# Solution-Set Geometry and Regularization Path of a Nonconvexly Regularized Convex Sparse Model 

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#### Abstract

The generalized minimax concave (GMC) penalty is a nonconvex sparse regularizer which can preserve the overall-convexity of the regularized least-squares problem. In this paper, we focus on a significant instance of the GMC model termed scaled GMC (sGMC), and present various notable findings on its solution-set geometry and regularization path. Our investigation indicates that while the sGMC penalty is a nonconvex extension of the LASSO penalty (i.e., the $\ell_{1}$-norm), the sGMC model preserves many celebrated properties of the LASSO model, hence can serve as a less biased surrogate of LASSO without losing its advantages. Specifically, for a fixed regularization parameter $\lambda$, we show that the solution-set geometry, solution uniqueness and sparseness of the sGMC model can be characterized in a similar elegant way to the LASSO model (see, e.g., Osborne et al. 2000, R. J. Tibshirani 2013). For a varying $\lambda$, we prove that the sGMC solution set is a continuous polytope-valued mapping of $\lambda$. Most noticeably, our study indicates that similar to LASSO, the minimum $\ell_{2}$-norm regularization path of the sGMC model is continuous and piecewise linear in $\lambda$. Based on these theoretical results, an efficient regularization path algorithm is proposed for the sGMC model, extending the well-known least angle regression (LARS) algorithm for LASSO. We prove the correctness and finite termination of the proposed algorithm under a mild assumption, and confirm its correctness-in-general-situation, efficiency, and practical utility through numerical experiments. Many results in this study also contribute to the theoretical research of LASSO.


## Index Terms

Sparse least-squares problems, compressed sensing, generalized minimax concave penalty, least angle regression, nonconvexly regularized convex models.

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## 1 Introduction

In the past two decades, the sparse least-squares problems have attracted significant attention in signal processing and machine learning [1], [2], [3], [4], [5], which are usually formulated as the following optimization program:

$$
\begin{equation*}
\underset{\boldsymbol{x} \in \mathbb{R}^{n}}{\operatorname{minimize}} J(\boldsymbol{x}):=\frac{1}{2}\|\boldsymbol{y}-\boldsymbol{A} \boldsymbol{x}\|_{2}^{2}+\lambda \Psi(\boldsymbol{x}) \tag{1}
\end{equation*}
$$

where $\boldsymbol{y} \in \mathbb{R}^{m}$ is the measured signal, $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ is the sensing matrix, $\Psi: \mathbb{R}^{n} \rightarrow \mathbb{R}$ is a sparseness-promoting penalty which approximates the $\ell_{0}$ pseudo-norm (i.e., the cardinality of nonzero components in $\boldsymbol{x}$ ), and $\lambda>0$ is the regularization parameter which trades off between sparsity and data fidelity.

A prominent choice for $\Psi$ is the $\ell_{1}$-norm, which is known as the tightest convex approximation of the $\ell_{0}$ pseudo-norm. The resultant convex program (1), also known as the LASSO [6] (or basis pursuit denoising [7]) problem, has been widely used [3], [8] for several favorable properties:

1) The cost function of LASSO is convex, which allows us to develop efficient and scalable algorithms [9], [10] that are provably convergent to a global minimizer of (1).
2) The solution set of LASSO has elegant geometric properties [6], [11], [12], [13], [14] that provides the LASSO model with reliable variable selection ability.
3) The minimum $\ell_{2}$-norm regularization path ${ }^{1}$ (or solution path) of LASSO is known to be piecewise linear and can be computed efficiently via the least angle regression (LARS) algorithm [11], [15], [16], [17], [14], which eases tuning of the regularization parameter $\lambda$.
Nevertheless, since the $\ell_{1}$-norm is coercive (i.e., $\|\boldsymbol{x}\|_{1}$ goes to infinity as $\|\boldsymbol{x}\|_{2} \rightarrow+\infty$ ), the LASSO solution tends to underestimate large-amplitude components, which leads to estimation bias. Although nonconvex regularizers such as SCAD [18] and MC [19] penalties have been proposed to yield less biased sparse regularization, they usually sacrifice the convexity of (1) and lose the favorable properties of LASSO.

To resolve this dilemma, the generalized minimax concave (GMC [20]) penalty has been proposed, as a generalization of the MC penalty (and the LASSO penalty), to yield less biased convex sparse regularization (see Sec. 2.2 for details). Notably, the GMC penalty itself is nonconvex, whereas it can preserve the overall-convexity of the cost function of (1), which makes an essential difference from earlier nonconvex sparse regularizers. Moreover, in contrast to the conventional LASSO model which solely convexifies the regularizer, the GMC model permits the regularizer to be nonconvex and convexifies the cost function as a whole. Hence the GMC model is a better convex approximation of the $\ell_{0}$-regularization model than LASSO and can possibly serve as a less biased surrogate of LASSO in certain applications.

Considerable efforts have been made to facilitate the application of the GMC model. For example, several extensions of GMC have been proposed to handle more complicated scenarios than sparse regression [21], [22], [23], [24], [25]. Moreover, exploiting the overall-convexity, a number of efficient and scalable algorithms have been developed for finding a global minimizer of the GMC model [20] or its extensions [21], [22], [23], [26], [27], [28], [25]. Nevertheless, up to now, little is known about the solution-set geometry and regularization path of the GMC model, which constitutes the motivation of the current study. In this paper, we consider a significant instance of the GMC models termed the scaled generalized minimax concave (sGMC) model (cf. Sec. 2.2 for details), and study its solution-set geometry and regularization path.

### 1.1 Contributions of This Study

We address the following important questions regarding the sGMC model:

1) For a fixed regularization parameter $\lambda$, what does the sGMC solution set look like? In particular, is the sGMC solution unique and is it guaranteed to be sparse?
2) For a varying $\lambda$, how do the sGMC solution set and the minimum $\ell_{2}$-norm sGMC solution change with $\lambda$ ?
3) Is there an efficient approach to calculate the minimum $\ell_{2}$-norm sGMC regularization path?

Our study provides very encouraging answers to these questions, showcasing that in spite of the nonconvex nature of the sGMC penalty, the sGMC model exhibits all the favorable properties of LASSO as delineated before.

Specifically, in the context of a fixed $\lambda$, we show that the sGMC solution set and its dual counterpart have similar geometric properties to those of LASSO solution sets [11], [14]. Indeed, the sGMC solution set can be expressed as a nonlinearly parameterized LASSO solution set (cf. Thm. 3 (a) in Sec. 4.1), thus it is nonempty, closed, convex and bounded. Hence while the sGMC solution is not unique in general for a given sensing matrix $\boldsymbol{A}$, there always exists uniquely the minimum $\ell_{2}$-norm sGMC solution. In addition, we show that similar to LASSO, as long as the entries of the sensing matrix $\boldsymbol{A}$ are drawn from a continuous probability distribution on $\mathbb{R}^{m \times n}$, the sGMC solution is almost surely unique with at most $\min \{m, n\}$ nonzero components, which verifies the variable selection ability of the sGMC model and the sparsity of the sGMC solution in a high dimensionally (i.e., $m \ll n$ ) probabilistic setting.

On the other hand, for a varying $\lambda$, we consider the Cartesian product of the primal and dual sGMC solution sets and term it the extended solution set of the sGMC model. We show that the extended sGMC solution set is a polytope changing continuously with $\lambda$, more precisely, it is a continuous (in set-valued variational analysis sense [29]) piecewise polytope-valued mapping of $\lambda$. Moreover, while the sGMC penalty is a nonconvex generalization of the LASSO penalty, our investigation reveals a counterintuitive observation: the minimum $\ell_{2}$-norm extended regularization path ${ }^{2}$ of the sGMC model remains to be piecewise linear with respect to $\lambda$. Explicit piecewise expressions of the extended sGMC solution set and minimum $\ell_{2}$-norm extended sGMC regularization path are obtained, where each piece is totally characterized by a discrete object called ex-equicorrelation pair (where "ex" is an abbreviation for "extended") which is an extension of similar notions introduced in [30], [31], [14].

Exploiting these theoretical results, we show that under the so-called "one-at-a-time" assumption (cf. [15, Thm. 1] and Assumption 1 in Sec. 7.3), all linear pieces (more precisely, their corresponding ex-equicorrelation pairs) in the minimum $\ell_{2}$ norm extended sGMC regularization path can be obtained in an iterative manner within finite steps. We term this iterative method the LARS-sGMC algorithm. The proposed algorithm extends the conventional LARS algorithm for LASSO [15]. Although our proof of the correctness of LARS-sGMC is established under the "one-at-a-time" assumption, numerical experiments demonstrate its correctness, efficiency, and practical utility in general situations.

Notably, since the sGMC model embraces LASSO as a special instance of it (cf. Sec. 2.2), all results in the present paper naturally apply to LASSO, and many of them hold substantial value even to the study of LASSO, for example:

[^0]1) In Sec. 4.3, we establish continuity [29] of the LASSO solution set as a set-valued mapping of $(\boldsymbol{y}, \lambda)$, which is the first set-valued variational analysis result for LASSO to the authors' knowledge.
2) In Sec. 6, we derive a concise piecewise linear expression of the minimum $\ell_{2}$-norm extended regularization path of the sGMC model (Thm. 6). To the best of the authors' knowledge, even for the LASSO case,

- Thm. 6 provides the first explicit expression of the number of linear pieces in a regularization path,
- the proof of Thm. 6 gives the first complete proof for showing piecewise linearity of a regularization path without requiring the "one-at-a-time" assumption of any regularization path algorithm.
See Sec. 6.3 for a comparison with previous works.


### 1.2 Paper Organization

The remainder of this paper is organized as follows. Sec. 2 introduces necessary mathematical preliminaries, and provides a brief introduction on the GMC model and the sGMC model. In Sec. 3, we study the geometry and solution uniqueness of the sGMC solution set and its dual counterpart. In Sec. 4, we formally define the extended solution set of the sGMC model and study its geometry as well as its continuity as a set-valued mapping of problem parameters. Sec. 5 introduces the notion of ex-equicorrelation pairs and reveals how it can be used to characterize the state of the sGMC model. Sec. 6 further derives the piecewise expressions of the extended solution set and minimum $\ell_{2}$-norm extended regularization path using ex-equicorrelation pairs. The LARS-sGMC algorithm is proposed in Sec. 7, along with its proof of correctness and complexity analysis. Sec. 8 reports some numerical results, demonstrating the correctness-in-general-situation, efficiency and practical utility of the LARS-sGMC algorithm. Sec. 9 closes this paper with some concluding remarks.

## 2 Preliminaries

### 2.1 Notation

### 2.1.1 Numbers, vectors and matrices

Let $\mathbb{N}, \mathbb{R}, \mathbb{R}_{+}, \mathbb{R}_{++}$respectively be the sets of nonnegative integers, real numbers, nonnegative real numbers and positive real numbers. For a real number set $I \subset \mathbb{R}$, we denote $\sup (I), \inf (I)$ and $\operatorname{int}(I)$ respectively as its supremum, infimum and interior; especially, when $I=\emptyset$, we adopt the convention $\sup (\emptyset):=-\infty$ and $\inf (\emptyset):=+\infty$. For $n$-dimensional Euclidean space $\mathbb{R}^{n},\|\cdot\|_{p}(p \geq 1)$ denotes the $\ell_{p}$-norm in $\mathbb{R}^{n}$. For a vector $\boldsymbol{x} \in \mathbb{R}^{n}$, we denote its $i$ th component by $x_{i}$ or $[\boldsymbol{x}]_{i}$, define its support as $\operatorname{supp}(\boldsymbol{x}):=\left\{i \in\{1, \ldots, n\} \mid x_{i} \neq 0\right\}$ and define its signs vector as $\operatorname{sign}(\boldsymbol{x}):=$ $\left[\operatorname{sign}\left(x_{1}\right), \operatorname{sign}\left(x_{2}\right), \ldots, \operatorname{sign}\left(x_{n}\right)\right]^{\top} \in\{+1,0,-1\}^{n}$. For a matrix $\boldsymbol{A} \in \mathbb{R}^{m \times n}, \operatorname{rank}(\boldsymbol{A})$ denotes its rank, $\boldsymbol{A}^{\top} \in \mathbb{R}^{n \times m}$ and $\boldsymbol{A}^{\dagger} \in \mathbb{R}^{n \times m}$ respectively denote its transpose and Moore-Penrose pseudoinverse, $\mathcal{N}(\boldsymbol{A})$ and $\mathcal{R}(\boldsymbol{A})$ respectively denote its null space and range (i.e., column space). For $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^{n}, \boldsymbol{x} \geq \boldsymbol{y}$ means that $x_{i} \geq y_{i}$ for every $i \in\{1, \ldots, n\}$; for $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{n \times n}, \boldsymbol{A} \succeq \boldsymbol{B}$ means that $\boldsymbol{A}-\boldsymbol{B}$ is positive semidefinite. The symbols $\mathbf{0}_{n}, \mathbf{1}_{n}, \mathbf{O}_{m \times n}$ and $\boldsymbol{I}_{n}$ respectively stand for the $n \times 1$ zero vector, the $n \times 1$ all-ones vector, the $m \times n$ zero matrix and the $n \times n$ identity matrix (when the ambient dimension is known, we may omit the subscripts). We denote the diagonal matrix with diagonal entries $d_{1}, \ldots, d_{n}$ by $\operatorname{diag}\left(d_{1}, \ldots, d_{n}\right)$, and denote the block diagonal matrix with diagonal blocks $\boldsymbol{A}_{1} \in \mathbb{R}^{m_{1} \times n_{1}}, \ldots, \boldsymbol{A}_{k} \in \mathbb{R}^{m_{k} \times n_{k}}$ by $\operatorname{blkdiag}\left(\boldsymbol{A}_{1}, \ldots, \boldsymbol{A}_{k}\right)$.

### 2.1.2 Slicing operation for vectors and matrices

For a vector $\boldsymbol{x}:=\left[x_{1}, \ldots, x_{n}\right]^{\top} \in \mathbb{R}^{n}$, a matrix $\boldsymbol{A}:=\left[\boldsymbol{a}_{1}, \ldots, \boldsymbol{a}_{n}\right] \in \mathbb{R}^{m \times n}$ and an index $\operatorname{set} \mathcal{I}=\left\{i_{1}, \ldots, i_{k}\right\} \subset\{1, \ldots, n\}$ with $i_{1}<i_{2}<\cdots<i_{k}$, we define $\boldsymbol{x}_{\mathcal{I}}:=[\boldsymbol{x}]_{\mathcal{I}}:=\left[x_{i_{1}}, \ldots, x_{i_{k}}\right]^{\top}, \boldsymbol{A}_{\mathcal{I}}:=\left[\boldsymbol{a}_{i_{1}}, \ldots, \boldsymbol{a}_{i_{k}}\right]$ and $\neg \mathcal{I}:=\{1, \ldots, n\} \backslash \mathcal{I}$, respectively. In particular, we define $\boldsymbol{A}_{\emptyset}:=\mathbf{0}_{m}$ and $\boldsymbol{x}_{\emptyset}:=0$ such that the slicing operation for the empty set does not lead to a void value, which avoids possible ambiguity in computations involving $\boldsymbol{A}_{\emptyset}$ and $\boldsymbol{x}_{\emptyset}$. The slicing operation will be repeatedly used in the study of ex-equicorrelation pairs (Sec. 5) and the LARS-sGMC algorithm (Sec. 7).

### 2.1.3 Functions and set-valued mappings

For an extended real-valued function $f: \mathbb{R}^{n} \rightarrow \mathbb{R} \cup\{+\infty\}$, the domain of $f$ is $\operatorname{dom} f:=\left\{\boldsymbol{x} \in \mathbb{R}^{n} \mid f(\boldsymbol{x})<+\infty\right\}$. For a convex function $f: \mathbb{R}^{n} \rightarrow \mathbb{R} \cup\{+\infty\}$, the subdifferential of $f$ at $\boldsymbol{x} \in \mathbb{R}^{n}$ is

$$
\partial f(\boldsymbol{x}):=\left\{\boldsymbol{u} \in \mathbb{R}^{n} \mid\left(\forall \boldsymbol{z} \in \mathbb{R}^{n}\right) \boldsymbol{u}^{\top}(\boldsymbol{z}-\boldsymbol{x})+f(\boldsymbol{x}) \leq f(\boldsymbol{z})\right\},
$$

if $\boldsymbol{u} \in \partial f(\boldsymbol{x})$, then $\boldsymbol{u}$ is called a subgradient of $f$ at $\boldsymbol{x}$. Especially, if $f$ is differentiable at $\boldsymbol{x} \in \mathbb{R}^{n}$, we denote $\nabla f(\boldsymbol{x})$ as the gradient of $f$ at $\boldsymbol{x}$, and the subdifferential of $f$ is given by $\partial f(\boldsymbol{x})=\{\nabla f(\boldsymbol{x})\}$. We denote the set of all proper, lower semicontinuous ${ }^{3}$ convex functions from $\mathbb{R}^{n}$ to $\mathbb{R} \cup\{+\infty\}$ by $\Gamma_{0}\left(\mathbb{R}^{n}\right)$. For two sets $X$ and $Y$, we adopt the notation $\mathcal{M}: X \rightrightarrows Y$ to denote a set-valued mapping (a.k.a. multifunction) from $X$ to the power set of $Y$, i.e., $\mathcal{M}$ maps a point in $X$ to a (possibly empty) subset of $Y$.
3. For $f: \mathbb{R}^{n} \rightarrow \mathbb{R} \cup\{+\infty\}, f$ is referred to as proper if $\operatorname{dom} f \neq \emptyset$, and lower semicontinuous if every lower level set of it is closed.

### 2.2 The GMC and sGMC Models

The formulation of the generalized minimax concave (GMC [20]) penalty is as follows:

$$
\begin{equation*}
\Psi_{\mathrm{GMC}}(\boldsymbol{x} ; \boldsymbol{B}):=\|\boldsymbol{x}\|_{1}-\min _{\boldsymbol{z} \in \mathbb{R}^{n}}\left(\|\boldsymbol{z}\|_{1}+\frac{1}{2}\|\boldsymbol{B}(\boldsymbol{x}-\boldsymbol{z})\|_{2}^{2}\right), \tag{2}
\end{equation*}
$$

where the subtrahend part is a debiasing function which cancels out the overpenalization effect caused by the $\ell_{1}$-norm, and the steering matrix $B \in \mathbb{R}^{q \times n}$ therein is a shape controlling parameter which adjusts the shape of $\Psi_{\mathrm{GMC}}$ to achieve overallconvexity of (1). Especially, if $\boldsymbol{B}^{\top} \boldsymbol{B}=\mathbf{O}_{n \times n}$, then the GMC penalty reduces to the $\ell_{1}$-norm; if $\boldsymbol{B}^{\top} \boldsymbol{B}=\operatorname{diag}\left(b_{1}^{2}, \ldots, b_{n}^{2}\right)$ with $b_{i} \neq 0$ for every $i \in\{1, \ldots, n\}$, then the GMC penalty is separable and reduces to the following expression:

$$
\Psi_{\mathrm{GMC}}(\boldsymbol{x} ; \boldsymbol{B})=\sum_{i=1}^{n} \psi_{\mathrm{MC}}\left(x_{i} ; b_{i}^{2}\right),
$$

where $\psi_{\mathrm{MC}}\left(\cdot ; b^{2}\right)$ with $b \neq 0$ is the minimax concave (MC [19]) penalty, which is a nearly unbiased nonconvex sparse regularizer formulated as follows:

$$
\psi_{\mathrm{MC}}\left(x ; b^{2}\right):= \begin{cases}|x|-\frac{1}{2} b^{2} x^{2}, & \text { if }|x| \leq 1 / b^{2} \\ \frac{1}{2 b^{2}}, & \text { otherwise }\end{cases}
$$

Hence the GMC penalty is a nonseparable multidimensional extension of the MC penalty, which accounts for its name.
Different from earlier nonconvex sparse regularizers [18], [19], although the GMC penalty itself is nonconvex in general, it has been proven in [20, Thm. 1] that as long as $\boldsymbol{B}$ satisfies the following linear matrix inequality

$$
\begin{equation*}
\boldsymbol{A}^{\top} \boldsymbol{A} \succeq \lambda \boldsymbol{B}^{\top} \boldsymbol{B} \tag{3}
\end{equation*}
$$

then the resultant GMC regularization model

$$
\begin{equation*}
\underset{\boldsymbol{x} \in \mathbb{R}^{n}}{\operatorname{minimize}} J_{\mathrm{GMC}}(\boldsymbol{x}):=\frac{1}{2}\|\boldsymbol{y}-\boldsymbol{A} \boldsymbol{x}\|_{2}^{2}+\lambda \Psi_{\mathrm{GMC}}(\boldsymbol{x} ; \boldsymbol{B}) \tag{4}
\end{equation*}
$$

is a convex program. A common and useful choice for $\boldsymbol{B}$ to satisfy (3) is the following expression [20, Eq. (48)]:

$$
\begin{equation*}
\boldsymbol{B}:=\sqrt{\rho / \lambda} \boldsymbol{A} \tag{5}
\end{equation*}
$$

where ${ }^{4} \rho \in[0,1)$. In this paper, we formally refer to the GMC model (4) using the steering matrix design (5) as the scaled generalized minimax concave (sGMC) model, i.e.,

$$
\begin{equation*}
\underset{\boldsymbol{x} \in \mathbb{R}^{n}}{\operatorname{minimize}} \quad J_{\mathrm{sGMC}}(\boldsymbol{x}):=\frac{1}{2}\|\boldsymbol{y}-\boldsymbol{A} \boldsymbol{x}\|_{2}^{2}+\lambda \Psi_{\mathrm{GMC}}(\boldsymbol{x} ; \sqrt{\rho / \lambda} \boldsymbol{A}) \tag{6}
\end{equation*}
$$

As will be shown later, by limiting the shape controlling parameter $\boldsymbol{B}$ to (5), the sGMC model exhibits many favorable properties which are not shared by general GMC models, which makes the sGMC model a significant instance of GMC.

Note that the sGMC model is uniquely determined by its problem parameters

$$
\begin{equation*}
(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda) \in \mathbb{R}^{m \times n} \times \mathbb{R}^{m} \times[0,1) \times \mathbb{R}_{++}=: \mathscr{P} \tag{7}
\end{equation*}
$$

where we define $\mathscr{P}$ as the space of problem parameters considered in this paper. Hence we can denote the sGMC model (6) by $\operatorname{sGMC}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$ for simplicity of presentation. Especially, if $\rho=0$, then (6) reduces to the LASSO model:

$$
\begin{equation*}
\underset{\boldsymbol{x} \in \mathbb{R}^{n}}{\operatorname{minimize}} J_{\mathrm{LASSO}}(\boldsymbol{x}):=\frac{1}{2}\|\boldsymbol{y}-\boldsymbol{A} \boldsymbol{x}\|_{2}^{2}+\lambda\|\boldsymbol{x}\|_{1} \tag{8}
\end{equation*}
$$

In this paper, we denote (8) by $\operatorname{LASSO}(\boldsymbol{A}, \boldsymbol{y}, \lambda)$, and remark that since $\operatorname{LASSO}(\boldsymbol{A}, \boldsymbol{y}, \lambda)$ is equivalent to $\operatorname{sGMC}(\boldsymbol{A}, \boldsymbol{y}, 0, \lambda)$, every theoretical result of the sGMC model introduced in this paper naturally applies to the LASSO model.

## 3 Geometry and Uniqueness of the Primal and Dual sGMC Solution Sets

In this section, we study the geometry and solution uniqueness of the sGMC solution set and its dual ${ }^{5}$ counterpart for a fixed parameter group $(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda) \in \mathscr{P}$.

The most notable results (Thm. 1 and 2) in this section can be summarized as follows: the primal and dual solution sets of $\operatorname{sGMC}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$ are both nonempty, closed, bounded and convex, and possess the following favorable properties:

1) Every sGMC solution $\boldsymbol{x}_{\mathrm{p}}$ (resp. dual sGMC solution $\boldsymbol{z}_{\mathrm{d}}$ ) gives the same linear fit $\boldsymbol{A} \boldsymbol{x}_{\mathrm{p}}$ (resp. $\boldsymbol{A} \boldsymbol{z}_{\mathrm{d}}$ ) and has the same $\ell_{1}$-norm $\left\|\boldsymbol{x}_{\mathrm{p}}\right\|_{1}$ (resp. $\left\|\boldsymbol{z}_{\mathrm{d}}\right\|_{1}$ ), where an upper bound of $\left\|\boldsymbol{x}_{\mathrm{p}}\right\|_{1}$ (resp. $\left\|\boldsymbol{z}_{\mathrm{d}}\right\|_{1}$ ) is derived.

[^1]2) Every sGMC solution gives the same value of data fidelity and the same value of sGMC penalty, hence all solutions of the sGMC model are equally good with respect to goodness of fit and and the sGMC sparseness measure.
3) If the entries of $\boldsymbol{A}$ are drawn from a continuous probability distribution on $\mathbb{R}^{m \times n}$, then the sGMC solution (resp. dual sGMC solution) is almost surely unique with at most $\min \{m, n\}$ nonzero elements.
It should be noted that the results listed here do not constitute the entirety of the solution-set geometry of the sGMC model. Subsequently, by jointly considering the primal and dual sGMC solution sets, we will derive some additional geometric properties of the sGMC solution set, respectively in Thm. 3 (a) (Sec. 4.1) and Thm. 7 (Sec. 6.2).

In the sequel, we first present some known facts (mainly from [14, Sec. 2.3]) about an $\ell_{1}$-regularization model with general convex loss functions. By rewritting the cost function of (6) as the sum of an involved convex loss function and the $\ell_{1}$-norm, we can apply the results introduced here, whereby elucidating the geometry of the sGMC solution set.

### 3.1 Solution Set of an $\ell_{1}$-Regularization Model with General Convex Loss Functions

Consider the following $\ell_{1}$-regularization model:

$$
\begin{equation*}
\underset{\boldsymbol{x} \in \mathbb{R}^{n}}{\operatorname{minimize}} J_{\mathrm{GL} 1}(\boldsymbol{x}):=f(\boldsymbol{A} \boldsymbol{x})+\lambda\|\boldsymbol{x}\|_{1} \tag{9}
\end{equation*}
$$

where the loss function $f: \mathbb{R}^{m} \rightarrow \mathbb{R}$ is differentiable, strictly convex and bounded from below, $A \in \mathbb{R}^{m \times n}$ is the sensing matrix and $\lambda>0$ is the regularization parameter. Then the solution set of (9) has the following geometric properties.
Fact 1 (Geometry of the $\ell_{1}$-regularization model (9)). The solution set of (9) is nonempty, closed, bounded and convex. Moreover, every solution of (9), say $\hat{\boldsymbol{x}}$, gives the same linear fit $\boldsymbol{A} \hat{\boldsymbol{x}}$ and has the same $\ell_{1}$-norm $\|\hat{\boldsymbol{x}}\|_{1}$, i.e., if $\hat{\boldsymbol{x}}_{1}, \hat{\boldsymbol{x}}_{2}$ are two solutions of (9), then $\boldsymbol{A} \hat{\boldsymbol{x}}_{1}=\boldsymbol{A} \hat{\boldsymbol{x}}_{2}$ and $\left\|\hat{\boldsymbol{x}}_{1}\right\|_{1}=\left\|\hat{\boldsymbol{x}}_{2}\right\|_{1}$.

Proof: Since $\|\cdot\|_{1}$ is coercive and $f(\boldsymbol{A} \cdot)$ is convex and bounded from below, $J_{\mathrm{GL} 1}$ is a continuous coercive convex function. Hence the solution set of (9) is nonempty and bounded from [32, Prop. 11.12 and 11.15]. The closedness and convexity of the solution set follow respectively from the continuity of $J_{\mathrm{GL1}}$ and [32, Prop. 11.6]. The uniqueness of $\boldsymbol{A} \hat{\boldsymbol{x}}$ and $\|\hat{\boldsymbol{x}}\|_{1}$ follows from [14, Para. 2 in Sec. 2.3].

Moreover, if the sensing matrix $\boldsymbol{A}$ is known to have columns in general position [33], then the solution of (9) is unique. The general-positionedness of a matrix is defined as follows.
Definition 1 (Matrices with columns in general position). We say $\boldsymbol{A}:=\left[\boldsymbol{a}_{1}, \ldots, \boldsymbol{a}_{n}\right] \in \mathbb{R}^{m \times n}$ has columns in general position if for every $\mathcal{I} \subset\{1, \ldots, n\}$ satisfying $|\mathcal{I}| \leq m$ and $\left(s_{j}\right)_{j \in \mathcal{I}} \subset\{ \pm 1\}$, the following holds:

$$
\begin{equation*}
(\forall i \in \mathcal{I}) \quad s_{i} \boldsymbol{a}_{i} \notin \operatorname{aff}\left\{s_{j} \boldsymbol{a}_{j}\right\}_{j \in \mathcal{I} \backslash\{i\}}, \tag{10}
\end{equation*}
$$

where for $\left\{\boldsymbol{b}_{1}, \ldots, \boldsymbol{b}_{n}\right\} \subset \mathbb{R}^{n}$, aff $\left\{\boldsymbol{b}_{1}, \ldots, \boldsymbol{b}_{n}\right\}$ denotes the affine hull of $\left\{\boldsymbol{b}_{1}, \ldots, \boldsymbol{b}_{n}\right\}$. Moreover, we define the set of $m \times n$ matrices that have columns in general position as $\mathcal{G}_{m \times n}$ :

$$
\mathcal{G}_{m \times n}:=\left\{\boldsymbol{A} \in \mathbb{R}^{m \times n} \mid \boldsymbol{A} \text { has columns in general position }\right\}
$$

Remark 1 (Mildness of the general-positioning condition). We note that for $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, " $\boldsymbol{A}$ has columns in general position" is a mild condition in practice, in the sense that if the entries of $\boldsymbol{A}$ follow a continuous probability distribution on $\mathbb{R}^{m \times n}$, then $\boldsymbol{A} \in \mathcal{G}_{m \times n}$ with probability one (to see this, one can verify that the logical complement of (10) leads to a set of matrices with Lebesgue measure zero).

Exploiting Def. 1, we can summarize the solution uniqueness condition of (9) into the following fact.
Fact 2 (Solution uniqueness of the $\ell_{1}$-regularization model (9)). If $\boldsymbol{A} \in \mathcal{G}_{m \times n}$, then (9) has a unique solution with at most $\min \{m, n\}$ nonzero components.

In particular, if the entries of $\boldsymbol{A}$ are drawn from a continuous probability distribution on $\mathbb{R}^{m \times n}$, then $\boldsymbol{A}$ is in $\mathcal{G}_{m \times n}$ with probability one, which implies that (9) almost surely (i.e., with probability one) has a unique solution with at most $\min \{m, n\}$ nonzero components.

Proof: See [14, Lemma 5 and the paragraph before it].
Moreover, since the LASSO model can be regarded as a special instance of (9), we can obtain the following results which will be useful in subsequent studies.
Fact 3 (Geometry and uniqueness of the LASSO solution set). For $(\boldsymbol{A}, \boldsymbol{y}, \lambda) \in \mathbb{R}^{m \times n} \times \mathbb{R}^{m} \times \mathbb{R}_{++}$, the solution set of $\operatorname{LASSO}(\boldsymbol{A}, \boldsymbol{y}, \lambda)$, denoted by $\mathcal{S}_{L A}(\boldsymbol{A}, \boldsymbol{y}, \lambda)$, is nonempty, closed, bounded and convex.

Moreover, for $\boldsymbol{x}_{L A} \in \mathcal{S}_{L A}(\boldsymbol{A}, \boldsymbol{y}, \lambda)$, the following holds:
(a) Every $\boldsymbol{x}_{L A}$ gives the same linear fit $\boldsymbol{A} \boldsymbol{x}_{L A}$ and the same $\ell_{1}$-norm $\left\|\boldsymbol{x}_{L A}\right\|_{1}$. In other words, the common linear fit $\boldsymbol{\beta}_{L A}$ and common $\ell_{1}$-norm $\gamma_{L A}$ of all LASSO solutions:

$$
\begin{aligned}
\boldsymbol{\beta}_{L A} & : \mathbb{R}^{m \times n} \times \mathbb{R}^{m} \times \mathbb{R}_{++} \rightarrow \mathbb{R}^{m}:(\boldsymbol{A}, \boldsymbol{y}, \lambda) \mapsto \boldsymbol{A} \boldsymbol{x}_{L A} \\
\gamma_{L A} & : \mathbb{R}^{m \times n} \times \mathbb{R}^{m} \times \mathbb{R}_{++} \rightarrow \mathbb{R}_{+}:(\boldsymbol{A}, \boldsymbol{y}, \lambda) \mapsto\left\|\boldsymbol{x}_{L A}\right\|_{1}
\end{aligned}
$$

are well-defined functions of the parameters $(\boldsymbol{A}, \boldsymbol{y}, \lambda)$.
(b) If $\boldsymbol{A} \in \mathcal{G}_{m \times n}$, then $\boldsymbol{x}_{L A}$ is unique and has at most $\min \{m, n\}$ nonzero components; as a result, if the entries of $\boldsymbol{A}$ are drawn from a continuous probability distribution, then $\boldsymbol{x}_{L A}$ is almost surely unique and has at most $\min \{m, n\}$ nonzero components.

Proof: By defining $f(\cdot):=\frac{1}{2}\|\boldsymbol{y}-\cdot\|_{2}^{2}$, (8) can be regarded as an instance of (9), hence Fact 1 and 2 naturally apply to $\operatorname{LASSO}(\boldsymbol{A}, \boldsymbol{y}, \lambda)$, which yields all the results.

Subsequently, we show that $\operatorname{sGMC}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$ can also be regarded as a special instance of (9) and hence we can apply the theoretical results introduced above.

### 3.2 On the (Primal) sGMC Solution Set

We can rewrite the cost function of the sGMC model (6) as follows:

$$
\begin{align*}
\frac{1}{2}\|\boldsymbol{y}-\boldsymbol{A} \boldsymbol{x}\|_{2}^{2}+\lambda \Psi_{\mathrm{GMC}}(\boldsymbol{x} ; \sqrt{\rho / \lambda} \boldsymbol{A}) & =\frac{1}{2}\|\boldsymbol{y}-\boldsymbol{A} \boldsymbol{x}\|_{2}^{2}+\lambda\|\boldsymbol{x}\|_{1}-\min _{\boldsymbol{z} \in \mathbb{R}^{n}}\left(\lambda\|\boldsymbol{z}\|_{1}+\frac{\rho}{2}\|\boldsymbol{A}(\boldsymbol{x}-\boldsymbol{z})\|_{2}^{2}\right) \\
& =\left[\frac{1}{2}\|\boldsymbol{y}-\boldsymbol{A} \boldsymbol{x}\|_{2}^{2}-\min _{\boldsymbol{z} \in \mathbb{R}^{n}}\left(\lambda\|\boldsymbol{z}\|_{1}+\frac{\rho}{2}\|\boldsymbol{A} \boldsymbol{x}-\boldsymbol{A} \boldsymbol{z}\|_{2}^{2}\right)\right]+\lambda\|\boldsymbol{x}\|_{1} \tag{11}
\end{align*}
$$

Define $\bar{f}: \mathbb{R}^{m} \rightarrow \mathbb{R}$ as follows:

$$
\begin{equation*}
\bar{f}(\boldsymbol{u}):=\frac{1}{2}\|\boldsymbol{y}-\boldsymbol{u}\|_{2}^{2}-\min _{\boldsymbol{z} \in \mathbb{R}^{n}}\left(\lambda\|\boldsymbol{z}\|_{1}+\frac{\rho}{2}\|\boldsymbol{u}-\boldsymbol{A} \boldsymbol{z}\|_{2}^{2}\right), \tag{12}
\end{equation*}
$$

then substituting the definition of $\bar{f}$ into (11) yields that

$$
\begin{equation*}
J_{\mathrm{sGMC}}(\boldsymbol{x})=\bar{f}(\boldsymbol{A} \boldsymbol{x})+\lambda\|\boldsymbol{x}\|_{1} . \tag{13}
\end{equation*}
$$

Hence $\operatorname{sGMC}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$ can be rewritten in the form of (9). In order to apply results introduced in Section 3.1 to $\operatorname{sGMC}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$, we only need to prove that $\bar{f}$ is differentiable, strictly convex and bounded from below. To show this, we first simplify the expression of $\bar{f}$. Define $\bar{g}: \mathbb{R}^{m} \rightarrow \mathbb{R}$ as

$$
\begin{equation*}
\bar{g}(\boldsymbol{u}):=\min _{\boldsymbol{z} \in \mathbb{R}^{n}}\left(\lambda\|\boldsymbol{z}\|_{1}+\frac{\rho}{2}\|\boldsymbol{u}-\boldsymbol{A} \boldsymbol{z}\|_{2}^{2}\right) \tag{14}
\end{equation*}
$$

then one can verify that $\bar{f}(\boldsymbol{u})=\frac{1}{2}\|\boldsymbol{y}-\boldsymbol{u}\|_{2}^{2}-\bar{g}(\boldsymbol{u})$. The following lemma presents three important properties of $\bar{g}$.
Lemma 1. For every $\lambda>0, \rho \in[0,1)$ and $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, the following holds for $\bar{g}$ defined in (14):
(a) $\bar{g}$ is differentiable.
(b) $\left(\frac{\rho}{2}\|\boldsymbol{y}-\cdot\|_{2}^{2}-\bar{g}(\cdot)\right)$ is convex.
(c) For every $\boldsymbol{u} \in \mathbb{R}^{m}, \bar{g}(\boldsymbol{u}) \leq \frac{\rho}{2}\|\boldsymbol{u}\|_{2}^{2}$.

Proof: See Appendix B.1.
Exploiting the properties of $\bar{g}$ described in Lemma 1, we can prove the differentiability, strict convexity and boundedness from below of $\bar{f}$. See the following proposition.
Proposition 1. For every $(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda) \in \mathscr{P}, \bar{f}$ defined in (12) is differentiable, strictly convex and bounded from below with the following lower bound:

$$
\left(\forall \boldsymbol{u} \in \mathbb{R}^{m}\right) \bar{f}(\boldsymbol{u}) \geq \frac{-\rho}{2(1-\rho)}\|\boldsymbol{y}\|_{2}^{2}
$$

Proof: Since $\bar{g}$ is differentiable from Lemma 1 (a), $\bar{f}(\boldsymbol{u})=\frac{1}{2}\|\boldsymbol{y}-\boldsymbol{u}\|_{2}^{2}-\bar{g}(\boldsymbol{u})$ is also differentiable. On the other hand, we can rewrite $\bar{f}$ as

$$
\bar{f}(\boldsymbol{u})=\frac{1}{2}\|\boldsymbol{y}-\boldsymbol{u}\|_{2}^{2}-\bar{g}(\boldsymbol{u})=\frac{(1-\rho)}{2}\|\boldsymbol{y}-\boldsymbol{u}\|_{2}^{2}+\left(\frac{\rho}{2}\|\boldsymbol{y}-\boldsymbol{u}\|_{2}^{2}-\bar{g}(\boldsymbol{u})\right) .
$$

Since $\rho<1, \frac{(1-\rho)}{2}\|\boldsymbol{y}-\cdot\|$ is strictly convex, combining which with Lemma 1 (b) yields the strict convexity of $\bar{f}$. Moreover, from Lemma 1 (c), for every $\boldsymbol{u} \in \mathbb{R}^{n}$, we have

$$
\bar{f}(\boldsymbol{u})=\frac{1}{2}\|\boldsymbol{y}-\boldsymbol{u}\|_{2}^{2}-\bar{g}(\boldsymbol{u}) \geq \frac{1}{2}\|\boldsymbol{y}-\boldsymbol{u}\|_{2}^{2}-\frac{\rho}{2}\|\boldsymbol{u}\|_{2}^{2}=\frac{(1-\rho)}{2}\left\|\boldsymbol{u}-\frac{\boldsymbol{y}}{1-\rho}\right\|_{2}^{2}-\frac{\rho}{2(1-\rho)}\|\boldsymbol{y}\|_{2}^{2} \geq-\frac{\rho}{2(1-\rho)}\|\boldsymbol{y}\|_{2}^{2},
$$

thus $\bar{f}$ is bounded from below.
Therefore, all results introduced in Section 3.1 apply to $\operatorname{sGMC}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$, which yield the following theorem about the geometry and uniqueness of the sGMC solution set.
Theorem 1 (Geometry and uniqueness of the (primal) sGMC solution set). For $(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda) \in \mathscr{P}$, the (primal) solution set of $\operatorname{sGMC}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$, denoted by $\mathcal{S}_{p}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$, is nonempty, closed, bounded and convex.

Moreover, for $\boldsymbol{x}_{p} \in \mathcal{S}_{p}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$, the following holds:
(a) Every $\boldsymbol{x}_{p}$ gives the same linear fit $\boldsymbol{A} \boldsymbol{x}_{p}$ and the same $\ell_{1}$-norm $\left\|\boldsymbol{x}_{p}\right\|_{1}$. In other words, the common linear fit $\boldsymbol{\beta}_{p}$ and common $\ell_{1}$-norm $\gamma_{p}$ of all (primal) sGMC solutions:

$$
\begin{aligned}
\boldsymbol{\beta}_{p}: \mathscr{P} & \rightarrow \mathbb{R}^{m}:(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda) \\
\gamma_{p}: \mathscr{P} & \mapsto \mathbb{R}_{+}:(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)
\end{aligned}>\left\|\boldsymbol{x}_{p}\right\|_{1} .
$$

are well-defined functions of the parameters $(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$.
Additionally, we have the following upper bound for $\gamma_{p}$ :

$$
\begin{equation*}
\gamma_{p}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda):=\left\|\boldsymbol{x}_{p}\right\|_{1} \leq \frac{1}{2 \lambda(1-\rho)}\|\boldsymbol{y}\|_{2}^{2} \tag{15}
\end{equation*}
$$

(b) For every two sGMC solutions $\boldsymbol{x}_{p}, \boldsymbol{x}_{p}^{\prime} \in \mathcal{S}_{p}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda), \boldsymbol{x}_{p}$ and $\boldsymbol{x}_{p}^{\prime}$ give the same data fidelity (cf. (6)), i.e.,

$$
\begin{equation*}
\frac{1}{2}\left\|\boldsymbol{y}-\boldsymbol{A} \boldsymbol{x}_{p}\right\|_{2}^{2}=\frac{1}{2}\left\|\boldsymbol{y}-\boldsymbol{A} \boldsymbol{x}_{p}^{\prime}\right\|_{2}^{2} \tag{16}
\end{equation*}
$$

and give the same sGMC penalty value (cf. (4)-(5)), i.e.,

$$
\begin{equation*}
\Psi_{G M C}\left(\boldsymbol{x}_{p} ; \sqrt{\rho / \lambda} \boldsymbol{A}\right)=\Psi_{G M C}\left(\boldsymbol{x}_{p}^{\prime} ; \sqrt{\rho / \lambda} \boldsymbol{A}\right) \tag{17}
\end{equation*}
$$

(c) If $\boldsymbol{A} \in \mathcal{G}_{m \times n}$ (cf. Def. 1), then $\boldsymbol{x}_{p}$ is unique and has at most $\min \{m, n\}$ nonzero components; as a result, if the entries of $\boldsymbol{A}$ are drawn from a continuous probability distribution, then $\boldsymbol{x}_{p}$ is unique and has at most $\min \{m, n\}$ nonzero components with probability one.

Proof: The results (a) and (c) follow from Fact 1, 2 and Prop. 1. Hence we only need to prove (15) and the result (b). From (13) and Prop. 1, the following holds

$$
\begin{equation*}
J_{\mathrm{sGMC}}\left(\boldsymbol{x}_{\mathrm{p}}\right) \geq \lambda\left\|\boldsymbol{x}_{\mathrm{p}}\right\|_{1}-\frac{\rho}{2(1-\rho)}\|\boldsymbol{y}\|_{2}^{2} \tag{18}
\end{equation*}
$$

Since $\boldsymbol{x}_{\mathrm{p}}$ is a solution of $\operatorname{sGMC}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$, we have

$$
J_{\mathrm{sGMC}}\left(\boldsymbol{x}_{\mathrm{p}}\right)=\min _{\boldsymbol{x} \in \mathbb{R}^{n}} J_{\mathrm{sGMC}}(\boldsymbol{x}) \leq J_{\mathrm{sGMC}}(\mathbf{0})
$$

Expanding the expression of $J_{\mathrm{SGMC}}(\mathbf{0})$ by (11) yields

$$
J_{\mathrm{sGMC}}\left(\boldsymbol{x}_{\mathrm{p}}\right) \leq \frac{1}{2}\|\boldsymbol{y}\|_{2}^{2}-\min _{\boldsymbol{z} \in \mathbb{R}^{n}}\left(\lambda\|\boldsymbol{z}\|_{1}+\frac{\rho}{2}\|\boldsymbol{A} \boldsymbol{z}\|_{2}^{2}\right)=\frac{1}{2}\|\boldsymbol{y}\|_{2}^{2}
$$

Combining the inequation above with (18) yields

$$
\lambda\left\|\boldsymbol{x}_{\mathrm{p}}\right\|_{1}-\frac{\rho}{2(1-\rho)}\|\boldsymbol{y}\|_{2}^{2} \leq \frac{1}{2}\|\boldsymbol{y}\|_{2}^{2}
$$

which implies (15).
For every two sGMC solutions $\boldsymbol{x}_{\mathrm{p}}, \boldsymbol{x}_{\mathrm{p}}^{\prime} \in \mathcal{S}_{\mathrm{p}}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$, we have $\boldsymbol{A} \boldsymbol{x}_{\mathrm{p}}=\boldsymbol{A} \boldsymbol{x}_{\mathrm{p}}^{\prime}$ from the result (a), hence (16) holds. Since $\boldsymbol{x}_{\mathrm{p}}$ and $\boldsymbol{x}_{\mathrm{p}}^{\prime}$ are both solutions of the sGMC model (6), we have $J_{\mathrm{sGMC}}\left(\boldsymbol{x}_{\mathrm{p}}\right)=J_{\mathrm{sGMC}}\left(\boldsymbol{x}_{\mathrm{p}}^{\prime}\right)$, combining which with (16) yields (17). Accordingly, the result (b) holds.

Remark 2 (Interpretation of Theorem 1). Thm. 1 indicates that the solution set of $\operatorname{sGMC}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$ is a nonempty closed bounded convex set lying in the intersection between an affine space and a contour of the $\ell_{1}$-norm (cf. Thm. 1 (a)), which gives a rough image on its shape.

Moreover, while the sGMC solution is not unique in general, the sGMC model has the following favorable properties:
(a) Every sGMC solution $\boldsymbol{x}_{p}$ satisfies $\left\|\boldsymbol{x}_{p}\right\|_{1} \leq \frac{1}{2 \lambda(1-\rho)}\|\boldsymbol{y}\|_{2}^{2}$ (cf. Thm. 1 (a)). Notice that this upper bound is independent of $\boldsymbol{A}$, hence the sGMC solution set is stable against ill-positioned sensing matrices.
(b) Every sGMC solution gives the same data fidelity and the same sGMC penalty value (cf. Thm. 1 (b)), hence all sGMC solutions are equally good with respect to goodness of fit and the sGMC sparseness measure.
(c) If $\boldsymbol{A}$ is stochastic, then the sGMC solution is almost surely unique and has at most $\min \{m, n\}$ nonzero components (cf. Thm. 1 (c)), which leads to a desirable sparse solution in a high-dimensional (i.e., $m \ll n$ ) setting.

### 3.3 On the Dual sGMC Solution Set

In this section, we further reformulate the sGMC model (6) as a convex concave min-max problem, based on which we define the dual solution set of the sGMC model. Employing the LASSO solution set as a bridge between the primal and dual sGMC solution sets, we can show that the dual sGMC solution set has similar geometric properties and uniqueness condition to that of the primal one.

We first present the definition of a dual sGMC solution. Substituting the expression (11) of the sGMC cost function into (6) and conducting some basic transformations, we can rewrite $\operatorname{sGMC}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$ as the following min-max problem:

$$
\begin{equation*}
\min _{\boldsymbol{x} \in \mathbb{R}^{n}} \max _{\boldsymbol{z} \in \mathbb{R}^{n}} \frac{1}{2}\|\boldsymbol{y}-\boldsymbol{A} \boldsymbol{x}\|_{2}^{2}+\lambda\|\boldsymbol{x}\|_{1}-\lambda\|\boldsymbol{z}\|_{1}-\frac{\rho}{2}\|\boldsymbol{A} \boldsymbol{x}-\boldsymbol{A} \boldsymbol{z}\|_{2}^{2} \tag{19}
\end{equation*}
$$

Define $G(\boldsymbol{x}, \boldsymbol{z})$ as the cost function in (19), i.e.,

$$
\begin{equation*}
G(\boldsymbol{x}, \boldsymbol{z}):=\frac{1}{2}\|\boldsymbol{y}-\boldsymbol{A} \boldsymbol{x}\|_{2}^{2}+\lambda\|\boldsymbol{x}\|_{1}-\lambda\|\boldsymbol{z}\|_{1}-\frac{\rho}{2}\|\boldsymbol{A} \boldsymbol{x}-\boldsymbol{A} \boldsymbol{z}\|_{2}^{2} \tag{20}
\end{equation*}
$$

then one can verify that $G(\cdot, \boldsymbol{z})$ is convex for every $\boldsymbol{z} \in \mathbb{R}^{n}$ and $G(\boldsymbol{x}, \cdot)$ is concave for every $\boldsymbol{x} \in \mathbb{R}^{n}$. Moreover, since $G(\boldsymbol{x}, \boldsymbol{z})$ is real-valued on $\mathbb{R}^{n} \times \mathbb{R}^{n}$, we have the following minimax equality from [34, Thm. 36.3]:

$$
\begin{equation*}
\min _{\boldsymbol{x} \in \mathbb{R}^{n}} \max _{\boldsymbol{z} \in \mathbb{R}^{n}} G(\boldsymbol{x}, \boldsymbol{z})=\max _{\boldsymbol{z} \in \mathbb{R}^{n}} \min _{\boldsymbol{x} \in \mathbb{R}^{n}} G(\boldsymbol{x}, \boldsymbol{z}) \tag{21}
\end{equation*}
$$

which implies the existence of a saddle point. Hence we can derive the following saddle point expressions, based on which we formally define the dual sGMC solution.
Proposition 2 (Primal and dual solutions of the sGMC model). For $(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda) \in \mathscr{P}, \boldsymbol{x}_{p} \in \mathbb{R}^{n}$ is a solution of $s G M C(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$ if and only if there exists $\boldsymbol{z}_{d} \in \mathbb{R}^{n}$ such that $\left(\boldsymbol{x}_{p}, \boldsymbol{z}_{d}\right) \in \mathbb{R}^{n} \times \mathbb{R}^{n}$ satisfies

$$
\begin{align*}
& \boldsymbol{x}_{p} \in \arg \min _{\boldsymbol{x} \in \mathbb{R}^{n}} \frac{1}{2}\|\boldsymbol{y}-\boldsymbol{A} \boldsymbol{x}\|_{2}^{2}-\frac{\rho}{2}\left\|\boldsymbol{A} \boldsymbol{x}-\boldsymbol{A} \boldsymbol{z}_{d}\right\|_{2}^{2}+\lambda\|\boldsymbol{x}\|_{1}  \tag{22}\\
& \boldsymbol{z}_{d} \in \arg \min _{\boldsymbol{z} \in \mathbb{R}^{n}} \frac{\rho}{2}\left\|\boldsymbol{A} \boldsymbol{x}_{p}-\boldsymbol{A} \boldsymbol{z}\right\|_{2}^{2}+\lambda\|\boldsymbol{z}\|_{1} \tag{23}
\end{align*}
$$

or equivalently, if and only if $\left(\boldsymbol{x}_{p}, \boldsymbol{z}_{d}\right) \in \mathbb{R}^{n} \times \mathbb{R}^{n}$ satisfies:
(a) $\boldsymbol{x}_{p}$ is a solution of $\operatorname{LASSO}\left(\boldsymbol{A}, \frac{\boldsymbol{y}-\rho \boldsymbol{A} \boldsymbol{z}_{d}}{(1-\rho)}, \frac{\lambda}{(1-\rho)}\right)$,
(b) $\boldsymbol{z}_{d}$ is a solution of $\operatorname{LASSO}\left(\sqrt{\rho} \boldsymbol{A}, \sqrt{\rho} \boldsymbol{A} \boldsymbol{x}_{p}, \lambda\right)$.

In this case, we call $\boldsymbol{x}_{p}$ a primal solution of $\operatorname{sGMC}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$ and call $\boldsymbol{z}_{d}$ its corresponding dual solution (vice visa), and we call $\left(\boldsymbol{x}_{p}, \boldsymbol{z}_{d}\right)$ a primal-dual solution pair of $\operatorname{sGMC}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$.

Proof: From (21), $\boldsymbol{x}_{\mathrm{p}}$ is a solution of $\operatorname{sGMC}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$ if and only if there exists $\boldsymbol{z}_{\mathrm{d}} \in \mathbb{R}^{n}$ such that $\left(\boldsymbol{x}_{\mathrm{p}}, \boldsymbol{z}_{\mathrm{d}}\right)$ is a saddle point of $G(\boldsymbol{x}, \boldsymbol{z})$, i.e.,

$$
\begin{align*}
& \boldsymbol{x}_{\mathrm{p}} \in \arg \min _{\boldsymbol{x} \in \mathbb{R}^{n}} G\left(\boldsymbol{x}, \boldsymbol{z}_{\mathrm{d}}\right),  \tag{24}\\
& \boldsymbol{z}_{\mathrm{d}} \in \arg \max _{\boldsymbol{z} \in \mathbb{R}^{n}} G\left(\boldsymbol{x}_{\mathrm{p}}, \boldsymbol{z}\right) . \tag{25}
\end{align*}
$$

Expanding the expression of $G\left(\boldsymbol{x}, \boldsymbol{z}_{\mathrm{d}}\right)$ in (24), $G\left(\boldsymbol{x}_{\mathrm{p}}, \boldsymbol{z}\right)$ in (25), and omitting constant terms yields (22) and (23). By properly scaling and rearranging the terms, one can verify the equivalence between (22), (23) and the LASSO problems.

