INTERMEDIATE LONG WAVE EQUATION IN NEGATIVE SOBOLEV SPACES

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Abstract. We study the intermediate long wave equation (ILW) in negative Sobolev spaces. In particular, despite the lack of scaling invariance, we identify the regularity $s = -\frac{1}{2}$ as the critical regularity for ILW with any depth parameter, by establishing the following two results. (i) By viewing ILW as a perturbation of the Benjamin-Ono equation (BO) and exploiting the complete integrability of BO, we establish a global-in-time a priori bound on the H^s-norm of a solution to ILW for $-\frac{1}{2} < s < 0$. (ii) By making use of explicit solutions, we prove that ILW is ill-posed in H^s for $s < -\frac{1}{2}$. Our results apply to both the real line case and the periodic case.

Contents

1. Introduction

We consider the intermediate long wave equation (ILW) on $\mathcal{M} = \mathbb{R}$ or $\mathbb{T} = (\mathbb{R}/\mathbb{Z})$:

$$\begin{cases} \partial_t u - \mathcal{G}_\delta \partial_x^2 u = \partial_x(u^2) \\ u|_{t=0} = u_0, \end{cases} \qquad (t,x) \in \mathbb{R} \times \mathcal{M} \tag{1.1}$$

for $0 < \delta < \infty$. The operator $\mathcal{G}_\delta$ is given by

$$\mathcal{G}_\delta = \mathcal{T}_\delta - \delta^{-1}\partial_x^{-1}, \tag{1.2}$$

where $\mathcal{T}_\delta$ is the Fourier multiplier operator with symbol

$$\widehat{\mathcal{T}_\delta f}(\xi) = -i\coth(\delta\xi)\widehat{f}(\xi), \quad \xi \in \widehat{\mathcal{M}}. \tag{1.3}$$

Here, $\widehat{\mathcal{M}}$ denotes the Pontryagin dual of $\mathcal{M}$, i.e. $\widehat{\mathcal{M}} = \mathbb{R}$ if $\mathcal{M} = \mathbb{R}$, and $\widehat{\mathcal{M}} = \mathbb{Z}$ if $\mathcal{M} = \mathbb{T}$. The ILW equation (1.1) was introduced in [14, 23] as a model describing the propagation of an internal wave at the interface of a stratified fluid of finite depth δ, with further applications in modeling wave phenomena in oceanography and meteorology. Furthermore, it appears as

2020 *Mathematics Subject Classification.* 35Q35, 37K10, 76B55.

Key words and phrases. intermediate long wave equation; Benjamin-Ono equation; a priori bound; complete integrability; ill-posedness.

an "intermediate" equation of finite depth $0 < \delta < \infty$ between the Benjamin-Ono equation (deep-water limit: $\delta \to \infty$) and the KdV equation (shallow-water limit: $\delta \to 0$), attracting wide attention from both the applied and theoretical scientific communities. Even from the purely analytical point of view, (1.1) is of great interest due to its rich structure; it is a dispersive equation, admitting soliton solutions. Moreover, it is completely integrable with an infinite number of conservation laws. See [35, 20] for an overview of these topics and more on the physical significance of ILW.

Despite recent popularity of the ILW equation and its deep connection to the well-known Benjamin-Ono and KdV equations, there remain many open questions in well-posedness of (1.1) and convergence as $\delta \to 0$ or ∞. In this paper, we focus on the former question; see [1, 32, 30, 11, 6] for the known well-posedness results for ILW. See also [1, 25, 26, 6, 7] for results on convergence issues from both deterministic and statistical viewpoints.

It is known (see [31, 22]) that, just like the Benjamin-Ono equation (see (1.4) below), ILW (1.1) is quasilinear in the sense that a contraction argument can not be used for constructing solutions, which makes the well-posedness question rather challenging, especially in a low-regularity setting.[1] In [11], Ifrim and Saut proved global well-posedness of ILW (1.1) in $L^2(\mathbb{R})$. In a recent preprint [6], the first, third, fourth, and fifth authors provided a unified argument for L^2-global well-posedness of (1.1) on both the real line and the circle. We point out that the basic strategy in [11, 6] is to view ILW (1.1) as a perturbation of the Benjamin-Ono equation (BO):

$$\partial_t u - \mathcal{H}\partial_x^2 u = \partial_x(u^2), \tag{1.4}$$

where $\mathcal{H}$ denotes the usual Hilbert transform with multiplier[2] $-i\,\mathrm{sgn}(\xi)$, $\xi \in \widehat{\mathcal{M}}$, and to suitably adapt the known well-posedness arguments for the BO equation [13, 29, 12]. We will elaborate this viewpoint further in the following.

Our main goal in this paper is to study issues related to well-posedness of ILW (1.1) in negative Sobolev spaces. It is well known that a scaling symmetry, if it exists, provides an important threshold (called a scaling critical regularity) on well-posedness for a dispersive equation. For example, BO (1.4) on the real line is known to be invariant under the following $\dot{H}^{-\frac{1}{2}}$-invariant scaling:

$$u_\lambda(t,x) = \lambda^{-1}u(\lambda^{-2}t, \lambda^{-1}x), \quad \lambda > 0. \tag{1.5}$$

This scaling symmetry induces the scaling critical regularity $s = -\frac{1}{2}$ for BO. While ILW does not enjoy a scaling symmetry, it was remarked in [6, Remark 4.2] that if u is a solution to ILW (1.1) on $\mathbb{R}$ (with the depth parameter δ), then the rescaled function u_λ in (1.5) solves (1.1) with the depth parameter $\lambda\delta$. Namely, *the family of the ILW equations with depth parameters* $0 < \delta < \infty$ *is invariant under the scaling* (1.5). This observation hints that the regularity $s = -\frac{1}{2}$ may be critical for ILW in an appropriate sense. We show that this is indeed the case by establishing the following results.

Theorem 1.1. *Let $\mathcal{M} = \mathbb{R}$ or $\mathbb{T}$ and $0 < \delta < \infty$. Then, the following statements hold.*

[1] We point out that, under the mean-zero assumption, the periodic BO and ILW posed on the circle $\mathbb{T}$ may be semilinear just like the KdV equation, at least in a smooth setting. See [28, Theorem 1.2], where, for $s \geq 0$, the solution map for BO (1.4) on $\mathbb{T}$ was shown to be real-analytic from the subspace $H^s_0(\mathbb{T}) \subset H^s(\mathbb{T})$, consisting of mean-zero functions, into itself. At this point, however, there is no known well-posedness argument via a contraction argument for the mean-zero periodic BO and ILW.

[2] On $\mathbb{T}$, we set $\mathrm{sgn}(0) = 0$.

(i) (global-in-time a priori bound). *Let* $-\frac{1}{2} < s < 0$. *Given* $u_0 \in H^\infty(\mathcal{M})$, *let* u *be the (unique) smooth solution to the ILW equation* (1.1). *Then, given small* $\varepsilon > 0$, *there exist positive constants* C_s *and* $A_{\delta,s} \sim \delta^{-2}(1 + \delta^{-|s|-\frac{1}{2}-\varepsilon})$, *independent of* $u_0 \in H^\infty(\mathcal{M})$, *such that*

$$\|u(t)\|_{H^s} \le C_s^{|s|+1} e^{A_{\delta,s}|t|} \Big(1 + 2C_s e^{A_{\delta,s}|t|} \|u_0\|_{H^s}\Big)^{\frac{2|s|}{1-2|s|}} \|u_0\|_{H^s} \tag{1.6}$$

for any $t \in \mathbb{R}$.

(ii) (ill-posedness) *Let* $s < -\frac{1}{2}$. *Then, the ILW equation* (1.1) *is ill-posed in* $H^s(\mathcal{M})$. *Moreover, when* $\mathcal{M} = \mathbb{T}$, *given any* $\alpha \in \mathbb{R}$, *the ILW equation* (1.1) *is ill-posed in* $H^s_\alpha(\mathbb{T})$, *where* $H^s_\alpha(\mathbb{T})$ *denotes the subspace of* $H^s(\mathbb{T})$ *consisting of functions with spatial mean* α.

