# Are all Weakly Convex and Decomposable Polyhedral Surfaces Infinitesimally Rigid?

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**Abstract**

It is conjectured that all decomposable (that is, interior can be triangulated without adding new vertices) polyhedra with vertices in convex position are infinitesimally rigid and only recently has it been shown that this is indeed true under an additional assumption of codecomposability (that is, the interior of the difference between the convex hull and the polyhedron itself can be triangulated without adding new vertices).

One major set of tools for studying infinitesimal rigidity happens to be the (negative) Hessian $M_T$ of the discrete Hilbert-Einstein functional. Besides its theoretical importance, it provides the necessary machinery to tackle the problem experimentally. To search for potential counterexamples to the conjecture, one constructs an explicit family of so-called $T$-polyhedra, all of which are weakly convex and decomposable, while being non-codecomposable. Since infinitesimal rigidity is equivalent to a non-degenerate $M_T$, one can let *Mathematica* search for the eigenvalues of $M_T$ and gather experimental evidence that such a flexible, weakly convex and decomposable $T$-polyhedron may not exist.

## Contents

# 1 Introduction

## 1.1 Structure of the paper

Already back in 1766, Leonhard Euler conjectured that "a closed spatial figure allows no changes, as long as it is not ripped apart". This lead to a lot of partial results, among them the well-known theorem by Cauchy assuring us that convex "closed spatial figures" are indeed all rigid, in a sense that will be made precise soon. However, the reason for why nobody could prove Euler's conjecture in its entirety turns out to be that it is simply *wrong*, as Connelly (1977) [2] demonstrated by providing a counterexample and gifting the world its first flexible and non-self-intersecting polyhedral surface in $\mathbb{R}^3$.

Since not all closed surfaces share rigidity, the natural question would be to seek minimal conditions under which the latter holds. This lead to the following conjecture, lying at the heart of this paper:

**Main conjecture**:
Every weakly convex and decomposable polyhedron is infinitesimally rigid.

Let us remark that the motivation for studying the conjecture stems from Izmestiev and Schlenker (2010) [11]. There it was shown that the statement is indeed true under the additional assumption of codecomposability and it was left as an open problem to determine whether this supplementary requirement is truly necessary. The conjecture was however also mentioned in Connelly and Schlenker (2010) [4] as Question 1.1. and even before that, in a paper by Schlenker (2005) [14] where it also originates from. The latter proves that the conjecture holds true for all polyhedra $P$ for which there exists an ellipsoid containing no vertices of $P$ but intersecting all its edges. Before continuing, we'll first need some definitions.

**Definition 1.1.1.** *Let $S$ be a triangulation of a compact, orientable surface with $V$ denoting the set of vertices and $E$ the set of edges. A* ***polyhedral surface*** *or* ***polyhedron*** *is the image of a map $S \longrightarrow \mathbb{R}^3$ that is affine on each edge and non-degenerate on the faces.*

**Definition 1.1.2** (Izmestiev, 2011, [9])**.** *Let $P \subset \mathbb{R}^3$ be a polyhedron with vertices $V = \{p_1, ..., p_n\}$. An* ***infinitesimal isometric deformation*** *of $P$ is a map $q : V \to \mathbb{R}^3$ such that*

$$\left.\frac{d}{dt}\right|_{t=0} \operatorname{dist}(p_i + tq_i, p_j + tq_j) = 0, \tag{1.1}$$

*for all edges $p_i p_j$ of $P$ and where $q(p_i) =: q_i$.*

**Definition 1.1.3.** *A polyhedron $P \subset \mathbb{R}^3$ is said to be* ***infinitesimally rigid*** *if every infinitesimal isometric deformation is trivial in first order, that is*

$$q_i = K(p_i),$$

*for $K$ a Killing field of $\mathbb{R}^3$. If there is a non-trivial infinitesimal isometric deformation, the polyhedron is said to be* ***infinitesimally flexible****.*

So, an infinitesimal deformation is just an assignment of vectors to each vertex of a polyhedron $P$. If moving the vertices in the assigned directions induces a zero first-order variation of the edge lengths, we speak of isometric infinitesimal deformation. Such a deformation is trivial if the Euclidean distance between every pair of points is preserved, that is, the motion is just a rigid motion in $\mathbb{R}^3$. Explicitly, one can verify that Equation (1.1) is equivalent to

$$\langle p_i - p_j, q_i - q_j \rangle = 0,$$

for all edges $p_i p_j$ of $P$.

**Definition 1.1.4.** *A polyhedron $P \subset \mathbb{R}^3$ is said to be **weakly convex** if every vertex $v$ of $P$ has a supporting plane that intersects $P$ at exactly $v$.*

For instance, consider a cube having an additional vertex in the center of one of its faces. This polyhedron is convex but not weakly convex. Such a vertex is called **flat vertex**. That weak convexity is indeed a necessary condition for infinitesimal rigidity will be illustrated in Section 3.3 by an example.

It should also be pointed out that weak convexity does not imply convexity. Indeed, the Schönhardt polyhedron, as shown in Figure 2.1, already provides us with a counterexample.

**Definition 1.1.5.** *A **triangulation** of a polyhedron $P$ is a partition of its interior into tetrahedra. Such a polyhedron is said to be **decomposable** if its interior can be triangulated without adding new vertices and **codecomposable** if the interior of its complement, that is, the difference between the convex hull of $P$ and $P$ itself, can be triangulated without adding new vertices.*

Let us remark that it is still an open problem to determine sufficient criteria a polyhedron has to satisfy in order to be decomposable. Note that in the discrete geometry literature, one rather encounters the word **tetrahedralizable** instead of decomposable.

The structure of the paper is now as follows. In order to explore the main conjecture, we would like to explicitly test a certain family of weakly convex, decomposable and non-codecomposable polyhedra for infinitesimal rigidity. This is done with the aid of the Schönhardt polyhedron, which is known to be infinitesimally flexible as shown by Izmestiev (2011) [9] for instance and the simplest non-decomposable polyhedron as demonstrated by Schönhardt (1928) [15]. Albeit not being able to provide a counterexample to the main conjecture, we'll suggest a recipe that enables one to decide with a computer whether a given polyhedron is infinitesimally rigid or not. This is done throughout Sections 2 and 3 with explicit computations given in the appendix.

In Section 3, an elementary proof of the fact that the $\pi/6$-twisted Schönhardt polyhedron is infinitesimally flexible is given. This is done by rederiving Wunderlich's (1965) [17] formula (Equation (3.1)) by purely geometric means. Lastly, we provide an example to illustrate why weak convexity is a necessary condition for infinitesimal rigidity and conclude with an outlook summarizing all of the experimental observations we collected so far. This leads to a new conjecture that all polyhedra belonging to a certain family and satisfying the assumptions of the main conjecture must be infinitesimally rigid. Moreover, we collect experimental evidence that for any infinitesimally flexible polyhedron in that family weak convexity and decomposability can not be achieved simultaneously.

## 1.2 Regge Calculus and the discretization of space

Before moving to the main part of the paper, we'll take a small detour to motivate the techniques that are used to study the infinitesimal rigidity of polyhedra.