Additionally, the minimax equality (21) implies that every dual solution also serves as a minimizer of some cost function. In the following corollary, we present an explicit expression of this "dual" cost function and we present a lower bound of it. This lower bound will be used later to derive an upper bound of the $\ell_{1}$-norm of all dual sGMC solutions in Thm. 2 .
Corollary 1. For $(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda) \in \mathscr{P}$, define the dual sGMC cost function $\bar{J}_{S G M C}: \mathbb{R}^{n} \rightarrow \mathbb{R}$ as:

$$
\bar{J}_{S G M C}(\boldsymbol{z}):=-\min _{\boldsymbol{x} \in \mathbb{R}^{n}} G(\boldsymbol{x}, \boldsymbol{z})=\lambda\|\boldsymbol{z}\|_{1}-\min _{\boldsymbol{x} \in \mathbb{R}^{n}}\left[\frac{1}{2}\|\boldsymbol{y}-\boldsymbol{A} \boldsymbol{x}\|_{2}^{2}-\frac{\rho}{2}\|\boldsymbol{A} \boldsymbol{x}-\boldsymbol{A} \boldsymbol{z}\|_{2}^{2}+\lambda\|\boldsymbol{x}\|_{1}\right]
$$

then the following holds:
(a) $\boldsymbol{z}_{d} \in \mathbb{R}^{n}$ is a dual solution of $\operatorname{sGMC}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$ if and only if $\boldsymbol{z}_{d}$ is a minimizer of $\bar{J}_{s G M C}$.
(b) For every $\boldsymbol{z} \in \mathbb{R}^{n}, \bar{J}_{S G M C}(\boldsymbol{z}) \geq \lambda\|\boldsymbol{z}\|_{1}-\frac{1}{2}\|\boldsymbol{y}\|_{2}^{2}$.

Proof: (a) From (24) and (25), $\boldsymbol{z}_{\mathrm{d}}$ is a dual solution of $\operatorname{sGMC}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$ if and only if

$$
\min _{\boldsymbol{x} \in \mathbb{R}^{n}} G\left(\boldsymbol{x}, \boldsymbol{z}_{\mathrm{d}}\right)=\max _{\boldsymbol{z} \in \mathbb{R}^{n}} \min _{\boldsymbol{x} \in \mathbb{R}^{n}} G(\boldsymbol{x}, \boldsymbol{z}) .
$$

Multiplying both sides of the equation above by -1 yields

$$
-\min _{\boldsymbol{x} \in \mathbb{R}^{n}} G\left(\boldsymbol{x}, \boldsymbol{z}_{\mathrm{d}}\right)=-\max _{\boldsymbol{z} \in \mathbb{R}^{n}} \min _{\boldsymbol{x} \in \mathbb{R}^{n}} G(\boldsymbol{x}, \boldsymbol{z}) \Longleftrightarrow-\min _{\boldsymbol{x} \in \mathbb{R}^{n}} G\left(\boldsymbol{x}, \boldsymbol{z}_{\mathrm{d}}\right)=\min _{\boldsymbol{z} \in \mathbb{R}^{n}}\left(-\min _{\boldsymbol{x} \in \mathbb{R}^{n}} G(\boldsymbol{x}, \boldsymbol{z})\right) \Longleftrightarrow \bar{J}_{\mathrm{SGMC}}\left(\boldsymbol{z}_{\mathrm{d}}\right)=\min _{\boldsymbol{z} \in \mathbb{R}^{n}} \bar{J}_{\mathrm{SGMC}}(\boldsymbol{z})
$$

(b) For every $\boldsymbol{z} \in \mathbb{R}^{n}$ we have the following

$$
\bar{J}_{\mathrm{SGMC}}(\boldsymbol{z})=\max _{\boldsymbol{x} \in \mathbb{R}^{n}}-G(\boldsymbol{x}, \boldsymbol{z}) \geq-G(\mathbf{0}, \boldsymbol{z})=\lambda\|\boldsymbol{z}\|_{1}-\frac{1}{2}\|\boldsymbol{y}\|_{2}^{2}+\frac{\rho}{2}\|\boldsymbol{A} \boldsymbol{z}\|_{2}^{2} \geq \lambda\|\boldsymbol{z}\|_{1}-\frac{1}{2}\|\boldsymbol{y}\|_{2}^{2}
$$

Combining the discussion above completes the proof.
Furthermore, exploiting the results introduced in Section 3.2, we can yield the following theorem about the geometry and uniqueness of the dual sGMC solution set, which indicates that the dual sGMC solution set has similar properties as the primal sGMC solution set.
Theorem 2 (Geometry and uniqueness of the dual sGMC solution set). For $(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda) \in \mathscr{P}$, the dual solution set of $s G M C(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$ (cf. Prop. 2), denoted by $\mathcal{S}_{d}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$, is nonempty, closed, bounded and convex.

Moreover, for $\boldsymbol{z}_{d} \in \mathcal{S}_{d}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$, the following holds:
(a) Every $\boldsymbol{z}_{d}$ gives the same linear fit $\boldsymbol{A} \boldsymbol{z}_{d}$ and the same $\ell_{1}$-norm $\left\|\boldsymbol{z}_{d}\right\|_{1}$. In other words, the common linear fit $\boldsymbol{\beta}_{d}$ and common $\ell_{1}$-norm $\gamma_{d}$ of all dual sGMC solutions:

$$
\begin{aligned}
& \boldsymbol{\beta}_{d}: \mathscr{P} \rightarrow \mathbb{R}^{m}:(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda) \mapsto \boldsymbol{A} \boldsymbol{z}_{d}, \\
& \gamma_{d}: \mathscr{P} \rightarrow \mathbb{R}_{+}:(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda) \mapsto\left\|\boldsymbol{z}_{d}\right\|_{1}
\end{aligned}
$$

are well-defined functions of the parameters $(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$.
Additionally, we have the following upper bound for $\gamma_{d}$ :

$$
\begin{equation*}
\gamma_{d}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda):=\left\|\boldsymbol{z}_{d}\right\|_{1} \leq \frac{1}{2 \lambda(1-\rho)}\|\boldsymbol{y}\|_{2}^{2} \tag{26}
\end{equation*}
$$

(b) If $\boldsymbol{A} \in \mathcal{G}_{m \times n}$ (cf. Def. 1), then $\boldsymbol{z}_{d}$ is unique and has at most $\min \{m, n\}$ nonzero components; as a result, if the entries of $\boldsymbol{A}$ are drawn from a continuous probability distribution, then $\boldsymbol{z}_{d}$ is unique and has at most $\min \{m, n\}$ nonzero components with probability one.

Proof: See Appendix B.2.

## 4 On The Extended sGMC Solution Set and Its Continuity Properties

In this section, we introduce the notion of the extended solution set of $\operatorname{sGMC}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$ which is defined as the set of primal-dual solution pairs (cf. Prop. 2) of the sGMC model.

By studying the geometry of the extended sGMC solution set, we obtain some additional properties of the primal sGMC solution set. Moreover, regarding the extended solution set as a set-valued mapping of the problem parameters $(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda) \in \mathscr{P}$, we study its continuity properties, which sheds some light on the numerical stability of the sGMC model with respect to $(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$, and paves the way ${ }^{6}$ for studying the piecewise expressions of the extended solution set mapping and the minimum $\ell_{2}$-norm extended regularization path in Sec. 6.

The most important results in this section are three fold:

1) Thm. 3 elucidates the geometry of the extended sGMC solution set. Noticeably, Thm. 3 (a) indicates that the extended sGMC solution set is the Cartesian product of two nonlinearly parameterized LASSO solution set, which reveals the essential relation between the sGMC model and the LASSO model.
2) Thm. 4 establishes the existence and uniqueness of the minimum $\ell_{2}$-norm extended solution, which will be one of the main players in subsequent studies. Additionally, some fundamental properties pertaining to the minimum $\ell_{2}$-norm extended solution are provided therein.
3) Thm. 5 establishes the continuity of the extended sGMC solution set as a set-valued mapping of the regularization parameter $\lambda$, which implies the continuity of the minimum $\ell_{2}$-norm extended regularization path in Cor. 3 .
Moreover, as a by-product of the proof of Thm. 5, we establish the set-valued continuity of the LASSO solution set with respect to $(\boldsymbol{y}, \lambda)$ in Lemma 4. Remarkably, this constitutes the first set-valued variational analysis result for LASSO, to the best of the authors' knowledge.

In the sequel, we first study the optimality condition of the min-max form (19) of the sGMC model, which leads to the definition of extended solution and extended solution set. Exploiting theoretical results from Sec. 3, we elucidate the underlying structure of the extended sGMC solution set.

[^2]
### 4.1 Definition and Geometry of the Extended Solution Set

Exploiting the optimality condition of the min-max problem introduced in (19), we can transform the sGMC model into a monotone inclusion problem [32] with respect to the primal-dual solution pair (cf. Prop. 2); see the following.

Proposition 3 (Optimality condition of the sGMC model). For $(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda) \in \mathscr{P}$, define $\boldsymbol{b}:=\left[\begin{array}{ll}\boldsymbol{y}^{\top} & \mathbf{0}_{m}{ }^{\top}\end{array}\right]^{\top}, \boldsymbol{C}:=\operatorname{blkdiag}(\boldsymbol{A}, \sqrt{\rho} \boldsymbol{A})$ and

$$
\boldsymbol{D}:=\left[\begin{array}{cc}
(1-\rho) \boldsymbol{I}_{m} & \sqrt{\rho} \boldsymbol{I}_{m} \\
-\sqrt{\rho} \boldsymbol{I}_{m} & \boldsymbol{I}_{m}
\end{array}\right]
$$

Then $\left(\boldsymbol{x}_{p}, \boldsymbol{z}_{d}\right) \in \mathbb{R}^{n} \times \mathbb{R}^{n}$ is a primal-dual solution pair (cf. Prop. 2 for definition) of $\operatorname{sGMC}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$ if and only if $\boldsymbol{w}_{e}:=$ $\left[\begin{array}{ll}\boldsymbol{x}_{p}{ }^{\top} & \boldsymbol{z}_{d}{ }^{\top}\end{array}\right]^{\top}$ satisfies

$$
\begin{equation*}
\boldsymbol{\xi}\left(\boldsymbol{w}_{e}\right):=\boldsymbol{C}^{\top}\left(\boldsymbol{b}-\boldsymbol{D} \boldsymbol{C} \boldsymbol{w}_{e}\right) \in \lambda \partial\left(\|\cdot\|_{1}\right)\left(\boldsymbol{w}_{e}\right) \tag{27}
\end{equation*}
$$

or equivalently, if and only if the following inclusion holds for every $i \in\{1,2, \ldots, 2 n\}$ :

$$
\xi_{i}\left(\boldsymbol{w}_{e}\right):=\boldsymbol{c}_{i}^{\top}\left(\boldsymbol{b}-\boldsymbol{D} \boldsymbol{C} \boldsymbol{w}_{e}\right) \in \begin{cases}\left\{\lambda \operatorname{sign}\left(\left[\boldsymbol{w}_{e}\right]_{i}\right)\right\}, & \text { if }\left[\boldsymbol{w}_{e}\right]_{i} \neq 0  \tag{OPT}\\ {[-\lambda, \lambda],} & \text { if }\left[\boldsymbol{w}_{e}\right]_{i}=0\end{cases}
$$

where $\boldsymbol{c}_{i}$ is the ith column vector of $\boldsymbol{C}$.
Proof: Let us define

$$
\begin{aligned}
g_{x}(\boldsymbol{x}) & :=\frac{1}{2}\|\boldsymbol{y}-\boldsymbol{A} \boldsymbol{x}\|_{2}^{2}-\frac{\rho}{2}\left\|\boldsymbol{A} \boldsymbol{x}-\boldsymbol{A} \boldsymbol{z}_{\mathrm{d}}\right\|_{2}^{2}+\lambda\|\boldsymbol{x}\|_{1} \\
g_{z}(\boldsymbol{z}) & :=\frac{\rho}{2}\left\|\boldsymbol{A} \boldsymbol{x}_{\mathrm{p}}-\boldsymbol{A} \boldsymbol{z}\right\|_{2}^{2}+\lambda\|\boldsymbol{z}\|_{1}
\end{aligned}
$$

then (24) and (25) are respectively equivalent to $\mathbf{0} \in \partial g_{x}\left(\boldsymbol{x}_{\mathrm{p}}\right)$ and $\mathbf{0} \in \partial g_{z}\left(\boldsymbol{z}_{\mathrm{d}}\right)$, expanding the expressions of which yields

$$
\left\{\begin{array}{l}
\mathbf{0} \in \boldsymbol{A}^{\top}\left(\boldsymbol{A}\left((1-\rho) \boldsymbol{x}_{\mathrm{p}}+\rho \boldsymbol{z}_{\mathrm{d}}\right)-\boldsymbol{y}\right)+\lambda \partial\left(\|\cdot\|_{1}\right)\left(\boldsymbol{x}_{\mathrm{p}}\right) \\
\mathbf{0} \in \rho \boldsymbol{A}^{\top} \boldsymbol{A}\left(\boldsymbol{z}_{\mathrm{d}}-\boldsymbol{x}_{\mathrm{p}}\right)+\lambda \partial\left(\|\cdot\|_{1}\right)\left(\boldsymbol{z}_{\mathrm{d}}\right) .
\end{array}\right.
$$

The inclusion above is further equivalent to (27). Expanding the expression of $\partial\left(\|\cdot\|_{1}\right)(\cdot)$ in (27) yields (OPT).
Based on Proposition 3, we define the extended solution and extended solution set of $\operatorname{sGMC}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$ as follows.
Definition 2 (Extended solution set of the sGMC model). For $(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda) \in \mathscr{P}, \boldsymbol{w}_{e}:=\left[\begin{array}{ll}\boldsymbol{x}_{p}{ }^{\top} & \boldsymbol{z}_{d}{ }^{\top}\end{array}\right]^{\top} \in \mathbb{R}^{2 n}$ is said to be an extended solution of $s G M C(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$ if $\left(\boldsymbol{x}_{p}, \boldsymbol{z}_{d}\right) \in \mathbb{R}^{n} \times \mathbb{R}^{n}$ is a primal-dual solution pair of $\operatorname{sGMC}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$, or equivalently, if $\boldsymbol{w}_{e}$ satisfies (OPT).

Moreover, we define the extended solution set of the sGMC model sGMC $(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$ as:

$$
\begin{equation*}
\mathcal{S}_{e}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)=\left\{\boldsymbol{w}_{e} \in \mathbb{R}^{2 n} \mid \boldsymbol{w}_{e} \text { satisfies }(O P T)\right\} . \tag{28}
\end{equation*}
$$

Recall the theoretical results in Sec. 3, we naturally have the following properties of $\mathcal{S}_{\mathrm{e}}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$.
Theorem 3 (Geometry and uniqueness of the extended sGMC solution set). For $(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda) \in \mathscr{P}$, the extended solution set of $s G M C(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$ (cf. Def. 2), denoted by $\mathcal{S}_{e}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$, is nonempty, closed, bounded and convex.

Moreover, the following holds:
(a) The following relation holds among $\mathcal{S}_{e}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$, the LASSO solution set mapping $\mathcal{S}_{L A}(\cdot)$ (cf. Fact 3), the primal sGMC solution set $\mathcal{S}_{p}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)\left(c f\right.$. Thm. 1) and the dual sGMC solution set $\mathcal{S}_{d}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$ (cf. Thm. 2):

$$
\begin{align*}
\mathcal{S}_{e}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda) & =\mathcal{S}_{L A}\left(\boldsymbol{A}, \frac{\boldsymbol{y}-\rho \boldsymbol{\beta}_{d}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)}{(1-\rho)}, \lambda\right) \times \mathcal{S}_{L A}\left(\sqrt{\rho} \boldsymbol{A}, \sqrt{\rho} \boldsymbol{\beta}_{p}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda), \lambda\right)  \tag{29}\\
& =\mathcal{S}_{p}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda) \times \mathcal{S}_{d}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda) \tag{30}
\end{align*}
$$

where $\boldsymbol{\beta}_{p}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$ and $\boldsymbol{\beta}_{d}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$ are respectively the common linear fits of all primal and dual solutions of $s G M C(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$ (cf. Thm. 1 (a) and 2 (a)).
(b) Every $\boldsymbol{w}_{e}:=\left[\begin{array}{ll}\boldsymbol{x}_{p}{ }^{\top} & \boldsymbol{z}_{d}{ }^{\top}\end{array}\right]^{\top} \in \mathcal{S}_{e}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$ gives the same extended linear fit $\boldsymbol{C} \boldsymbol{w}_{e}$ and the same $\ell_{1}-$ norm $\left\|\boldsymbol{w}_{e}\right\|_{1}$. In other words, the common extended linear fit of all extended sGMC solutions

$$
\boldsymbol{\beta}_{e}: \mathscr{P} \rightarrow \mathbb{R}^{2 m}:(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda) \mapsto \boldsymbol{C} \boldsymbol{w}_{e}:=\left[\begin{array}{c}
\boldsymbol{A} \boldsymbol{x}_{p} \\
\sqrt{\rho} \boldsymbol{A} \boldsymbol{z}_{d}
\end{array}\right]
$$

and the common $\ell_{1}$-norm of all extended sGMC solutions

$$
\gamma_{e}: \mathscr{P} \rightarrow \mathbb{R}_{+}:(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda) \mapsto\left\|\boldsymbol{w}_{e}\right\|_{1}:=\left\|\boldsymbol{x}_{p}\right\|_{1}+\left\|\boldsymbol{z}_{d}\right\|_{1}
$$

are well-defined functions of the parameters $(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$.

Additionally, we have the following relations

$$
\begin{align*}
\boldsymbol{\beta}_{e}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda) & =\left[\begin{array}{c}
\boldsymbol{\beta}_{p}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda) \\
\sqrt{\rho} \boldsymbol{\beta}_{d}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)
\end{array}\right]  \tag{31}\\
\gamma_{e}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda) & =\gamma_{p}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)+\gamma_{d}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda) \leq \frac{1}{\lambda(1-\rho)}\|\boldsymbol{y}\|_{2}^{2} \tag{32}
\end{align*}
$$

where $\boldsymbol{\beta}_{p}$ and $\gamma_{p}$ are respectively the common linear fit and $\ell_{1}$-norm of all primal sGMC solutions (cf. Thm. 1 (a)), $\boldsymbol{\beta}_{d}$ and $\gamma_{d}$ are respectively the common linear fit and $\ell_{1}$-norm of all dual sGMC solutions (cf. Thm. 2 (a)).
(c) If $\boldsymbol{A} \in \mathcal{G}_{m \times n}$ (cf. Def. 1), then $\mathcal{S}_{e}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$ has a unique element with at most $2 \min \{m, n\}$ nonzero elements; as a result, if the entries of $\boldsymbol{A}$ are drawn from a continuous probability distribution, then $\mathcal{S}_{e}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$ almost surely has a unique element with at most $2 \min \{m, n\}$ nonzero components.

Proof: (29) and (30) follow directly from Prop. 2. Hence from Thm. 1 and $2, \mathcal{S}_{\mathrm{e}}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$ is nonempty, closed, bounded and convex, and the result (a) holds. The results (b) and (c) follow from Thm. 1, 2 and Prop. 3.

The geometric properties of the extended sGMC solution set naturally guarantee the existence and uniqueness of the minimum $\ell_{2}$-norm extended solution; see the following theorem for some fundamental properties of it.

Theorem 4 (Basic properties of the minimum $\ell_{2}$-norm extended solution). For $(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda) \in \mathscr{P}$, there exists uniquely a minimum $\ell_{2}$-norm element in $\mathcal{S}_{e}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$, denoted by

$$
\boldsymbol{w}_{\star}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda):=\left[\begin{array}{l}
\boldsymbol{x}_{\star}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)  \tag{33}\\
\boldsymbol{z}_{\star}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)
\end{array}\right]
$$

where $\boldsymbol{x}_{\star}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda), \boldsymbol{z}_{\star}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda) \in \mathbb{R}^{n}$.
Moreover, the following holds:
(a) $\boldsymbol{x}_{\star}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$ and $\boldsymbol{z}_{\star}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$ are respectively the unique minimum $\ell_{2}$-norm elements in the primal sGMC solution set $\mathcal{S}_{p}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)\left(c f\right.$. Thm. 1) and the dual sGMC solution set $\mathcal{S}_{d}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$ (cf. Thm. 2).
(b) Define $\lambda_{\max }^{\star}=\max _{i \in\{1,2, \ldots, n\}}\left|\boldsymbol{a}_{i}^{\top} \boldsymbol{y}\right|$, then we have

$$
\left(\forall \lambda \geq \lambda_{\max }^{\star}\right) \quad \boldsymbol{w}_{\star}(\lambda)=\mathbf{0}_{2 n} .
$$

Note that the value of $\lambda_{\max }^{\star}$ is independent of the shape-controlling parameter $\rho$ of the sGMC model.
Proof: See Appendix C.1.
Subsequently, we study the continuity of the extended sGMC solution set $\mathcal{S}_{\mathrm{e}}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$ with respect to the problem parameters $(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda) \in \mathscr{P}$. For the sake of clarity, we first introduce some necessary knowledge for set-valued variational analysis in Sec. 4.2, exploiting which we can establish the continuity of $\mathcal{S}_{\mathrm{e}}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$ with respect to $\lambda$ in Sec. 4.3.

### 4.2 Selected Elements of Set-Valued Variational Analysis

In this subsection, we present a brief introduction on useful notions and theorems for analyzing continuity properties of set-valued mappings. To make this introduction self-contained, we start from defining the convergence of a sequence of sets. In the following, we introduce the "outer limit" and "inner limit" of a sequence of sets, which are respectively analogies of the upper and lower limits of a real sequence.
Definition 3 (Outer and inner limits of a sequence of sets [29, Def. 4.1 and the paragraph after it]). For a sequence of sets $\left(S_{k}\right)_{k \in \mathbb{N}}$ with $S_{k} \subset \mathbb{R}^{n}$ for every $k \in \mathbb{N}$, the outer limit of $\left(S_{k}\right)_{k \in \mathbb{N}}$ is defined as

$$
\limsup _{k \rightarrow \infty} S_{k}:=\left\{\boldsymbol{x} \in \mathbb{R}^{n} \mid \forall i \in \mathbb{N}, \exists \boldsymbol{x}_{k_{i}} \in S_{k_{i}} \text { with }\left(S_{k_{i}}\right)_{i \in \mathbb{N}} \text { being a subsequence of }\left(S_{k}\right)_{k \in \mathbb{N}} \text { and } \lim _{i \rightarrow \infty} \boldsymbol{x}_{k_{i}}=\boldsymbol{x}\right\}
$$

and the inner limit of $\left(S_{k}\right)_{k \in \mathbb{N}}$ is defined as

$$
\liminf _{k \rightarrow \infty} S_{k}:=\left\{\boldsymbol{x} \in \mathbb{R}^{n} \mid \forall k \in \mathbb{N}, \exists \boldsymbol{x}_{k} \in S_{k} \text { such that } \lim _{k \rightarrow \infty} \boldsymbol{x}_{k}=\boldsymbol{x}\right\} .
$$

In words, $\lim \sup _{k} S_{k}$ consists of all possible cluster points of $\left(\boldsymbol{x}_{k}\right)_{k \in \mathbb{N}}$ with $\boldsymbol{x}_{k} \in S_{k}$ for all $k$, whereas $\liminf _{k} S_{k}$ consists of all possible limit points of such sequences ${ }^{7}$. From the definition, we naturally have $\liminf _{k} S_{k} \subset \limsup _{k} S_{k}$.
7. We note that there is an equivalent characterization of the inner limit and outer limit, which draws a connection between 'semilimits' of sequences of sets and that of real numbers (see [29, Exercise 4.2 (a)]):

$$
\begin{aligned}
\liminf _{k \rightarrow \infty} S_{k} & =\left\{\boldsymbol{x} \mid \limsup _{k \rightarrow \infty} d\left(\boldsymbol{x}, S_{k}\right)=0\right\} \\
\limsup _{k \rightarrow \infty} S_{k} & =\left\{\boldsymbol{x} \mid \liminf _{k \rightarrow \infty} d\left(\boldsymbol{x}, S_{k}\right)=0\right\}
\end{aligned}
$$

where $d(\boldsymbol{x}, S)$ is the distance between the point $\boldsymbol{x}$ and the set $S$.

Recalling that a real-valued function $f$ is continuous at $x \in \mathbb{R}$ if and only if $f(x)=\limsup _{z \rightarrow x} f(z)=\liminf _{z \rightarrow x} f(z)$, we can define the continuity of set-valued mappings in a similar way; see the following definition.
Definition 4 (Outer and inner semicontinuity of a set-valued mapping [29, Def. 5.4]). For $D \subset \mathbb{R}^{m}, \mathcal{M}: D \rightrightarrows \mathbb{R}^{n}$ is outer semicontinuous (osc) at $\overline{\boldsymbol{\theta}} \in D$ relative to $D$ if for every $\left(\boldsymbol{\theta}_{k}\right)_{k \in \mathbb{N}} \subset D$ with $\lim _{k \rightarrow \infty} \boldsymbol{\theta}_{k}=\overline{\boldsymbol{\theta}}$, we have

$$
\limsup _{k \rightarrow \infty} \mathcal{M}\left(\boldsymbol{\theta}_{k}\right) \subset \mathcal{M}(\overline{\boldsymbol{\theta}}) .
$$

On the other hand, $\mathcal{M}$ is inner semicontinuous (isc) at $\overline{\boldsymbol{\theta}} \in D$ relative to $D$ if for every $\left(\boldsymbol{\theta}_{k}\right)_{k \in \mathbb{N}} \subset D$ with $\lim _{k \rightarrow \infty} \boldsymbol{\theta}_{k}=\overline{\boldsymbol{\theta}}$,

$$
\liminf _{k \rightarrow \infty} \mathcal{M}\left(\boldsymbol{\theta}_{k}\right) \supset \mathcal{M}(\overline{\boldsymbol{\theta}})
$$

$\mathcal{M}$ is called continuous at $\overline{\boldsymbol{\theta}} \in D$ relative to $D$ if it is both osc and isc at $\overline{\boldsymbol{\theta}}$ relative to $D$, i.e., if for every $\left(\boldsymbol{\theta}_{k}\right)_{k \in \mathbb{N}} \subset D$ with $\lim _{k \rightarrow \infty} \boldsymbol{\theta}_{k}=\overline{\boldsymbol{\theta}}$,

$$
\limsup _{k \rightarrow \infty} \mathcal{M}\left(\boldsymbol{\theta}_{k}\right)=\liminf _{k \rightarrow \infty} \mathcal{M}\left(\boldsymbol{\theta}_{k}\right)=\mathcal{M}(\overline{\boldsymbol{\theta}})
$$

If $\mathcal{M}$ is osc (resp. isc, continuous) at every point in $D$ relative to $D$, we say $\mathcal{M}$ is osc (resp. isc, continuous) relative to $D$.
Notice that single-valued mappings can be viewed as special set-valued mappings whose values are singletons. Then one can verify that in the context of single-valued mappings, Def. 4 coincides with the conventional definition of continuity of functionals (cf. [35, Def. Thm. 4.2 and 4.6]), which demonstrates the reasonableness of Def. 4.

In the sequel, we introduce two useful lemmas about semicontinuous mappings, which will play critical roles in the analysis of the extended sGMC solution set mapping. Our first lemma (Lemma 2) indicates that the composition of a osc (resp. isc) set-valued mapping and a continuous single-valued mapping is osc (resp. isc) under mild conditions.

Lemma 2. For $D_{T} \subset \mathbb{R}^{p}, D_{M} \subset \mathbb{R}^{m}$, suppose that the following holds for $T: D_{T} \rightarrow \mathbb{R}^{m}, \mathcal{M}: D_{M} \rightrightarrows \mathbb{R}^{n}$ :
(a) $T$ is continuous relative to $D_{T}$,
(b) For every $\boldsymbol{\theta} \in D_{T}, T(\boldsymbol{\theta}) \in D_{M}$.
(c) $\mathcal{M}$ is osc (resp. isc) relative to $D_{M}$.

Then $\mathcal{M} \circ T(\cdot):=\mathcal{M}(T(\cdot))$ is osc (resp. isc) relative to $D_{T}$.
Proof: For $\overline{\boldsymbol{\theta}} \in D_{T}$, let $\left(\boldsymbol{\theta}_{k}\right)_{k \in \mathbb{N}} \subset D_{T}$ be an arbitrary sequence convergent to $\overline{\boldsymbol{\theta}}$. Then from the assumptions (a) and (b), we have $T\left(\boldsymbol{\theta}_{k}\right) \in D_{M}$ for every $k$ and $\lim _{k \rightarrow \infty} T\left(\boldsymbol{\theta}_{k}\right)=T(\overline{\boldsymbol{\theta}}) \in D_{M}$. If $\mathcal{S}$ is osc relative to $D_{M}$, from Def. 4 we have

$$
\limsup _{k \rightarrow \infty} \mathcal{M}\left(T\left(\boldsymbol{\theta}_{k}\right)\right) \subset \mathcal{M}(T(\overline{\boldsymbol{\theta}}))
$$

hence $\mathcal{M} \circ T(\cdot)$ is osc relative to $D_{T}$. Similarly, if $\mathcal{M}$ is isc relative to $D_{T}$, then $\mathcal{M} \circ T(\cdot)$ is isc relative to $D_{T}$.
Our second lemma (Lemma 3) shows that the solution set mapping of a linear program is continuous with respect to the right-hand side data of the problem.
Lemma 3. Given $\boldsymbol{A} \in \mathbb{R}^{m \times n}, \boldsymbol{Q} \in \mathbb{R}^{k \times n}$, for $\boldsymbol{\zeta} \in \mathbb{R}^{m}$ and $\boldsymbol{\eta} \in \mathbb{R}^{k}$, define $\mathcal{L}_{\boldsymbol{A}, \boldsymbol{Q}}: \mathbb{R}^{m} \times \mathbb{R}^{k} \rightrightarrows \mathbb{R}^{n}$ as:

$$
\mathcal{L}_{\boldsymbol{A}, \boldsymbol{Q}}(\boldsymbol{\zeta}, \boldsymbol{\eta}):=\left\{\boldsymbol{x} \in \mathbb{R}^{n} \mid \boldsymbol{A} \boldsymbol{x}=\boldsymbol{\zeta}, \boldsymbol{Q} \boldsymbol{x} \leq \boldsymbol{\eta}\right\}
$$

Then $\mathcal{L}_{\boldsymbol{A}, \boldsymbol{Q}}(\cdot, \cdot)$ is continuous relative to its domain $\operatorname{dom} \mathcal{L}_{\boldsymbol{A}, \boldsymbol{Q}}:=\left\{(\boldsymbol{\zeta}, \boldsymbol{\eta}) \in \mathbb{R}^{m} \times \mathbb{R}^{k} \mid \mathcal{L}_{\boldsymbol{A}, \boldsymbol{Q}}(\boldsymbol{\zeta}, \boldsymbol{\eta}) \neq \emptyset\right\}$.
Proof: See Appendix C.2.

### 4.3 Continuity of the Extended Solution Set Mapping

Subsequently, we apply the theoretical results above to the extended solution set mapping and analyze its continuity. We firstly show the outer semicontinuity of $\mathcal{S}_{\mathrm{e}}(\cdot)$ relative to $\mathscr{P}$, which can be derived directly from (OPT) and Thm. 3 .
Proposition 4 (Outer semicontinuity of the extended sGMC solution set mapping). For an arbitrary sequence of parameters $\left(\boldsymbol{A}_{k}, \boldsymbol{y}_{k}, \rho_{k}, \lambda_{k}\right)_{k \in \mathbb{N}} \subset \mathscr{P}$ convergent to $(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda) \in \mathscr{P}$, let $\left(\boldsymbol{w}_{k}\right)_{k \in \mathbb{N}} \subset \mathbb{R}^{2 n}$ be a sequence of points satisfying

$$
\boldsymbol{w}_{k} \in \mathcal{S}_{e}\left(\boldsymbol{A}_{k}, \boldsymbol{y}_{k}, \rho_{k}, \lambda_{k}\right),
$$

then the following holds:
(a) $\left(\boldsymbol{w}_{k}\right)_{k}$ is bounded.
(b) For every cluster point of $\left(\boldsymbol{w}_{k}\right)_{k}$, say $\boldsymbol{w}_{\infty}$, we have $\boldsymbol{w}_{\infty} \in \mathcal{S}_{e}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$.

In other words, $\mathcal{S}_{e}(\cdot)$ is osc relative to $\mathscr{P}$.
Proof: See Appendix C.3.
Exploiting Prop. 4, we can further establish the single-valued continuity of the common extended linear fit value $\boldsymbol{\beta}_{\mathrm{e}}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$ and the common $\ell_{1}$-norm $\gamma_{\mathrm{e}}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$ of all extended sGMC solutions (cf. Thm. 3 (b)) with respect to the problem parameters $(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$; see the following corollary.

Corollary 2 (Continuity of the two single-valued mappings $\boldsymbol{\beta}_{\mathrm{e}}(\cdot)$ and $\gamma_{\mathrm{e}}(\cdot)$ ). The mappings of the common extended linear fit $\boldsymbol{\beta}_{e}(\cdot)$ and common $\ell_{1}$-norm $\gamma_{e}(\cdot)$ of all extended sGMC solutions (cf. Thm. $3(b)$ ) are continuous relative to $\mathscr{P}$.

Proof: See Appendix C.4.
Recall that the LASSO model is a special instance of the sGMC model, all results above naturally apply to the LASSO model, as summarized in the following remark.

Remark 3 (Outer semicontinuity of the LASSO solution set mapping). Since for every $(\boldsymbol{A}, \boldsymbol{y}, \lambda) \in \mathbb{R}^{m \times n} \times \mathbb{R}^{m} \times \mathbb{R}_{++}$, $\operatorname{LASSO}(\boldsymbol{A}, \boldsymbol{y}, \lambda)$ is equivalent to $\operatorname{sGMC}(\boldsymbol{A}, \boldsymbol{y}, 0, \lambda)$, we can derive from Prop. 4 and Cor. 2 that:
(a) The LASSO solution set mapping $\mathcal{S}_{L A}(\cdot)$ (cf. Fact 3) is osc relative to $\mathbb{R}^{m \times n} \times \mathbb{R}^{m} \times \mathbb{R}_{++}$.
(b) The mappings of the common linear fit $\boldsymbol{\beta}_{L A}(\cdot)$ and common $\ell_{1}$-norm $\gamma_{L A}(\cdot)$ of all LASSO solutions (cf. Fact 3) are continuous relative to $\mathbb{R}^{m \times n} \times \mathbb{R}^{m} \times \mathbb{R}_{++}$.

On the other hand, we should remark that it is typically harder to verify the inner semicontinuity of a set-valued mapping than its outer semicontinuity [29, p. 155, para. 2]. Nevertheless, by leveraging (29) in Thm. 3 and Remark 3, we manage to prove the inner semicontinuity of $\mathcal{S}_{\mathrm{e}}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$ with respect to $\lambda$, whereby establishing the continuity of $\mathcal{S}_{\mathrm{e}}(\boldsymbol{A}, \boldsymbol{y}, \rho, \cdot)$. Our strategy for proving continuity is to repeatedly decompose a complicated set-valued mapping into the composition of a simpler continuous set-valued mapping and a continuous single-valued mapping, and use Lemma 2 to transfer continuity. More precisely,

1) we first prove the continuity of the LASSO solution set mapping with respect to $(\boldsymbol{y}, \lambda)$ by connecting it with the continuous solution set mapping of a linear program (cf. Lemma 3) and a continuous single-valued mapping involving $\boldsymbol{\beta}_{\mathrm{LA}}(\cdot)$ and $\gamma_{\mathrm{LA}}(\cdot)$ (cf. Remark 3),
2) notice that in Thm. 3 (a), $\mathcal{S}_{\mathrm{e}}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$ is the composition of the LASSO solution set mapping and a single-valued mapping involving $\boldsymbol{\beta}_{\mathrm{e}}(\cdot)$ (which is continuous from Cor. 2), we can prove the continuity of $\mathcal{S}_{\mathrm{e}}(\boldsymbol{A}, \boldsymbol{y}, \rho, \cdot)$ by the continuity of the LASSO solution set mapping.
Hereafter, we present the aforementioned results step by step.
We first show the continuity of the LASSO solution set mapping with respect to the parameters $(\boldsymbol{y}, \lambda)$.
Lemma 4 (Continuity of the LASSO solution set mapping in $(\boldsymbol{y}, \lambda)$ ). Given $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, the LASSO solution set mapping, i.e., $\mathcal{S}_{L A}(\boldsymbol{A}, \cdot, \cdot)$ defined in Fact 3 , is continuous relative to $\mathbb{R}^{m} \times \mathbb{R}_{++}$. In words, the LASSO solution set $\mathcal{S}_{L A}(\boldsymbol{A}, \boldsymbol{y}, \lambda)$ changes continuously with $(\boldsymbol{y}, \lambda) \in \mathbb{R}^{m} \times \mathbb{R}_{++}$.

Proof: See Appendix C.5.
To our knowledge, Lemma 4 is the first set-valued variational analysis result for the LASSO model. Moreover, if one reads carefully the proof of Lemma 4, one may realize that it may be possible to prove the "Lipschitz continuity" of $\mathcal{S}_{\mathrm{LA}}(\boldsymbol{A}, \cdot, \cdot)$ if we derive a stronger version of Lemma 2 for Lipschitz continuous mappings and use Fact 8 in Appendix A. However, although the "Lipschitz continuity" of $\mathcal{S}_{\mathrm{LA}}(\boldsymbol{A}, \cdot, \cdot)$ is of strong interest (especially for studying the numerical stability of LASSO), it is out of the scope of the current paper and we would like to leave it for future work.

Rewritting $\mathcal{S}_{\mathrm{e}}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$ as the Cartesian product of two LASSO solution sets (cf. Thm. 3 (a)), we can prove the continuity of $\mathcal{S}_{\mathrm{e}}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$ with respect to $\lambda$ using Lemma 4 , which constitutes the main result of this subsection.
Theorem 5 (Continuity of the extended sGMC solution set mapping in $\lambda$ ). Given $(\boldsymbol{A}, \boldsymbol{y}, \rho) \in \mathbb{R}^{m \times n} \times \mathbb{R}^{m} \times[0,1$ ), $\mathcal{S}_{e}(\boldsymbol{A}, \boldsymbol{y}, \rho, \cdot)$ (see Def. 2) is continuous relative to $\mathbb{R}_{++}$. In words, the extended sGMC solution set $\mathcal{S}_{e}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$ changes continuously with $\lambda \in \mathbb{R}_{++}$.

Proof: See Appendix C.6.
Thm. 5 further implies the following corollary, which establishes the continuity of the minimum $\ell_{2}$-norm extended regularization path of $\operatorname{sGMC}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$.

Corollary 3 (Continuity of the minimum $\ell_{2}$-norm extended regularization path). Given $(\boldsymbol{A}, \boldsymbol{y}, \rho) \in \mathbb{R}^{m \times n} \times \mathbb{R}^{m} \times[0,1$ ), the minimum $\ell_{2}$-norm extended regularization path, i.e., $\boldsymbol{w}_{\star}(\boldsymbol{A}, \boldsymbol{y}, \rho, \cdot)$ defined in (33), is continuous relative to $\mathbb{R}_{++}$. In words, the minimum $\ell_{2}$-norm extended solution, i.e., $\boldsymbol{w}_{\star}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$, changes continuously with $\lambda \in \mathbb{R}_{++}$.

Proof: Define $\mathcal{M}(\cdot):=\mathcal{S}_{\mathrm{e}}(\boldsymbol{A}, \boldsymbol{y}, \rho, \cdot)$. Then from Thm. 3, $\mathcal{M}$ is convex-valued with $\operatorname{dom} \mathcal{M}=\mathbb{R}_{++}$. Additionally, $\mathcal{M}$ is continuous relative to $\operatorname{dom} \mathcal{M}=\mathbb{R}_{++}$from Thm. 5, and for every $\lambda>0, \boldsymbol{w}_{\star}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$ is the projection of $\mathbf{0}_{2 n}$ on $\mathcal{M}(\lambda)$ from Thm. 4. Hence from Fact 9 in Appendix $\mathrm{A}, \boldsymbol{w}_{\star}(\boldsymbol{A}, \boldsymbol{y}, \rho, \cdot)$ is continuous relative to $\mathbb{R}_{++}$.

In the rest of this paper, we will delve deeper in how the extended solution set $\mathcal{S}_{\mathrm{e}}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$ and the minimum $\ell_{2}$-norm extended solution $\boldsymbol{w}_{\star}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$ change with $\lambda$. Notice that in the sections that follow, we are only interested in the varying regularization parameter $\lambda$, hence we make the following remark to simplify the notation.
Remark 4. In subsequent sections, we assume that the value of $(\boldsymbol{A}, \boldsymbol{y}, \rho) \in \mathbb{R}^{m \times n} \times \mathbb{R}^{m} \times[0,1)$ is fixed. Hence for simplicity of presentation, in the sequel, we abbreviate every function of $(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda) \in \mathscr{P}$ as a function of $\lambda$, for example:

$$
\begin{aligned}
\mathcal{S}_{e}(\lambda) & :=\mathcal{S}_{e}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda) \\
\boldsymbol{w}_{\star}(\lambda) & :=\boldsymbol{w}_{\star}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)
\end{aligned}
$$

and we abbreviate sGMC $(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$ as $s G M C(\lambda)$.

## 5 Characterizing the sGMC Model with Ex-Equicorrelation Pairs

In this section, we introduce ex-equicorrelation pair, a discrete object for characterizing the states of the extended solution set $\mathcal{S}_{\mathrm{e}}(\lambda)$ and the minimum $\ell_{2}$-norm extended solution $\boldsymbol{w}_{\star}(\lambda)$ at $\lambda$. It will be shown that all information about the continuous mappings $\mathcal{S}_{\mathrm{e}}(\cdot)$ and $\boldsymbol{w}_{\star}(\cdot)$ are essentially embraced in the finite state space termed ex-equicorrelation space which comprise all ex-equicorrelation pairs. Ex-equicorrelation pairs and ex-equicorrelation space will be central tools for deriving the piecewise expressions of $\boldsymbol{w}_{\star}(\lambda)$ and $\mathcal{S}_{\mathrm{e}}(\lambda)$ in Sec. 6, as well as for developing the LARS-sGMC algorithm in Sec. 7 .

The main content of this section is summarized as follows. Given the values of $(\boldsymbol{A}, \boldsymbol{y}, \rho) \in \mathbb{R}^{m \times n} \times \mathbb{R}^{m} \times[0,1)$, the ex-equicorrelation space $\mathcal{V}$ is defined as the set of all possible combinations $(\mathcal{E}, s)$ of an index set $\mathcal{E} \subset\{1,2, \ldots, 2 n\}$ and a signs vector $s \in\{-1,0,+1\}^{2 n}$ that satisfy certain constraints (cf. Prop. 5). All information about the mappings $\mathcal{S}_{\mathrm{e}}(\cdot)$ and $\boldsymbol{w}_{\star}(\cdot)$ are embraced in $\mathcal{V}$ in the sense that for every $\lambda>0$,

1) $\mathcal{S}_{\mathrm{e}}(\lambda)$ always coincides with the value of a polytope-valued mapping $\mathcal{S}_{\mathrm{POLY}}(\mathcal{E}, \boldsymbol{s}, \lambda)$ of $\lambda$ for some $(\mathcal{E}, \boldsymbol{s}) \in \mathcal{V}$, and $\mathcal{S}_{\mathrm{e}}(\cdot)$ can be equivalently represented by an ex-equicorrelation-pair-valued mapping of $\lambda$ (cf. Cor. 5),
2) $\boldsymbol{w}_{\star}(\lambda)$ always coincides with the value of an affine function $\hat{\boldsymbol{w}}_{\mathrm{EQ}}(\mathcal{E}, \boldsymbol{s}, \lambda)$ of $\lambda$ for some $(\mathcal{E}, \boldsymbol{s}) \in \mathcal{V}$, and $\boldsymbol{w}_{\star}(\cdot)$ can be equivalently represented by another ex-equicorrelation-pair-valued mapping of $\lambda$ (cf. Cor. 6).
As a direct application of these results, in the end of this section (cf. Remark 7), we present an intuitive proof for piecewise linearity of $\boldsymbol{w}_{\star}(\lambda)$, which may possibly be considered as a counterintuive fact if one recalls nonconvexity of the sGMC penalty (cf. (2)).

### 5.1 Ex-Equicorrelation Space and Ex-Equicorrelation Pairs

In this section, we start from introducing the notions of ex-equicorrelation space and ex-equicorrelation pairs. Firstly, we revisit the optimality condition (OPT) of the sGMC model (cf. Prop. 3), from which we point out the connection between the extended solution set $\mathcal{S}_{\mathrm{e}}(\lambda)$ and two discrete-valued functions of $\lambda$ (cf. Def. 5). The possible values of the pair of these two discrete functions further lead to the definitions of the ex-equicorrelation space and ex-equicorrelation pairs.

Recall that given $\lambda>0, \boldsymbol{w}_{\mathrm{e}} \in \mathbb{R}^{2 n}$ is in $\mathcal{S}_{\mathrm{e}}(\lambda)$ if and only if for every $i \in\{1, \ldots, 2 n\}, \boldsymbol{w}_{\mathrm{e}}$ satisfies that

$$
\xi_{i}\left(\boldsymbol{w}_{\mathrm{e}}\right):=\boldsymbol{c}_{i}^{\top}\left(\boldsymbol{b}-\boldsymbol{D} \boldsymbol{C} \boldsymbol{w}_{\mathrm{e}}\right) \in \begin{cases}\left\{\lambda \operatorname{sign}\left(\left[\boldsymbol{w}_{\mathrm{e}}\right]_{i}\right)\right\}, & \text { if }\left[\boldsymbol{w}_{\mathrm{e}}\right]_{i} \neq 0  \tag{OPT}\\ {[-\lambda, \lambda],} & \text { if }\left[\boldsymbol{w}_{\mathrm{e}}\right]_{i}=0\end{cases}
$$

Observing (OPT) carefully, one can see that the behavior of $\boldsymbol{w}_{\mathrm{e}}$ in (OPT) highly depends on the value of $\boldsymbol{\xi}\left(\boldsymbol{w}_{\mathrm{e}}\right):=$ $\boldsymbol{C}^{\top}\left(\boldsymbol{b}-\boldsymbol{D} \boldsymbol{C} \boldsymbol{w}_{\mathrm{e}}\right)$, which evaluates the correlation between the column vectors of the "extended sensing matrix" $\boldsymbol{C}$ and the "extended residual vector" $\left(\boldsymbol{b}-\boldsymbol{D C} \boldsymbol{w}_{\mathrm{e}}\right)$. Moreover, we can summarize (OPT) verbally into the following two rules.

Remark 5 (Two rules of (OPT)). For every $\lambda>0$ and arbitrary $\boldsymbol{w}_{e} \in \mathcal{S}_{e}(\lambda)$, the following two rules always hold:
(a) For every nonzero-component index $i \in \operatorname{supp}\left(\boldsymbol{w}_{e}\right)$ of an extended solution $\boldsymbol{w}_{e} \in \mathcal{S}_{e}(\lambda)$,

- the magnitude of the ith correlation coefficient (i.e., $\left.\left|\xi_{i}\left(\boldsymbol{w}_{e}\right)\right|\right)$ must attain the threshold $\lambda$,
- the ith correlation sign (i.e., $\operatorname{sign}\left(\xi_{i}\left(\boldsymbol{w}_{e}\right)\right)$ ) must align with the ith component sign (i.e., $\left.\operatorname{sign}\left(\left[\boldsymbol{w}_{e}\right]_{i}\right)\right)$.
(b) For every zero-component index $i \in \neg \operatorname{supp}\left(\boldsymbol{w}_{e}\right)$ of an extended solution $\boldsymbol{w}_{e} \in \mathcal{S}_{e}(\lambda)$,
- the magnitude of the ith correlation coefficient (i.e., $\left|\xi_{i}\left(\boldsymbol{w}_{e}\right)\right|$ ) should not exceed the threshold $\lambda$.

One can verify that the rule (a) above relates the support of $\boldsymbol{w}_{\mathrm{e}}$ with the set of indices that attains correlation-threshold equality $^{8}$ (i.e., $\left|\xi_{i}\left(\boldsymbol{w}_{\mathrm{e}}\right)\right|=\lambda$ ), and relates the signs vector of $\boldsymbol{w}_{\mathrm{e}}$ with the signs of the corresponding correlation coefficients. Motivated by this relationship, for an arbitrary vector $\boldsymbol{w} \in \mathbb{R}^{2 n}$ (where $\boldsymbol{w}$ does not have to be an extended solution), we formally define its set of indices satisfying correlation-threshold equality as the ex-equicorrelation set $\mathcal{E}(\lambda, \boldsymbol{w})$ (where "ex" is an abbreviation for "extended"), and define the masked version ${ }^{9}$ of the correlation signs vector (i.e., $\operatorname{sign}\left(\boldsymbol{\xi}\left(\boldsymbol{w}_{\mathrm{e}}\right)\right)$ ) on $\mathcal{E}(\lambda, \boldsymbol{w})$ as the ex-equicorrelation signs vector $\boldsymbol{s}(\lambda, \boldsymbol{w})$.
Definition 5. For $\lambda>0$ and $\boldsymbol{w} \in \mathbb{R}^{2 n}$, we define the ex-equicorrelation set $\mathcal{E}(\lambda, \boldsymbol{w})$ and the ex-equicorrelation signs vector $\boldsymbol{s}(\lambda, \boldsymbol{w}):=\left[s_{1}(\lambda, \boldsymbol{w}), \cdots, s_{2 n}(\lambda, \boldsymbol{w})\right]^{\top}$ as:

$$
\begin{aligned}
\mathcal{E}(\lambda, \boldsymbol{w}) & :=\left\{i \in\{1, \ldots, 2 n\}| | \xi_{i}(\boldsymbol{w}) \mid=\lambda\right\}, \\
s_{i}(\lambda, \boldsymbol{w}) & := \begin{cases}\operatorname{sign}\left(\xi_{i}(\boldsymbol{w})\right), & \text { if } i \in \mathcal{E}(\lambda, \boldsymbol{w}), \\
0, & \text { otherwise } .\end{cases}
\end{aligned}
$$

We remark that the terms "equicorrelation set" and "equicorrelation signs vector" were firstly coined in the study of the regularization path of the generalized LASSO model [30], though the idea of studying similar notions appears earlier in the studies of conventional LASSO regularization path [15]. Def. 5 of ex-equicorrelation set and ex-equicorrelation signs vector are natural variants and extensions of previous works.
8. It should be noted that from the rule (b), even for a zero-component index $i \in \neg \operatorname{supp}\left(\boldsymbol{w}_{\mathrm{e}}\right)$, it is possible for $\left|\xi_{i}\left(\boldsymbol{w}_{\mathrm{e}}\right)\right|$ to attain the threshold $\lambda$, hence the correlation-threshold equality condition is a necessary but not sufficient condition for the $i$ th component $\left[\boldsymbol{w}_{\mathrm{e}}\right]_{i}$ to be nonzero.
9. For a vector $\boldsymbol{x} \in \mathbb{R}^{n}$ and an index set $\mathcal{I} \subset\{1,2, \ldots, n\}$, the masked version of $\boldsymbol{x}$ on $\mathcal{I}$ is the unique vector $\boldsymbol{z} \in \mathbb{R}^{n}$ that satisfies $[\boldsymbol{z}]_{\mathcal{I}}=[\boldsymbol{x}]_{\mathcal{I}}$ and $[\boldsymbol{z}]_{\neg \mathcal{I}}=\mathbf{0}$.

However, it should be noted that while the two discrete-valued functions $\mathcal{E}(\lambda, \boldsymbol{w})$ and $\boldsymbol{s}(\lambda, \boldsymbol{w})$ play crucial roles in subsequent derivation, they are not the main characters. Instead, the most interested objects will be the finite state space termed ex-equicorrelation space $\mathcal{V}$ (cf. Prop. 5) that $(\mathcal{E}(\lambda, \boldsymbol{w}), \boldsymbol{s}(\lambda, \boldsymbol{w}))$ takes its value from, as well as all of the discrete states (termed ex-equicorrelation pairs) in $\mathcal{V}$. To introduce their definitions, in the following, we first present a lemma revealing the general relation among $\mathcal{E}(\lambda, \boldsymbol{w}), \boldsymbol{s}(\lambda, \boldsymbol{w})$ and the correlation vector $\boldsymbol{\xi}(\boldsymbol{w})$.
Lemma 5. For $\lambda>0$ and $\boldsymbol{w} \in \mathbb{R}^{2 n}$, we have the following relations among $\mathcal{E}:=\mathcal{E}(\lambda, \boldsymbol{w}), \boldsymbol{s}:=\boldsymbol{s}(\lambda, \boldsymbol{w})$ and $\boldsymbol{\xi}(\boldsymbol{w})$ :
(a) $\mathcal{E}=\operatorname{supp}(s)$.
(b) $[\boldsymbol{\xi}(\boldsymbol{w})]_{\mathcal{E}}=\boldsymbol{C}_{\mathcal{E}}{ }^{\top}(\boldsymbol{b}-\boldsymbol{D C \boldsymbol { w }})=\lambda[\boldsymbol{s}]_{\mathcal{E}}$.

