r}/2$ is the vector potential of the applied magnetic field $\mathbf{B}$, and h is the Zeeman field results from the proximity to the ferromagnetic insulator. We ignore the orbital effect of the magnetic field from the ferromagnetic insulator [S2] as well as the Zeeman effect from the weak externally applied magnetic field, and therefore treat h and $\mathbf{B}$ as independent. Δ is the proximity-induced superconducting pairing potential. We work in units where the reduced Planck's constant $\hbar = 1$ and the electric charge $e = 1$.

The dynamics of superconducting quasiparticles in the current model driven by the applied magnetic field can be investigated using the wave-packet approach. The vector potential $\mathbf{A}(\mathbf{r})$ in the Hamiltonian above is assumed to be slowly varying in the spread of the wave packet. This allows us to study the wave packet dynamics by approximating the vector potential $A(\mathbf{r})$ by its value at the wave-packet center $\mathbf{r}_c$, i.e., $\mathbf{A}(\mathbf{r}_c)$. The resulting effective Hamiltonian, which governs the wave packet dynamics, can be further simplified by transforming to the momentum space:

$$c^\dagger_\sigma(\mathbf{k}) = \int d^2\mathbf{r} e^{i(\mathbf{k}+\mathbf{A}(\mathbf{r}_c))\cdot\mathbf{r}} c^\dagger_\sigma(\mathbf{r}), \tag{S2}$$

and introducing the Nambu spinor

$$\Psi(\mathbf{k}) = \begin{bmatrix} c_\uparrow(\mathbf{k}) & c_\downarrow(\mathbf{k}) & c^\dagger_\downarrow(-\mathbf{k}) & -c^\dagger_\uparrow(-\mathbf{k}) \end{bmatrix}^T. \tag{S3}$$

After the transformation, the approximated Hamiltonian assumes the form

$$\begin{aligned} H = &\frac{1}{2} \sum_{\sigma=\uparrow,\downarrow} \int \frac{d^2\mathbf{k}}{(2\pi)^2} \Psi^\dagger(\mathbf{k}) h_{\mathsf{BdG}}(\mathbf{k}) \Psi(\mathbf{k}), \\ h_{\mathsf{BdG}}(\mathbf{k}) = &\left[\xi_k + \alpha(\sigma_x k_y - \sigma_y k_x)\right] \tau_z + h\sigma_z + \frac{1}{2}\Delta(\tau_x + i\tau_y) + \frac{1}{2}\Delta^*(\tau_x - i\tau_y). \end{aligned} \tag{S4}$$

Here $\xi_k = k^2/2m - \mu$ and τ_i denotes the Pauli matrix acting in the Nambu space.

This Hamiltonian can be diagonalized by applying the Bogoliubov–de Gennes (BdG) transformation:

$$\gamma_s(\mathbf{k}) = \Phi^\dagger_s(\mathbf{k}) \Psi(\mathbf{k}), \qquad s = \pm 1. \tag{S5}$$

Here $\Phi_s(\mathbf{k}) = \left[u_{\uparrow s}(\mathbf{k}) \;\; u_{\downarrow s}(\mathbf{k}) \;\; v_{\downarrow s}(\mathbf{k}) \;\; v_{\uparrow s}(\mathbf{k})\right]^T$ is the eigenfunction of $h_{\mathsf{BdG}}(\mathbf{k})$ with positive eigenenergy $E_s(k)$ and it satisfies

$$h_{\mathsf{BdG}}(\mathbf{k})\Phi_s(\mathbf{k}) = E_s(k)\Phi_s(\mathbf{k}). \tag{S6}$$

The transformed Hamiltonian assumes the form

$$H = \sum_{s=\pm 1} \int \frac{d^2\mathbf{k}}{(2\pi)^2} E_s(k)\gamma_s^\dagger(\mathbf{k})\gamma_s(\mathbf{k}) + \text{const.}. \tag{S7}$$

Solving the BdG equation Eq. S5, we find the quasiparticle energy

$$E_s(k) = \sqrt{(\alpha k)^2 + |\Delta|^2 + h^2 + \xi_k^2 + 2s\sqrt{\xi_k^2\left((\alpha k)^2 + h^2\right) + |\Delta|^2 h^2}}. \tag{S8}$$

For $\xi_k \neq h$ or $s \neq -1$, the BdG eigenfunction $\Phi_s(\mathbf{k})$ is given by

$$\begin{aligned}
u_{\uparrow s}(\mathbf{k}) &= D_s(k)\left(E_s(k)h\xi_k + s(E_s(k) + h + \xi_k)\sqrt{(\alpha k)^2\xi_k^2 + h^2\left(|\Delta|^2 + \xi_k^2\right)} + |\Delta|^2 h + h\xi_k^2 + \xi_k(\alpha^2 k^2 + h^2)\right) e^{i(\chi - \phi_{\mathbf{k}} + \pi/2)/2},\\
u_{\downarrow s}(\mathbf{k}) &= D_s(k)\alpha k\left(\xi_k(\xi_k + E_s(k)) + s\sqrt{(\alpha k)^2\xi_k^2 + h^2\left(|\Delta|^2 + \xi_k^2\right)}\right) e^{i(\chi + \phi_{\mathbf{k}} - \pi/2)/2},\\
v_{\downarrow s}(\mathbf{k}) &= D_s(k)|\Delta|\left(h(h + E_s(k)) + s\sqrt{(\alpha k)^2\xi_k^2 + h^2\left(|\Delta|^2 + \xi_k^2\right)}\right) e^{-i(\chi + \phi_{\mathbf{k}} - \pi/2)/2},\\
v_{\uparrow s}(\mathbf{k}) &= D_s(k)|\Delta|\alpha k(\xi_k - h)e^{-i(\chi - \phi_{\mathbf{k}} + \pi/2)/2}.
\end{aligned} \tag{S9}$$

Here $D_s(k)$ is the normalization constant, $\chi \equiv \arg(\Delta)$ indicates the phase of the superconducting gap Δ, and $\phi_{\mathbf{k}} \equiv \arctan(k_y/k_x)$ denotes the polar angle of the momentum $\mathbf{k}$. When $\xi_k = h$ and $s = -1$, the BdG eigenfunction is no longer described by Eq. S9 and instead takes the form:

$$\begin{aligned}
u_{\uparrow,-1}(\mathbf{k}) &= 0, & u_{\downarrow,-1}(\mathbf{k}) &= D_{-1}(k)|\Delta|e^{i(\chi + \phi_{\mathbf{k}} - \pi/2)/2},\\
v_{\downarrow,-1}(\mathbf{k}) &= -D_{-1}(k)\alpha k e^{-i(\chi + \phi_{\mathbf{k}} - \pi/2)/2}, & v_{\uparrow,-1}(\mathbf{k}) &= D_{-1}(k)E_{-1}(k)e^{-i(\chi - \phi_{\mathbf{k}} + \pi/2)/2}.
\end{aligned} \tag{S10}$$

II. WAVE PACKET

We construct a quasiparticle wave packet $|W_s\rangle$ by applying the quasiparticle creation operator $\gamma_s^\dagger(\mathbf{k})$ to the ground state $|G\rangle$ [S3, S4]:

$$|W_s\rangle = \int \frac{d^2\mathbf{k}}{(2\pi)^2} w_s(\mathbf{k}, t)\gamma_s^\dagger(\mathbf{k})\,|G\rangle\,. \tag{S11}$$