In his paper "General relativity without coordinates" (1961) [13], Tullio Regge developed a way to discretize $N$-dimensional Riemannian manifolds using a collection of $N$-dimensional building blocks whose intrinsic geometry (their metric) is Euclidean (that is, flat). This is known as *Regge Calculus*. Besides the original paper, see for example chapter 42 of Misner et al. (1973) [12].

Apart from being interesting for gravitational physics by providing the necessary tools to evaluate the curvature of Lorentzian manifolds in an intrinsic and manageable way, it also constitutes one of the cornerstones of the mathematical formulation of infinitesimal rigidity, considering that the $N$-dimensional building blocks mentioned above are Euclidean simplices $S_N$, as depicted in Figure 1.1.

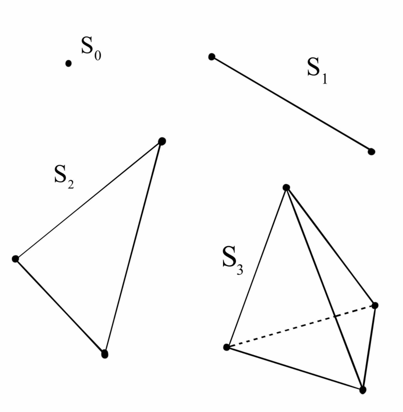



Figure 1.1: A few Euclidean simplices

Joined facet to facet, the initially smooth manifold transforms into a discrete association of simplices. What is particularly well-suited for computations is that the characterization of any such discrete manifold skeleton only requires the specification of the edge lengths of the simplices and the gluing rules for connecting them. By choosing a collection of sufficiently small simplices, any smooth manifold can be approximated to arbitrarily high precision with an assembly of that kind. The fact that edge lengths suffice to specify the intrinsic geometry, has been exploited so far to conclude that

**curvature lies concentrated into simplices of dimension** $N - 2$.

These simplices are the so-called **bones** (or, in the language of discrete geometry, the 1-skeleton) of the structure and turn out to be of uttermost value in our context. Given a weakly convex polyhedron $P \subset \mathbb{R}^3$ and a triangulation of its interior into $S_3$ simplices such that every vertex of every interior tetrahedron coincides with a vertex of the polyhedron, we will restrain ourselves from moving the vertices of $P$ (and therefore altering its boundary lengths) and modify the metric inside of it. Before making all of this precise in the next subsection, this is roughly speaking done by varying the lengths of the bones (which are just $S_1$ simplices) of $P$, that is the collection of all interior edges of the triangulation. The edge lengths of the polyhedron itself are not altered, only the ones of the individual tetrahedra. As a result, this induces a total angle of $\omega_i$ around each such interior edge that potentially differs now from $2\pi$. This is done by summing up the individual angles of the now modified tetrahedra that constituted the interior triangulation while respecting the initial gluing. Defining the total curvature of an interior edge $i$ as $\kappa_i = 2\pi - \omega_i$, Regge Calculus enables us to express it in terms of edge lengths. In order to extend this to infinitesimal rigidity, it is necessary to borrow another function from physics.

## 1.3 The discrete Hilbert-Einstein functional

Following Izmestiev (2014) [10], we briefly recall the most important results surrounding the discrete Hilbert-Einstein functional.

**Definition 1.3.1.** *Let $P \subset \mathbb{R}^3$ be a polyhedron and $T$ a triangulation of $P$ with interior edges $e_1, ..., e_n$. Then $\mathcal{D}_{\mathcal{P},\mathcal{T}}$ is defined as the collection of $n$-tuples of the form $l := (l_1, ..., l_n) \in \mathbb{R}^n_{>0}$ which, for every simplex $\sigma$ of $T$, are such that replacing the lengths of the edges of $\sigma$ which are interior edges of the triangulation by the corresponding $l_j$'s in $l$ induces a non-degenerate simplex.*

**Definition 1.3.2.** *Let $e'_1, ..., e'_r$ denote the boundary edges of $P$, $l'_j$ the length of the edge $e'_j$ and $\alpha_j$ the dihedral angle of $P$ at $e'_j$, for $j \in \{1, ..., r\}$. Moreover, for $i \in \{1, ..., n\}$, let $\omega_i$ be the total angle*

*around the interior edge $e_i$ and $\kappa_i := 2\pi - \omega_i$ the singular curvature along it.*
*In that case, the* ***discrete Hilbert-Einstein functional*** *is defined by*

$$\begin{aligned} \mathrm{HE} : \mathcal{D}_{\mathcal{P},\mathcal{T}} &\longrightarrow \mathbb{R} \\ l &\longmapsto \sum_{i=1}^{n} l_i \kappa_i + \sum_{j=1}^{r} l'_j(\pi - \alpha_j). \end{aligned}$$

Note that this functional can be viewed as the discrete analog of twice the total scalar curvature of $P$ plus half of the total mean curvature of $\partial P$, hence its name. In combination with the 3-dimensional Euclidean Schläfli formula (which is valid for each individual simplex),

$$\sum_e l_e \, \mathrm{d}\alpha_e = 0,$$

where the sum runs over all edges of $P$, $l_e$ denoting the length and $\alpha_e$ the dihedral angle at each edge, the first-order variation of the Hilbert-Einstein functional reduces to

$$\mathrm{dHE} = \sum_{i=1}^{n} \kappa_i \, \mathrm{d}l_i.$$

Note that this expression takes in tangent vectors to $\mathcal{D}_{\mathcal{P},\mathcal{T}}$ as input, so that it expresses the first-order variation of the interior edge lengths of a triangulation $T$ of a polyhedron $P$ with $n$ interior edges. Most importantly, the Hessian $(\frac{\partial^2 \mathrm{HE}}{\partial l_i \partial l_j})$ of HE is equal to the Jacobian of the map $(l_i)_{i=1}^n \to (\kappa_i)_{i=1}^n$. Since differentiation eradicates the constant of $2\pi$ in the curvature term, one has

$$M_T := \left( \frac{\partial \omega_i}{\partial l_j} \right) = - \left( \frac{\partial^2 \mathrm{HE}}{\partial l_i \partial l_j} \right).$$

Observe moreover that $M_T$ must be symmetric, given that it equals minus the Hessian of HE. This matrix plays an important role in the theory, as illustrated by the following two results. Given that symmetric matrices are especially well suited for computations, it also provides us with the necessary tools to tackle the problem experimentally.

