THE CONTINUUM LIMIT OF NON-LOCAL FOLLOW-THE-LEADER MODELS

HELGE HOLDEN AND NILS HENRIK RISEBRO

ABSTRACT. We study a generalized Follow-the-Leader model where the driver considers the position of an arbitrary but finite number of vehicles ahead, as well as the position of the vehicle directly behind the driver. It is proved that this model converges to the classical Lighthill–Whitham–Richards model for traffic flow when traffic becomes dense. This also underscores the robustness of the Lighthill–Whitham–Richards model.

1. Introduction

We study a generalized Follow-the-Leader (FtL) model for unidirectional traffic of M_ℓ vehicles on a single lane road given by

$$
\begin{aligned}
\dot{x}_i(t) &= \sum_{j=0}^{N} c_j v\Big(\frac{\ell}{x_{i+j+1}(t)-x_{i+j}(t)}\Big) \\
&\quad + \kappa\bigg(v\Big(\frac{\ell}{x_{i+1}(t)-x_i(t)}\Big) - v\Big(\frac{\ell}{x_i(t)-x_{i-1}(t)}\Big)\bigg), \quad i=1,\dots,M_\ell. \qquad (1.1)
\end{aligned}
$$

Here the position of the ith vehicle, each of length ℓ, is $x_i(t)$ at time t. The system (1.1) is closed by posing appropriate periodic boundary conditions, see later.

The velocity function v is a decreasing function that vanishes at maximum capacity of the road. Each driver considers the distances to the N vehicles ahead and the one vehicle right behind. The impact of more distant vehicles is less pronounced, and thus we assume that the constants c_j decrease, i.e., $c_j \ge c_{j+1} \ge 0$. In addition, the driver considers the distance to the vehicle right behind, and if that becomes too short, the driver will speed up, and the coefficient κ measures this influence.

We show that as the length of each vehicle becomes smaller, $\ell \to 0$ and the number of vehicles increases, $M_\ell \to \infty$, while N is kept fixed, the distribution of vehicles will approach the solution of the classical Lighthill–Whitham–Richards (LWR) model [16, 17]

$$
\rho_t + (\rho v(\rho))_x = 0, \qquad (1.2)
$$

where the density, or rather saturation, ρ is approximated by $\ell/(x_{i+1}(t)-x_i(t))$. This is of course a scalar hyperbolic conservation law [13]. The result is independent of the finer details given by N and the coefficients c_j, κ, and shows the robustness

---

Date: May 8, 2024.

2010 *Mathematics Subject Classification.* Primary: 35L02; Secondary: 35Q35, 82B21.

Key words and phrases. Follow-the-Leader model, Lighthill–Whitham–Richards model, traffic flow, continuum limit.

Both authors were supported by the Swedish Research Council under grant no. 2021-06594 while they were in residence at Institut Mittag-Leffler in Djursholm, Sweden during the fall semester of 2023. The research of HH was supported in part by the project *IMod — Partial differential equations, statistics and data: An interdisciplinary approach to data-based modelling*, project number 65114, from the Research Council of Norway.

of the LWR model. As long as N remains unaltered in the limiting process, this limit is distinct from the widely studied problem of non-local to local limit for conservation law, see, e.g., [4, 5].

The model (1.1) is based on the following anticipated behavior: In general, the longer the distance to the vehicle in front, the faster the drivers are willing to drive. However, each driver considers the distance between successive vehicles ahead of the driver. More weight is given to the vehicles close to the driver. By the same token, the driver can look in the rear mirror and assess the distance to the vehicle immediately behind. If that distance is shorter than the distance to the vehicle just in front, the driver will speed up. This generalization of the FtL model is more realistic than the traditional one, as it takes into account finitely many vehicles ahead of the driver, and includes the observed fact that also the behavior of the vehicle right behind you influences your actions.