Here the superconducting ground state $|G\rangle$ satisfies $\gamma_{s'}(\mathbf{k})\,|G\rangle = 0$ for all $\mathbf{k}$ and s'. Note that we consider weak enough perturbation and concentrate only on the contribution from the lower band $s = -1$ which intersects with the chemical potential. The envelop function $w_s(\mathbf{k}, t)$ satisfies the normalization condition

$$\int \frac{d^2\mathbf{k}}{(2\pi)^2}|w_s(\mathbf{k}, t)|^2 = 1, \tag{S12}$$

and is sharply peaked around the wave-packet center $\mathbf{k}_c$ defined by

$$\mathbf{k}_c = \int \frac{d^2\mathbf{k}}{(2\pi)^2}|w_s(\mathbf{k}, t)|^2\mathbf{k}. \tag{S13}$$

The integral of any smooth function $f(\mathbf{k})$ weighted by $|w_s(\mathbf{k}, t)|^2$ can therefore be approximated by the value of $f(\mathbf{k})$ at $\mathbf{k} = \mathbf{k}_c$:

$$\int \frac{d^2\mathbf{k}}{(2\pi)^2}|w_s(\mathbf{k}, t)|^2 f(\mathbf{k}) \approx f(\mathbf{k}_c). \tag{S14}$$

The wave-packet center coordinate in the real space is defined as [S5]

$$\begin{aligned}\mathbf{r}_c =& i\int_{\mathbf{k}} \left(w_s(\mathbf{k},t)\Phi_s(\mathbf{k})\right)^\dagger \partial_{\mathbf{k}} \left(w_s(\mathbf{k},t)\Phi_s(\mathbf{k})\right)\\ =& \partial_{\mathbf{k}_c}\theta_w(\mathbf{k}_c,t) + i\sum_\sigma \left(u^*_{\sigma s}(\mathbf{k}_c)\partial_{\mathbf{k}_c}u_{\sigma s}(\mathbf{k}_c) + v^*_{\sigma s}(\mathbf{k}_c)\partial_{\mathbf{k}_c}v_{\sigma s}(\mathbf{k}_c)\right).\end{aligned} \tag{S15}$$

Here, θ_w stands for the phase of the envelope function $\theta_w(\mathbf{k},t) \equiv -\arg w_s(\mathbf{k},t)$.

We then introduce the moment-space Berry connection $\mathcal{A}_{ks}$ for the BdG wavefunction $\Phi_s(\mathbf{k}_c)$:

$$\mathcal{A}_{ks} \equiv i\Phi^\dagger_s(\mathbf{k}_c)\partial_{\mathbf{k}_c}\Phi_s(\mathbf{k}_c) = i\sum_\sigma \left(u^*_{\sigma s}(\mathbf{k}_c)\partial_{\mathbf{k}_c}u_{\sigma s}(\mathbf{k}_c) + v^*_{\sigma s}(\mathbf{k}_c)\partial_{\mathbf{k}_c}v_{\sigma s}(\mathbf{k}_c)\right). \tag{S16}$$

Using Eq. S9, we find that $\mathcal{A}_{ks}$ can be rewritten as

$$\mathcal{A}_{ks} = -\frac{1}{2}\rho_{0s}(\mathbf{k}_c)\partial_{\mathbf{k}_c}\chi + \frac{1}{2}\rho_{1s}(\mathbf{k}_c)\partial_{\mathbf{k}_c}\phi_{\mathbf{k}_c}. \tag{S17}$$

Here $\rho_{0s}(\mathbf{k})$ and $\rho_{1s}(\mathbf{k})$ are defined as

$$\rho_{0s}(\mathbf{k}) \equiv \sum_\sigma \left(|u_{\sigma s}(\mathbf{k})|^2 - |v_{\sigma s}(\mathbf{k})|^2\right), \qquad \rho_{1s}(\mathbf{k}) \equiv \sum_\sigma \zeta_\sigma \left(|u_{\sigma s}(\mathbf{k})|^2 - |v_{\sigma s}(\mathbf{k})|^2\right), \tag{S18}$$

with $\zeta_\uparrow = -\zeta_\downarrow = 1$. Their physical meanings will become apparent below. Making use of Eqs. S16 and S17, one can express the real-space wave-packet center coordinate $\mathbf{r}_c$ as

$$\mathbf{r}_c = \partial_{\mathbf{k}_c}\theta_w(\mathbf{k}_c,t) + \mathcal{A}_{ks} = \partial_{\mathbf{k}_c}\theta_w(\mathbf{k}_c,t) - \frac{1}{2}\rho_{0s}(\mathbf{k}_c)\partial_{\mathbf{k}_c}\chi + \frac{1}{2}\rho_{1s}(\mathbf{k}_c)\partial_{\mathbf{k}_c}\phi_{\mathbf{k}_c}. \tag{S19}$$

Let us now consider the wave-packet average of the total charge Q, total spin S, and the charge dipole moment $\mathbf{d}$ defined as

$$\begin{aligned}&O = \langle W_s|\hat{O}|W_s\rangle - \langle\Omega|\hat{O}|\Omega\rangle, \qquad O = Q, S, \mathbf{d},\\ &\hat{O} = \sum_\sigma \int d\mathbf{r}\, g_O(\mathbf{r})c^\dagger_\sigma(\mathbf{r})c_\sigma(\mathbf{r}), \qquad g_O(\mathbf{r}) = \begin{cases}1, & O = Q,\\ \zeta_\sigma/2, & O = S,\\ \mathbf{r}-\mathbf{r}_c, & O = \mathbf{d}.\end{cases}\end{aligned} \tag{S20}$$

In terms of the quasiparticle operator $\gamma_s(\mathbf{k})$, $\hat{O} = \hat{Q}, \hat{S}, \hat{\mathbf{d}}$ can be rewritten as

$$\hat{O} = \sum_\sigma\sum_{s_1,s_2}\int_{\mathbf{r}}\int_{\mathbf{k}_1,\mathbf{k}_2} g_O(\mathbf{r})e^{-i(\mathbf{k}_1-\mathbf{k}_2)\cdot\mathbf{r}}\left[u^*_{\sigma s_1}(\mathbf{k}_1)u_{\sigma s_2}(\mathbf{k}_2)\gamma^\dagger_{s_1}(\mathbf{k}_1)\gamma_{s_2}(\mathbf{k}_2) + v_{\sigma s_1}(-\mathbf{k}_1)v^*_{\sigma s_2}(-\mathbf{k}_2)\gamma_{s_1}(-\mathbf{k}_1)\gamma^\dagger_{s_2}(-\mathbf{k}_2)\right] + ..., \tag{S21}$$

where ... represents terms involving $\gamma\gamma$ or $\gamma^\dagger\gamma^\dagger$ whose contribution vanishes after taking the expectation value. For brevity, we have introduced the shorthand notation $\int_{\mathbf{k}} \equiv \int d^2\mathbf{k}/(2\pi)^2$.