**Theorem 1.3.3.** *Let $P$ be a convex polyhedron and $T$ a triangulation admitting $m$ interior and $k$ flat vertices. The dimension of $\ker(M_T)$ is $3m + k$ and $M_T$ has $m$ negative eigenvalues.*

*Proof.* Izmestiev and Schlenker (2010) [11]. □

So, if the convex polyhedron in question can be triangulated without interior and flat vertices, $M_T$ is positive definite. Another result that will be of great use to us is the following:

**Lemma 1.3.4.** *Let $P$ be a polyhedron admitting a triangulation $T$ without interior vertices. Then $P$ is infinitesimally rigid if and only if $M_T$ is non-degenerate.*

*Proof.* Bobenko and Izmestiev (2008) [1]. □

While Izmestiev and Schlenker proved Theorem 1.3.3 in [11], they obtained the following result as a consequence:

**Theorem 1.3.5.** *If a polyhedron is weakly convex, decomposable, and weakly codecomposable with triangular faces then it must be infinitesimally rigid.*

Here, **weakly codecomposable** denotes any polyhedron $P$ which sits inside a convex polyhedron $Q$ such that the vertices of $P$ form a subset of the vertices of $Q$ and the complement of $P$ in $Q$ (that is, the difference between $Q$ and $P$) can be triangulated without adding new vertices.

Recall that codecomposability is stronger in the sense that the complement of the polyhedron with respect to its convex hull can be triangulated without adding new vertices.

To see how Theorem 1.3.5 follows from Theorem 1.3.3, notice that for a weakly convex, decomposable, and weakly codecomposable polyhedron $P$, there must exist a convex polyhedron $P_c$ such that $P_c$ shares all its vertices with $P$, and a triangulation $T$ of $P$ that is contained in a triangulation $T_c$ of $P_c$, where the vertices of $T_c$ are precisely those of $P_c$. Now, it can be shown that $M_T$ must be a principal minor of $M_{Tc}$. Thus, by Theorem 1.3.3, $M_{Tc}$ is positive definite and therefore $M_T$ must be as well. Since positive definite matrices are invertible, Lemma 1.3.4 yields the desired result.

# 2 An empirical approach

## 2.1 The flexible Schönhardt polyhedron

Depicted in Figure 2.1 is the so-called **Schönhardt polyhedron**, a polyhedron named after his discoverer Erich Schöhnhardt and having the special property of being weakly convex, non-decomposable and (infinitesimally) flexible as shown by Izmestiev (2011) [9]. In fact, it is the simplest (in the sense of fewest vertices) flexible polyhedron which doesn't admit a triangulation without interior vertices as was verified by Schönhardt (1928) [15], making it the perfect footing onto which to construct polyhedra violating the codecomposability assumption of Theorem 1.3.5.

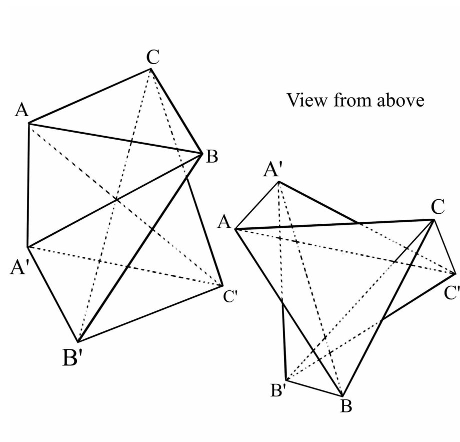



Figure 2.1: The Schönhardt polyhedron

As a figurative example, consider a Schönhardt polyhedron with vertices

$$A = (1,0,1) \qquad B = \left(\cos\tfrac{2\pi}{3}, \sin\tfrac{2\pi}{3}, 1\right) \qquad C = \left(\cos\tfrac{4\pi}{3}, \sin\tfrac{4\pi}{3}, 1\right)$$

$$A' = \left(\cos\tfrac{\pi}{6}, \sin\tfrac{\pi}{6}, -1\right) \qquad B' = \left(\cos\tfrac{5\pi}{6}, \sin\tfrac{5\pi}{6}, -1\right) \qquad C' = (0,-1,-1),$$

in $\mathbb{R}^3$, having edges $AA'$, $AC'$, $BB'$, $BA'$, $CC'$ and $CB'$ and faces $AA'C'$, $AA'B$, $BA'B'$, $BB'C$, $CB'C'$, $CC'A$, $ABC$ and $A'B'C'$. Later on, we will have to remove faces $ABC$ and $A'B'C'$ from the list since we'll restrict ourselves to weakly convex polyhedra of genus one with triangular faces (for the sake of simplicity) and admitting the Schönhardt polyhedron as their complement (in the sense that their convex hull contains the Schönhardt polyhedron).

In practice, the following three major steps will be employed to discern infinitesimally rigid polyhedra from infinitesimally flexible ones:

1. Triangulate the polyhedron. This triangulation $T$ will have $n \in \mathbb{N}$ interior edges.

2. Calculate all the dihedral angles between the faces of the simplices making up $T$ and meeting at an interior edge of the triangulation of $P$. The total angle around an interior edge will be the sum of all such dihedral angles.

3. Determine the eigenvalues of $M_T$ (a $n \times n$ matrix). If zero is not an eigenvalue, $P$ must be infinitesimally rigid.

Let us remark that it is in practice not feasible to search for infinitesimal flexible polyhedra through the linear system of equations (1.1) proposed in Definition 1.1.2. In fact, this method would require one to stumble exactly upon the right polyhedron and would involve many issues coming from rounding errors. However, when working with $M_T$, this is not the case since it suffices to just find two polyhedra having different signatures. This would then be enough to conclude that a polyhedron with zero-determinant $M_T$ (which would then be infinitesimally flexible) must exist between them, without having to construct it explicitly.

Since we will explicitly construct $P$ (instead of studying random polyhedra), step one is not something we will have to worry about much. What is of greater importance is to find a way to determine the total angle around the interior edges $e_i$, for $i = 1, ..., n$, of $P$ as a function of the edge lengths $l_i$ of the $e_i$ (remember, the entries of $M_T$ are the derivatives of total angle around each edge with respect to edge length). This is not as straightforward as it sounds at first since expressing dihedral angles of tetrahedra as a function of edge lengths is sometimes quite cumbersome and unnecessary complicates the process even further (for example, using trigonometry and Heron's formula requires calculating the areas of the faces, which we neither need nor want). A more efficient method to obtain the desired dihedral angles turns out to be through Cayley-Menger determinants.

## 2.2 The Cayley-Menger determinant

A sextuple of the form $S = (e_{12}, e_{13}, e_{14}, e_{23}, e_{24}, e_{34})$ determines a non-degenerate Euclidean tetrahedron (that is, not all points of $S$ are lying in the same plane) if and only if the following two conditions are satisfied:

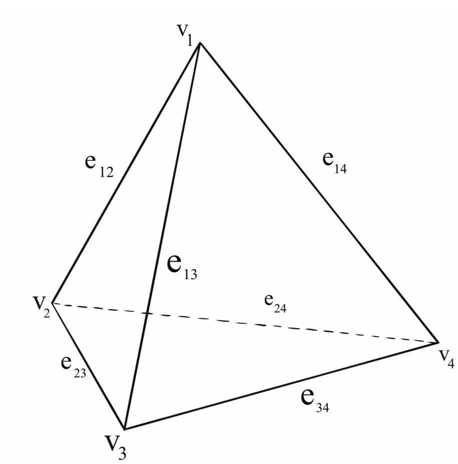



Figure 2.2: A tetrahedron

1. All face triplets of $S$ are of the form $F = (e_{ij}, e_{ik}, e_{jk})$, where $F$ satisfies $e_{ij} < e_{ik} + e_{jk}$, $e_{ik} < e_{ij} + e_{jk}$ and $e_{jk} < e_{ij} + e_{ik}$;