It follows from the proof of Theorem 1.1 (ii) that if the solution map $\Phi : H^s(\mathcal{M}) \to C([-T,T]; H^s(\mathcal{M}))$, sending initial data u_0 to solutions $u = \Phi(u_0)$ of ILW (1.1), extended to $s < -\frac{1}{2}$, then it would be discontinuous at $u_0 = -2\pi\delta_0$ for any $T > 0$, where δ_0 denotes the Dirac delta function. The known global well-posedness of ILW in $L^2(\mathcal{M})$ [11, 6] and the ill-posedness result in $H^s(\mathcal{M})$ for $s < -\frac{1}{2}$ (Theorem 1.1 (ii)) leave the gap $-\frac{1}{2} \le s < 0$. While the a priori bound in Theorem 1.1 (i) indicates that well-posedness should extend, at least, to the range $-\frac{1}{2} < s < 0$, the actual well-posedness of ILW in the range $-\frac{1}{2} \le s < 0$ on either geometry is completely open. In view of the positive and negative results in Theorem 1.1, we propose that $s = -\frac{1}{2}$ is the critical regularity for the ILW equation (1.1), which is in particular independent of the depth parameter δ.

Let us briefly discuss the strategy for proving Theorem 1.1. As for the ill-posedness claim in Theorem 1.1 (ii), we follow closely the strategy in [5, 4] for ill-posedness of BO in $H^s(\mathcal{M})$, $s < -\frac{1}{2}$; see also [15]. Namely, we make use of explicit traveling wave solutions to ILW (1.1) which approximate, at time $t = 0$, (a constant multiple of) the Dirac delta function as the speed of the wave diverges to infinity. On the circle, the BO equation (1.4) is known to be ill-posed in the critical space $H^{-\frac{1}{2}}(\mathbb{T})$ whose proof heavily relies on the complete integrability via the use of the Birkhoff map; see [9, Section 7]. It would be of interest to investigate if a similar ill-posedness result in $H^{-\frac{1}{2}}(\mathbb{T})$ holds for ILW.

Let us now turn to Theorem 1.1 (i). While ILW is known to be completely integrable, we do not make use of its integrable structure (which is not well understood) to prove Theorem 1.1(i). We instead view ILW (1.1) as a perturbation of BO (1.4) (just as in [11, 6]) and exploit the integrable structure of the BO equation. Define $\mathcal{Q}_\delta$ by

$$\mathcal{Q}_\delta = (\mathcal{T}_\delta - \mathcal{H})\partial_x, \tag{1.7}$$

where $\mathcal{T}_\delta$ is as in (1.3). Then, in view of (1.2), we can write ILW (1.1) as

$$\partial_t u - \mathcal{H}\partial_x^2 u = \partial_x(u^2) - \delta^{-1}\partial_x u + \mathcal{Q}_\delta \partial_x u. \tag{1.8}$$

As seen in [11, 6],[3] the operator $\mathcal{Q}_\delta$ enjoys a strong smoothing property (see Lemma 2.1), which allows us to view the last term in (1.8) as a perturbation in a suitable sense.

[3]In [11, 6], a Galilean transform (see (1.11) below) was applied to remove the linear term $-\delta^{-1}\partial_x u$ in (1.8). While we could apply the same Galilean transform to remove this term and study a renormalized ILW, it turns out that such a procedure is not necessary for establishing an a priori bound, since the generator for this linear term (i.e. $\delta^{-1}M(u)$, where $M(u)$ is the mass defined in (3.8)) Poisson-commutes with the key quantity $\beta_s(\kappa; u)$ defined in (3.9); see Lemma 3.1.

There have been two successful approaches to the well-posedness study of BO, exploiting its complete integrability. In [9], Gérard, Kappeler, and Topalov proved sharp global well-posedness of the periodic BO in $H^s(\mathbb{T})$, $s > -\frac{1}{2}$, by building a suitable Birkhoff map. In a recent preprint [16], Killip, Laurens, and Vişan applied the method of commuting flows [17] to the BO equation (1.4) and proved sharp[4] global well-posedness in $H^s(\mathcal{M})$, $s > -\frac{1}{2}$, on both the real line and the circle. In the following, we use the completely integrable structure for BO as presented in the latter work [16]. In [16], for $\kappa \gg 1$, the authors constructed the quantity $\beta_s(\kappa; u)$ (see (3.9) below) which is conserved under the flow of BO and is equivalent to the H^s-norm, provided that $-\frac{1}{2} < s < 0$; see Lemma 3.1 below. While this quantity $\beta_s(\kappa; u)$ is not conserved under the flow of ILW (1.8), the only non-zero contribution to its time derivative comes from the last term $\mathcal{Q}_\delta \partial_x u$ in (1.8). As this term is linear and enjoys sufficient smoothing, we can apply a Gronwall argument to control the growth of $\beta_s(\kappa; u)$. This explains the reason for the time-dependent growth in (1.6).

We conclude this introduction with several remarks.

Remark 1.2. (i) While we expect that there are time-independent a priori bounds for ILW, it seems that one would have to develop an appropriate completely integrable structure of ILW for this purpose. We chose not to pursue this direction to exemplify the point that ILW can be thought of as a perturbation of BO to obtain the a priori bound. We note that (the proof of) the a priori bound in Theorem 1.1 also holds for suitable (potentially) non-integrable variants of BO; see, for example, Part (ii) of this remark and also Remark 1.3.

(ii) A close look at the proof of Theorem 1.1 (i) (see (3.22) in the proof of Lemma 3.2) shows that we only need smoothing[5] of order $\frac{3}{2} - s + \varepsilon$ (for some $\varepsilon > 0$) from the operator $\mathcal{Q}_\delta$ in proving the a priori bound (1.6). This in particular implies that if we instead consider the following variant of the BO equation:

$$\partial_t u - \mathcal{H}\partial_x^2 u = \partial_x(u^2) + c_1 \partial_x u + c_2 I \partial_x u, \tag{1.9}$$

where $c_1, c_2 \in \mathbb{R}$ and I is a linear operator with smoothing of order $\frac{3}{2} - s + \varepsilon$ for some $\varepsilon > 0$, then a slight modification of the proof of Theorem 1.1 (i) yields an analogous a priori bound on the H^s-norm of a solution to (1.9) for $-\frac{1}{2} < s < 0$. We point out that, under a weaker assumption on I being smoothing of order 1, a slight modification of the argument in [6] yields global well-posedness of (1.9) in $L^2(\mathcal{M})$. Our proof of ill-posedness in Theorem 1.1 (ii) relies on the explicit solutions to ILW (1.1) and thus it does not extend to (1.9).

(iii) By using a differencing technique as in [18], we expect that our approach of building a (time-dependent) a priori bound (as in Theorem 1.1 (i)) will extend to positive regularities. See also [32, 6] for persistency-of-regularity arguments, controlling the H^s-norms of solutions to (1.1), at least for $0 < s < 1$.

(iv) Following [18], a quantity based on a series expansion of the perturbation determinant was used in [36] to establish an a priori bound on the H^s-norm of a solution to BO for $-\frac{1}{2} < s < 0$. See also [34, 19]. For our purpose, however, we find the quantity $\beta_s(\kappa; u)$ in (3.9) more convenient especially because it does not involve a series expansion.

(v) We point out a similarity between our argument for establishing the a priori bound (Theorem 1.1 (i)) and the work of Laurens [24] who studied low-regularity well-posedness of the KdV

[4]Except for the endpoint $s = -\frac{1}{2}$ on the real line.

[5]Namely, mapping $L^2(\mathcal{M})$ into $H^{\frac{3}{2}-s+\varepsilon}(\mathcal{M})$.

equation with a space-time potential. The presence of the potential also broke the conservation laws and thus a Gronwall argument was needed to control their growth.