We then make use of the following identities

$$\begin{aligned}\langle W_s|\gamma^\dagger_{s_1}(\mathbf{k}_1)\gamma_{s_2}(\mathbf{k}_2)|W_s\rangle =& \int_{\mathbf{k}'_1,\mathbf{k}'_2} w^*_s(\mathbf{k}'_1,t)w_s(\mathbf{k}'_2,t)\langle\Omega|\gamma_s(\mathbf{k}'_1)\gamma^\dagger_{s_1}(\mathbf{k}_1)\gamma_{s_2}(\mathbf{k}_2)\gamma^\dagger_s(\mathbf{k}'_2)|\Omega\rangle\\ =& w^*_s(\mathbf{k}_1,t)w_s(\mathbf{k}_2,t)\delta_{s,s_2}\delta_{s,s_1},\\ \langle W_s|\gamma_{s_1}(\mathbf{k}_1)\gamma^\dagger_{s_2}(\mathbf{k}_2)|W_s\rangle =& \int_{\mathbf{k}'_1,\mathbf{k}'_2} w^*_s(\mathbf{k}'_1,t)w_s(\mathbf{k}'_2,t)\langle\Omega|\gamma_s(\mathbf{k}'_1)\gamma_{s_1}(\mathbf{k}_1)\gamma^\dagger_{s_2}(\mathbf{k}_2)\gamma^\dagger_s(\mathbf{k}'_2)|\Omega\rangle\\ =& \delta_{\mathbf{k}_1,\mathbf{k}_2}\delta_{s_1,s_2} - \delta_{s,s_1}\delta_{s,s_2}w^*_s(\mathbf{k}_2,t)w_s(\mathbf{k}_1,t),\end{aligned} \tag{S22}$$

where the normalization condition Eq. S12 has been used. This leads to

$$\begin{aligned}\langle W_s|\hat{O}|W_s\rangle - \langle G|\hat{O}|G\rangle =& \sum_\sigma\int_{\mathbf{r}}\int_{\mathbf{k}_1,\mathbf{k}_2} e^{-i(\mathbf{k}_1-\mathbf{k}_2)\cdot\mathbf{r}}g_O(\mathbf{r})\\ &\times\left[w^*_s(\mathbf{k}_1,t)w_s(\mathbf{k}_2,t)u^*_{\sigma s}(\mathbf{k}_1)u_{\sigma s}(\mathbf{k}_2) - w^*_s(-\mathbf{k}_2,t)w_s(-\mathbf{k}_1,t)v_{\sigma s}(-\mathbf{k}_1)v^*_{\sigma s}(-\mathbf{k}_2)\right].\end{aligned} \tag{S23}$$

For $\hat{O} = \hat{Q}$ or $\hat{S}$, it is easy to see that the above equation can be further simplified to

$$Q = \int_{\mathbf{k}} |w_s(\mathbf{k},t)|^2 \rho_{0s}(\mathbf{k}) = \rho_{0s}(\mathbf{k}_c), \qquad S = \frac{1}{2}\int_{\mathbf{k}} |w_s(\mathbf{k},t)|^2 \rho_{1s}(\mathbf{k}) = \frac{1}{2}\rho_{1s}(\mathbf{k}_c), \tag{S24}$$

where ρ_{0s} and ρ_{1s} have been defined in Eq. S18.

For the charge dipole moment $\mathbf{d}$, one can express $\mathbf{r}e^{-i(\mathbf{k}_1-\mathbf{k}_2)\cdot\mathbf{r}}$ in the Eq. S23 as $-i\partial_{\mathbf{k}_2}e^{-i(\mathbf{k}_1-\mathbf{k}_2)\cdot\mathbf{r}}$, which leads to

$$\begin{aligned}
\langle W_s|\,\hat{\mathbf{d}}\,|W_s\rangle - \langle\Omega|\,\hat{\mathbf{d}}\,|\Omega\rangle =& \sum_\sigma \int_{\mathbf{r}}\int_{\mathbf{k}_1,\mathbf{k}_2} \left(-i\partial_{\mathbf{k}_2}e^{-i(\mathbf{k}_1-\mathbf{k}_2)\cdot\mathbf{r}}\right) w_s^*(\mathbf{k}_1,t)w_s(\mathbf{k}_2,t)\left[u^*_{\sigma s}(\mathbf{k}_1)u_{\sigma s}(\mathbf{k}_2) - v^*_{\sigma s}(\mathbf{k}_1)v_{\sigma s}(\mathbf{k}_2)\right] - \mathbf{r}_c Q\\
=& i\sum_\sigma \int_{\mathbf{k}} w_s^*(\mathbf{k},t)(\partial_{\mathbf{k}}w_s(\mathbf{k},t))\left(|u_{\sigma s}(\mathbf{k})|^2 - |v_{\sigma s}(\mathbf{k})|^2\right) + i\sum_\sigma\int_{\mathbf{k}} |w_s(\mathbf{k},t)|^2\left(u^*_{\sigma s}(\mathbf{k})\partial_{\mathbf{k}}u_{\sigma s}(\mathbf{k}) - v^*_{\sigma s}(\mathbf{k})\partial_{\mathbf{k}}v_{\sigma s}(\mathbf{k})\right) - \mathbf{r}_c Q\\
=& \int_{\mathbf{k}} |w_s(\mathbf{k},t)|^2\left(\rho_{0s}(\mathbf{k})\partial_{\mathbf{k}}\theta_w(\mathbf{k},t) - \partial_{\mathbf{k}}\chi/2 + \rho_{2s}(\mathbf{k})\partial_{\mathbf{k}}\phi_{\mathbf{k}}/2\right) - \mathbf{r}_c Q\\
&+ \frac{i}{2}\sum_\sigma\int_{\mathbf{k}}(\partial_{\mathbf{k}}|w_s(\mathbf{k},t)|^2)\left(|u_{\sigma s}(\mathbf{k})|^2 - |v_{\sigma s}(\mathbf{k})|^2\right) + \frac{i}{2}\sum_\sigma\int_{\mathbf{k}}|w_s(\mathbf{k},t)|^2\partial_{\mathbf{k}}\left(|u_{\sigma s}(\mathbf{k})|^2 - |v_{\sigma s}(\mathbf{k})|^2\right)\\
=& \rho_{0s}(\mathbf{k}_c)\partial_{\mathbf{k}_c}\theta_w(\mathbf{k}_c,t) - \partial_{\mathbf{k}_c}\chi/2 + \rho_{2s}(\mathbf{k}_c)\partial_{\mathbf{k}_c}\phi_{\mathbf{k}_c}/2 - \mathbf{r}_c\rho_{0s}(\mathbf{k}_c),
\end{aligned} \tag{S25}$$

where

$$\rho_{2s}(\mathbf{k}) \equiv \sum_\sigma \zeta_\sigma\left(|u_{\sigma s}(\mathbf{k})|^2 + |v_{\sigma s}(\mathbf{k})|^2\right). \tag{S26}$$

Inserting the expression for $\mathbf{r}_c$ in Eq. S19, we find the wave packet average of the charge dipole moment is given by

$$\mathbf{d} = \frac{1}{2}\left(\rho^2_{0s}(\mathbf{k}_c) - 1\right)\partial_{\mathbf{k}_c}\chi + \frac{1}{2}\left(\rho_{2s}(\mathbf{k}_c) - \rho_{0s}(\mathbf{k}_c)\rho_{1s}(\mathbf{k}_c)\right)\partial_{\mathbf{k}_c}\phi_{\mathbf{k}_c}. \tag{S27}$$