2. The determinant $D$ of the following matrix is strictly greater than 0:

$$\mathrm{CM} := \begin{pmatrix} 0 & e_{12}^2 & e_{13}^2 & e_{14}^2 & 1 \\ e_{12}^2 & 0 & e_{23}^2 & e_{24}^2 & 1 \\ e_{13}^2 & e_{23}^2 & 0 & e_{34}^2 & 1 \\ e_{14}^2 & e_{24}^2 & e_{34}^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

The first condition simply states that the tetrahedron has four faces with non-negative edge lengths satisfying the triangle inequality. The determinant $D$ is the Cayley-Menger determinant. Its geometric significance becomes apparent when considering how it relates to the volume $V$ of the tetrahedron, that is,

$$D = 288V^2.$$

See Fiedler (2011) [5] for a proof. In order to compute dihedral angles, it is necessary to look at a specific minor of the matrix CM. Given any edge $e_{ij}$ of the tetrahedron, the term $e_{ij}^2$ appears twice in CM, namely in the two rows and columns $i$ and $j$. To obtain the desired cofactor, defined by

$$D_{ij} := (-1)^{k+l} \cdot (k,l) \text{ - minor of CM},$$

it suffices to localize the two terms $e_{ij}^2$ in CM and delete row $k$ and column $l$ which does not contain the term $e_{ij}^2$, evaluate the determinant and multiply by $(-1)^{k+l}$. It is important that the 5th row and column of CM does not take part in this excision process.

As an example, in order to obtain $D_{12}$, delete row 3 and column 4 of CM (or, equivalently, row 4 and column 3), calculate the determinant of this smaller matrix and multiply it by minus one, since $3 + 4 = 7$.

Denoting the interior dihedral angle at the edge $e_{ij}$ by $\alpha_{ij}$, the following relationship between $D$ and $\alpha_{ij}$ can be derived:

$$\alpha_{ij} = \arccos\left(\frac{D_{ij}}{\sqrt{2e_{ij}^2 D + D_{ij}^2}}\right) \tag{2.1}$$

There are many ways to prove this, see for instance Fiedler (2011) [5].

# 3 Rotational flexibility

## 3.1 When, how and why does a certain polyhedron twist?

The first part of this section is devoted to the Schönhardt polyhedron. In particular, we would like to understand which properties discern it from other, infinitesimally rigid, polyhedra. Let us therefore start by recalling

**Cauchy's rigidity theorem**: If two convex polyhedra in $\mathbb{R}^3$ have pairwise congruent faces, then the two polyhedra must themselves be congruent.

What is meant by requiring **congruence** is that the two faces (or sets of points in general) can be transformed into each other by means of isometries, that is, rigid motions in $\mathbb{R}^3$ which are combinations of translations, rotations and reflections, with no changes in size allowed. Even though the untwisted and twisted states of the Schönhardt polyhedron have the same polyhedral net, they form a pair of incongruent octahedra. Yes, two corresponding faces are congruent to each other and two corresponding edges have the same convexity character (one concave edge in each side), however, the twisted and untwisted states of the Schönhardt polyhedron do not constitute convex polyhedra and hence Cauchy's theorem is not applicable to begin with.

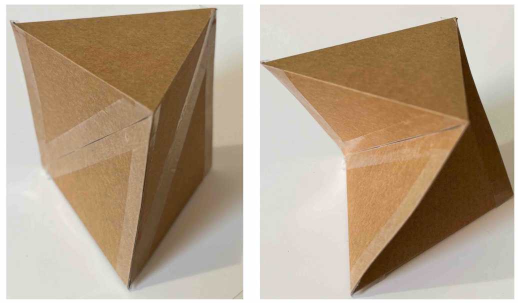


Figure 3.1: Cardboard model of the Schönhardt polyhedron

As is shown in Figure 3.1, the untwisted Schönhardt polyhedron is not an upright triangular prism, but a so-called concave, triangular gyroprism. Let us also remark that the two states of the Schönhardt polyhedron can not be continuously deformed into each other. This means in particular that if the model depicted in Figure 3.1 was made of a perfectly stiff material, one wouldn't be able to twist it from one state to the other without disassembling and rebuilding it. Now, if one were to play around with it for a while, one would make the following **experimental observation** (**EO**):

**EO1**: There is no flex of the Schönhardt polyhedron that does not involve a twist.

Even though this seems to be a trivial statement, it is far from being a superfluous one.

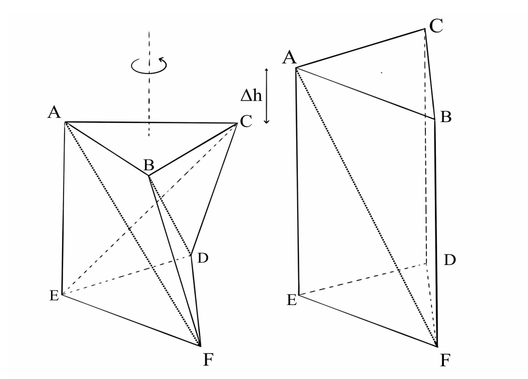



Figure 3.2: Twisting the Schönhardt polyhedron

Still and all, this is not the end of the story. Observe how the motion of untwisting the Schönhardt polyhedron induces a height augmentation $\Delta h$ of the structure and that every vertex of the top triangle is prescribed to move on a cylinder (while the bottom basis is kept in a fixed position). It is then natural to believe that this $\Delta h$ can be expressed in terms of some of the vital polyhedral characteristics (such as height, angle and edge lengths of the two equilateral triangles). Moreover, one can easily convince oneself, by trying to construct a model for instance, that the polyhedral net must not be made out of a perfect rectangle but rather a parallelogram. We can denote this little overhang by $m$, as depicted in Figure 3.3.

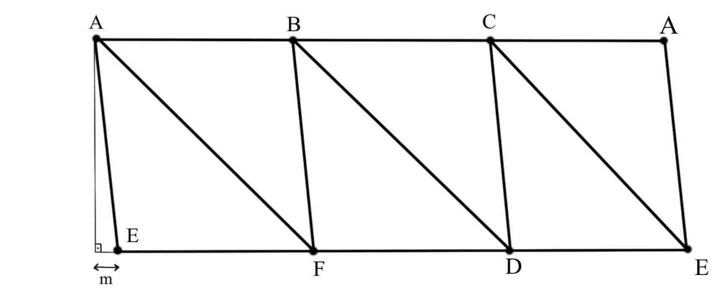



Figure 3.3: The Schönhardt polyhedron's net with top and bottom faces removed

As already remarked, we can keep the bottom basis fixed and imagine an infinitesimal flex. During that process, all edge lengths are kept at constant lengths and so the vertex $A$ (for instance) of the top basis is not only constrained to move on a cylinder of radius $r$, but also on a circle of radius $c$, (corresponding to the edge length $EA$). Figure 3.4 describes how the (infinitesimal) twist is done:

