

THE DISTRIBUTION OF POLYNOMIALS IN MONOTONE INDEPENDENT ELEMENTS

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ABSTRACT. Building on the work of Arizmendi and Celestino (2021), we derive the $*$ -distributions of polynomials in monotone independent and infinitesimally monotone independent elements. For non-zero complex numbers α and β , we derive explicitly the $*$ -distribution of $p_{\alpha,\beta} = \alpha ab + \beta ba$ whenever a and b are monotone or infinitesimally monotone independent elements. This encompasses both cases of the commutator and anti-commutator. This approach can be pushed to study more general polynomials. As applications, we derive the limiting distribution with respect to the partial trace of polynomials in a certain class of random matrices.

1. INTRODUCTION

In non-commutative probability theory, diverse notions of independence arise through the examination of purely algebraic representations of fundamental concepts in classical probability theory. Five notions of independence were identified to satisfy natural axioms of probability: tensor (classical), free, Boolean, monotone, and its mirror image, anti-monotone. These notions of independence were classified by Speicher [31] and Muraki [23, 24] and also by Ben Ghorbal and Schürmann [6] in the framework of category theory.

Following Voiculescu's introduction of the concept of free independence, numerous extensions and generalizations of freeness have arisen within the non-commutative setting. Among these, one notable generalization is infinitesimal freeness, see [15] and [8]. This extension has found practical applications in investigating the occurrence of spikes in diverse deformed random matrix models, as shown in [3, 30, 9]. Consequently, extensions of infinitesimal Boolean and monotone independence have also been introduced in [27, 17].

Just like the classical counterpart, these notions of independence enable the computation of probability distributions for sums and products of independent random variables when we have knowledge of their individual distributions. This is achieved through the corresponding operations of additive and multiplicative convolutions. For more details, we refer to [36, 35] for free convolutions, [32, 16] for Boolean convolutions, and [22, 7] for monotone convolutions. Similarly, analogous convolutions, known as infinitesimal additive and multiplicative convolutions, have been developed in [5, 19, 28, 34] for the free case and [27, 33] for the Boolean and monotone cases.

Beyond sums and products, there is a keen interest in determining distributions of general polynomials in independent variables. A natural starting point involves deriving the distributions of two particular polynomials: the commutator $i(x_1x_2 - x_2x_1)$, and the anti-commutator, $x_1x_2 + x_2x_1$, for independent variables x_1 and x_2 . While the distribution of the

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commutator in the free setting was derived combinatorially using cumulants in prior works [25], the distribution of the anti-commutator was only obtained recently in [14, 26]. This is also the case for Boolean independent variables where the distributions of both the commutator and anti-commutator were determined recently in [21]. Furthermore, the infinitesimal free and Boolean distributions of commutators and anti-commutators were, under certain assumptions, derived recently in [21].

On the other hand, distributions of commutators and anti-commutators for monotone independent variables and infinitesimally monotone independent remain unexplored. One of our primary objectives in this paper is to address this gap. As polynomials, we study closely linear spans in Section 3 for which we obtain explicit results. In fact, let $(\mathcal{C}, \varphi)$ be a $*$ -probability space and $a, b \in \mathcal{C}$ such that $\mathcal{A}_a < \mathring{\mathcal{A}}_b$ where $\mathcal{A}_a$ is the $*$ -algebra generated by a and $\mathring{\mathcal{A}}_b$ is the $*$ -algebra generated by b and 1. We determine explicitly in Theorem 3.1 the $*$ -distribution of $p_{\alpha, \beta} = \alpha ab + \beta ba$, where α and β are non-zero complex numbers. Theorem 3.1 yields for the particular case where $\varphi(b) = 0$ that for any $k \geq 1$,

$$\varphi(p_{\alpha, \beta}^k) = \begin{cases} \sqrt{\alpha\beta\varphi(b^2)}^k \varphi(a^k) & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

The framework of Theorem 3.1 encompasses both cases of the commutator $q = i(ab - ba)$ and anti-commutator $p = ab + ba$ and yields for each $k \geq 1$,

$$\begin{aligned} \varphi(p^k) &= \frac{1}{2\sqrt{\varphi(b^2)}} \left([\varphi(b) + \sqrt{\varphi(b^2)}]^{k+1} - [\varphi(b) - \sqrt{\varphi(b^2)}]^{k+1} \right) \varphi(a^k); \\ \varphi(q^k) &= \frac{1}{2} (1 - (-1)^{k+1}) \left(\sqrt{\varphi(b^2) - \varphi(b)^2} \right)^k \varphi(a^k). \end{aligned}$$

Moreover, using the upper triangular technique developed in [34, 27], we extend our results to the infinitesimal monotone setting where we study linear spans in Section 4 and general polynomials as described in Remark 4.5.

In the work of Lenczewski [18], random matrix models for asymptotic monotone independence with respect to the partial trace were given and were further discussed in [12, 10]. On the other hand, Mingo and Tseng [21] introduced recently a constructive approach for generating random matrix models for which asymptotic monotone independence holds. We extend in this paper the utility of our results to derive the distribution of such random matrices with respect to the partial trace, as elaborated in Section 5.

Our approach builds upon the work of Arizmendi and Celestino [1] who study distributions of polynomials with cyclic monotone elements. The notion of cyclic monotone independence was introduced by Collins, Hasebe, and Sakuma [11] to describe the joint limiting spectral distribution of some pairs of matrices. They derived spectral formulas for certain polynomials of degree 2 and 3 in cyclic monotone elements, including the commutator and anti-commutator. In [1], Arizmendi and Celestino offered a novel approach for finding spectral distributions for polynomials in cyclic monotone elements. A pivotal aspect of their method lies in the observation that many polynomials can be expressed as the $(1, 1)$ -entry of matrix products, allowing for the derivation of the distribution by examining the corresponding product matrices. Building on this idea, we extend this approach to study polynomials in monotone independent elements. Moreover, this approach can be further pushed to investigate more general polynomials which lead to computing the precise distributions of a wide range of polynomials in random matrices as Proposition 5.4.

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2. PRELIMINARIES

2.1. Monotone Independence. A *non-commutative probability space* (or ncps for short) is a pair $(\mathcal{C}, \varphi)$ such that $\mathcal{C}$ is a unital algebra and φ is a linear functional on $\mathcal{C}$ that $\varphi(1) = 1$. Moreover, a ncps $(\mathcal{C}, \varphi)$ is called a **-probability space* if we further assume $\mathcal{C}$ is a *-algebra and $\varphi(a^*a) \geq 0$ for all $a \in \mathcal{C}$.

For $a_1, \dots, a_k \in \mathcal{C}$, the **-distribution* of $\{a_1, \dots, a_k\}$ is the set of all possible joints moments

$$\mu_{a_1, \dots, a_k} = \{\varphi(a_{i_1}^{m_1} \cdots a_{i_n}^{m_n}) \mid n \geq 1, 1 \leq i_1, \dots, i_n \leq k \text{ and } m_1, \dots, m_n \in \{1, *\}\}.$$

Definition 2.1. Suppose that $(\mathcal{C}, \varphi)$ is a *-probability space, and $\mathcal{A}$ and $\mathcal{B}$ are *-subalgebras of $\mathcal{C}$ that are not necessarily unital. We say that $(\mathcal{A}, \mathcal{B})$ is *monotone independent* and write $\mathcal{A} < \mathcal{B}$ if

$$(2.1) \quad \varphi(b_0 a_1 b_1 \cdots a_n b_n) = \varphi(a_1 \cdots a_n) \varphi(b_0) \varphi(b_1) \cdots \varphi(b_n),$$

for all $a_1, \dots, a_n \in \mathcal{A}$, $b_1, \dots, b_{n-1} \in \mathcal{B}$ and b_0, b_n are either 1 or elements in $\mathcal{B}$.

For two sets A_1 and A_2 of elements in $\mathcal{C}$, we say (A_1, A_2) is monotone independent and write $A_1 < A_2$ if $\mathcal{A}_1 < \mathring{\mathcal{A}}_2$ where $\mathcal{A}_1$ is the *-algebra generated by A_1 and $\mathring{\mathcal{A}}_2$ is the *-algebra generated by A_2 and 1.

Remark 2.2. Note that under the notion of monotone independence of pair algebras $(\mathcal{A}_1, \mathcal{A}_2)$, we usually don't consider $\mathcal{A}_1$ to be unital; otherwise, for all $a \in \mathcal{A}_2$, we have

$$\varphi(a^n) = \varphi(1 \cdot a \cdot 1 \cdot a \cdots 1 \cdot a) = \varphi(a)^n \varphi(1 \cdot 1 \cdots 1 \cdot 1) = \varphi(a)^n,$$

which is not the case that we are interested in.

2.2. Infinitesimally Monotone Independence. Let $(\mathcal{C}, \varphi)$ be a *-probability space and $\varphi' : \mathcal{C} \rightarrow \mathbb{C}$ be a linear functional such that $\varphi'(1) = 0$ and $\varphi(a^*) = \overline{\varphi(a)}$ for all $a \in \mathcal{C}$. Then we call the triple $(\mathcal{C}, \varphi, \varphi')$ an *infinitesimal *-probability space*.