III. SEMICLASSICAL DYNAMICS

A. Lagrangian

To study the semiclassical dynamics of the wave packet, we first evaluate the Lagrangian

$$L = \langle W_s|\left(i\frac{d}{dt} - H\right)|W_s\rangle - \langle\Omega|\left(i\frac{d}{dt} - H\right)|\Omega\rangle. \tag{S28}$$

The part which contains the Hamiltonian is given by

$$\begin{aligned}
\langle W_s|\,H\,|W_s\rangle - \langle\Omega|\,H\,|\Omega\rangle =& \int_{\mathbf{k}_1,\mathbf{k}_2} w_s^*(\mathbf{k}_1,t)w_s(\mathbf{k}_2,t)\,\langle\Omega|\,\gamma_s(\mathbf{k}_1)\left(\int_{\mathbf{k}}\sum_{s'}E_{s'}(k)\gamma^\dagger_{s'}(\mathbf{k})\gamma_{s'}(\mathbf{k})\right)\gamma_s^\dagger(\mathbf{k}_2)\,|\Omega\rangle\\
=& \int_{\mathbf{k}}|w_s(\mathbf{k},t)|^2E_s(k) = E_s(k_c),
\end{aligned} \tag{S29}$$

while the part which involves the time derivative reduces to

$$\begin{aligned}
\langle W_s|\,i\frac{d}{dt}\,|W_s\rangle =& i\int_{\mathbf{k}} w_s^*(\mathbf{k},t)\frac{\partial}{\partial t}w_s(\mathbf{k},t) + i\frac{d\mathbf{r}_c}{dt}\cdot\int_{\mathbf{k}} w_s^*(\mathbf{k},t)\frac{\partial}{\partial\mathbf{r}_c}w_s(\mathbf{k},t)\\
&+ i\frac{d\mathbf{r}_c}{dt}\cdot\int_{\mathbf{k}_1,\mathbf{k}_2} w_s^*(\mathbf{k}_1,t)w_s(\mathbf{k}_2,t)\,\langle\Omega|\,\gamma_s(\mathbf{k}_1)(\frac{\partial}{\partial\mathbf{r}_c}\gamma_s^\dagger(\mathbf{k}_2))\,|\Omega\rangle.
\end{aligned} \tag{S30}$$

Note that the time dependence of the quasiparticle creation operator $\gamma_s^\dagger$ comes from the time dependence of the wave-packet center $\mathbf{r}_c$.

Using the BdG transformation Eq. S5, the term $\langle W_s| i\frac{d}{dt} |W_s\rangle$ can be further divided into three parts:

$$
\begin{aligned}
&\langle W_s| i\frac{d}{dt} |W_s\rangle = L_1 + L_2 + L_3, \\
&L_1 = i\int_{\mathbf{k}} w_s^*(\mathbf{k},t)\frac{\partial}{\partial t}w_s(\mathbf{k},t) + i\frac{d\mathbf{r}_c}{dt}\cdot\int_{\mathbf{k}} w_s^*(\mathbf{k},t)\frac{\partial}{\partial \mathbf{r}_c}w_s(\mathbf{k},t) \\
&L_2 = i\frac{d\mathbf{r}_c}{dt}\cdot\int_{\mathbf{k}_1,\mathbf{k}_2} w_s^*(\mathbf{k}_1,t)w_s(\mathbf{k}_2,t) \\
&\times \langle\Omega|\gamma_s(\mathbf{k}_1)\left[(\frac{\partial}{\partial \mathbf{r}_c}u_{\uparrow s}(\mathbf{k}_2))c_\uparrow^\dagger(\mathbf{k}_2) + (\frac{\partial}{\partial \mathbf{r}_c}u_{\downarrow s}(\mathbf{k}_2))c_\downarrow^\dagger(\mathbf{k}_2) + (\frac{\partial}{\partial \mathbf{r}_c}v_{\downarrow s}(\mathbf{k}_2))c_\downarrow(-\mathbf{k}_2) - (\frac{\partial}{\partial \mathbf{r}_c}v_{\uparrow s}(\mathbf{k}_2))c_\uparrow(-\mathbf{k}_2)\right]|\Omega\rangle, \\
&L_3 = i\frac{d\mathbf{r}_c}{dt}\cdot\int_{\mathbf{k}_1,\mathbf{k}_2} w_s^*(\mathbf{k}_1,t)w_s(\mathbf{k}_2,t) \\
&\times \langle\Omega|\gamma_s(\mathbf{k}_1)\left[u_{\uparrow s}(\mathbf{k}_2)(\frac{\partial}{\partial \mathbf{r}_c}c_\uparrow^\dagger(\mathbf{k}_2)) + u_{\downarrow s}(\mathbf{k}_2)(\frac{\partial}{\partial \mathbf{r}_c}c_\downarrow^\dagger(\mathbf{k}_2)) + v_{\downarrow s}(\mathbf{k}_2)(\frac{\partial}{\partial \mathbf{r}_c}c_\downarrow(-\mathbf{k}_2)) - v_{\uparrow s}(\mathbf{k}_2)(\frac{\partial}{\partial \mathbf{r}_c}c_\uparrow(-\mathbf{k}_2))\right]|\Omega\rangle.
\end{aligned}
\tag{S31}
$$

Here L_1 can be expressed in terms of the phase of the envelope function $w_s(\mathbf{k},t)$ at the wave packet center $\mathbf{k}=\mathbf{k}_c$, i.e., $\theta_w(\mathbf{k}_c,t)$,

$$
L_1 = \int_{\mathbf{k}} |w_s(\mathbf{k},t)|^2(\frac{\partial}{\partial t}\theta_w(\mathbf{k},t) + \frac{d\mathbf{r}_c}{dt}\cdot\frac{\partial}{\partial \mathbf{r}_c}\theta_w(\mathbf{k},t)) = \frac{\partial}{\partial t}\theta_w(\mathbf{k}_c,t) + \frac{d\mathbf{r}_c}{dt}\cdot\frac{\partial}{\partial \mathbf{r}_c}\theta_w(\mathbf{k}_c,t), \tag{S32}
$$

Using the expression for the wave-packet center $\mathbf{r}_c$ in Eq. S19, one can rewrite this equation as

$$
L_1 = \frac{d}{dt}\theta_w(\mathbf{k}_c,t) - \frac{d\mathbf{k}_c}{dt}\cdot\frac{\partial}{\partial \mathbf{k}_c}\theta_w(\mathbf{k}_c,t) = \frac{d}{dt}\theta_w(\mathbf{k}_c,t) - \dot{\mathbf{k}}_c\cdot(\mathbf{r}_c - \mathcal{A}_{ks}). \tag{S33}
$$