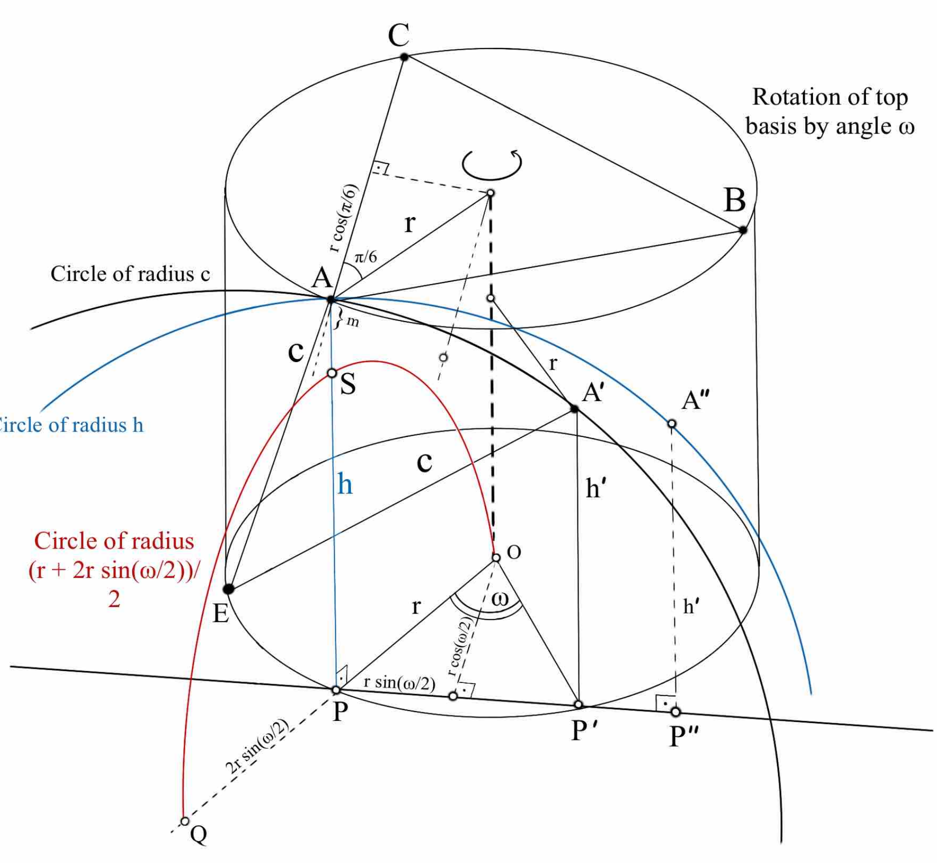



Figure 3.4: Twisting the Schönhardt polyhedron

**Proposition 3.1.1.** *With notation as in Figures 3.3 and 3.4, the difference of the square of the* *heights of the twisted and untwisted Schönhardt polyhedron denoted, respectively, by $h$ and $h'$, is given by*

$$h^2 - h'^2 = 2r^2 \sin\left(\frac{\omega}{2}\right). \tag{3.1}$$

*Proof.* Note that this formula already appears in Wunderlich (1965) [17], albeit derived by different means. Here we present a purely geometric proof using Figure 3.4. Hollow points indicate that they do not belong to the set of vertices of the Schönhardt polyhedron.
Now, since $ABC$ is an equilateral triangle, all its angles must be equal to $\pi/3$. Thus, by construction, the two segments of length $r\cos\left(\frac{\pi}{6}\right)$ and $r\cos\left(\frac{\omega}{2}\right)$ are parallel and their difference is equal to $m$, that is,

$$m = r\cos\left(\frac{\omega}{2}\right) - r\cos\left(\frac{\pi}{6}\right) = r\left(\cos\left(\frac{\omega}{2}\right) - \cos\left(\frac{\pi}{6}\right)\right).$$

With the fact that cosine is bounded by 1 and $1 - \cos\left(\frac{\pi}{6}\right) = 1 - \sqrt{3}/2 \approx 0.134$, we find that the overhang $m$ must satisfy

$$0 \leq m \leq 0.134 \cdot r.$$

On the other hand, observe that

$$\text{distance}(P, P') = 2r\sin\left(\frac{\omega}{2}\right).$$

Constructing the blue circle of radius $h$ and center $P$ gives birth, upon projecting the point $A'$ onto the blue circle and connecting it to the line that passes through the points $P$ and $P'$, to the right angled triangle $PA''P''$. In order to find the distance between $P$ and $P''$, we can extend the line segment connecting $O$ and $P$ by a length of $2r\sin\left(\frac{\omega}{2}\right)$. The trick consists in taking $2r\sin\left(\frac{\omega}{2}\right) + r$ to be the diameter of a new circle. This circle (depicted in red) has of course radius $\left(2r\sin\left(\frac{\omega}{2}\right) + r\right)/2$ (which is the arithmetic mean of $2r\sin\left(\frac{\omega}{2}\right)$ and $r$) and intersects the segment $h$ at the point $S$ which, by the geometric properties of semi-circles, is at a distance of $\sqrt{2r^2\sin\left(\frac{\omega}{2}\right)}$ from the point $P$. The length of $\sqrt{2r^2\sin\left(\frac{\omega}{2}\right)}$ corresponds to the geometric mean of $2r\sin\left(\frac{\omega}{2}\right)$ and $r$. The only thing that is left to do is to notice that

$$\text{distance}(P, P'') = \text{distance(P, S)}$$

and apply the Pythagorean theorem to the triangle $PA''P''$

$$h^2 = h'^2 + \left(\sqrt{2r^2\sin\left(\frac{\omega}{2}\right)}\right)^2,$$

that is,

$$h^2 - h'^2 = 2r^2\sin\left(\frac{\omega}{2}\right).$$

□

It remains to show that the Schönhardt polyhedron is indeed infinitesimally flexible.

**Theorem 3.1.2.** *The $\pi/6$-twisted Schönhardt polyhedron is infinitesimally flexible.*

*Proof.* By Definition 1.1.3, we know that a polyhedron is infinitesimally rigid if every isometric infinitesimal deformation is trivial in first order, so that each motion corresponds to a rigid body motion. If $\nu = \{v_1, \ldots, v_n\}$ denotes the set of vertices of the polyhedron $P$, any such motion can be expressed through a map $q : \nu \to \mathbb{R}^3$ satisfying

$$\left.\frac{d}{dt}\right|_{t=0} \text{distance}(v_i + tq(v_i), v_j + tq(v_j)) = 0 \tag{3.2}$$

for every edge $v_i v_j$ of $P$. This is equivalent to

$$\langle v_i - v_j, q_i - q_j \rangle = 0 \tag{3.3}$$

because (3.2) forces $2(v_i + tq(v_i)q_i + 2(v_j + tq(v_j)q_j = 0$ which, at $t = 0$, is just $v_i q_i + v_j q_j = 0$. So, in order to show infinitesimal flexibility, we need to find at least one infinitesimal isometric deformation that is non-trivial in first order.