For $a_1, \dots, a_k \in \mathcal{C}$, the *infinitesimal *-distribution* of $\{a_1, \dots, a_k\}$ is the pair $(\mu_{a_1, \dots, a_k}, \mu'_{a_1, \dots, a_k})$ where $\mu_{a_1, \dots, a_k}$ is the *-distribution of $\{a_1, \dots, a_k\}$ and

$$\mu'_{a_1, \dots, a_k} = \{\varphi'(a_{i_1}^{m_1} \cdots a_{i_n}^{m_n}) \mid n \geq 1, 1 \leq i_1, \dots, i_n \leq k \text{ and } m_1, \dots, m_n \in \{1, *\}\}.$$

Definition 2.3. Let $(\mathcal{C}, \varphi, \varphi')$ be an infinitesimal $*$ -probability space and let $\mathcal{A}$ and $\mathcal{B}$ be $*$ -subalgebras of $\mathcal{C}$ that are not necessary unital. We say that $(\mathcal{A}, \mathcal{B})$ is *infinitesimally monotone independent* and write $\mathcal{A} \ll \mathcal{B}$ if

$$\begin{aligned} \varphi(b_0 a_1 b_1 \cdots a_n b_n) &= \varphi(a_1 \cdots a_n) \varphi(b_0) \varphi(b_1) \cdots \varphi(b_n), \\ \varphi'(b_0 a_1 b_1 \cdots a_n b_n) &= \varphi'(a_1 \cdots a_n) \varphi(b_0) \varphi(b_1) \cdots \varphi(b_n) \\ &\quad + \varphi(a_1 \cdots a_n) \sum_{j=0}^n \varphi(b_0) \cdots \varphi(b_{j-1}) \varphi'(b_j) \varphi(b_{j+1}) \cdots \varphi(b_n), \end{aligned}$$

for all $a_1, \dots, a_n \in \mathcal{A}$, $b_1, \dots, b_{n-1} \in \mathcal{B}$ and b_0, b_n are either 1 or elements in $\mathcal{B}$.

For two sets A_1 and A_2 in $\mathcal{C}$, we say that (A_1, A_2) is infinitesimally monotone independent and write $A_1 \ll A_2$ if $\mathcal{A}_1 \ll \mathcal{A}_2$ with $\mathcal{A}_1$ being the $*$ -algebra generated by A_1 and $\mathcal{A}_2$ be the $*$ -algebra generated by A_2 and 1.

Note that the infinitesimal monotone independence can be understood as monotone independence with amalgamation. To be precise, let $(\mathcal{C}, \varphi, \varphi')$ be an infinitesimal $*$ -probability space and define

$$\tilde{\mathcal{C}} = \left\{ \begin{bmatrix} a & a' \\ 0 & a \end{bmatrix} \mid a, a' \in \mathcal{C} \right\}, \quad \tilde{\mathbb{C}} = \left\{ \begin{bmatrix} c & c' \\ 0 & c \end{bmatrix} \mid c, c' \in \mathbb{C} \right\}.$$

Finally, let $\tilde{\varphi} : \tilde{\mathcal{C}} \rightarrow \tilde{\mathbb{C}}$ be the linear functional defined by

$$\tilde{\varphi} \left(\begin{bmatrix} a & a' \\ 0 & a \end{bmatrix} \right) = \begin{bmatrix} \varphi(a) & \varphi'(a) + \varphi(a') \\ 0 & \varphi(a) \end{bmatrix}.$$

Then the triple $(\tilde{\mathcal{C}}, \tilde{\mathbb{C}}, \tilde{\varphi})$ forms a $\tilde{\mathbb{C}}$ -valued probability space (see [34]). The correspondence to independence with amalgamation is illustrated by the following theorem by Perales and Tseng [27].

Theorem 2.4. Let $(\mathcal{C}, \varphi, \varphi')$ be an infinitesimal $*$ -probability space and let $\mathcal{A}$ and $\mathcal{B}$ be $*$ -subalgebras of $\mathcal{C}$ that are not necessarily unital. Then $(\mathcal{A}, \mathcal{B})$ is infinitesimally monotone independent if and only if $(\tilde{\mathcal{A}}, \tilde{\mathcal{B}})$ is monotone independent with respect to $\tilde{\varphi}$ where

$$\tilde{\mathcal{A}} = \left\{ \begin{bmatrix} a & a' \\ 0 & a \end{bmatrix} \mid a, a' \in \mathcal{A} \right\} \text{ and } \tilde{\mathcal{B}} = \left\{ \begin{bmatrix} a & a' \\ 0 & a \end{bmatrix} \mid a, a' \in \mathcal{B} \right\}.$$

2.3. Random Matrices for Monotone independence.

Definition 2.5. Let $(\mathcal{C}_n, \varphi_n)_n$ be a sequence of ncps and (a_n, b_n) be a pair of elements in $\mathcal{C}_n$ for each n . We say that the sequence $(a_n, b_n)_n$ is *asymptotically monotone independent* with respect to φ_n if there exist a ncps $(\mathcal{C}, \varphi)$ and elements $a, b \in \mathcal{C}$ with $a < b$ such that (a_n, b_n) converges to (a, b) in distribution; that is,

$$\lim_{n \rightarrow \infty} \varphi_n(p(a_n, b_n)) = \varphi(p(a, b)) \text{ for any polynomial } p$$

and $a < b$ with respect to φ .

We present the random matrix model given by Lenczewski [18] where asymptotic monotone independence holds with respect to the partial trace. Let A_N be an $N \times N$ random matrix

and $N_0 \leq N$ be a fixed natural number. A_N can be represented as a composition of four block matrices in the following manner:

$$A_N = \begin{bmatrix} A^{(1,1)} & A^{(1,2)} \\ A^{(2,1)} & A^{(2,2)} \end{bmatrix},$$

where $A^{(1,1)}$ corresponds to a square submatrix of size $N_0 \times N_0$, $A^{(2,2)}$ to a square submatrix of size $(N - N_0) \times (N - N_0)$, $A^{(1,2)}$ to a submatrix of size $N_0 \times (N - N_0)$, and $A^{(2,1)}$ to a submatrix of size $(N - N_0) \times N_0$. The partial trace ψ_N of A_N is then defined by

$$(2.2) \quad \psi_N(A_N) := \frac{1}{N_0}(\mathbb{E} \circ \text{Tr}_{N_0})(A_N^{(1,1)})$$

where Tr_{N_0} is the non-normalized trace. Here we note that the partial trace ψ_N depends on N_0 . Lenczewski showed that if A_N and B_N are independent GUE matrices, then (T_{A_N}, B_N) is asymptotically monotone independent with respect to ψ_N where

$$T_{A_N} = \begin{bmatrix} 0 & A^{(1,2)} \\ A^{(2,1)} & 0 \end{bmatrix}.$$

In [21], Mingo and Tseng considered the notion of infinitesimal freeness and showed a method for utilizing infinitesimal idempotents to construct monotone independent random matrix models. By this construction, they recover the random matrix models in [18] as particular cases. We illustrate this construction for the reader's convenience and start by recalling some notation.

Let $(\mathcal{C}, \varphi, \varphi')$ be an infinitesimal probability space, and let $j \in \mathcal{C}$ be an *infinitesimal idempotent* element, that is, $\varphi(j^k) = 0$ for all $k \geq 1$ and $j = j^2$. Assume furthermore that $\varphi'(j) \neq 0$ and let $\mathcal{A}$ be a unital subalgebra of $\mathcal{C}$ that is infinitesimally free from $\{j\}$. Finally, define

$$\psi(a) = \frac{1}{\varphi'(j)}\varphi'(aj) \text{ for all } a \in \mathcal{C},$$

and note that $\psi(a) = \varphi(a)$ for all $a \in \mathcal{A}$. It is clear that ψ is a linear functional on $\mathcal{C}$ with $\psi(1) = 1$, and hence, $(\mathcal{C}, \psi)$ is a ncps. Finally, set $j^{(-1)} := j$ and $j^{(1)} := 1 - j$, and define $\mathcal{J}_a(\mathcal{A})$ to be the algebra generated by

$$\{j^{(\varepsilon_1)} a_1 j^{(\varepsilon_2)} a_2 \cdots j^{(\varepsilon_{k-1})} a_{k-1} j^{(\varepsilon_k)} \mid k \geq 0, \varepsilon_1, \dots, \varepsilon_k \in \{\pm 1\} \text{ with } \varepsilon_1 \neq \cdots \neq \varepsilon_k, a_1, \dots, a_k \in \mathcal{A}\}.$$

We refer to [15, 19] for more details on infinitesimal free independence and recall now the following result in [21, Theorem 5.5].