Converting the electron operator c_σ into the quasiparticle operator γ_s in L_2, we obtain

$$
\begin{aligned}
L_2 = i\frac{d\mathbf{r}_c}{dt}\cdot\int_{\mathbf{k}_1,\mathbf{k}_2} w_s^*(\mathbf{k}_1,t)w_s(\mathbf{k}_2,t)\langle\Omega|\gamma_s(\mathbf{k}_1)
&\begin{bmatrix}
\left(\frac{\partial}{\partial \mathbf{r}_c}u_{\uparrow s}(\mathbf{k}_2)\right)\sum_{s'}\left(u_{\uparrow s'}^*(\mathbf{k}_2)\gamma_{s'}^\dagger(\mathbf{k}_2) - v_{\uparrow s'}(-\mathbf{k}_2)\gamma_{s'}(-\mathbf{k}_2)\right) \\
+\left(\frac{\partial}{\partial \mathbf{r}_c}u_{\downarrow s}(\mathbf{k}_2)\right)\sum_{s'}\left(u_{\downarrow s'}^*(\mathbf{k}_2)\gamma_{s'}^\dagger(\mathbf{k}_2) + v_{\downarrow s'}(-\mathbf{k}_2)\gamma_{s'}(-\mathbf{k}_2)\right) \\
+\left(\frac{\partial}{\partial \mathbf{r}_c}v_{\downarrow s}(\mathbf{k}_2)\right)\sum_{s'}\left(v_{\downarrow s'}^*(\mathbf{k}_2)\gamma_{s'}^\dagger(\mathbf{k}_2) + u_{\downarrow s'}(-\mathbf{k}_2)\gamma_{s'}(-\mathbf{k}_2)\right) \\
+\left(\frac{\partial}{\partial \mathbf{r}_c}v_{\uparrow s}(\mathbf{k}_2)\right)\sum_{s'}\left(v_{\uparrow s'}^*(\mathbf{k}_2)\gamma_{s'}^\dagger(\mathbf{k}_2) - u_{\uparrow s'}(-\mathbf{k}_2)\gamma_{s'}(-\mathbf{k}_2)\right)
\end{bmatrix}|\Omega\rangle \\
&= i\frac{d\mathbf{r}_c}{dt}\cdot\int_{\mathbf{k}}|w_s(\mathbf{k},t)|^2\sum_\sigma\left(u_{\sigma s}^*(\mathbf{k})\frac{\partial}{\partial \mathbf{r}_c}u_{\sigma s}(\mathbf{k}) + v_{\sigma s}^*(\mathbf{k})\frac{\partial}{\partial \mathbf{r}_c}v_{\sigma s}(\mathbf{k})\right) \\
&= \dot{\mathbf{r}}_c\cdot\mathcal{A}_{rs}.
\end{aligned}
\tag{S34}
$$

Here $\mathcal{A}_{rs}$ represents the real-space Berry connection for the BdG wavefunction $\Phi_s(\mathbf{k}_c)$, and is defined analogous to its momentum-space counterpart $\mathcal{A}_{ks}$:

$$
\mathcal{A}_{rs} \equiv i\Phi_s^\dagger(\mathbf{k}_c)\partial_{\mathbf{r}_c}\Phi_s(\mathbf{k}_c) = i\sum_\sigma\left(u_{\sigma s}^*(\mathbf{k}_c)\partial_{\mathbf{r}_c}u_{\sigma s}(\mathbf{k}_c) + v_{\sigma s}^*(\mathbf{k}_c)\partial_{\mathbf{r}_c}v_{\sigma s}(\mathbf{k}_c)\right). \tag{S35}
$$

It is straightforward to see from Eq. S17 that this equation can be rewritten as

$$
\mathcal{A}_{rs} = -\frac{1}{2}\rho_{0s}(\mathbf{k}_c)\partial_{\mathbf{r}_c}\chi + \frac{1}{2}\rho_{1s}(\mathbf{k}_c)\partial_{\mathbf{r}_c}\phi_{\mathbf{k}_c}. \tag{S36}
$$

For the evaluation of L_3, note that

$$
\begin{aligned}
\frac{d\mathbf{r}_c}{dt}\cdot\frac{\partial}{\partial \mathbf{r}_c}c_\sigma(\pm\mathbf{k}) &= \frac{d\mathbf{r}_c}{dt}\cdot\frac{\partial}{\partial \mathbf{r}_c}\left(\int d^2\mathbf{r}e^{-i(\pm\mathbf{k}+\mathbf{A}(\mathbf{r}_c))\cdot\mathbf{r}}c_\sigma(\mathbf{r})\right) = -i\frac{d\mathbf{r}_c}{dt}\cdot\int_{\mathbf{r}}\frac{\partial\mathbf{A}(\mathbf{r}_c)\cdot\mathbf{r}}{\partial \mathbf{r}_c}e^{-i(\pm\mathbf{k}+\mathbf{A}(\mathbf{r}_c))\cdot\mathbf{r}}c_\sigma(\mathbf{r}) \\
&= \frac{d\mathbf{A}(\mathbf{r}_c)}{dt}\cdot\int d^2\mathbf{r}\left(\pm\frac{\partial}{\partial \mathbf{k}}e^{-i(\pm\mathbf{k}+\mathbf{A}(\mathbf{r}_c))\cdot\mathbf{r}}\right)c_\sigma(\mathbf{r}) = \pm\frac{d\mathbf{A}(\mathbf{r}_c)}{dt}\cdot\frac{\partial}{\partial \mathbf{k}}c_\sigma(\pm\mathbf{k}),
\end{aligned}
\tag{S37}
$$

where we have used Eq. S2. This leads to

$$\begin{aligned}
L_3 =& i\frac{d}{dt}\mathbf{A}(\mathbf{r}_c)\cdot\int_{\mathbf{k}_1,\mathbf{k}_2} w_s^*(\mathbf{k}_1,t)w_s(\mathbf{k}_2,t)\\
&\times\langle\Omega|\,\gamma_s(\mathbf{k}_1)\left[u_{\uparrow s}(\mathbf{k}_2)(\frac{\partial}{\partial\mathbf{k}_2}c_\uparrow^\dagger(\mathbf{k}_2))+u_{\downarrow s}(\mathbf{k}_2)(\frac{\partial}{\partial\mathbf{k}_2}c_\downarrow^\dagger(\mathbf{k}_2))-v_{\downarrow s}(\mathbf{k}_2)(\frac{\partial}{\partial\mathbf{k}_2}c_\downarrow(-\mathbf{k}_2))+v_{\uparrow s}(\mathbf{k}_2)(\frac{\partial}{\partial\mathbf{k}_2}c_\uparrow(-\mathbf{k}_2))\right]|\Omega\rangle\\
=&-i\frac{d}{dt}\mathbf{A}(\mathbf{r}_c)\cdot\int_{\mathbf{k}} w_s^*(\mathbf{k},t)\left(\partial_{\mathbf{k}}w_s(\mathbf{k},t)\right)\left(|u_{\sigma s}(\mathbf{k})|^2-|v_{\sigma s}(\mathbf{k})|^2\right)\\
&-i\frac{d}{dt}\mathbf{A}(\mathbf{r}_c)\cdot\int_{\mathbf{k}}|w_s(\mathbf{k},t)|^2\left(u_{\sigma s}^*(\mathbf{k})\partial_{\mathbf{k}}u_{\sigma s}(\mathbf{k})-v_{\sigma s}^*(\mathbf{k})\partial_{\mathbf{k}}v_{\sigma s}(\mathbf{k})\right).
\end{aligned} \tag{S38}$$

Comparing this equation with the expression for the electric dipole moment $\mathbf{d}$ (See Eqs. S25 and S27), it is easy to see that L_3 can be expressed as

$$L_3 = -\dot{\mathbf{A}}(\mathbf{r}_c)\cdot(\rho_{0s}(\mathbf{k}_c)\mathbf{r}_c+\mathbf{d}). \tag{S39}$$