Proof: The results follow directly from Def. 5.
Exploiting Lemma 5, it can be shown that for every $\lambda>0$ and $\boldsymbol{w} \in \mathbb{R}^{2 n}$, the pair of $(\mathcal{E}(\lambda, \boldsymbol{w}), \boldsymbol{s}(\lambda, \boldsymbol{w}))$ can only take discrete values from certain finite state space, whereby we can formally define the ex-equicorrelation space and ex-equicorrelation pair, i.e., the main characters in this section.

Proposition 5 (Ex-equicorrelation space and pairs). Define the ex-equicorrelation space $\mathcal{V}$ as:

$$
\mathcal{V}:=\left\{(\mathcal{E}, s) \mid s \in\{+1,0,-1\}^{2 n}, \mathcal{E}=\operatorname{supp}(\boldsymbol{s}),[s]_{\mathcal{E}} \in \mathcal{R}\left(\boldsymbol{C}_{\mathcal{E}}{ }^{\top}\right)\right\}
$$

Then $\mathcal{V}$ is a nonempty finite state space with $1 \leq|\mathcal{V}| \leq 3^{2 n}$, which contains all possible values of the pair of ex-equicorrelation set and ex-equicorrelation signs vector, i.e.,

$$
\left(\forall \lambda>0, \forall \boldsymbol{w} \in \mathbb{R}^{2 n}\right)(\mathcal{E}(\lambda, \boldsymbol{w}), \boldsymbol{s}(\lambda, \boldsymbol{w})) \in \mathcal{V}
$$

Hereafter, for every discrete state $(\mathcal{E}, s) \in \mathcal{V}$, we call $(\mathcal{E}, s)$ an ex-equicorrelation pair.
Proof: From the definition of $\mathcal{V}$, one can verify that $\left(\emptyset, \mathbf{0}_{2 n}\right) \in \mathcal{V}$, hence $|\mathcal{V}| \geq 1$; on the other hand, for every $(\mathcal{E}, s) \in \mathcal{V}$, the value of $\mathcal{E}=\operatorname{supp}(s)$ is determined by $s$. Since the value of $s$ can only be taken from $\{-1,0,+1\}^{2 n}$ which has $3^{2 n}$ elements, we have $|\mathcal{V}| \leq 3^{2 n}$.

For $\lambda>0$ and $\boldsymbol{w} \in \mathbb{R}^{2 n}$, let $(\mathcal{E}, \boldsymbol{s}):=(\mathcal{E}(\lambda, \boldsymbol{w}), \boldsymbol{s}(\lambda, \boldsymbol{w}))$. Then $\boldsymbol{s} \in\{-1,0,+1\}^{2 n}$ follows directly from Def. 5 . From Lemma 5 (a), we have $\mathcal{E}=\operatorname{supp}(s)$. Lemma 5 (b) yields

$$
\boldsymbol{C}_{\mathcal{E}}^{\top}(\boldsymbol{b}-\boldsymbol{D C w})=\lambda[\boldsymbol{s}]_{\mathcal{E}} \in \mathcal{R}\left(\boldsymbol{C}_{\mathcal{E}}^{\top}\right)
$$

hence we always have $(\mathcal{E}(\lambda, \boldsymbol{w}), \boldsymbol{s}(\lambda, \boldsymbol{w})) \in \mathcal{V}$.

### 5.2 Two Important Ex-Equicorrelation-Pair-Valued Mappings

Next, we further study the values of $(\mathcal{E}(\lambda, \cdot), s(\lambda, \cdot))$ for extended solutions, whereby we introduce two important ex-equicorrelation-pair-valued mappings of $\lambda$ that will play crucial roles in the characterization of $\mathcal{S}_{\mathrm{e}}(\lambda)$ and $\boldsymbol{w}_{\star}(\lambda)$.

We first show that for every extended solution $\boldsymbol{w}_{\mathrm{e}} \in \mathcal{S}_{\mathrm{e}}(\lambda)$, the value of $\left(\mathcal{E}\left(\lambda, \boldsymbol{w}_{\mathrm{e}}\right), \boldsymbol{s}\left(\lambda, \boldsymbol{w}_{\mathrm{e}}\right)\right)$ gives the same exequicorrelation pair $\left(\mathcal{E}_{\mathrm{e}}(\lambda), s_{\mathrm{e}}(\lambda)\right) \in \mathcal{V}$, which leads to the first important ex-equicorrelation-pair-valued mapping in this section. As will be shown later in Cor. $5,\left(\mathcal{E}_{\mathrm{e}}(\lambda), s_{\mathrm{e}}(\lambda)\right)$ indeed encodes the whole information about $\mathcal{S}_{\mathrm{e}}(\lambda)$.
Lemma 6. Given $\lambda>0$, then for every $\boldsymbol{w}_{e} \in \mathcal{S}_{e}(\lambda),\left(\mathcal{E}\left(\lambda, \boldsymbol{w}_{e}\right), \boldsymbol{s}\left(\lambda, \boldsymbol{w}_{e}\right)\right)$ gives the same ex-equicorrelation pair in $\mathcal{V}$. In words, the pair of common ex-equicorrelation set and ex-equicorrelation signs vector for all $\boldsymbol{w}_{e} \in \mathcal{S}_{e}(\lambda)$ :

$$
\left(\mathcal{E}_{e}(\lambda), \boldsymbol{s}_{e}(\lambda)\right):=\left(\mathcal{E}\left(\lambda, \boldsymbol{w}_{e}\right), \boldsymbol{s}\left(\lambda, \boldsymbol{w}_{e}\right)\right) \in \mathcal{V}
$$

is a well-defined ex-equicorrelation-pair-valued mapping of $\lambda$.
Proof: From Def. 5, given $\boldsymbol{w}_{\mathrm{e}} \in \mathcal{S}_{\mathrm{e}}(\lambda)$, the values of $\mathcal{E}\left(\lambda, \boldsymbol{w}_{\mathrm{e}}\right)$ and $\boldsymbol{s}\left(\lambda, \boldsymbol{w}_{\mathrm{e}}\right)$ are uniquely determined by the correlation vector $\boldsymbol{\xi}\left(\boldsymbol{w}_{\mathrm{e}}\right):=\boldsymbol{C}^{\top}\left(\boldsymbol{b}-\boldsymbol{D} \boldsymbol{C} \boldsymbol{w}_{\mathrm{e}}\right)$. Moreover, from Thm. 3 (b) in Sec. 4.1, every $\boldsymbol{w}_{\mathrm{e}} \in \mathcal{S}_{\mathrm{e}}(\lambda)$ gives the same extended linear fit $\boldsymbol{C} \boldsymbol{w}_{\mathrm{e}}=\boldsymbol{\beta}_{\mathrm{e}}(\lambda)$, hence the result holds.

Secondly, as mentioned in the paragraph just after Remark 5, we show that for every $\boldsymbol{w}_{\mathrm{e}} \in \mathcal{S}_{\mathrm{e}}(\lambda)$, there exists a connection between $\left(\operatorname{supp}\left(\boldsymbol{w}_{\mathrm{e}}\right), \operatorname{sign}\left(\boldsymbol{w}_{\mathrm{e}}\right)\right)$ and $\left(\mathcal{E}\left(\lambda, \boldsymbol{w}_{\mathrm{e}}\right), \boldsymbol{s}\left(\lambda, \boldsymbol{w}_{\mathrm{e}}\right)\right)$. Specifically, for every $\boldsymbol{w}_{\mathrm{e}} \in \mathcal{S}_{\mathrm{e}}(\lambda)$ (cf. Prop. 6),

- the value of $\left(\operatorname{supp}\left(\boldsymbol{w}_{\mathrm{e}}\right), \operatorname{sign}\left(\boldsymbol{w}_{\mathrm{e}}\right)\right)$ also gives an ex-equicorrelation pair in $\mathcal{V}$,
- the relation between the two ex-equicorrelation pairs $\left(\operatorname{supp}\left(\boldsymbol{w}_{\mathrm{e}}\right), \operatorname{sign}\left(\boldsymbol{w}_{\mathrm{e}}\right)\right)$ and $\left(\mathcal{E}\left(\lambda, \boldsymbol{w}_{\mathrm{e}}\right), \boldsymbol{s}\left(\lambda, \boldsymbol{w}_{\mathrm{e}}\right)\right)$ can be expressed compactly using a partial order ${ }^{10}$ on $\mathcal{V}$.
These results naturally lead to the definition of another important ex-equicorrelation-pair-valued mapping of $\lambda$, i.e.,

$$
\left(\operatorname{supp}\left(\boldsymbol{w}_{\star}(\lambda)\right), \operatorname{sign}\left(\boldsymbol{w}_{\star}(\lambda)\right)\right) \in \mathcal{V}
$$

and elucidate its relation with $\left(\mathcal{E}_{\mathrm{e}}(\lambda), s_{\mathrm{e}}(\lambda)\right)$ defined in Lemma 6 . As will be shown in Cor. 6 , the ex-equicorrelation-pairvalued mapping $\left(\operatorname{supp}\left(\boldsymbol{w}_{\star}(\lambda)\right), \operatorname{sign}\left(\boldsymbol{w}_{\star}(\lambda)\right)\right)$ indeed encodes the whole information about $\boldsymbol{w}_{\star}(\lambda)$.

[^3]

Fig. 1: An illustration for the correspondence between the two rules of (OPT) (cf. Remark 5) and the definition of the cutting polytope (see Def. 7).

For convenience of subsequent analysis, we present the aforementioned results in a non-straightforward order. We first define a super set $\overline{\mathcal{V}}$ of $\mathcal{V}$ which contains all possible values of $(\operatorname{supp}(\boldsymbol{w}), \operatorname{sign}(\boldsymbol{w}))$ for every $\boldsymbol{w} \in \mathbb{R}^{2 n}$. The partial order on $\mathcal{V}$ will then be defined on this super set $\overline{\mathcal{V}}$, and the claimed results will be introduced as a property of the partial order.
Definition 6 (A super set of the ex-equicorrelation space). Define the finite state space $\overline{\mathcal{V}}$ as

$$
\overline{\mathcal{V}}:=\left\{(\mathcal{E}, s) \mid s \in\{+1,0,-1\}^{2 n}, \mathcal{E}=\operatorname{supp}(s)\right\}
$$

Then since $\overline{\mathcal{V}}$ is defined simply by removing the condition $[s]_{\mathcal{E}} \in \mathcal{R}\left(\boldsymbol{C}_{\mathcal{E}}{ }^{\top}\right)$ in the definition of $\mathcal{V}$ in Prop. 5, we naturally have $\overline{\mathcal{V}} \supset \mathcal{V}$. Moreover, one can verify that

$$
\left(\forall \boldsymbol{w} \in \mathbb{R}^{2 n}\right) \quad(\operatorname{supp}(\boldsymbol{w}), \operatorname{sign}(\boldsymbol{w})) \in \overline{\mathcal{V}}
$$

Proposition 6 (A partial order on the super set of the ex-equicorrelation space). For every $\left(\mathcal{E}_{1}, s_{1}\right),\left(\mathcal{E}_{2}, s_{2}\right) \in \overline{\mathcal{V}}$, we say $\left(\mathcal{E}_{1}, \boldsymbol{s}_{1}\right) \preceq\left(\mathcal{E}_{2}, \boldsymbol{s}_{2}\right)$ if

$$
\mathcal{E}_{1} \subset \mathcal{E}_{2} \text { and }\left[s_{1}\right]_{\mathcal{E}_{1}}=\left[s_{2}\right]_{\mathcal{E}_{1}} .
$$

In words, $\left(\mathcal{E}_{1}, s_{1}\right) \preceq\left(\mathcal{E}_{2}, s_{2}\right)$ if the components of $s_{1}$ and $s_{2}$ coincide on the smaller support $\mathcal{E}_{1}$.
Then $\preceq$ is a partial order on $\overline{\mathcal{V}}$, and the following holds:
(a) $\mathcal{V}$ is downward closed ${ }^{11}$ with respect to $\preceq$.
(b) for every $\lambda>0$ and $\boldsymbol{w}_{e} \in \mathcal{S}_{e}(\lambda)$, we have

$$
\begin{equation*}
\mathcal{V} \ni\left(\operatorname{supp}\left(\boldsymbol{w}_{e}\right), \operatorname{sign}\left(\boldsymbol{w}_{e}\right)\right) \preceq\left(\mathcal{E}_{e}(\lambda), s_{e}(\lambda)\right) \in \mathcal{V} \tag{34}
\end{equation*}
$$

where $\left(\mathcal{E}_{e}(\lambda), s_{e}(\lambda)\right)$ is the pair of common ex-equicorrelation set and ex-equicorrelation signs vector for all extended solutions in $\mathcal{S}_{e}(\lambda)$ (cf. Lemma 6).

Proof: See Appendix D.1.
As a direct result of Prop. 6, we can formally introduce the second important ex-equicorrelation-pair-valued mapping $\left(\operatorname{supp}\left(\boldsymbol{w}_{\star}(\lambda)\right), \operatorname{sign}\left(\boldsymbol{w}_{\star}(\lambda)\right)\right)$ in this section and can elucidate its relation with $\left(\mathcal{E}_{\mathrm{e}}(\lambda), \boldsymbol{s}_{\mathrm{e}}(\lambda)\right)($ Lemma 6) as follows.
Corollary 4. For every $\lambda>0$, we have

$$
\mathcal{V} \ni\left(\operatorname{supp}\left(\boldsymbol{w}_{\star}(\lambda)\right), \operatorname{sign}\left(\boldsymbol{w}_{\star}(\lambda)\right)\right) \preceq\left(\mathcal{E}_{e}(\lambda), \boldsymbol{s}_{e}(\lambda)\right) \in \mathcal{V}
$$

In words, $\left(\operatorname{supp}\left(\boldsymbol{w}_{\star}(\lambda)\right), \operatorname{sign}\left(\boldsymbol{w}_{\star}(\lambda)\right)\right)$ is a well-defined ex-equicorrelation-pair-valued mapping of $\lambda$ that is always "no larger than" $\left(\mathcal{E}_{e}(\lambda), s_{e}(\lambda)\right)$ (cf. Lemma 6) with respect to the partial order $\preceq$ defined in Prop. 6.

In the following two subsections, we will characterize the extended solution set $\mathcal{S}_{\mathrm{e}}(\lambda)$ and the minimum $\ell_{2}$-norm extended solution $\boldsymbol{w}_{\star}(\lambda)$ using ex-equicorrelation pairs, whereby establishing equivalence between $\mathcal{S}_{\mathrm{e}}(\lambda)$, $\boldsymbol{w}_{\star}(\lambda)$ and the two ex-equicorrelation-pair-valued mappings introduced above.

### 5.3 Characterizing the Extended sGMC Solution Set Using Ex-Equicorrelation Pairs

In this section, we present a characterization of the extended solution set $\mathcal{S}_{\mathrm{e}}(\lambda)$ using ex-equicorrelation pairs.
It will be shown that (cf. Cor. 5): for every $(\mathcal{E}, s) \in \mathcal{V}$, we can define a polytope-valued mapping $\mathcal{S}_{\mathrm{POLY}}(\mathcal{E}, s, \lambda)$ of $\lambda$ termed cutting polytope (Def. 7) to approximate $\mathcal{S}_{\mathrm{e}}(\lambda)$, such that as $\lambda$ varies in $\mathbb{R}_{++}$,

1) the value of $\mathcal{S}_{\mathrm{e}}(\lambda)$ is always switching within the finite collection of cutting polytopes $\left\{\mathcal{S}_{\text {POLY }}(\mathcal{E}, s, \lambda)\right\}_{(\mathcal{E}, s) \in \mathcal{V}}$,
11. For a partially ordered set $(X, \preceq)$ and a subset $L \subset X$, if for every $l \in L$ and $x \in X, x \preceq l$ implies $x \in L$, then we say $L$ is downward closed with respect to $\preceq$. In other words, $L$ is closed under going down.
2) the set-valued mapping $\mathcal{S}_{\mathrm{e}}(\cdot)$ can be equivalently represented by the ex-equicorrelation-pair-valued mapping $\left(\mathcal{E}_{\mathrm{e}}(\cdot), \boldsymbol{s}_{\mathrm{e}}(\cdot)\right)$ defined in Lemma 6.
These results explain why the ex-equicorretion space and ex-equicorrelation pairs are of central interest by showing that: every $(\mathcal{E}, s) \in \mathcal{V}$ encodes an elemental behavior pattern $\mathcal{S}_{\text {POLY }}(\mathcal{E}, s, \cdot)$ of the extended solution set mapping $\mathcal{S}_{e}(\cdot)$.

We firstly present the definition (Def. 7) of cutting polytope. The justification for it will be given right after Def. 7.
Definition 7 (Cutting Polytope). Given $(\mathcal{E}, s) \in \mathcal{V}$, for every ${ }^{12} \lambda \in \mathbb{R}$, we define the cutting polytope of $(\mathcal{E}, s)$ at $\lambda$ (cf. Remark 6 (b) for the meaning of the name), denoted by $\mathcal{S}_{P O L Y}(\mathcal{E}, s, \lambda)$, as the set of all vectors $\boldsymbol{w} \in \mathbb{R}^{2 n}$ simultaneously satisfying the following system of linear equations:

$$
\begin{cases}(\forall i \in \mathcal{E}) & \xi_{i}(\boldsymbol{w}):=\boldsymbol{c}_{i}^{\top}(\boldsymbol{b}-\boldsymbol{D} \boldsymbol{C} \boldsymbol{w})=\lambda[\boldsymbol{s}]_{i}  \tag{EQ-a}\\ (\forall i \in \neg \mathcal{E}) & {[\boldsymbol{w}]_{i}=0,}\end{cases}
$$

as well as the following system of linear inequations:

$$
\begin{cases}(\forall i \in \mathcal{E}) & {[\boldsymbol{s}]_{i}[\boldsymbol{w}]_{i} \geq 0}  \tag{NQ-a}\\ (\forall i \in \neg \mathcal{E}) & \left|\xi_{i}(\boldsymbol{w})\right|:=\left|\boldsymbol{c}_{i}^{\top}(\boldsymbol{b}-\boldsymbol{D} \boldsymbol{C} \boldsymbol{w})\right| \leq \lambda\end{cases}
$$

In particular, for $\boldsymbol{w} \in \mathbb{R}^{2 n}$, if $\boldsymbol{w}$ satisfies $(E Q)$ (resp. (NQ)) individually, then we say $\boldsymbol{w}$ satisfies $(\mathcal{E}, \boldsymbol{s})-(E Q)(r e s p .(\mathcal{E}, \boldsymbol{s})-(N Q)$ ) at $\lambda$ for the sake of clarity.

One can verify that for every $(\mathcal{E}, \boldsymbol{s}) \in \mathcal{V}$ and $\lambda \in \mathbb{R}, \mathcal{S}_{\text {POLY }}(\mathcal{E}, \boldsymbol{s}, \lambda)$ is a (possibly empty) closed convex polytope.
The conditions (EQ) and (NQ) in Def. 7 can be interpreted as follows (see Fig. 1 for a visual illustration):

1) (EQ-a) and (NQ-a) are the conditions for all indices in $\mathcal{E}$, which is an imitation of the rule (a) of (OPT) (i.e., the rule for all nonzero-component indices, see Remark 5 in Sec. 5.1). Specifically,

- (EQ-a) imitates the correlation-threshold equality,
- (NQ-a) forces the correlation sign to align with the component sign.

2) (EQ-b) and (NQ-b) are the conditions for all indices in $\neg \mathcal{E}$, which is an imitation of the rule (b) of (OPT) (i.e., the rule for all zero-component indices). To be precise,

- (EQ-b) forces the component value to be zero,
- (NQ-b) ensures that the magnitude of the correlation coefficient will not exceed the threshold $\lambda$.

According to this interpretation, one can imagine that for every $(\mathcal{E}, \boldsymbol{s}) \in \mathcal{V}$ and $\lambda>0, \mathcal{S}_{\mathrm{POLY}}(\mathcal{E}, \boldsymbol{s}, \lambda)$ gives an approximation of $\mathcal{S}_{\mathrm{e}}(\lambda)$. In the following proposition, we demonstrate that the cutting polytope $\mathcal{S}_{\mathrm{POLY}}(\mathcal{E}, s, \lambda)$ is a fine approximation of $\mathcal{S}_{\mathrm{e}}(\lambda)$, and present several useful properties of it.
Proposition 7 (Properties of cutting polytopes). For every $\lambda>0$, the following holds:
(a) For every $(\mathcal{E}, s) \in \mathcal{V}, \mathcal{S}_{\text {POLY }}(\mathcal{E}, s, \lambda) \subset \mathcal{S}_{e}(\lambda)$.
(b) There exists $(\mathcal{E}, s) \in \mathcal{V}$ satisfying $(\mathcal{E}, s)=\left(\mathcal{E}_{e}(\lambda), s_{e}(\lambda)\right)$, such that

$$
\mathcal{S}_{P O L Y}(\mathcal{E}, s, \lambda)=\mathcal{S}_{e}(\lambda)
$$

where $\left(\mathcal{E}_{e}(\lambda), s_{e}(\lambda)\right)$ is the pair of common ex-equicorrelation set and common ex-equicorrelation signs vector for all extended solutions in $\mathcal{S}_{e}(\lambda)$ (cf. Lemma 6).
(c) For $(\mathcal{E}, \boldsymbol{s}) \in \mathcal{V}$ and $\boldsymbol{w} \in \mathcal{S}_{e}(\lambda), \boldsymbol{w} \in \mathcal{S}_{P O L Y}(\mathcal{E}, \boldsymbol{s}, \lambda)$ if and only if the following generalized inequality holds:

$$
\begin{equation*}
(\operatorname{supp}(\boldsymbol{w}), \operatorname{sign}(\boldsymbol{w})) \preceq(\mathcal{E}, \boldsymbol{s}) \preceq\left(\mathcal{E}_{e}(\lambda), \boldsymbol{s}_{e}(\lambda)\right), \tag{37}
\end{equation*}
$$

where the partial order $\preceq$ is defined in Prop. 6.
(d) $\operatorname{For}(\mathcal{E}, \boldsymbol{s}) \in \mathcal{V}$ and $\boldsymbol{w} \in \mathcal{S}_{\text {POLY }}(\mathcal{E}, \boldsymbol{s}, \lambda)$, if $\boldsymbol{w}$ satisfies $(\mathcal{E}, \boldsymbol{s})$-(NQ-a) strictly at $\lambda$, i.e.,

$$
(\forall i \in \mathcal{E})[\boldsymbol{s}]_{i}[\boldsymbol{w}]_{i}>0
$$

then the following equality holds:

$$
(\mathcal{E}, \boldsymbol{s})=(\operatorname{supp}(\boldsymbol{w}), \operatorname{sign}(\boldsymbol{w}))
$$

If $\boldsymbol{w}$ satisfies $(\mathcal{E}, \boldsymbol{s})-(N Q-b)$ strictly at $\lambda$, i.e.,

$$
(\forall i \in \neg \mathcal{E})\left|\xi_{i}(\boldsymbol{w})\right|:=\left|\boldsymbol{c}_{i}^{\top}(\boldsymbol{b}-\boldsymbol{D C w})\right|<\lambda,
$$

then the following equality holds:

$$
(\mathcal{E}, \boldsymbol{s})=\left(\mathcal{E}_{e}(\lambda), \boldsymbol{s}_{e}(\lambda)\right)
$$

Proof: See Appendix D.2.
12. Here we do not require $\lambda$ to be positive. This unconstrained setting for $\lambda$ simplifies subsequent derivation in Appendices E. 1 and E.1, where we will solve certain variants of (NQ-a) and (NQ-b) with respect to $\lambda$.

Remark 6 (Interpretation of Proposition 7). (a) From Prop. 7 (a), for every $\lambda>0, \mathcal{S}_{\text {POLY }}(\cdot, \cdot, \lambda$ ) is an oracle which takes an ex-equicorrelation pair as input and outputs an inner polytope approximation of $\mathcal{S}_{e}(\lambda)$.
(b) If the input ex-equicorrelation pair $(\mathcal{E}, \boldsymbol{s})$ happens to coincide with $\left(\mathcal{E}_{e}(\lambda), \boldsymbol{s}_{e}(\lambda)\right)$, then $\mathcal{S}_{\text {POLY }}(\mathcal{E}, \boldsymbol{s}, \lambda)$ is exactly $\mathcal{S}_{e}(\lambda)$ (Prop. 7 (b)); otherwise $\mathcal{S}_{\text {POLY }}(\mathcal{E}, s, \lambda)$ cuts a partial polytope in $\mathcal{S}_{e}(\lambda)$, which accounts for its name.
(c) Prop. 7 (c) indicates that for every $(\mathcal{E}, s) \in \mathcal{V}$, if its cutting polytope contains an extended solution $\boldsymbol{w} \in \mathcal{S}_{e}(\lambda)$, then $(\mathcal{E}, \boldsymbol{s})$ must satisfy the generalized inequality (37).
(d) Prop. 7 (d) further draws a connection between the equality condition of (37) regarding $(\mathcal{E}, \boldsymbol{s})$ and the strict inequality condition of $(N Q)$ regarding $\boldsymbol{w} \in \mathcal{S}_{P O L Y}(\mathcal{E}, \boldsymbol{s}, \lambda)$.
Especially, we remark that Prop. 7 (c) and (d) will play critical roles in the derivation of the piecewise expressions of $\mathcal{S}_{e}(\lambda)$ and $\boldsymbol{w}_{\star}(\lambda)$ (Prop. 11) and the proof of correctness of the LARS-sGMC algorithm (Thm. 8).

Recall that $\mathcal{V}$ is a finite set, hence there are only finite cutting polytopes, combining which with Prop. 7 (b), one can imagine that the shape of $\mathcal{S}_{\mathrm{e}}(\lambda)$ can be classified into finite patterns regardless of the value of $\lambda$, and can verify the equivalence between $\mathcal{S}_{\mathrm{e}}(\cdot)$ and $\left(\mathcal{E}_{\mathrm{e}}(\cdot), \boldsymbol{s}_{\mathrm{e}}(\cdot)\right)$.
Corollary 5 (Cutting polytope collection). As $\lambda$ varies in $\mathbb{R}_{++}$, the value of $\mathcal{S}_{e}(\lambda)$ is always switching within the finite collection of cutting polytopes $\left\{\mathcal{S}_{\text {POLY }}(\mathcal{E}, s, \lambda)\right\}_{(\mathcal{E}, s) \in \mathcal{V}}$, i.e.,

$$
(\forall \lambda>0) \quad \mathcal{S}_{e}(\lambda) \in\left\{\mathcal{S}_{P O L Y}(\mathcal{E}, s, \lambda)\right\}_{(\mathcal{E}, s) \in \mathcal{V}}
$$

Moreover, for $\left(\mathcal{E}_{e}(\lambda), s_{e}(\lambda)\right) \in \mathcal{V}$ defined in Lemma 6, the following holds:

$$
(\forall \lambda>0) \quad \mathcal{S}_{e}(\lambda)=\mathcal{S}_{P O L Y}\left(\mathcal{E}_{e}(\lambda), s_{e}(\lambda), \lambda\right),
$$

which means that the ex-equicorrelation-pair-valued mapping $\left(\mathcal{E}_{e}(\cdot), s_{e}(\cdot)\right)$ and the set-valued mapping $\mathcal{S}_{e}(\cdot)$ reciprocally contains the whole information about each other.

Proof: The result follows directly from Prop. 7 (b).

### 5.4 Characterizing the Minimum $\ell_{2}$-Norm Extended sGMC Solution Using Ex-Equicorrelation Pairs

In this section, we further present a characterization of the minimum $\ell_{2}$-norm extended solution $\boldsymbol{w}_{\star}(\lambda)$ using ex-equicorrelation pairs.

It will be shown that: for every $(\mathcal{E}, s) \in \mathcal{V}$, we can define an affine function $\hat{\boldsymbol{w}}_{\mathrm{EQ}}(\mathcal{E}, \boldsymbol{s}, \lambda)$ of $\lambda$ termed trajectory function (Prop. 8) to approximate $\boldsymbol{w}_{\star}(\lambda)$, such that as $\lambda$ varies in $\mathbb{R}_{++}$,

1) the value of $\boldsymbol{w}_{\star}(\lambda)$ is always switching within the finite collection of trajectory functions $\left\{\hat{\boldsymbol{w}}_{\mathrm{EQ}}(\mathcal{E}, \boldsymbol{s}, \lambda)\right\}_{(\mathcal{E}, s) \in \mathcal{V}}$,
2) the vector-valued mapping $\boldsymbol{w}_{\star}(\cdot)$ can be equivalently represented by the ex-equicorrelation-pair-valued mapping $\left(\operatorname{supp}\left(\boldsymbol{w}_{\star}(\cdot), \operatorname{sign}\left(\boldsymbol{w}_{\star}(\cdot)\right)\right)\right)$ introduced in Cor. 4.
These results demonstrate the importance of ex-equicorrelation space and ex-equicorrelation pairs by showing that: every $(\mathcal{E}, \boldsymbol{s}) \in \mathcal{V}$ encodes an elemental behavior pattern $\hat{\boldsymbol{w}}_{E Q}(\mathcal{E}, \boldsymbol{s}, \cdot)$ of the minimum $\ell_{2}$-norm extended regularization path $\boldsymbol{w}_{\star}(\cdot)$.

Moreover, for every $(\mathcal{E}, s) \in \mathcal{V}$, since $\hat{\boldsymbol{w}}_{\mathrm{EQ}}(\mathcal{E}, s, \cdot)$ is affine, we can interpret the value of $\hat{\boldsymbol{w}}_{\mathrm{EQ}}(\mathcal{E}, s, \lambda)$ as the position of a uniformly moving train at time $\lambda$, whereby we can deduce the piecewise linear motion of $\boldsymbol{w}_{\star}(\lambda)$ with $\lambda$ (cf. Remark 7).

We first introduce the notion of trajectory function.
Cor. 5 implies that for every $\lambda>0$, the extended solution set $\mathcal{S}_{\mathrm{e}}(\lambda)$ must coincide with a cutting polytope $\mathcal{S}_{\text {POLY }}(\mathcal{E}, s, \lambda)$ for $(\mathcal{E}, s):=\left(\mathcal{E}_{\mathrm{e}}(\lambda), s_{\mathrm{e}}(\lambda)\right) \in \mathcal{V}$. Hence as long as we have access to the value of this $(\mathcal{E}, s) \in \mathcal{V}$, we can obtain the minimum $\ell_{2}$-norm extended solution $\boldsymbol{w}_{\star}(\lambda)$ by solving a convex program:

$$
\begin{array}{ll}
\underset{\boldsymbol{w} \in \mathbb{R}^{2 n}}{\operatorname{minimize}} & \frac{1}{2}\|\boldsymbol{w}\|_{2}^{2} \\
\text { subject to } & \boldsymbol{w} \in \mathcal{S}_{\mathrm{POLY}}(\mathcal{E}, \boldsymbol{s}, \lambda)
\end{array}
$$

From Def. 7, for arbitrary $(\mathcal{E}, s) \in \mathcal{V}$, the convex optimization problem above is a quadratic program with linear inequality constraints (i.e., (NQ) in Def. 7), which is much simpler than the original sGMC model (6) but still cannot be solved in closed-form. To yield closed-form approximations of $\boldsymbol{w}_{\star}(\lambda)$, here we analyze individually the tractable part in Def. 7, i.e., the linear equality constraints (EQ). We regard the least-squares solution of (EQ) as an easily-obtainable approximation of $\boldsymbol{w}_{\star}(\lambda)$, and define its closed-form expression (which is a function of $\lambda$ ) as the trajectory function of $(\mathcal{E}, s)$.
Proposition 8 (Trajectory function). Given $(\mathcal{E}, s) \in \mathcal{V}$, for every $\lambda \in \mathbb{R}$, the solution set of $(\mathcal{E}, s)-(E Q)$ at $\lambda$ is always nonempty (see Def. 7 for the expression of (EQ)).

Moreover, define ${ }^{13}$ the mapping $\hat{\boldsymbol{w}}_{E Q}: \mathcal{V} \times \mathbb{R} \rightarrow \mathbb{R}^{2 n}$ as

$$
\hat{\boldsymbol{w}}_{E Q}:(\mathcal{E}, \boldsymbol{s}, \lambda) \mapsto \text { the unique least squares solution of }(\mathcal{E}, \boldsymbol{s})-(E Q) \text { at } \lambda,
$$

13. In align with Def. 7, here we do not require $\lambda$ to be positive.


Fig. 2: An illustration for the piecewise linearity of $\boldsymbol{w}_{\star}(\lambda)$.
then $\hat{\boldsymbol{w}}_{E Q}(\mathcal{E}, \boldsymbol{s}, \lambda)$ can be expressed in closed-form as

$$
\left\{\begin{array}{l}
{\left[\hat{\boldsymbol{w}}_{E Q}(\mathcal{E}, \boldsymbol{s}, \lambda)\right]_{\mathcal{E}}=\left(\boldsymbol{C}_{\mathcal{E}}^{\top} \boldsymbol{D} \boldsymbol{C}_{\mathcal{E}}\right)^{\dagger}\left(\boldsymbol{C}_{\mathcal{E}}^{\top} \boldsymbol{b}-\lambda[\boldsymbol{s}]_{\mathcal{E}}\right),} \\
{\left[\hat{\boldsymbol{w}}_{E Q}(\mathcal{E}, \boldsymbol{s}, \lambda)\right]_{\neg \mathcal{E}}=\mathbf{0}}
\end{array}\right.
$$

Notice that given $(\mathcal{E}, s) \in \mathcal{V}, \hat{\boldsymbol{w}}_{E Q}(\mathcal{E}, s, \lambda)$ is affine in $\lambda$.
Hereafter, we term the affine function $\hat{\boldsymbol{w}}_{E Q}(\mathcal{E}, s, \cdot)$ the trajectory function of $(\mathcal{E}, s)$ (the meaning of this name will be explained later in the paragraph right after Prop. 9).

Proof: See Appendix D.3.
From the definition of the trajectory function, for every $(\mathcal{E}, s) \in \mathcal{V}, \hat{\boldsymbol{w}}_{\mathrm{EQ}}(\mathcal{E}, \boldsymbol{s}, \lambda)$ seems to be a very rough approximation of $\boldsymbol{w}_{\star}(\lambda)$. Nevertheless, we can prove that as long as we have access to a proper $(\mathcal{E}, \boldsymbol{s}) \in \mathcal{V}, \hat{\boldsymbol{w}}_{\mathrm{EQ}}(\mathcal{E}, \boldsymbol{s}, \lambda)$ produces exactly the value of $\boldsymbol{w}_{\star}(\lambda)$. See the following.
Proposition 9 (A key property of trajectory functions). For every $\lambda>0$, there exists $(\mathcal{E}, \boldsymbol{s}) \in \mathcal{V}$ satisfying

$$
(\mathcal{E}, \boldsymbol{s})=\left(\operatorname{supp}\left(\boldsymbol{w}_{\star}(\lambda)\right), \operatorname{sign}\left(\boldsymbol{w}_{\star}(\lambda)\right)\right)
$$

such that $\hat{\boldsymbol{w}}_{E Q}(\mathcal{E}, \boldsymbol{s}, \lambda)=\boldsymbol{w}_{\star}(\lambda)$.
Proof: See Appendix D.4.
The result of Prop. 9 may not seem that useful at first glance: to obtain the value of $\boldsymbol{w}_{\star}(\lambda)$ by the trajectory function $\hat{\boldsymbol{w}}_{\mathrm{EQ}}(\mathcal{E}, \boldsymbol{s}, \lambda)$, Prop. 9 requires to know the support and signs vector of $\boldsymbol{w}_{\star}(\lambda)$, which is almost as difficult as solving $\boldsymbol{w}_{\star}(\lambda)$ directly. However, if one reconsiders the claim of Prop. 9 in the world of the ex-equicorrelation space $\mathcal{V}$, one may realize its real power: since $\mathcal{V}$ is a finite set, there are only finite trajectory functions $\hat{\boldsymbol{w}}_{\mathrm{EQ}}(\mathcal{E}, \boldsymbol{s}, \cdot)$, hence Prop. 9 implies that the regularization path $\boldsymbol{w}_{\star}(\cdot)$ can only take its value from a finite collection of affine functions $\hat{\boldsymbol{w}}_{\mathrm{EQ}}(\mathcal{E}, s, \cdot)$. Accordingly, each trajectory function $\hat{\boldsymbol{w}}_{\mathrm{EQ}}(\mathcal{E}, \boldsymbol{s}, \cdot)$ represents a pattern of trajectory of $\boldsymbol{w}_{\star}(\cdot)$, which accounts for its name.

Corollary 6 (Trajectory function collection). As $\lambda$ varies in $\mathbb{R}_{++}$, the path of $\boldsymbol{w}_{\star}(\lambda)$ is always switching within the finite collection of trajectory functions $\left\{\hat{\boldsymbol{w}}_{E Q}(\mathcal{E}, s, \lambda)\right\}_{(\mathcal{E}, \boldsymbol{s}) \in \mathcal{V}^{\prime}}$, i.e.,

$$
(\forall \lambda>0) \boldsymbol{w}_{\star}(\lambda) \in\left\{\hat{\boldsymbol{w}}_{E Q}(\mathcal{E}, \boldsymbol{s}, \lambda)\right\}_{(\mathcal{E}, \boldsymbol{s}) \in \mathcal{V}} .
$$

Moreover, for $\left(\operatorname{supp}\left(\boldsymbol{w}_{\star}(\lambda)\right), \operatorname{sign}\left(\boldsymbol{w}_{\star}(\lambda)\right)\right) \in \mathcal{V}(c f$. Cor. 4), the following holds:

$$
(\forall \lambda>0) \quad \boldsymbol{w}_{\star}(\lambda)=\hat{\boldsymbol{w}}_{E Q}\left(\operatorname{supp}\left(\boldsymbol{w}_{\star}(\lambda)\right), \operatorname{sign}\left(\boldsymbol{w}_{\star}(\lambda)\right), \lambda\right),
$$

which means that the ex-equicorrelation-pair-valued mapping $\left(\operatorname{supp}\left(\boldsymbol{w}_{\star}(\cdot)\right), \operatorname{sign}\left(\boldsymbol{w}_{\star}(\cdot)\right)\right)$ and the vector-valued mapping $\boldsymbol{w}_{\star}(\cdot)$ mutually contain each other's whole information.

Proof: The result follows directly from Prop. 9.
Moreover, combining Cor. 5 with the affine nature of $\hat{\boldsymbol{w}}_{\mathrm{EQ}}(\mathcal{E}, s, \cdot)$, one may realize that we can already yield the piecewise linearity of $\boldsymbol{w}_{\star}(\lambda)$. See the following intuition.
Remark 7 (Intuition behind piecewise linearity of $\boldsymbol{w}_{\star}(\lambda)$ ). Let us interpret $\lambda>0$ as a time index, then (see Fig. 2 for a visual illustration):

- since $\boldsymbol{w}_{\star}(\cdot)$ is continuous (Cor. 3), $\boldsymbol{w}_{\star}(\lambda)$ can be interpreted as the position of a person in $\mathbb{R}^{2 n}$ at time $\lambda$,
- since $\hat{\boldsymbol{w}}_{E Q}(\mathcal{E}, \boldsymbol{s}, \cdot)$ is affine for every $(\mathcal{E}, \boldsymbol{s}) \in \mathcal{V}$ (Prop. 8), $\hat{\boldsymbol{w}}_{E Q}(\mathcal{E}, \boldsymbol{s}, \lambda)$ can be interpreted as the position of a train in $\mathbb{R}^{2 n}$ at time $\lambda$, where $(\mathcal{E}, s)$ is the train number of this uniformly moving train on a straight track.

From Cor. 6 , for every time $\lambda>0$, the "passenger" $\boldsymbol{w}_{\star}(\lambda)$ is always aboard on a "train" $\hat{\boldsymbol{w}}_{E Q}(\mathcal{E}, \boldsymbol{s}, \lambda)$ for some $(\mathcal{E}, \boldsymbol{s}) \in \mathcal{V}$, hence one can deduce that $\boldsymbol{w}_{\star}(\cdot)$ must be piecewise linear.

However, it should be noted that the "passenger-train" intuition introduced above only implies the piecewise linear trajectory of the "passenger" $\boldsymbol{w}_{\star}(\cdot)$, but provides no information about its "boarding schedule", i.e., we do not know which "trains" (trajectory functions) $\boldsymbol{w}_{\star}(\lambda)$ will select at time $\lambda$. This motivates our study on the "trajectory selection rule" of $\boldsymbol{w}_{\star}(\lambda)$ in the subsequent section, which further leads to the piecewise expressions of $\boldsymbol{w}_{\star}(\lambda)$ and $\mathcal{S}_{\text {e }}(\lambda)$.

## 6 Piecewise Expressions of The Minimum $\ell_{2}$-Norm Extended Regularization Path and The Extended Solution Set Mapping

In this section, we delve deeper in the interplay between the minimum $\ell_{2}$-norm extended regularization path $\boldsymbol{w}_{\star}(\lambda)$ and the collection of trajectory functions $\left\{\hat{\boldsymbol{w}}_{\mathrm{EQ}}(\mathcal{E}, s, \lambda)\right\}_{(\mathcal{E}, s) \in \mathcal{V}}$ introduced in Sec. 5.4 , whereby we elucidate the "trajectory selection rule" of $\boldsymbol{w}_{\star}(\lambda)$ (recall Remark 7) and derive an explicit piecewise linear expression of it. As a by-product, we also obtain the expression of the extended solution set $\mathcal{S}_{\mathrm{e}}(\lambda)$ as a piecewise polytope-valued mapping of $\lambda$.

The main results of this section are summarized as follows:

1) Sec. 6.1 elucidates the trajectory selection rule of $\boldsymbol{w}_{\star}(\lambda)$ (cf. Remark 9). Specifically, it will be shown that for every $(\mathcal{E}, \boldsymbol{s}) \in \mathcal{V}, \boldsymbol{w}_{\star}(\lambda)=\hat{\boldsymbol{w}}_{\mathrm{EQ}}(\mathcal{E}, \boldsymbol{s}, \lambda)$ almost if and only if the following selection condition holds:

$$
\begin{equation*}
\hat{\boldsymbol{w}}_{\mathrm{EQ}}(\mathcal{E}, \boldsymbol{s}, \lambda) \text { satisfies }(\mathcal{E}, \boldsymbol{s})-(\mathrm{NQ}) \text { at } \lambda . \tag{SEL}
\end{equation*}
$$

Given $(\mathcal{E}, s) \in \mathcal{V}$, the set of $\lambda>0$ satisfying (SEL) is an interval $I_{\text {SEL }}(\mathcal{E}, s)$ termed selection interval with a closed-form expression. All selection intervals with nonempty interior define a partition of $\mathbb{R}_{++}$, whereby we can elucidate the whole "boarding schedule" of $\boldsymbol{w}_{\star}(\lambda)$ for all $\lambda>0$, answering the question raised in the last paragraph of Sec. 5 .
2) Sec. 6.2 presents the expressions of $\boldsymbol{w}_{\star}(\lambda)$ and $\mathcal{S}_{\mathrm{e}}(\lambda)$ (cf. Thm. 6 and 7 ). It will be shown that $\boldsymbol{w}_{\star}(\lambda)$ and $\mathcal{S}_{\mathrm{e}}(\lambda)$ are both piecewise simple functions with the same piece segmentation of $\lambda$, where every segment of $\lambda$ coincides with a selection interval $I_{\text {SEL }}(\mathcal{E}, s)$ with nonempty interior, and within the segment $\lambda \in \operatorname{int}\left(I_{\text {SEL }}(\mathcal{E}, s)\right)$ :

- the expression of $\boldsymbol{w}_{\star}(\lambda)$ is affine in $\lambda$ with

$$
\boldsymbol{w}_{\star}(\lambda)=\hat{\boldsymbol{w}}_{\mathrm{EQ}}(\mathcal{E}, \boldsymbol{s}, \lambda)
$$

whilst the support and signs vector of $\boldsymbol{w}_{\star}(\lambda)$ are constant with $\left(\operatorname{supp}\left(\boldsymbol{w}_{\star}(\lambda)\right), \operatorname{sign}\left(\boldsymbol{w}_{\star}(\lambda)\right)\right)=(\mathcal{E}, \boldsymbol{s})$,

- the expression of $\mathcal{S}_{\mathrm{e}}(\lambda)$ is a polytope with

$$
\mathcal{S}_{\mathrm{e}}(\lambda)=\mathcal{S}_{\mathrm{POLY}}(\mathcal{E}, s, \lambda)
$$

whilst the common ex-equicorrelation set and ex-equicorrelation signs vector for all extended solutions in $\mathcal{S}_{\mathrm{e}}(\lambda)$ are constant with $\left(\mathcal{E}_{\mathrm{e}}(\lambda), \boldsymbol{s}_{\mathrm{e}}(\lambda)\right)=(\mathcal{E}, \boldsymbol{s})$.
We should note that the strategy adopted in the present paper for proving continuity and piecewise linearity of a regularization path is very different from that in previous works for the LASSO model [15], [14] or its variants [30], [31]. Indeed, our strategy provides a more rigorous analysis than previous ones and leads to several valuable new findings. See Sec. 6.3 for a thorough comparison with related works.

### 6.1 Trajectory Selection Rule of $\boldsymbol{w}_{\star}(\lambda)$

In this section, we study the "trajectory selection rule" of $\boldsymbol{w}_{\star}(\lambda)$. By the intuition of passenger and trains (Remark 7), this problem can be studied in two perspectives:

1) Passenger-centered view: for time $\lambda>0$, which "train" $\hat{\boldsymbol{w}}_{\mathrm{EQ}}(\mathcal{E}, \boldsymbol{s}, \lambda)$ will the "passenger" $\boldsymbol{w}_{\star}(\lambda)$ select at $\lambda$ ?
2) Train-centered view: for train number $(\mathcal{E}, s) \in \mathcal{V}$, what is the period of $\lambda$ that $\hat{\boldsymbol{w}}_{\mathrm{EQ}}(\mathcal{E}, \boldsymbol{s}, \lambda)$ is selected by $\boldsymbol{w}_{\star}(\lambda)$ ?

Both views are intuitive. However, here we would like to focus on the latter one, which turns out to have a very elegant answer.

We first derive the condition for $\hat{\boldsymbol{w}}_{\mathrm{EQ}}(\mathcal{E}, \boldsymbol{s}, \lambda)$ to be selected by $\boldsymbol{w}_{\star}(\lambda)$ at $\lambda$. Recall that $\hat{\boldsymbol{w}}_{\mathrm{EQ}}(\mathcal{E}, \boldsymbol{s}, \lambda)$ is the least-squares solution of $(\mathcal{E}, s)$-(EQ) at $\lambda$ (Prop. 8), i.e., the tractable part in the definition of $\mathcal{S}_{\text {POLY }}(\mathcal{E}, s, \lambda)$ (Def. 7), hence it is natural to consider the condition of whether $\hat{\boldsymbol{w}}_{\mathrm{EQ}}(\mathcal{E}, s, \lambda)$ satisfies the ignored constraint (NQ), which directly leads to the definition of the following selection condition.

Definition 8 (Selection condition). Given $(\mathcal{E}, \boldsymbol{s}) \in \mathcal{V}$, we define the selection condition of $(\mathcal{E}, \boldsymbol{s})$ at $\lambda>0$ as:

$$
\begin{equation*}
\hat{\boldsymbol{w}}_{E Q}(\mathcal{E}, \boldsymbol{s}, \lambda) \text { satisfies }(\mathcal{E}, \boldsymbol{s})-(N Q) \text { at } \lambda \tag{SEL}
\end{equation*}
$$

or equivalently, the following holds:

$$
\left\{\begin{array}{l}
(\forall i \in \mathcal{E})  \tag{SEL-a}\\
(\forall i \in \neg \mathcal{E})
\end{array}\right.
$$

$$
\begin{align*}
& {[\boldsymbol{s}]_{i}\left[\hat{\boldsymbol{w}}_{E Q}(\mathcal{E}, \boldsymbol{s}, \lambda)\right]_{i} \geq 0} \\
& \quad-\lambda \leq \xi_{i}\left(\hat{\boldsymbol{w}}_{E Q}(\mathcal{E}, \boldsymbol{s}, \lambda)\right) \leq \lambda, \tag{SEL-b}
\end{align*}
$$

where $\hat{\boldsymbol{w}}_{E Q}(\mathcal{E}, \boldsymbol{s}, \cdot)$ is the trajectory function in Prop. 8.
Notice that given $(\mathcal{E}, s), \hat{\boldsymbol{w}}_{\mathrm{EQ}}(\mathcal{E}, s, \cdot)$ is an affine function. Hence if we fix $(\mathcal{E}, s) \in \mathcal{V}$, then every inequation in (SEL) is nothing but a one-dimensional linear inequation with respect to $\lambda$, which means all of them can be solved in closed-form with respect to $\lambda$. Taking the intersection of their solution sets yields the set of $\lambda>0$ satisfying (SEL), which leads to the definition of the selection interval $I_{\text {SEL }}(\mathcal{E}, s)$.

Proposition 10 (Selection interval). Given $(\mathcal{E}, s) \in \mathcal{V}$, then the following holds:
(a) For every $i \in \mathcal{E}$, the set of $\lambda \in \mathbb{R}$ satisfying (SEL-a) is a (possibly empty) closed interval. Denote it by $I_{i}^{a}(\mathcal{E}, s)$.
(b) For every $i \in \neg \mathcal{E}$, the set of $\lambda \in \mathbb{R}$ satisfying (SEL-b) is a (possibly empty) closed interval. Denote it by $I_{i}^{b}(\mathcal{E}, s)$.

Moreover, define the selection interval $I_{S E L}(\mathcal{E}, s)$ as the set of $\lambda>0$ satisfying (SEL), then $I_{S E L}(\mathcal{E}, s)$ is a (possibly empty) interval in $\mathbb{R}_{++}$that can be expressed as:

$$
I_{S E L}(\mathcal{E}, s)=\left[l_{S E L}(\mathcal{E}, s), r_{S E L}(\mathcal{E}, s)\right] \backslash\{0,+\infty\}
$$

where the endpoints $l_{\text {SEL }}(\mathcal{E}, \boldsymbol{s})$ and $r_{\text {SEL }}(\mathcal{E}, \boldsymbol{s})$ are:

$$
\begin{aligned}
& l_{S E L}(\mathcal{E}, \boldsymbol{s})=\max \left\{0, \max _{i \in \mathcal{E}} l_{i}^{a}(\mathcal{E}, s), \max _{i \in \neg \mathcal{E}} l_{i}^{b}(\mathcal{E}, \boldsymbol{s})\right\}, \\
& r_{S E L}(\mathcal{E}, \boldsymbol{s})=\min \left\{\min _{i \in \mathcal{E}} r_{i}^{a}(\mathcal{E}, \boldsymbol{s}), \min _{i \in \neg \mathcal{E}} r_{i}^{b}(\mathcal{E}, \boldsymbol{s})\right\}
\end{aligned}
$$

with $l_{i}^{a}(\mathcal{E}, s), r_{i}^{a}(\mathcal{E}, s), l_{i}^{b}(\mathcal{E}, s)$ and $\left.r_{i}^{b}(\mathcal{E}, \boldsymbol{s})\right)$ defined as:

$$
\begin{aligned}
l_{i}^{a}(\mathcal{E}, s):=\inf I_{i}^{a}(\mathcal{E}, s), & r_{i}^{a}(\mathcal{E}, s):=\sup I_{i}^{a}(\mathcal{E}, s) \\
l_{i}^{b}(\mathcal{E}, s):=\inf I_{i}^{b}(\mathcal{E}, s), & r_{i}^{b}(\mathcal{E}, s):=\sup I_{i}^{b}(\mathcal{E}, s)
\end{aligned}
$$

Notice that $l_{i}^{a}(\mathcal{E}, s), r_{i}^{a}(\mathcal{E}, s), l_{i}^{b}(\mathcal{E}, s), r_{i}^{b}(\mathcal{E}, s) \in \mathbb{R} \cup\{ \pm \infty\}$ can all be computed in closed-form (cf. Lemma 9 in Appendix E. 1 and Lemma 10 in Appendix E.1).

Proof: See Appendix E.1.
In the sequel, we study the "trajectory selection rule" of $\boldsymbol{w}_{\star}(\lambda)$ exploiting the notion of the selection interval. Our main result is composed of two parts. The first part (Prop. 11) elucidates the behavior of $(\mathcal{E}, \boldsymbol{s})$ (i.e., its associated trajectory function $\hat{\boldsymbol{w}}_{\mathrm{EQ}}(\mathcal{E}, \boldsymbol{s}, \lambda)$ and cutting polytope $\left.\mathcal{S}_{\mathrm{POLY}}(\mathcal{E}, \boldsymbol{s}, \lambda)\right)$ inside and outside $I_{\mathrm{SEL}}(\mathcal{E}, \boldsymbol{s})$.