Theorem 2.6. Suppose $\mathcal{A}$ and $\mathcal{B}$ are unital subalgebras of $\mathcal{C}$ such that $\{\mathcal{A}, \mathcal{B}\}$ are infinitesimally free from j in $(\mathcal{C}, \varphi, \varphi')$. If $\mathcal{A}$ and $\mathcal{B}$ are free with respect to φ , then $(\mathcal{J}_a(\mathcal{A}), \mathcal{B})$ is monotone independent with respect to ψ .

Let us recall a natural framework of an infinitesimal $*$ -probability space for random matrices. Let $A_N^{(1)}, \dots, A_N^{(k)}$ be $N \times N$ random matrices, and consider the linear map φ_N on $\mathbb{C}\langle X_1, \dots, X_k \rangle$, the algebra of polynomials in k non-commuting indeterminates, defined by

$$\varphi_N(p) = \frac{1}{N}(\mathbb{E} \circ \text{Tr}_N)(p(A_N^{(1)}, \dots, A_N^{(k)})).$$

If for all $p \in \mathbb{C}\langle X_1, \dots, X_k \rangle$, $\varphi(p) := \lim_{N \rightarrow \infty} \varphi_N(p)$ exists as well as $\varphi'(p) = \lim_{N \rightarrow \infty} \varphi'_N(p)$ where $\varphi'_N(p) := N[\varphi_N(p) - \varphi(p)]$, then (φ, φ') is said to be the asymptotic infinitesimal law of $\{A_N^{(1)}, \dots, A_N^{(k)}\}$.

We recall now real and complex Wigner matrices and consider certain assumptions on their moments. We will provide applications of our main results to polynomials in such matrices in Section 5. We denote by W_N the $N \times N$ Hermitian matrix defined by $W_N = \frac{1}{\sqrt{N}}(W_{i,j}^{(N)})_{1 \leq i,j \leq N}$ with $W_{i,j}^{(N)} = \overline{W_{j,i}^{(N)}}$ for $j > i$. For each $N \in \mathbb{N}$, $(W_{i,j}^{(N)})_{1 \leq i \leq j \leq N}$ is a family of independent random variables such that the diagonal entries are real-valued centered random variables with common variance, say s^2 . For real Wigner matrices, we assume that off-diagonal entries are real-valued variables satisfying for all $i < j$:

$$\mathbb{E}[W_{i,j}^{(N)}] = 0 \quad \text{and} \quad \mathbb{E}[|W_{i,j}^{(N)}|^2] = \sigma^2.$$

For the complex Wigner matrices, we assume that the off-diagonal entries are complex-valued and satisfy, in addition to the above conditions, $\mathbb{E}[(W_{i,j}^{(N)})^2] = 0$ for all $i < j$. It is well-known that the limiting spectral distribution of a Wigner matrix is the semicircular distribution. Since the latter is compactly supported, this is equivalent to the convergence of the moments, i.e., for all $k \geq 1$,

$$\lim_{N \rightarrow \infty} \frac{1}{N} (\mathbb{E} \circ \text{Tr}_N)(W_N^k) = \begin{cases} C_m & \text{if } k = 2m \text{ is even,} \\ 0 & \text{if } k \text{ is odd,} \end{cases}$$

where C_m is the m -th Catalan number. Moreover, a family of independent real or complex Wigner matrices is known to be asymptotically free with respect to φ_N , see for instance [2][Theorem 5.4.2]. Now assuming furthermore that for $i < j$, $\mathbb{E}[|W_{i,j}^{(N)}|^4] = \alpha$ and

$$\sup_{N \in \mathbb{N}} \sup_{1 \leq i \leq k \leq N} \mathbb{E}[|W_{i,j}^{(N)}|^\ell] < \infty \quad \text{for all } \ell \in \mathbb{N},$$

then Au [3] proved that W_N has an infinitesimal distribution generalizing the result of [13] for matrices whose entries are identically distributed. Moreover, Au proves that a family of independent Wigner matrices satisfying the above moment conditions is asymptotically infinitesimally free from finite rank matrices. Indeed, Lemma 3.4 and Corollary 3.11 in [3] yield the following:

Theorem 2.7. Let $(W_N^{(i)})_{i \in I}$ be a family of independent $N \times N$ real or complex Wigner matrices satisfying the above moment conditions and denote by $E_N^{(j,k)}$ the unit matrix in the (j,k) -th coordinate. Then for any fixed N_0 , $(W_N^{(i)})_{i \in I}$ and $(E_N^{(j,k)})_{1 \leq j,k \leq N_0}$ are asymptotically infinitesimally free with respect to (φ_N, φ'_N) .

We note that in a more recent paper, Au proves asymptotic infinitesimal freeness for a more general class of matrices, see [4, Proposition 3.6].

It is noteworthy that the limit operators of $(E_N^{(j,j)})_{1 \leq j \leq N_0}$ are infinitesimal idempotents. Consequently, we can construct many asymptotically monotone independent random matrix models by combining Theorem 2.6 and Theorem 2.7. In particular, choose A_N and B_N to be asymptotically free and asymptotically infinitesimally free from $j_N = \sum_{j=1}^{N_0} E_N^{j,j}$. It is easy to see that $\psi_N(\cdot) = \frac{1}{\varphi'_N(j_N)} \varphi'_N(\cdot j_N)$ is the partial trace in (2.2) and that $T_{A_N} = j_N A_N (1 - j_N) + (1 - j_N) A_N j_N$. This yields that (T_{A_N}, B_N) is asymptotically monotone independent with respect to the partial trace. Moreover, the limiting distribution of T_{A_N} with respect to ψ_N can then be computed following [21, Proposition 5.3] which, whenever

A_N and j_N are asymptotically infinitesimally free, yields that

$$(2.3) \lim_{N \rightarrow \infty} \psi_N(T_{A_N}^k) = \begin{cases} \left[\lim_{N \rightarrow \infty} ((\mathbb{E} \circ \text{Tr}_N)(A_N^2) - (\mathbb{E} \circ \text{Tr}_N)(A_N)^2) \right]^{k/2} & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd,} \end{cases}$$

where tr_N denotes the normalized trace.

3. DISTRIBUTION OF POLYNOMIALS IN MONOTONE INDEPENDENT ELEMENTS

In this section, we extend the approach in [1] to the monotone setting and study distributions of polynomials in monotone independent elements. For two monotone independent elements a and b , we start by studying distributions of linear spans of ab and ba for which we derive explicit expressions. As particular cases, we determine the distributions of the commutator and anti-commutator. Then, in Remark 3.5, we discuss more general polynomials in monotone independent elements.

The Linear Span of ab and ba . Let $(\mathcal{C}, \varphi)$ be a $*$ -probability space. For any a and b in $\mathcal{C}$, we denote by $p_{\alpha, \beta} := p_{\alpha, \beta}(a, b)$, the linear span of ab and ba , given by $p_{\alpha, \beta} = \alpha ab + \beta ba$ for some non-zero complex numbers α and β .

We recall that for a given $x \in \mathcal{A}$, we set $\mathcal{A}_x$ to be the $*$ -algebra generated by x , and $\mathring{\mathcal{A}}_x$ to be the $*$ -algebra generated by x and 1.

Theorem 3.1. Assume that $\mathcal{A}_a < \mathring{\mathcal{A}}_b$, then for each $k \geq 1$, we have for any $k \in \mathbb{N}$,

$$\varphi(p_{\alpha, \beta}^k) = \frac{\varphi(a^k)}{2^{k+1}\gamma} \left[((\alpha + \beta)x + \gamma)^{k+1} - ((\alpha + \beta)x - \gamma)^{k+1} \right]$$

where $x = \varphi(b)$ and $\gamma = \sqrt{(\alpha - \beta)^2 \varphi(b)^2 + 4\alpha\beta\varphi(b^2)}$.

Remark 3.2. Note that by monotone independence, the $*$ -distribution of $p_{\alpha, \beta}$ depends on the distribution of b only through its first and second moments $\varphi(b)$ and $\varphi(b^2)$. Now, if we assume that $\varphi(b) = 0$, then for any $k \geq 1$,

$$\varphi(p_{\alpha, \beta}^k) = \frac{1}{2} \sqrt{\alpha\beta\varphi(b^2)}^k \varphi(a^k) + \frac{(-1)^k}{2} \sqrt{\alpha\beta\varphi(b^2)}^k \varphi(a^k) = \begin{cases} \sqrt{\alpha\beta\varphi(b^2)}^k \varphi(a^k) & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

Furthermore, if a is an even operator, then we have the $*$ -distribution of $p_{\alpha, \beta}$ is a mere scaling of the distribution of a .

Note that, as a direct consequence, we obtain the $*$ -distribution of anti-commutators and commutators in monotone independent variables by taking $(\alpha, \beta) = (1, 1)$ or $(\alpha, \beta) = (i, -i)$ respectively. More precisely, we obtain the following moment expressions.