Remark 1.3. In [23], the equation for the motion of an internal wave in a finite depth fluid was derived with two depth parameters δ_j, $j = 1, 2$, where δ_1 and δ_2 represent the depths of the upper and lower fluids, respectively, and is given by

$$\partial_t u - c_1 \mathcal{G}_{\delta_1} \partial_x^2 u - c_2 \mathcal{G}_{\delta_2} \partial_x^2 u = \partial_x(u^2), \tag{1.10}$$

where $c_1, c_2 > 0$. By applying the Galilean transform[6]

$$v(t, x) = u(t, x + \gamma t), \quad \gamma := \tfrac{c_1}{\delta_1} + \tfrac{c_2}{\delta_2}, \tag{1.11}$$

we see that v satisfies the renormalized equation:

$$\partial_t v - c_1 \mathcal{T}_{\delta_1} \partial_x^2 v - c_2 \mathcal{T}_{\delta_2} \partial_x^2 v = \partial_x(v^2). \tag{1.12}$$

Then, we rewrite (1.12) as

$$\partial_t v - (c_1 + c_2)\mathcal{H}\partial_x^2 v = \partial_x(v^2) + c_1 \mathcal{Q}_{\delta_1} \partial_x^2 v + c_2 \mathcal{Q}_{\delta_2} \partial_x^2 v. \tag{1.13}$$

By viewing (1.13) as a perturbation of the following BO equation:

$$\partial_t v - (c_1 + c_2)\mathcal{H}\partial_x^2 v = \partial_x(v^2), \tag{1.14}$$

a slight modification of the argument in [6] yields global well-posedness of (1.12) (and of (1.10)) in $L^2(\mathcal{M})$, and, moreover, the solutions converge to solutions of (1.14) as $\min(\delta_1, \delta_2) \to \infty$. Similarly, a slight modification of the proof of Theorem 1.1 (i) yields an a priori bound on the H^s-norm of a solution to (1.10) for $-\frac{1}{2} < s < 0$. We point out that (1.10) is not expected to be completely integrable.

2. Notations

We write $A \lesssim B$ to denote that there exists $C > 0$ such that $A \le CB$, and $A \ll B$ when $A \le CB$ with sufficiently small $C > 0$.

Next, we go over our convention for Fourier transforms, following [16]. On the real line, we write

$$\widehat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-i\xi x} dx \quad \text{and} \quad f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \widehat{f}(\xi) e^{i\xi x} d\xi, \tag{2.1}$$

while on the circle, we set

$$\widehat{f}(\xi) = \int_{\mathbb{T}} f(x) e^{-i\xi x} dx \quad \text{and} \quad f(x) = \sum_{\xi \in 2\pi\mathbb{Z}} \widehat{f}(\xi) e^{i\xi x}.$$

Then, Plancherel's identity takes the form

$$\|f\|_{L^2(\mathbb{R})} = \|\widehat{f}\|_{L^2(\mathbb{R})} \quad \text{and} \quad \|f\|_{L^2(\mathbb{T})} = \|\widehat{f}\|_{L^2(2\pi\mathbb{Z})} = \bigg(\sum_{\xi \in 2\pi\mathbb{Z}} |\widehat{f}(\xi)|^2 \bigg)^{\frac{1}{2}},$$

[6]We point out that the Galilean transform (1.11) is needed only for proving L^2-global well-posedness of (1.10), following the argument in [6], and that it is not needed to establish an a priori bound in $H^s(\mathcal{M})$, $-\frac{1}{2} < s < 0$.

and $\widehat{\partial_x f}(\xi) = i\xi \widehat{f}(\xi)$. Given $s \in \mathbb{R}$ and $\kappa > 0$, we define the L^2-based Sobolev spaces $H^s_\kappa(\mathbb{R})$ and $H^s_\kappa(\mathbb{T})$ via

$$\|f\|_{H^s_\kappa(\mathbb{R})} = \bigg(\int \langle \xi \rangle_\kappa^{2s} |\widehat{f}(\xi)|^2 d\xi \bigg)^{\frac{1}{2}} \quad \text{and} \quad \|f\|_{H^s_\kappa(\mathbb{T})} = \bigg(\sum_{\xi \in 2\pi\mathbb{Z}} \langle \xi \rangle_\kappa^{2s} |\widehat{f}(\xi)|^2 \bigg)^{\frac{1}{2}},$$

where $\langle \xi \rangle_\kappa = (\kappa^2 + |\xi|^2)^{\frac{1}{2}}$; see also [21, 33]. When $\kappa = 1$, $H^s_\kappa(\mathcal{M})$ reduces to the standard L^2-based Sobolev space $H^s(\mathcal{M})$. For $s < 0$, the H^s_κ-norm of f is decreasing in κ, which plays an important role in proving (3.10) below; see [16, the proof of Lemma 4.3]. Moreover, for $s < 0$ and $\kappa \geq 1$, we have

$$\|f\|_{H^s} \leq \kappa^{-s} \|f\|_{H^s_\kappa}. \tag{2.2}$$

We define the Cauchy-Szegő projector Π_+ by setting

$$\widehat{\Pi_+ f}(\xi) = \mathbf{1}_{[0,\infty)}(\xi) \cdot \widehat{f}(\xi).$$

Then, the Hardy space $H^s_+(\mathcal{M})$ is defined by $H^s_+(\mathcal{M}) = \Pi_+ H^s(\mathcal{M})$. Recall that when $\mathcal{M} = \mathbb{R}$, functions in $H^s_+(\mathbb{R})$ are the boundary values (on the real line) of holomorphic functions on the upper half-plane, and that when $\mathcal{M} = \mathbb{T}$, functions in $H^s_+(\mathbb{T})$ are the boundary values (on the circle) of holomorphic functions on the unit disc.

Next, we record the following smoothing property of the operator $\mathcal{Q}_\delta \partial_x$.

Lemma 2.1. *Let $\mathcal{M} = \mathbb{R}$ or $\mathbb{T}$ and $0 < \delta < \infty$. Then, given $s_1, s_2 \in \mathbb{R}$ with $s_1 \leq s_2$, there exists $C_{s_1 - s_2} > 0$, independent of $0 < \delta < \infty$, such that*

$$\|\mathcal{Q}_\delta \partial_x f\|_{H^{s_2}(\mathcal{M})} \leq C_{s_1 - s_2}\, \delta^{-2} (1 + \delta^{s_1 - s_2}) \|f\|_{H^{s_1}(\mathcal{M})}. \tag{2.3}$$

Proof. We proceed as in the proof of [6, Lemma 2.3]; see also [11, Lemma 2.2]. With a slight abuse of notation, let $\widehat{\mathcal{Q}_\delta}(\xi)$ denote the multiplier for the operator $\mathcal{Q}_\delta$ in (1.7). Then, we have

$$\widehat{\mathcal{Q}_\delta}(\xi) = \xi \big(\coth(\delta \xi) - \operatorname{sgn}(\xi) \big) = \frac{2|\xi|}{e^{2|\delta\xi|} - 1}$$

for $\xi \neq 0$. Then, (2.3) follows from noting that $x^\sigma \leq C_\sigma (e^{2x} - 1)$ for any $x \geq 0$, provided that $\sigma \geq 1$. $\square$

3. Global-in-time a priori bound

In this section, we present the proof of Theorem 1.1 (i) by viewing ILW (1.8) as a perturbation of the BO equation (1.4) and exploiting the completely integrable structure of BO.

3.1. **Completely integrable structure of the BO equation.** In this subsection, we recall from [16] the completely integrable structure of the BO equation (1.4) and relevant results. Note that our convention for the signs in (1.4) differ from that in [16]. In order to translate between the two, one should use the map $u \mapsto -u$. First, recall that BO is a Hamiltonian PDE with the Hamiltonian:

$$\text{Hamiltonian:} \;\; H_{\text{BO}}(u) = \frac{1}{2} \int_{\mathcal{M}} u \mathcal{H} \partial_x u dx + \frac{1}{3} \int_{\mathcal{M}} u^3 dx, \tag{3.1}$$

where the Poisson bracket is given by

$$\{F, G\} = \int_{\mathcal{M}} \frac{\partial F}{\partial u} \partial_x \frac{\partial G}{\partial u} dx. \tag{3.2}$$

Moreover, BO is completely integrable with a Lax pair $(L, B) = (L_u, B_u)$ given by

$$L = -i\partial_x + \Pi_+ u \quad \text{and} \quad B = -i\partial_x^2 + 2\partial_x \Pi_+ u - 2(\partial_x \Pi_+ u) \tag{3.3}$$

such that $\partial_t L = [B, L]$, when u is a solution to BO. We also denote by $L_0 = -i\partial_x$ the Lax operator with the zero potential. The following lemma summarizes the basic properties of the Lax operator L (see Part (i) below for the definition of $L = L_u$ with a proper domain) and its resolvent from [16]; see Propositions 3.2, 4.1, 4.3, and 4.7 and Lemma 4.11 in [16].