Proposition 11 (Behavior of an ex-equicorrelation pair inside and outside the selection interval). Given $(\mathcal{E}, s) \in \mathcal{V}$, suppose that $\lambda>0$, then the following holds:
(a) For every $\lambda \in \operatorname{int}\left(I_{\text {SEL }}(\mathcal{E}, s)\right)$, we have

$$
\begin{aligned}
\boldsymbol{w}_{\star}(\lambda) & =\hat{\boldsymbol{w}}_{E Q}(\mathcal{E}, \boldsymbol{s}, \lambda) \\
\mathcal{S}_{e}(\lambda) & =\mathcal{S}_{P O L Y}(\mathcal{E}, \boldsymbol{s}, \lambda)
\end{aligned}
$$

and $(\mathcal{E}, \boldsymbol{s})$ satisfies the following equality

$$
(\mathcal{E}, \boldsymbol{s})=\left(\operatorname{supp}\left(\boldsymbol{w}_{\star}(\lambda)\right), \operatorname{sign}\left(\boldsymbol{w}_{\star}(\lambda)\right)\right)=\left(\mathcal{E}_{e}(\lambda), \boldsymbol{s}_{e}(\lambda)\right)
$$

where $\left(\mathcal{E}_{e}(\lambda), s_{e}(\lambda)\right)$ is the pair of common ex-equicorrelation set and ex-equicorrelation signs vector for all extended solutions in $\mathcal{S}_{e}(\lambda)$ (Lemma 6).
(b) For every $\lambda \notin I_{\text {SEL }}(\mathcal{E}, s)$, we have

$$
\boldsymbol{w}_{\star}(\lambda) \neq \hat{\boldsymbol{w}}_{E Q}(\mathcal{E}, \boldsymbol{s}, \lambda) .
$$

Proof: See Appendix E.2.
Remark 8 (Interpretation of Proposition 11). For every $(\mathcal{E}, s) \in \mathcal{V}$, its selection interval $I_{\text {SEL }}(\mathcal{E}, s)$ represents the range of $\lambda$ where $(\mathcal{E}, s)$ holds value to the sGMC model:
(a) $(\mathcal{E}, s)$ is well-behaved inside its selection interval, in the sense that for every $\lambda \in \operatorname{int}\left(I_{\text {SEL }}(\mathcal{E}, s)\right)$,

- the trajectory function $\hat{\boldsymbol{w}}_{E Q}(\mathcal{E}, \boldsymbol{s}, \lambda)$ is selected by the minimum $\ell_{2}$-norm extended solution $\boldsymbol{w}_{\star}(\lambda)$ (Cor. 6),
- the cutting polytope $\mathcal{S}_{\text {POLY }}(\mathcal{E}, s, \lambda)$ coincides with the extended solution set $\mathcal{S}_{e}(\lambda)$ (Cor. 5).

Moreover, $(\mathcal{E}, s)$ itself coincides with the values of the two important ex-equicorrelation-pair-valued mappings introduced in Sec. 5.2 at $\lambda$.
(b) When $\lambda \notin I_{S E L}(\mathcal{E}, \boldsymbol{s}),(\mathcal{E}, \boldsymbol{s})$ is out of interest since $\hat{\boldsymbol{w}}_{E Q}(\mathcal{E}, \boldsymbol{s}, \lambda)$ will not be selected by $\boldsymbol{w}_{\star}(\lambda)$.

Combining these results, one can deduce that the selection interval $I_{\text {SEL }}(\mathcal{E}, \boldsymbol{s})$ is almost equal to the set of $\lambda>0$ satisfying $\boldsymbol{w}_{\star}(\lambda)=\hat{\boldsymbol{w}}_{E Q}(\mathcal{E}, \boldsymbol{s}, \lambda)$ (exceptions may only hold at the endpoints of $I_{S E L}(\mathcal{E}, \boldsymbol{s})$ ), hence $I_{S E L}(\mathcal{E}, \boldsymbol{s})$ can be interpreted as the "boarding period" of $\boldsymbol{w}_{\star}(\lambda)$ on $\hat{\boldsymbol{w}}_{E Q}(\mathcal{E}, \boldsymbol{s}, \lambda)$.

Notice that for $(\mathcal{E}, s) \in \mathcal{V}$, if $\operatorname{int}\left(I_{\text {SEL }}(\mathcal{E}, s)\right)=\emptyset$, then we either have $I_{\mathrm{SEL}}(\mathcal{E}, s)=\emptyset$ or $I_{\mathrm{SEL}}(\mathcal{E}, s)=\{\bar{\lambda}\}$ for some $\bar{\lambda}>0$. In this case, Prop. 11 implies that $\hat{\boldsymbol{w}}_{\mathrm{EQ}}(\mathcal{E}, s, \lambda)$ may be selected by $\boldsymbol{w}_{\star}(\lambda)$ for at most one "time instant" of $\lambda \in \mathbb{R}_{++}$, which
is negligible to the whole regularization path $\boldsymbol{w}_{\star}(\cdot)$. Hence only selection intervals with nonempty interior are significant in the study of $\boldsymbol{w}_{\star}(\cdot)$.

In the following, we present the second part of the "trajectory selection rule" of $\boldsymbol{w}_{\star}(\lambda)$, showing that selection intervals with nonempty interior are well distributed throughout $\mathbb{R}_{++}$, more precisely, they define a partition of $\mathbb{R}_{++}$.

Proposition 12 (Distribution of selection intervals with nonempty interior). Define $\mathcal{V}_{\star}$ as the set of ex-equicorrelation pairs whose selection interval has nonempty interior, i.e.,

$$
\mathcal{V}_{\star}:=\left\{(\mathcal{E}, s) \in \mathcal{V} \mid \operatorname{int}\left(I_{S E L}(\mathcal{E}, s)\right) \neq \emptyset\right\}
$$

Then the following holds:
(a) For $\left(\mathcal{E}_{1}, \boldsymbol{s}_{1}\right),\left(\mathcal{E}_{2}, \boldsymbol{s}_{2}\right) \in \mathcal{V}_{\star}$, if $\left(\mathcal{E}_{1}, \boldsymbol{s}_{1}\right) \neq\left(\mathcal{E}_{2}, \boldsymbol{s}_{2}\right)$, then

$$
\operatorname{int}\left(I_{S E L}\left(\mathcal{E}_{1}, s_{1}\right)\right) \cap \operatorname{int}\left(I_{S E L}\left(\mathcal{E}_{2}, s_{2}\right)\right)=\emptyset
$$

(b) $\left\{I_{\text {SEL }}(\mathcal{E}, \boldsymbol{s}) \mid(\mathcal{E}, \boldsymbol{s}) \in \mathcal{V}_{\star}\right\}$ is a cover of $\mathbb{R}_{++}$.

In summary, all selection intervals with nonempty interior, i.e., $\left\{I_{S E L}(\mathcal{E}, s)\right\}_{(\mathcal{E}, s) \in \mathcal{V}_{\star}}$, define a partition of $\mathbb{R}_{++}$.
Proof: See Appendix E.3.
Combining Prop. 11 and Prop. 12, we can summarize the "trajectory selection rule" of $\boldsymbol{w}_{\star}(\lambda)$ as follows.
Remark 9 (Trajectory selection rule of $\boldsymbol{w}_{\star}(\lambda)$ ). (a) For $(\mathcal{E}, \boldsymbol{s}) \in \mathcal{V}$ and $\lambda>0, \boldsymbol{w}_{\star}(\lambda)=\hat{\boldsymbol{w}}_{E Q}(\mathcal{E}, \boldsymbol{s}, \lambda)$ is almost equivalent to the selection condition (SEL):

$$
\begin{equation*}
\hat{\boldsymbol{w}}_{E Q}(\mathcal{E}, \boldsymbol{s}, \lambda) \text { satisfies }(\mathcal{E}, \boldsymbol{s})-(N Q) \text { at } \lambda, \tag{SEL}
\end{equation*}
$$

where exceptions may only happen when $\lambda$ is an endpoint of the selection interval $I_{S E L}(\mathcal{E}, \boldsymbol{s})$, i.e., the set of $\lambda>0$ satisfying (SEL).
(b) For all $(\mathcal{E}, \boldsymbol{s}) \in \mathcal{V}$, their selection intervals satisfy:

- interiors of selection intervals of every two different ex-equicorrelation pairs do not intersect,
- the collection of all selection intervals with nonempty interior covers every time index $\lambda \in \mathbb{R}_{++}$.

From Remark 9, one can imagine that the whole "boarding schedule" of $\boldsymbol{w}_{\star}(\cdot)$ can be obtained by combining all selection intervals with nonempty interior, i.e., $\left\{I_{\text {SEL }}(\mathcal{E}, s)\right\}_{(\mathcal{E}, s) \in \mathcal{V}_{\star}}$, where $\mathcal{V}_{\star}$ is defined in Prop. 12. Combining this with Prop. 11 (a), we finally arrive at the main theorems of Sec. 6, i.e., piecewise expressions of $\boldsymbol{w}_{\star}(\lambda)$ and $\mathcal{S}_{\mathrm{e}}(\lambda)$.

### 6.2 Piecewise Expressions of $\boldsymbol{w}_{\star}(\lambda)$ and $\mathcal{S}_{\mathbf{e}}(\lambda)$

This section presents the piecewise expressions of the minimum $\ell_{2}$-norm extended regularization path $\boldsymbol{w}_{\star}(\lambda)$ and the extended solution set mapping $\mathcal{S}_{\mathrm{e}}(\lambda)$.

For convenience of presentation, in the sequel, we first index every ex-equicorrelation pair in $\mathcal{V}_{\star}$ (i.e., every $(\mathcal{E}, s) \in \mathcal{V}$ whose selection interval $I_{\text {SEL }}(\mathcal{E}, s)$ has nonempty interior) by a natural sort induced by Prop. 12. The piecewise expressions of $\boldsymbol{w}_{\star}(\lambda)$ and $\mathcal{S}_{\mathrm{e}}(\lambda)$ will be presented using the indexed ex-equicorrelation pairs in $\mathcal{V}_{\star}$.
Definition 9 (A natural sort on $\mathcal{V}_{\star}$ ). We sort all ex-equicorrelation pairs in the set $\mathcal{V}_{\star}$ defined in Prop. 12, i.e.,

$$
\mathcal{V}_{\star}:=\left\{(\mathcal{E}, s) \in \mathcal{V} \mid \operatorname{int}\left(I_{S E L}(\mathcal{E}, s)\right) \neq \emptyset\right\}
$$

by the following procedure:

- arrange all of selection intervals $I_{\text {SEL }}(\mathcal{E}, s)(c f . \operatorname{Prop} .10)$ with $(\mathcal{E}, s) \in \mathcal{V}_{\star}$ on $\mathbb{R}_{++}$,
- for the selection interval located $k$ th $(k=0,1, \ldots)$ on $\mathbb{R}_{++}$from right to left, we define its associated ex-equicorrelation pair in $\mathcal{V}_{\star}$ as $\left(\mathcal{E}_{k}^{\star}, s_{k}^{\star}\right)$.
Since $\left\{I_{S E L}(\mathcal{E}, s)\right\}_{(\mathcal{E}, s) \in \mathcal{V}_{\star}}$ defines a partition of $\mathbb{R}_{++}$(Prop. 12), the sort above is well-defined.
The expression of $\boldsymbol{w}_{\star}(\lambda)$ as a piecewise linear function of $\lambda$ is given below, followed by some analysis on its properties.
Theorem 6 (Piecewise linear expression of $\boldsymbol{w}_{\star}(\lambda)$ ). The minimum $\ell_{2}$-norm extended regularization path $\boldsymbol{w}_{\star}(\lambda)$ is a continuous piecewise linear function of $\lambda$ with $\left|\mathcal{V}_{\star}\right|\left(\leq|\mathcal{V}| \leq 3^{2 n}\right)$ linear pieces.

Moreover, for every $k \in\left\{0,1, \ldots,\left|\mathcal{V}_{\star}\right|-1\right\}$, we have:
(a) for every $\lambda \in I_{\text {SEL }}\left(\mathcal{E}_{k}^{\star}, s_{k}^{\star}\right)$,

$$
\begin{equation*}
\boldsymbol{w}_{\star}(\lambda)=\hat{\boldsymbol{w}}_{E Q}\left(\mathcal{E}_{k}^{\star}, \boldsymbol{s}_{k}^{\star}, \lambda\right), \tag{38}
\end{equation*}
$$

which constitutes the expression of the $k$ th linear piece (located from right to left on $\mathbb{R}_{++}$) of $\boldsymbol{w}_{\star}(\cdot)$.
(b) for every $\lambda \in \operatorname{int}\left(I_{S E L}\left(\mathcal{E}_{k}^{\star}, s_{k}^{\star}\right)\right)$,

$$
\begin{equation*}
\left(\mathcal{E}_{k}^{\star}, \boldsymbol{s}_{k}^{\star}\right)=\left(\operatorname{supp}\left(\boldsymbol{w}_{\star}(\lambda)\right), \operatorname{sign}\left(\boldsymbol{w}_{\star}(\lambda)\right)\right), \tag{39}
\end{equation*}
$$

which means that the combination of support and signs vector of $\boldsymbol{w}_{\star}(\cdot)$ keeps constant within its $k$ th linear piece.
The meanings of the notations used above are as follows:


Fig. 3: A visual illustration of the minimum $\ell_{2}$-norm extended regularization path $\boldsymbol{w}_{\star}(\lambda)$ of the sGMC model. The red solid line is the trajectory of $\boldsymbol{w}_{\star}(\cdot)$, whereas the black solid line is the positive real axis of $\lambda \in \mathbb{R}_{++}$.

- $\left(\mathcal{E}_{k}^{\star}, s_{k}^{\star}\right)$ is the $k$ th element in $\mathcal{V}_{\star}$ (cf. Def. 9),
- $\hat{\boldsymbol{w}}_{E Q}\left(\mathcal{E}_{k}^{\star}, \boldsymbol{s}_{k}^{\star}, \cdot\right)$ is the trajectory function in Prop. 8,
- $I_{\text {SEL }}\left(\mathcal{E}_{k}^{\star}, s_{k}^{\star}\right)$ is the selection interval in Prop. 10.

Proof: From Prop. 11 (a), (38) and (39) hold for every $\lambda \in \operatorname{int}\left(I_{\text {SEL }}\left(\mathcal{E}_{k}^{\star}, s_{k}^{\star}\right)\right)$. Since $\operatorname{int}\left(I_{\text {SEL }}\left(\mathcal{E}_{k}^{\star}, s_{k}^{\star}\right)\right) \neq \emptyset$ (Def. 9) and $\boldsymbol{w}_{\star}(\cdot)$ is continuous (Cor. 3 in Sec. 4.3), we derive that (38) holds for every $\lambda \in I_{\text {SEL }}\left(\mathcal{E}_{k}^{\star}, s_{k}^{\star}\right)$.

Since $\left\{I_{\text {SEL }}\left(\mathcal{E}_{k}^{\star}, s_{k}^{\star}\right) \mid k \in\left\{0,1, \ldots,\left|\mathcal{V}_{\star}\right|-1\right\}\right\}$ defines a partition of $\mathbb{R}_{++}($Prop. 11$), \boldsymbol{w}_{\star}(\cdot)$ is piecewise linear with $\left|\mathcal{V}_{\star}\right|$ segments. We have $\left|\mathcal{V}_{\star}\right| \leq|\mathcal{V}| \leq 3^{2 n}$ from Prop. 5.

Remark 10 (Number of linear pieces in $\boldsymbol{w}_{\star}(\lambda)$ ). Thm. 6 establishes a one-to-one correspondence between linear pieces in $\boldsymbol{w}_{\star}(\lambda)$ and ex-equicorrelation pairs in $\mathcal{V}_{\star}$, which directly implies the following equation:

$$
\begin{equation*}
\text { number of linear pieces in } \boldsymbol{w}_{\star}(\lambda)=\left|\left\{(\mathcal{E}, \boldsymbol{s}) \in \mathcal{V} \mid \operatorname{int}\left(I_{\text {SEL }}(\mathcal{E}, \boldsymbol{s})\right) \neq \emptyset\right\}\right| . \tag{40}
\end{equation*}
$$

Notice that given $(\mathcal{E}, s) \in \mathcal{V}, I_{S E L}(\mathcal{E}, s)$ can be computed in closed-form. To the best of the authors' knowledge, even for the LASSO case, (40) is the first ${ }^{14}$ explicit expression for the number of linear pieces in a regularization path.

Remark 11 (Support and signs vector of $\boldsymbol{w}_{\star}(\lambda)$ ). Notice that in addition to piecewise linearity of $\boldsymbol{w}_{\star}(\lambda)$, Thm. 6 (b) also indicates that the $\operatorname{supp}\left(\boldsymbol{w}_{\star}(\lambda)\right)$ and $\operatorname{sign}\left(\boldsymbol{w}_{\star}(\lambda)\right)$ are piecewise constant in $\lambda$ with the same change points. Moreover, since every linear piece of $\boldsymbol{w}_{\star}(\cdot)$ is dominated by a different ex-equicorrelation pair, we can deduce that: as $\lambda$ varies monotonically in $\mathbb{R}_{++}$, the same combination of $\operatorname{supp}\left(\boldsymbol{w}_{\star}(\lambda)\right)$ and $\operatorname{sign}\left(\boldsymbol{w}_{\star}(\lambda)\right)$ never repeats.

In general, it is difficult to compute the $k$ th linear piece of $\boldsymbol{w}_{\star}(\lambda)$ numerically (cf. Sec. 7.1). Nevertheless, exploiting results from preceding sections, we can always obtain the closed-form expression of the zeroth linear piece of $\boldsymbol{w}_{\star}(\lambda)$.
Corollary 7 (Zeroth linear piece of $\boldsymbol{w}_{\star}(\lambda)$ ). Given the parameters $(\boldsymbol{A}, \boldsymbol{y}, \rho) \in \mathbb{R}^{m \times n} \times \mathbb{R}^{m} \times[0,1)$ in the sGMC model (6), the following holds for $\boldsymbol{w}_{\star}(\lambda)$ :

$$
\left(\mathcal{E}_{0}^{\star}, \boldsymbol{s}_{0}^{\star}\right)=\left(\emptyset, \mathbf{0}_{2 n}\right) \text { with } I_{S E L}\left(\mathcal{E}_{0}^{\star}, s_{0}^{\star}\right)=\left[\lambda_{\max }^{\star},+\infty\right)
$$

where $\lambda_{\max }^{\star}:=\max _{i \in\{1,2, \ldots, n\}}\left|\boldsymbol{a}_{i}^{\top} \boldsymbol{y}\right|$. From Thm. 6 , this implies that the zeroth linear piece of $\boldsymbol{w}_{\star}(\cdot)$ is:

$$
\left(\forall \lambda \in\left[\lambda_{\max }^{\star},+\infty\right)\right) \boldsymbol{w}_{\star}(\lambda)=\mathbf{0}_{2 n} .
$$

Proof: From Thm. 4 (b) in Sec. 4.1 and Thm. 6 (b), we have $\left(\mathcal{E}_{0}^{\star}, \boldsymbol{s}_{0}^{\star}\right)=\left(\emptyset, \mathbf{0}_{2 n}\right)$, then the expression of $I_{\text {SEL }}\left(\mathcal{E}_{0}^{\star}, \boldsymbol{s}_{0}^{\star}\right)$ can be computed in closed-form by Prop. 10 in Sec. 6.1.

Summarizing the results above, we can obtain Fig. 3, illustrating the shape of the whole regularization path of $\boldsymbol{w}_{\star}(\lambda)$.
Finally, as a by-product of the trajectory selection rule of $\boldsymbol{w}_{\star}(\lambda)$ (Prop. 11 (a)), we can also obtain the expression of $\mathcal{S}_{\mathrm{e}}(\lambda)$ as a piecewise polytope-valued mapping of $\lambda$.
Theorem 7 (Piecewise expression of $\mathcal{S}_{\mathrm{e}}(\lambda)$ ). For every $k \in\left\{0,1,2, \ldots,\left|\mathcal{V}_{\star}\right|-1\right\}$, the following holds:
(a) for every $\lambda \in I_{\text {SEL }}\left(\mathcal{E}_{k}^{\star}, s_{k}^{\star}\right)$,

$$
\begin{equation*}
\mathcal{S}_{e}(\lambda)=\mathcal{S}_{P O L Y}\left(\mathcal{E}_{k}^{\star}, s_{k}^{\star}, \lambda\right) \tag{41}
\end{equation*}
$$

which constitutes the expression of $\mathcal{S}_{e}(\lambda)$ within the kth linear piece (located from right to left on $\mathbb{R}_{++}$) of $\boldsymbol{w}_{\star}(\cdot)$.
(b) for every $\lambda \in \operatorname{int}\left(I_{\text {SEL }}\left(\mathcal{E}_{k}^{\star}, s_{k}^{\star}\right)\right)$,

$$
\begin{equation*}
\left(\mathcal{E}_{k}^{\star}, \boldsymbol{s}_{k}^{\star}\right)=\left(\mathcal{E}_{e}(\lambda), \boldsymbol{s}_{e}(\lambda)\right), \tag{42}
\end{equation*}
$$

which means that the common ex-equicorrelation set $\mathcal{E}_{e}(\lambda)$ and ex-equicorrelation signs vector $\boldsymbol{s}_{e}(\lambda)$ of all extended solutions in $\mathcal{S}_{e}(\lambda)$ (cf. Lemma 6 in Sec. 5.2) keeps constant within the kth linear piece of $\boldsymbol{w}_{\star}(\cdot)$.
14. In previous works [17], [14], only upper bounds of the linear piece number (which are exponential in the variable number $n$ ) are derived.

The meanings of the notations used above are as follows:

- $\left(\mathcal{E}_{k}^{\star}, s_{k}^{\star}\right)$ is the $k$ th element in $\mathcal{V}_{\star}$ (cf. Def. 9),
- $\mathcal{S}_{\text {POLY }}\left(\mathcal{E}_{k}^{\star}, \boldsymbol{s}_{k}^{\star}, \cdot\right)$ is the cutting polytope in Def. 7,
- $I_{S E L}\left(\mathcal{E}_{k}^{\star}, s_{k}^{\star}\right)$ is the selection interval in Prop. 10.

Proof: From Prop. 11 (a), (41) holds for every $\lambda \in \operatorname{int}\left(I_{\text {SEL }}\left(\mathcal{E}_{k}^{\star}, s_{k}^{\star}\right)\right)$. Since $\operatorname{int}\left(I_{\text {SEL }}\left(\mathcal{E}_{k}^{\star}, s_{k}^{\star}\right)\right) \neq \emptyset$ (Def. 9) and $\mathcal{S}_{\mathrm{e}}(\cdot)$ is continuous (Thm. 5 in Sec. 4.3), we further have $\mathcal{S}_{\mathrm{e}}(\lambda)=\mathcal{S}_{\text {POLY }}(\mathcal{E}, \boldsymbol{s}, \lambda)$ for every $\lambda \in I_{\text {SEL }}\left(\mathcal{E}_{k}^{\star}, s_{k}^{\star}\right)$.

### 6.3 Comparison with Related Works

In Sec. 4, 5 and 6, we have proven the continuity and piecewise linearity of the minimum $\ell_{2}$-norm extended regularization path and have derived its explicit piecewise expression. Here we would like to make clear the difference between our proof and that in related works.

We first review some notable previous studies on the piecewise linear regularization path of LASSO or its variants. Efron et al. [15] developed ${ }^{15}$ the well known LARS algorithm for computing a LASSO regularization path, where continuity and piecewise linearity of the regularization path are established as a by-product of the correctness of LARS. However, the proof therein requires the LASSO solution to be unique ${ }^{16}$ and relies on the so-called "one-at-a-time" assumption (cf. [15, Thm. 1] and Assumption 1). Rosset and Zhu [37] extended the result of LARS to a more general $\ell_{1}$-regularization model with the family of "almost quadratic" loss functions. They developed a piecewise linear solution algorithm, but the derivation implicitly used the same assumptions as [15]. R. J. Tibshirani and Taylor [31], [30] extended many results of LARS to the generalized LASSO model. R. J. Tibshirani also improved the proof of the conventional LARS-LASSO algorithm and developed many new results [14]. The proofs in [31], [30], [14] do not require the solution to be unique, but implicitly used the "one-at-a-time" assumption. Recently, Berk et al. [38] proved the local Lipschitz continuity of the LASSO solution with respect to $(\boldsymbol{y}, \lambda)$, assuming the solution to be unique.

There also have been some studies on the regularization path of more complicated extensions of LASSO, which are not piecewise linear. However, in this case, even proving the continuity of the regularization path is difficult. Xiao et al. [39] proposed a regularization path algorithm for the group LASSO model, which involves solving an ODE system in every iteration. They established the continuity of the unique regularization path assuming the loss function to be strictly convex and twice differentiable. Yukawa and Amari [40] studied the path of critical points of the nonconvex $\ell_{p}$-regularized $(0<p<1)$ least squares problem and confirmed its discontinuity. Mishkin and Pilanci [41] established the almost everywhere continuity of the regularization path for group LASSO, which can be improved to everywhere continuity under the solution uniqueness assumption.

The present work falls into the class of studies on piecewise linear regularization paths. However, different from previous works, our proof of Thm. 6 is established from a totally geometric perspective without using any assumption on any regularization path algorithm. Specifically, we regard the proof of continuity and piecewise linearity of the minimum $\ell_{2}$-norm extended regularization path $\boldsymbol{w}_{\star}(\lambda)$ as two independent issues, and tackle them with different mathematical tools:

1) To prove the continuity of $\boldsymbol{w}_{\star}(\lambda)$, we leverage tools from set-valued variational analysis and prove the continuity of the extended solution set mapping $\mathcal{S}_{\mathrm{e}}(\lambda)$. The continuity of $\boldsymbol{w}_{\star}(\lambda)$ is implied as a direct result (Sec. 4.3).
2) To prove the piecewise linearity of $\boldsymbol{w}_{\star}(\lambda)$, we start from analyzing the optimality condition (OPT), whereby we observe two rules (cf. Remark 5 in Sec. 5.1) and notice the discrete nature of $\mathcal{S}_{\mathrm{e}}(\lambda)$. Motivated by this,

- we define the ex-equicorrelation space $\mathcal{V}$, and take the ex-equicorrelation pairs in it as major tools for characterizing the sGMC model (Sec. 5.1),
- for every $(\mathcal{E}, \boldsymbol{s}) \in \mathcal{V}$, we introduce the notion of cutting polytope $\mathcal{S}_{\text {POLY}}(\mathcal{E}, \boldsymbol{s}, \lambda)$ to approximate $\mathcal{S}_{\mathrm{e}}(\lambda)$ (Sec. 5.3), and introduce the notion of trajectory function $\hat{\boldsymbol{w}}_{\mathrm{EQ}}(\mathcal{E}, \boldsymbol{s}, \lambda)$ to approximate $\boldsymbol{w}_{\star}(\lambda)$ (Sec. 5.4),
- interpreting $\boldsymbol{w}_{\star}(\lambda)$ and $\hat{\boldsymbol{w}}_{\mathrm{EQ}}(\mathcal{E}, \boldsymbol{s}, \lambda)$ respectively as the positions of passenger and train at time $\lambda$ (Remark 7), we yield the piecewise linearity of $\boldsymbol{w}_{\star}(\cdot)$,
- by studying the condition for $\hat{\boldsymbol{w}}_{\mathrm{EQ}}(\mathcal{E}, \boldsymbol{s}, \lambda)$ to be selected by $\boldsymbol{w}_{\star}(\lambda)$ (i.e., selection condition (SEL)), we obtain the "boarding period" of $\boldsymbol{w}_{\star}(\lambda)$ on $\hat{\boldsymbol{w}}_{\mathrm{EQ}}(\mathcal{E}, \boldsymbol{s}, \lambda)$ (i.e., selection interval $I_{\mathrm{SEL}}(\mathcal{E}, \boldsymbol{s})$ ), which yields the piecewise expression of $\boldsymbol{w}_{\star}(\lambda)$ (Sec. 6.1 and 6.2).
By this novel proof strategy, we manage to get rid of the "one-at-a-time" assumption ${ }^{17}$ which previous works rely on. Furthermore, our proof reveals many geometric properties of the sGMC model (as well as the LASSO model) that are unknown before, such as the set-valued continuity of the sGMC solution set mapping (Thm. 5) and the explicit expression of linear piece number in $\boldsymbol{w}_{\star}(\lambda)$ (Remark 10).

[^4]
## 7 Least Angle Regression for the sGMC Models

In this section, we propose an extension of the least angle regression (LARS [15]) algorithm for computing the whole minimum $\ell_{2}$-norm extended regularization path $\boldsymbol{w}_{\star}(\lambda)$ of the sGMC model, and we present a theoretical analysis on its conditional correctness and complexity.

The proposed LARS-sGMC algorithm resembles the conventional LARS algorithm. However, in contrast to previous works [36], [15], [14] which develop the regularization path algorithm as a homotopy method, in this paper we derive the LARS-sGMC algorithm from a different perspective, considering it as a solution algorithm to a discrete search problem. See the section that follows.

### 7.1 A Preliminary Discussion on the Computational Task

From Thm. 6, the whole regularization path of $\boldsymbol{w}_{\star}(\lambda)$ can be obtained by connecting all trajectory functions $\hat{\boldsymbol{w}}_{\mathrm{EQ}}\left(\mathcal{E}_{k}^{\star}, \boldsymbol{s}_{k}^{\star}, \lambda\right)$ $\left(k=0,1, \ldots,\left|\mathcal{V}_{\star}\right|-1\right)$ within their corresponding selection intervals $I_{\mathrm{SEL}}\left(\mathcal{E}_{k}^{\star}, s_{k}^{\star}\right)$. Hence to obtain the whole information about $\boldsymbol{w}_{\star}(\lambda)$, we only need to find all elements in $\mathcal{V}_{\star}$, i.e., all ex-equicorrelation pairs $(\mathcal{E}, \boldsymbol{s}) \in \mathcal{V}$ whose selection interval $I_{\mathrm{SEL}}(\mathcal{E}, s)$ has nonempty interior (cf. Prop. 12 in Sec. 6.1).

This result is intriguing from a computational perspective because it reveals an equivalence between the following two computational problems of totally different types:

1) The continuous optimization problem of finding the minimum $\ell_{2}$-norm primal and dual solutions of the sGMC model (cf. Thm. 4 in Sec. 4.1) for all $\lambda>0$.
2) The discrete search problem of finding all ex-equicorrelation pairs $(\mathcal{E}, s) \in \mathcal{V}$ whose selection interval $I_{\text {SEL }}(\mathcal{E}, s)$ has nonempty interior.
It is evident that the former problem generally requires infinite computations ${ }^{18}$, whereas the computational complexity of the latter one is finite. Hence while the second discrete search problem may be NP-hard from its combinatorial nature, we still would prefer to take it as the task of computation.

Recalling the definition of the ex-equicorrelation space $\mathcal{V}$ (Prop. 5 in Sec. 5.1), one may immediately figure out the following brute force algorithm for solving the aforementioned discrete search problem.
Procedure 1 (A brute force algorithm for finding all ex-equicorrelation pairs in $\mathcal{V}_{\star}$ ). Explore all $s \in\{+1,0,-1\}^{2 n}$, for each $s$, set $\mathcal{E}:=\operatorname{supp}(s)$ and check

1) whether $(\mathcal{E}, \boldsymbol{s})$ is in $\mathcal{V}$, i.e., whether $[s]_{\mathcal{E}} \in \mathcal{R}\left(\boldsymbol{C}_{\mathcal{E}}{ }^{\top}\right)$,
2) whether $I_{\text {SEL }}(\mathcal{E}, s)$ has nonempty interior.

When $(\mathcal{E}, \boldsymbol{s})$ satisfies both conditions above, we confirm that $(\mathcal{E}, \boldsymbol{s})$ is in $\mathcal{V}_{\star}$, otherwise we discard such $(\mathcal{E}, \boldsymbol{s})$.
Evidently, the computational complexity of such brute force algorithm is $\mathcal{O}\left(3^{2 n}\right)$, which is finite but no chance.
Compared to searching for $(\mathcal{E}, s) \in \mathcal{V}_{\star}$ randomly in the whole ex-equicorrelation space $\mathcal{V}$, it definitely leads to more efficient algorithms if we can exploit the underlying structure of $\mathcal{V}_{\star}$. Recall the one-to-one correspondence between all ex-equicorrelation pairs in $\mathcal{V}_{\star}$ and linear pieces in the regularization path $\boldsymbol{w}_{\star}(\cdot)$ (cf. $\mathrm{Thm} .6 \mathrm{in} \mathrm{Sec.6.2)}$, consider the following problem, which intends to find all elements in $\mathcal{V}_{\star}$ in an iterative manner.

Problem 1. Given the parameters $(\boldsymbol{A}, \boldsymbol{y}, \rho) \in \mathbb{R}^{m \times n} \times \mathbb{R}^{m} \times[0,1)$ of the sGMC model (6), suppose we have access to the value of $\left(\mathcal{E}_{k}^{\star}, s_{k}^{\star}\right) \in \mathcal{V}_{\star}$ for some possibly unknown index $k$, can we efficiently obtain its adjacent ex-equicorrelation pair $\left(\mathcal{E}_{k+1}^{\star}, s_{k+1}^{\star}\right)$ or $\left(\mathcal{E}_{k-1}^{\star}, \boldsymbol{s}_{k-1}^{\star}\right)$ ?

In the sequel, we present a closed-form estimate for the solution of Problem 1. The process of computing such estimate constitutes a single iteration of the LARS-sGMC algorithm.

### 7.2 Derivation of the LARS-sGMC Iteration

Let the input of the LARS-sGMC iteration be

$$
(\mathcal{E}, s)=\left(\mathcal{E}_{k}^{\star}, s_{k}^{\star}\right) \in \mathcal{V}_{\star} \text { with } k \in\left\{0,1, \ldots,\left|\mathcal{V}_{\star}\right|-2\right\}
$$

where $k$ may be unknown. The goal of the LARS-sGMC iteration is to generate an estimate ${ }^{19}\left(\mathcal{E}_{+}, \boldsymbol{s}_{+}\right)$for $\left(\mathcal{E}_{k+1}^{\star}, s_{k+1}^{\star}\right)$.
Recall that $\left(\mathcal{E}_{k}^{\star}, s_{k}^{\star}\right)$ and $\left(\mathcal{E}_{k+1}^{\star}, \boldsymbol{s}_{k+1}^{\star}\right)$ respectively determine two adjacent linear pieces in $\boldsymbol{w}_{\star}(\lambda)$ (cf. Fig. 3), one can imagine that the transition from $\left(\mathcal{E}_{k}^{\star}, \boldsymbol{s}_{k}^{\star}\right)$ to $\left(\mathcal{E}_{k+1}^{\star}, \boldsymbol{s}_{k+1}^{\star}\right)$ can be derived from the trajectory selection rule of $\boldsymbol{w}_{\star}(\lambda)$ (cf. Remark 9 in Sec. 6.1) around the change point:

$$
\begin{equation*}
\lambda_{+}:=l_{\mathrm{SEL}}\left(\mathcal{E}_{k}^{\star}, s_{k}^{\star}\right)=r_{\mathrm{SEL}}\left(\mathcal{E}_{k+1}^{\star}, s_{k+1}^{\star}\right) . \tag{43}
\end{equation*}
$$

Let $\epsilon>0$ be an infinitesimally small number. Then as $\lambda$ decreases from $\lambda_{+}+\epsilon$ to $\lambda_{+}-\epsilon$, the following happens:

[^5]

Fig. 4: An illustration of the computation of $\mathcal{I}_{\text {delete }}^{+}$and $\mathcal{I}_{\text {insert }}^{+}$in Procedure 3. In this example, we assume $n=3$ and $\mathcal{E}_{k}^{\star}=\{1,2,4,5\}$. The black solid line represents the positive real axis of $\lambda \in \mathbb{R}_{++}$, whilst the black dashed line is the position of $\lambda_{+}:=l_{\text {SEL }}\left(\mathcal{E}_{k}^{\star}, s_{k}^{\star}\right)=r_{\text {SEL }}\left(\mathcal{E}_{k}^{\star}, s_{k}^{\star}\right)$. Every red (resp. purple) solid line represents an interval $I_{i}^{\mathrm{a}}(\mathcal{E}, s)$ (resp. $I_{i}^{\mathrm{b}}(\mathcal{E}, s)$ ) for some $i \in \mathcal{E}_{k}^{\star}$ (resp. $i \in \neg \mathcal{E}_{k}^{\star}$ ), where only the left endpoint of the interval is plotted as a solid dot.

1) For $\lambda=\lambda_{+}+\epsilon,(\mathcal{E}, s)=\left(\mathcal{E}_{k}^{\star}, s_{k}^{\star}\right)$ satisfies the selection condition (SEL) (cf. Def. 8 in Sec. 6.1) at $\lambda$, i.e.,

$$
\begin{cases}(\forall i \in \mathcal{E}) & {[\boldsymbol{s}]_{i}\left[\hat{\boldsymbol{w}}_{\mathrm{EQ}}(\mathcal{E}, \boldsymbol{s}, \lambda)\right]_{i} \geq 0}  \tag{SEL-a}\\ (\forall i \in \neg \mathcal{E}) & \left|\xi_{i}\left(\hat{\boldsymbol{w}}_{\mathrm{EQ}}(\mathcal{E}, \boldsymbol{s}, \lambda)\right)\right| \leq \lambda\end{cases}
$$

2) As $\lambda$ decreases infinitesimally from $\lambda_{+}+\epsilon$ to $\lambda_{+}-\epsilon$, there emerges some $i \in \mathcal{I}_{\text {break }}^{\text {a }}$ or $i \in \mathcal{I}_{\text {break }}^{\text {b }}$ with

$$
\begin{aligned}
& \mathcal{I}_{\text {break }}^{\text {a }}:=\left\{i \in \mathcal{E} \mid[\boldsymbol{s}]_{i}\left[\hat{\boldsymbol{w}}_{\mathrm{EQ}}\left(\mathcal{E}, \boldsymbol{s}, \lambda_{+}-\epsilon\right)\right]_{i}<0\right\} \\
& \mathcal{I}_{\text {break }}^{\mathrm{b}}:=\left\{i \in \neg \mathcal{E}| | \xi_{i}\left(\hat{\boldsymbol{w}}_{\mathrm{EQ}}\left(\mathcal{E}, \boldsymbol{s}, \lambda_{+}-\epsilon\right)\right) \mid>\lambda_{+}-\epsilon\right\},
\end{aligned}
$$

such that for every $i \in \mathcal{I}_{\text {break }}^{\mathrm{a}}$ (resp. $i \in \mathcal{I}_{\text {break }}^{\mathrm{b}}$ ), $(\mathcal{E}, s)$ breaks the $i$ th (SEL-a) (resp. the $i$ th (SEL-b)) condition at $\lambda=\lambda_{+}-\epsilon$.
3) For $\lambda=\lambda_{+}-\epsilon,(\mathcal{E}, s)$ has to transit from $\left(\mathcal{E}_{k}^{\star}, s_{k}^{\star}\right)$ to $\left(\mathcal{E}_{k+1}^{\star}, s_{k+1}^{\star}\right)$ to maintain the validity of (SEL) at $\lambda$.

Based on this observation, given $(\mathcal{E}, s)=\left(\mathcal{E}_{k}^{\star}, s_{k}^{\star}\right)$, if we can modify the value of $(\mathcal{E}, s)$ such that it satisfies (SEL) at $\lambda_{+}-\epsilon$, then the modified value of $(\mathcal{E}, \boldsymbol{s})$ is exactly $\left(\mathcal{E}_{k+1}^{\star}, s_{k+1}^{\star}\right)$. A greedy modification of $(\mathcal{E}, \boldsymbol{s})$ is to remove every condition in (SEL) that breaks at $\lambda_{+}-\epsilon$ from the expression of (SEL), i.e., remove every $i \in \mathcal{I}_{\text {break }}^{\text {a }}$ (resp. $i \in \mathcal{I}_{\text {break }}^{\text {b }}$ ) from $\mathcal{E}$ (resp. $\neg \mathcal{E}$ ). See the following procedure.

Procedure 2 (A deletion-insertion process). Given the input ex-equicorrelation pair $(\mathcal{E}, s)=\left(\mathcal{E}_{k}^{\star}, s_{k}^{\star}\right)$, define

$$
\begin{aligned}
& \mathcal{I}_{\text {delete }}^{+}:=\mathcal{I}_{\text {break }}^{a}=\left\{i \in \mathcal{E} \mid[\boldsymbol{s}]_{i}\left[\hat{\boldsymbol{w}}_{E Q}\left(\mathcal{E}, \boldsymbol{s}, \lambda_{+}-\epsilon\right)\right]_{i}<0\right\}, \\
& \mathcal{I}_{\text {insert }}^{+}:=\mathcal{I}_{\text {break }}^{b}=\left\{i \in \neg \mathcal{E}| | \xi_{i}\left(\hat{\boldsymbol{w}}_{E Q}\left(\mathcal{E}, \boldsymbol{s}, \lambda_{+}-\epsilon\right)\right) \mid>\lambda_{+}-\epsilon\right\} .
\end{aligned}
$$

We conduct the following deletion and insertion operations on $(\mathcal{E}, s)$ to fit $(\mathcal{E}, s)$ into (SEL) at $\lambda_{+}-\epsilon$ :

- Deletion: for every $i \in \mathcal{I}_{\text {delete, }}^{+}$, we delete $i$ from $\mathcal{E}$ to remove the ith (SEL-a) condition that breaks at $\lambda_{+}-\epsilon$, and set $[\boldsymbol{s}]_{i}=0$ in response to the change of $\mathcal{E}$.
- Insertion: for every $i \in \mathcal{I}_{\text {insert }}^{+}$, we insert $i$ into $\mathcal{E}$ (i.e., delete $i$ from $\neg \mathcal{E}$ ) to remove the $i$ th (SEL-b) condition that breaks at $\lambda_{+}-\epsilon$, and set

$$
[\boldsymbol{s}]_{i}=\operatorname{sign}\left(\xi_{i}\left(\boldsymbol{w}_{\star}\left(\lambda_{+}-\epsilon\right)\right)\right)=\operatorname{sign}\left(\xi_{i}\left(\boldsymbol{w}_{\star}\left(\lambda_{+}\right)\right)\right)=\operatorname{sign}\left(\xi_{i}\left(\hat{\boldsymbol{w}}_{E Q}\left(\mathcal{E}_{k}^{\star}, \boldsymbol{s}_{k}^{\star}, \lambda_{+}\right)\right)\right)
$$

to align the component sign $[\boldsymbol{s}]_{i}$ of $\left[\boldsymbol{w}_{\star}\left(\lambda_{+}-\epsilon\right)\right]_{i}$ with the correlation sign $\operatorname{sign}\left(\xi_{i}\left(\boldsymbol{w}_{\star}\left(\lambda_{+}-\epsilon\right)\right)\right)(c f$. Fig. 1$)$.
Let the modified value of $(\mathcal{E}, s)$ be $\left(\mathcal{E}_{+}, \boldsymbol{s}_{+}\right)$, then we set $\left(\mathcal{E}_{+}, \boldsymbol{s}_{+}\right)$as the output of the current LARS-sGMC iteration, i.e., an estimate of $\left(\mathcal{E}_{k+1}^{\star}, s_{k+1}^{\star}\right)$.

Recall that for every $i \in \mathcal{E}$ (resp. $i \in \neg \mathcal{E}$ ), the interval of $\lambda \in \mathbb{R}$ satisfying the $i$ th (SEL-a) (resp. (SEL-b)) condition can be computed in closed-form (cf. Prop. $10 \mathrm{in} \mathrm{Sec}. \mathrm{6.1)} .\mathrm{Exploiting} \mathrm{the} \mathrm{results} \mathrm{therein} ,\mathrm{we} \mathrm{can} \mathrm{obtain} \mathrm{a} \mathrm{closed-form} \mathrm{expression}$ of $\mathcal{I}_{\text {delete }}^{+}$and $\mathcal{I}_{\text {insert }}^{+}$in Procedure 2, whereby the output $\left(\mathcal{E}_{+}, s_{+}\right)$can be computed. See the following procedure.
Procedure 3 (A sketch of the LARS-sGMC iteration). Given the input ex-equicorrelation pair $(\mathcal{E}, s)=\left(\mathcal{E}_{k}^{\star}, s_{k}^{\star}\right)$, we obtain the output $\left(\mathcal{E}_{+}, s_{+}\right)$of Procedure 2 by the following steps (cf. Fig. 4 for a visual illustration):

```
Algorithm 1 A single iteration of the LARS-sGMC algorithm
Problem Parameters: \((\boldsymbol{A}, \boldsymbol{y}, \rho) \in \mathbb{R}^{m \times n} \times \mathbb{R}^{m} \times[0,1)\)
Input: \((\mathcal{E}, \boldsymbol{s})=\left(\mathcal{E}_{k}^{\star}, s_{k}^{\star}\right) \in \mathcal{V}_{\star}\) with \(k \in\left\{0,1, \ldots,\left|\mathcal{V}_{\star}\right|-2\right\}\) (the index \(k\) may be unknown)
Output: the estimate \(\left(\mathcal{E}_{+}, s_{+}\right)\)of \(\left(\mathcal{E}_{k+1}^{\star}, s_{k+1}^{\star}\right)\)
    \(\triangleright\) Compute \(b, C\) and \(D\) (Prop. 3 in Sec. 4.1)
    \(\boldsymbol{b} \leftarrow\left[\begin{array}{ll}\boldsymbol{y}^{\top} & \mathbf{0}_{m}{ }^{\top}\end{array}\right]^{\top}\)
    \(\boldsymbol{C} \leftarrow \operatorname{blkdiag}(\boldsymbol{A}, \sqrt{\rho} \boldsymbol{A})\)
    \(\boldsymbol{D} \leftarrow\left[\begin{array}{cc}(1-\rho) \boldsymbol{I}_{m} & \sqrt{\rho} \boldsymbol{I}_{m} \\ -\sqrt{\rho} \boldsymbol{I}_{m} & \boldsymbol{I}_{m}\end{array}\right]\)
    \(\triangleright\) Obtain the expressions of \(\hat{\boldsymbol{w}}_{\mathrm{EQ}}(\mathcal{E}, \boldsymbol{s}, \lambda)=\boldsymbol{p}-\boldsymbol{q} \lambda\) and \(\xi_{i}\left(\hat{\boldsymbol{w}}_{\mathrm{EQ}}(\mathcal{E}, \boldsymbol{s}, \lambda)\right)=\boldsymbol{c}_{i}^{\top}(\boldsymbol{u}-\boldsymbol{v} \lambda)\) (Appendix E.1)
    \([\boldsymbol{p}]_{\mathcal{E}} \leftarrow\left(\boldsymbol{C}_{\mathcal{E}}{ }^{\top} \boldsymbol{D} \boldsymbol{C}_{\mathcal{E}}\right)^{\dagger} \boldsymbol{C}_{\mathcal{E}}^{\top} \boldsymbol{b}\)
    \([\boldsymbol{q}]_{\mathcal{E}} \leftarrow\left(\boldsymbol{C}_{\mathcal{E}}{ }^{\top} \boldsymbol{D} \boldsymbol{C}_{\mathcal{E}}\right)^{\dagger}[s]_{\mathcal{E}}\)
    \(\boldsymbol{u} \leftarrow \boldsymbol{b}-\boldsymbol{D} \boldsymbol{C}_{\mathcal{E}}[\boldsymbol{p}]_{\mathcal{E}}\)
    \(\boldsymbol{v} \leftarrow-D C_{\mathcal{E}}[\boldsymbol{q}]_{\mathcal{E}}\)
    \(\triangleright\) Step 1: compute the left endpoint \(l_{i}^{\text {a }}\) of the solution interval of the \(i\) th (SEL-a) condition ( \((78)\) in Appendix E.1)
    for \(i \in \mathcal{E}\) do
        \(i \in \mathcal{E}\) do \(\boldsymbol{o}_{i}\),
\(l_{i}^{\text {a }} \leftarrow \begin{cases}\frac{[\boldsymbol{p}]_{i}}{[\boldsymbol{q}]_{i}}, & \text { if }[\boldsymbol{s}]_{i}[\boldsymbol{q}]_{i}<0, \\ -\infty, & \text { otherwise. }\end{cases}\)
    end for
    \(\triangleright\) Step 2: compute the left endpoint \(l_{i}^{\mathrm{b}}\) of the solution interval of the \(i\) th (SEL-b) condition ((79) in Appendix E.1)
    for \(i \in \neg \mathcal{E}\) do
        \(i \in \neg \mathcal{E} \mathbf{d o}_{\mathbf{d o}_{\top}}\),
\(l_{i}^{\mathrm{b}} \leftarrow \begin{cases}\frac{\boldsymbol{c}_{i}}{\boldsymbol{c}_{i}^{\top} \boldsymbol{u}}, & \text { if } \boldsymbol{c}_{i}^{\top} \boldsymbol{u}>0, \\ \frac{\boldsymbol{c}_{i}^{\top} \boldsymbol{u}}{\boldsymbol{c}_{i}^{\top} \boldsymbol{v}-1}, & \text { if } \boldsymbol{c}_{i}^{\top} \boldsymbol{u}<0, \\ 0, & \text { otherwise. }\end{cases}\)
    end for
    \(\triangleright\) Step 3: compute the trajectory switching time \(\lambda_{+}\), the deleted index set \(\mathcal{I}_{\text {delete }}^{+}\)and the inserted index set \(\mathcal{I}_{\text {insert }}^{+}\)
    \(\lambda_{+} \leftarrow \max \left\{0, \max _{i \in \mathcal{E}} l_{i}^{\mathrm{a}}, \max _{i \in \neg \mathcal{E}} l_{i}^{\mathrm{b}}\right\}\)
    \(\mathcal{I}_{\text {delete }}^{+} \leftarrow\left\{i \in \mathcal{E} \mid l_{i}^{\mathrm{a}}=\lambda_{+}\right\}\)
    \(\mathcal{I}_{\text {insert }}^{+} \leftarrow\left\{i \in \neg \mathcal{E} \mid l_{i}^{b}=\lambda_{+}\right\}\)
    \(\triangleright\) Step 4: compute the estimate \(\left(\mathcal{E}_{+}, s_{+}\right)\)by the deletion-insertion process
    \(\mathcal{E}_{+} \leftarrow\left(\mathcal{E} \backslash \mathcal{I}_{\text {delete }}^{+}\right) \cup \mathcal{I}_{\text {insert }}^{+}\)
    for \(i=1,2, \ldots, 2 n\) do
        \(\left[\boldsymbol{s}_{+}\right]_{i} \leftarrow \begin{cases}0, & \text { if } i \in \mathcal{I}_{\text {delete }}^{+}, \\ \operatorname{sign}\left(\boldsymbol{c}_{i}^{\top}\left(\boldsymbol{u}-\boldsymbol{v} \lambda_{+}\right)\right), & \text {if } i \in \mathcal{I}_{\text {insert }}^{+}, \\ {[\boldsymbol{s}]_{i},} & \text { otherwise },\end{cases}\)
    end for
```

- Step 1: for every $i \in \mathcal{E}$, compute the left endpoint $l_{i}^{a}(\mathcal{E}, s)$ of the interval $I_{i}^{a}(\mathcal{E}, s)$ of $\lambda \in \mathbb{R}$ satisfying the $i$ th (SEL-a) condition (cf. (78) in Appendix E.1).
- Step 2: for every $i \in \neg \mathcal{E}$, compute the left endpoint $l_{i}^{b}(\mathcal{E}, \boldsymbol{s})$ of the interval $I_{i}^{b}(\mathcal{E}, \boldsymbol{s})$ of $\lambda \in \mathbb{R}$ satisfying the ith (SEL-b) condition (cf. (79) in Appendix E.1).
- Step 3: compute the trajectory switching time

$$
\lambda_{+}:=l_{S E L}(\mathcal{E}, s)=\max \left\{0, \max _{i \in \mathcal{E}} l_{i}^{a}(\mathcal{E}, s), \max _{i \in \neg \mathcal{E}} l_{i}^{b}(\mathcal{E}, s)\right\}
$$

as well as the deleted and inserted index sets

$$
\mathcal{I}_{\text {delete }}^{+}:=\left\{i \in \mathcal{E} \mid l_{i}^{a}(\mathcal{E}, s)=\lambda_{+}\right\}, \quad \mathcal{I}_{\text {insert }}^{+}:=\left\{i \in \neg \mathcal{E} \mid l_{i}^{b}(\mathcal{E}, s)=\lambda_{+}\right\} .
$$

- Step 4: compute $\left(\mathcal{E}_{+}, s_{+}\right)$by the following expression

$$
\mathcal{E}_{+}:=\left(\mathcal{E} \backslash \mathcal{I}_{\text {delete }}^{+}\right) \cup \mathcal{I}_{\text {insert }}^{+}, \quad\left[s_{+}\right]_{i}:= \begin{cases}0, & \text { if } i \in \mathcal{I}_{\text {delete }}^{+}, \\ \operatorname{sign}\left(\xi_{i}\left(\hat{\boldsymbol{w}}_{E Q}\left(\mathcal{E}, s, \lambda_{+}\right)\right)\right), & \text {if } i \in \mathcal{I}_{\text {insert }}^{+}, \\ {[s]_{i},} & \text { otherwise },\end{cases}
$$

TABLE 1: Time and space complexity of a single LARS-sGMC iteration (Algorithm 1).

| Object of computation | Time complexity | Space complexity |
| :---: | :---: | :---: |
| $(\boldsymbol{p}, \boldsymbol{q})$ (line 6-7) | $\mathcal{O}\left(\|\mathcal{E}\|^{2} m+\|\mathcal{E}\|^{3}\right)$ | $\mathcal{O}(n)$ |
| $(\boldsymbol{u}, \boldsymbol{v})$ (line 8-9) | $\mathcal{O}(m\|\mathcal{E}\|)$ | $\mathcal{O}(m)$ |
| $\left(l_{i}^{a}\right)_{i \in \mathcal{E}}$ (line 11-13) | $\mathcal{O}(\|\mathcal{E}\|)$ | $\mathcal{O}(\|\mathcal{E}\|)$ |
| $\left(l_{i}^{b}\right)_{i \in \neg \mathcal{E}}$ (line 15-17) | $\mathcal{O}(m\|\neg \mathcal{E}\|)$ | $\mathcal{O}(\|\mathcal{E}\|)$ |
| $\mathcal{I}_{\text {delete }}^{+}$and $\mathcal{I}_{\text {insert }}^{+}$(line 19-21) | $\mathcal{O}(n)$ | $\mathcal{O}(n)$ |
| $\left(\mathcal{E}_{+}, \boldsymbol{s}_{+}\right)$(line 23-26) | $\mathcal{O}\left(\left\|\mathcal{I}_{\text {delete }}^{+}\right\|+\left\|\mathcal{I}_{\text {insert }}^{+}\right\|\right)$ | $\mathcal{O}(n)$ |
| In total | $\mathcal{O}\left(m n+\|\mathcal{E}\|^{2} m+\|\mathcal{E}\|^{3}\right)$ | $\mathcal{O}(m+n)$ |

Applying the results of Appendix E. 1 to Procedure 3 yields Algorithm 1, which constitutes a single iteration of the proposed LARS-sGMC algorithm. Comparing Algorithm 1 with the conventional LARS algorithm for LASSO (cf. [14, Sec. 3.1]), one can verify that the conventional LARS algorithm is a special instance of LARS-sGMC with $\rho=0$.