Corollary 3.3. Consider $a, b \in \mathcal{C}$ such that $\mathcal{A}_a < \mathring{\mathcal{A}}_b$. Let $p = ab + ba$ and $q = i(ab - ba)$, then for each $k \geq 1$,

$$\begin{aligned} \varphi(p^k) &= \frac{1}{2\sqrt{\varphi(b^2)}} \left([\varphi(b) + \sqrt{\varphi(b^2)}]^{k+1} - [\varphi(b) - \sqrt{\varphi(b^2)}]^{k+1} \right) \varphi(a^k); \\ \varphi(q^k) &= \frac{1}{2} (1 - (-1)^{k+1}) \left(\sqrt{\varphi(b^2) - \varphi(b)^2} \right)^k \varphi(a^k). \end{aligned}$$

In order to derive explicit expressions for the distributions of spans of ab and ba , we prove first the following proposition that plays a crucial role in bridging the gap between matrices with monotone entries and the distribution of polynomials. For a given algebra $\mathcal{D}$, we represent the unital $*$ -algebra generated by $\mathcal{D}$ as $\mathring{\mathcal{D}}$.

Proposition 3.4. Let $(\mathcal{C}, \varphi)$ be a $*$ -probability space. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are $*$ -subalgebras of $\mathcal{C}$ such that $\mathcal{A} < \mathring{\mathcal{B}}$. Consider $A_k = [a_{i,j}^{(k)}]_{i,j \in [n]} \in M_n(\mathcal{A})$ for each $k = 1, \dots, m$ and $B_s = [b_{i,j}^{(s)}]_{i,j \in [n]} \in M_n(\mathcal{B})$ for each $s = 0, 1, \dots, m$. Then we have

$$(Tr_n \otimes \varphi)(D_0 A_1 B_1 A_2 B_2 \cdots B_{m-1} A_m D_m) = (Tr_n \otimes \varphi)(D'_0 A_1 B'_1 A_2 B'_2 \cdots B'_{m-1} A_m D'_m)$$

where $D_0 = I_n$ or B_0 , $D_m = I_n$ or B_m , $B'_k = \left(\varphi(b_{i,j}^{(k)})_{i,j \in [n]} \right) \in M_n(\mathbb{C})$, for each k , and $D'_0 = I_n$ or B'_0 , $D'_m = I_n$ or B'_m .

Proof. Without loss of generality, we may assume $D_0 = I_n$ and $D_m = B_m$ (the proof of all other cases are similar). Note that by monotone independence with respect to φ , we obtain

$$\begin{aligned} & (Tr_n \otimes \varphi)(A_1 B_1 A_2 B_2 \cdots A_m B_m) \\ &= \sum_{i_1, \dots, i_{2m} \in [n]} \varphi \left(a_{i_1, i_2}^{(1)} b_{i_2, i_3}^{(1)} a_{i_3, i_4}^{(2)} b_{i_4, i_5}^{(2)} \cdots a_{i_{2m-1}, i_{2m}}^{(m)} b_{i_{2m}, i_1}^{(m)} \right) \\ &= \sum_{i_1, \dots, i_{2m} \in [n]} \varphi \left(a_{i_1, i_2}^{(1)} \varphi(b_{i_2, i_3}^{(1)}) a_{i_3, i_4}^{(2)} \varphi(b_{i_4, i_5}^{(2)}) \cdots a_{i_{2m-1}, i_{2m}}^{(m)} \varphi(b_{i_{2m}, i_1}^{(m)}) \right) \\ &= (Tr_n \otimes \varphi)(A_1 B'_1 A_2 B'_2 \cdots A_m B'_m). \end{aligned}$$

□

Having this in hand, we are ready to prove the above results.

Proof of Theorem 3.1. Let

$$B_0 = \begin{bmatrix} \alpha & b \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}, \quad \text{and} \quad B_1 = \begin{bmatrix} b & 0 \\ \beta & 0 \end{bmatrix}.$$

Then it is obvious that $B_0 A B_1 = \text{Diag}(\alpha ab + \beta ba, 0)$. Note that following Proposition 3.4, for any $k \geq 1$, we have

$$\begin{aligned} \varphi(p_{\alpha, \beta}^k) &= (Tr_2 \otimes \varphi) \left((B_0 A B_1)^k \right) \\ &= (Tr_2 \otimes \varphi) \left(B_0 (A B_1 B_0)^{k-1} A B_1 \right) \\ &= (Tr_2 \otimes \varphi) \left(\begin{bmatrix} \alpha & \varphi(b) \\ 0 & 0 \end{bmatrix} \left(\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \begin{bmatrix} \alpha \varphi(b) & \varphi(b^2) \\ \alpha \beta & \beta \varphi(b) \end{bmatrix} \right)^{k-1} \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \begin{bmatrix} \varphi(b) & 0 \\ \beta & 0 \end{bmatrix} \right) \\ &= (Tr_2 \otimes \varphi) \left(\begin{bmatrix} a^k & 0 \\ 0 & a^k \end{bmatrix} \begin{bmatrix} \alpha & \varphi(b) \\ 0 & 0 \end{bmatrix} \left(\begin{bmatrix} \alpha \varphi(b) & \varphi(b^2) \\ \alpha \beta & \beta \varphi(b) \end{bmatrix} \right)^{k-1} \begin{bmatrix} \varphi(b) & 0 \\ \beta & 0 \end{bmatrix} \right). \end{aligned}$$

We observe that for all $x, y \in \mathbb{C}$, $\begin{bmatrix} \alpha x & y \\ \alpha \beta & \beta x \end{bmatrix} = S D S^{-1}$ where

$$D = \begin{bmatrix} \frac{(\alpha + \beta)x - \gamma}{2} & 0 \\ 0 & \frac{(\alpha + \beta)x + \gamma}{2} \end{bmatrix}, \quad S = \begin{bmatrix} \frac{(\alpha - \beta)x - \gamma}{2\alpha\beta} & \frac{(\alpha - \beta)x + \gamma}{2\alpha\beta} \\ 1 & 1 \end{bmatrix}, \quad S^{-1} = \begin{bmatrix} \frac{-\alpha\beta}{\gamma} & \frac{\gamma + (\alpha - \beta)x}{2\gamma} \\ \frac{\alpha\beta}{\gamma} & \frac{\gamma - (\alpha - \beta)x}{2\gamma} \end{bmatrix}$$

and $\gamma = \sqrt{\varphi(b)^2(\alpha - \beta)^2 + 4\alpha\beta\varphi(b^2)}$. Then

$$\begin{aligned} & \begin{bmatrix} \alpha & x \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha x & y \\ \alpha\beta & \beta x \end{bmatrix}^{k-1} \begin{bmatrix} x & 0 \\ \beta & 0 \end{bmatrix} \\ &= \begin{bmatrix} \frac{(\alpha-\beta)x-\gamma}{2\beta} + x & \frac{(\alpha-\beta)x+\gamma}{2\beta} + x \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{((\alpha+\beta)x-\gamma)^{k-1}}{2^{k-1}} & 0 \\ 0 & \frac{((\alpha+\beta)x+\gamma)^{k-1}}{2^{k-1}} \end{bmatrix} \begin{bmatrix} \frac{-(\beta^2+\alpha\beta)x+\beta\gamma}{2\gamma} & 0 \\ \frac{(\beta^2+\alpha\beta)x+\beta\gamma}{2\gamma} & 0 \end{bmatrix} \\ &= \frac{1}{2^{k+1}\gamma} \left[((\alpha+\beta)x-\gamma)^k (-(\alpha+\beta)x+\gamma) + ((\alpha+\beta)x+\gamma)^k ((\alpha+\beta)x-\gamma) \right] \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

Thus, we complete the proof by concluding that

$$\varphi(p_{\alpha,\beta}^k) = \frac{\varphi(a^k)}{2^{k+1}\gamma} \left[((\alpha+\beta)x+\gamma)^{k+1} - ((\alpha+\beta)x-\gamma)^{k+1} \right].$$

□

Remark 3.5. Let $(\mathcal{C}, \varphi)$ be a $*$ -probability space. Let p be any polynomial generated by elements of $\mathcal{A}$ and $\mathcal{B}$ with $\mathcal{A} < \mathcal{B}$. Note that p can be expressed in the form

$$p = \sum_{j=1}^n b_j y_j b'_j$$

where $n \geq 1$, $b_1, b'_1, \dots, b_n, b'_n \in \text{alg}(\mathcal{B}, 1_{\mathcal{C}})$, $y_1, \dots, y_n \in \mathcal{Q}$ with $\mathcal{Q}$ being the set defined by

$$(3.1) \quad \mathcal{Q} = \{a_0 b_1 a_1 \cdots b_k a_k \mid k \geq 0, a_0, \dots, a_k \in \mathcal{A} \text{ and } b_1, \dots, b_k \in \mathcal{B}\}.$$

To soothe the notation, we write $\tilde{y} = a_0 \varphi(b_1) a_1 \cdots \varphi(b_k) a_k$ whenever $y = a_0 b_1 a_1 \cdots b_k a_k$ for some $a_1, \dots, a_k \in \mathcal{A}$ and $b_1, \dots, b_k \in \mathcal{B}$. Then by monotone independence, we get for any $m \in \mathbb{N}$,

$$(3.2) \quad \varphi(p^m) = \sum_{j_1, \dots, j_m \in [n]} \varphi(\tilde{y}_{j_1} \cdots \tilde{y}_{j_m}) \varphi(b_{j_1}) \varphi(b'_{j_1} b_{j_2}) \varphi(b'_{j_2} b_{j_3}) \cdots \varphi(b'_{j_{m-1}} b_{j_m}) \varphi(b'_{j_m}).$$

We provide in the following proposition an example to demonstrate how to compute the law of a given polynomial.