Lemma 3.1. *Let $\mathcal{M} = \mathbb{R}$ or $\mathbb{T}$, $-\frac{1}{2} < s < 0$, and $\sigma = \frac{1}{2}(\frac{1}{2} + s) \in (0, \frac{1}{4})$. Then, there exists a constant $C_s \geq 1$ such that whenever $u \in H^s(\mathcal{M})$ satisfies*

$$\kappa \geq C_s \big(1 + \|u\|_{H^s_\kappa}\big)^{\frac{1}{2\sigma}} \tag{3.4}$$

for some $\kappa \geq 1$, the following statements hold true.

(i) *There exists a unique self-adjoint, semi-bounded operator $L = L_u$ associated to the quadratic form*

$$f \mapsto \langle f, L_0 f\rangle_{L^2} + \int_{\mathcal{M}} u(x)|f(x)|^2 dx$$

with $H^{\frac{1}{2}}_+(\mathcal{M})$ as the domain for the quadratic form, where $\langle \cdot, \cdot\rangle_{L^2}$ denotes the L^2-inner product given by $\langle f, g\rangle_{L^2} = \int_{\mathcal{M}} \overline{f(x)} g(x) dx$. The resolvent $R(\kappa; u) = (L + \kappa)^{-1}$ exists and maps $H^{-\frac{1}{2}}_+(\mathcal{M})$ into $H^{\frac{1}{2}}_+(\mathcal{M})$.

(ii) *Let $m(\kappa; u) = -R(\kappa; u)\Pi_+ u$. Then, we have*

$$\|m(\kappa; u)\|_{H^{s+1}_\kappa} \lesssim \|u\|_{H^s_\kappa} \quad \text{and} \quad \|m(\kappa; u)\|_{H^s} \lesssim \kappa^{-1}\|u\|_{H^s}. \tag{3.5}$$

Moreover, if $u \in H^\infty(\mathcal{M})$, then $m \in H^\infty(\mathcal{M})$.

(iii) *The quantity $\beta(\kappa; u)$ defined by*

$$\beta(\kappa; u) = -\int u(x) m(x; \kappa, u) dx = \langle \Pi_+ u, R(\kappa; u)\Pi_+ u\rangle_{L^2}$$

is finite, real-valued, and real-analytic as a function of u, and satisfies

$$\frac{\partial \beta}{\partial u} = -\big(m + \overline{m} + |m|^2\big). \tag{3.6}$$

Moreover, we have

$$\{\beta(\kappa; u); H_{\mathrm{BO}}(u)\} = \{\beta(\kappa; u); M(u)\} = 0, \tag{3.7}$$

where $M(u)$ is the mass defined by

$$M(u) = \frac{1}{2}\int_{\mathcal{M}} u^2 dx. \tag{3.8}$$

Finally, by setting

$$\beta_s(\kappa; u) = \int_\kappa^\infty \tau^{2s} \beta(\tau; u) d\tau, \tag{3.9}$$

we have

$$C_s^{-1}\|u\|^2_{H^s_\kappa} \leq \beta_s(\kappa; u) \leq C_s \|u\|^2_{H^s_\kappa}. \tag{3.10}$$

Recalling that when $s < 0$, the H^s_κ-norm is decreasing in $\kappa \geq 1$, we see that if, given $u \in H^s(\mathcal{M})$, the condition (3.4) is satisfied for $\kappa = \kappa_0$ for some $\kappa_0 \geq 1$, then (3.4) holds for any $\kappa \geq \kappa_0$. In view of (3.7) and (3.9), we see that $\beta_s(\kappa; u)$ is conserved under the BO dynamics.

3.2. **Proof of Theorem 1.1 (i).** In this subsection, we present the proof of Theorem 1.1 (i). Let us state a lemma, where, under some assumption, we control the growth of $\beta_s(\kappa; u)$ via a Gronwall argument.

Lemma 3.2. *Let $\mathcal{M} = \mathbb{R}$ or $\mathbb{T}$ and $0 < \delta < \infty$. Given $-\frac{1}{2} < s < 0$, let C_s and $\sigma = \frac{1}{2}(\frac{1}{2} + s)$ be as in Lemma 3.1. Let u be a smooth global solution to ILW (1.1) such that*

$$\kappa \geq \sup_{t\in[0,T]} C_s\big(1 + c_0\|u(t)\|_{H^s}\big)^{\frac{1}{2\sigma}} \tag{3.11}$$

for some $\kappa \geq 1$, $T > 0$, and $c_0 \geq 1$. Then, there exists $A_{\delta,s} > 0$, independent of $\kappa \geq 1$, $T > 0$, and $c_0 \geq 1$, such that

$$\beta_s(\kappa; u(t)) \leq e^{A_{\delta,s}t}\beta_s(\kappa; u(0)) \tag{3.12}$$

for any $0 \leq t \leq T$.

In the following, we first present the proof of Theorem 1.1 (i) by assuming Lemma 3.2 whose proof is presented at the end of this section.

Proof of Theorem 1.1 (i). We only consider the case $t \geq 0$. Fix $T > 0$, and set

$$c_0 = C_s e^{A_{\delta,s}T} \geq 1 \tag{3.13}$$

where $C_s > 0$ is as in Lemma 3.1 and $A_{\delta,s} > 0$ is as in Lemma 3.2. Given $u_0 \in H^\infty(\mathcal{M})$, fix $\kappa \geq 1$ such that

$$\kappa \geq C_s(1 + 2c_0\|u_0\|_{H^s_\kappa})^{\frac{1}{2\sigma}}, \tag{3.14}$$

where $\sigma = \frac{1}{2}(\frac{1}{2} + s)$ is as in Lemma 3.1. Then, it follows from the continuity in time of u with values in $H^s_\kappa(\mathcal{M})$ (recall $c_0 \geq 1$) that there exists $0 < T_0 \leq T$ such that

$$\|u(t)\|_{H^s_\kappa} \leq 2c_0\|u_0\|_{H^s_\kappa} \tag{3.15}$$

for any $0 \leq t \leq T_0$. It follows from (3.14) and (3.15) that the condition (3.11) in Lemma 3.2 is satisfied on $[0, T_0]$. Hence, by applying Lemma 3.2 with (3.10) and (3.13), we have

$$\|u(t)\|_{H^s_\kappa} \leq c_0\|u_0\|_{H^s_\kappa} \tag{3.16}$$

for any $0 \leq t \leq T_0$. Therefore, by a continuity argument, we conclude that (3.16) holds on the entire interval $[0, T]$.

Finally, by choosing

$$\kappa = C_s(1 + 2c_0\|u_0\|_{H^s})^{\frac{1}{2\sigma}}, \tag{3.17}$$

we obtain from (2.2), (3.17), (3.16), and the monotonicity of the H^s_κ-norm in κ that

$$\begin{aligned}
\|u(t)\|_{H^s} &\leq \kappa^{|s|}\|u(t)\|_{H^s_\kappa}\\
&\leq C_s^{|s|+1}e^{A_{\delta,s}T}\Big(1 + 2C_s e^{A_{\delta,s}T}\|u_0\|_{H^s}\Big)^{\frac{2|s|}{1-2|s|}}\|u_0\|_{H^s}
\end{aligned}$$

for $0 \leq t \leq T$, from which we conclude (1.6) for any $t \geq 0$. □

We conclude this section by presenting the proof of Lemma 3.2.