In the case of the Schönhardt polyhedron, $\nu = \{A, B, C, D, E, F\}$. Assume furthermore that the top basis is twisted by an angle of $\theta = \pi/6$ with respect to the bottom basis. Here, $\theta$ denotes the total rotation, that is, the angle $\omega$ (as depicted in Figure 3.4) and the smaller angle stemming from the overhang $m$ added together. The reason for choosing the particular value of $\pi/6$ is that this corresponds to the unique choice of "twisting angle" $\theta$ that makes the polyhedron infinitesimally flexible.

To see why, assume for simplicity that the side lengths of the equilateral triangles $ABC$ and $DEF$ are equal to 1 and apply the Pythagorean theorem to the triangle $AFE$ such to obtain

$$AF^2 = 1^2 + \left( \sqrt{\frac{\sin(\pi/3 + \theta)}{\cos(\pi/6)}} \right)^2$$

and

$$AF^2 = 1 + \frac{2}{\sqrt{3}} \sin(\pi/3 + \theta).$$

This function hits its maximum at $\theta = \pi/6$ and therefore forces its derivative to have a zero at that value. In other words, the edge length $AF$ does not change (up to first order) with respect to $\theta$, or, by imagining that the twist is executed in a uniform and symmetric manner, $AF$ is kept constant with respect to time (similar holds of course for the other diagonals $BD$ and $CE$). Of course, this will then be used to study infinitesimal isometric deformations.

Let $q(D) = q(E) = q(F) = 0$ or, equivalently, keep the bottom basis of the polyhedron at a fixed position and apply a non-zero velocity vector (pointing to the outside of the polyhedron) to the remaining vertices such that $q(A)$ is orthogonal to the plane $AEF$ and $q(B)$ and $q(C)$ are the images of $q(A)$ under rotation by an angle of $2\pi/3$ and $4\pi/3$ around the axis of the cylinder of radius $r$. Clearly, the edge lengths of the triangle $ABC$ are kept constant since no transformation is applied to its vertices. Since $q(A)$ is just an infinitesimal rotation around the edge $EF$ of the vertex $A$, the side lengths of the triangle $EAF$ are preserved up to first order.

By symmetry, $q(B)$ and $q(C)$ have a similar effect on the vertices $B$ and $C$ and thus the side lengths of the triangles $DFB$ and $CDE$ are also infinitesimally preserved. Taking our particular choice of $\theta = \pi/6$ into account, the planes $AEF$, $BDF$ and $CDE$ must pass through the center of the triangle $ABC$ and so $q(A)$, $q(B)$ and $q(C)$ are tangent to the cylinder of radius $r$ and axis passing through the center of $ABC$. This on its own means, by taking the symmetry of the motion into consideration, that $ABC$ does indeed *twist* around the axis of the cylinder and therefore naturally preserves its side lengths.

Hence, we have found a non-trivial infinitesimal isometric transformation (corresponding to a twist) and can conclude that this Schönhardt polyhedron (of angle $\theta = \pi/6$) is indeed infinitesimally flexible. □

Remember, our initial goal is to build a weakly convex and decomposable structure violating the codecomposability hypothesis of the main conjecture and, in the (quite utopian) best-case scenario, obtain an infinitesimally flexible polyhedron that will therefore disprove it.

Now, the absence of a suitable triangulation for the Schönhardt polyhedron is due to the fact that any such tetrahedron would have vertices contained in the top and bottom basis of the latter and therefore edges that coincide with the diagonals of the Schönhardt polyhedron. However, the Schönhardt polyhedron has no internal diagonals and so any simplex of a potential triangulation could never lie entirely inside, thus the impossibility to decompose it.

Before investigating more complicated arrangements, it will prove useful to set up some terminology.

## 3.2 Some weakly convex, non-codecomposable polyhedra

Let us start by remarking that whenever we mention notions of rigidity and flexibility in what follows, we are referring to infinitesimal rigidity and infinitesimal flexibility.

Since we would like to examine the family of weakly convex, decomposable polyhedra having the Schönhardt polyhedron as its complement, we set up some terminology and refer to them as $T$**-polyhedra**, $T$ standing for *twist* (even though most of them will be perfectly rigid), Every such polyhedron can be partitioned into three regions:

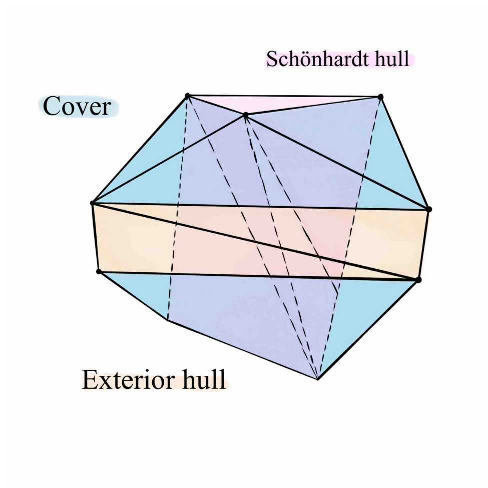



Figure 3.5: A $T$-polyhedron

The **cover** at the top and bottom, the **Schönhardt hull** lying in the middle of the structure and the **exterior hull**, a polygonal ring connected to the cover. Notice how vertices belonging to the cover must necessarily be connected to vertices of the Schönhardt hull. Of course, nothing prevents the exterior hull from being trivial (in the sense of nonexistent) such that all vertices of the upper cover are immediately connected to the bottom cover. However, polyhedra of that kind will not be of great use to us since they most certainly will never twist. To see why, observe the following:

**EO2**: Any potential infinitesimal twist of a $T$-polyhedron will preserve the geometry of the cover, that is, the cover can only perform an infinitesimal rigid body motion.

In other words, the distance between vertices of the cover and vertices of the Schönhardt hull to which they are connected remains constant during any potential twist. Thus, if the exterior hull is trivial, the rigidity of the cover will prevent the polyhedron from twisting. One could speak of *induced* rigidity stemming from the exterior hull and spreading onto the whole $T$-polyhedron. In that spirit:

**EO3**: If the exterior hull is rigid (can not be twisted), then the $T$-polyhedron is itself rigid (can not be twisted either).

So, in the case of convex exterior hulls, it is not necessary to calculate any eigenvalues or Cayley-Menger determinants since **EO3** allows us to immediately conclude that the polyhedron is rigid, which already rules out quite a few candidates. Alright, but there is even more.

With **EO1** in mind, note how the rotational flexibility of the Schönhardt polyhedron forces the exterior hull to not only have rotational flexibility as well, but to rotate in one and the same manner. For instance, constructing a $T$-polyhedron with an exterior hull that rotates to the right and a Schönhardt hull that twists to the left would produce a perfectly rigid structure even though the individual "building pieces" are not. The same is (probably?) true for exterior hulls that would potentially flex in some other way that does not involve a twist.