Proposition 3.6. Suppose $(\mathcal{C}, \varphi)$ is a $*$ -probability space, and a, b_1, b_2, c_1 , and c_2 are elements in $\mathcal{C}$ such that $(\{a\}, \{b_1, b_2, c_1, c_2\})$ is monotone independent. Assume $\varphi(b_1) = \tilde{b}_1$, $\varphi(b_2) = \tilde{b}_2$, $\varphi(c_1) = \tilde{c}_1$, $\varphi(c_2) = \tilde{c}_2$, and

$$\varphi(b_i b_j) = \begin{cases} b_{1,1} & \text{if } (i, j) = (1, 1), \\ b_{2,2} & \text{if } (i, j) = (2, 2), \\ b_{1,2} & \text{if } (i, j) = (1, 2) \text{ or } (2, 1). \end{cases}$$

We set $p = b_1 a c_1 a b_1 + b_2 a c_2 a b_2$. Then the distribution of p is

$$\alpha \mu_{a^2} \delta_{\vartheta+\zeta} + (1-\alpha) \mu_{a^2} \delta_{\vartheta-\zeta}$$

where μ_{a^2} is the distribution of a^2 , δ denotes the dirac measure, $\alpha = \frac{1}{2\zeta}(\zeta - \vartheta + \tilde{c}_1 \tilde{b}_1^2 + \tilde{c}_2 \tilde{b}_1^2)$ with

$$\vartheta = \frac{1}{2}(b_{2,2} \tilde{c}_2 + b_{1,1} \tilde{c}_1) \text{ and } \zeta = \frac{1}{2} \sqrt{(b_{2,2} \tilde{c}_2 - b_{1,1} \tilde{c}_1)^2 + 4b_{1,2}^2 \tilde{c}_2 \tilde{c}_1}.$$

Proof. By Remark 3.5, we have

$$\begin{aligned}\varphi(p^k) &= \varphi\left([b_1 \cdot a\varphi(c_1)a \cdot b_1 + b_2 \cdot a\varphi(c_2)a \cdot b_2]^k\right) \\ &= \sum_{j_1, \dots, j_k=1}^2 \varphi(a_{j_1} \cdots a_{j_k})\varphi(b_{j_1})\varphi(b_{j_1}b_{j_2})\varphi(b_{j_2}b_{j_3}) \cdots \varphi(b_{j_{k-1}}b_{j_k})\varphi(b_{j_k})\end{aligned}$$

where $a_1 = \varphi(c_1)a^2$ and $a_2 = \varphi(c_2)a^2$. Note that for all $(j_1, \dots, j_k) \in [2]^k$,

$$\varphi(a_{j_1} \cdots a_{j_k}) = \varphi(a^{2k})\varphi(c_{j_1})\varphi(c_{j_2}) \cdots \varphi(c_{j_k}).$$

Thus, if for each $i \in [2]$, we set $\alpha_1^{(i)} = \varphi(b_i)$, and

$$\alpha_k^{(i)} = \sum_{j_1, \dots, j_{k-1} \in [2]} \varphi(b_i \cdot b_{j_1})\varphi(b_{j_1}b_{j_2}) \cdots \varphi(b_{j_{k-2}}b_{j_{k-1}})\varphi(b_{j_{k-1}})\varphi(c_{j_1})\varphi(c_{j_2}) \cdots \varphi(c_{j_{k-1}}) \text{ for } k \geq 2$$

then we obtain

$$\varphi(p^k) = \varphi(a^{2k})\alpha_k \quad \text{where} \quad \alpha_k := \varphi(b_1)\varphi(c_1)\alpha_k^{(1)} + \varphi(b_2)\varphi(c_2)\alpha_k^{(2)}.$$

We observe that for each $l \geq 2$,

$$\alpha_l^{(2)} = \varphi(b_2b_1)\varphi(c_1)\alpha_{l-1}^{(1)} + \varphi(b_2b_2)\varphi(c_2)\alpha_{l-1}^{(2)} = b_{1,2}\tilde{c}_1\alpha_{l-1}^{(1)} + b_{2,2}\tilde{c}_2\alpha_{l-1}^{(2)},$$

which when iterated we obtain for each $l \geq 3$,

$$\begin{aligned}\alpha_l^{(2)} &= b_{1,2}\tilde{c}_1\alpha_{l-1}^{(1)} + b_{1,2}\tilde{c}_1b_{2,2}\tilde{c}_2\alpha_{l-2}^{(1)} + b_{2,2}^2\tilde{c}_2^2\alpha_{l-2}^{(2)} \\ &= b_{1,2}\tilde{c}_1\left(\alpha_{l-1}^{(1)} + b_{2,2}\tilde{c}_2\alpha_{l-2}^{(1)} + b_{2,2}^2\tilde{c}_2^2\alpha_{l-3}^{(1)}\right) + b_{2,2}^3\tilde{c}_2^3\alpha_{l-3}^{(2)} \\ &= b_{1,2}\tilde{c}_1\left(\alpha_{l-1}^{(1)} + \sum_{j=1}^{l-3} b_{2,2}^j\tilde{c}_2^j\alpha_{l-(j+1)}^{(1)}\right) + b_{2,2}^{l-2}\tilde{c}_2^{l-2}(b_{1,2}\tilde{b}_1\tilde{c}_1 + b_{2,2}\tilde{b}_2\tilde{c}_2).\end{aligned}$$

Using the expression obtained for $\alpha_{k-1}^{(2)}$, we have get for any $k \geq 2$,

$$\begin{aligned}\alpha_k^{(1)} &= \varphi(b_1b_1)\varphi(c_1)\alpha_{k-1}^{(1)} + \varphi(b_1b_2)\varphi(c_2)\alpha_{k-1}^{(2)} \\ &= b_{1,1}\tilde{c}_1\alpha_{k-1}^{(1)} + b_{1,2}\tilde{c}_2\alpha_{k-1}^{(2)} \\ &= b_{1,1}\tilde{c}_1\alpha_{k-1}^{(1)} + b_{1,2}\tilde{c}_1\left(\tilde{c}_2\alpha_{k-2}^{(1)} + \sum_{j=1}^{k-4} b_{2,2}^j\tilde{c}_2^{j+1}\alpha_{k-j-2}^{(1)} + b_{2,2}^{k-3}\tilde{c}_2^{k-2}\tilde{b}_1\right) \\ &\quad + b_{1,2}b_{2,2}^{k-2}\tilde{c}_2^{k-1}\tilde{b}_2.\end{aligned}$$

Then we observe

$$\alpha_k^{(1)} - b_{2,2}\tilde{c}_2\alpha_{k-1}^{(1)} = b_{1,1}\tilde{c}_1\alpha_{k-1}^{(1)} + (b_{1,2}^2 - b_{2,2}b_{1,1})\tilde{c}_2\tilde{c}_1\alpha_{k-2}^{(1)},$$

and hence we obtain the following linear homogeneous recurrences

$$\alpha_k^{(1)} - (b_{2,2}\tilde{c}_2 + b_{1,1}\tilde{c}_1)\alpha_{k-1}^{(1)} + (b_{2,2}b_{1,1} - b_{1,2}^2)\tilde{c}_2\tilde{c}_1\alpha_{k-2}^{(1)} = 0.$$

Thus by solving the characteristic equation $x^2 - (b_{2,2}\tilde{c}_2 + b_{1,1}\tilde{c}_1)x + (b_{2,2}b_{1,1} - b_{1,2}^2)\tilde{c}_2\tilde{c}_1 = 0$, we have $x = \vartheta \pm \zeta$ as a solution with

$$\vartheta = \frac{1}{2}(b_{2,2}\tilde{c}_2 + b_{1,1}\tilde{c}_1) \text{ and } \zeta = \frac{1}{2}\sqrt{(b_{2,2}\tilde{c}_2 - b_{1,1}\tilde{c}_1)^2 + 4b_{1,2}^2\tilde{c}_2\tilde{c}_1}.$$

This yields that

$$\alpha_k^{(1)} = s_1^{(1)}(\vartheta + \zeta)^k + s_2^{(1)}(\vartheta - \zeta)^k$$

for some constants $s_1^{(1)}$ and $s_2^{(1)}$ that determined by the initial conditions, and similarly

$$\alpha_k^{(2)} = s_1^{(2)}(\vartheta + \zeta)^k + s_2^{(2)}(\vartheta - \zeta)^k$$

for some constants $s_1^{(2)}$ and $s_2^{(2)}$. Hence, putting the above terms together, we infer that for any $k \geq 1$,

$$\alpha_k = (\tilde{b}_1 \tilde{c}_1 s_1^{(1)} + \tilde{b}_2 \tilde{c}_2 s_1^{(2)})(\vartheta + \zeta)^k + (\tilde{b}_1 \tilde{c}_1 s_2^{(1)} + \tilde{b}_2 \tilde{c}_2 s_2^{(2)})(\vartheta - \zeta)^k.$$