Proof of Lemma 3.2. Let $H_{\mathrm{ILW},\delta}$ be the Hamiltonian for ILW (1.1) (with the Poisson bracket in (3.2)):

$$H_{\mathrm{ILW},\delta}(u) = \frac{1}{2}\int_{\mathcal{M}} u\mathcal{G}_\delta\partial_x u dx + \frac{1}{3}\int_{\mathcal{M}} u^3 dx.$$

In view of (1.2), (1.7) (3.1), and (3.8), we then have

$$H_{\mathrm{ILW},\delta}(u) = H_{\mathrm{BO}}(u) - \delta^{-1}M(u) + H_{\mathcal{Q}_\delta}(u), \quad \text{where } \ H_{\mathcal{Q}_\delta}(u) = \frac{1}{2}\int_{\mathcal{M}} u\mathcal{Q}_\delta u dx. \tag{3.18}$$

Noting that $\mathcal{Q}_\delta$ is a symmetric operator, we have

$$\frac{\partial H_{\mathcal{Q}_\delta}}{\partial u} = \mathcal{Q}_\delta u. \tag{3.19}$$

Fix $\kappa \geq 1$ and $T > 0$. Let u be a smooth global solution to (1.1), satisfying (3.11). Since $s < 0$ and $\kappa \geq 1$, we have $\|u(t)\|_{H^s_\kappa} \leq \|u(t)\|_{H^s}$. Thus, the hypothesis (3.11) implies that (3.4) is satisfied. In particular, $\beta_s(\kappa; u(t))$ is well defined for every $t \in [0,T]$ and all the results of Lemma 3.1 hold. Recalling that $\partial_t F(u(t)) = \{F, H_{\mathrm{ILW},\delta}\}(u(t))$ for a smooth function $F(u)$ (see [10, Lemma 2.8]), it follows from (3.9), (3.18), (3.7), and (3.2) with (3.6) and (3.19) that

$$\begin{aligned}
\frac{d}{dt}\beta_s(\kappa; u(t)) &= \int_\kappa^\infty \tau^{2s}\frac{d}{dt}\beta(\tau; u(t))d\tau \\
&= \int_\kappa^\infty \tau^{2s}\{\beta(\tau), H_{\mathrm{ILW},\delta}\}(u(t))d\tau \\
&= \int_\kappa^\infty \tau^{2s}\{\beta(\tau), H_{\mathcal{Q}_\delta}\}(u(t))d\tau \\
&= -\int_\kappa^\infty \tau^{2s}\int_{\mathcal{M}}(m(\tau; u(t)) + \overline{m}(\tau; u(t)) + |m(\tau; u(t))|^2)\mathcal{Q}_\delta\partial_x u(t) dx d\tau \\
&=: I_1 + I_2 + I_3,
\end{aligned} \tag{3.20}$$

where I_1, I_2, and I_3 represent the contributions from $m(\tau; u(t))$, $\overline{m}(\tau; u(t))$, and $|m(\tau; u(t))|^2$, respectively. From Cauchy-Schwarz's inequality (on the Fourier side), (3.5), Lemma 2.1, (2.2), and (3.10), we have

$$\begin{aligned}
|I_1| + |I_2| &\lesssim \int_\kappa^\infty \tau^{2s}\|m(\tau; u(t))\|_{H^s}\|\mathcal{Q}_\delta\partial_x u(t)\|_{H^{-s}}\, d\tau \\
&\lesssim \delta^{-2}(1 + \delta^{-2|s|})\|u(t)\|^2_{H^s}\int_\kappa^\infty \tau^{2s-1}d\tau \\
&\lesssim \delta^{-2}(1 + \delta^{-2|s|})\kappa^{2|s|}\|u(t)\|^2_{H^s_\kappa}\kappa^{-2|s|} \\
&\lesssim \delta^{-2}(1 + \delta^{-2|s|})\|u(t)\|^2_{H^s_\kappa} \\
&\lesssim \delta^{-2}(1 + \delta^{-2|s|})\beta_s(\kappa; u(t)).
\end{aligned} \tag{3.21}$$

Similarly, from Hölder's inequality, the Sobolev embedding theorem (with small $\varepsilon > 0$), Lemma 2.1, (2.2), $\langle\xi\rangle_\tau^{s+1} \geq \tau^{s+1}$ (recall that $s > -\frac{1}{2}$), (3.5), and the monotonicity of the

H^s_κ-norm (in κ), we have

$$\begin{aligned}
|I_3| &\le \|\mathcal{Q}_\delta \partial_x u(t)\|_{H^{\frac12+\varepsilon}} \int_\kappa^\infty \tau^{2s} \|m(\tau; u(t))\|_{L^2}^2 d\tau \\
&\lesssim \delta^{-2}(1+\delta^{-|s|-\frac12-\varepsilon})\kappa^{|s|}\|u(t)\|_{H^s_\kappa} \int_\kappa^\infty \tau^{2s-2(s+1)} \|m(\tau; u(t))\|_{H^{s+1}_\tau}^2 \, d\tau \\
&\lesssim \delta^{-2}(1+\delta^{-|s|-\frac12-\varepsilon})\kappa^{|s|}\|u(t)\|_{H^s_\kappa} \int_\kappa^\infty \tau^{-2} \|u(t)\|_{H^s_\tau}^2 \, d\tau \\
&\lesssim \delta^{-2}(1+\delta^{-|s|-\frac12-\varepsilon})\kappa^{|s|}\|u(t)\|_{H^s_\kappa}^3 \int_\kappa^\infty \tau^{-2} d\tau \\
&\lesssim \delta^{-2}(1+\delta^{-|s|-\frac12-\varepsilon}) \frac{\|u(t)\|_{H^s_\kappa}^3}{\kappa^{1+s}}.
\end{aligned} \tag{3.22}$$

By separately considering the cases $\|u(t)\|_{H^s_\kappa} < 1$ and $\|u(t)\|_{H^s_\kappa} \ge 1$ (where, in the latter case, we use (3.11) with $2\sigma \le 1+s$ which follows from the definition of σ in Lemma 3.1), the fact that $\kappa, c_0 \ge 1$, and (3.10), we have

$$\begin{aligned}
|I_3| &\lesssim \delta^{-2}(1+\delta^{-|s|-\frac12-\varepsilon})\|u(t)\|_{H^s_\kappa}^2 \\
&\lesssim \delta^{-2}(1+\delta^{-|s|-\frac12-\varepsilon})\beta_s(\kappa; u(t)).
\end{aligned} \tag{3.23}$$

Hence, from (3.20), (3.21), and (3.23), we have

$$\frac{d}{dt}\beta_s(\kappa; u(t)) \le C\delta^{-2}(1+\delta^{-|s|-\frac12-\varepsilon})\beta_s(\kappa; u(t)) =: A_{\delta,s}\beta_s(\kappa; u(t)),$$

where $A_{\delta,s}$ is independent of $\kappa \ge 1$, $T > 0$, and $c_0 \ge 1$. Then, the desired bound (3.12) follows from Gronwall's inequality. □

4. Ill-posedness

In this section, we prove ill-posedness of ILW (1.1) in $H^s(\mathcal{M})$ for $s < -\frac12$ (Theorem 1.1 (ii)). In Subsection 4.1, we discuss the real line case, while we treat the periodic case in Subsection 4.2.