In the spirit of the discussion from the previous section, recall how the motion of untwisting the Schönhardt polyhedron induces a height augmentation $\Delta h$ of the structure. This implies that any exterior hull giving rise to a flexible $T$-polyhedron would have to vary in height by the same amount $\Delta h$. Thus:

**EO4**: A flexible $T$-polyhedron is characterized by the height variation $\Delta h$ of the Schönhardt hull. Any rotational motion of the exterior hull must not only be performed in the same direction as the one of the Schönhardt hull, but vary in height by the same amount of $\Delta h$.

Being equipped with **EO4** and the contrapositive of **EO3**, a natural choice of $T$-polyhedra to test would be the ones obtained by replacing the exterior hull by a second Schönhardt polyhedron. After all, we can be assured that the exterior hull has the same "rotational properties" as the Schönhardt hull and calibrating the edge lengths of the equilateral triangles constituting it with respect to the height of the twisted exterior hull (such that to verify **EO4**), this would supply us with a nice and flexible polyhedron. See Figure 3.6, with top and bottom cover removed for a better visualization.

Figure 3.6: Nested Schönhardt polyhedra

Even though this might seem like we completed our task by finding a flexible $T$-polyhedron (and the thought does indeed admit quite a compelling attraction), we are unfortunately anything but done. The problem lies in the sole fact that our promising candidate is actually not a $T$-polyhedron!

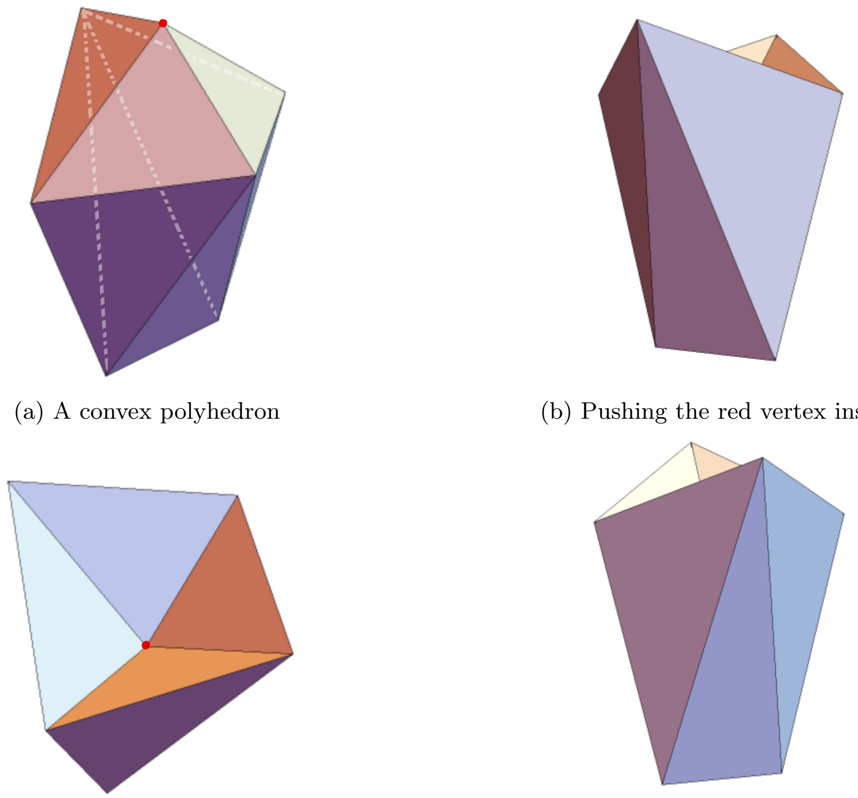
To see why, remember that $T$-polyhedra are by construction assumed to be decomposable and that calculating eigenvalues of $M_T$ requires us to find a triangulation of the polyhedron as a preliminary. Since there must exist some simplex of any potential triangulation of the preceding polyhedron that would have to share one of its faces with one of the triangular faces of the Schönhardt hull, the remaining fourth vertex of the simplex would be forced to lie on a vertex of the exterior hull, creating thereby forbidden intersections.

Figure 3.7: Resulting non-decomposable polyhedron

Since the possibly easiest choice of exterior hull that permits rotations did not provide us with the result we sought, it is time to look at more complicated (and less obvious) examples with the help of a computer. This purely computational approach will however not be explored in depth here (see Appendix).

## 3.3 Seemingly incompatible assumptions

It proves not to be particularly difficult to find counterexamples to variations of the main conjecture obtained by dropping one of the assumptions. To see why weak convexity is necessary, consider the polyhedron in Figure 3.8a. Since it is decomposable via a certain triangulation $T$, one can calculate the eigenvalues of the associated matrix $M_T$. Of course, such a computation is not needed in this particular case since Cauchy's Rigidity Theorem assures us that due to its convexity, the polyhedron must be rigid. As a check, it can be computationally confirmed that $M_T$ does indeed admit only positive eigenvalues.



(a) A convex polyhedron

(b) Pushing the red vertex inside

(c) View from above

(d) Weak convexity is lost

Figure 3.8: A non weakly convex example

Things look quite different when investigating the polyhedron obtained by "pushing" the top vertex of the previous one towards the interior. Even though the triangulations of both polyhedra admit the same number of simplices, the first one is (weakly) convex whereas the second one is definitely not. In the latter case, we find that $M_T$ admits a negative real and zero as eigenvalues, which implies, using Lemma 1.3.4, that the polyhedron is not infinitesimally rigid, hence infinitesimally flexible. So, weak convexity is not an assumption that should be dropped from the main conjecture.

Coming back to $T$-polyhedra, it is indeed the case that the triangulation issue we encountered before can be resolved, and that by shifting the Schönhardt hull to an appropriate height (for a given exterior hull, there are two possible choices for the height of the Schönhardt hull that would make its $\Delta h$ be identical to the $\Delta h$ of the exterior hull).

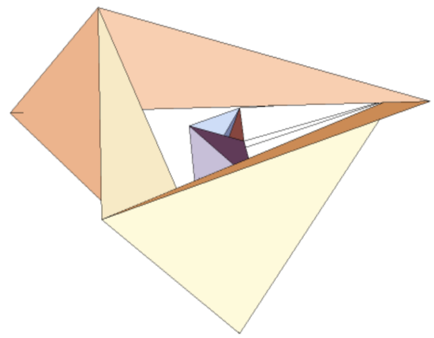


Figure 3.9: Resolving non-decomposability

Unfortunately, this process makes the polyhedron lose weak convexity and admittedly, it does seem just like flexibility forces us to choose between decomposability or weak convexity, an exclusive or. Thus, let us make one final (and quite reckless) **experimental conjecture**:

**EC1**: There is no $T$-polyhedron that twists, that is, any weakly convex, decomposable polyhedron having the Schönhardt polyhedron as its complement is infinitesimally rigid.

### 3.4 Conclusions and outlook

As the reader may have remarked, the conducted *Mathematica* - calculations (although very interesting) did not provide us with any additional information. Taking into account that the polyhedra studied in this paper were rather small (none of them exceeded 18 vertices) and predictable, most of the computations could have been replaced by a straightforward application of a certain theorem and we were not able to develop the powerful formalism involving $M_T$ to its full potential. This could of course be achieved by writing a better program that allows to test millions of more complicated examples. Maybe such an expanded and more sophisticated search would lead towards the almighty one that might disprove the main conjecture.