Corollary 8. If $\rho=0$, then Algorithm 1 reduces to one iteration of the conventional LARS algorithm for LASSO.
Notice that if we believe in the correctness of the LARS-sGMC iteration (i.e., for the input $(\mathcal{E}, s)=\left(\mathcal{E}_{k}^{\star}, s_{k}^{\star}\right) \in \mathcal{V}_{\star}$, the output $\left(\mathcal{E}_{+}, s_{+}\right)$of Algorithm 1 equals to $\left(\mathcal{E}_{k+1}^{\star}, s_{k+1}^{\star}\right)$ ), then we can further take the value of $\left(\mathcal{E}_{+}, s_{+}\right)$as the input of Algorithm 1 and compute an estimate of $\left(\mathcal{E}_{k+2}^{\star}, \boldsymbol{s}_{k+2}^{\star}\right)$. Hence as long as we have an initial element in $\mathcal{V}_{\star}$, we can compute estimates for a consecutive sequence of ex-equicorrelation pairs in $\mathcal{V}_{\star}$. In the following, we discuss about how to obtain this initial input for the LARS-sGMC algorithm.

Remark 12 (How to obtain the initial input of the LARS-sGMC algorithm). We recommend the following two approaches to obtain an initial ex-equicorrelation pair in $\mathcal{V}_{\star}$ :

1) From Cor. 7 in Sec. 6.2, we always have

$$
\left(\mathcal{E}_{0}^{\star}, \boldsymbol{s}_{0}^{\star}\right)=\left(\emptyset, \mathbf{o}_{2 n}\right)
$$

regardless of the problem parameters. Hence we can always take this known $\left(\mathcal{E}_{0}^{\star}, s_{0}^{\star}\right)$ as the initial input.
2) For arbitrary $\bar{\lambda}>0$, we can compute an extended solution $\boldsymbol{w}_{e} \in \mathcal{S}_{e}(\bar{\lambda})$ by some iterative algorithm. Then from Thm. 7 (b) in Sec. 6.2 and Lemma 6 in Sec. 5.2, we have

$$
\left(\mathcal{E}_{k}^{\star}, \boldsymbol{s}_{k}^{\star}\right):=\left(\mathcal{E}\left(\bar{\lambda}, \boldsymbol{w}_{e}\right), \boldsymbol{s}\left(\bar{\lambda}, \boldsymbol{w}_{e}\right)\right)
$$

as long as $\bar{\lambda}$ is not an endpoint of $I_{S E L}\left(\mathcal{E}_{k}^{\star}, \boldsymbol{s}_{k}^{\star}\right)$, where $k$ is the index of the linear piece that $\boldsymbol{w}_{\star}(\bar{\lambda})$ lies in, $\mathcal{E}(\cdot, \cdot)$ and $\boldsymbol{s}(\cdot, \cdot)$ are the ex-equicorrelation set and ex-equicorrelation signs vector (cf. Def. 5 in Sec. 5.1).

### 7.3 Property Analysis of the LARS-sGMC Iteration

In this section, we analyze the properties of the LARS-sGMC iteration, including its correctness and complexity.
We first prove the correctness of a single LARS-sGMC iteration under the following "one-at-a-time" assumption.
Assumption 1. Given the problem parameters $(\boldsymbol{A}, \boldsymbol{y}, \rho) \in \mathbb{R}^{m \times n} \times \mathbb{R}^{m} \times[0,1)$, let

$$
(\mathcal{E}, s)=\left(\mathcal{E}_{k}^{\star}, s_{k}^{\star}\right) \in \mathcal{V}_{\star} \text { with } k \in\left\{0,1, \ldots,\left|\mathcal{V}_{\star}\right|-2\right\}
$$

be the input of the LARS-sGMC iteration (Algorithm 1). We say the kth "one-at-a-time" assumption holds if

$$
\left|\mathcal{I}_{\text {delete }}^{+}\right|+\left|\mathcal{I}_{\text {insert }}^{+}\right|=1 .
$$

In words, the kth "one-at-a-time" assumption holds if the deletion-insertion process (Procedure 2) of the LARS-sGMC iteration involves only one index.

Remark 13 (Mildness of the "one-at-a-time" assumption). From line 19-21 of Algorithm 1, the kth "one-at-a-time" assumption holds if and only if in line 19:

$$
\lambda_{+} \leftarrow \max \left\{0, \max _{i \in \mathcal{E}} l_{i}^{a}, \max _{i \in \neg \mathcal{E}} l_{i}^{b}\right\}
$$

the maximum in the RHS is attained by a unique index $i \in \mathcal{E}$ or $i \in \neg \mathcal{E}$. One can imagine that if $\left(l_{i}^{a}\right)_{i \in \mathcal{E}}$ and $\left(l_{i}^{b}\right)_{i \in \neg \mathcal{E}}$ follow a continuous joint probability distribution on $\mathbb{R}^{2 n}$, then the "one-at-a-time" assumption holds with probability one. Hence informally speaking, Assumption 1 is a mild assumption.
Theorem 8 (Conditional correctness of the LARS-sGMC iteration). Given the parameters $(\boldsymbol{A}, \boldsymbol{y}, \rho) \in \mathbb{R}^{m \times n} \times \mathbb{R}^{m} \times[0,1)$ of the sGMC model (6), let

$$
(\mathcal{E}, \boldsymbol{s})=\left(\mathcal{E}_{k}^{\star}, \boldsymbol{s}_{k}^{\star}\right) \in \mathcal{V}_{\star} \text { with } k \in\left\{0,1, \ldots,\left|\mathcal{V}_{\star}\right|-2\right\}
$$

be the input of the LARS-sGMC iteration (Algorithm 1). Suppose the kth "one-at-a-time" assumption holds, then

$$
\left(\mathcal{E}_{+}, \boldsymbol{s}_{+}\right)=\left(\mathcal{E}_{k+1}^{\star}, s_{k+1}^{\star}\right) \in \mathcal{V}_{\star}
$$

Proof: See Appendix F.
We should note that while Thm. 8 is established under Assumption 1, extensive experiments (cf. Sec. 8.1) indicate that the correctness of the LARS-sGMC iteration remains to hold even if the "one-at-a-time" assumption breaks. However, it is difficult to prove this empirical observation to our current knowledge, and we would like to leave it for future work.

In the sequel, we present a complexity analysis of the LARS-sGMC algorithm.
Firstly, from Thm. 8, it is evident that under the "one-at-a-time" assumption, it takes at most $\left|\mathcal{V}_{\star}\right|-1$ LARS-sGMC iterations to compute the whole regularization path of $\boldsymbol{w}_{\star}(\lambda)$.

Corollary 9 (Iteration complexity for computing the whole regularization path $\boldsymbol{w}_{\star}(\lambda)$ ). Suppose the kth "one-at-a-time" assumption holds for every $k \in\left\{0,1, \ldots,\left|\mathcal{V}_{\star}\right|-2\right\}$. Then starting from $\left(\mathcal{E}_{0}^{\star}, \boldsymbol{s}_{0}^{\star}\right)=\left(\emptyset, \mathbf{0}_{2 n}\right)$, it takes

$$
\left|\mathcal{V}_{\star}\right|-1\left(\leq|\mathcal{V}|-1 \leq 3^{2 n}-1\right)
$$

LARS-sGMC iterations (Algorithm 1) to compute all ex-equicorrelation pairs in $\mathcal{V}_{\star}$, i.e., the whole information about the minimum $\ell_{2}$-norm extended regularization path $\boldsymbol{w}_{\star}(\lambda)$.

Proof: The result follows directly from Thm. 6 in Sec. 6.2 and Thm. 8.
Remark 14 (On the worst-case complexity of the LARS-sGMC algorithm). From Cor. 9, the worst-case iteration complexity for Algorithm 1 to compute the whole regularization path $\boldsymbol{w}_{\star}(\lambda)$ is $\mathcal{O}\left(3^{2 n}\right)$, which is the same as the brute-force algorithm (Procedure 1). Hence one may concern with the efficiency of the LARS-sGMC algorithm. However, empirical results (cf. Fig. 7 in Sec. 8.1) indicate that the value of $\lambda_{+}$in Algorithm 1 decreases almost exponentially with iteration, hence the worst-case complexity rarely occurs in practice.

Additionally, Table 1 presents the time and space complexity of a single LARS-sGMC iteration. From Table 1, it is evident that the LARS-sGMC iteration is space efficient. However, since $|\mathcal{E}| \leq 2 n$, the time complexity of the LARS-sGMC iteration can be up to $\mathcal{O}\left(n^{3}\right)$, which is not so efficient when $n$ is large. Fortunately, such case is unlikely to happen when we apply Algorithm 1 in practice. See the following remark.

Remark 15 (Computational efficiency of the LARS-sGMC iteration). For the input $(\mathcal{E}, s)=\left(\mathcal{E}_{k}^{\star}, s_{k}^{\star}\right)$ of the LARS-sGMC iteration, $|\mathcal{E}|$ is exactly the $\ell_{0}$ pseudo-norm of $\boldsymbol{w}_{\star}(\lambda)$ within the $k$ th linear piece (cf. Thm. 6 (b)). Notice that we are only interested in sparse solutions of the sGMC model in practice, hence one can expect that Algorithm 1 is only applied in cases where $|\mathcal{E}|<2 n$, i.e., the time complexity of Algorithm 1 should be much lower than $\mathcal{O}\left(n^{3}\right)$ in application.

Moreover, from Thm. 3 (c) in Sec. 4.1, if the sensing matrix $\boldsymbol{A}$ is generated randomly, then $|\mathcal{E}| \leq 2 m$ holds with probability one, in which case the time complexity of Algorithm 1 is at most $\mathcal{O}\left(m^{3}\right)$. Accordingly, the time complexity of Algorithm 1 is acceptable in the standard setting of sparse recovery problems where $m \ll n$.

## 8 Numerical Experiments

This section evaluates the performance of the proposed LARS-sGMC algorithm (Algorithm 1) on synthetic data. The experimental results demonstrate correctness-in-general-situation, efficiency and practical utility of the LARS-sGMC algorithm, and naturally verify some theoretical results about the geometry of the sGMC model. Moreover, some empirical observations introduced in this section may lead to interesting topics for future theoretical research.

All experiments were were performed using MATLAB (R2021a) on a computer with Intel(R) Core(TM) i7-7700HQ CPU $2.80,8 \mathrm{~GB}$ (RAM), under Windows 10, 64Bits.

### 8.1 Correctness of Algorithm 1 in General Situations

In Thm. 8, we have proven correctness of the LARS-sGMC iteration under the "one-at-a-time" assumption (Assumption 1). However, when we apply Algorithm 1 in practice, the situation may deviate from the ideal case considered in Thm. 8 in the sense that:

1) the "one-at-a-time" assumption may possibly break,
2) since the LARS-sGMC algorithm is conducted in an iterative manner, computation error may accumulate with iteration. Notice that $\lambda_{+}$(cf. (43) in Sec. 7.2) in Algorithm 1 is supposed to decrease monotonically with the LARSsGMC iteration, this accumulated error may be non-negligible when $\lambda_{+}$is very small.
Aforementioned issues constitute realistic concerns on the correctness of the LARS-sGMC algorithm in general situations. To examine correctness of the LARS-sGMC algorithm and verify its stability to computation error, we implement the following perturbed version of LARS-sGMC iteration.

Remark 16 (A perturbed LARS-sGMC iteration). Recall that in line 19-21 of Algorithm 1:

$$
\begin{aligned}
& \lambda_{+} \leftarrow \max \left\{0, \max _{i \in \mathcal{E}} l_{i}^{a}, \max _{i \in \mathcal{E}} l_{i}^{b}\right\}, \\
& \mathcal{I}_{\text {delete }}^{+} \leftarrow\left\{i \in \mathcal{E} \mid l_{i}^{a}=\lambda_{+}\right\}, \\
& \mathcal{I}_{\text {insert }}^{+} \leftarrow\left\{i \in \neg \mathcal{E} \mid l_{i}^{b}=\lambda_{+}\right\},
\end{aligned}
$$



Fig. 5: Number of inserted and deleted variables in each LARS-sGMC iteration with different levels of perturbation.


Fig. 6: Relative error of $\boldsymbol{w}_{\star}(\lambda)$ generated by the LARS-sGMC algorithm with different levels of perturbation.


Fig. 7: Value of $\lambda_{+} / \lambda_{\max }^{\star}$ versus iteration number.


Fig. 8: Numbers of nonzero components in $\boldsymbol{w}_{\star}\left(\lambda_{+}\right)$and $\boldsymbol{x}_{\star}\left(\lambda_{+}\right)$versus iteration number.
the "one-at-a-time" assumption holds if and only if the maximum in RHS of the expression of $\lambda_{+}$is attained by a unique index $i \in \mathcal{E}$ or $i \in \neg \mathcal{E}$. To facilitate $\left|\mathcal{I}_{\text {delete }}^{+}\right|$and $\left|\mathcal{I}_{\text {insert }}^{+}\right|$to be larger than one (i.e., to break Assumption 1), we replace line 20-21 of Algorithm 1 by the following perturbed version:

$$
\begin{aligned}
& \mathcal{I}_{\text {delete }}^{+} \leftarrow\left\{i \in \mathcal{E} \mid l_{i}^{a} \geq(1-\delta) \lambda_{+}\right\}, \\
& \mathcal{I}_{\text {insert }}^{+} \leftarrow\left\{i \in \neg \mathcal{E} \mid l_{i}^{b} \geq(1-\delta) \lambda_{+}\right\},
\end{aligned}
$$

where $0 \leq \delta<1$ is a number that determines the intensity of the perturbation. Notice that when $\delta=0$, the perturbed LARS-sGMC iteration reduces to Algorithm 1.

The experimental data is generated as follows: the entries of the measured signal $\boldsymbol{y} \in \mathbb{R}^{100}$ and the sensing matrix $\boldsymbol{A} \in \mathbb{R}^{100 \times 500}$ follow i.i.d. standard normal distribution (note that $\boldsymbol{y}$ does not have to be the multiplication of $\boldsymbol{A}$ and some sparse signal), and the shape-controlling parameter $\rho \in[0,1)$ is set as 0.5 . We test the perturbed LARS-sGMC algorithm (Remark 16) on a random instance, where the perturbation parameter $\delta$ is respectively set to $0,0.01$ and 0.05 to yield zero perturbation, mild perturbation and larger perturbation in each iteration. For every value of $\delta$, we start the LARS-sGMC algorithm with initial input $\left(\mathcal{E}_{0}^{\star}, \boldsymbol{s}_{0}^{\star}\right)=\left(\emptyset, \mathbf{0}_{2 n}\right)(\mathrm{cf}$. Remark 12$)$ and repeatedly conduct the perturbed LARS-sGMC iteration until $\lambda_{+} \leq 10^{-6}$ (in which case we regard $\lambda_{+} \approx 0$ and believe that the LARS-sGMC algorithm has computed all linear pieces in $\left.\boldsymbol{w}_{\star}(\lambda)\right)$.

Fig. 5 plots the deleted variable number $\left|\mathcal{I}_{\text {delete }}^{+}\right|$and the inserted variable number $\left|\mathcal{I}_{\text {insert }}^{+}\right|$versus iteration number, where we plot $\left|\mathcal{I}_{\text {insert }}^{+}\right|$as a positive integer and plot $\left|\mathcal{I}_{\text {delete }}^{+}\right|$as a negative one to distinguish them. From Fig. 5, one can verify that the "one-at-a-time" assumption holds empirically if no perturbation is introduced in Algorithm 1, which confirms the mildness of Assumption 1 (cf. Remark 13). When $\delta \neq 0$, Assumption 1 does not hold in general.

Fig. 6 evaluates the quality of the extended solution path $\boldsymbol{w}_{\star}(\lambda)$ generated by LARS-sGMC. We obtain the ground truth of $\boldsymbol{w}_{\star}(\lambda)$ by the forward-backward splitting (FBS) algorithm proposed in [20], which solves the GMC model with a fixed $\lambda$
in a single run. The relative error of $\boldsymbol{w}_{\star}(\lambda)$ is calculated as follows:

$$
\text { Relative error }:=\frac{\left\|\boldsymbol{w}_{\star}^{\mathrm{LARS}}(\lambda)-\boldsymbol{w}_{\star}^{\mathrm{FBS}}(\lambda)\right\|_{2}}{\left\|\boldsymbol{w}_{\star}^{\mathrm{FBS}}(\lambda)\right\|_{2}}
$$

where $\boldsymbol{w}_{\star}^{\mathrm{LARS}}(\lambda)$ and $\boldsymbol{w}_{\star}^{\mathrm{FBS}}(\lambda)$ are respectively the solutions obtained by LARS-sGMC and FBS [20]. We plot the relative error with $\lambda=\frac{i}{100} \lambda_{\text {max }}^{\star}$, where $i=1,2, \ldots, 99$. From Fig. 6 , one can verify that

1) when $\delta=0$, the relative error of $\boldsymbol{w}_{\star}(\lambda)$ generated by the original LARS-sGMC algorithm is always smaller than $10^{-10}$, which confirms that the computation error does not accumulate and is negligible even for very small $\lambda_{+}$,
2) when $\delta=0.01$, the "one-at-a-time" assumption does not hold (cf. Fig. 5), whereas the relative error of $\boldsymbol{w}_{\star}(\lambda)$ is smaller than $10^{-3}$ for most values of $\lambda$, which implies that the correctness of Algorithm 1 is likely to hold in general situations even if Assumption 1 breaks.
3) when $\delta=0.05$, as the value of $\lambda$ decreases from $\lambda_{\text {max }}^{\star}$, the relative error of $\boldsymbol{w}_{\star}(\lambda)$ first exceeds $10^{-1}$, then decreases and keeps smaller than $10^{-1}$, which implies that there exists certain error-correction mechanism in the LARSsGMC iteration such that the computation error will not accumulate with the LARS-sGMC iteration, as long as the computation error is not too large.
Noticeably, when $\delta>0$, it is evident that the sequence of ex-equicorrelation pairs generated by the perturbed LARssGMC algorithm is not the correct solution to the discrete search problem discussed in Sec. 7.1, whereas the extended regularization path $\boldsymbol{w}_{\star}(\lambda)$ generated by it can have very small relative error. This implies that for every $\left(\mathcal{E}_{k}^{\star}, s_{k}^{\star}\right) \in \mathcal{V}_{\star}$, there certainly exists multiple ex-equicorrelation pairs $(\mathcal{E}, \boldsymbol{s}) \in \mathcal{V}$ whose trajectory function $\hat{\boldsymbol{w}}_{\mathrm{EQ}}(\mathcal{E}, \boldsymbol{s}, \lambda)$ approximates $\hat{\boldsymbol{w}}_{\mathrm{EQ}}\left(\mathcal{E}_{k}^{\star}, s_{k}^{\star}, \lambda\right)$ when $\lambda$ is around the selection interval $I_{\mathrm{SEL}}\left(\mathcal{E}_{k}^{\star}, s_{k}^{\star}\right)$. This observation may lead to interesting topics for future research.

Fig. 7 depicts $\lambda_{+} / \lambda_{\max }^{\star}$ versus iteration number. From Fig. 7, when $\delta=0$, the value of $\lambda_{+}$decreases almost exponentially with iteration, which dispels the concern on the worst-case complexity of the LARS-sGMC algorithm (cf. Remark 14). Moreover, the larger the perturbation parameter $\delta$ is, the faster $\lambda_{+}$decreases, and the sooner the perturbed LARs-sGMC algorithm terminates. Hence the perturbation technique introduced in Remark 16 can be viewed as a straightforward approach for accelerating the LARS algorithm. Notice that when $\delta>0, \lambda_{+}$may not decrease monotonically with iteration.

Fig. 8 depicts numbers of nonzero components in $\boldsymbol{w}_{\star}\left(\lambda_{+}\right)$and $\boldsymbol{x}_{\star}\left(\lambda_{+}\right)$versus iteration number. From Fig. 8 , the number of nonzero components in $\boldsymbol{w}_{\star}\left(\lambda_{+}\right)$and that in $\boldsymbol{x}_{\star}\left(\lambda_{+}\right)$respectively do not exceed $200(=2 m)$ and $100(=m)$, where $m$ is the number of rows in the sensing matrix $\boldsymbol{A} \in \mathbb{R}^{100 \times 500}$. This observation verifies our claim on the solution sparseness of the sGMC model (cf. Thm. 1 (c) in Sec. 3.2 and Thm. 3 (c) in Sec. 4.1). Moreover, Fig. 8 indicates that for all $\delta$, the values of the $\ell_{0}$ pseudo-norm of $\boldsymbol{w}_{\star}\left(\lambda_{+}\right)$and $\boldsymbol{x}_{\star}\left(\lambda_{+}\right)$grow sublinearly with iteration, where the larger $\delta$ is, the faster the $\ell_{0}$ pseudo-norm values grow. This again demonstrates that the perturbation technique in Remark 16 can be used to accelerate the LARS algorithm.

### 8.2 Efficiency and Practical Utility of Algorithm 1

Subsequently, we demonstrate the efficiency and practical utility of the LARS-sGMC algorithm in the problem of finding optimal regularization parameter of the sGMC model.

Practitioners usually face two difficulties in parameter tuning of regularization models:

1) How to efficiently obtain solutions of the regularization model with different regularization parameters $\lambda$ ?
2) Given a solution of the regularization model, how to evaluate its quality without knowing the true signal?

In this section, we will show that for the sGMC model, the first difficulty can be resolved by the LARS-sGMC algorithm, whilst the second one can be tackled using proper model selection criterion [42]. In the following remark, we elaborate on how to select the optimal regularization parameter of the sGMC model using an approximate version of Bayesian information criterion (BIC [43], [12], [44]).

Remark 17 (Regularization parameter selection using an approximate BIC). Let $\boldsymbol{A} \times \boldsymbol{y} \times \rho \in \mathbb{R}^{m \times n} \times \mathbb{R}^{m} \times[0,1$ ) be the problem parameters of the sGMC model (6). For $\lambda>0$, let $\boldsymbol{x}_{\star}(\lambda)$ be the minimum $\ell_{2}$-norm solution of the sGMC model with regularization parameter $\lambda$. We evaluate the quality of $\boldsymbol{x}_{\star}(\lambda)$ using the BIC for LASSO proposed in $[12,(2.16)]$ :

$$
B I C\left(\boldsymbol{x}_{\star}(\lambda)\right):=\frac{\left\|\boldsymbol{y}-\boldsymbol{A} \boldsymbol{x}_{\star}(\lambda)\right\|_{2}^{2}}{m \sigma^{2}}+\frac{\log _{e}(m)}{m}\left\|\boldsymbol{x}_{\star}(\lambda)\right\|_{0}
$$

where $\|\cdot\|_{0}$ is the $\ell_{0}$ pseudo-norm, $\sigma^{2}>0$ can be interpreted as the variance of measurement noise on every component of $\boldsymbol{y}$ (cf. [12, the paragraph just before (1.2)]). We define the optimal value of $\lambda$ recommended by BIC as:

$$
\lambda_{B I C}:=\min _{\lambda>0} B I C\left(\boldsymbol{x}_{\star}(\lambda)\right) .
$$

Notice that $\lambda_{\text {BIC }}$ is determined by the hyperparameter $\sigma^{2}$.
We test the LARS-sGMC algorithm (equipped with approximate BIC) on a more realistic setting. The sensing matrix $A \in \mathbb{R}^{500 \times 3670}$ is formed by randomly downsampling and normalizing 3670 images of flowers ${ }^{20}$, where each downsampled
20. url: http:/ /download.tensorflow.org/example_images/flower_photos.tgz. Every color image is transformed into grayscale.


Fig. 9: NMSE versus average computation time for various algorithms.


Fig. 10: NMSE of $\boldsymbol{x}_{\star}\left(\lambda_{\text {BIC }}\right)$ versus the hyperparameter $\sigma^{2}$.
image is vectorized into a $500 \times 1$ vector and set as a column vector of $\boldsymbol{A}$. We generate the measured signal $\boldsymbol{y} \in \mathbb{R}^{500}$ with the following model:

$$
\boldsymbol{y}=\boldsymbol{A} \boldsymbol{x}_{\text {true }}+\boldsymbol{\epsilon}
$$

where $\boldsymbol{x}_{\text {true }} \in \mathbb{R}^{3670}$ is a sparse signal generated by the "sprandn" function in Matlab, $\boldsymbol{\epsilon}$ is the white Gaussian noise with variance $\sigma_{\epsilon}^{2}$. The signal-to-noise ratio (SNR), defined as SNR $:=10 \log _{10}\left(\left\|\boldsymbol{A} \boldsymbol{x}_{\text {true }}\right\|_{2}^{2} /\left(500 \sigma_{\epsilon}^{2}\right)\right)$, is set as 10 dB . We set the number of nonzero components $S$ in $\boldsymbol{x}_{\text {true }}$ respectively as 10, 50 and 250 to yield extremely sparse, moderately sparse and nearly dense ${ }^{21}$ true signals. Our goal is to find the optimal value of $\lambda>0$ such that the minimum $\ell_{2}$-norm solution $\boldsymbol{x}_{\star}(\lambda)$ obtained by LARS-sGMC optimally estimates $\boldsymbol{x}_{\text {true }}$. The shape-controlling parameter $\rho$ of the sGMC model is set as $0,0.25$, $0.5,0.75$ and 0.99 respectively.

We use the LARS-sGMC algorithm with initial input $\left(\mathcal{E}_{0}^{\star}, \boldsymbol{s}_{0}^{\star}\right)=\left(\emptyset, \mathbf{0}_{2 n}\right)$ to compute the regularization path $\boldsymbol{x}_{\star}(\lambda)$ of the sGMC model, and select $\lambda_{\text {BIC }}$ defined in Remark 17 as an estimate of the optimal value of $\lambda$. We vary the hyperparameter $\sigma^{2}$ of BIC from $0.1 \sigma_{\epsilon}^{2}$ to $10 \sigma_{\epsilon}^{2}$ with 21 logarithmically spaced points, and evaluate the performance of the resultant value of $\lambda_{\text {BIC }}$ by the normalized mean squared error (NMSE) of $\boldsymbol{x}_{\star}\left(\lambda_{\text {BIC }}\right)$ :

$$
\text { NMSE }:=20 \log _{10} \frac{\left\|\boldsymbol{x}_{\star}\left(\lambda_{\text {BIC }}\right)-\boldsymbol{x}_{\text {true }}\right\|_{2}}{\left\|\boldsymbol{x}_{\text {true }}\right\|_{2}}
$$

All results were averaged over 100 Monte Carlo trials.
For every problem instance, we compare the efficiency of the LARS-sGMC algorithm with the following algorithms:

1) The conventional LARS-LASSO algorithm [15], [14]: we implement the LARS-LASSO algorithm as an instance of LARS-sGMC by setting $\rho=0$ and discarding operations involving the dual part in the extended solution.
2) The forward-backward splitting method for the GMC model (FBS-GMC [20, Prop. 15]): it is known that FBS-GMC is one of the fastest algorithms for solving the sGMC model with a fixed $\lambda$ (cf. [25, Fig. 3]). In our experiment, we run FBS-GMC with the true optimal value of $\lambda$ that achieves lowest NMSE of $\boldsymbol{x}_{\star}(\lambda)$ (this optimal $\lambda$ is obtained by LARS-sGMC). The stepsize is set as $\gamma:=1.98 \max (1, \rho /(1-\rho))\left\|\boldsymbol{A}^{\top} \boldsymbol{A}\right\|_{2}$.
Fig. 9 plots NMSE of the estimate of $\boldsymbol{x}_{\text {true }}$ in every iteration of aforementioned algorithms versus computation time (for LARS-type algorithms, we compute NMSE of $x_{\star}\left(\lambda_{+}\right)$of the current iteration), where one circle mark represents
[^6]10 iterations of an algorithm. From Fig. 9, one can verify that the time needed for LARS-sGMC to compute the sGMC regularization path from $\lambda_{\max }^{\star}$ to the optimal $\lambda$ is of the same order as the time needed for FBS-GMC to solve the sGMC model with a fixed $\lambda$, which demonstrates the efficiency of LARS-sGMC. Additionally, while the LARS-LASSO algorithm finds the optimal $\lambda$ for the LASSO model faster than LARS-sGMC, it can only produce a suboptimal estimate. Noticeably, when the underlying true signal is sufficiently sparse (e.g., $S=10$ ), the performance gain of the sGMC model with optimal $\rho$ over the LASSO model can be up to around 6 dB , and the time needed for LARS-sGMC to find the optimal $\lambda$ can be almost the same as the convergence time of a single run of FBS-GMC, which demonstrates applicability of LARS-sGMC in sparse recovery problems with highly redundant dictionary matrices.

Fig. 10 plots NMSE of $\boldsymbol{x}_{\star}\left(\lambda_{\text {BIC }}\right)$ versus the hyperparameter $\sigma^{2}$ of BIC. From Fig. 10 , when $\sigma^{2}$ is close to $\sigma_{\epsilon}^{2}$, while the NMSE of $\boldsymbol{x}_{\star}\left(\lambda_{\text {BIC }}\right)$ does not necessarily approach optimum, the estimation performance is usually satisfactory. Hence in practice, the strategy of equipping LARS-sGMC with model selection criterion can at least find a satisfactory $\lambda$ efficiently. Fig. 10 also indicates that the optimal shape-controlling parameter $\rho$ of the sGMC model tends to decrease as the number of nonzero components in $\boldsymbol{x}_{\text {true }}$ increases. This verifies the intuition that larger $\rho$ yields more aggressive debiasing effect in the sGMC penalty, i.e., the sGMC model approximates the $\ell_{0}$-regularization model better (cf. Sec. 2.2).

## 9 Conclusion

In this paper, we presented theoretical results on the solution-set geometry and regularization path of the scaled generalized minimax concave (sGMC) model, demonstrating that while the sGMC penalty is a nonconvex extension of the LASSO penalty, the sGMC model can preserve many celebrated properties of the LASSO model, hence can serve as a less biased surrogate of LASSO.

Specifically, for a fixed regularization parameter $\lambda$, we have shown that the primal and dual sGMC solution sets are both nonempty, closed, bounded, convex sets, and every sGMC solution $\boldsymbol{x}_{\mathrm{p}}$ (resp. dual solution $\boldsymbol{z}_{\mathrm{d}}$ ) gives the same value of $\boldsymbol{A} \boldsymbol{x}_{\mathrm{p}}$ (resp. $\boldsymbol{A} \boldsymbol{z}_{\mathrm{d}}$ ) and the same value of $\left\|\boldsymbol{x}_{\mathrm{p}}\right\|_{1}$ (resp. $\left\|\boldsymbol{z}_{\mathrm{d}}\right\|_{1}$ ). We have derived upper bounds for $\left\|\boldsymbol{x}_{\mathrm{p}}\right\|_{1}$ and $\left\|\boldsymbol{z}_{\mathrm{d}}\right\|_{1}$. Under additional general-positioning assumption on the sensing matrix, we further proved that both the primal solution and dual solution of the sGMC model are almost surely unique and have at most $\min \{m, n\}$ nonzero components. As a side contribution, we have established a connection between the primal-dual sGMC solution sets and the LASSO solution sets.

For a varying $\lambda$, we have shown that the extended sGMC solution set $\mathcal{S}_{\mathrm{e}}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$ is a continuous piecewise polytopevalued mapping of $\lambda$, and the minimum $\ell_{2}$-norm extended sGMC solution $\boldsymbol{w}_{\star}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$ is a continuous piecewise linear function of $\lambda$. Explicit piecewise expressions of $\mathcal{S}_{\mathrm{e}}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$ and $\boldsymbol{w}_{\star}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$ have been derived using the notion of ex-equicorrelation pairs. Moreover, exploiting the theoretical findings, we proposed an iterative algorithm termed LARSsGMC to compute the whole minimum $\ell_{2}$-norm extended regularization path, which extends the conventional LARS algorithm for LASSO. We have proven the correctness and finite termination of LARS-sGMC under the "one-at-a-time" assumption, and have demonstrated its correctness-in-general-situation, efficiency and practical utility through numerical experiments on synthetic data.

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## Appendix A

## Other Useful Facts

Fact 4 ([34, Thm. 25.1]). Let $f: \mathbb{R}^{n} \rightarrow \mathbb{R}$ be a convex function, and let $\boldsymbol{x} \in \mathbb{R}^{n}$ be a point where $f$ is finite. If $f$ has a unique subgradient at $\boldsymbol{x}$, then $f$ is differentiable at $\boldsymbol{x}$.

Fact 5 ([32, Prop. 16.59]). Let $F \in \Gamma_{0}\left(\mathbb{R}^{m} \times \mathbb{R}^{n}\right)$, and set:

$$
f: \mathbb{R}^{m} \rightarrow[-\infty,+\infty]: \boldsymbol{u} \mapsto \inf _{\boldsymbol{z} \in \mathbb{R}^{n}} F(\boldsymbol{u}, \boldsymbol{z}) .
$$

Suppose that $f$ is proper and that $(\overline{\boldsymbol{u}}, \overline{\boldsymbol{z}}) \in \mathbb{R}^{m} \times \mathbb{R}^{n}$ satisfies $f(\overline{\boldsymbol{u}})=F(\overline{\boldsymbol{u}}, \overline{\boldsymbol{z}})$, and let $\boldsymbol{v} \in \mathbb{R}^{m}$. Then $\boldsymbol{v} \in \partial f(\overline{\boldsymbol{u}}) \Longleftrightarrow\left(\boldsymbol{v}, \mathbf{0}_{n}\right) \in$ $\partial F(\overline{\boldsymbol{u}}, \overline{\boldsymbol{z}})$.
Fact 6 ([32, Prop. 9.3]). Let $\left(f_{i}\right)_{i \in I}$ be a family of convex functions from $\mathbb{R}^{n}$ to $[-\infty,+\infty]$. Then $\sup _{i \in I} f_{i}$ is convex.
Fact 7 ([46, Prop. 6.4.12(e)]). Let $\left(a_{k}\right)_{k \in \mathbb{N}}$ be a sequence of real numbers, if its limit superior $L^{+}$(resp. limit inferior $L^{-}$) is finite, then it is a cluster point of $\left(a_{k}\right)_{k \in \mathbb{N}}$, i.e., there exists a subsequence of $\left(a_{k}\right)_{k \in \mathbb{N}}$ convergent to $L^{+}$(resp. $L^{-}$).
Fact 8 ([47, Thm. 2.2]). Given $\boldsymbol{A} \in \mathbb{R}^{m \times n}, \boldsymbol{Q} \in \mathbb{R}^{k \times n}$, for $\boldsymbol{y} \in \mathbb{R}^{m}$ and $\boldsymbol{z} \in \mathbb{R}^{k}$, define $\mathcal{L}_{\boldsymbol{A}, \boldsymbol{Q}}: \mathbb{R}^{m} \times \mathbb{R}^{k} \rightrightarrows \mathbb{R}^{n}$ as:

$$
\mathcal{L}_{\boldsymbol{A}, \boldsymbol{Q}}(\boldsymbol{y}, \boldsymbol{z}):=\left\{\boldsymbol{x} \in \mathbb{R}^{n} \mid \boldsymbol{A} \boldsymbol{x}=\boldsymbol{y}, \boldsymbol{Q} \boldsymbol{x} \leq \boldsymbol{z}\right\} .
$$

Then $\mathcal{L}_{\boldsymbol{A}, \boldsymbol{Q}}(\cdot, \cdot)$ is "Lipschitz continuous" on its domain

$$
\operatorname{dom} \mathcal{L}_{\boldsymbol{A}, \boldsymbol{Q}}:=\left\{(\boldsymbol{y}, \boldsymbol{z}) \in \mathbb{R}^{m} \times \mathbb{R}^{k} \mid \mathcal{L}_{\boldsymbol{A}, \boldsymbol{Q}}(\boldsymbol{y}, \boldsymbol{z}) \neq \emptyset\right\}
$$

More precisely, there exists $\mu_{\mathcal{L}}>0$ and a well-defined norm $\|\cdot\|_{\mathcal{L}}$ on $\mathbb{R}^{m+k}$ satisfying: for every $\left(\boldsymbol{y}_{1}, \boldsymbol{z}_{1}\right),\left(\boldsymbol{y}_{2}, \boldsymbol{z}_{2}\right) \in \operatorname{dom} \mathcal{L}_{\boldsymbol{A}, \boldsymbol{Q}}$, for every $\boldsymbol{x}_{1} \in \mathcal{L}_{\boldsymbol{A}, \boldsymbol{Q}}\left(\boldsymbol{y}_{1}, \boldsymbol{z}_{1}\right)$, there exists an $\boldsymbol{x}_{2} \in \mathcal{L}\left(\boldsymbol{y}_{2}, \boldsymbol{z}_{2}\right)$ closest to $\boldsymbol{x}_{1}$ in the $\ell_{\infty}$-norm such that

$$
\left\|\boldsymbol{x}_{1}-\boldsymbol{x}_{2}\right\|_{\infty} \leq \mu_{\mathcal{L}}\left\|\left[\begin{array}{l}
\boldsymbol{y}_{1}-\boldsymbol{y}_{2} \\
\boldsymbol{z}_{1}-\boldsymbol{z}_{2}
\end{array}\right]\right\|_{\mathcal{L}}
$$

Fact 9 ([29, Example 5.57]). Let $\mathcal{M}: \mathbb{R}^{n} \rightrightarrows \mathbb{R}^{m}$ be continuous relative to $\operatorname{dom} \mathcal{M}:=\left\{\boldsymbol{\theta} \in \mathbb{R}^{n} \mid \mathcal{M}(\boldsymbol{\theta}) \neq \emptyset\right\}$ and convex-valued. For each $\boldsymbol{u} \in \mathbb{R}^{m}$, define $\boldsymbol{s}_{\boldsymbol{u}}: \mathbb{R}^{n} \rightarrow \mathbb{R}^{m}$ by taking $\boldsymbol{s}_{\boldsymbol{u}}(\boldsymbol{\theta})$ to be the projection $P_{\mathcal{M}(\boldsymbol{\theta})}(\boldsymbol{u})$ of $\boldsymbol{u}$ on $\mathcal{M}(\boldsymbol{\theta})$. Then $\boldsymbol{s}_{\boldsymbol{u}}$ is continuous relative to $\operatorname{dom} \mathcal{M}$.

Fact 10 ([48, Thm 2.1]). Consider the following square matrix

$$
\boldsymbol{R}:=\left[\begin{array}{ll}
\boldsymbol{A} & \boldsymbol{B} \\
\boldsymbol{C} & \boldsymbol{D}
\end{array}\right]
$$

where $\boldsymbol{A}$ and $\boldsymbol{D}$ are square matrices, $\boldsymbol{B}$ and $\boldsymbol{C}$ are of compatible dimension. Suppose that $\boldsymbol{A}$ is invertible. Then $\boldsymbol{R}$ is invertible if and only if $\left(\boldsymbol{D}-\boldsymbol{C} \boldsymbol{A}^{-1} \boldsymbol{B}\right)$ is invertible.
Lemma 7. For integers $m>0, r_{1} \geq 0$ and $r_{2} \geq 0$, let $\boldsymbol{P}$ be a block diagonal matrix such that $\boldsymbol{P}:=\operatorname{blkdiag}\left(\boldsymbol{P}_{1}, \boldsymbol{P}_{2}\right)$ with $\boldsymbol{P}_{1} \in \mathbb{R}^{m \times r_{1}}, \boldsymbol{P}_{2} \in \mathbb{R}^{m \times r_{2}}$. Let

$$
\boldsymbol{\Lambda}_{1}:=\left[\begin{array}{ll}
a \boldsymbol{I}_{m} & \\
& d \boldsymbol{I}_{m}
\end{array}\right], \quad \boldsymbol{\Lambda}_{2}:=\left[\begin{array}{ll} 
& b \boldsymbol{I}_{m} \\
c \boldsymbol{I}_{m} &
\end{array}\right]
$$

and $\boldsymbol{\Lambda}:=\boldsymbol{\Lambda}_{1}+\boldsymbol{\Lambda}_{2}$ with $a>0, d>0$ and $b c \leq 0$.
Then we have $\mathcal{R}\left(\boldsymbol{P}^{\top} \boldsymbol{\Lambda} \boldsymbol{P}\right)=\mathcal{R}\left(\boldsymbol{P}^{\top}\right)$.
Proof: To prove the result, we only need to prove

$$
\mathcal{N}\left(\boldsymbol{P}^{\top} \boldsymbol{\Lambda}^{\top} \boldsymbol{P}\right)=\mathcal{N}(\boldsymbol{P})
$$

If $\boldsymbol{P} \boldsymbol{x}=\mathbf{0}$, then we immediately have $\boldsymbol{P}^{\top} \boldsymbol{\Lambda}^{\top} \boldsymbol{P} \boldsymbol{x}=\mathbf{0}$, thus $\mathcal{N}(\boldsymbol{P}) \subset \mathcal{N}\left(\boldsymbol{P}^{\top} \boldsymbol{\Lambda}^{\top} \boldsymbol{P}\right)$. Reversely, if $\boldsymbol{P}^{\top} \boldsymbol{\Lambda}^{\top} \boldsymbol{P} \boldsymbol{x}=\mathbf{0}$, then:

$$
\boldsymbol{P}^{\top}\left(\boldsymbol{\Lambda}_{1}^{\top}+\boldsymbol{\Lambda}_{2}^{\top}\right) \boldsymbol{P} \boldsymbol{x}=\mathbf{0} \quad \Longrightarrow \quad \boldsymbol{P}^{\top} \boldsymbol{\Lambda}_{1}^{\top} \boldsymbol{P} \boldsymbol{x}=-\boldsymbol{P}^{\top} \boldsymbol{\Lambda}_{2}^{\top} \boldsymbol{P} \boldsymbol{x} \quad \Longrightarrow \quad \boldsymbol{P}^{\top} \boldsymbol{P} \boldsymbol{x}=-\boldsymbol{P}^{\top} \boldsymbol{\Lambda}_{1}^{-1} \boldsymbol{\Lambda}_{2}^{\top} \boldsymbol{P} \boldsymbol{x}
$$

multiply both sides of the equation above by $\boldsymbol{P}^{\top \dagger}$ yields

$$
\boldsymbol{P}^{\top \dagger} \boldsymbol{P}^{\top} \boldsymbol{P} \boldsymbol{x}=-\boldsymbol{P}^{\top \dagger} \boldsymbol{P}^{\top} \boldsymbol{\Lambda}_{1}^{-1} \boldsymbol{\Lambda}_{2}^{\top} \boldsymbol{P} \boldsymbol{x} \quad \Longrightarrow \quad \boldsymbol{P} \boldsymbol{P}^{\dagger} \boldsymbol{P} \boldsymbol{x}=-\boldsymbol{P} \boldsymbol{P}^{\dagger} \boldsymbol{\Lambda}_{1}^{-1} \boldsymbol{\Lambda}_{2}^{\top} \boldsymbol{P} \boldsymbol{x} \quad \Longrightarrow \quad \boldsymbol{P} \boldsymbol{x}=-\boldsymbol{P} \boldsymbol{P}^{\dagger} \boldsymbol{\Lambda}_{1}^{-1} \boldsymbol{\Lambda}_{2}^{\top} \boldsymbol{P} \boldsymbol{x}
$$

where the second implication follows from the symmetricity of $\boldsymbol{P} \boldsymbol{P}^{\dagger}$ and the third follows from the definition of $\boldsymbol{P}^{\dagger}$. The equation above further implies

$$
\begin{equation*}
\left(\boldsymbol{I}_{2 m}+\boldsymbol{P} \boldsymbol{P}^{\dagger} \boldsymbol{\Lambda}_{1}^{-1} \boldsymbol{\Lambda}_{2}^{\top}\right) \boldsymbol{P} \boldsymbol{x}=\mathbf{0} \tag{44}
\end{equation*}
$$

If $\left(\boldsymbol{I}_{2 m}+\boldsymbol{P} \boldsymbol{P}^{\dagger} \boldsymbol{\Lambda}_{1}^{-1} \boldsymbol{\Lambda}_{2}^{\top}\right)$ is invertible, then we have $\boldsymbol{P} \boldsymbol{x}=\mathbf{0}$, which implies $\mathcal{N}\left(\boldsymbol{P}^{\top} \boldsymbol{\Lambda}^{\top} \boldsymbol{P}\right) \subset \mathcal{N}(\boldsymbol{P})$ and completes the proof. Next we prove that $\left(\boldsymbol{I}_{2 m}+\boldsymbol{P} \boldsymbol{P}^{\dagger} \boldsymbol{\Lambda}_{1}^{-1} \boldsymbol{\Lambda}_{2}^{\top}\right)$ is invertible.

One can verify that $\left(\boldsymbol{I}_{2 m}+\boldsymbol{P} \boldsymbol{P}^{\dagger} \boldsymbol{\Lambda}_{1}^{-1} \boldsymbol{\Lambda}_{2}^{\top}\right)$ has the following block structure

$$
\left(\boldsymbol{I}_{2 m}+\boldsymbol{P} \boldsymbol{P}^{\dagger} \boldsymbol{\Lambda}_{1}^{-1} \boldsymbol{\Lambda}_{2}^{\top}\right)=\left[\begin{array}{cc}
\boldsymbol{I}_{m} & \frac{c}{a} \boldsymbol{P}_{1} \boldsymbol{P}_{1}^{\dagger} \\
\frac{b}{d} \boldsymbol{P}_{2} \boldsymbol{P}_{2}^{\dagger} & \boldsymbol{I}_{m}
\end{array}\right]
$$

Since $\boldsymbol{I}_{m}$ is invertible, by Fact 10, $\left(\boldsymbol{I}_{2 m}+\boldsymbol{P} \boldsymbol{P}^{\dagger} \boldsymbol{\Lambda}_{1}^{-1} \boldsymbol{\Lambda}_{2}^{\top}\right)$ is invertible if and only if

$$
\boldsymbol{Q}:=\boldsymbol{I}_{m}+\frac{(-b c)}{a d} \boldsymbol{P}_{2} \boldsymbol{P}_{2}^{\dagger} \boldsymbol{P}_{1} \boldsymbol{P}_{1}^{\dagger}
$$

is invertible. We can prove the invertibility of $\boldsymbol{Q}$ by contradiction. Suppose that $Q$ is singular, then there exists a nonzero vector $\boldsymbol{z} \in \mathcal{N}(\boldsymbol{Q})$ such that $\boldsymbol{Q} \boldsymbol{z}=\mathbf{0}$, which yields

$$
\begin{equation*}
\boldsymbol{z}=\frac{b c}{a d} \boldsymbol{P}_{2} \boldsymbol{P}_{2}^{\dagger} \boldsymbol{P}_{1} \boldsymbol{P}_{1}^{\dagger} \boldsymbol{z} \in \mathcal{R}\left(\boldsymbol{P}_{2}\right) \tag{45}
\end{equation*}
$$

Thus we have $\boldsymbol{z}=P_{\mathcal{R}\left(\boldsymbol{P}_{2}\right)}(\boldsymbol{z})=\boldsymbol{P}_{2} \boldsymbol{P}_{2}^{\dagger} \boldsymbol{z}$ from $\boldsymbol{z} \in \mathcal{R}\left(\boldsymbol{P}_{2}\right)$, substituting which into the RHS of (45) yields

$$
\boldsymbol{z}=\frac{b c}{a d} \boldsymbol{P}_{2} \boldsymbol{P}_{2}^{\dagger} \boldsymbol{P}_{1} \boldsymbol{P}_{1}^{\dagger} \boldsymbol{P}_{2} \boldsymbol{P}_{2}^{\dagger} \boldsymbol{z} \quad \Longrightarrow \quad\left(\boldsymbol{I}_{m}+\frac{(-b c)}{a d} \boldsymbol{P}_{2} \boldsymbol{P}_{2}^{\dagger} \boldsymbol{P}_{1} \boldsymbol{P}_{1}^{\dagger} \boldsymbol{P}_{2} \boldsymbol{P}_{2}^{\dagger}\right) \boldsymbol{z}=\mathbf{0}
$$

Since $\boldsymbol{P}_{1} \boldsymbol{P}_{1}^{\dagger}$ and $\boldsymbol{P}_{2} \boldsymbol{P}_{2}^{\dagger}$ are positive semidefinite matrices, the coefficient matrix of the linear equation above is positive definite, hence we have $\boldsymbol{z}=\mathbf{0}$, which leads to a contradiction. Thus $\boldsymbol{Q}$ is invertible, which further implies that $\left(\boldsymbol{I}_{2 m}+\boldsymbol{P} \boldsymbol{P}^{\dagger} \boldsymbol{\Lambda}_{1}^{-1} \boldsymbol{\Lambda}_{2}^{\top}\right)$ is invertible.

Combining the discussion above completes the proof.

## Appendix B <br> Proofs for Section 3

## B. 1 Proof of Lemma 1

Proof: (a) We first prove the differentiability of $\bar{g}$. Define

$$
\bar{G}(\boldsymbol{u}, \boldsymbol{z}):=\lambda\|\boldsymbol{z}\|_{1}+\frac{\rho}{2}\|\boldsymbol{u}-\boldsymbol{A} \boldsymbol{z}\|_{2}^{2},
$$

then one can verify that $\bar{G} \in \Gamma_{0}\left(\mathbb{R}^{m} \times \mathbb{R}^{n}\right)$ and $\bar{g}(\boldsymbol{u})=\min _{\boldsymbol{z} \in \mathbb{R}^{n}} \bar{G}(\boldsymbol{u}, \boldsymbol{z})$. For $\overline{\boldsymbol{u}} \in \mathbb{R}^{m}$, let $\overline{\boldsymbol{z}} \in \mathbb{R}^{n}$ be a global minimizer of $\bar{G}(\overline{\boldsymbol{u}}, \cdot)$, then from Fact 5 in Appendix A, $\boldsymbol{v} \in \partial \bar{g}(\overline{\boldsymbol{u}})$ if and only if $\left(\boldsymbol{v}, \mathbf{0}_{n}\right) \in \partial \bar{G}(\overline{\boldsymbol{u}}, \overline{\boldsymbol{z}})$, i.e.,

$$
\begin{aligned}
& \boldsymbol{v} \in \partial(\bar{G}(\cdot, \overline{\boldsymbol{z}}))(\overline{\boldsymbol{u}})=\{\rho(\overline{\boldsymbol{u}}-\boldsymbol{A} \overline{\boldsymbol{z}})\}, \\
& \mathbf{0}_{n} \in \partial(\bar{G}(\overline{\boldsymbol{u}}, \cdot))(\overline{\boldsymbol{z}}) .
\end{aligned}
$$

The monotone inclusions above are further equivalent to

$$
\begin{gathered}
\boldsymbol{v}=\rho(\overline{\boldsymbol{u}}-\boldsymbol{A} \overline{\boldsymbol{z}}), \\
\overline{\boldsymbol{z}} \in \arg \min _{\boldsymbol{z} \in \mathbb{R}^{n}} \lambda\|\boldsymbol{z}\|_{1}+\frac{1}{2}\|\sqrt{\rho} \overline{\boldsymbol{u}}-\sqrt{\rho} \boldsymbol{A} \boldsymbol{z}\|_{2}^{2},
\end{gathered}
$$

which implies that

$$
\partial \bar{g}(\overline{\boldsymbol{u}})=\{\rho(\overline{\boldsymbol{u}}-\boldsymbol{A} \overline{\boldsymbol{z}}) \mid \overline{\boldsymbol{z}} \text { is a solution of } \operatorname{LASSO}(\sqrt{\rho} \boldsymbol{A}, \sqrt{\rho} \overline{\boldsymbol{u}}, \lambda)\} .
$$

According to Fact 3 in Section 3.1, every solution $\overline{\boldsymbol{z}}$ of $\operatorname{LASSO}(\sqrt{\rho} \boldsymbol{A}, \sqrt{\rho} \overline{\boldsymbol{u}}, \lambda)$ gives the same value of $\boldsymbol{A} \overline{\boldsymbol{z}}$, hence $\partial \bar{g}(\overline{\boldsymbol{u}})$ is single-valued, which implies that $\bar{g}$ is differentiable at $\bar{u}$ from Fact 4 in Appendix A. The arbitrariness of $\bar{u}$ proves the differentiability of $\bar{g}$ on $\mathbb{R}^{m}$.
(b) Define $\bar{h}(\boldsymbol{u}):=\frac{\rho}{2}\|\boldsymbol{y}-\boldsymbol{u}\|_{2}^{2}-\bar{g}(\boldsymbol{u})$. Then we have

$$
\begin{aligned}
\bar{h}(\boldsymbol{u}) & =\frac{\rho}{2}\|\boldsymbol{y}-\boldsymbol{u}\|_{2}^{2}-\min _{z \in \mathbb{R}^{n}}\left(\lambda\|\boldsymbol{z}\|_{1}+\frac{\rho}{2}\|\boldsymbol{u}-\boldsymbol{A} \boldsymbol{z}\|_{2}^{2}\right) \\
& =\max _{\boldsymbol{z} \in \mathbb{R}^{n}} \frac{\rho}{2}\left(\|\boldsymbol{y}-\boldsymbol{u}\|_{2}^{2}-\|\boldsymbol{u}-\boldsymbol{A} \boldsymbol{z}\|_{2}^{2}\right)-\lambda\|\boldsymbol{z}\|_{1} \\
& =\max _{\boldsymbol{z} \in \mathbb{R}^{n}} \rho(\boldsymbol{A} \boldsymbol{z}-\boldsymbol{y})^{\top} \boldsymbol{u}+\frac{\rho}{2}\left(\|\boldsymbol{y}\|_{2}^{2}-\|\boldsymbol{A} \boldsymbol{z}\|_{2}^{2}\right)-\lambda\|\boldsymbol{z}\|_{1} .
\end{aligned}
$$

For given $\boldsymbol{z} \in \mathbb{R}^{n}$, let us define

$$
h_{\boldsymbol{z}}(\boldsymbol{u}):=\rho(\boldsymbol{A} \boldsymbol{z}-\boldsymbol{y})^{\top} \boldsymbol{u}+\frac{\rho}{2}\left(\|\boldsymbol{y}\|_{2}^{2}-\|\boldsymbol{A} \boldsymbol{z}\|_{2}^{2}\right)-\lambda\|\boldsymbol{z}\|_{1},
$$

then one can verify that $h_{\boldsymbol{z}}(\cdot)$ is convex and

$$
\bar{h}(\boldsymbol{u})=\max _{\boldsymbol{z} \in \mathbb{R}^{n}} h_{\boldsymbol{z}}(\boldsymbol{u})
$$

Hence $\bar{h}$ is convex from Fact 6 in Appendix A.
(c) Since $\bar{g}(\boldsymbol{u})=\min _{\boldsymbol{z} \in \mathbb{R}^{n}} \bar{G}(\boldsymbol{u}, \boldsymbol{z})$, we have

$$
\bar{g}(\boldsymbol{u}) \leq \bar{G}\left(\boldsymbol{u}, \mathbf{0}_{n}\right)=\frac{\rho}{2}\|\boldsymbol{u}\|_{2}^{2}
$$

Combining the discussion above completes the proof.