Now we observe that $\alpha_0 = 1$ and $\alpha_1 = \tilde{c}_1 \tilde{b}_1^2 + \tilde{c}_2 \tilde{b}_2^2$. Thus,

$$\begin{cases} \tilde{b}_1 \tilde{c}_1 (s_1^{(1)} + s_2^{(1)}) + \tilde{b}_2 \tilde{c}_2 (s_1^{(2)} + s_2^{(2)}) = 1 \\ \tilde{b}_1 \tilde{c}_1 [s_1^{(1)}(\vartheta + \zeta) + s_2^{(1)}(\vartheta - \zeta)] + \tilde{b}_2 \tilde{c}_2 [s_1^{(2)}(\vartheta + \zeta) + s_2^{(2)}(\vartheta - \zeta)] = \tilde{c}_1 \tilde{b}_1^2 + \tilde{c}_2 \tilde{b}_2^2. \end{cases}$$

As a result, we obtain

$$\begin{aligned} \tilde{b}_1 \tilde{c}_1 s_1^{(1)} + \tilde{b}_2 \tilde{c}_2 s_1^{(2)} &= \frac{1}{2\zeta}(\zeta - \vartheta + \tilde{c}_1 \tilde{b}_1^2 + \tilde{c}_2 \tilde{b}_2^2) \\ \tilde{b}_1 \tilde{c}_1 s_2^{(1)} + \tilde{b}_2 \tilde{c}_2 s_2^{(2)} &= \frac{1}{2\zeta}(\zeta + \vartheta - \tilde{c}_1 \tilde{b}_1^2 - \tilde{c}_2 \tilde{b}_2^2) \end{aligned}$$

by solving a system of equations which concludes the proof by obtaining for any $k \geq 1$,

$$\begin{aligned} \varphi(p^k) &= \varphi(a^{2k})\alpha_k \\ &= \frac{\varphi(a^{2k})}{2\zeta}(\zeta - \vartheta + \tilde{c}_1 \tilde{b}_1^2 + \tilde{c}_2 \tilde{b}_2^2)(\vartheta + \zeta)^k + \frac{\varphi(a^{2k})}{2\zeta}(\zeta + \vartheta - \tilde{c}_1 \tilde{b}_1^2 - \tilde{c}_2 \tilde{b}_2^2)(\vartheta - \zeta)^k. \end{aligned}$$

□

4. DISTRIBUTION OF POLYNOMIALS IN INFINITESIMALLY MONOTONE INDEPENDENT ELEMENTS

Let $(\mathcal{C}, \varphi, \varphi')$ be an infinitesimal $*$ -probability space and consider its associated $\tilde{\mathbb{C}}$ -valued probability space $(\tilde{\mathcal{C}}, \tilde{\mathbb{C}}, \tilde{\varphi})$, described in Subsection 2.2. Note that by Theorem 2.4 and the fact that $\tilde{\mathbb{C}}$ is commutative, we extend our approach to the framework of $(\tilde{\mathcal{C}}, \tilde{\mathbb{C}}, \tilde{\varphi})$. This allows passing directly many results on $*$ -distributions of polynomials in monotone independent variables to the infinitesimal setting.

For two infinitesimally monotone elements a and b , we start by illustrating, in Section 4, how this allows us to derive an explicit expression of the infinitesimal distribution of linear spans of ab and ba , covering the cases of the commutator and anti-commutator. Then, we discuss in Remark 4.5, more general polynomials in infinitesimal monotone independent elements.

Linear spans in ab and ba . The commutativity of $\tilde{\mathbb{C}}$ allows deriving explicit expressions of the infinitesimal $*$ -distribution of linear spans. Let a and b be two elements in $\mathcal{C}$ such that $a \ll b$ and set $A = \text{Diag}(a, a)$ and $B = \text{Diag}(b, b)$. Then, by Theorem 2.4, we have that $A < B$ with respect to $\tilde{\varphi}$. We obtain, following the same lines of proof as in Theorem 3.1, the explicit distribution of the linear span $P_{\alpha,\beta} := P_{\alpha,\beta}(A, B) = \alpha AB + \beta BA$ for any non-zero complex numbers α and β . More precisely, setting $X = \tilde{\varphi}(B)$ and $\Gamma = \sqrt{(\alpha - \beta)^2 \tilde{\varphi}(B)^2 + 4\alpha\beta \tilde{\varphi}(B^2)}$, we obtain for each $k \geq 1$,

$$(4.1) \quad \tilde{\varphi}(P_{\alpha,\beta}^k) = \frac{\tilde{\varphi}(A^k) \Gamma^{-1}}{2^{k+1}} \left[((\alpha + \beta)X + \Gamma)^{k+1} - ((\alpha + \beta)X - \Gamma)^{k+1} \right].$$

The key point that enables deriving such an expression following the same lines as in the scalar case is the fact that $\tilde{\mathbb{C}}$ is commutative, and more precisely, $X\Gamma = \Gamma X$. Note that by comparing the $(1, 2)$ -entry of both sides of (4.1), we obtain explicitly the k th moment of $p_{\alpha,\beta}$ with respect to φ' .

Remark 4.1. Note that we can also compute $\varphi'(p_{\alpha,\beta}^k)$ by formally differentiating $\varphi(p_{\alpha,\beta}^k)$ in Theorem 3.1. However, we adopt the method in [34, 27] by translating problems in the infinitesimal setting to independence over the commutative algebra $\tilde{\mathbb{C}}$. This methodology proves beneficial for addressing a range of other problems in infinitesimal independence, which motivates our choice to illustrate it for computing the infinitesimal distribution.

Theorem 4.2. Let a and b be two elements in an infinitesimal $*$ -probability space $(\mathcal{C}, \varphi, \varphi')$ such that $\mathcal{A}_a \ll \mathcal{A}_b$. For given non-zero complex numbers α and β , let $p_{\alpha,\beta} = \alpha ab + \beta ba$. Then for each $k \geq 0$,

$$\begin{aligned} \varphi'(p_{\alpha,\beta}^k) &= \frac{\varphi(a^k)}{2^{k+1}\gamma} \left[((\alpha + \beta)\varphi(b) + \gamma)^k \left((\alpha + \beta)((k+1)\varphi'(b) - \varphi(b)\omega/\gamma^2) + k\omega/\gamma \right) \right. \\ &\quad \left. - ((\alpha + \beta)\varphi(b) - \gamma)^k \left((\alpha + \beta)((k+1)\varphi'(b) - \varphi(b)\omega/\gamma^2) - k\omega/\gamma \right) \right] \\ &\quad + \frac{\varphi'(a^k)}{2^{k+1}\gamma} \left[((\alpha + \beta)\varphi(b) + \gamma)^{k+1} - ((\alpha + \beta)\varphi(b) - \gamma)^{k+1} \right], \end{aligned}$$

where

$$(4.2) \quad \gamma = \sqrt{(\alpha - \beta)^2 \varphi(b)^2 + 4\alpha\beta \varphi(b^2)} \quad \text{and} \quad \omega = (\alpha - \beta)^2 \varphi(b) \varphi'(b) + 2\alpha\beta \varphi'(b^2).$$

Remark 4.3. Note that the infinitesimal $*$ -distribution of $p_{\alpha,\beta}$ only depends on that of a and on $\varphi(b), \varphi(b^2), \varphi'(b)$ and $\varphi'(b^2)$. In the particular case where $\varphi(b) = \varphi'(b) = 0$, this reduces to

$$\varphi'(p_{\alpha,\beta}^k) = \left[\frac{k}{4} \varphi'(b^2) \varphi(b^2)^{k/2-1} \varphi(a^k) + \frac{\varphi'(a^k) \varphi(b^2)^{k/2}}{2} \right] (\alpha\beta)^{k/2} (1 + (-1)^k).$$

Again, as a direct consequence of Theorem 4.2, we obtain the infinitesimal $*$ -distribution of anti-commutators and commutators by taking $(\alpha, \beta) = (1, 1)$ or $(\alpha, \beta) = (i, -i)$ respectively.