Combining everything, we obtain

$$L = -E_s(k_c)-\dot{\mathbf{k}}_c\cdot(\mathbf{r}_c-\mathcal{A}_{ks})+\dot{\mathbf{r}}_c\cdot\mathcal{A}_{rs}-\dot{\mathbf{A}}(\mathbf{r}_c)\cdot(\rho_{0s}(\mathbf{k}_c)\mathbf{r}_c+\mathbf{d}). \tag{S40}$$

Here we have dropped terms which can be expressed as total time derivatives and as a result do not enter the equation of motion. Using $\mathbf{A}(\mathbf{r}_c)=\frac{1}{2}\mathbf{B}\times\mathbf{r}_c$, we find that the Lagrangian can rewritten as

$$L = -E_s(k_c)-\dot{\mathbf{k}}_c\cdot\mathbf{r}_c+\dot{\mathbf{k}}_c\cdot\mathcal{A}_{ks}+\dot{\mathbf{r}}_c\cdot\tilde{\mathcal{A}}_{rs}. \tag{S41}$$

Here the new real-space Berry connection $\tilde{\mathcal{A}}_{rs}$ is defined as

$$\tilde{\mathcal{A}}_{rs}\equiv\mathcal{A}_{rs}+\rho_{0s}(\mathbf{k}_c)\mathbf{A}(\mathbf{r}_c)+\frac{1}{2}\mathbf{B}\times\mathbf{d}. \tag{S42}$$

B. Equation of motion

Inserting the expression for the Lagrangian in Eq. S41 into the Euler–Lagrange equation

$$\frac{\partial L}{\partial r_c^i}=\frac{d}{dt}\frac{\partial L}{\partial\dot{r}_c^i},\qquad \frac{\partial L}{\partial k_c^i}=\frac{d}{dt}\frac{\partial L}{\partial\dot{k}_c^i},\qquad i=x,y, \tag{S43}$$

one can derive the equation of motion for the wave-packet center:

$$\begin{aligned}
\dot{k}_c^i =& -\partial_{r_c^i}E_s(k_c)+\Omega_{rks}^{ij}\dot{k}_c^j+\Omega_{rrs}^{ij}\dot{r}_c^j,\\
\dot{r}_c^i =& \partial_{k_c^i}E_s(k_c)-\Omega_{kks}^{ij}\dot{k}_c^j-\Omega_{krs}^{ij}\dot{r}_c^j.
\end{aligned} \tag{S44}$$

Here we have employed the notation that repeated indices implies summation. The Berry curvatures Ω are defined as derivatives of the Berry connections $\mathcal{A}_{ks}$ and $\tilde{\mathcal{A}}_{rs}$:

$$\begin{aligned}
&\Omega_{kks}^{ij}\equiv\partial_{k_c^i}\mathcal{A}_{ks}^j-\partial_{k_c^j}\mathcal{A}_{ks}^i,\qquad \Omega_{rrs}^{ij}\equiv\partial_{r_c^i}\tilde{\mathcal{A}}_{rs}^j-\partial_{r_c^j}\tilde{\mathcal{A}}_{rs}^i,\\
&\Omega_{rks}^{ij}\equiv-\Omega_{krs}^{ji}\equiv\partial_{r_c^i}\mathcal{A}_{ks}^j-\partial_{k_c^j}\tilde{\mathcal{A}}_{rs}^i.
\end{aligned} \tag{S45}$$

The equation of motion Eq. S44 can be rewritten in a vector form:

$$\begin{aligned}
\dot{\mathbf{k}}_c =& -\nabla_{\mathbf{r}_c}E_s(k_c)+\dot{\mathbf{r}}_c\times\mathbf{\Omega}_{rs}+\nabla_{\mathbf{r}_c}\left(\dot{\mathbf{k}}_c\cdot\mathcal{A}_{ks}\right)-\left(\dot{\mathbf{k}}_c\cdot\nabla_{\mathbf{k}_c}\right)\tilde{\mathcal{A}}_{rs},\\
\dot{\mathbf{r}}_c =& \nabla_{\mathbf{k}_c}E_s(k_c)-\dot{\mathbf{k}}_c\times\mathbf{\Omega}_{ks}-\nabla_{\mathbf{k}_c}\left(\dot{\mathbf{r}}_c\cdot\tilde{\mathcal{A}}_{rs}\right)+\left(\dot{\mathbf{r}}_c\cdot\nabla_{\mathbf{r}_c}\right)\mathcal{A}_{ks}.
\end{aligned} \tag{S46}$$

Here the vector forms of the Berry curvatures $\mathbf{\Omega}_{rs}$ and $\mathbf{\Omega}_{ks}$ are defined as

$$\mathbf{\Omega}_{ks}=\nabla_{\mathbf{k}_c}\times\mathcal{A}_{ks},\qquad \mathbf{\Omega}_{rs}=\nabla_{\mathbf{r}_c}\times\tilde{\mathcal{A}}_{rs}, \tag{S47}$$

and they are related to their tensor counterparts Ω_{rrs} and Ω_{kks} by:

$$\mathbf{\Omega}^l_{rs/ks} = \frac{1}{2}\epsilon_{lij}\Omega^{ij}_{rrs/kks}, \qquad \Omega^{ij}_{rrs/kks} = \quad \epsilon_{ijl}\mathbf{\Omega}^l_{rs/ks}, \tag{S48}$$

with ϵ_{ijl} being the Levi-Civita symbol. Note that in the present paper, we consider a 2D system, and therefore only the z component of the vector Berry curvature $\mathbf{\Omega}$ is nonvanishing.

Using Eqs. S17, S36, and S42, we obtain

$$\begin{aligned} \mathbf{\Omega}_{ks} &= -\frac{1}{2}\nabla_{\mathbf{k}_c}\rho_{0s}(\mathbf{k}_c)\times\nabla_{\mathbf{k}_c}\chi + \frac{1}{2}\nabla_{\mathbf{k}_c}\rho_{1s}(\mathbf{k}_c)\times\nabla_{\mathbf{k}_c}\phi_{\mathbf{k}_c}, \\ \mathbf{\Omega}_{rs} &= -\nabla_{\mathbf{r}_c}\rho_{0s}(\mathbf{k}_c)\times(\nabla_{\mathbf{r}_c}\chi/2 - \mathbf{A}(\mathbf{r}_c)) + \rho_{0s}(\mathbf{k}_c)\mathbf{B} + \frac{1}{2}\nabla_{\mathbf{r}_c}\times(\mathbf{B}\times\mathbf{d}) + \frac{1}{2}\nabla_{\mathbf{r}_c}\rho_{1s}(\mathbf{k}_c)\times\nabla_{\mathbf{r}_c}\phi_{\mathbf{k}_c}. \end{aligned} \tag{S49}$$

The tensor-form of the phase-space Berry curvature Ω_{rk} is given by,

$$\begin{aligned} \Omega^{ij}_{rks} = &-\frac{1}{2}\left(\partial_{r^i_c}\rho_{0s}(\mathbf{k}_c)\partial_{k^j_c}\chi - \partial_{k^j_c}\rho_{0s}(\mathbf{k}_c)\partial_{r^i_c}\chi\right) + \frac{1}{2}\left(\partial_{r^i_c}\rho_{1s}(\mathbf{k}_c)\partial_{k^j_c}\phi_{\mathbf{k}_c} - \partial_{k^j_c}\rho_{1s}(\mathbf{k}_c)\partial_{r^i_c}\phi_{\mathbf{k}_c}\right) \\ &- \partial_{k^j_c}\rho_{0s}(\mathbf{k}_c)\mathbf{A}^i(\mathbf{r}_c) - \frac{1}{2}\partial_{k^j_c}(\mathbf{B}\times\mathbf{d})^i. \end{aligned} \tag{S50}$$