## B. 2 Proof of Theorem 2

Proof: We first prove (26). From Cor. 1 (b), we have

$$
\begin{equation*}
\bar{J}_{\mathrm{sGMC}}\left(\boldsymbol{z}_{\mathrm{d}}\right) \geq \lambda\left\|\boldsymbol{z}_{\mathrm{d}}\right\|_{1}-\frac{1}{2}\|\boldsymbol{y}\|_{2}^{2} . \tag{46}
\end{equation*}
$$

From Cor. 1 (a), we further have

$$
\begin{align*}
\bar{J}_{\mathrm{SGMC}}\left(\boldsymbol{z}_{\mathrm{d}}\right) & =\min _{\boldsymbol{z} \in \mathbb{R}^{n}} \bar{J}_{\mathrm{SGMC}}(\boldsymbol{z}) \leq \bar{J}_{\mathrm{sGMC}}(\mathbf{0}) \\
& =-\min _{\boldsymbol{x} \in \mathbb{R}^{n}}\left[\frac{1}{2}\|\boldsymbol{y}-\boldsymbol{A} \boldsymbol{x}\|_{2}^{2}-\frac{\rho}{2}\|\boldsymbol{A} \boldsymbol{x}\|_{2}^{2}+\lambda\|\boldsymbol{x}\|_{1}\right] \\
& \leq-\min _{\boldsymbol{x} \in \mathbb{R}^{n}}\left[\frac{1}{2}\|\boldsymbol{y}-\boldsymbol{A} \boldsymbol{x}\|_{2}^{2}-\frac{\rho}{2}\|\boldsymbol{A} \boldsymbol{x}\|_{2}^{2}\right] \\
& \leq-\min _{\boldsymbol{u} \in \mathbb{R}^{m}}\left[\frac{1}{2}\|\boldsymbol{y}-\boldsymbol{u}\|_{2}^{2}-\frac{\rho}{2}\|\boldsymbol{u}\|_{2}^{2}\right]  \tag{47}\\
& =\frac{\rho}{2(1-\rho)}\|\boldsymbol{y}\|_{2}^{2}, \tag{48}
\end{align*}
$$

where (48) is obtained by solving the quadratic program in (47) in closed-form. Combining (46) and (48) yields (26).
Next we prove the results (a) and (b). From Prop. 2 (b) and Thm. 1 (a), we have

$$
\begin{equation*}
\mathcal{S}_{\mathrm{d}}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)=\mathcal{S}_{\mathrm{LA}}\left(\sqrt{\rho} \boldsymbol{A}, \sqrt{\rho} \boldsymbol{\beta}_{\mathrm{p}}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda), \lambda\right) \tag{49}
\end{equation*}
$$

where $\mathcal{S}_{\mathrm{LA}}(\cdot)$ is the LASSO solution set mapping defined in Fact $3, \boldsymbol{\beta}_{\mathrm{p}}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$ is the common linear fit of all primal sGMC solutions defined in Thm. 1 (a).

Applying Fact 3 in Sec. 3.1 to (49), one can verify that

- $\mathcal{S}_{\mathrm{d}}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$ is nonempty, closed, bounded and convex,
- the results (a) and (b) hold for $\rho \in(0,1)$.

When $\rho=0, \sqrt{\rho} \boldsymbol{A}=\mathbf{O}_{m \times n}$ does not have columns in general position even for $\boldsymbol{A} \in \mathcal{G}_{m \times n}$. However, in this case, from (23) in Prop. 2, $\boldsymbol{z}_{\mathrm{d}}=\mathbf{0}_{n}$ is evidently the unique dual solution, hence the results (a) and (b) still hold.

Combining the discussion above completes the proof.

## Appendix C

## Proofs for Section 4

## C. 1 Proof of Theorem 4

Proof: Since $\mathcal{S}_{\mathrm{e}}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$ is nonempty, closed and convex from Thm. 3 (a), the projection of $\mathbf{0}$ onto $\mathcal{S}_{\mathrm{e}}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$ exists and is unique from [32, Thm. 3.16], which guarantees the existence and uniqueness of $\boldsymbol{w}_{\star}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$.
(a) Since the primal sGMC solution set $\mathcal{S}_{\mathrm{p}}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$ (resp. the dual sGMC solution set $\mathcal{S}_{\mathrm{d}}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$ ) is nonempty, closed and convex from Thm. 1 (resp. Thm. 2), there exists uniquely a minimum $\ell_{2}$-norm element in $\mathcal{S}_{\mathrm{p}}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$ (resp. $\left.\mathcal{S}_{\mathrm{d}}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)\right)$ from [32, Thm. 3.16].

From (30) in Thm. 3 (a), $\mathcal{S}_{\mathrm{e}}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$ is the Cartesian product of $\mathcal{S}_{\mathrm{p}}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$ and $\mathcal{S}_{\mathrm{d}}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$, hence if $\boldsymbol{w} \boldsymbol{w}_{\star}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$ achieves minimum $\ell_{2}$-norm in $\mathcal{S}_{\mathrm{e}}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$, its first half part and second half part must achieve minimum $\ell_{2}$-norm respectively in $\mathcal{S}_{\mathrm{p}}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$ and $\mathcal{S}_{\mathrm{d}}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$. Accordingly, the result (a) holds.
(b) From Def. 2 in Sec. 4.1, $\mathbf{0}_{2 n} \in \mathcal{S}_{\mathrm{e}}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$ means that " $\mathbf{0}_{2 n}$ satisfies (OPT)". Substituting $\boldsymbol{w}_{\mathrm{e}}:=\mathbf{0}_{2 n}$ into both sides of (OPT) yields

$$
(\forall i \in\{1,2, \ldots, 2 n\}) \quad \xi_{i}\left(\mathbf{0}_{2 n}\right):=\boldsymbol{c}_{i}^{\top} \boldsymbol{b} \in \begin{cases}\{0\}, & \text { if } 0 \neq 0 \\ {[-\lambda, \lambda],} & \text { if } 0=0\end{cases}
$$

which naturally reduces to

$$
(\forall i \in\{1,2, \ldots, 2 n\}) \quad \xi_{i}\left(\mathbf{0}_{2 n}\right):=\boldsymbol{c}_{i}^{\top} \boldsymbol{b} \in[-\lambda, \lambda] .
$$

Recall that $\boldsymbol{b}:=\left[\begin{array}{ll}\boldsymbol{y}^{\top} & \mathbf{0}_{m}{ }^{\top}\end{array}\right]^{\top}, \boldsymbol{C}:=\operatorname{blkdiag}(\boldsymbol{A}, \sqrt{\rho} \boldsymbol{A})$, the inclusion above is equivalent to

$$
\begin{array}{ll}
(\forall i \in\{1,2, \ldots, n\}) & \xi_{i}\left(\mathbf{0}_{2 n}\right)=\boldsymbol{a}_{i}^{\top} \boldsymbol{y} \in[-\lambda, \lambda], \\
(\forall i \in\{n+1, \ldots, 2 n\}) & \xi_{i}\left(\mathbf{0}_{2 n}\right)=0 \in[-\lambda, \lambda] .
\end{array}
$$

Hence one can verify that if $\lambda \geq \lambda_{\text {max }}^{\star}$, then $\mathbf{0}_{2 n}$ satisfies (OPT), hence $\mathbf{0}_{2 n} \in \mathcal{S}_{\mathrm{e}}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$, which definitely achieves minimum $\ell_{2}$-norm in $\mathcal{S}_{\mathrm{e}}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$, thus $\boldsymbol{w}_{\star}(\lambda)=\mathbf{0}_{2 n}$.

## C. 2 Proof of Lemma 3

Proof: For $\overline{\boldsymbol{\theta}}:=(\overline{\boldsymbol{\zeta}}, \overline{\boldsymbol{\eta}}) \in \operatorname{dom} \mathcal{L}_{\boldsymbol{A}, \boldsymbol{Q}}$, let $\left(\boldsymbol{\theta}_{k}\right)_{k \in \mathbb{N}}:=\left(\boldsymbol{\zeta}_{k}, \boldsymbol{\eta}_{k}\right)_{k \in \mathbb{N}} \subset \operatorname{dom} \mathcal{L}_{\boldsymbol{A}, \boldsymbol{Q}}$ be a sequence convergent to $\overline{\boldsymbol{\theta}}$.
We first prove the outer semicontinuity of $\mathcal{L}_{\boldsymbol{A}, \boldsymbol{Q}}(\cdot, \cdot)$. Let $\overline{\boldsymbol{x}}$ be an arbitrary point in $\limsup _{k} \mathcal{L}_{\boldsymbol{A}, \boldsymbol{Q}}\left(\boldsymbol{\theta}_{k}\right)$, then from Def. 3, $\overline{\boldsymbol{x}}$ is a cluster point of some sequence $\left(\boldsymbol{x}_{k}\right)_{k \in \mathbb{N}}$ satisfying $\boldsymbol{x}_{k} \in \mathcal{L}_{\boldsymbol{A}, \boldsymbol{Q}}\left(\boldsymbol{\theta}_{k}\right)$. Hence there exists a subsequence of $\left(\boldsymbol{x}_{k}\right)_{k \in \mathbb{N}}$, say $\left(\boldsymbol{x}_{k_{i}}\right)_{i \in \mathbb{N}}$, that converges to $\overline{\boldsymbol{x}}$. Since $\boldsymbol{x}_{k_{i}} \in \mathcal{L}_{\boldsymbol{A}, \boldsymbol{Q}}\left(\boldsymbol{\theta}_{k_{i}}\right)$, we have $\boldsymbol{A} \boldsymbol{x}_{k_{i}}=\boldsymbol{\zeta}_{k_{i}}$ and $\boldsymbol{Q} \boldsymbol{x}_{k_{i}} \leq \boldsymbol{\eta}_{k_{i}}$. Taking the limit $i \rightarrow \infty$ on these two inequations yields $\boldsymbol{A} \overline{\boldsymbol{x}}=\overline{\boldsymbol{\zeta}}$ and $\boldsymbol{Q} \overline{\boldsymbol{x}} \leq \overline{\boldsymbol{\eta}}$, which implies that $\overline{\boldsymbol{x}} \in \mathcal{L}_{\boldsymbol{A}, \boldsymbol{Q}}(\overline{\boldsymbol{\theta}})$. Thus we have $\lim \sup _{k} \mathcal{L}_{\boldsymbol{A}, \boldsymbol{Q}}\left(\boldsymbol{\theta}_{k}\right) \subset \mathcal{L}_{\boldsymbol{A}, \boldsymbol{Q}}(\overline{\boldsymbol{\theta}})$. From Def. 4, we conclude that $\mathcal{L}_{\boldsymbol{A}, \boldsymbol{Q}}(\cdot, \cdot)$ is osc relative to dom $\mathcal{L}_{\boldsymbol{A}, \boldsymbol{Q}}$.

Next we prove the inner semicontinuity of $\mathcal{L}_{\boldsymbol{A}, \boldsymbol{Q}}(\cdot, \cdot)$. Let $\overline{\boldsymbol{x}}$ be an arbitrary point in $\mathcal{L}_{\boldsymbol{A}, \boldsymbol{Q}}(\overline{\boldsymbol{\theta}})$. Then by Fact 8 in Appendix A, for every $k$, we can find $\boldsymbol{x}_{k} \in \mathcal{L}_{\boldsymbol{A}, \boldsymbol{Q}}\left(\boldsymbol{\theta}_{k}\right)$ such that

$$
\left\|\overline{\boldsymbol{x}}-\boldsymbol{x}_{k}\right\|_{\infty} \leq \mu_{\mathcal{L}}\left\|\left[\begin{array}{l}
\overline{\boldsymbol{\zeta}}-\boldsymbol{\zeta}_{k} \\
\overline{\boldsymbol{\eta}}-\boldsymbol{\eta}_{k}
\end{array}\right]\right\|_{\mathcal{L}}
$$

Taking the limit $k \rightarrow \infty$ on the inequation above, one can verify that $\lim _{k \rightarrow \infty} \boldsymbol{x}_{k}=\overline{\boldsymbol{x}}$. Hence from Def. 3, we have $\overline{\boldsymbol{x}} \in \liminf _{k} \mathcal{L}_{\boldsymbol{A}, \boldsymbol{Q}}(\overline{\boldsymbol{\theta}})$, which further implies that $\mathcal{L}_{\boldsymbol{A}, \boldsymbol{Q}}(\overline{\boldsymbol{\theta}}) \subset \liminf _{k} \mathcal{L}_{\boldsymbol{A}, \boldsymbol{Q}}\left(\boldsymbol{\theta}_{k}\right)$. From Def. 4, we conclude that $\mathcal{L}_{\boldsymbol{A}, \boldsymbol{Q}}(\cdot, \cdot)$ is isc relative to $\operatorname{dom} \mathcal{L}_{\boldsymbol{A}, \boldsymbol{Q}}$.

## C. 3 Proof of Proposition 4

Proof: (a) From Thm. 3 (b), we have

$$
(\forall k \in \mathbb{N}) \quad\left\|\boldsymbol{w}_{k}\right\|_{1} \leq \frac{1}{\lambda_{k}\left(1-\rho_{k}\right)}\left\|\boldsymbol{y}_{k}\right\|_{2}^{2}
$$

Since $\left(\boldsymbol{A}_{k}, \boldsymbol{y}_{k}, \rho_{k}, \lambda_{k}\right)$ converges to $(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$, for large enough $k$ we have

$$
\left\|\boldsymbol{w}_{k}\right\|_{1} \leq \frac{1}{\lambda(1-\rho)}\|\boldsymbol{y}\|_{2}^{2}+1
$$

which verifies the boundedness of $\left(\boldsymbol{w}_{k}\right)_{k \in \mathbb{N}}$.
(b) The boundedness of $\left(\boldsymbol{w}_{k}\right)_{k \in \mathbb{N}}$ guarantees the existence of a cluster point $\boldsymbol{w}_{\infty}$. Hence by passing to a subsequence, we can assume $\lim _{k \rightarrow \infty} \boldsymbol{w}_{k}=\boldsymbol{w}_{\infty}$ without loss of generality. From (OPT), for every $k$ and $i \in\{1, \ldots, 2 n\}$, we have

$$
\begin{equation*}
\boldsymbol{c}_{k, i}\left(\boldsymbol{b}_{k}-\boldsymbol{D}_{k} \boldsymbol{C}_{k} \boldsymbol{w}_{k}\right) \in\left[-\lambda_{k}, \lambda_{k}\right] \tag{50}
\end{equation*}
$$

where $\left(\boldsymbol{b}_{k}, \boldsymbol{D}_{k}, \boldsymbol{C}_{k}\right)$ are the counterpart parts of $(\boldsymbol{b}, \boldsymbol{D}, \boldsymbol{C})$ in (OPT) for $\operatorname{sGMC}\left(\boldsymbol{A}_{k}, \boldsymbol{y}_{k}, \rho_{k}, \lambda_{k}\right)$, and $\boldsymbol{c}_{k, i}$ is the $i$ th column vector of $\boldsymbol{C}_{k}$. Taking $k \rightarrow \infty$ on both sides of (50) yields

$$
\begin{equation*}
\boldsymbol{c}_{i}^{\top}\left(\boldsymbol{b}-\boldsymbol{D C} \boldsymbol{w}_{\infty}\right) \in[-\lambda, \lambda] \tag{51}
\end{equation*}
$$

For every $i \in\{1, \ldots, 2 n\}$ such that $\left[\boldsymbol{w}_{\infty}\right]_{i} \neq 0$, we have $\operatorname{sign}\left(\left[\boldsymbol{w}_{k}\right]_{i}\right)=\operatorname{sign}\left(\left[\boldsymbol{w}_{\infty}\right]_{i}\right)$ for large enough $k$, hence

$$
\boldsymbol{c}_{k, i}^{\top}\left(\boldsymbol{b}_{k}-\boldsymbol{D}_{k} \boldsymbol{C}_{k} \boldsymbol{w}_{k}\right)=\lambda_{k} \operatorname{sign}\left(\left[\boldsymbol{w}_{k}\right]_{i}\right)=\lambda_{k} \operatorname{sign}\left(\left[\boldsymbol{w}_{\infty}\right]_{i}\right)
$$

from (OPT). Taking $k \rightarrow \infty$ on both sides of the equation above yields $\boldsymbol{c}_{i}^{\top}\left(\boldsymbol{b}-\boldsymbol{D} \boldsymbol{C} \boldsymbol{w}_{\infty}\right)=\lambda \operatorname{sign}\left(\left[\boldsymbol{w}_{\infty}\right]_{i}\right)$. Combining this with (51), one can verify that $\boldsymbol{w}_{\infty}$ satisfies (OPT) for $\operatorname{sGMC}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$, hence $\boldsymbol{w}_{\infty} \in \mathcal{S}_{\mathrm{e}}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$.

## C. 4 Proof of Corollary 2

Proof: For $(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda) \in \mathscr{P}$, let $\left(\boldsymbol{A}_{k}, \boldsymbol{y}_{k}, \rho_{k}, \lambda_{k}\right)_{k \in \mathbb{N}}$ be an arbitrary sequence in $\mathscr{P}$ convergent to $(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$. Note that for every $k \in \mathbb{N}$, we can find some $\boldsymbol{w}_{k} \in \mathcal{S}_{\mathrm{e}}\left(\boldsymbol{A}_{k}, \boldsymbol{y}_{k}, \rho_{k}, \lambda_{k}\right)$, hence from Thm. 3 (b) we have

$$
\begin{aligned}
\boldsymbol{\beta}_{k} & :=\boldsymbol{\beta}_{\mathrm{e}}\left(\boldsymbol{A}_{k}, \boldsymbol{y}_{k}, \rho_{k}, \lambda_{k}\right)=\boldsymbol{C}_{k} \boldsymbol{w}_{k} \\
\gamma_{k} & :=\gamma_{\mathrm{e}}\left(\boldsymbol{A}_{k}, \boldsymbol{y}_{k}, \rho_{k}, \lambda_{k}\right)=\left\|\boldsymbol{w}_{k}\right\|_{1} .
\end{aligned}
$$

where $\boldsymbol{C}_{k}:=\operatorname{blkdiag}\left(\boldsymbol{A}_{k}, \boldsymbol{A}_{k}\right)$. We further define

$$
\begin{aligned}
\boldsymbol{\beta}_{\infty} & :=\boldsymbol{\beta}_{\mathrm{e}}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda), & \gamma_{\infty} & :=\gamma_{\mathrm{e}}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda) \\
a_{k} & :=\left\|\boldsymbol{\beta}_{k}-\boldsymbol{\beta}_{\infty}\right\|_{2}, & b_{k} & :=\left|\gamma_{k}-\gamma_{\infty}\right|
\end{aligned}
$$

Then to prove Cor. 2, we only need to prove

$$
\lim _{k \rightarrow \infty} a_{k}=\lim _{k \rightarrow \infty} b_{k}=0
$$

Since $a_{k} \geq 0, b_{k} \geq 0$, it is sufficient to prove

$$
\limsup _{k \rightarrow \infty} a_{k}=\limsup _{k \rightarrow \infty} b_{k}=0
$$

From Fact 7, there exists a subsequence $\left(a_{k_{i}}\right)_{i \in \mathbb{N}}$ of $\left(a_{k}\right)_{k \in \mathbb{N}}$ such that $\lim _{i \rightarrow \infty} a_{k_{i}}=\limsup \sup _{k \rightarrow \infty} a_{k}$. Since $\left(\boldsymbol{w}_{k_{i}}\right)_{i \in \mathbb{N}}$ is bounded from Prop. 4 (a), by further passing to a subsequence (if necessary), we can assume that $\left(\boldsymbol{w}_{k_{i}}\right)_{i \in \mathbb{N}}$ is convergent and $\lim _{i \rightarrow \infty} \boldsymbol{w}_{k_{i}}=\boldsymbol{w}_{a, \infty}$. Then from Prop. 4 (b) we have $\boldsymbol{w}_{a, \infty} \in \mathcal{S}_{\mathrm{e}}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$, which implies $\boldsymbol{\beta}_{\infty}=\boldsymbol{C} \boldsymbol{w}_{a, \infty}$ with $\boldsymbol{C}:=\operatorname{blkdiag}(\boldsymbol{A}, \boldsymbol{A})$. Accordingly, we have

$$
\limsup _{k \rightarrow \infty} a_{k}=\lim _{i \rightarrow \infty} a_{k_{i}}=\lim _{i \rightarrow \infty}\left\|\boldsymbol{C}_{k_{i}} \boldsymbol{w}_{k_{i}}-\boldsymbol{C} \boldsymbol{w}_{a, \infty}\right\|_{2}=0
$$

Similarly, there exists a subsequence $\left(b_{k_{j}}\right)_{j \in \mathbb{N}}$ of $\left(b_{k}\right)_{k \in \mathbb{N}}$ such that $\lim _{j \rightarrow \infty} b_{k_{j}}=\limsup _{k \rightarrow \infty} b_{k}$. By further passing to a subsequence (if necessary), we can assume that $\lim _{j \rightarrow \infty} \boldsymbol{w}_{k_{j}}=\boldsymbol{w}_{b, \infty}$ for some $\boldsymbol{w}_{b, \infty} \in \mathcal{S}_{\mathrm{e}}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)$. Hence we have $\gamma_{\infty}=\left\|\boldsymbol{w}_{b, \infty}\right\|_{1}$ and

$$
\limsup _{k \rightarrow \infty} b_{k}=\lim _{j \rightarrow \infty} b_{k_{j}}=\lim _{k \rightarrow \infty}\left|\left\|\boldsymbol{w}_{k_{j}}\right\|_{1}-\left\|\boldsymbol{w}_{b, \infty}\right\|_{1}\right|=0
$$

Combining the discussion above completes the proof.

## C. 5 Proof of Lemma 4

Proof: For $(\boldsymbol{y}, \lambda) \in \mathbb{R}^{m} \times \mathbb{R}_{++}$, let $\boldsymbol{x}_{\mathrm{p}} \in \mathcal{S}_{\mathrm{LA}}(\boldsymbol{A}, \boldsymbol{y}, \lambda)$, then $\boldsymbol{x} \in \mathbb{R}^{n}$ is in $\mathcal{S}_{\mathrm{LA}}(\boldsymbol{A}, \boldsymbol{y}, \lambda)$ if and only if

$$
\frac{1}{2}\|\boldsymbol{y}-\boldsymbol{A} \boldsymbol{x}\|_{2}^{2}+\lambda\|\boldsymbol{x}\|_{1} \leq \frac{1}{2}\left\|\boldsymbol{y}-\boldsymbol{A} \boldsymbol{x}_{\mathrm{p}}\right\|_{2}^{2}+\lambda\left\|\boldsymbol{x}_{\mathrm{p}}\right\|_{1}
$$

From Fact 3 (a) in Sec. 3.1, the optimality condition above is further equivalent to

$$
\begin{equation*}
\boldsymbol{A} \boldsymbol{x}=\boldsymbol{\beta}_{\mathrm{LA}}(\boldsymbol{A}, \boldsymbol{y}, \lambda),\|\boldsymbol{x}\|_{1} \leq \gamma_{\mathrm{LA}}(\boldsymbol{A}, \boldsymbol{y}, \lambda) \tag{52}
\end{equation*}
$$

where $\boldsymbol{\beta}_{\mathrm{LA}}(\cdot)$ and $\gamma_{\mathrm{LA}}(\cdot)$ are respectively the mappings of the linear fit value of LASSO solutions and the $\ell_{1}$-norm value of LASSO solutions. Notice that the $\ell_{1}$-norm can be rewritten as

$$
\begin{equation*}
\|\boldsymbol{x}\|_{1}=\sum_{i=1}^{n} \operatorname{sign}\left([\boldsymbol{x}]_{i}\right)[\boldsymbol{x}]_{i}=\max _{\boldsymbol{q} \in\{ \pm 1\}^{n}} \boldsymbol{q}^{\top} \boldsymbol{x} \tag{53}
\end{equation*}
$$

by substituting (53) into (52), one can verify that $\boldsymbol{x} \in \mathcal{S}_{\mathrm{LA}}(\boldsymbol{A}, \boldsymbol{y}, \lambda)$ is equivalent to

$$
\begin{equation*}
\boldsymbol{A} \boldsymbol{x}=\boldsymbol{\beta}_{\mathrm{LA}}(\boldsymbol{A}, \boldsymbol{y}, \lambda), \boldsymbol{Q} \boldsymbol{x} \leq \gamma_{\mathrm{LA}}(\boldsymbol{A}, \boldsymbol{y}, \lambda) \mathbf{1}_{2^{n}} \tag{54}
\end{equation*}
$$

where $\boldsymbol{Q} \in \mathbb{R}^{2^{n} \times n}$ is the matrix with its rows being all possible vectors in $\{ \pm 1\}^{n}$. Define

$$
\begin{aligned}
T & : \mathbb{R}^{m} \times \mathbb{R}_{++} \rightarrow \mathbb{R}^{m} \times \mathbb{R}^{2^{n}}:(\boldsymbol{y}, \lambda) \mapsto\left(\boldsymbol{\beta}_{\mathrm{LA}}(\boldsymbol{A}, \boldsymbol{y}, \lambda), \gamma_{\mathrm{LA}}(\boldsymbol{A}, \boldsymbol{y}, \lambda) \mathbf{1}_{2^{n}}\right) \\
\mathcal{M} & : \operatorname{dom} \mathcal{L}_{\boldsymbol{A}, \boldsymbol{Q}} \rightrightarrows \mathbb{R}^{n}:(\boldsymbol{\zeta}, \boldsymbol{\eta}) \mapsto \mathcal{L}_{\boldsymbol{A}, \boldsymbol{Q}}(\boldsymbol{\zeta}, \boldsymbol{\eta})
\end{aligned}
$$

where $\mathcal{L}_{\boldsymbol{A}, \boldsymbol{Q}}$ is defined as in Lemma 3. Then from (54), for every $(\boldsymbol{y}, \lambda) \in \mathbb{R}^{m} \times \mathbb{R}_{++}$, we have $\mathcal{S}_{\mathrm{LA}}(\boldsymbol{A}, \boldsymbol{y}, \lambda)=\mathcal{M}(T(\boldsymbol{y}, \lambda))$, thus $\mathcal{S}_{\mathrm{LA}}(\boldsymbol{A}, \cdot, \cdot)=\mathcal{M} \circ T(\cdot, \cdot)$.

Moreover, we can derive the following:
(a) from Remark 3 (b), $T$ is continuous relative to $\mathbb{R}^{m} \times \mathbb{R}_{++}$,
(b) for every $(\boldsymbol{y}, \lambda) \in \mathbb{R}^{m} \times \mathbb{R}_{++}$, from the nonemptiness of $\mathcal{S}_{\mathrm{LA}}(\boldsymbol{A}, \boldsymbol{y}, \lambda)$ (cf. Fact 3), we have $T(\boldsymbol{y}, \lambda) \in \operatorname{dom} \mathcal{L}_{\boldsymbol{A}, \boldsymbol{Q}}$,
(c) from Lemma $3, \mathcal{S}$ is continuous relative to $\operatorname{dom} \mathcal{L}_{\boldsymbol{A}, \boldsymbol{Q}}$.

Hence from Lemma 2, $\mathcal{M} \circ T$ is continuous relative to $\mathbb{R}^{m} \times \mathbb{R}_{++}$, which implies the continuity of $\mathcal{S}_{\mathrm{LA}}(\boldsymbol{A}, \cdot, \cdot)$ relative to $\mathbb{R}^{m} \times \mathbb{R}_{++}$.

## C. 6 Proof of Theorem 5

Proof: From (29) in Thm. 3, for every $\lambda \in \mathbb{R}_{++}$we have

$$
\mathcal{S}_{\mathrm{e}}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)=\mathcal{S}_{\mathrm{LA}}\left(\boldsymbol{A}, \frac{\boldsymbol{y}-\rho \boldsymbol{\beta}_{\mathrm{d}}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)}{(1-\rho)}, \lambda\right) \times \mathcal{S}_{\mathrm{LA}}\left(\sqrt{\rho} \boldsymbol{A}, \sqrt{\rho} \boldsymbol{\beta}_{\mathrm{p}}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda), \lambda\right)
$$

where $\mathcal{S}_{\mathrm{LA}}(\cdot)$ is the LASSO solution set mapping (cf. Fact 3 ), $\boldsymbol{\beta}_{\mathrm{p}}(\cdot)$ and $\boldsymbol{\beta}_{\mathrm{d}}(\cdot)$ are respectively the mappings of the common linear fit of all primal sGMC solutions and that of all dual solutions (cf. Thm. 1 and 2). Define the following mappings:

$$
\begin{aligned}
& T_{1}: \mathbb{R}_{++} \rightarrow \mathbb{R}^{m} \times \mathbb{R}_{++}: \lambda \mapsto\left(\frac{\boldsymbol{y}-\rho \boldsymbol{\beta}_{\mathrm{d}}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)}{(1-\rho)}, \lambda\right) \\
& T_{2}: \mathbb{R}_{++} \rightarrow \mathbb{R}^{m} \times \mathbb{R}_{++}: \lambda \mapsto\left(\sqrt{\rho} \boldsymbol{\beta}_{\mathrm{p}}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda), \lambda\right), \\
& \mathcal{M}_{1}: \mathbb{R}^{m} \times \mathbb{R}_{++} \rightrightarrows \mathbb{R}^{n}:(\boldsymbol{\zeta}, \eta) \mapsto \mathcal{S}_{\mathrm{LA}}(\boldsymbol{A}, \boldsymbol{\zeta}, \eta), \\
& \mathcal{M}_{2}: \mathbb{R}^{m} \times \mathbb{R}_{++} \rightrightarrows \mathbb{R}^{n}:(\boldsymbol{\zeta}, \eta) \mapsto \mathcal{S}_{\mathrm{LA}}(\sqrt{\rho} \boldsymbol{A}, \boldsymbol{\zeta}, \eta)
\end{aligned}
$$

Then one can verify that

$$
\begin{equation*}
\mathcal{S}_{\mathrm{e}}(\boldsymbol{A}, \boldsymbol{y}, \rho, \lambda)=\mathcal{M}_{1}\left(T_{1}(\lambda)\right) \times \mathcal{M}_{2}\left(T_{2}(\lambda)\right) \tag{55}
\end{equation*}
$$

Moreover, we can derive the following:
(a) from Cor. 2 and (31) in Thm. 3, $T_{1}$ and $T_{2}$ are both continuous relative to $\mathbb{R}_{++}$,
(b) from the definition of $T_{1}$ and $T_{2}$, for every $\lambda \in \mathbb{R}_{++}, T_{1}(\lambda), T_{2}(\lambda) \in \mathbb{R}^{m} \times \mathbb{R}_{++}$,
(c) from Lemma $4, \mathcal{M}_{1}$ and $\mathcal{M}_{2}$ are both continuous relative to $\mathbb{R}^{m} \times \mathbb{R}_{++}$.

Thus from Lemma $2, \mathcal{M}_{1} \circ T_{1}(\cdot)$ and $\mathcal{M}_{2} \circ T_{2}(\cdot)$ are both continuous relative to $\mathbb{R}_{++}$, combining which with (55) yields the continuity of $\mathcal{S}_{\mathrm{e}}(\boldsymbol{A}, \boldsymbol{y}, \rho, \cdot)$ relative to $\mathbb{R}_{++}$.

## Appendix D

## Proofs for Section 5

## D. 1 Proof of Proposition 6

Proof: From the definition of $\preceq$, one can verify that $\preceq$ satisfies reflexivity, transitivity and antisymmetry, hence $\preceq$ is a partial order on $\overline{\mathcal{V}}$ (cf. [32, Sec. 1.3]).
(a) Suppose we have $\left(\mathcal{E}_{1}, s_{1}\right) \preceq\left(\mathcal{E}_{2}, s_{2}\right)$ with $\left(\mathcal{E}_{2}, s_{2}\right) \in \mathcal{V}$. Then from the definition of $\mathcal{V}$ in Prop. 5, we have $\left[s_{2}\right]_{\mathcal{E}_{2}} \in$ $\mathcal{R}\left(\boldsymbol{C}_{\mathcal{E}_{2}}{ }^{\top}\right)$, hence there exists $\boldsymbol{z} \in \mathbb{R}^{2 m}$ such that

$$
\left[s_{2}\right]_{\mathcal{E}_{2}}=\boldsymbol{C}_{\mathcal{E}_{2}}{ }^{\top} \boldsymbol{z}=\left[\boldsymbol{C}^{\top} \boldsymbol{z}\right]_{\mathcal{E}_{2}} .
$$

Since $\mathcal{E}_{1} \subset \mathcal{E}_{2}$ and $\left[s_{1}\right]_{\mathcal{E}_{1}}=\left[s_{2}\right]_{\mathcal{E}_{1}}$, one can deduce that

$$
\left[s_{1}\right]_{\mathcal{E}_{1}}=\left[s_{2}\right]_{\mathcal{E}_{1}}=\left[\boldsymbol{C}^{\top} \boldsymbol{z}\right]_{\mathcal{E}_{1}}=\boldsymbol{C}_{\mathcal{E}_{1}}^{\top} \boldsymbol{z} \in \mathcal{R}\left(\boldsymbol{C}_{\mathcal{E}_{1}}^{\top}\right)
$$

which implies that $\left(\mathcal{E}_{1}, s_{1}\right) \in \mathcal{V}$. Hence $\mathcal{V}$ is downward closed with respect to $\preceq$.
(b) For $\lambda>0$ and $\boldsymbol{w}_{\mathrm{e}} \in \mathcal{S}_{\mathrm{e}}(\lambda)$, from the rule (a) of (OPT) in Remark 5 (Sec. 5.1), one can verify that

$$
\begin{aligned}
\operatorname{supp}\left(\boldsymbol{w}_{\mathrm{e}}\right) & \subset \mathcal{E}_{\mathrm{e}}(\lambda) \\
{\left[\operatorname{sign}\left(\boldsymbol{w}_{\mathrm{e}}\right)\right]_{\operatorname{supp}\left(\boldsymbol{w}_{\mathrm{e}}\right)} } & =\left[\boldsymbol{s}_{\mathrm{e}}(\lambda)\right]_{\operatorname{supp}\left[\boldsymbol{w}_{\mathrm{e}}\right]} .
\end{aligned}
$$

Thus the generalized inequality (34) naturally holds. Since $\preceq$ is downward closed on $\mathcal{V}$ and $\left(\mathcal{E}_{\mathrm{e}}(\lambda), s_{\mathrm{e}}(\lambda)\right) \in \mathcal{V}$ from Lemma 6 , we naturally have $\left(\operatorname{supp}\left(\boldsymbol{w}_{\mathrm{e}}\right), \operatorname{sign}\left(\boldsymbol{w}_{\mathrm{e}}\right)\right) \in \mathcal{V}$.

## D. 2 Proof of Proposition 7

Proof: (a) $\operatorname{For}(\mathcal{E}, \boldsymbol{s}) \in \mathcal{V}$ and $\lambda>0$, let $\boldsymbol{w} \in \mathcal{S}_{\mathrm{POLY}}(\mathcal{E}, \boldsymbol{s}, \lambda)$, then $\boldsymbol{w}$ satisfies $(\mathcal{E}, \boldsymbol{s})$-(EQ) and $(\mathcal{E}, \boldsymbol{s})$-(NQ) simultaneously at $\lambda$, i.e.,

$$
\begin{align*}
& \begin{cases}(\forall i \in \mathcal{E}) & \xi_{i}(\boldsymbol{w}):=\boldsymbol{c}_{i}^{\top}(\boldsymbol{b}-\boldsymbol{D} \boldsymbol{C} \boldsymbol{w})=\lambda[\boldsymbol{s}]_{i} \\
(\forall i \in \neg \mathcal{E}) & {[\boldsymbol{w}]_{i}=0,}\end{cases}  \tag{EQ-a}\\
& \begin{cases}(\forall i \in \mathcal{E}) & {[\boldsymbol{s}]_{i}[\boldsymbol{w}]_{i} \geq 0,} \\
(\forall i \in \neg \mathcal{E}) & \left|\xi_{i}(\boldsymbol{w})\right|:=\left|\boldsymbol{c}_{i}^{\top}(\boldsymbol{b}-\boldsymbol{D} \boldsymbol{C} \boldsymbol{w})\right| \leq \lambda\end{cases} \tag{EQ-b}
\end{align*}
$$

To prove $\boldsymbol{w} \in \mathcal{S}_{\mathrm{e}}(\lambda)$, we only need to show that $\boldsymbol{w}$ satisfies the $i$ th (OPT) condition for every $i \in\{1,2, \ldots, 2 n\}$ (Def. 2 in Sec. 4.1), i.e.,

$$
\xi_{i}(\boldsymbol{w}):=\boldsymbol{c}_{i}^{\top}(\boldsymbol{b}-\boldsymbol{D C} \boldsymbol{w}) \in \begin{cases}\left\{\lambda \operatorname{sign}\left([\boldsymbol{w}]_{i}\right)\right\}, & \text { if }[\boldsymbol{w}]_{i} \neq 0  \tag{OPT}\\ {[-\lambda, \lambda],} & \text { if }[\boldsymbol{w}]_{i}=0\end{cases}
$$

For every $i \in \neg \mathcal{E}$, (EQ-b) and (NQ-b) imply that $[\boldsymbol{w}]_{i}=0$ and $\left|\xi_{i}(\boldsymbol{w})\right| \leq \lambda$, hence $\boldsymbol{w}$ satisfies the $i$ th (OPT) condition.
On the other hand, for every $i \in \mathcal{E}$, $\operatorname{since} \mathcal{E}=\operatorname{supp}(s)$ from Prop. 5, we have $[s]_{i} \in\{ \pm 1\}$, which further implies from (EQ-a) that $\left|\xi_{i}(\boldsymbol{w})\right|=\lambda$. Hence we can derive that

- if $[\boldsymbol{w}]_{i}=0$, then since $\left|\xi_{i}(\boldsymbol{w})\right|=\lambda \in[-\lambda, \lambda], \boldsymbol{w}$ satisfies the $i$ th (OPT) condition,
- if $[\boldsymbol{w}]_{i} \neq 0$, then from (NQ-a) and $[\boldsymbol{s}]_{i} \in\{ \pm 1\}$ we have $[\boldsymbol{s}]_{i}=\operatorname{sign}\left([\boldsymbol{w}]_{i}\right)$, combining which with (EQ-a) implies

$$
\xi_{i}(\boldsymbol{w})=\lambda[\boldsymbol{s}]_{i}=\lambda \operatorname{sign}\left([\boldsymbol{w}]_{i}\right) .
$$

Thus $\boldsymbol{w}$ satisfies the $i$ th (OPT) condition.
Accordingly, $\boldsymbol{w}$ satisfies the $i$ th (OPT) condition for every $i \in\{1,2, \ldots, 2 n\}$, which yields $\boldsymbol{w} \in \mathcal{S}_{\mathrm{e}}(\lambda)$. Hence we conclude that $\mathcal{S}_{\text {POLY }}(\mathcal{E}, s, \lambda) \subset \mathcal{S}_{\mathrm{e}}(\lambda)$.
(b) For $\lambda>0$, define $\left(\mathcal{E}_{\mathrm{e}}, s_{\mathrm{e}}\right):=\left(\mathcal{E}_{\mathrm{e}}(\lambda), s_{\mathrm{e}}(\lambda)\right)$, then we have $\left(\mathcal{E}_{\mathrm{e}}, s_{\mathrm{e}}\right) \in \mathcal{V}$ directly from Lemma 6 and Prop. 5. From the result (a), we have $\mathcal{S}_{\text {POLY }}\left(\mathcal{E}_{\mathrm{e}}, \boldsymbol{s}_{\mathrm{e}}, \lambda\right) \subset \mathcal{S}_{\mathrm{e}}(\lambda)$, hence we only need to prove $\mathcal{S}_{\text {POLY }}\left(\mathcal{E}_{\mathrm{e}}, \boldsymbol{s}_{\mathrm{e}}, \lambda\right) \supset \mathcal{S}_{\mathrm{e}}(\lambda)$.

Let $\boldsymbol{w} \in \mathcal{S}_{\mathrm{e}}(\lambda)$, then $\boldsymbol{w}$ satisfies (OPT), and we have

$$
\begin{equation*}
(\mathcal{E}(\lambda, \boldsymbol{w}), \boldsymbol{s}(\lambda, \boldsymbol{w}))=\left(\mathcal{E}_{\mathrm{e}}(\lambda), s_{\mathrm{e}}(\lambda)\right)=:\left(\mathcal{E}_{\mathrm{e}}, \boldsymbol{s}_{\mathrm{e}}\right) \tag{56}
\end{equation*}
$$

from Lemma 6. To prove $\boldsymbol{w} \in \mathcal{S}_{\text {POLY }}\left(\mathcal{E}_{\mathrm{e}}, \boldsymbol{s}_{\mathrm{e}}, \lambda\right)$, we only need to show that $\boldsymbol{w}$ satisfies $\left(\mathcal{E}_{\mathrm{e}}, \boldsymbol{s}_{\mathrm{e}}\right)-(\mathrm{EQ})$ and $\left(\mathcal{E}_{\mathrm{e}}, \boldsymbol{s}_{\mathrm{e}}\right)-(\mathrm{NQ})$ at $\lambda$. From (56) and Lemma 5 (b), we can derive

$$
\begin{equation*}
\left(\forall i \in \mathcal{E}_{\mathrm{e}}\right) \quad \xi_{i}(\boldsymbol{w})=\lambda\left[\boldsymbol{s}_{\mathrm{e}}\right]_{i} \tag{57}
\end{equation*}
$$

Moreover, from (56) and Def. 5, we have

$$
\left(\forall i \in \neg \mathcal{E}_{\mathrm{e}}\right) \quad\left|\xi_{i}(\boldsymbol{w})\right| \neq \lambda,
$$

combining which with the $i$ th (OPT) condition implies

$$
\begin{equation*}
\left(\forall i \in \neg \mathcal{E}_{\mathrm{e}}\right)[\boldsymbol{w}]_{i}=0 \tag{58}
\end{equation*}
$$

Putting (57) and (58) together, we can verify that $\boldsymbol{w}$ satisfies $\left(\mathcal{E}_{\mathrm{e}}, \boldsymbol{s}_{\mathrm{e}}\right)$-(EQ) at $\lambda$.
In addition, for every $i \in \mathcal{E}_{\mathrm{e}}$, we have

- if $i \notin \operatorname{supp}(\boldsymbol{w})$, then we naturally have $\left[s_{\mathrm{e}}\right]_{i}[\boldsymbol{w}]_{i}=0$,
- if $i \in \operatorname{supp}(\boldsymbol{w})$, then from (OPT) we have $\xi_{i}(\boldsymbol{w})=\lambda \operatorname{sign}\left([\boldsymbol{w}]_{i}\right)$, combining which with (57) yields $\left[\boldsymbol{s}_{\mathrm{e}}\right]_{i}=[\operatorname{sign}(\boldsymbol{w})]_{i}$, hence $\left[s_{\mathrm{e}}\right]_{i}[\boldsymbol{w}]_{i}>0$,
thus we verify that $\boldsymbol{w}$ satisfies $\left(\mathcal{E}_{\mathrm{e}}, \boldsymbol{s}_{\mathrm{e}}\right)$-(NQ-a) at $\lambda$. Moreover, (OPT) implies that $\xi_{i}(\boldsymbol{w}) \in[-\lambda, \lambda]$ for every $i \in\{1,2, \ldots, 2 n\}$, hence $\boldsymbol{w}$ satisfies $\left(\mathcal{E}_{\mathrm{e}}, \boldsymbol{s}_{\mathrm{e}}\right)$-(NQ-b) at $\lambda$. Accordingly, $\boldsymbol{w}$ satisfies $\left(\mathcal{E}_{\mathrm{e}}, \boldsymbol{s}_{\mathrm{e}}\right)$-(NQ) at $\lambda$.

Combining the discussion above yields $\mathcal{S}_{\text {POLY }}\left(\mathcal{E}_{\mathrm{e}}, s_{\mathrm{e}}, \lambda\right) \supset \mathcal{S}_{\mathrm{e}}(\lambda)$. Hence we have $\mathcal{S}_{\mathrm{POLY}}\left(\mathcal{E}_{\mathrm{e}}, s_{\mathrm{e}}, \lambda\right)=\mathcal{S}_{\mathrm{e}}(\lambda)$.
(c) $\operatorname{For}(\mathcal{E}, \boldsymbol{s}) \in \mathcal{V}$ and $\boldsymbol{w} \in \mathcal{S}_{\mathrm{e}}(\lambda)$, we have

$$
\begin{equation*}
(\operatorname{supp}(\boldsymbol{w}), \operatorname{sign}(\boldsymbol{w})) \preceq\left(\mathcal{E}_{\mathrm{e}}(\lambda), \boldsymbol{s}_{\mathrm{e}}(\lambda)\right)=(\mathcal{E}(\lambda, \boldsymbol{w}), \boldsymbol{s}(\lambda, \boldsymbol{w})) \tag{59}
\end{equation*}
$$

directly from Prop. 6 and Lemma 6.
If $\boldsymbol{w} \in \mathcal{S}_{\text {POLY }}(\mathcal{E}, \boldsymbol{s}, \lambda)$, then from (EQ-a), we have

$$
\begin{array}{r}
\mathcal{E} \subset \mathcal{E}(\lambda, \boldsymbol{w})=\mathcal{E}_{\mathrm{e}}(\lambda), \\
\lambda[\boldsymbol{s}]_{\mathcal{E}}=[\boldsymbol{\xi}(\boldsymbol{w})]_{\mathcal{E}}=\lambda[\boldsymbol{s}(\lambda, \boldsymbol{w})]_{\mathcal{E}}=\lambda\left[\boldsymbol{s}_{\mathrm{e}}(\lambda)\right]_{\mathcal{E}}
\end{array}
$$

where the second equality in the latter equation follows from Lemma 5 (b) and $\mathcal{E} \subset \mathcal{E}(\lambda, \boldsymbol{w})$. Hence one can derive $(\mathcal{E}, \boldsymbol{s}) \preceq\left(\mathcal{E}_{\mathrm{e}}(\lambda), s_{\mathrm{e}}(\lambda)\right)$. From (EQ-b), we have $\operatorname{supp}(\boldsymbol{w}) \subset \mathcal{E}$. Combining these results with (59) implies

$$
\begin{aligned}
(\operatorname{supp}(\boldsymbol{w}), \operatorname{sign}(\boldsymbol{w})) & \preceq\left(\mathcal{E}_{\mathrm{e}}(\lambda), \boldsymbol{s}_{\mathrm{e}}(\lambda)\right), \\
(\mathcal{E}, \boldsymbol{s}) & \preceq\left(\mathcal{E}_{\mathrm{e}}(\lambda), s_{\mathrm{e}}(\lambda)\right), \\
\operatorname{supp}(\boldsymbol{w}) & \subset \mathcal{E}
\end{aligned}
$$

thus from the definition of $\preceq$ on $\overline{\mathcal{V}}$ (cf. Prop. 6), we have

$$
(\operatorname{supp}(\boldsymbol{w}), \operatorname{sign}(\boldsymbol{w})) \preceq(\mathcal{E}, \boldsymbol{s}) \preceq\left(\mathcal{E}_{\mathrm{e}}(\lambda), s_{\mathrm{e}}(\lambda)\right)
$$

Reversely, if $(\mathcal{E}, s)$ satisfies the generalized inequality above, then we can derive the following:

- since $\operatorname{supp}(\boldsymbol{w}) \subset \mathcal{E}$, we have $[\boldsymbol{w}]_{\neg \mathcal{E}}=\mathbf{0}$, hence $\boldsymbol{w}$ satisfies $(\mathcal{E}, \boldsymbol{s})$-(EQ-b) at $\lambda$,
- since $\boldsymbol{w} \in \mathcal{S}_{\mathrm{e}}(\lambda)$, we have $[\boldsymbol{\xi}(\boldsymbol{w})]_{\mathcal{E}_{\mathrm{e}}(\lambda)}=\lambda\left[\boldsymbol{s}_{\mathrm{e}}(\lambda)\right]_{\mathcal{E}_{\mathrm{e}}(\lambda)}$ from Lemma 5 (b) and Lemma 6, combining which with $(\mathcal{E}, \boldsymbol{s}) \preceq\left(\mathcal{E}_{\mathrm{e}}(\lambda), \boldsymbol{s}_{\mathrm{e}}(\lambda)\right)$ yields

$$
[\boldsymbol{\xi}(\boldsymbol{w})]_{\mathcal{E}}=\lambda\left[\boldsymbol{s}_{\mathrm{e}}(\lambda)\right]_{\mathcal{E}}=\lambda[\boldsymbol{s}]_{\mathcal{E}}
$$

hence $\boldsymbol{w}$ satisfies $(\mathcal{E}, \boldsymbol{s})$-(EQ-a) at $\lambda$,

- since $\boldsymbol{w} \in \mathcal{S}_{\mathrm{e}}(\lambda)$, from (OPT), we have $\left|\xi_{i}(\boldsymbol{w})\right| \leq \lambda$ for every $i \in\{1, \ldots, 2 n\}$, which implies that $\boldsymbol{w}$ satisfies $(\mathcal{E}, \boldsymbol{s})$ -(NQ-b) at $\lambda$,
- since $\boldsymbol{w} \in \mathcal{S}_{\mathrm{e}}(\lambda)$, from the result (b), we can derive that $\boldsymbol{w}$ satisfies $\left(\mathcal{E}_{\mathrm{e}}(\lambda), \boldsymbol{s}_{\mathrm{e}}(\lambda)\right)-(\mathrm{NQ}-\mathrm{a})$ at $\lambda$, i.e.,

$$
\left(\forall i \in \mathcal{E}_{\mathbf{e}}(\lambda)\right) \quad\left[s_{\mathbf{e}}(\lambda)\right]_{i}[\boldsymbol{w}]_{i} \geq 0
$$

combining which with $(\mathcal{E}, \boldsymbol{s}) \subset\left(\mathcal{E}_{\mathrm{e}}(\lambda), \boldsymbol{s}_{\mathrm{e}}(\lambda)\right)$ yields that $\boldsymbol{w}$ satisfies $(\mathcal{E}, \boldsymbol{s})$-(NQ-a) at $\lambda$.
Hence $\boldsymbol{w}$ satisfies $(\mathcal{E}, \boldsymbol{s})$-(EQ) and $(\mathcal{E}, \boldsymbol{s})$-(NQ) simultaneously at $\lambda$, i.e., $\boldsymbol{w} \in \mathcal{S}_{\mathrm{POLY}}(\mathcal{E}, \boldsymbol{s}, \lambda)$.
(d) For $\boldsymbol{w} \in \mathcal{S}_{\mathrm{POLY}}(\mathcal{E}, \boldsymbol{s}, \lambda)$, if $\boldsymbol{w}$ satisfies $(\mathcal{E}, \boldsymbol{s})$-(NQ-a) strictly $\lambda$, then we have $(\mathcal{E}, \boldsymbol{s})=(\operatorname{supp}(\boldsymbol{w}), \operatorname{sign}(\boldsymbol{w}))$ directly from (EQ-b) and strict inequality of (NQ-a).