4.1. **Ill-posedness on the real line.** We first recall the following traveling wave solutions for ILW (1.1); see [14, 3, 2]. Given $c > 0$, let $a = a(c) \in \big(0, \frac{\pi}{\delta}\big)$ be the unique solution of the equation

$$a\delta \cot(a\delta) = 1 - c\delta.$$

Then, u_c defined by

$$u_c(t,x) = \frac{-a\sin(a\delta)}{\cosh(a(x-ct)) + \cos(a\delta)} \tag{4.1}$$

satisfies (1.1). Our strategy is to follow the work [5] by Biagioni and Linares for the BO equation and to take $c \to \infty$ (which is equivalent to $\frac{\pi}{a} \to \delta$). From the formula [8, (6) on p. 30]:

$$\int_{\mathbb{R}} \frac{e^{-i\xi x}}{\cosh(ax) + \cos(a\delta)} dx = \frac{2\pi}{a\sin(a\delta)} \frac{\sinh(\delta\xi)}{\sinh(\frac{\pi\xi}{a})},$$

we see that

$$\widehat{u}_c(0,\xi) = -\sqrt{2\pi}\frac{\sinh(\delta\xi)}{\sinh(\frac{\pi\xi}{a})} \tag{4.2}$$

with the understanding that

$$\frac{\sinh(\delta\xi)}{\sinh(\frac{\pi\xi}{a})}\bigg|_{\xi=0} = \lim_{\xi\to 0}\frac{\sinh(\delta\xi)}{\sinh(\frac{\pi\xi}{a})} = \frac{a\delta}{\pi}. \tag{4.3}$$

Observe that $\widehat{u}_c(0,\xi)$ enjoys the following properties: (i) it is bounded for $|\xi|\le 1$, (ii) since $\delta < \frac{\pi}{a}$, it decays exponentially as $|\xi|\to\infty$, and (iii) for each fixed $\xi\in\mathbb{R}$, we have

$$\widehat{u}_c(0,\xi) \to -\sqrt{2\pi}$$

as $c\to\infty$ (i.e. $\frac{\pi}{a}\to\delta$). These three properties with the dominated convergence theorem ensure that,[7] as $c\to\infty$, $u_c(0)\to -2\pi\delta_0$ in $H^s(\mathbb{R})$ for any $s<-\frac{1}{2}$, where δ_0 denotes the Dirac delta function on $\mathbb{R}$. Together with (4.1), this convergence implies

$$\lim_{c\to\infty}\|u_c(t)\|_{H^s(\mathbb{R})} = \lim_{c\to\infty}\|u_c(0)\|_{H^s(\mathbb{R})} = 2\pi\|\delta_0\|_{H^s(\mathbb{R})} \tag{4.4}$$

for any $t\in\mathbb{R}$. On the other hand, for any test function $\psi\in C_c^\infty(\mathbb{R})$ and any fixed $t\neq 0$, we have

$$\langle u_c(t),\psi\rangle_{L^2} = \int_{\mathbb{R}} u_c(0,x)\psi(x+ct)dx = \int_{\mathbb{R}}\frac{-\sin(a\delta)}{\cosh(x)+\cos(a\delta)}\psi(\tfrac{x}{a}+ct)dx \to 0$$

as $c\to\infty$, since $u_c(0)$ decays exponentially as $|x|\to\infty$. In particular, for $t\neq 0$, $u_c(t)$ converges to 0 in the distributional sense as $c\to\infty$, which implies that $u_c(t)$ does not converge in $H^s(\mathbb{R})$ in view of (4.4). This completes the proof of Theorem 1.1(ii) in the real line case.

4.2. **Ill-posedness on the circle.** We go over the following construction of a periodic traveling wave solution for (1.1) in [27]. We start with the profile $u_c(0)$ in (4.1) for the traveling wave solution on the real line and apply the Poisson summation formula which, with our convention for the Fourier transforms, reads as

$$\sum_{n\in\mathbb{Z}} f(x+n) = \sqrt{2\pi}\sum_{\xi\in 2\pi\mathbb{Z}}\mathcal{F}_{\mathbb{R}}(f)(\xi)e^{i\xi x}, \tag{4.5}$$

where $f:\mathbb{R}\to\mathbb{R}$ and its Fourier transform $\mathcal{F}_{\mathbb{R}}(f)$ decay sufficiently rapidly. Let U_c be the periodization of $u_c(0)$ in (4.1). Then, from (4.5) and (4.2), we have

$$U_c(x) = \sum_{n\in\mathbb{Z}}\frac{-a\sin(a\delta)}{\cosh(a(x+n))+\cos(a\delta)} = -2\pi\sum_{\xi\in 2\pi\mathbb{Z}}\frac{\sinh(\delta\xi)}{\sinh(\frac{\pi\xi}{a})}e^{i\xi x}, \tag{4.6}$$

where $a\in\big(0,\frac{\pi}{\delta}\big)$. Here, we used the convention (4.3).

We now verify that there exists a choice for both $c=c(\delta,a)$ and a constant of integration $B=B(\delta,a)$ such that U_c solves

$$-cU_c + \delta^{-1}U_c - \mathcal{T}_\delta\partial_x U_c - U_c^2 = B, \tag{4.7}$$

where $\mathcal{T}_\delta$ is as in (1.3). Fix $x\in\mathbb{R}$. For $n\in\mathbb{Z}$, define $b_n=b_n(x)$ and $d_n=d_n(x)$ by

$$b_n = \frac{1}{\cosh(a(x+n))+\cos(a\delta)} \quad\text{and}\quad d_n = b_n\sinh(a(x+n)). \tag{4.8}$$

Note that

$$U_c = -a\sin(a\delta)\sum_{n\in\mathbb{Z}} b_n. \tag{4.9}$$

[7] Recall our convention (2.1) for the Fourier transform.

From (4.6) and (1.3), we have

$$\mathcal{T}_\delta U_c(x) = 2\pi i \sum_{\xi\in 2\pi\mathbb{Z}} \frac{\cosh(\delta\xi)}{\sinh(\frac{\pi}{a}\xi)} e^{i\xi x} = -\sum_{n\in\mathbb{Z}} \frac{a\sinh(a(x+n))}{\cosh(a(x+n))+\cos(a\delta)}, \tag{4.10}$$

where the second equality follows from (4.5) and [8, (7) on p. 88]. Thus, from (4.10) with (4.8), we have

$$\begin{aligned} -\mathcal{T}_\delta \partial_x U_c(x) &= a^2 \sum_{n\in\mathbb{Z}} \frac{1+\cos(a\delta)\cosh(a(x+n))}{\big[\cosh(a(x+n))+\cos(a\delta)\big]^2} \\ &= \sum_{n\in\mathbb{Z}} a^2\big[1+\cos(a\delta)\cosh(a(x+n))\big] b_n^2. \end{aligned} \tag{4.11}$$

Next, we compute U_c^2. By (4.9), we have

$$U_c^2 = \sum_{n\in\mathbb{Z}} a^2\sin^2(a\delta) b_n^2 + a^2\sin^2(a\delta) \sum_{\substack{n,m\in\mathbb{Z}\\ n\neq m}} b_n b_m. \tag{4.12}$$

In order to compute the second term above, we use the following identity (see [27, (A 1) on p. 622]):

$$\begin{aligned} 2b_n b_m = &-\frac{\cos(a\delta)}{\sinh^2(\frac{a}{2}(m-n))+\sin^2(a\delta)}(b_n+b_m) \\ &+ \frac{\coth(\frac{a}{2}(n-m))}{\sinh^2(\frac{a}{2}(m-n))+\sin^2(a\delta)}(d_n-d_m) \end{aligned} \tag{4.13}$$

for all $n, m\in\mathbb{Z}$, $n\neq m$. Putting $n = k+\ell$ and $m = k-\ell$, it follows from (4.13) that

$$\begin{aligned} 2\sum_{\substack{n,m\in\mathbb{Z}\\ n\neq m}} b_n b_m = &-\lim_{N\to\infty} \sum_{\ell\in\mathbb{Z}\setminus\{0\}} \frac{\cos(a\delta)}{\sinh^2(\frac{a}{2}\ell)+\sin^2(a\delta)} \sum_{k=-N}^{N} (b_{k+\ell}+b_{k-\ell}) \\ &+ \lim_{N\to\infty} \sum_{\ell\in\mathbb{Z}\setminus\{0\}} \frac{\coth(\frac{a}{2}\ell)}{\sinh^2(\frac{a}{2}\ell)+\sin^2(a\delta)} \sum_{k=-N}^{N} (d_{k+\ell}-d_{k-\ell}). \end{aligned} \tag{4.14}$$