Inserting Eq. S49 into Eq. S46 leads to the following equation of motion

$$\begin{aligned} \dot{\mathbf{k}}_c = &-\nabla_{\mathbf{r}_c}E_s(k_c) + \dot{\mathbf{r}}_c\times\left[-\nabla_{\mathbf{r}_c}\rho_{0s}(\mathbf{k}_c)\times\left(\frac{1}{2}\nabla_{\mathbf{r}_c}\chi - \mathbf{A}(\mathbf{r}_c)\right) + \rho_{0s}(\mathbf{k}_c)\mathbf{B} + \frac{1}{2}\nabla_{\mathbf{r}_c}\times(\mathbf{B}\times\mathbf{d}) + \frac{1}{2}\nabla_{\mathbf{r}_c}\rho_{1s}(\mathbf{k}_c)\times\nabla_{\mathbf{r}_c}\phi_{\mathbf{k}_c}\right] \\ &-\frac{1}{2}\nabla_{\mathbf{r}_c}\left[\dot{\mathbf{k}}_c\cdot(\rho_{0s}(\mathbf{k}_c)\nabla_{\mathbf{k}_c}\chi - \rho_{1s}(\mathbf{k}_c)\nabla_{\mathbf{k}_c}\phi_{\mathbf{k}_c})\right] + \left(\dot{\mathbf{k}}_c\cdot\nabla_{\mathbf{k}_c}\right)\left(\rho_{0s}(\mathbf{k}_c)\left(\frac{1}{2}\nabla_{\mathbf{r}_c}\chi - \mathbf{A}(\mathbf{r}_c)\right) - \frac{1}{2}\mathbf{B}\times\mathbf{d} - \frac{1}{2}\rho_{1s}(\mathbf{k}_c)\nabla_{\mathbf{r}_c}\phi_{\mathbf{k}_c}\right), \\ \dot{\mathbf{r}}_c = &\nabla_{\mathbf{k}_c}E_s(k_c) - \dot{\mathbf{k}}_c\times\left(-\frac{1}{2}\nabla_{\mathbf{k}_c}\rho_{0s}(\mathbf{k}_c)\times\nabla_{\mathbf{k}_c}\chi + \frac{1}{2}\nabla_{\mathbf{k}_c}\rho_{1s}(\mathbf{k}_c)\times\nabla_{\mathbf{k}_c}\phi_{\mathbf{k}_c}\right) \\ &+\nabla_{\mathbf{k}_c}\left[\dot{\mathbf{r}}_c\cdot\left(\rho_{0s}(\mathbf{k}_c)\left(\frac{1}{2}\nabla_{\mathbf{r}_c}\chi - \mathbf{A}(\mathbf{r}_c)\right) - \frac{1}{2}\mathbf{B}\times\mathbf{d} - \frac{1}{2}\rho_{1s}(\mathbf{k}_c)\nabla_{\mathbf{r}_c}\phi_{\mathbf{k}_c}\right)\right] - \frac{1}{2}\left(\dot{\mathbf{r}}_c\cdot\nabla_{\mathbf{r}_c}\right)\left(\rho_{0s}(\mathbf{k}_c)\nabla_{\mathbf{k}_c}\chi - \rho_{1s}(\mathbf{k}_c)\nabla_{\mathbf{k}_c}\phi_{\mathbf{k}_c}\right). \end{aligned} \tag{S51}$$

IV. BERRY CURVATURE CORRECTION TO THE PHASE-SPACE DENSITY OF STATES

Due to the presence of the Berry curvatures, the equation of motion Eq. S44 exhibits a noncanonical structure. This leads to the breakdown of the conservation of the phase-space volume. In particular, the phase-space volume element $\Delta V = \Delta\mathbf{k}\Delta\mathbf{r}$ is no longer constant in time and instead evolves according to [S6]

$$\frac{1}{\Delta V}\frac{d\Delta V}{dt} = \nabla_{\mathbf{r}}\cdot\dot{\mathbf{r}} + \nabla_{\mathbf{k}}\cdot\dot{\mathbf{k}}. \tag{S52}$$

One therefore needs to introduce a modified phase-space density of states $D(\mathbf{r},\mathbf{k})$ such that the number of states in volume element ΔV, i.e., $D(\mathbf{r},\mathbf{k})\Delta V$, remains constant over time. In terms of the Berry curvatures, up to an inessential coefficient, the modified phase-space density of states $D(\mathbf{r},\mathbf{k})$ acquires a form [S4, S7]

$$D = \sqrt{\det\begin{bmatrix}\Omega_{rrs} & \Omega_{rks} - I \\ \Omega_{krs} + I & \Omega_{kks}\end{bmatrix}}. \tag{S53}$$

Since the external perturbation is assumed to be slowly varying in real space, one can perform an expansion in terms of the spatial gradient and keep only the leading order terms [S8]:

$$D(\mathbf{r},\mathbf{k}) = 1 + \mathrm{Tr}\,\Omega_{krs} - \mathbf{\Omega}_{rs}\cdot\mathbf{\Omega}_{ks}. \tag{S54}$$

Making use of Eqs. S45, S36, S42, S17 and S49, we obtain the Berry curvature correction to the phase-space density of states $\delta D \equiv D - 1$

$$\begin{aligned} \delta D = &-\nabla_{\mathbf{k}}\rho_{0s}\cdot\mathbf{p}_s - \frac{1}{2}\mathbf{B}\cdot(\nabla_{\mathbf{k}}\times\mathbf{d}) + \frac{1}{2}\nabla_{\mathbf{r}}\rho_{0s}\cdot\nabla_{\mathbf{k}}\chi + \frac{1}{2}\nabla_{\mathbf{k}}\rho_{1s}\cdot\nabla_{\mathbf{r}}\phi_{\mathbf{k}} - \frac{1}{2}\nabla_{\mathbf{r}}\rho_{1s}\cdot\nabla_{\mathbf{k}}\phi_{\mathbf{k}} \\ &-\frac{1}{2}\left(\nabla_{\mathbf{k}}\rho_{0s}\times\nabla_{\mathbf{k}}\chi - \nabla_{\mathbf{k}}\rho_{1s}\times\nabla_{\mathbf{k}}\phi_{\mathbf{k}}\right)\cdot\left(\nabla_{\mathbf{r}}\rho_{0s}(\mathbf{k})\times\mathbf{p}_s - \rho_{0s}\mathbf{B} - \frac{1}{2}\nabla_{\mathbf{r}}\times(\mathbf{B}\times\mathbf{d}) - \frac{1}{2}\nabla_{\mathbf{r}}\rho_{1s}(\mathbf{k})\times\nabla_{\mathbf{r}}\phi_{\mathbf{k}}\right). \end{aligned} \tag{S55}$$

where $\mathbf{p}_s$ denotes the supercurrent

$$\mathbf{p}_s \equiv \frac{1}{2}\nabla_{\mathbf{r}}\chi - \mathbf{A}. \tag{S56}$$