On the other hand, suppose $\boldsymbol{w}$ satisfies $(\mathcal{E}, s)$-(NQ-a) strictly at $\lambda$. From the result (a), we have $\boldsymbol{w} \in \mathcal{S}_{\mathrm{e}}(\lambda)$. Hence from (EQ-a) and the strict inequality of (NQ-b), one can derive

$$
\mathcal{E}=\mathcal{E}(\lambda, \boldsymbol{w})=\mathcal{E}_{\mathrm{e}}(\lambda)
$$

by Def. 5 and Lemma 6 . Combining the equality above with (EQ-a) and notice that $(\mathcal{E}, s) \in \mathcal{V}$, we can further yield

$$
\boldsymbol{s}=\boldsymbol{s}(\lambda, \boldsymbol{w})=s_{\mathrm{e}}(\lambda)
$$

by Def. 5 and Lemma 6 . Hence $(\mathcal{E}, s)=\left(\mathcal{E}_{\mathrm{e}}(\lambda), s_{\mathrm{e}}(\lambda)\right)$.

## D. 3 Proof of Proposition 8

Proof: We first show that the solution set of $(\mathcal{E}, \boldsymbol{s})$-(EQ) at $\lambda$ is always nonempty for every $\lambda \in \mathbb{R}$. Notice that $\boldsymbol{w} \in \mathbb{R}^{2 n}$ satisfies $(\mathcal{E}, \boldsymbol{s})$-(EQ) at $\lambda$ if and only if

$$
\left\{\begin{array}{l}
\boldsymbol{C}_{\mathcal{E}}^{\top}\left(\boldsymbol{b}-\boldsymbol{D} \boldsymbol{C}_{\mathcal{E}}[\boldsymbol{w}]_{\mathcal{E}}\right)=\lambda[\boldsymbol{s}]_{\mathcal{E}},  \tag{60}\\
{[\boldsymbol{w}]_{\neg \mathcal{E}}=\mathbf{0}}
\end{array}\right.
$$

Hence the solution set of $(\mathcal{E}, s)-(\mathrm{EQ})$ at $\lambda$ is nonempty if and only if the following equation with respect to $[\boldsymbol{w}]_{\mathcal{E}}$ :

$$
\begin{equation*}
\boldsymbol{C}_{\mathcal{E}}^{\top}\left(\boldsymbol{b}-\boldsymbol{D} \boldsymbol{C}_{\mathcal{E}}[\boldsymbol{w}]_{\mathcal{E}}\right)=\lambda[\boldsymbol{s}]_{\mathcal{E}} \quad \Longleftrightarrow \quad \boldsymbol{C}_{\mathcal{E}}^{\top} \boldsymbol{D} \boldsymbol{C}_{\mathcal{E}}[\boldsymbol{w}]_{\mathcal{E}}=\boldsymbol{C}_{\mathcal{E}}^{\top} \boldsymbol{b}-\lambda[\boldsymbol{s}]_{\mathcal{E}} \tag{61}
\end{equation*}
$$

is compatible. Since $(\mathcal{E}, s) \in \mathcal{V}$, we have $[s]_{\mathcal{E}} \in \mathcal{R}\left(\boldsymbol{C}_{\mathcal{E}}{ }^{\top}\right)$ from Prop. 5. Hence we have

$$
\boldsymbol{C}_{\mathcal{E}}^{\top} \boldsymbol{b}-\lambda[\boldsymbol{s}]_{\mathcal{E}} \in \mathcal{R}\left(\boldsymbol{C}_{\mathcal{E}}^{\top}\right)=\mathcal{R}\left(\boldsymbol{C}_{\mathcal{E}}^{\top} \boldsymbol{D} \boldsymbol{C}_{\mathcal{E}}\right)
$$

where the equality holds from Lemma 7 in Appendix A. Accordingly, (61) is compatible, which implies that the solution set of $(\mathcal{E}, s)$-(EQ) at $\lambda$ is nonempty.

Next we derive the expression of the least squares solution of $(\mathcal{E}, s)-(\mathrm{EQ})$ at $\lambda$. Combining (60) and (61), one can verify that $\boldsymbol{w}$ satisfies $(\mathcal{E}, \boldsymbol{s})$-(EQ) at $\lambda$ if and only if

$$
\left\{\begin{array}{l}
\boldsymbol{C}_{\mathcal{E}}^{\top} \boldsymbol{D} \boldsymbol{C}_{\mathcal{E}}[\boldsymbol{w}]_{\mathcal{E}}=\boldsymbol{C}_{\mathcal{E}}^{\top} \boldsymbol{b}-\lambda[\boldsymbol{s}]_{\mathcal{E}},  \tag{62}\\
{[\boldsymbol{w}]_{\neg \mathcal{E}}=\mathbf{0}}
\end{array}\right.
$$

Hence we can derive that the minimum norm solution $\hat{\boldsymbol{w}}_{\mathrm{EQ}}$ should satisfy that

$$
\left\{\begin{array}{l}
{\left[\hat{\boldsymbol{w}}_{\mathrm{EQ}}\right]_{\mathcal{E}}=\left(\boldsymbol{C}_{\mathcal{E}}{ }^{\top} \boldsymbol{D} \boldsymbol{C}_{\mathcal{E}}\right)^{\dagger}\left(\boldsymbol{C}_{\mathcal{E}}^{\top} \boldsymbol{b}-\lambda[\boldsymbol{s}]_{\mathcal{E}}\right),} \\
{\left[\hat{\boldsymbol{w}}_{\mathrm{EQ}}\right]_{\neg \mathcal{E}}=\mathbf{0},}
\end{array}\right.
$$

which yields the closed-form expression of $\hat{\boldsymbol{w}}_{\mathrm{EQ}}(\mathcal{E}, \boldsymbol{s}, \lambda)$.

## D. 4 Proof of Proposition 9

Proof: Define $\left(\mathcal{E}_{\star}, \boldsymbol{s}_{\star}\right):=\left(\operatorname{supp}\left(\boldsymbol{w}_{\star}(\lambda)\right), \operatorname{sign}\left(\boldsymbol{w}_{\star}(\lambda)\right)\right)$, then we have $\left(\mathcal{E}_{\star}, \boldsymbol{s}_{\star}\right) \in \mathcal{V}$ directly from Prop. 6, which implies that $\left(\mathcal{E}_{\star}, \boldsymbol{s}_{\star}\right)$ is an ex-equicorrelation pair. Define $\boldsymbol{w}_{\star}:=\boldsymbol{w}_{\star}(\lambda)$ for simplicity. Since $\left(\mathcal{E}_{\star}, \boldsymbol{s}_{\star}\right) \in \mathcal{V}$ and $\boldsymbol{w}_{\star} \in \mathcal{S}_{\mathrm{e}}(\lambda)$, from Prop. 7 (c) we have

$$
\begin{equation*}
\boldsymbol{w}_{\star} \in \mathcal{S}_{\mathrm{POLY}}\left(\mathcal{E}_{\star}, \boldsymbol{s}_{\star}, \lambda\right) \subset \mathcal{S}_{\mathrm{e}}(\lambda) \tag{63}
\end{equation*}
$$

where the second inclusion follows from Prop. 7 (a). Hence $\boldsymbol{w}_{\star}$ satisfies $\left(\mathcal{E}_{\star}, s_{\star}\right)-(E Q)$ at $\lambda$, which is equivalent to

$$
\left\{\begin{array}{l}
\boldsymbol{C}_{\mathcal{E}_{\star}}^{\top} \boldsymbol{D} \boldsymbol{C}_{\mathcal{E}_{\star}}\left[\boldsymbol{w}_{\star}\right]_{\mathcal{E}_{\star}}=\boldsymbol{C}_{\mathcal{E}_{\star}}{ }^{\top} \boldsymbol{b}-\lambda\left[\boldsymbol{s}_{\star}\right]_{\mathcal{E}_{\star}},  \tag{64}\\
{\left[\boldsymbol{w}_{\star}\right]_{\neg \mathcal{E}_{\star}}=\mathbf{0}}
\end{array}\right.
$$

from (62). One can verify that $\boldsymbol{w}_{\star}$ is the least squares solution of (64) (i.e., $\boldsymbol{w}_{\star}=\hat{\boldsymbol{w}}_{\mathrm{EQ}}\left(\mathcal{E}_{\star}, \boldsymbol{s}_{\star}, \lambda\right)$ ), if and only if $\left[\boldsymbol{w}_{\star}\right]_{\mathcal{E}_{\star}}$ satisfies the following:

$$
\begin{equation*}
\left[\boldsymbol{w}_{\star}\right]_{\mathcal{E}_{\star}} \in \mathcal{N}\left(\boldsymbol{C}_{\mathcal{E}_{\star}}^{\top} \boldsymbol{D} \boldsymbol{C}_{\mathcal{E}_{\star}}\right)^{\perp} . \tag{65}
\end{equation*}
$$

Hereafter, we will show that the inclusion (65) holds.
From Thm. 4 in Sec. 4.1, $\boldsymbol{w}_{\star}$ is the unique minimum norm element in $\mathcal{S}_{\mathrm{e}}(\lambda) \supset \mathcal{S}_{\mathrm{POLY}}\left(\mathcal{E}_{\star}, \boldsymbol{s}_{\star}, \lambda\right) \ni \boldsymbol{w}_{\star}$. Hence $\boldsymbol{w}_{\star}$ is the unique solution of the following convex program:

$$
\begin{array}{ll}
\underset{\boldsymbol{w} \in \mathbb{R}^{2 n}}{\operatorname{minimize}} & \frac{1}{2}\|\boldsymbol{w}\|_{2}^{2} \\
\text { subject to } & \boldsymbol{w} \in \mathcal{S}_{\mathrm{POLY}}\left(\mathcal{E}_{\star}, s_{\star}, \lambda\right) .
\end{array}
$$

From Def. 7, the convex program above is equivalent to

$$
\begin{array}{cl}
\underset{\boldsymbol{w} \in \mathbb{R}^{2 n}}{\operatorname{minimize}} & \frac{1}{2}\|\boldsymbol{w}\|_{2}^{2} \\
\text { subject to } & \boldsymbol{C}_{\mathcal{E}_{\star}}^{\top}(\boldsymbol{b}-\boldsymbol{D} \boldsymbol{C} \boldsymbol{w})=\lambda\left[\boldsymbol{s}_{\star}\right]_{\mathcal{E}_{\star}} \\
& {[\boldsymbol{w}]_{\neg \mathcal{E}_{\star}}=\mathbf{0}} \\
& {\left[\boldsymbol{s}_{\star}\right]_{\mathcal{E}_{\star}} \odot[\boldsymbol{w}]_{\mathcal{E}_{\star}} \geq \mathbf{0}} \\
& -\lambda \mathbf{1} \leq \boldsymbol{C}_{\neg \mathcal{E}_{\star}}^{\top}(\boldsymbol{b}-\boldsymbol{D} \boldsymbol{C} \boldsymbol{w}) \leq \lambda \mathbf{1} \tag{66e}
\end{array}
$$

where $\odot$ is the Hadamard product (a.k.a. entrywise product). From (66c), we know that every solution $\boldsymbol{w}$ of (66) satisfies $[\boldsymbol{w}]_{\neg \mathcal{E}_{\star}}=\mathbf{0}$. Define $\boldsymbol{z}:=[\boldsymbol{w}]_{\mathcal{E}_{\star}}$ and notice that $\boldsymbol{C} \boldsymbol{w}=\boldsymbol{C}_{\mathcal{E}_{\star}} \boldsymbol{z}$, then (66) is further equivalent to $[\boldsymbol{w}]_{\neg \mathcal{E}_{\star}}=\mathbf{0}$ and

$$
\begin{array}{cl}
\underset{\boldsymbol{z} \in \mathbb{R}^{\left|\mathcal{E}_{\star}\right|}}{\operatorname{minimize}} & \frac{1}{2}\|\boldsymbol{z}\|_{2}^{2} \\
\text { subject to } & \boldsymbol{C}_{\mathcal{E}_{\star}}^{\top}\left(\boldsymbol{b}-\boldsymbol{D} \boldsymbol{C}_{\mathcal{E}_{\star}} \boldsymbol{z}\right)=\lambda\left[\boldsymbol{s}_{\star}\right] \mathcal{E}_{\star}, \\
& {\left[\boldsymbol{s}_{\star}\right]_{\mathcal{E}_{\star}} \odot \boldsymbol{z \geq \mathbf { 0 }}} \\
& \boldsymbol{C}_{\neg \mathcal{E}_{\star}}{ }^{\top}\left(\boldsymbol{D} \boldsymbol{C}_{\mathcal{E}_{\star}} \boldsymbol{z}-\boldsymbol{b}\right) \leq \lambda \mathbf{1} \\
& \boldsymbol{C}_{\neg \mathcal{E}_{\star}}{ }^{\top}\left(\boldsymbol{b}-\boldsymbol{D} \boldsymbol{C}_{\mathcal{E}_{\star}} \boldsymbol{z}\right) \leq \lambda \mathbf{1} \tag{67e}
\end{array}
$$

The Lagrangian function of (67) is

$$
\begin{aligned}
L(\boldsymbol{z}, \boldsymbol{\mu}, \boldsymbol{\nu})= & \frac{1}{2}\|\boldsymbol{z}\|_{2}^{2}+\boldsymbol{\mu}^{\top}\left(\boldsymbol{C}_{\mathcal{E}_{\star}}^{\top}\left(\boldsymbol{b}-\boldsymbol{D} \boldsymbol{C}_{\mathcal{E}_{\star}} \boldsymbol{z}\right)-\lambda\left[\boldsymbol{s}_{\star}\right]_{\mathcal{E}_{\star}}\right) \\
& -\boldsymbol{\nu}_{1}^{\top}\left(\left[\boldsymbol{s}_{\star}\right]_{\mathcal{E}_{\star}} \odot \boldsymbol{z}\right)+\boldsymbol{\nu}_{2}^{\top}\left(\boldsymbol{C}_{\neg \mathcal{E}_{\star}}{ }^{\top}\left(\boldsymbol{D} \boldsymbol{C}_{\mathcal{E}_{\star}} \boldsymbol{z}-\boldsymbol{b}\right)-\lambda \mathbf{1}\right)+\boldsymbol{\nu}_{3}^{\top}\left(\boldsymbol{C}_{\neg \mathcal{E}_{\star}}^{\top}\left(\boldsymbol{b}-\boldsymbol{D} \boldsymbol{C}_{\mathcal{E}_{\star}} \boldsymbol{z}\right)-\lambda \mathbf{1}\right)
\end{aligned}
$$

where $\boldsymbol{\mu}$ and $\boldsymbol{\nu}:=\left(\boldsymbol{\nu}_{1}, \boldsymbol{\nu}_{2}, \boldsymbol{\nu}_{3}\right)$ are the Lagrange multipliers. Hence from the KKT condition, we can yield the following necessary conditions of optimality (cf. [49, Sec. 5.5.3]):

$$
\begin{align*}
\nabla_{\boldsymbol{z}} L(\boldsymbol{z}, \boldsymbol{\mu}, \boldsymbol{\nu}) & =\mathbf{0} .  \tag{68a}\\
\boldsymbol{\nu}_{1} \odot\left(\left[\boldsymbol{s}_{\star}\right]_{\mathcal{E}_{\star}} \odot \boldsymbol{z}\right) & =\mathbf{0} . \tag{68b}
\end{align*}
$$

Expanding the expression of $\nabla_{\boldsymbol{z}} L(\boldsymbol{z}, \boldsymbol{\mu}, \boldsymbol{\nu})$ in (68a) yields

$$
\begin{equation*}
\boldsymbol{z}=\boldsymbol{C}_{\mathcal{E}_{\star}}{ }^{\top} \boldsymbol{D}^{\top}\left(\boldsymbol{C}_{\mathcal{E}_{\star}} \boldsymbol{\mu}+\boldsymbol{C}_{\neg \mathcal{E}_{\star}}\left(\boldsymbol{\nu}_{3}-\boldsymbol{\nu}_{2}\right)\right)+\left[\boldsymbol{s}_{\star}\right]_{\mathcal{E}_{\star}} \odot \boldsymbol{\nu}_{1} \tag{69}
\end{equation*}
$$

Notice that if $\boldsymbol{z}$ is the solution of (67), then since $\mathcal{E}_{\star}:=\operatorname{supp}\left(\boldsymbol{w}_{\star}\right)$, we have

$$
\boldsymbol{z}=\left[\boldsymbol{w}_{\star}\right]_{\mathcal{E}_{\star}}=\left[\boldsymbol{w}_{\star}\right]_{\operatorname{supp}\left(\boldsymbol{w}_{\star}\right)}
$$

which means that every component in $\boldsymbol{z}$ is nonzero. Since $\left(\mathcal{E}_{\star}, s_{\star}\right) \in \mathcal{V}$, every component in $\left[s_{\star}\right]_{\mathcal{E}_{\star}}$ is also nonzero, hence from (68b) we can derive that $\boldsymbol{\nu}_{1}=\mathbf{0}$, substituting which into (69) yields

$$
\boldsymbol{z}=\boldsymbol{C}_{\mathcal{E}_{\star}}{ }^{\top} \boldsymbol{D}^{\top}\left(\boldsymbol{C}_{\mathcal{E}_{\star}} \boldsymbol{\mu}+\boldsymbol{C}_{\neg \mathcal{E}_{\star}}\left(\boldsymbol{\nu}_{3}-\boldsymbol{\nu}_{2}\right)\right) \in \mathcal{R}\left(\boldsymbol{C}_{\mathcal{E}_{\star}}{ }^{\top}\right) .
$$

Combining the inclusion above with $\boldsymbol{z}=\left[\boldsymbol{w}_{\star}\right]_{\mathcal{E}_{\star}}$ and Lemma 7 in Appendix A yields

$$
\left[\boldsymbol{w}_{\star}\right]_{\mathcal{E}_{\star}} \in \mathcal{R}\left(\boldsymbol{C}_{\mathcal{E}_{\star}}^{\top} \boldsymbol{D}^{\top} \boldsymbol{C}_{\mathcal{E}_{\star}}\right)=\mathcal{N}\left(\boldsymbol{C}_{\mathcal{E}_{\star}}^{\top} \boldsymbol{D} \boldsymbol{C}_{\mathcal{E}_{\star}}\right)^{\perp}
$$

Hence the inclusion (65) holds, which implies that $\boldsymbol{w}_{\star}(\lambda)$ is the least squares solution of $\left(\mathcal{E}_{\star}, s_{\star}\right)$-(EQ) at $\lambda$. Hence we have $\boldsymbol{w}_{\star}(\lambda)=\hat{\boldsymbol{w}}_{\mathrm{EQ}}\left(\mathcal{E}_{\star}, \boldsymbol{s}_{\star}, \lambda\right)$ from Prop. 8.

## Appendix E

## Proofs for Section 6

## E. 1 Proof of Proposition 10

In this appendix, we derive the closed-form expression of the selection interval $I_{\text {SEL }}(\mathcal{E}, s)$ defined in Prop. 10, which naturally proves the theoretical results claimed in Prop. 10.

## E.1.1 Outline and Main Results of the Derivation

For the sake of clarity, we initially present an outline of the derivation, along with a lemma that will be used in subsequent studies. The computational details will be elaborated upon in following sections.

Let us fix $(\mathcal{E}, \boldsymbol{s}) \in \mathcal{V}$, then the solution set of (SEL) with respect to $\lambda>0$ is the set of $\lambda \in \mathbb{R}$ satisfying:

$$
\begin{cases}(\forall i \in \mathcal{E}) & {[\boldsymbol{s}]_{i}\left[\hat{\boldsymbol{w}}_{\mathrm{EQ}}(\mathcal{E}, \boldsymbol{s}, \lambda)\right]_{i} \geq 0}  \tag{SEL-a}\\ (\forall i \in \neg \mathcal{E}) & -\lambda \leq \xi_{i}\left(\hat{\boldsymbol{w}}_{\mathrm{EQ}}(\mathcal{E}, \boldsymbol{s}, \lambda)\right) \leq \lambda, \\ & \lambda>0,\end{cases}
$$

which is exactly the selection interval $I_{\text {SEL }}(\mathcal{E}, \boldsymbol{s})$ in Prop. 10.
Since $\hat{\boldsymbol{w}}_{\mathrm{EQ}}(\mathcal{E}, s, \cdot)$ is affine with a closed-form expression (cf. Prop. 8), we can obtain the closed-form expression of $I_{\text {SEL }}(\mathcal{E}, s)$ by the following four steps:

1) Step 1: we rewrite $\hat{\boldsymbol{w}}_{\mathrm{EQ}}(\mathcal{E}, \boldsymbol{s}, \lambda)$ and $\xi_{i}\left(\hat{\boldsymbol{w}}_{\mathrm{EQ}}(\mathcal{E}, \boldsymbol{s}, \lambda)\right)$ as affine functions of $\lambda$ more explicitly (cf. Appendix E.1):

$$
\begin{align*}
\hat{\boldsymbol{w}}_{\mathrm{EQ}}(\mathcal{E}, \boldsymbol{s}, \lambda) & =\boldsymbol{p}(\mathcal{E})-\lambda \boldsymbol{q}(\mathcal{E}, \boldsymbol{s})  \tag{70}\\
\xi_{i}\left(\hat{\boldsymbol{w}}_{\mathrm{EQ}}(\mathcal{E}, \boldsymbol{s}, \lambda)\right) & =\boldsymbol{c}_{i}^{\top}(\boldsymbol{u}(\mathcal{E})-\lambda \boldsymbol{v}(\mathcal{E}, \boldsymbol{s})) \tag{71}
\end{align*}
$$

2) Step 2: for every $i \in \mathcal{E}$, we substitute (70) into the $i$ th (SEL-a) inequation, leading to (cf. Appendix E.1):

$$
\begin{equation*}
[\boldsymbol{s}]_{i}[\boldsymbol{q}(\mathcal{E}, \boldsymbol{s})]_{i} \lambda \leq[\boldsymbol{s}]_{i}[\boldsymbol{p}(\mathcal{E})]_{i} \tag{72}
\end{equation*}
$$

Solving (72) with respect to $\lambda \in \mathbb{R}$ yields a (possibly empty) closed interval, denoted by $I_{i}^{\mathrm{a}}(\mathcal{E}, s)$.
3) Step 3: for every $i \in \neg \mathcal{E}$, we substitute (71) into the $i$ th (SEL-b) inequation, leading to (cf. Appendix E.1):

$$
\begin{equation*}
-\lambda \leq \boldsymbol{c}_{i}^{\top} \boldsymbol{u}(\mathcal{E})-\boldsymbol{c}_{i}^{\top} \boldsymbol{v}(\mathcal{E}, \boldsymbol{s}) \lambda \leq \lambda \tag{73}
\end{equation*}
$$

Solving (73) with respect to $\lambda \in \mathbb{R}$ yields a (possibly empty) closed interval, denoted by $I_{i}^{\mathrm{b}}(\mathcal{E}, s)$.
4) Step 4: putting the results above together, we have

$$
I_{\mathrm{SEL}}(\mathcal{E}, s)=(0,+\infty) \cap\left(\cap_{i \in \mathcal{E}} I_{i}^{\mathrm{a}}(\mathcal{E}, s)\right) \cap\left(\cap_{i \in \neg \mathcal{E}} I_{i}^{\mathrm{b}}(\mathcal{E}, s)\right)
$$

hence we can obtain the expressions of the left endpoint $l_{\text {SEL }}(\mathcal{E}, s)$ and right endpoint $r_{\text {SEL }}(\mathcal{E}, s)$ of $I_{\text {SEL }}(\mathcal{E}, s)$.
Summarizing the discussion above, we can obtain the closed-form expression of $I_{\text {SEL }}(\mathcal{E}, s)$ in Prop. 10.
Additionally, we can derive the following lemma regarding the behavior of $\hat{\boldsymbol{w}}_{\mathrm{EQ}}(\mathcal{E}, \boldsymbol{s}, \lambda)$ for $\lambda \in \operatorname{int}\left(I_{\mathrm{SEL}}(\mathcal{E}, \boldsymbol{s})\right)$. This result will be used in the proof of Prop. 11.
Lemma 8 (Behavior of the trajectory function inside its selection interval). Given $(\mathcal{E}, s) \in \mathcal{V}$, then for every $\lambda \in \operatorname{int}\left(I_{\text {SEL }}(\mathcal{E}, s)\right)$, the following holds:

$$
\hat{\boldsymbol{w}}_{E Q}(\mathcal{E}, \boldsymbol{s}, \lambda) \text { satisfies }(\mathcal{E}, \boldsymbol{s})-(N Q) \text { strictly at } \lambda,
$$

i.e., $\boldsymbol{w}:=\hat{\boldsymbol{w}}_{E Q}(\mathcal{E}, \boldsymbol{s}, \lambda)$ satisfies

$$
\begin{cases}(\forall i \in \mathcal{E}) & {[\boldsymbol{s}]_{i}[\boldsymbol{w}]_{i}>0} \\ (\forall i \in \neg \mathcal{E}) & \left|\xi_{i}(\boldsymbol{w})\right|:=\left|\boldsymbol{c}_{i}^{\top}(\boldsymbol{b}-\boldsymbol{D} \boldsymbol{C} \boldsymbol{w})\right|<\lambda\end{cases}
$$

Proof: The result follows directly from the step 2, 3, 4 in the outline of the derivation above.
In the sequel, we provide additional computational details omitted in the preceding outline of derivation. Specifically, we elaborate on the computation of steps 1 to 3 .

## E.1.2 Step 1: Rewriting $\hat{\boldsymbol{w}}_{E Q}(\mathcal{E}, \boldsymbol{s}, \lambda)$ and $\boldsymbol{\xi}\left(\hat{\boldsymbol{w}}_{E Q}(\mathcal{E}, \boldsymbol{s}, \lambda)\right)$

From Prop. 8 in Sec. 5.4, we can rewrite the expression of $\hat{\boldsymbol{w}}_{\mathrm{EQ}}(\mathcal{E}, \boldsymbol{s}, \lambda)$ as follows:

$$
\begin{equation*}
\hat{\boldsymbol{w}}_{\mathrm{EQ}}(\mathcal{E}, \boldsymbol{s}, \lambda)=\boldsymbol{p}(\mathcal{E})-\lambda \boldsymbol{q}(\mathcal{E}, \boldsymbol{s}) \tag{70}
\end{equation*}
$$

where $\boldsymbol{p}(\mathcal{E})$ is the unique vector satisfying

$$
\left\{\begin{array}{l}
{[\boldsymbol{p}(\mathcal{E})]_{\mathcal{E}}=\left(\boldsymbol{C}_{\mathcal{E}}{ }^{\top} \boldsymbol{D} \boldsymbol{C}_{\mathcal{E}}\right)^{\dagger} \boldsymbol{C}_{\mathcal{E}}^{\top} \boldsymbol{b}}  \tag{74}\\
{[\boldsymbol{p}(\mathcal{E})]_{\neg \mathcal{E}}=\mathbf{0}}
\end{array}\right.
$$

and $\boldsymbol{q}(\mathcal{E}, \boldsymbol{s})$ is the unique vector satisfying

$$
\left\{\begin{array}{l}
{[\boldsymbol{q}(\mathcal{E}, \boldsymbol{s})]_{\mathcal{E}}=\left(\boldsymbol{C}_{\mathcal{E}}^{\top} \boldsymbol{D} \boldsymbol{C}_{\mathcal{E}}\right)^{\dagger}[\boldsymbol{s}]_{\mathcal{E}}}  \tag{75}\\
{[\boldsymbol{q}(\mathcal{E}, \boldsymbol{s})]_{\neg \mathcal{E}}=\mathbf{0}}
\end{array}\right.
$$

For every $i \in\{1,2, \ldots, 2 n\}$, substituting (70) into

$$
\xi_{i}\left(\hat{\boldsymbol{w}}_{\mathrm{EQ}}(\mathcal{E}, \boldsymbol{s}, \lambda)\right):=\boldsymbol{c}_{i}^{\top}\left(\boldsymbol{b}-\boldsymbol{D} \boldsymbol{C} \hat{\boldsymbol{w}}_{\mathrm{EQ}}(\mathcal{E}, \boldsymbol{s}, \lambda)\right),
$$

we can rewrite the expression of $\xi_{i}\left(\hat{\boldsymbol{w}}_{\mathrm{EQ}}(\mathcal{E}, \boldsymbol{s}, \lambda)\right)$ as follows:

$$
\begin{equation*}
\xi_{i}\left(\hat{\boldsymbol{w}}_{\mathrm{EQ}}(\mathcal{E}, \boldsymbol{s}, \lambda)\right)=\boldsymbol{c}_{i}^{\top}(\boldsymbol{u}(\mathcal{E})-\lambda \boldsymbol{v}(\mathcal{E}, \boldsymbol{s})), \tag{71}
\end{equation*}
$$

where $\boldsymbol{u}(\mathcal{E})$ and $\boldsymbol{v}(\mathcal{E}, \boldsymbol{s})$ are defined as

$$
\begin{align*}
u(\mathcal{E}) & :=\boldsymbol{b}-\boldsymbol{D} \boldsymbol{C}_{\mathcal{E}}[\boldsymbol{p}(\mathcal{E})]_{\mathcal{E}}  \tag{76}\\
\boldsymbol{v}(\mathcal{E}, s) & :=-\boldsymbol{D} \boldsymbol{C}_{\mathcal{E}}[\boldsymbol{q}(\mathcal{E}, s)]_{\mathcal{E}} \tag{77}
\end{align*}
$$

with $\boldsymbol{p}(\mathcal{E})$ and $\boldsymbol{q}(\mathcal{E}, \boldsymbol{s})$ respectively defined in (74) and (75).
E.1.3 Step 2: Computing the Solution Interval $I_{i}^{a}(\mathcal{E}, s)$ of the $i$ th (SEL-a) Inequation

For every $i \in \mathcal{E}$, substituting (70) into the $i$ th (SEL-a) inequation yields

$$
\begin{equation*}
[\boldsymbol{s}]_{i}[\boldsymbol{q}(\mathcal{E}, \boldsymbol{s})]_{i} \lambda \leq[\boldsymbol{s}]_{i}[\boldsymbol{p}(\mathcal{E})]_{i} \tag{72}
\end{equation*}
$$

Let us fix $(\mathcal{E}, s) \in \mathcal{V}$ and solve (72) with respect to $\lambda \in \mathbb{R}$.

Since (72) is a linear inequation with respect to $\lambda$, its solution set should be a (possibly empty) closed interval, say $I_{i}^{\mathrm{a}}(\mathcal{E}, s)$, where its endpoints are defined as

$$
\begin{aligned}
l_{i}^{\mathrm{a}}(\mathcal{E}, s) & :=\inf \{\lambda \in \mathbb{R} \mid \lambda \text { satisfies }(72)\}, \\
r_{i}^{\mathrm{a}}(\mathcal{E}, \boldsymbol{s}) & :=\sup \{\lambda \in \mathbb{R} \mid \lambda \text { satisfies }(72)\}
\end{aligned}
$$

The closed-form expressions of $l_{i}^{\mathrm{a}}(\mathcal{E}, \boldsymbol{s})$ and $r_{i}^{\mathrm{a}}(\mathcal{E}, \boldsymbol{s})$ can be obtained as follows (notice that $[\boldsymbol{s}]_{i} \neq 0$ from $i \in \mathcal{E}$ ):

1) If $[\boldsymbol{s}]_{i}[\boldsymbol{q}(\mathcal{E}, s)]_{i}>0$, then we have

$$
l_{i}^{\mathrm{a}}(\mathcal{E}, \boldsymbol{s})=-\infty, \quad r_{i}^{\mathrm{a}}(\mathcal{E}, \boldsymbol{s})=[\boldsymbol{p}(\mathcal{E})]_{i} /[\boldsymbol{q}(\mathcal{E}, \boldsymbol{s})]_{i}
$$

2) If $[\boldsymbol{s}]_{i}[\boldsymbol{q}(\mathcal{E}, s)]_{i}<0$, then we have

$$
l_{i}^{\mathrm{a}}(\mathcal{E}, \boldsymbol{s})=[\boldsymbol{p}(\mathcal{E})]_{i} /[\boldsymbol{q}(\mathcal{E}, \boldsymbol{s})]_{i}, \quad r_{i}^{\mathrm{a}}(\mathcal{E}, \boldsymbol{s})=+\infty
$$

3) If $[\boldsymbol{q}(\mathcal{E}, \boldsymbol{s})]_{i}=0$, then:

- when $[\boldsymbol{s}]_{i}[\boldsymbol{p}(\mathcal{E})]_{i} \geq 0$, we have

$$
l_{i}^{\mathrm{a}}(\mathcal{E}, s)=-\infty, \quad r_{i}^{\mathrm{a}}(\mathcal{E}, s)=+\infty
$$

- when $[\boldsymbol{s}]_{i}[\boldsymbol{p}(\mathcal{E})]_{i}<0$, we have

$$
l_{i}^{\mathrm{a}}(\mathcal{E}, s)=+\infty, \quad r_{i}^{\mathrm{a}}(\mathcal{E}, s)=-\infty
$$

Combining the discussion above with the definition of the $i$ th (SEL-a) inequation yields the following lemma.
Lemma 9. For $(\mathcal{E}, \boldsymbol{s}) \in \mathcal{V}, \lambda \in \mathbb{R}$ and $i \in \mathcal{E}$, we have

$$
\begin{equation*}
[\boldsymbol{s}]_{i}\left[\hat{\boldsymbol{w}}_{E Q}(\mathcal{E}, \boldsymbol{s}, \lambda)\right]_{i} \geq 0 \tag{SEL-a}
\end{equation*}
$$

if and only if $l_{i}^{a}(\mathcal{E}, \boldsymbol{s}) \leq \lambda \leq r_{i}^{a}(\mathcal{E}, s)$.
Moreover, the following holds:
(a) If $l_{i}^{a}(\mathcal{E}, \boldsymbol{s})<\lambda<r_{i}^{a}(\mathcal{E}, \boldsymbol{s})$, then $\hat{\boldsymbol{w}}_{E Q}(\mathcal{E}, \boldsymbol{s}, \lambda)$ satisfies the ith $(\mathcal{E}, \boldsymbol{s})-(N Q-a)$ inequation strictly at $\lambda$, i.e.,

$$
[\boldsymbol{s}]_{i}\left[\hat{\boldsymbol{w}}_{E Q}(\mathcal{E}, \boldsymbol{s}, \lambda)\right]_{i}>0
$$

(b) If $\lambda=l_{i}^{a}(\mathcal{E}, \boldsymbol{s})$ or $\lambda=r_{i}^{a}(\mathcal{E}, \boldsymbol{s})$, then $\hat{\boldsymbol{w}}_{E Q}(\mathcal{E}, \boldsymbol{s}, \lambda)$ attains equality for the $i$ ith $(\mathcal{E}, \boldsymbol{s})-(N Q-a)$ inequation at $\lambda$, i.e.,

$$
[\boldsymbol{s}]_{i}\left[\hat{\boldsymbol{w}}_{E Q}(\mathcal{E}, \boldsymbol{s}, \lambda)\right]_{i}=0
$$

The expressions of $l_{i}^{a}(\mathcal{E}, \boldsymbol{s})$ and $r_{i}^{a}(\mathcal{E}, \boldsymbol{s})$ are as follows: for $\boldsymbol{p}(\mathcal{E})$ defined in (74) and $\boldsymbol{q}(\mathcal{E}, \boldsymbol{s})$ defined in (75),

- if $[\boldsymbol{q}(\mathcal{E}, \boldsymbol{s})]_{i}=0$ and $[\boldsymbol{s}]_{i}[\boldsymbol{p}(\mathcal{E})]_{i}<0$, then

$$
l_{i}^{a}(\mathcal{E}, s):=+\infty, \quad r_{i}^{a}(\mathcal{E}, s):=-\infty
$$

in which case we have $\left[l_{i}^{a}(\mathcal{E}, s), r_{i}^{a}(\mathcal{E}, s)\right]=\emptyset$.

- if $[\boldsymbol{q}(\mathcal{E}, \boldsymbol{s})]_{i} \neq 0$ or $[\boldsymbol{s}]_{i}[\boldsymbol{p}(\mathcal{E})]_{i} \geq 0$, then

$$
\begin{align*}
l_{i}^{a}(\mathcal{E}, \boldsymbol{s}) & := \begin{cases}\frac{[\boldsymbol{p}(\mathcal{E})]_{i}}{[\boldsymbol{q}(\mathcal{E}, \boldsymbol{s})]_{i}}, & \text { if }[\boldsymbol{s}]_{i}[\boldsymbol{q}(\mathcal{E}, \boldsymbol{s})]_{i}<0, \\
-\infty, & \text { otherwise }\end{cases}  \tag{78}\\
r_{i}^{a}(\mathcal{E}, \boldsymbol{s}) & := \begin{cases}\frac{[\boldsymbol{p}(\mathcal{E})]_{i}}{[\boldsymbol{q}(\mathcal{E}, \boldsymbol{s})]_{i}}, & \text { if }[\boldsymbol{s}]_{i}[\boldsymbol{q}(\mathcal{E}, \boldsymbol{s})]_{i}>0 \\
+\infty, & \text { otherwise }\end{cases}
\end{align*}
$$

in which case we have $\left[l_{i}^{a}(\mathcal{E}, s), r_{i}^{a}(\mathcal{E}, s)\right] \neq \emptyset$.

## E.1.4 Step 2: Computing the Solution Interval $I_{i}^{b}(\mathcal{E}, s)$ of the $i$ th (SEL-b) Inequation

For every $i \in \neg \mathcal{E}$, substituting (71) into the $i$ th (SEL-b) inequation yields

$$
\begin{equation*}
-\lambda \leq \boldsymbol{c}_{i}^{\top} \boldsymbol{u}(\mathcal{E})-\boldsymbol{c}_{i}^{\top} \boldsymbol{v}(\mathcal{E}, \boldsymbol{s}) \lambda \leq \lambda \tag{73}
\end{equation*}
$$

Let us fix $(\mathcal{E}, \boldsymbol{s}) \in \mathcal{V}$ and solve (73) with respect to $\lambda \in \mathbb{R}$.
Consider the 2 D space $(\lambda, f) \in \mathbb{R}^{2}$, then it is evident that solving (73) is equivalent to finding the intersection between the line $f=\boldsymbol{c}_{i}^{\top} \boldsymbol{u}(\mathcal{E})-\boldsymbol{c}_{i}^{\top} \boldsymbol{v}(\mathcal{E}, s) \lambda$ and the region $-\lambda \leq f \leq \lambda$. Hence one can imagine that the solution set of (73) should be a (possibly empty) closed interval, say $I_{i}^{\mathrm{b}}(\mathcal{E}, s)$, where its endpoints are defined as

$$
\begin{aligned}
l_{i}^{\mathrm{b}}(\mathcal{E}, s) & :=\inf \{\lambda \in \mathbb{R} \mid \lambda \text { satisfies }(73)\} \\
r_{i}^{\mathrm{b}}(\mathcal{E}, s) & :=\sup \{\lambda \in \mathbb{R} \mid \lambda \text { satisfies }(73)\}
\end{aligned}
$$

Exploiting the geometric interpretation above, the closed-form expressions of $l_{i}^{\mathrm{b}}(\mathcal{E}, \boldsymbol{s})$ and $r_{i}^{\mathrm{b}}(\mathcal{E}, \boldsymbol{s})$ can be obtained as follows:

1) If $\boldsymbol{c}_{i}^{\top} \boldsymbol{u}(\mathcal{E})>0$, then:

- when $\boldsymbol{c}_{i}^{\top} \boldsymbol{v}(\mathcal{E}, \boldsymbol{s}) \leq-1$, we have

$$
l_{i}^{\mathrm{b}}(\mathcal{E}, s)=+\infty, \quad r_{i}^{\mathrm{b}}(\mathcal{E}, s)=-\infty
$$

- when $-1<\boldsymbol{c}_{i}^{\top} \boldsymbol{v}(\mathcal{E}, s) \leq 1$, we have

$$
l_{i}^{\mathrm{b}}(\mathcal{E}, \boldsymbol{s})=\frac{\boldsymbol{c}_{i}^{\top} \boldsymbol{u}(\mathcal{E})}{\boldsymbol{c}_{i}^{\top} \boldsymbol{v}(\mathcal{E}, \boldsymbol{s})+1}, \quad r_{i}^{\mathrm{b}}(\mathcal{E}, \boldsymbol{s})=+\infty
$$

- when $\boldsymbol{c}_{i}^{\top} \boldsymbol{v}(\mathcal{E}, \boldsymbol{s})>1$, we have

$$
l_{i}^{\mathrm{b}}(\mathcal{E}, s)=\frac{\boldsymbol{c}_{i}^{\top} \boldsymbol{u}(\mathcal{E})}{\boldsymbol{c}_{i}^{\top} \boldsymbol{v}(\mathcal{E}, \boldsymbol{s})+1}, \quad r_{i}^{\mathrm{b}}(\mathcal{E}, s)=\frac{\boldsymbol{c}_{i}^{\top} \boldsymbol{u}(\mathcal{E})}{\boldsymbol{c}_{i}^{\top} \boldsymbol{v}(\mathcal{E}, \boldsymbol{s})-1}
$$

2) If $\boldsymbol{c}_{i}^{\top} \boldsymbol{u}(\mathcal{E})<0$, then:

- when $\boldsymbol{c}_{i}^{\top} \boldsymbol{v}(\mathcal{E}, s) \geq 1$, we have

$$
l_{i}^{\mathrm{b}}(\mathcal{E}, s)=+\infty, \quad r_{i}^{\mathrm{b}}(\mathcal{E}, s)=-\infty
$$

- when $1>\boldsymbol{c}_{i}^{\top} \boldsymbol{v}(\mathcal{E}, \boldsymbol{s}) \geq-1$, we have

$$
l_{i}^{\mathrm{b}}(\mathcal{E}, s)=\frac{\boldsymbol{c}_{i}^{\top} \boldsymbol{u}(\mathcal{E})}{\boldsymbol{c}_{i}^{\top} \boldsymbol{v}(\mathcal{E}, s)-1}, \quad r_{i}^{\mathrm{b}}(\mathcal{E}, s)=+\infty
$$

- when $\boldsymbol{c}_{i}^{\top} \boldsymbol{v}(\mathcal{E}, \boldsymbol{s})<-1$, we have

$$
l_{i}^{\mathrm{b}}(\mathcal{E}, s)=\frac{\boldsymbol{c}_{i}^{\top} \boldsymbol{u}(\mathcal{E})}{\boldsymbol{c}_{i}^{\top} \boldsymbol{v}(\mathcal{E}, s)-1}, \quad r_{i}^{\mathrm{b}}(\mathcal{E}, s)=\frac{\boldsymbol{c}_{i}^{\top} \boldsymbol{u}(\mathcal{E})}{\boldsymbol{c}_{i}^{\top} \boldsymbol{v}(\mathcal{E}, s)+1}
$$

3) If $\boldsymbol{c}_{i}^{\top} \boldsymbol{u}(\mathcal{E})=0$, then:

- when $-1 \leq \boldsymbol{c}_{i}^{\top} \boldsymbol{v}(\mathcal{E}, s) \leq 1$, we have

$$
l_{i}^{\mathrm{b}}(\mathcal{E}, s)=0, \quad r_{i}^{\mathrm{b}}(\mathcal{E}, s)=+\infty
$$

- when $\left|\boldsymbol{c}_{i}^{\top} \boldsymbol{v}(\mathcal{E}, s)\right|>1$, we have

$$
l_{i}^{\mathrm{b}}(\mathcal{E}, s)=0, \quad r_{i}^{\mathrm{b}}(\mathcal{E}, s)=0
$$

Combining the discussion above with the definition of the $i$ th (SEL-b) inequation yields the following lemma.
Lemma 10. For $(\mathcal{E}, s) \in \mathcal{V}, \lambda \in \mathbb{R}$ and $i \in \neg \mathcal{E}$, we have

$$
\begin{equation*}
-\lambda \leq \xi_{i}\left(\hat{\boldsymbol{w}}_{E Q}(\mathcal{E}, \boldsymbol{s}, \lambda)\right) \leq \lambda \tag{SEL-a}
\end{equation*}
$$

if and only if $l_{i}^{b}(\mathcal{E}, s) \leq \lambda \leq r_{i}^{b}(\mathcal{E}, s)$.
Moreover, the following holds:
(a) If $l_{i}^{b}(\mathcal{E}, s)<\lambda<r_{i}^{b}(\mathcal{E}, s)$, then $\hat{\boldsymbol{w}}_{E Q}(\mathcal{E}, s, \lambda)$ satisfies the $i$ th $(\mathcal{E}, s)$-(NQ-b) inequation strictly at $\lambda$, i.e.,

$$
\left|\xi_{i}\left(\hat{\boldsymbol{w}}_{E Q}(\mathcal{E}, \boldsymbol{s}, \lambda)\right)\right|<\lambda
$$

(b) If $\lambda=l_{i}^{b}(\mathcal{E}, s)$ or $\lambda=r_{i}^{b}(\mathcal{E}, s)$, then $\hat{\boldsymbol{w}}_{E Q}(\mathcal{E}, s, \lambda)$ attains equality for the $i$ th $(\mathcal{E}, s)-(N Q-b)$ inequation at $\lambda$, i.e.,

$$
\left|\xi_{i}\left(\hat{\boldsymbol{w}}_{E Q}(\mathcal{E}, \boldsymbol{s}, \lambda)\right)\right|=\lambda
$$

The expressions of $l_{i}^{b}(\mathcal{E}, \boldsymbol{s})$ and $r_{i}^{b}(\mathcal{E}, \boldsymbol{s})$ are as follows: for $\boldsymbol{u}(\mathcal{E})$ defined in (76) and $\boldsymbol{v}(\mathcal{E}, \boldsymbol{s})$ defined in (77),

- if $\left|\boldsymbol{c}_{i}^{\top} \boldsymbol{v}(\mathcal{E})\right| \geq 1$ and $\boldsymbol{c}_{i}^{\top} \boldsymbol{u}(\mathcal{E}) \boldsymbol{c}_{i}^{T} \boldsymbol{v}(\mathcal{E}, \boldsymbol{s})<0$, then

$$
l_{i}^{b}(\mathcal{E}, s)=+\infty, \quad r_{i}^{b}(\mathcal{E}, s)=-\infty
$$

in which case we have $\left[l_{i}^{b}(\mathcal{E}, s), r_{i}^{b}(\mathcal{E}, s)\right]=\emptyset$.

- if $\left|\boldsymbol{c}_{i}^{\top} \boldsymbol{v}(\mathcal{E})\right|<1$ or $\boldsymbol{c}_{i}^{\top} \boldsymbol{u}(\mathcal{E}) \boldsymbol{c}_{i}^{T} \boldsymbol{v}(\mathcal{E}, s) \geq 0$, then

$$
\begin{align*}
& l_{i}^{b}(\mathcal{E}, \boldsymbol{s}):= \begin{cases}\frac{\boldsymbol{c}_{i}^{\top} \boldsymbol{u}(\mathcal{E})}{\boldsymbol{c}_{i}^{\top} \boldsymbol{v}(\mathcal{E}, \boldsymbol{s})+1}, & \text { if } \boldsymbol{c}_{i}^{\top} \boldsymbol{u}(\mathcal{E})>0, \\
\frac{\boldsymbol{c}_{i}^{\top} \boldsymbol{u}(\mathcal{E})}{\boldsymbol{c}_{i}^{\top} \boldsymbol{v}(\mathcal{E}, \boldsymbol{s})-1}, & \text { if } \boldsymbol{c}_{i}^{\top} \boldsymbol{u}(\mathcal{E})<0, \\
0, & \text { otherwise },\end{cases}  \tag{79}\\
& r_{i}^{b}(\mathcal{E}, \boldsymbol{s}):= \begin{cases}\frac{\boldsymbol{c}_{i}^{\top} \boldsymbol{u}(\mathcal{E})}{\boldsymbol{c}_{i}^{\top} \boldsymbol{v}(\mathcal{E}, \boldsymbol{s})-1}, & \text { if } \min \left\{\boldsymbol{c}_{i}^{\top} \boldsymbol{u}(\mathcal{E}), \boldsymbol{c}_{i}^{\top} \boldsymbol{v}(\mathcal{E}, \boldsymbol{s})-1\right\}>0, \\
\frac{\boldsymbol{c}_{i}^{\top} \boldsymbol{u}(\mathcal{E})}{\boldsymbol{c}_{i}^{\top} \boldsymbol{v}(\mathcal{E}, \boldsymbol{s})+1}, & \text { if } \max \left\{\boldsymbol{c}_{i}^{\top} \boldsymbol{u}(\mathcal{E}), \boldsymbol{c}_{i}^{\top} \boldsymbol{v}(\mathcal{E}, \boldsymbol{s})+1\right\}<0, \\
0, & \text { if } \boldsymbol{c}_{i}^{\top} \boldsymbol{u}(\mathcal{E})=0 \text { and }\left|\boldsymbol{c}_{i}^{\top} \boldsymbol{v}(\mathcal{E}, \boldsymbol{s})\right|>1, \\
+\infty, & \text { otherwise },\end{cases}
\end{align*}
$$

in which case we have $\left[l_{i}^{b}(\mathcal{E}, s), r_{i}^{b}(\mathcal{E}, s)\right] \neq \emptyset$.

## E. 2 Proof of Proposition 11

Before proving Prop. 11, we present the following lemma, introducing several equivalent statements of (SEL) (Def. 8).
Lemma 11. For $(\mathcal{E}, s) \in \mathcal{V}$ and $\lambda>0$, consider the selection condition of $(\mathcal{E}, \boldsymbol{s})$ at $\lambda$ :

$$
\begin{equation*}
\hat{\boldsymbol{w}}_{E Q}(\mathcal{E}, \boldsymbol{s}, \lambda) \text { satisfies }(\mathcal{E}, \boldsymbol{s})-(N Q) \text { at } \lambda . \tag{SEL}
\end{equation*}
$$

Then (SEL) is equivalent to the following statements, which are arranged in a gradually stronger order:
(a) $\hat{\boldsymbol{w}}_{E Q}(\mathcal{E}, \boldsymbol{s}, \lambda) \in \mathcal{S}_{e}(\lambda)$,
(b) $\hat{\boldsymbol{w}}_{E Q}(\mathcal{E}, \boldsymbol{s}, \lambda) \in \mathcal{S}_{P O L Y}(\mathcal{E}, \boldsymbol{s}, \lambda) \subset \mathcal{S}_{e}(\lambda)$,
(c) $\hat{\boldsymbol{w}}_{E Q}(\mathcal{E}, \boldsymbol{s}, \lambda)$ is the unique minimum $\ell_{2}$-norm element in $\mathcal{S}_{\text {POLY }}(\mathcal{E}, \boldsymbol{s}, \lambda) \subset \mathcal{S}_{e}(\lambda)$,
where $\mathcal{S}_{\text {POLY }}(\mathcal{E}, s, \lambda)$ is the cutting polytope in Def. 7.
Proof: For simplicity, we define $\hat{\boldsymbol{w}}_{\mathrm{EQ}}:=\hat{\boldsymbol{w}}_{\mathrm{EQ}}(\mathcal{E}, s, \lambda)$.
Since $\hat{\boldsymbol{w}}_{\mathrm{EQ}}$ naturally satisfies $(\mathcal{E}, \boldsymbol{s})$-(EQ) at $\lambda$ from Prop. 8 , one can see that the statement (b) is equivalent to (SEL). In the following, we prove the equivalence between the statement (b) and the statements (a)(c).
(I) We first prove the equivalence between the statements (a) and (b). If $\hat{\boldsymbol{w}}_{\mathrm{EQ}} \in \mathcal{S}_{\mathrm{POLY}}(\mathcal{E}, \boldsymbol{s}, \lambda)$, then we have $\hat{\boldsymbol{w}}_{\mathrm{EQ}} \in \mathcal{S}_{\mathrm{e}}(\lambda)$ directly from Prop. 7 (a). Hence (b) implies (a).