We hope that this note will be considered helpful then.

## A Finding the eigenvalues of $M_T$ with *Mathematica*

In order to demonstrate how *Mathematica* extracts the eigenvalues of $M_T$, it is best to study a simple example. The first step consists in defining the combinatorics of the polyhedron. For instance, an octahedron can be encoded as:

```
    Q =
Polyhedron[{A1 = {1,0,0}, A2 = {-1,0,0}, A3 = {0,1,0},
A4 = {0,-1,0}, A5 = {0,0,1}, A6 = {0,0,-1}},
{{1,3,5},{2,5,4},{4,1,5},{6,1,3},
{3,6,2},{2,6,4},{4,6,1},{3,2,5}}]
```

Since this octahedron can be triangulated by means of four equivalent simplices only one dihedral angle needs to be determined, the remaining ones being identical in this case. Thus, we choose the vertices A2, A4, A5, and A6, and compute the resulting Cayley-Menger determinant:

```
    CM1 =
```

```
Det[{{0,EuclideanDistance[A2,A5]^2, EuclideanDistance[A5,A4]^2,
        EuclideanDistance[A5,A6]^2,1},
    {EuclideanDistance[A2,A5]^2, 0, EuclideanDistance[A2,A4]^2,
        EuclideanDistance[A2,A6]^2, 1},
    {EuclideanDistance[A5,A4]^2, EuclideanDistance[A2,A4]^2,
        0, EuclideanDistance[A4,A6]^2, 1},
    {EuclideanDistance[A5,A6]^2, EuclideanDistance[A2,A6]^2,
        EuclideanDistance[A4,A6]^2, 0,1},
    {1,1,1,1,0}}]
```

Now, in order to use Equation (2.1), one starts by computing the minor of the matrix CM1 associated to the edge length of the simplex that corresponds to the interior edge length of the triangulation. Concretely, this is the edge between vertices A5 and A6. Then,

```
    De2 =
Det[{{0, EuclideanDistance[A5,A4]^2, EuclideanDistance[A5,A6]^2,1},
    {EuclideanDistance[A2,A5]^2, EuclideanDistance[A2,A4]^2,
        EuclideanDistance[A2,A6]^2, 1},
    {EuclideanDistance[A5,A4]^2, 0, EuclideanDistance[A4,A6]^2, 1},
    {EuclideanDistance[A5,A6]^2, EuclideanDistance[A4,A6]^2, 0,1}}]
```

in combination with Equation (2.1) and

```
    ArcCos[De2/(Sqrt[2 * EuclideanDistance[A5,A6]^2 * CM + (De2)^2])]
```

yields the expected dihedral angle of $\alpha_0 := \pi/2$. This is the initial dihedral angle.

Of course, the total angle around the edge $A5, A6$ is the sum of the individual dihedral angles. Since they are all equal, we obtain $4 \cdot \pi/2 = 2\pi$, which is not much of a surprise. In particular, since we haven't deformed the metric inside of the polyhedron yet, everything is nice and Euclidean and dihedral angles naturally sum up to $2\pi$. A useful fact to remember.

Recalling that the entries of $M_T$ are the derivatives of the total angle around each edge with respect to the corresponding edge length, we are faced with a complication since differentiating the expression obtained by Equation (2.1) slows the calculations down considerably. Thus as a first step and in order to facilitate computations, we'll make the following approximation:

$$\frac{\partial \omega_i}{\partial l_j} \approx \frac{\omega_{if} - \omega_{i0}}{l_{jf} - l_{j0}}.$$

In other words, the interior edge lengths of the triangulation receive a tiny length variation, say $\epsilon$ (of course, $\epsilon > 0$), which (eventually) induces a change in the total angle around the modified edge and all the remaining interior edges as well. As remarked earlier, prior to the change of interior edge lengths the geometry is perfectly Euclidean. Hence $\omega_{i0} = 2\pi$, even without calculating any CM determinants. In that vein, we can reformulate the approximated derivative as

$$\frac{\omega_{if} - \omega_{i0}}{l_{jf} - l_{j0}} = \frac{\omega_{if} - 2\pi}{l_{j0} + \epsilon - l_{j0}} = \frac{\omega_{if} - 2\pi}{\epsilon},$$

which gains evermore on accuracy the smaller the $\epsilon$ is. The only quantity that is left to be determined is now $\omega_{if}$. Coming back to our example, we can pick $\epsilon = 0.00000001$ and repeat the process from before while taking care that the edge length between vertices $A5$ and $A6$ has now gained on length (all the other edge lengths are kept constant). The Cayley-Menger determinant in this case is

```
    CM2 =
Det[{{0, EuclideanDistance[A2,A5]^2, EuclideanDistance[A5,A4]^2,
```

```
(EuclideanDistance[A5,A6] + 0.00000001)^2, 1},
{EuclideanDistance[A2,A5]^2, 0, EuclideanDistance[A2, A4]^2,
EuclideanDistance[A2,A6]^2, 1},
{EuclideanDistance[A5,A4]^2, EuclideanDistance[A2, A4]^2,
0, EuclideanDistance[A4,A6]^2, 1},
{(EuclideanDistance[A5,A6] + 0.00000001)^2,
EuclideanDistance[A2,A6]^2, EuclideanDistance[A4, A6]^2, 0, 1},
{1,1,1,1,0}}]
```

and with the appropriate minor

```
    De3 =
Det[{{0, EuclideanDistance[A5,A4]^2,
        (EuclideanDistance[A5,A6] + 0.00000001)^2,1},
    {EuclideanDistance[A2,A5]^2, EuclideanDistance[A2,A4]^2,
        EuclideanDistance[A2,A6]^2, 1},
    {EuclideanDistance[A5,A4]^2, 0, EuclideanDistance[A4,A6]^2, 1},
    {(EuclideanDistance[A5,A6] + 0.00000001)^2, EuclideanDistance[A4,A6]^2,0,1}}]
```

one can use once again Equation (2.1) to obtain

```
    ArcCos[De3/(Sqrt[2 * EuclideanDistance[A5, A6]^2 * CM2 + (De3)2])] = 1.5708.
```

With this, the matrix $M_T$ becomes

$$M_T = \left(\frac{\omega_{if} - 2\pi}{\epsilon}\right) = \left(\frac{\omega_{if} - 2\pi}{0.00000001}\right) = \left(\frac{4 \cdot 1.5708 - 2\pi}{0.00000001}\right) = (1469.28)\,.$$

It being a $1 \times 1$ matrix in this example, the eigenvalues are easily read off and we can conclude, with the aid of Lemma 1.3.4, that the polyhedron is indeed infinitesimally rigid.

Since this particular polyhedron is convex, it is true that we could have just applied Cauchy's theorem to conclude the same, and that without having to do any calculations. However, the purpose of this example was to give a simple outline of the method, nothing more and nothing less.

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