Since $b_k > 0$ for any $k\in\mathbb{Z}$, the monotone convergence theorem implies

$$\begin{aligned} &\lim_{N\to\infty} \sum_{\ell\neq 0} \frac{\cos(a\delta)}{\sinh^2(\frac{a}{2}\ell)+\sin^2(a\delta)} \sum_{k=-N}^{N} (b_{k+\ell}+b_{k-\ell}) \\ &\quad = 4\bigg(\sum_{\ell=1}^{\infty} \frac{\cos(a\delta)}{\sinh^2(\frac{a}{2}\ell)+\sin^2(a\delta)}\bigg) \sum_{k\in\mathbb{Z}} b_k. \end{aligned} \tag{4.15}$$

As for the second term on the right-hand side of (4.14), by noting that, for each $x\in\mathbb{R}$, $\operatorname{sgn}(k) d_k(x) \to 1$ as $|k|\to\infty$, we have, for each fixed $\ell\in\mathbb{Z}\setminus\{0\}$ and $x\in\mathbb{R}$,

$$\sum_{k=-N}^{N} (d_{k+\ell}(x)-d_{k-\ell}(x)) = \sum_{k=N-\ell+1}^{N+\ell} d_k(x) - \sum_{k=-N-\ell}^{-N+\ell-1} d_k(x) \longrightarrow 4\ell, \tag{4.16}$$

as $N \to \infty$. Putting (4.14), (4.15), and (4.16) together, we have

$$\sum_{\substack{n,m\in\mathbb{Z}\\ n\neq m}} b_n b_m = -2\bigg(\sum_{\ell=1}^{\infty}\frac{\cos(a\delta)}{\sinh^2(\frac{a}{2}\ell)+\sin^2(a\delta)}\bigg)\sum_{k\in\mathbb{Z}} b_k + 4\sum_{\ell=1}^{\infty}\frac{\ell\coth(\frac{a}{2}\ell)}{\sinh^2(\frac{a}{2}\ell)+\sin^2(a\delta)}, \tag{4.17}$$

where we suppressed the x-dependence. See [27, (18) on p. 622]. Hence, from (4.12), (4.17), and (4.9), we obtain

$$U_c^2 = \sum_{n\in\mathbb{Z}} a^2\sin^2(a\delta)b_n^2 + VU_c + D, \tag{4.18}$$

where $V = V(\delta, a)$ and $D = D(\delta, a)$ are given by

$$\begin{aligned} V &= \sum_{\ell=1}^{\infty}\frac{a\sin(2a\delta)}{\sinh^2(\frac{a}{2}\ell)+\sin^2(a\delta)},\\ D &= 4a^2\sin^2(a\delta)\sum_{\ell=1}^{\infty}\frac{\ell\coth(\frac{a}{2}\ell)}{\sinh^2(\frac{a}{2}\ell)+\sin^2(a\delta)}. \end{aligned} \tag{4.19}$$

By substituting (4.11) and (4.18) into (4.7), we obtain

$$(-c+\delta^{-1}-V)U_c + \sum_{n\in\mathbb{Z}} a^2\big[1+\cos(a\delta)\cosh(a(x+n)) - \sin^2(a\delta)\big]b_n^2 = B + D.$$

Using (4.9) with (4.8), this becomes

$$\begin{aligned} &\sum_{n\in\mathbb{Z}} ab_n^2\big[a + a\cos(a\delta)\cosh(a(x+n)) - a\sin^2(a\delta)\\ &\qquad - (-c+\delta^{-1}-V)\sin(a\delta)(\cosh(a(x+n))+\cos(a\delta))\big] = B + D. \end{aligned} \tag{4.20}$$

Noting that the right-hand side of (4.20) is independent of $x \in \mathbb{T}$ (but still depends on δ and a), we now impose the following three conditions:

$$\begin{aligned} a\cos(a\delta) &= (-c+\delta^{-1}-V)\sin(a\delta),\\ a - a\sin^2(a\delta) &= (-c+\delta^{-1}-V)\sin(a\delta)\cos(a\delta),\\ B &= -D. \end{aligned} \tag{4.21}$$

Note that the last condition in (4.21) should be interpreted as a definition of $B = B(\delta, a)$ in terms of $D = D(\delta, a)$ in (4.19). In view of the first condition in (4.21), we choose c such that

$$-c + \delta^{-1} - V = a\cot(a\delta). \tag{4.22}$$

such that the first condition in (4.21) is satisfied. It is easy to check that with this choice of c, the second condition in (4.21) is also satisfied. From (4.19) and (4.22), we have

$$c = \delta^{-1} - a\cot(a\delta) - \sum_{\ell=1}^{\infty}\frac{a\sin(2a\delta)}{\sinh^2(\frac{a}{2}\ell)+\sin^2(a\delta)},$$

which shows that $c \to \infty$, as $a \to \frac{\pi}{\delta}$.

Having constructed the periodic traveling wave $u_c(t,x) := U_c(x - ct)$ (with $c = c(\delta, a)$) as above, we can proceed as in Subsection 4.1 to prove ill-posedness of ILW (1.1) on the circle. In view of (4.6), by arguing as in Subsection 4.1, we see that, as $c \to \infty$ (i.e. $a \to \frac{\pi}{\delta}$), $u_c|_{t=0} = U_c$

converges to $-2\pi\delta_0$ in $H^s(\mathbb{T})$ for $s < -\frac{1}{2}$, where δ_0 is the Dirac delta function on $\mathbb{T}$. On the other hand, we have

$$\widehat{u}_c(t, 2\pi) = \int_{\mathbb{T}} U_c(x - ct) e^{-2\pi i x} dx = -2\pi e^{-2\pi i c t} \frac{\sinh(2\pi\delta)}{\sinh(2\pi \frac{\pi}{a})}.$$

As $a \to \frac{\pi}{\delta}$ (and thus $c \to \infty$), we have $\frac{\sinh(2\pi\delta)}{\sinh(2\pi\frac{\pi}{a})} \to 1$ but the exponential $e^{-2\pi i c t}$ diverges for $t \neq 0$. Hence, for $t \neq 0$, $u_c(t)$ does not converge in the distributional sense (and in particular in $H^s(\mathbb{T})$). This proves ill-posedness of ILW (1.1) in $H^s(\mathbb{T})$ for $s < -\frac{1}{2}$.

Next, let us briefly discuss ill-posedness in $H^s_\alpha(\mathbb{T})$ for given $\alpha \in \mathbb{R}$. Given $c > 0$, let μ_c denote the spatial mean of U_c in (4.6). In view of (4.3), we have

$$\mu_c = -2a\delta \longrightarrow -2\pi = \text{the spatial mean of } -2\pi\delta_0, \tag{4.23}$$

as $c \to \infty$ (and hence $a\delta \to \pi$). Given $\gamma \in \mathbb{R}$, define a Galilean transform Γ_γ by

$$\Gamma_\gamma(u)(t, x) = u(t, x - 2\gamma t) - \gamma. \tag{4.24}$$

Note that if u is a solution to (1.1), then so is $\Gamma_\gamma(u)$ for any $\gamma \in \mathbb{R}$.

Fix $\alpha \in \mathbb{R}$. Given $c > 0$, let $u_c(t, x) = U_c(x - ct)$ be the traveling wave solution constructed above. Then, by setting $v_{c,\alpha} = \Gamma_{\mu_c - \alpha}(u_c)$, it follows from the discussion above with (4.23) and (4.24) that (i) $v_{c,\alpha}(t) \in H^s_\alpha(\mathbb{T})$ for any $t \in \mathbb{R}$, (ii) $v_{c,\alpha}(0)$ converges to $-2\pi\delta_0 + (2\pi + \alpha)$ in $H^s_\alpha(\mathbb{T})$ for $s < -\frac{1}{2}$, and (iii) we have

$$\widehat{v}_{c,\alpha}(t, 2\pi) = -2\pi e^{-2\pi i (c + 2\mu_c - \alpha)t} \frac{\sinh(2\pi\delta)}{\sinh(2\pi\frac{\pi}{a})}$$

which is divergent as $c \to \infty$ for any $t \neq 0$. This proves ill-posedness of ILW (1.1) in $H^s_\alpha(\mathbb{T})$ for $s < -\frac{1}{2}$ and $\alpha \in \mathbb{R}$. This concludes the proof of Theorem 1.1 (ii) on the circle.