We now separate δD into three different components: the external magnetic field $\mathbf{B}$ dependent part δD_1, the supercurrent $\mathbf{p}_s$ dependent part δD_2, and the remaining part δD_3:

$$\begin{aligned}
\delta D_1(\mathbf{r},\mathbf{k}) &= -\frac{1}{2}\mathbf{B}\cdot(\nabla_{\mathbf{k}}\times\mathbf{d}) + \frac{1}{2}\left(\nabla_{\mathbf{k}}\rho_{0s}\times\nabla_{\mathbf{k}}\chi - \nabla_{\mathbf{k}}\rho_{1s}\times\nabla_{\mathbf{k}}\phi_{\mathbf{k}}\right)\cdot\left[\rho_{0s}\mathbf{B} + \frac{1}{2}\nabla_{\mathbf{r}}\times(\mathbf{B}\times\mathbf{d})\right],\\
\delta D_2(\mathbf{r},\mathbf{k}) &= -\nabla_{\mathbf{k}}\rho_{0s}\cdot\mathbf{p}_s - \frac{1}{2}\left(\nabla_{\mathbf{k}}\rho_{0s}\times\nabla_{\mathbf{k}}\chi - \nabla_{\mathbf{k}}\rho_{1s}\times\nabla_{\mathbf{k}}\phi_{\mathbf{k}}\right)\cdot\left(\nabla_{\mathbf{r}}\rho_{0s}\times\mathbf{p}_s\right),\\
\delta D_3(\mathbf{r},\mathbf{k}) &= \frac{1}{2}\left(\nabla_{\mathbf{r}}\rho_{0s}\cdot\nabla_{\mathbf{k}}\chi + \nabla_{\mathbf{k}}\rho_{1s}\cdot\nabla_{\mathbf{r}}\phi_{\mathbf{k}} - \nabla_{\mathbf{r}}\rho_{1s}\cdot\nabla_{\mathbf{k}}\phi_{\mathbf{k}}\right) + \frac{1}{4}\left(\nabla_{\mathbf{k}}\rho_{0s}\times\nabla_{\mathbf{k}}\chi - \nabla_{\mathbf{k}}\rho_{1s}\times\nabla_{\mathbf{k}}\phi_{\mathbf{k}}\right)\cdot\left(\nabla_{\mathbf{r}}\rho_{1s}\times\nabla_{\mathbf{r}}\phi_{\mathbf{k}}\right).
\end{aligned} \tag{S57}$$

We then drop the higher order terms in the spatial gradient $\nabla_{\mathbf{r}}$ in the equation above, which leads to

$$\begin{aligned}
\delta D_1(\mathbf{r},\mathbf{k}) &= -\frac{1}{2}\mathbf{B}\cdot(\nabla_{\mathbf{k}}\times\mathbf{d}) + \frac{1}{2}\rho_{0s}\left(\nabla_{\mathbf{k}}\rho_{0s}\times\nabla_{\mathbf{k}}\chi - \nabla_{\mathbf{k}}\rho_{1s}\times\nabla_{\mathbf{k}}\phi_{\mathbf{k}}\right)\cdot\mathbf{B},\\
\delta D_2(\mathbf{r},\mathbf{k}) &= -\nabla_{\mathbf{k}}\rho_{0s}\cdot\mathbf{p}_s,\\
\delta D_3(\mathbf{r},\mathbf{k}) &= 0.
\end{aligned} \tag{S58}$$

Inserting the expression for $\mathbf{d}$ in Eq. S27, δD_1 can be further simplified to

$$\delta D_1(\mathbf{r},\mathbf{k}) = -\frac{1}{4}\mathbf{B}\cdot\left[\left(\nabla_{\mathbf{k}}\rho_{2s} + \rho_{0s}\nabla_{\mathbf{k}}\rho_{1s} - \rho_{1s}\nabla_{\mathbf{k}}\rho_{0s}\right)\times\nabla_{\mathbf{k}}\phi_{\mathbf{k}}\right]. \tag{S59}$$

V. SUPPLEMENTARY FIGURES

In this section, we provide two additional figures to demonstrate the effect of Berry curvature on the normal metal-to-superconductor tunneling conductance $G(\omega)$ and the spectral function $A(\mathbf{k},\omega)$. Specifically, in Fig. S2, we plot the BC contribution to the tunneling conductance per unit magnetic field strength $\delta G(\omega)/B$ together with the bare signal $G_0(\omega)$. This plot is generated in the same way as Fig. 2 in the main text but with a different Zeeman field strength $h = 1.5$. Fig. S2 compares the uncorrected spectral function $A_0(\mathbf{k},\omega)$ with the Berry curvature contribution per unit magnetic field strength $\delta A(\mathbf{k},\omega)/B$, and is the same to Fig. 3 in the main text but with a larger spin-orbit coupling $\alpha = 4$.

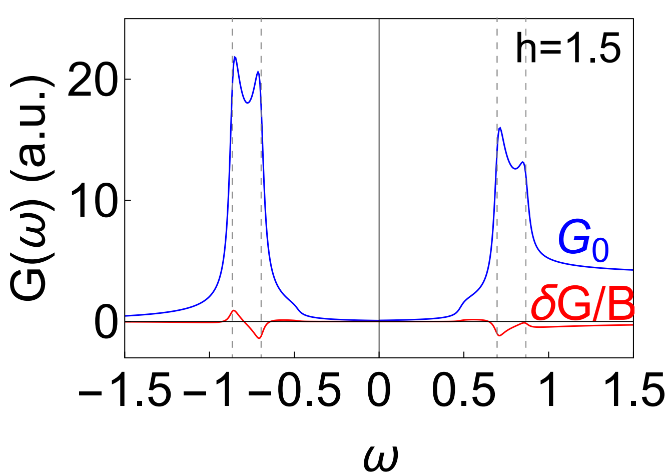

FIG. S1: Same as the two panels in Fig. 2 in the main text but for $h = 1.5$.

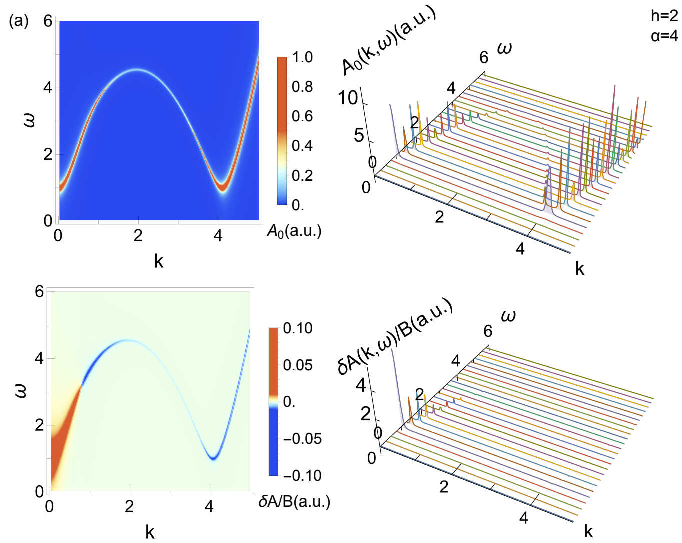

FIG. S2: Same as Fig. 3 in the main text but for $\alpha = 4$.

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* Electronic address: liao2@kth.se
† Electronic address: yhsu2@nd.edu
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