Corollary 4.4. Let $p = ab + ba$ and $q = i(ab - ba)$, then for each $k \geq 1$, we have

$$\begin{aligned} \varphi'(p^k) &= \frac{\varphi(a^k)}{4\sqrt{\varphi(b^2)}\varphi(b^2)} \left[(\varphi(b) + \sqrt{\varphi(b^2)})^k \left(2(k+1)\varphi'(b)\varphi(b^2) - \varphi'(b^2)[\varphi(b) - k\sqrt{\varphi(b^2)}] \right) \right. \\ &\quad \left. + (\varphi(b) - \sqrt{\varphi(b^2)})^k \left(2(k+1)\varphi'(b)\varphi(b^2) - \varphi'(b^2)[\varphi(b) + k\sqrt{\varphi(b^2)}] \right) \right] \\ &\quad + \frac{\varphi'(a^k)}{2\sqrt{\varphi(b^2)}} \left([\varphi(b) + \sqrt{\varphi(b^2)}]^{k+1} - [\varphi(b) - \sqrt{\varphi(b^2)}]^{k+1} \right) \end{aligned}$$

and

$$\begin{aligned} \varphi'(q^k) &= \frac{\varphi(a^k)k}{4} (1 + (-1)^k) \left(\sqrt{\varphi(b^2) - \varphi(b)^2} \right)^{k-2} (\varphi'(b^2) - 2\varphi(b)\varphi'(b)) \\ &\quad + \frac{\varphi'(a^k)}{2} (1 + (-1)^k) \left(\sqrt{\varphi(b^2) - \varphi(b)^2} \right)^k. \end{aligned}$$

Proof of Theorem 4.2. We start by noting that

$$\tilde{\varphi}(P_{\alpha,\beta}^k) = \tilde{\varphi} \left(\begin{bmatrix} p_{\alpha,\beta}^k & 0 \\ 0 & p_{\alpha,\beta}^k \end{bmatrix} \right) = \begin{bmatrix} \varphi(p_{\alpha,\beta}^k) & \varphi'(p_{\alpha,\beta}^k) \\ 0 & \varphi(p_{\alpha,\beta}^k) \end{bmatrix}.$$

Thus in order to derive an expression for $\varphi'(p_{\alpha,\beta}^k)$, we shall compute the $(1,2)$ -entry of the right hand side of (4.1). In order to do this, let us note for any $z, z' \in \mathbb{C}$ and matrix

$$M = \begin{bmatrix} z & z' \\ 0 & z \end{bmatrix} \in \tilde{\mathbb{C}},$$

$$M^{1/2} = \sqrt{M} = \begin{bmatrix} \sqrt{z} & \frac{z'}{2\sqrt{z}} \\ 0 & \sqrt{z} \end{bmatrix} \text{ and } M^{-1} = \begin{bmatrix} z^{-1} & -z^{-2}z' \\ 0 & z^{-1} \end{bmatrix}.$$

Setting $x = \varphi(b)$ and $y = \varphi'(b)$, we observe that

$$X = \tilde{\varphi}(B) = \begin{bmatrix} x & y \\ 0 & x \end{bmatrix} \quad \text{and} \quad \Gamma = \sqrt{(\alpha - \beta)^2 \tilde{\varphi}(B)^2 + 4\alpha\beta \tilde{\varphi}(B^2)} = \begin{bmatrix} \gamma & \frac{\omega}{\gamma} \\ 0 & \gamma \end{bmatrix},$$

where γ and ω are defined in (4.2), and finally obtain that

$$\frac{1}{2^{k+1}} \Gamma^{-1} = \frac{1}{2^{k+1}} \begin{bmatrix} \gamma^{-1} & -\gamma^{-2} \cdot \omega/\gamma \\ 0 & \gamma^{-1} \end{bmatrix} = \frac{1}{2^{k+1}\gamma} \begin{bmatrix} 1 & -\omega/\gamma^2 \\ 0 & 1 \end{bmatrix}.$$

Note that

$$\begin{aligned} ((\alpha + \beta)X - \Gamma)^{k+1} &= \begin{bmatrix} ((\alpha + \beta)x - \gamma)^{k+1} & (k+1)((\alpha + \beta)x - \gamma)^k((\alpha + \beta)y - \omega/\gamma) \\ 0 & ((\alpha + \beta)x - \gamma)^{k+1} \end{bmatrix} \\ &= ((\alpha + \beta)x - \gamma)^k \begin{bmatrix} (\alpha + \beta)x - \gamma & (k+1)((\alpha + \beta)y - \omega/\gamma) \\ 0 & (\alpha + \beta)x - \gamma \end{bmatrix}. \end{aligned}$$

Therefore,

$$\begin{aligned}
& \frac{1}{2^{k+1}} \Gamma^{-1} ((\alpha + \beta)X - \Gamma)^{k+1} \\
&= \frac{((\alpha + \beta)x - \gamma)^k}{2^{k+1}\gamma} \begin{bmatrix} 1 & -\omega/\gamma^2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} (\alpha + \beta)x - \gamma & (k+1)((\alpha + \beta)y - \omega/\gamma) \\ 0 & (\alpha + \beta)x - \gamma \end{bmatrix} \\
&= \frac{((\alpha + \beta)x - \gamma)^k}{2^{k+1}\gamma} \begin{bmatrix} (\alpha + \beta)x - \gamma & (k+1)((\alpha + \beta)y - \omega/\gamma) - \omega/\gamma^2((\alpha + \beta)x - \gamma) \\ 0 & (\alpha + \beta)x - \gamma \end{bmatrix} \\
&= \frac{((\alpha + \beta)x - \gamma)^k}{2^{k+1}\gamma} \begin{bmatrix} (\alpha + \beta)x - \gamma & (\alpha + \beta)((k+1)y - x\omega/\gamma^2) - k\omega/\gamma \\ 0 & (\alpha + \beta)x - \gamma \end{bmatrix}
\end{aligned}$$

Similarly, we also have

$$\begin{aligned}
& \frac{1}{2^{k+1}} \Gamma^{-1} ((\alpha + \beta)X + \Gamma)^{k+1} \\
&= \frac{((\alpha + \beta)x + \gamma)^k}{2^{k+1}\gamma} \begin{bmatrix} (\alpha + \beta)x + \gamma & (\alpha + \beta)((k+1)y - x\omega/\gamma^2) + k\omega/\gamma \\ 0 & (\alpha + \beta)x + \gamma \end{bmatrix}.
\end{aligned}$$

Thus, the right hand side of Equation (4.1) is

$$\begin{aligned}
& \frac{\varphi(a^k)}{2^{k+1}\gamma} \left[((\alpha + \beta)\varphi(b) + \gamma)^k \left((\alpha + \beta)((k+1)\varphi'(b) - \varphi(b)\omega/\gamma^2) + k\omega/\gamma \right) \right. \\
& \quad \left. - ((\alpha + \beta)\varphi(b) - \gamma)^k \left((\alpha + \beta)((k+1)\varphi'(b) - \varphi(b)\omega/\gamma^2) - k\omega/\gamma \right) \right] \\
& \quad + \frac{\varphi'(a^k)}{2^{k+1}\gamma} \left[((\alpha + \beta)\varphi(b) + \gamma)^{k+1} - ((\alpha + \beta)\varphi(b) - \gamma)^{k+1} \right]
\end{aligned}$$

which completes the proof. $\square$

Remark 4.5. (*General Polynomials*) The commutativity of $\tilde{\mathcal{C}}$ allows again deriving infinitesimal $*$ -distributions of general polynomials. Following the notation in Remark 3.5 and assuming $\mathcal{A} \ll \mathcal{B}$, let p be any polynomials of the form $p = \sum_{j=1}^n b_j y_j b'_j$ with $b_1, b'_1, \dots, b_n, b'_n \in \mathcal{B}$ and $y_1, \dots, y_n \in \mathcal{Q}$. We denote by

$$\tilde{\mathcal{Q}} = \{A_0 B_1 A_1 \cdots B_k A_k \mid k \geq 0, a_0, \dots, a_k \in \mathcal{A} \text{ and } b_1, \dots, b_k \in \mathcal{B}\}$$

where $A_i = \text{Diag}(a_i, a_i)$ and $B_j = \text{Diag}(b_j, b_j)$ for each $0 \leq i \leq n, 1 \leq j \leq n$. For each element $Y = A_0 B_1 A_1 \cdots B_k A_k$ in $\tilde{\mathcal{Q}}$, we set $\tilde{Y} = A_0 \tilde{\varphi}(B_1) A_1 \cdots \tilde{\varphi}(B_k) A_k$. The for $P = \sum_{j=1}^n B_j Y_j B'_j$, we have by Remark 3.5 for any $m \in \mathbb{N}$,

$$\begin{aligned}
(4.3) \quad \tilde{\varphi}(P^m) &= \tilde{\varphi}\left(\left[\sum_{j=1}^n B_j \tilde{Y}_j B'_j\right]^m\right) \\
&= \sum_{j_1, \dots, j_m \in [n]} \tilde{\varphi}(\tilde{Y}_{j_1} \cdots \tilde{Y}_{j_m}) \tilde{\varphi}(B_{j_1}) \tilde{\varphi}(B'_{j_1} B_{j_2}) \tilde{\varphi}(B'_{j_2} B_{j_3}) \cdots \tilde{\varphi}(B'_{j_{m-1}} B_{j_m}) \tilde{\varphi}(B_{j_m}),
\end{aligned}$$