Reversely, if $\hat{\boldsymbol{w}}_{\mathrm{EQ}} \in \mathcal{S}_{\mathrm{e}}(\lambda)$, then since $\hat{\boldsymbol{w}}_{\mathrm{EQ}}$ satisfies $(\mathcal{E}, \boldsymbol{s})$-(EQ-a) at $\lambda$ from its definition (Prop. 8), we have $\xi_{i}\left(\hat{\boldsymbol{w}}_{\mathrm{EQ}}\right)=$ $\lambda[s]_{i}$ for every $i \in \mathcal{E}$, which implies that

$$
\begin{array}{r}
\mathcal{E} \subset \mathcal{E}\left(\lambda, \hat{\boldsymbol{w}}_{\mathrm{EQ}}\right)=\mathcal{E}_{\mathrm{e}}(\lambda) \\
{[\boldsymbol{s}]_{\mathcal{E}}=\left[\boldsymbol{s}\left(\lambda, \hat{\boldsymbol{w}}_{\mathrm{EQ}}\right)\right]_{\mathcal{E}}=\left[\boldsymbol{s}_{\mathrm{e}}(\lambda)\right]_{\mathcal{E}}}
\end{array}
$$

Hence we have $(\mathcal{E}, s) \preceq\left(\mathcal{E}_{\mathrm{e}}(\lambda), s_{\mathrm{e}}(\lambda)\right)$. Moreover, since $\hat{\boldsymbol{w}}_{\mathrm{EQ}}$ satisfies $(\mathcal{E}, \boldsymbol{s})$-(EQ-b) at $\lambda$ from its definition (Prop. 8), we have $\operatorname{supp}\left(\hat{\boldsymbol{w}}_{\mathrm{EQ}}\right) \subset \mathcal{E}$. Combining the discussion above with Prop. 6 in Sec. 5.2, we obtain the following relations:

$$
\begin{aligned}
(\mathcal{E}, \boldsymbol{s}) & \preceq\left(\mathcal{E}_{\mathrm{e}}(\lambda), \boldsymbol{s}_{\mathrm{e}}(\lambda)\right), \\
\operatorname{supp}\left(\hat{\boldsymbol{w}}_{\mathrm{EQ}}\right) & \subset \mathcal{E} \\
\left(\operatorname{supp}\left(\hat{\boldsymbol{w}}_{\mathrm{EQ}}\right), \operatorname{sign}\left(\hat{\boldsymbol{w}}_{\mathrm{EQ}}\right)\right. & \preceq\left(\mathcal{E}_{\mathrm{e}}(\lambda), \boldsymbol{s}_{\mathrm{e}}(\lambda)\right) .
\end{aligned}
$$

One can verify that the relations above yields

$$
\left(\operatorname{supp}\left(\hat{\boldsymbol{w}}_{\mathrm{EQ}}\right), \operatorname{sign}\left(\hat{\boldsymbol{w}}_{\mathrm{EQ}}\right)\right) \preceq(\mathcal{E}, \boldsymbol{s}) \preceq\left(\mathcal{E}_{\mathrm{e}}(\lambda), \boldsymbol{s}_{\mathrm{e}}(\lambda)\right) .
$$

Since $\hat{\boldsymbol{w}}_{\mathrm{EQ}} \in \mathcal{S}_{\mathrm{e}}(\lambda)$ and $(\mathcal{E}, \boldsymbol{s}) \in \mathcal{V}$, the generalized inequality above implies $\hat{\boldsymbol{w}}_{\mathrm{EQ}} \in \mathcal{S}_{\mathrm{POLY}}(\mathcal{E}, \boldsymbol{s}, \lambda)$ from Prop. 7 (c). Hence the statements (a) and (b) are equivalent.
(II) Next we prove the equivalence between the statements (b) and (c). Notice that from Prop. $8, \hat{w}_{\mathrm{EQ}}(\mathcal{E}, \boldsymbol{s}, \lambda)$ is the minimum $\ell_{2}$-norm element in the solution set of $(\mathcal{E}, \boldsymbol{s})$-(EQ) at $\lambda$, which is a super set of $\mathcal{S}_{\mathrm{POLY}}(\mathcal{E}, \boldsymbol{s}, \lambda)$. Hence $\hat{\boldsymbol{w}}_{\mathrm{EQ}}(\mathcal{E}, \boldsymbol{s}, \lambda)$ is the minimum $\ell_{2}$-norm element in $\mathcal{S}_{\mathrm{POLY}}(\mathcal{E}, \boldsymbol{s}, \lambda)$ if and only if $\hat{\boldsymbol{w}}_{\mathrm{EQ}}(\mathcal{E}, \boldsymbol{s}, \lambda) \in \mathcal{S}_{\mathrm{POLY}}(\mathcal{E}, \boldsymbol{s}, \lambda)$, which leads to the equivalence between the statements (b) and (c).

Next, we prove Prop. 11 using Lemma 11.
Proof of Proposition 11: Given $(\mathcal{E}, s) \in \mathcal{V}$, from the definition of the selection interval $I_{\text {SEL }}(\mathcal{E}, s)$ (Prop. 10), ( $\left.\mathcal{E}, s\right)$ satisfies (SEL) at $\lambda>0$ if and only if $\lambda \in I_{\text {SEL }}(\mathcal{E}, s)$. Hence we can prove the results (a) and (b) as follows.
(a) For every $\lambda \in \operatorname{int}\left(I_{\text {SEL }}(\mathcal{E}, s)\right)$, we have

$$
\hat{\boldsymbol{w}}_{\mathrm{EQ}}(\mathcal{E}, \boldsymbol{s}, \lambda) \in \mathcal{S}_{\mathrm{POLY}}(\mathcal{E}, \boldsymbol{s}, \lambda) \subset \mathcal{S}_{\mathrm{e}}(\lambda)
$$

from Lemma 11. Moreover, from Lemma 8 in Appendix E.1, $\hat{\boldsymbol{w}}_{\mathrm{EQ}}(\mathcal{E}, \boldsymbol{s}, \lambda)$ satisfies $(\mathcal{E}, \boldsymbol{s})$-(NQ-b) strictly at $\lambda$, hence one can derive the following

$$
(\mathcal{E}, s)=\left(\mathcal{E}_{\mathrm{e}}(\lambda), s_{\mathrm{e}}(\lambda)\right)
$$

from the properties of cutting polytope (Prop. 7 (d)).
From Lemma 11 (c), $\hat{\boldsymbol{w}}_{\mathrm{EQ}}(\mathcal{E}, s, \lambda)$ is the minimum $\ell_{2}$-norm element in

$$
\mathcal{S}_{\mathrm{POLY}}(\mathcal{E}, s, \lambda)=\mathcal{S}_{\mathrm{POLY}}\left(\mathcal{E}_{\mathrm{e}}(\lambda), s_{\mathrm{e}}(\lambda), \lambda\right)=\mathcal{S}_{\mathrm{e}}(\lambda)
$$

where the last equality follows from Prop. $7(\mathrm{~b})$. Thus we have $\hat{\boldsymbol{w}}_{\mathrm{EQ}}(\mathcal{E}, \boldsymbol{s}, \lambda)=\boldsymbol{w}_{\star}(\lambda)$ from the definition of $\boldsymbol{w}_{\star}(\lambda)$.
In addition, from Lemma 8 in Appendix E.1, for every $\lambda \in \operatorname{int}\left(I_{\mathrm{SEL}}(\mathcal{E}, \boldsymbol{s})\right)$, $\hat{\boldsymbol{w}}_{\mathrm{EQ}}(\mathcal{E}, \boldsymbol{s}, \lambda) \in \mathcal{S}_{\mathrm{POLY}}(\mathcal{E}, \boldsymbol{s}, \lambda)$ satisfies $(\mathcal{E}, s)$-(NQ-a) strictly at $\lambda$, hence one can derive

$$
(\mathcal{E}, \boldsymbol{s})=\left(\operatorname{supp}\left(\hat{\boldsymbol{w}}_{\mathrm{EQ}}(\mathcal{E}, \boldsymbol{s}, \lambda)\right), \operatorname{sign}\left(\hat{\boldsymbol{w}}_{\mathrm{EQ}}(\mathcal{E}, \boldsymbol{s}, \lambda)\right)\right)
$$

from Prop. $7(\mathrm{~d})$. Since $\hat{\boldsymbol{w}}_{\mathrm{EQ}}(\mathcal{E}, \boldsymbol{s}, \lambda)=\boldsymbol{w}_{\star}(\lambda)$, we have

$$
(\mathcal{E}, \boldsymbol{s})=\left(\operatorname{supp}\left(\boldsymbol{w}_{\star}(\lambda)\right), \operatorname{sign}\left(\boldsymbol{w}_{\star}(\lambda)\right)\right)
$$

(b) For $\lambda \notin I_{\text {SEL }}(\mathcal{E}, s)$, Lemma 11 implies that

$$
\hat{\boldsymbol{w}}_{\mathrm{EQ}}(\mathcal{E}, \boldsymbol{s}, \lambda) \notin \mathcal{S}_{\mathrm{e}}(\lambda),
$$

hence we have $\hat{\boldsymbol{w}}_{\mathrm{EQ}}(\mathcal{E}, \boldsymbol{s}, \lambda) \neq \boldsymbol{w}_{\star}(\lambda) \in \mathcal{S}_{\mathrm{e}}(\lambda)$.

## E. 3 Proof of Proposition 12

Proof: (a) We prove the result by contradiction. Suppose there exists $\bar{\lambda} \in \operatorname{int}\left(I_{\text {SEL }}\left(\mathcal{E}_{1}, \boldsymbol{s}_{1}\right)\right) \cap \operatorname{int}\left(I_{\text {SEL }}\left(\mathcal{E}_{2}, \boldsymbol{s}_{2}\right)\right)$, then from Prop. 11 (a), we have

$$
\left(\mathcal{E}_{\mathrm{e}}(\bar{\lambda}), \boldsymbol{s}_{\mathrm{e}}(\bar{\lambda})\right)=\left(\mathcal{E}_{1}, \boldsymbol{s}_{1}\right) \neq\left(\mathcal{E}_{2}, \boldsymbol{s}_{2}\right)=\left(\mathcal{E}_{\mathrm{e}}(\bar{\lambda}), \boldsymbol{s}_{\mathrm{e}}(\bar{\lambda})\right)
$$

which leads to a contradiction. Hence the result (a) holds.
(b) Define the following:

$$
\begin{aligned}
& I_{\star}:=\cup_{(\mathcal{E}, s) \in \mathcal{V}_{\star}} I_{\mathrm{SEL}}(\mathcal{E}, s) \\
& \bar{I}_{\star}:=\cup_{(\mathcal{E}, s) \in \mathcal{V} \backslash \mathcal{V}_{\star}} I_{\mathrm{SEL}}(\mathcal{E}, s)
\end{aligned}
$$

From Cor. 6 and Prop. 11 (b), one can derive that

$$
\cup_{(\mathcal{E}, s) \in \mathcal{V}} I_{\mathrm{SEL}}(\mathcal{E}, \boldsymbol{s})=\mathbb{R}_{++}
$$

Hence we have $I_{\star} \cup \bar{I}_{\star}=\mathbb{R}_{++}$, which implies $I_{\star} \supset \mathbb{R}_{++} \backslash \bar{I}_{\star}$.
Notice that from the definition of $\mathcal{V}_{\star}$ in Prop. 12, for every $(\mathcal{E}, s) \in \mathcal{V} \backslash \mathcal{V}_{\star}, I_{\mathrm{SEL}}(\mathcal{E}, s)$ is either empty or only contains a single positive number, hence $\bar{I}_{\star}$ is either empty or contains finite positive numbers. If $\bar{I}_{\star}=\emptyset$, then we naturally have $I_{\star} \supset \mathbb{R}_{++} \backslash \bar{I}_{\star}=\mathbb{R}_{++}$, which directly proves the result (b).

Suppose that $\bar{I}_{\star} \neq \emptyset$, then without loss of generality, we can assume that

$$
\bar{I}_{\star}=\left\{\lambda_{1}, \lambda_{2}, \ldots, \lambda_{K}\right\} \subset \mathbb{R}_{++}
$$

where $K$ is a positive integer. For every $k \in\{1,2, \ldots, K\}$, if there exists $(\mathcal{E}, \boldsymbol{s}) \in \mathcal{V}_{\star}$ such that $\lambda_{k} \in \operatorname{int}\left(I_{\text {SEL }}(\mathcal{E}, \boldsymbol{s})\right)$, then we naturally have $\lambda_{k} \in I_{\star}$; otherwise, since $I_{\star} \cup \bar{I}_{\star}=\mathbb{R}_{++}$, one can verify that there must exist $(\mathcal{E}, s) \in \mathcal{V}_{\star}$, such that one of the endpoints of $I_{\text {SEL }}(\mathcal{E}, s)$ coincides with $\lambda_{k}$. Recall that in Prop. 10, all endpoints of $I_{\mathrm{SEL}}(\mathcal{E}, s)$ except 0 and $+\infty$ are closed, hence we have $\lambda_{k} \in I_{\star}$. Accordingly, we have $\lambda_{k} \in I_{\star}$ for every $k \in\{1,2, \ldots, K\}$, combining which with $I_{\star} \supset \mathbb{R}_{++} \backslash \bar{I}_{\star}$ yields $I_{\star}=\mathbb{R}_{++}$, i.e., the result (b) holds.

## Appendix F

## Proof of Theorem 8 in Section 7

## F. 1 A Sketch of the Proof

For the sake of clarity, we initially provide an outline of the proof of Thm. 8. The omitted details within this proof sketch will be elaborated upon in following sections.

Recall the following definition (cf. (43) in Sec. 7.2) of the trajectory switching time $\lambda_{+}$of $\boldsymbol{w}_{\star}(\lambda)$ :

$$
\begin{equation*}
\lambda_{+}:=l_{\mathrm{SEL}}\left(\mathcal{E}_{k}^{\star}, s_{k}^{\star}\right)=r_{\mathrm{SEL}}\left(\mathcal{E}_{k+1}^{\star}, s_{k+1}^{\star}\right) . \tag{43}
\end{equation*}
$$

Hence one can imagine that

$$
\begin{equation*}
\boldsymbol{w}_{\star}^{+}:=\boldsymbol{w}_{\star}\left(\lambda_{+}\right) \tag{80}
\end{equation*}
$$

can be used as a bridge for characterizing the relation among

- the input $(\mathcal{E}, \boldsymbol{s})=\left(\mathcal{E}_{k}^{\star}, s_{k}^{\star}\right) \in \mathcal{V}_{\star}$,
- the output $\left(\mathcal{E}_{+}, s_{+}\right)$,
- the desired result $\left(\mathcal{E}_{k+1}^{\star}, s_{k+1}^{\star}\right) \in \mathcal{V}_{\star}$
of the LARS-sGMC iteration (Algorithm 1).
Exploiting the relation between $\boldsymbol{w}_{\star}^{+}$and ex-equicorrelation pairs involved in the LARS-sGMC iteration, we can establish the correctness of the LARS-sGMC iteration under the "one-at-a-time" assumption. See the following proof sketch.

A sketch of the proof of Theorem 8: We establish the equality between $\left(\mathcal{E}_{+}, \boldsymbol{s}_{+}\right)$and $\left(\mathcal{E}_{k+1}^{\star}, \boldsymbol{s}_{k+1}^{\star}\right)$ by four steps.
Step 1: we derive a value range for $\left(\mathcal{E}_{k+1}^{\star}, s_{k+1}^{\star}\right)$. From (43), we have $\lambda_{+} \in I_{\text {SEL }}\left(\mathcal{E}_{k+1}^{\star}, \boldsymbol{s}_{k+1}^{\star}\right)$, which implies

$$
\begin{equation*}
\boldsymbol{w}_{\star}^{+} \in \mathcal{S}_{\mathrm{POLY}}\left(\mathcal{E}_{k+1}^{\star}, \boldsymbol{s}_{k+1}^{\star}, \lambda_{+}\right) \tag{81}
\end{equation*}
$$

from Thm. 7 in Sec. 6.2. Exploiting the properties of cutting polytopes (Prop. 7 (c) in Sec. 5.3), we can derive the following value range for $\left(\mathcal{E}_{k+1}^{\star}, s_{k+1}^{\star}\right)$ from (81):

$$
\begin{equation*}
\left(\operatorname{supp}\left(\boldsymbol{w}_{\star}^{+}\right), \operatorname{sign}\left(\boldsymbol{w}_{\star}^{+}\right)\right) \preceq\left(\mathcal{E}_{k+1}^{\star}, \boldsymbol{s}_{k+1}^{\star}\right) \preceq\left(\mathcal{E}_{\mathrm{e}}\left(\lambda_{+}\right), \boldsymbol{s}_{\mathrm{e}}\left(\lambda_{+}\right)\right) . \tag{82}
\end{equation*}
$$

Step 2: we establish the relation between $(\mathcal{E}, s)$ and the lower/upper bounds of $\left(\mathcal{E}_{k+1}^{\star}, s_{k+1}^{\star}\right)$. As will be shown in Lemma 12, we can establish the following inclusion:

$$
\boldsymbol{w}_{\star}^{+} \in \mathcal{S}_{\mathrm{POLY}}\left(\mathcal{E}, \boldsymbol{s}, \lambda_{+}\right),
$$

i.e., $\boldsymbol{w}_{\star}^{+}$simultaneously satisfies $(\mathcal{E}, \boldsymbol{s})$-(EQ) and $(\mathcal{E}, \boldsymbol{s})-(\mathrm{NQ})$ at $\lambda_{+}$(cf. Def. 7 in Sec. 5.3). Moreover, the following holds:
(a) For every $i \in\left(\mathcal{E} \backslash \mathcal{I}_{\text {delete }}^{+}\right)$, $\boldsymbol{w}_{\star}^{+}$satisfies the $i$ th $(\mathcal{E}, s)$-(NQ-a) inequation strictly at $\lambda_{+}$.
(b) For every $i \in\left(\neg \mathcal{E} \backslash \mathcal{I}_{\text {insert }}^{+}\right)$, $\boldsymbol{w}_{\star}^{+}$satisfies the $i$ th $(\mathcal{E}, \boldsymbol{s})$-(NQ-b) inequation strictly at $\lambda_{+}$.

Combining the results above (Lemma 12) with Prop. 7 (d) in Sec. 5.3, we can derive the following relation between $(\mathcal{E}, s)$ and the lower/upper bounds of (82):

- If $\left|\mathcal{I}_{\text {delete }}^{+}\right|=0$, then $(\mathcal{E}, \boldsymbol{s})=\left(\operatorname{supp}\left(\boldsymbol{w}_{\star}^{+}\right), \operatorname{sign}\left(\boldsymbol{w}_{\star}^{+}\right)\right)$.
- If $\left|\mathcal{I}_{\text {insert }}^{+}\right|=0$, then $(\mathcal{E}, \boldsymbol{s})=\left(\mathcal{E}_{\mathrm{e}}\left(\lambda_{+}\right), \boldsymbol{s}_{\mathrm{e}}\left(\lambda_{+}\right)\right)$.

Step 3: we establish the relation between $\left(\mathcal{E}_{+}, s_{+}\right)$and the lower/upper bounds of $\left(\mathcal{E}_{k+1}^{\star}, s_{k+1}^{\star}\right)$. As will be shown in Lemma 13, we can establish the following inclusion:

$$
\boldsymbol{w}_{\star}^{+} \in \mathcal{S}_{\mathrm{POLY}}\left(\mathcal{E}_{+}, \boldsymbol{s}_{+}, \lambda_{+}\right)
$$

i.e., $\boldsymbol{w}_{\star}^{+}$simultaneously satisfies $\left(\mathcal{E}_{+}, s_{+}\right)-(\mathrm{EQ})$ and $\left(\mathcal{E}_{+}, \boldsymbol{s}_{+}\right)-(\mathrm{NQ})$ at $\lambda_{+}$(cf. Def. $7 \mathrm{in} \mathrm{Sec.5.3)} \mathrm{}. \mathrm{Moreover} ,\mathrm{we} \mathrm{have:}$
(a) For every $i \in\left(\mathcal{E}_{+} \backslash \mathcal{I}_{\text {insert }}^{+}\right)$, $\boldsymbol{w}_{\star}^{+}$satisfies the $i$ th $\left(\mathcal{E}_{+}, \boldsymbol{s}_{+}\right)$-(NQ-a) inequation strictly at $\lambda_{+}$.
(b) For every $i \in\left(\neg \mathcal{E}_{+} \backslash \mathcal{I}_{\text {delete }}^{+}\right)$, $\boldsymbol{w}_{\star}^{+}$satisfies the $i$ th $\left(\mathcal{E}_{+}, \boldsymbol{s}_{+}\right)$-(NQ-b) inequation strictly at $\lambda_{+}$.

Combining the results above (Lemma 13) with Prop. 7 (d) in Sec. 5.3, we can derive the following relation between $\left(\mathcal{E}_{+}, s_{+}\right)$ and the lower/upper bounds of (82):

- If $\left|\mathcal{I}_{\text {insert }}^{+}\right|=0$, then $\left(\mathcal{E}_{+}, \boldsymbol{s}_{+}\right)=\left(\operatorname{supp}\left(\boldsymbol{w}_{\star}^{+}\right), \operatorname{sign}\left(\boldsymbol{w}_{\star}^{+}\right)\right)$.
- If $\left|\mathcal{I}_{\text {delete }}^{+}\right|=0$, then $\left(\mathcal{E}_{+}, s_{+}\right)=\left(\mathcal{E}_{\mathrm{e}}\left(\lambda_{+}\right), \boldsymbol{s}_{\mathrm{e}}\left(\lambda_{+}\right)\right)$.

Step 4: we apply the $k$ th "one-at-a-time" assumption to narrow down the range of $\left(\mathcal{E}_{k+1}^{\star}, s_{k+1}^{\star}\right)$. From the $k$ th "one-at-a-time" assumption, we have

$$
\left|\mathcal{I}_{\text {delete }}^{+}\right|+\left|\mathcal{I}_{\text {insert }}^{+}\right|=1
$$

Furthermore, exploiting the results of the step 2 and step 3 in this proof sketch, we can show that

- if $\left|\mathcal{I}_{\text {delete }}^{+}\right|=1$ and $\left|\mathcal{I}_{\text {insert }}^{+}\right|=0$, then

$$
\begin{aligned}
(\mathcal{E}, \boldsymbol{s}) & =\left(\mathcal{E}_{\mathrm{e}}\left(\lambda_{+}\right), \boldsymbol{s}_{\mathrm{e}}\left(\lambda_{+}\right)\right) \\
\left(\mathcal{E}_{+}, \boldsymbol{s}_{+}\right) & =\left(\operatorname{supp}\left(\boldsymbol{w}_{\star}^{+}\right), \operatorname{sign}\left(\boldsymbol{w}_{\star}^{+}\right)\right),
\end{aligned}
$$

- if $\left|\mathcal{I}_{\text {delete }}^{+}\right|=0$ and $\left|\mathcal{I}_{\text {insert }}^{+}\right|=1$, then

$$
\begin{aligned}
(\mathcal{E}, \boldsymbol{s}) & =\left(\operatorname{supp}\left(\boldsymbol{w}_{\star}^{+}\right), \operatorname{sign}\left(\boldsymbol{w}_{\star}^{+}\right)\right), \\
\left(\mathcal{E}_{+}, \boldsymbol{s}_{+}\right) & =\left(\mathcal{E}_{\mathrm{e}}\left(\lambda_{+}\right), \boldsymbol{s}_{\mathrm{e}}\left(\lambda_{+}\right)\right) .
\end{aligned}
$$

Combining the results above with the value range (82) of $\left(\mathcal{E}_{k+1}^{\star}, s_{k+1}^{\star}\right)$ in the step 1 , one can verify that either

$$
(\mathcal{E}, \boldsymbol{s}) \preceq\left(\mathcal{E}_{k+1}^{\star}, \boldsymbol{s}_{k+1}^{\star}\right) \preceq\left(\mathcal{E}_{+}, \boldsymbol{s}_{+}\right)
$$

or $\left(\mathcal{E}_{+}, s_{+}\right) \preceq\left(\mathcal{E}_{k+1}^{\star}, s_{k+1}^{\star}\right) \preceq(\mathcal{E}, s)$ holds. From the "one-at-a-time" assumption, the difference between $\mathcal{E}$ and $\mathcal{E}_{+}$is only one index, hence by the definition of $\preceq$ (Prop. 6 in Sec. 5.2), either of the generalized inequalities above implies that

$$
\left(\mathcal{E}_{k+1}^{\star}, \boldsymbol{s}_{k+1}^{\star}\right)=(\mathcal{E}, \boldsymbol{s}) \text { or }\left(\mathcal{E}_{+}, \boldsymbol{s}_{+}\right)
$$

Notice that

$$
(\mathcal{E}, \boldsymbol{s})=\left(\mathcal{E}_{k}^{\star}, \boldsymbol{s}_{k}^{\star}\right) \neq\left(\mathcal{E}_{k+1}^{\star}, \boldsymbol{s}_{k+1}^{\star}\right)
$$

we conclude that $\left(\mathcal{E}_{k+1}^{\star}, s_{k+1}^{\star}\right)=\left(\mathcal{E}_{+}, s_{+}\right)$.
Next, we provide additional proof details that were omitted in the preceding proof sketch. Specifically, we will prove the results stated in the step 2 and step 3.

## F. 2 Missing Details in Step 2 of the Proof Sketch

We first present the following Lemma 12, which proves the result stated in step 2 of the proof sketch in Appendix F and establishes some preliminary results for proving step 3.
Lemma 12. Let the input of the LARS-sGMC iteration be

$$
(\mathcal{E}, s)=\left(\mathcal{E}_{k}^{\star}, s_{k}^{\star}\right) \text { with } k \in\left\{0,1, \ldots,\left|\mathcal{V}_{\star}\right|-2\right\}
$$

and let $\lambda_{+}$be defined as in (43). Then we have

$$
\begin{equation*}
\boldsymbol{w}_{\star}^{+}:=\boldsymbol{w}_{\star}\left(\lambda_{+}\right) \in \mathcal{S}_{P O L Y}\left(\mathcal{E}, \boldsymbol{s}, \lambda_{+}\right) \tag{83}
\end{equation*}
$$

i.e., $\boldsymbol{w}_{\star}^{+}$simultaneously satisfies $(\mathcal{E}, \boldsymbol{s})-(E Q)$ and $(\mathcal{E}, \boldsymbol{s})-(N Q)$ at $\lambda_{+}$(cf. Def. 7 in Sec. 5.3). Moreover, the following holds:
(a1) For every $i \in\left(\mathcal{E} \backslash \mathcal{I}_{\text {delete }}^{+}\right) \subset \mathcal{E}, \boldsymbol{w}_{\star}^{+}$satisfies the $i$ th $(\mathcal{E}, \boldsymbol{s})$-(NQ-a) inequation strictly at $\lambda_{+}$, i.e.,

$$
\left(\forall i \in \mathcal{E} \backslash \mathcal{I}_{\text {delete }}^{+}\right) \quad[s]_{i}\left[\boldsymbol{w}_{\star}^{+}\right]_{i}>0
$$

(a2) For every $i \in \mathcal{I}_{\text {delete }}^{+} \subset \mathcal{E}, \boldsymbol{w}_{\star}^{+}$attains equality for the ith $(\mathcal{E}, s)-(N Q-a)$ inequation at $\lambda_{+}$, i.e.,

$$
\left(\forall i \in \mathcal{I}_{\text {delete }}^{+}\right) \quad[\boldsymbol{s}]_{i}\left[\boldsymbol{w}_{\star}^{+}\right]_{i}=0
$$

(b1) For every $i \in\left(\neg \mathcal{E} \backslash \mathcal{I}_{\text {insert }}^{+}\right) \subset \neg \mathcal{E}, \boldsymbol{w}_{\star}^{+}$satisfies the ith $(\mathcal{E}, s)$-(NQ-b) inequation strictly at $\lambda_{+}$, i.e.,

$$
\left(\forall i \in \neg \mathcal{E} \backslash \mathcal{I}_{\text {insert }}^{+}\right) \quad\left|\xi_{i}\left(\boldsymbol{w}_{\star}^{+}\right)\right|<\lambda_{+}
$$

(b2) For every $i \in \mathcal{I}_{\text {insert }}^{+} \subset \neg \mathcal{E}, \boldsymbol{w}_{\star}^{+}$attains equality for the $i$ th $(\mathcal{E}, s)-(N Q-b)$ inequation at $\lambda_{+}$, i.e.,

$$
\left(\forall i \in \mathcal{I}_{\text {insert }}^{+}\right) \quad\left|\xi_{i}\left(\boldsymbol{w}_{\star}^{+}\right)\right|=\lambda_{+}
$$

Proof: From the definition (43) of $\lambda_{+}$:

$$
\begin{equation*}
\lambda_{+}:=l_{\mathrm{SEL}}\left(\mathcal{E}_{k}^{\star}, s_{k}^{\star}\right)=r_{\mathrm{SEL}}\left(\mathcal{E}_{k+1}^{\star}, \boldsymbol{s}_{k+1}^{\star}\right), \tag{43}
\end{equation*}
$$

we have $\lambda_{+} \in I_{\text {SEL }}\left(\mathcal{E}_{k}^{\star}, s_{k}^{\star}\right)$, which implies

$$
\begin{align*}
& \boldsymbol{w}_{\star}^{+}=\hat{\boldsymbol{w}}_{\mathrm{EQ}}\left(\mathcal{E}_{k}^{\star}, \boldsymbol{s}_{k}^{\star}, \lambda_{+}\right)=\hat{\boldsymbol{w}}_{\mathrm{EQ}}\left(\mathcal{E}, \boldsymbol{s}, \lambda_{+}\right)  \tag{84}\\
& \boldsymbol{w}_{\star}^{+} \in \mathcal{S}_{\mathrm{POLY}}\left(\mathcal{E}_{k}^{\star}, \boldsymbol{s}_{k}^{\star}, \lambda_{+}\right)=\mathcal{S}_{\mathrm{POLY}}\left(\mathcal{E}, \boldsymbol{s}, \lambda_{+}\right)
\end{align*}
$$

from Thm. 6 (a) and Thm. 7 (a) in Sec. 6.2. Hence (83) holds.
Since $(\mathcal{E}, s)=\left(\mathcal{E}_{k}^{\star}, s_{k}^{\star}\right) \in \mathcal{V}_{\star}$, from the definition of $\mathcal{V}_{\star}$ (Prop. 12 in Sec. 6.1), we have

$$
\lambda_{+}:=l_{\mathrm{SEL}}(\mathcal{E}, s)<r_{\mathrm{SEL}}(\mathcal{E}, s)
$$

combining which with Prop 10 in Sec. 6.1 yields:

$$
\begin{array}{ll}
(\forall i \in \mathcal{E}) & l_{i}^{\mathrm{a}}(\mathcal{E}, \boldsymbol{s}) \leq \lambda_{+}<r_{i}^{\mathrm{a}}(\mathcal{E}, \boldsymbol{s}) \\
(\forall i \in \neg \mathcal{E}) & l_{i}^{\mathrm{b}}(\mathcal{E}, \boldsymbol{s}) \leq \lambda_{+}<r_{i}^{\mathrm{b}}(\mathcal{E}, \boldsymbol{s})
\end{array}
$$

In the sequel, we prove the result (a) and (b) using the aforementioned results.
(a) Recall the definition of $\mathcal{I}_{\text {delete }}^{+}$in Procedure 3:

$$
\mathcal{I}_{\text {delete }}^{+}:=\left\{i \in \mathcal{E} \mid l_{i}^{\mathrm{a}}(\mathcal{E}, s)=\lambda_{+}\right\} .
$$

Hence all indices in $\mathcal{E}$ can be divided into two disjoint parts $\mathcal{E} \backslash \mathcal{I}_{\text {delete }}^{+}$and $\mathcal{I}_{\text {delete, }}^{+}$, and we can derive the following results:

- for every $i \in\left(\mathcal{E} \backslash \mathcal{I}_{\text {delete }}^{+}\right)$, we have $l_{i}^{\text {a }}(\mathcal{E}, s) \neq \lambda_{+}$, combining which with (85) yields

$$
l_{i}^{a}(\mathcal{E}, s)<\lambda_{+}<r_{i}^{\mathrm{a}}(\mathcal{E}, \boldsymbol{s})
$$

Thus from (84) and Lemma 9 (a) in Appendix E.1, $\boldsymbol{w}_{\star}^{+}=\hat{\boldsymbol{w}}_{\mathrm{EQ}}\left(\mathcal{E}, \boldsymbol{s}, \lambda_{+}\right)$satisfies the $i$ th $(\mathcal{E}, \boldsymbol{s})$-(NQ-a) inequation strictly at $\lambda_{+}$. Hence the result (a1) holds.

- for every $i \in \mathcal{I}_{\text {delete }}^{+}$, we have $l_{i}^{\mathrm{a}}(\mathcal{E}, \boldsymbol{s})=\lambda_{+}$. Thus from (84) and Lemma 9 (b) in Appendix E.1, $\boldsymbol{w}_{\star}^{+}=\hat{\boldsymbol{w}}_{\mathrm{EQ}}\left(\mathcal{E}, \boldsymbol{s}, \lambda_{+}\right)$ attains equality for the $i$ th $(\mathcal{E}, s)-(\mathrm{NQ}-\mathrm{a})$ inequation at $\lambda_{+}$. Hence the result (a2) holds.
(b) Recall the definition of $\mathcal{I}_{\text {insert }}^{+}$in Procedure 3:

$$
\mathcal{I}_{\text {insert }}^{+}:=\left\{i \in \neg \mathcal{E} \mid l_{i}^{\mathrm{b}}(\mathcal{E}, s)=\lambda_{+}\right\} .
$$

Hence all indices in $\neg \mathcal{E}$ can be divided into two disjoint parts $\neg \mathcal{E} \backslash \mathcal{I}_{\text {insert }}^{+}$and $\mathcal{I}_{\text {insert }}^{+}$, and we can derive the following results:

- for every $i \in\left(\neg \mathcal{E} \backslash \mathcal{I}_{\text {insert }}^{+}\right)$, we have $l_{i}^{\mathrm{b}}(\mathcal{E}, s) \neq \lambda_{+}$, combining which with (86) yields

$$
l_{i}^{\mathrm{b}}(\mathcal{E}, s)<\lambda_{+}<r_{i}^{\mathrm{b}}(\mathcal{E}, s)
$$

Thus from (84) and Lemma 10 (a) in Appendix E.1, $\boldsymbol{w}_{\star}^{+}=\hat{\boldsymbol{w}}_{\mathrm{EQ}}\left(\mathcal{E}, \boldsymbol{s}, \lambda_{+}\right)$satisfies the $i$ th $(\mathcal{E}, \boldsymbol{s})$-(NQ-b) inequation strictly at $\lambda_{+}$. Hence the result (b1) holds.

- for every $i \in \mathcal{I}_{\text {insert }}^{+}$, we have $l_{i}^{\mathrm{b}}(\mathcal{E}, \boldsymbol{s})=\lambda_{+}$. Thus from (84) and Lemma 10 (b) in Appendix E.1, $\boldsymbol{w}_{\star}^{+}=\hat{\boldsymbol{w}}_{\mathrm{EQ}}\left(\mathcal{E}, \boldsymbol{s}, \lambda_{+}\right)$ attains equality for the $i$ th $(\mathcal{E}, s)-(\mathrm{NQ}-\mathrm{b})$ inequation at $\lambda_{+}$. Hence the result (b2) holds.
Combining the discussion above completes the proof.


Fig. 11: An illustration of the relation between $\mathcal{E}$ and $\mathcal{E}_{+}$.

## F. 3 Missing Details in Step 3 of the Proof Sketch

In this section, we present the following Lemma 13, proving the result stated in step 2 of the proof sketch in Appendix F.

Lemma 13. Let the input of the LARS-sGMC iteration be

$$
(\mathcal{E}, \boldsymbol{s})=\left(\mathcal{E}_{k}^{\star}, s_{k}^{\star}\right) \text { with } k \in\left\{0,1, \ldots,\left|\mathcal{V}_{\star}\right|-2\right\}
$$

let the output be $\left(\mathcal{E}_{+}, s_{+}\right)$and let $\lambda_{+}$be defined as in (43). Then $\left(\mathcal{E}_{+}, s_{+}\right) \in \mathcal{V}$, and we have

$$
\begin{equation*}
\boldsymbol{w}_{\star}^{+}:=\boldsymbol{w}_{\star}\left(\lambda_{+}\right) \in \mathcal{S}_{P O L Y}\left(\mathcal{E}_{+}, \boldsymbol{s}_{+}, \lambda_{+}\right) \tag{87}
\end{equation*}
$$

i.e., $\boldsymbol{w}_{\star}^{+}$simultaneously satisfies $\left(\mathcal{E}_{+}, \boldsymbol{s}_{+}\right)-(E Q)$ and $\left(\mathcal{E}_{+}, \boldsymbol{s}_{+}\right)-(N Q)$ at $\lambda_{+}$(cf. Def. 7 in Sec. 5.3).

Moreover, the following holds:
(a1) For every $i \in\left(\mathcal{E}_{+} \backslash \mathcal{I}_{\text {insert }}^{+}\right) \subset \mathcal{E}_{+}, \boldsymbol{w}_{\star}^{+}$satisfies the $i$ ith $\left(\mathcal{E}_{+}, \boldsymbol{s}_{+}\right)$-(NQ-a) inequation strictly at $\lambda_{+}$, i.e.,

$$
\left(\forall i \in \mathcal{E}_{+} \backslash \mathcal{I}_{\text {insert }}^{+}\right) \quad\left[s_{+}\right]_{i}\left[\boldsymbol{w}_{\star}^{+}\right]_{i}>0
$$

(a2) For every $i \in \mathcal{I}_{\text {insert }}^{+} \subset \mathcal{E}_{+}, \boldsymbol{w}_{\star}^{+}$attains equality for the $i$ th $\left(\mathcal{E}_{+}, s_{+}\right)$-(NQ-a) inequation at $\lambda_{+}$, i.e.,

$$
\left(\forall i \in \mathcal{I}_{\text {insert }}^{+}\right) \quad\left[s_{+}\right]_{i}\left[\boldsymbol{w}_{\star}^{+}\right]_{i}=0
$$

(b1) For every $i \in\left(\neg \mathcal{E}_{+} \backslash \mathcal{I}_{\text {delete }}^{+}\right) \subset \neg \mathcal{E}_{+}, \boldsymbol{w}_{\star}^{+}$satisfies the $i$ ith $\left(\mathcal{E}_{+}, \boldsymbol{s}_{+}\right)-(N Q-b)$ inequation strictly at $\lambda_{+}$, i.e.,

$$
\left(\forall i \in \neg \mathcal{E}_{+} \backslash \mathcal{I}_{\text {delete }}^{+}\right) \quad\left|\xi_{i}\left(\boldsymbol{w}_{\star}^{+}\right)\right|<\lambda_{+} .
$$

(b2) For every $i \in \mathcal{I}_{\text {delete }}^{+} \subset \neg \mathcal{E}_{+}, \boldsymbol{w}_{\star}^{+}$attains equality for the $i$ th $\left(\mathcal{E}_{+}, s_{+}\right)$-(NQ-b) inequation at $\lambda_{+}$, i.e.,

$$
\left(\forall i \in \mathcal{I}_{\text {insert }}^{+}\right) \quad\left|\xi_{i}\left(\boldsymbol{w}_{\star}^{+}\right)\right|=\lambda_{+}
$$

Proof: We first prove $\left(\mathcal{E}_{+}, s_{+}\right) \in \mathcal{V}$ by the definition of the ex-equicorrelation space $\mathcal{V}$ (cf. Prop. 5 in Sec. 5.1).
Substituting $\boldsymbol{w}_{\star}^{+}=\hat{\boldsymbol{w}}_{E Q}\left(\mathcal{E}, s, \lambda_{+}\right)$(cf. (84)) into the expression of $\left(\mathcal{E}_{+}, \boldsymbol{s}_{+}\right)$in Procedure 3 yields

$$
\begin{align*}
\mathcal{E}_{+} & :=\left(\mathcal{E} \backslash \mathcal{I}_{\text {delete }}^{+}\right) \cup \mathcal{I}_{\text {insert }}^{+},  \tag{88}\\
{\left[\boldsymbol{s}_{+}\right]_{i} } & := \begin{cases}0, & \text { if } i \in \mathcal{I}_{\text {delete }}^{+}, \\
\operatorname{sign}\left(\xi_{i}\left(\boldsymbol{w}_{\star}^{+}\right)\right), & \text {if } i \in \mathcal{I}_{\text {insert }}^{+}, \\
{[\boldsymbol{s}]_{i},} & \text { otherwise. },\end{cases} \tag{89}
\end{align*}
$$

We have $\boldsymbol{s}_{+} \in\{+1,0,-1\}^{2 n}$ directly from (89). Moreover, from Lemma 12 (b2), for every $i \in \mathcal{I}_{i \text { insert }}^{+}$, we have $\left|\xi_{i}\left(\boldsymbol{w}_{\star}^{+}\right)\right|=\lambda_{+}>$ 0 , hence $\left[s_{+}\right]_{i} \neq 0$, combining which with the description of the deletion-insertion process (Procedure 2) yields $\operatorname{supp}\left(s_{+}\right)=\mathcal{E}_{+}$. Thus to prove $\left(\mathcal{E}_{+}, \boldsymbol{s}_{+}\right) \in \mathcal{V}$, we only need to show $\left[\boldsymbol{s}_{+}\right]_{\mathcal{E}_{+}} \in \mathcal{R}\left(\boldsymbol{C}_{\mathcal{E}_{+}}{ }^{\top}\right)$. From (83), for every $i \in \mathcal{E}$, $\boldsymbol{w}_{\star}^{+}$satisfies the $i$ ith $(\mathcal{E}, \boldsymbol{s})-(E Q-a)$ equation at $\lambda_{+}$, i.e.,

$$
\begin{equation*}
(\forall i \in \mathcal{E}) \quad \xi_{i}\left(\boldsymbol{w}_{\star}^{+}\right):=\boldsymbol{c}_{i}^{\top}\left(\boldsymbol{b}-\boldsymbol{D} \boldsymbol{C} \boldsymbol{w}_{\star}^{+}\right)=\lambda_{+}[\boldsymbol{s}]_{i}, \tag{90}
\end{equation*}
$$

In addition, from Lemma 12 (b2) and (89), we have

$$
\begin{equation*}
\left(\forall i \in \mathcal{I}_{\text {insert }}^{+}\right) \quad \xi_{i}\left(\boldsymbol{w}_{\star}^{+}\right):=\boldsymbol{c}_{i}^{\top}\left(\boldsymbol{b}-\boldsymbol{D} \boldsymbol{C} \boldsymbol{w}_{\star}^{+}\right)=\lambda_{+}\left[\boldsymbol{s}_{+}\right] . \tag{91}
\end{equation*}
$$

Combining (90), (91) with (88) and (89) yields

$$
\left(\forall i \in \mathcal{E}_{+}\right) \quad \xi_{i}\left(\boldsymbol{w}_{\star}^{+}\right):=\boldsymbol{c}_{i}^{\top}\left(\boldsymbol{b}-\boldsymbol{D} \boldsymbol{C} \boldsymbol{w}_{\star}^{+}\right)=\lambda_{+}\left[\boldsymbol{s}_{+}\right]_{i},
$$

which further implies

$$
\left[\boldsymbol{s}_{+}\right]_{\mathcal{E}_{+}}=\frac{1}{\lambda_{+}} \boldsymbol{C}_{\mathcal{E}_{+}}{ }^{\top}\left(\boldsymbol{b}-\boldsymbol{D} \boldsymbol{C} \boldsymbol{w}_{\star}^{+}\right) \in \mathcal{R}\left(\boldsymbol{C}_{\mathcal{E}_{+}}{ }^{\top}\right)
$$

Hence we conclude that $\left(\mathcal{E}_{+}, s_{+}\right) \in \mathcal{V}$.
In the sequel, we prove (87) and the result (a)(b).
(a) First, for every $i \in \mathcal{E}_{+}$, we will show that:

- the result (a1) and (a2) hold, which implies that $\boldsymbol{w}_{\star}^{+}$satisfies the ith $\left(\mathcal{E}_{+}, s_{+}\right)-(N Q-a)$ inequation at $\lambda_{+}$,
- $\boldsymbol{w}_{\star}^{+}$satisfies the $i$ th $\left(\mathcal{E}_{+}, \boldsymbol{s}_{+}\right)$-(EQ-a) equation at $\lambda_{+}$.

Recall that $\mathcal{I}_{\text {delete }}^{+} \subset \mathcal{E}$ and $\mathcal{I}_{\text {insert }}^{+} \subset \neg \mathcal{E}$ (cf. Procedure 3), hence from (88), all indices in $\mathcal{E}_{+}$can be divided into two disjoint parts $\left(\mathcal{E}_{+} \backslash \mathcal{I}_{\text {insert }}^{+}\right)=\left(\mathcal{E} \backslash \mathcal{I}_{\text {delete }}^{+}\right)$and $\mathcal{I}_{\text {insert }}^{+}$.

Part one of $\mathcal{E}_{+}$: for every $i \in\left(\mathcal{E}_{+} \backslash \mathcal{I}_{\text {insert }}^{+}\right)=\left(\mathcal{E} \backslash \mathcal{I}_{\text {delete }}^{+}\right)$, it is evident that $i \notin\left(\mathcal{I}_{\text {delete }}^{+} \cup \mathcal{I}_{\text {insert }}^{+}\right)$, hence we have $\left[s_{+}\right]_{i}=[s]_{i}$ from (89), combining which with Lemma 12 (a1) yields

$$
\left(\forall i \in \mathcal{E}_{+} \backslash \mathcal{I}_{\text {insert }}^{+}\right) \quad\left[\boldsymbol{s}_{+}\right]_{i}\left[\boldsymbol{w}_{\star}^{+}\right]_{i}>0,
$$

i.e., the result (a1) holds.

In addition, notice that $\left(\mathcal{E}_{+} \backslash \mathcal{I}_{\text {insert }}^{+}\right)=\left(\mathcal{E} \backslash \mathcal{I}_{\text {delete }}^{+}\right) \subset \mathcal{E}$, hence from (83), $\boldsymbol{w}_{\star}^{+}$satisfies the $i$ th $(\mathcal{E}, s)$-(EQ-a) equation at $\lambda_{+}$, namely

$$
\left(\forall i \in \mathcal{E}_{+} \backslash \mathcal{I}_{\text {insert }}^{+}\right) \quad \xi_{i}\left(\boldsymbol{w}_{\star}^{+}\right)=\lambda_{+}[\boldsymbol{s}]_{i}=\lambda_{+}\left[\boldsymbol{s}_{+}\right]_{i}
$$

Thus $\boldsymbol{w}_{\star}^{+}$satisfies the ith $\left(\mathcal{E}_{+}, \boldsymbol{s}_{+}\right)-(E Q-a)$ equation at $\lambda_{+}$.
Part two of $\mathcal{E}_{+}$: for every $i \in \mathcal{I}_{\text {insert }}^{+}$since $\mathcal{I}_{\text {insert }}^{+} \subset \neg \mathcal{E}$, from (83), $\boldsymbol{w}_{\star}^{+}$satisfies the $i$ ith $(\mathcal{E}, s)-(E Q-b)$ equation at $\lambda_{+}$, i.e.,

$$
\left(\forall i \in \mathcal{I}_{\text {insert }}^{+}\right) \quad\left[\boldsymbol{w}_{\star}^{+}\right]_{i}=0
$$

Hence the result (a2) naturally holds.
In addition, from Lemma 12 (b2) and (89), we have

$$
\left(\forall i \in \mathcal{I}_{\text {insert }}^{+}\right) \quad \xi_{i}\left(\boldsymbol{w}_{\star}^{+}\right)=\lambda_{+}\left[\boldsymbol{s}_{+}\right]_{i} .
$$

Thus $\boldsymbol{w}_{\star}^{+}$satisfies the $i$ th $\left(\mathcal{E}_{+}, s_{+}\right)-(E Q-a)$ equation at $\lambda_{+}$.
(b) Next, for every $i \in \neg \mathcal{E}_{+}$, we will show that:

- the result (b1) and (b2) hold, which implies that $\boldsymbol{w}_{\star}^{+}$satisfies the ith $\left(\mathcal{E}_{+}, s_{+}\right)-(N Q-b)$ inequation at $\lambda_{+}$,
- $\boldsymbol{w}_{\star}^{+}$satisfies the $i$ th $\left(\mathcal{E}_{+}, \boldsymbol{s}_{+}\right)-(E Q-b)$ equation at $\lambda_{+}$.

Recall that $\mathcal{I}_{\text {delete }}^{+} \subset \mathcal{E}$ and $\mathcal{I}_{\text {insert }}^{+} \subset \neg \mathcal{E}$ (cf. Procedure 3), hence from (88), we can derive that (cf. Fig. 11)

$$
\neg \mathcal{E}_{+}=\left(\neg \mathcal{E} \backslash \mathcal{I}_{\text {insert }}^{+}\right) \cup \mathcal{I}_{\text {delete }}^{+},
$$

which implies that all indices in $\neg \mathcal{E}_{+}$can be divided into two disjoint parts $\left(\neg \mathcal{E}_{+} \backslash \mathcal{I}_{\text {delete }}^{+}\right)=\left(\neg \mathcal{E} \backslash \mathcal{I}_{\text {insert }}^{+}\right)$and $\mathcal{I}_{\text {delete }}^{+}$.
Part one of $\neg \mathcal{E}_{+}$: for every

$$
i \in\left(\neg \mathcal{E}_{+} \backslash \mathcal{I}_{\text {delete }}^{+}\right)=\left(\neg \mathcal{E} \backslash \mathcal{I}_{\text {insert }}^{+}\right)
$$

from Lemma 12 (b1) we have

$$
\left(\forall i \in \neg \mathcal{E}_{+} \backslash \mathcal{I}_{\text {delete }}^{+}\right) \quad\left|\xi_{i}\left(\boldsymbol{w}_{\star}^{+}\right)\right|<\lambda_{+},
$$

which implies that the result (b1) holds.
In addition, since $\left(\neg \mathcal{E}_{+} \backslash \mathcal{I}_{\text {delete }}^{+}\right)=\left(\neg \mathcal{E} \backslash \mathcal{I}_{\text {insert }}^{+}\right) \subset \neg \mathcal{E}$, hence from (83), $\boldsymbol{w}_{\star}^{+}$satisfies the $i$ th $(\mathcal{E}, \boldsymbol{s})-(E Q-b)$ equation at $\lambda_{+}$, namely

$$
\left(\forall i \in \neg \mathcal{E}_{+} \backslash \mathcal{I}_{\text {delete }}^{+}\right) \quad\left[\boldsymbol{w}_{\star}^{+}\right]_{i}=0
$$

Thus $\boldsymbol{w}_{\star}^{+}$satisfies the $i$ th $\left(\mathcal{E}_{+}, \boldsymbol{s}_{+}\right)-(E Q-b)$ equation at $\lambda_{+}$.
Part two of $\neg \mathcal{E}_{+}$: for every $i \in \mathcal{I}_{\text {delete }}^{+}$, since $\mathcal{I}_{\text {delete }}^{+} \subset \mathcal{E}$, from (83), $\boldsymbol{w}_{\star}^{+}$satisfies the $i$ th $(\mathcal{E}, \boldsymbol{s})-(E Q-a)$ equation at $\lambda_{+}$, i.e.,

$$
\left(\forall i \in \mathcal{I}_{\text {delete }}^{+}\right) \quad \xi_{i}\left(\boldsymbol{w}_{\star}^{+}\right)=\lambda_{+}[\boldsymbol{s}]_{i} .
$$

Since we have $[s]_{i} \in\{1,-1\}$ from $\mathcal{I}_{\text {delete }}^{+} \subset \mathcal{E}$, the equation above naturally implies the result (b2).
In addition, from Lemma 12 (a2), we have

$$
\left(\forall i \in \mathcal{I}_{\text {delete }}^{+}\right) \quad[\boldsymbol{s}]_{i}\left[\boldsymbol{w}_{\star}^{+}\right]_{i}=0
$$

Since we have $[s]_{i} \in\{1,-1\}$ from $\mathcal{I}_{\text {delete }}^{+} \subset \mathcal{E}$, the equation above naturally implies

$$
\left(\forall i \in \mathcal{I}_{\text {delete }}^{+}\right) \quad\left[\boldsymbol{w}_{\star}^{+}\right]_{i}=0
$$

i.e., $\boldsymbol{w}_{\star}^{+}$satisfies the ith $\left(\mathcal{E}_{+}, \boldsymbol{s}_{+}\right)-(E Q-b)$ equation at $\lambda_{+}$.

Combining the discussion above completes the proof of (87), the result (a) and the result (b).

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[^0]:    1. A regularization path (or solution path) of a sparse least-squares problem (1) is the solution of (1) as a function of the regularization parameter $\lambda$.
    2. As will be shown in Thm. 4 (a) in Sec. 4.1, the minimum $\ell_{2}$-norm extended solution is exactly the concatenation of the minimum $\ell_{2}$-norm solution and the minimum $\ell_{2}$-norm dual solution of the sGMC model.
[^1]:    4. Although [20, Eq. (48)] permits $\rho$ to be 1, Prop. 1 in the present paper requires $\rho$ to be smaller than 1, which is a preliminary proposition for proving many subsequent results. Hence we assume $\rho<1$ in the sGMC model.
    5. While the dual sGMC solution set may not currently pique individual interest, many subsequent results can be derived only by considering the primal and dual sGMC solution sets together. Hence it is necessary to study the dual sGMC solution set here.
[^2]:    6. As will be shown in Thm. 6 in Sec. 6.2 , the piecewise linearity of the minimum $\ell_{2}$-norm extended regularization path is established merely using the results of ex-equicorrelation pairs in Sec. 5, which naturally implies its continuity within each linear piece. Nevertheless, continuity of the minimum $\ell_{2}$-norm extended regularization path at change points can only be proven through set-valued variational analysis approaches, hence it is necessary to study set-valued continuity of the extended solution set here.
[^3]:    10. We note that the partial order introduced in Prop. 6 will play a critical role in Prop. 7 (c) (d) (Sec. 5.3), which is useful in the derivation of piecewise expressions of $\mathcal{S}_{\mathrm{e}}(\lambda)$ and $\boldsymbol{w}_{\star}(\lambda)$ (Prop. 11 in Sec. 6.1) and in the proof of correctness of the LARS-sGMC algorithm (Thm. 8 in Sec. 7.3).
[^4]:    15. Before LARS, Osborne et al. [36] proposed a homotopy method to compute the LASSO regularization path, which is essentially the same idea as LARS, but the proof of correctness of the algorithm was not given explicitly.
    16. The original proof in [15] assumed that the sensing matrix has full column rank (cf. [15, p. 413, second paragraph]) to ensure solution uniqueness, which can be relaxed to the general-positioning condition as in Fact 3.
    17. However, the "one-at-a-time" assumption will still be used in Sec. 7, in order to prove the correctness of the LARS-sGMC algorithm.
[^5]:    18. For every $\lambda>0$, it takes infinite iterations for an iterative algorithm to converge to an exact solution of the sGMC model, and such procedure has to be repeated for every $\lambda \in \mathbb{R}_{++}$. Hence indeed, the complexity of computing the whole regularization path is "infinity of infinity".
    19. For simplicity, here we only consider the computation of $\left(\mathcal{E}_{k+1}^{\star}, \boldsymbol{s}_{k+1}^{\star}\right)$. However, we note that $\left(\mathcal{E}_{k-1}^{\star}, \boldsymbol{s}_{k-1}^{\star}\right)$ can be computed in a similar way to the approach introduced here.
[^6]:    21. It is a well known fact that in the noiseless setting, the number of measurements $m$ needed to reconstruct a sparse signal with $S$ nonzero components is at least $2 S$ (cf. [45, p. 49]). Hence we refer to the case where $S=250=m / 2$ as "nearly dense" because it is the most dense signal that can be reconstructed using $m=500$ noiseless measurements.