Acknowledgements. This material is based upon work supported by the Swedish Research Council under grant no. 2016-06596 while the first, fourth, and fifth authors were in residence at Institut Mittag-Leffler in Djursholm, Sweden during the program “Order and Randomness in Partial Differential Equations” in Fall, 2023. A.C., J.F., G.L., and T.O. were supported by the European Research Council (grant no. 864138 “SingStochDispDyn”). G.L. was also supported by the EPSRC New Investigator Award (grant no. EP/S033157/1). D.P. was supported by a Trond Mohn Foundation grant.

References

[1] L. Abdelouhab, J.L. Bona, M. Felland, J.-C. Saut, *Nonlocal models for nonlinear, dispersive waves*, Phys. D 40 (1989), no. 3, 360–392.

[2] J.P. Albert, *Positivity properties and uniqueness of solitary wave solutions of the intermediate long-wave equation*, Evolution equations (Baton Rouge, LA, 1992), 11–20, Lecture Notes in Pure and Appl. Math., 168, Dekker, New York, 1995.

[3] J.P. Albert, J.F. Toland, *On the exact solutions of the intermediate long-wave equation*, Differential Integral Equations 7 (1994), no. 3-4, 601–612.

[4] J. Angulo Pava, S. Hakkaev, *Ill-posedness for periodic nonlinear dispersive equations*, Electron. J. Differential Equations 2010, No. 119, 19 pp.

[5] H. Biagioni, F. Linares, *Ill-posedness for the derivative Schrödinger and generalized Benjamin-Ono equations*, Trans. Amer. Math. Soc. 353 (2001), no. 9, 3649–3659.

[6] A. Chapouto, G. Li, T. Oh, D. Pilod, *Deep-water limit of the intermediate long wave equation in L^2*, arXiv:2311.07997 [math.AP].

[7] A. Chapouto, G. Li, T. Oh, *Deep-water and shallow-water limits of statistical equilibria for the intermediate long wave equation*, in preparation.

[8] A. Erdélyi, W. Magnus, F. Oberhettinger, F. G. Tricomi, *Tables of integral transforms. Vol. I. Based, in part, on notes left by Harry Bateman*, McGraw-Hill Book Co., Inc., New York-Toronto-London, 1954. xx+391 pp.

[9] P. Gérard, T. Kappeler, P. Topalov, *Sharp well-posedness results of the Benjamin-Ono equation in $H^s(\mathbb{T};\mathbb{R})$ and qualitative properties of its solution*, Acta Math. 231 (2023), no. 1, 31–88.

[10] B. Grébert, *Birkhoff normal form and Hamiltonian PDEs*, Partial differential equations and applications, 1–46, Sémin. Congr., 15, Soc. Math. France, Paris, 2007.

[11] M. Ifrim, J.-C. Saut, *The lifespan of small data solutions for Intermediate Long Wave equation (ILW)*, arXiv:2305.05102 [math.AP].

[12] M. Ifrim, D. Tataru, *Well-posedness and dispersive decay of small data solutions for the Benjamin-Ono equation*, Ann. Sci. Éc. Norm. Supér. 52 (2019), no. 2, 297–335.

[13] A. Ionescu, C. Kenig, *Global well-posedness of the Benjamin-Ono equation in low-regularity spaces*, J. Amer. Math. Soc. 20 (2007), no. 3, 753–798.

[14] R.I. Joseph, *Solitary waves in a finite depth fluid,* J. Phys. A 10 (1977), no. 12, 225–227.

[15] C. Kenig, G. Ponce, L. Vega, *On the ill-posedness of some canonical dispersive equations*, Duke Math. J. 106 (2001), no. 3, 617–633.

[16] R. Killip, T. Laurens, M. Vişan, *Sharp well-posedness of for the Benjamin-Ono equation*, arXiv:2304.00124 [math.AP].

[17] R. Killip, M. Vişan, *KdV is well-posed in H^{-1}*, Ann. of Math. 190 (2019), no. 1, 249–305.

[18] R. Killip, M. Vişan, X. Zhang, *Low regularity conservation laws for integrable PDE*, Geom. Funct. Anal. 28 (2018), no. 4, 1062–1090.

[19] F. Klaus, R. Schippa, *A priori estimates for the derivative nonlinear Schrödinger equation*, Funkcial. Ekvac. 65 (2022), no. 3, 329–346.

[20] C. Klein, J.-C. Saut, *Nonlinear dispersive equations–inverse scattering and PDE methods*, Applied Mathematical Sciences, 209. Springer, Cham, [2021], ©2021. xx+580 pp.

[21] H. Koch, D. Tataru, *Energy and local energy bounds for the 1-d cubic NLS equation in $H^{-\frac{1}{4}}$*, Ann. Inst. H. Poincaré C Anal. Non Linéaire 29 (2012), no. 6, 955–988.

[22] H. Koch, N. Tzvetkov, *Nonlinear wave interactions for the Benjamin-Ono equation*, Int. Math. Res. Not. 2005, no. 30, 1833–1847.

[23] T. Kubota, D.R.S. Ko, L.D. Dobbs, *Weakly-nonlinear, long internal gravity waves in stratified fluids of finite depth*, J. Hydronautics 12 (1978), no. 4, 157–165.

[24] T. Laurens, *Global well-posedness for $H^{-1}(\mathbb{R})$ perturbations of KdV with exotic spatial asymptotics,* Comm. Math. Phys. 397 (2023), no. 3, 1387–1439.

[25] G. Li, *Deep-water and shallow-water limits of the generalised intermediate long wave equation*, arXiv:2207.12088 [math.AP].

[26] G. Li, T. Oh, G. Zheng, *On the deep-water and shallow-water limits of the intermediate long wave equation from a statistical viewpoint*, arXiv:2211.03243 [math.AP].

[27] T. Miloh, *On periodic and solitary wavelike solutions of the intermediate long-wave equation*, J. Fluid Mech. 211 (1990), 617–627.

[28] L. Molinet, *Global well-posedness in L^2 for the periodic Benjamin-Ono equation*, Amer. J. Math. 130 (2008), no. 3, 635–683.

[29] L. Molinet, D. Pilod, *The Cauchy problem for the Benjamin-Ono equation in L^2 revisited*, Anal. PDE 5 (2012), no. 2, 365–395.

[30] L. Molinet, D. Pilod, S. Vento, *On well-posedness for some dispersive perturbations of Burgers' equation*, Ann. Inst. H. Poincaré C Anal. Non Linéaire 35 (2018), no. 7, 1719–1756.

[31] L. Molinet, J.-C. Saut, N. Tzvetkov, *Ill-posedness issues for the Benjamin-Ono and related equations*, SIAM J. Math. Anal. 33 (2001), no. 4, 982–988.

[32] L. Molinet, S. Vento, *Improvement of the energy method for strongly nonresonant dispersive equations and applications*, Anal. PDE 8 (2015), no. 6, 1455–1495.

[33] T. Oh, Y. Wang, *Global well-posedness of the periodic cubic fourth order NLS in negative Sobolev spaces*, Forum Math. Sigma 6 (2018), e5, 80 pp.

[34] T. Oh, Y. Wang, *Global well-posedness of the one-dimensional cubic nonlinear Schrödinger equation in almost critical spaces*, J. Differential Equations 269 (2020), no. 1, 612–640.

[35] J.-C. Saut, *Benjamin-Ono and intermediate long wave equations: modeling, IST and PDE. Nonlinear dispersive partial differential equations and inverse scattering*, 95–160, Fields Inst. Commun., 83, Springer, New York, [2019], ©2019.

[36] B. Talbut, *Low regularity conservation laws for the Benjamin-Ono equation*, Math. Res. Lett. 28 (2021), no. 3, 889–905.

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