It remains to track the $(1, 2)$ -entry of the above equality. In the particular case where $p = \sum_{k=1}^n b_k a_k b'_k$, then

$$\varphi'(p^m) = \sum_{\substack{j_1, \dots, j_m \in [n] \\ (u_1, \dots, u_{m+1}) \in \mathcal{W}_m}} \varphi^{(u_1)}(a_{j_1} a_{j_2} \cdots a_{j_m}) \varphi^{(u_2)}(b_{j_1}) \varphi^{(u_3)}(b'_{j_1} b_{j_2}) \cdots \varphi^{(u_m)}(b'_{j_{m-1}} b_{j_m}) \varphi^{(u_{m+1})}(b'_{j_m}),$$

where $\varphi^{(0)} = \varphi$ and $\varphi^{(1)} = \varphi'$ and for each $m \geq 0$,

$$\mathcal{W}_m = \{(1, 0, 0, \dots, 0), (0, 1, 0, \dots, 0), \dots, (0, 0, \dots, 0, 1)\} \subset \mathbb{R}^{m+1}.$$

5. APPLICATIONS TO RANDOM MATRICES

We show in the section the utility of our results to derive the distributions of some random matrices with respect to the partial trace. Let us first recall that the partial trace is defined for a given $N \times N$ random matrix A_N by

$$\psi_N(A_N) := \frac{1}{N_0} (\mathbb{E} \circ \text{Tr}_{N_0})(A^{(1,1)})$$

where Tr_{N_0} is the non-normalized trace and $A^{(1,1)}$ is the first block of A_N . To draw the connection to random matrices, we recall the notation in Subsection 2.3 and the results stated therein and prove the following:

Proposition 5.1. Suppose A_N and B_N are independent $N \times N$ Wigner matrices as defined in Section 2.3, then $(T_{Q(A_N)}, B_N)_N$ is asymptotically monotone independent with respect to ψ_N for any polynomial Q whose constant term is zero.

Proof. Let Q be a polynomial that has no constant term. We note that A_N and B_N are asymptotically free with respect to $\mathbb{E} \circ \text{Tr}_N$ where we recall that tr_N is the normalized trace. Also note that $T_{Q(A_N)} = j_N Q(A_N) (1 - j_N) + (1 - j_N) Q(A_N) j_N$ where $j_N = \sum_{j=1}^{N_0} E_N^{j,j}$ and that by Theorem 2.7, $Q(A_N)$ and B_N are asymptotically infinitesimally free from j_N . Then it follows by Theorem 2.6 that $(T_{Q(A_N)}, B_N)_N$ is asymptotically monotone independent with respect to ψ_N . $\square$

Proposition 5.2. Let A_N and B_N be independent $N \times N$ Wigner matrices and let for some $m \geq 0$ and non-zero $\alpha \in \mathbb{C}$, P_N be the polynomial defined by

$$(5.1) \quad P_N = \alpha T_{A_N^m} B_N + \bar{\alpha} B_N T_{A_N^m}.$$

Then the limiting distribution of P_N with respect to ψ_N is $\mu = \frac{1}{2}(\delta_\omega + \delta_{-\omega})$ where δ is the dirac measure and $\omega = \sqrt{|\alpha| D_m}$ with

$$D_m = \begin{cases} C_m - C_{m/2}^2 & \text{if } m \text{ is even,} \\ C_m & \text{if } m \text{ is odd,} \end{cases}$$

and C_k being the k -th Catalan number.

Proof. By Proposition 5.1, $(T_{A_N^m}, B_N)_N$ is asymptotically monotone independent with respect to ψ_N . In addition, we note that the limiting distribution of $T_{A_N^m}$ with respect to ψ_N can be easily computed following [21, Proposition 5.3]. Indeed, we have for any $k \geq 1$,

$$\lim_{N \rightarrow \infty} \psi_N(T_{A_N^m}^k) = \begin{cases} \left[\lim_{N \rightarrow \infty} ((\mathbb{E} \circ \text{Tr}_N)(A_N^{2m}) - (\mathbb{E} \circ \text{Tr}_N)(A_N^m)^2) \right]^{k/2} & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

By the Wigner semicircular law, we infer that

$$\lim_{N \rightarrow \infty} \psi_N(T_{A_N}^k) = \begin{cases} D_m^{k/2} & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

Finally, observing that $\lim_{N \rightarrow \infty} (\mathbb{E} \circ \text{Tr}_N)(B_N) = 0$ and $\lim_{N \rightarrow \infty} (\mathbb{E} \circ \text{Tr}_N)(B_N^2) = 1$, we get by applying Theorem 3.1 and taking into account Remark 3.2 that for any $k \geq 1$,

$$\lim_{N \rightarrow \infty} \psi_N(P_N^k) = \begin{cases} (|\alpha| D_m)^{k/2} & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

□

Remark 5.3. The above result also holds for A_N being an $N \times N$ GUE matrix, and B_N an entry permutation of A_N that is self-adjoint. In particular, B_N can be chosen to be the transpose of A_N . Indeed, by [20] and [29], we have that $(A_N, B_N)_N$ is asymptotically free and by combining Theorems 2.6 and 2.7, we obtain that $(T_{A_N}, B_N)_N$ is asymptotically monotone independent with respect to ψ_N . Then by analogous arguments, we obtain the same limiting *-distribution of P_N with respect to ψ_N as in Proposition 5.2.

Finally, we give another example around the polynomial in Proposition 3.6 and show how our results easily extend to computing the distribution with respect to the partial trace of such a polynomial in Wigner matrices.

Proposition 5.4. Let A_N and B_N be $N \times N$ independent Wigner matrices and consider the polynomial given by

$$P_N = B_N^{2n} T_{A_N} B_N^{2h} T_{A_N} B_N^{2n} + B_N^{2m} T_{A_N} B_N^{2s} T_{A_N} B_N^{2m}$$

for some $n, m, h, s \geq 1$. Then P_N converges in distribution with respect to ψ_N to $\mu = \alpha \delta_{\vartheta+\zeta} + (1-\alpha) \delta_{\vartheta-\zeta}$ where δ denotes the dirac measure, $\alpha = \frac{1}{2\zeta}(\zeta - \vartheta + C_h C_n^2 + C_s C_m^2)$ with

$$\vartheta = \frac{1}{2}(C_{2m} C_s + C_{2n} C_h), \quad \zeta = \frac{1}{2} \sqrt{(C_{2m} C_s - C_{2n} C_h)^2 + 4b_{1,2}^2 C_s C_h},$$

and C_k being the k -th Catalan number.

Proof. It is notable that $(T_{A_N}, B_N)_N$ is asymptotically monotone independent with respect to ψ_N by Proposition 5.1. Then there is a *-probability space $(\mathcal{C}, \psi)$ and elements a and b in $\mathcal{C}$ such that $a < b$ and (T_{A_N}, B_N) converges to (a, b) in *-distribution. In other words, for any $k \geq 1$,

$$\lim_{N \rightarrow \infty} \psi_N(P_N^k) = \psi(p^k) \text{ where } p = b_1 a c_1 a b_1 + b_2 a c_2 a b_2.$$

with $b_1 = b^{2n}, b_2 = b^{2m}, c_1 = b^{2h}, c_2 = b^{2s}$. Following the notation in Proposition 3.6, we have $\tilde{b}_1 = C_n, \tilde{b}_2 = C_m, \tilde{c}_1 = C_h, \tilde{c}_2 = C_s$, and

$$(5.2) \quad b_{1,2} = \begin{cases} C_{2n} & \text{if } (i, j) = (1, 1), \\ C_{2m} & \text{if } (i, j) = (2, 2), \\ C_{n+m} & \text{if } (i, j) = (1, 2) \text{ or } (2, 1). \end{cases}$$

Then by applying Proposition 3.6, we conclude that

$$\psi(p^k) = \frac{1}{2\zeta}(\zeta - \vartheta + C_h C_n^2 + C_s C_m^2)(\vartheta + \zeta)^k + \frac{1}{2\zeta}(\zeta + \vartheta - C_h C_n^2 - C_s C_m^2)(\vartheta - \zeta)^k$$

where

$$\vartheta = \frac{1}{2}(C_{2m}C_s + C_{2n}C_h), \quad \text{and} \quad \zeta = \frac{1}{2}\sqrt{(C_{2m}C_s - C_{2n}C_h)^2 + 4C_{n+m}^2C_sC_h}.$$

□

Remark 5.5. $\mu = \alpha\delta_{\vartheta+\zeta} + (1 - \alpha)\delta_{\vartheta-\zeta}$ in Proposition 5.4 is a probability measure. One could also check that indeed $\alpha \in [0, 1]$ using the fact that

$$2C_n^2C_m^2 \leq C_{n+m}^2 \leq C_{2n}C_{2m} \text{ for all } n, m \geq 1.$$

Note that the first inequality can be deduced by induction, while the second one follows from the Cauchy-Schwartz inequality with respect to the inner product on polynomials defined by

$$\langle f, g \rangle := \int f(x)g(x)d\mu \text{ where } d\mu(t) = \frac{1}{2\pi}\sqrt{4 - t^2}dt.$$

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