More specifically, we are given coefficients $c_j \geq 0$ that indicate the weight given to the velocity between the jth and $(j+1)$th vehicle (as counted from the ith vehicle). We assume that the influence drops with the distance, thus $c_j \geq c_{j+1} \geq 0$. The driver is willing to consider N vehicles. We have $\sum_{j=0}^{N} c_j = 1$, and for convenience we put $c_N = 0$. Similarly, if the distance to the vehicle immediately behind is shorter than the distance to the vehicle directly in front, the driver will speed up. The impact of this is scaled by the numerical parameter $\kappa > 0$. This results in the model (1.1).

There are two main classes of mathematical models for traffic flow, namely discrete models based on car-following on the one hand, and, on the other hand, continuum models based on the assumption of dense traffic for which the flow can be described by a density, resulting in "traffic hydrodynamics" models. The dichotomy between microscopic and macroscopic models, or discrete and continuum models, is of course one of the fundamental outstanding problems of mathematical physics. The problem here is a considerably easier than the general problem, however, it allows for a rigorous analysis of the limit, as the number "particles" tends to infinity.

There is a wide range of car-following models dating from the late fifties and early sixties, see [3]. We here generalize a Follow-the-Leader model based on what is called safety-distance models or collision avoidance models, as a feature of this model is that it is collision-free. On the other hand, the LWR model [16, 17] has been, and still is, the prevalent continuum model. A consequence of the analysis in the present paper is that even for the generalized FtL model presented here, the scaling limit remains the LWR model, and this offers yet another justification for the LWR model.

By now there are several ways to show that the standard FtL model converges to the LWR model, the first one being [10]. The realization that the Follow-the-Leader is nothing but a semi-discrete approximation of the LWR in Lagrangian coordinates simplified the proof considerably, see [14, 15]. See also [1, 2, 7, 9, 18, 12, 19, 11] and references therein. There are also proofs in the setting of traffic on a network [6, 8].

Let us now describe the content of this paper. We introduce the short-hand notation $\Delta_\pm a_i = \pm(a_{i\pm 1} - a_i)$ and $\bar{a}_i = \sum_{j=0}^{N} c_j a_{i+j}$. If we define $v_i = v(\ell/(x_{i+1}(t) - x_i(t))) = v(\ell/\Delta_+ x_i(t))$, the model (1.1) takes the compact form

$$\dot{x}_i(t) = \bar{v}_i + \kappa \Delta_- v_i. \tag{1.3}$$

It turns out that it is convenient to introduce the Lagrangian variable $y_i(t) = \Delta_+ x_i(t)/\ell$, and then we find the following equation

$$\ell \dot{y}_i(t) = \Delta_+ \bar{V}_i + \kappa \Delta_+ \Delta_- V_i, \tag{1.4}$$

with $V(y) = v(1/y)$, and hence $V_i = V(y_i) = v_i$. Note that for $N = 1$ we recover the traditional FtL model.

The proof that the FtL model converges to the LWR model proceeds as follows: To avoid technical complications associated with boundary terms, we consider the periodic case. More precisely, we impose in equation (1.4) that

$$y_{i+M_\ell} = y_i, \quad i \in \mathbb{Z}.$$

We first analyze the equation for y_i. By introducing a spatially piecewise constant function y_ℓ using the values y_i and a fixed grid in space of size $M_\ell = 1/\ell$, cf. (3.7), we can show that

$$|y_\ell(t,\,\cdot\,)|_{BV([0,1])} \le |y_\ell(0,\,\cdot\,)|_{BV([0,1])}\,, \quad \|y_\ell(t,\,\cdot\,) - y_\ell(s,\,\cdot\,)\|_{L^1([0,1])} \le C\,|t-s|\,.$$

Furthermore, if we have another solution $z_i(t)$ of (1.4), we get stability in the sense that

$$\|y_\ell(t,\,\cdot\,) - z_\ell(t,\,\cdot\,)\|_{L^1([0,1])} \le \|y_\ell(0,\,\cdot\,) - z_\ell(0,\,\cdot\,)\|_{L^1([0,1])}\,.$$

This suffices to obtain strong convergence $y_\ell \to y$ in $C([0,T]; L^1([0,1]))$ as $\ell \to 0$. However, as we are interested in the Eulerian formulation in terms of the density ρ, we simply define $\rho_i(t) = 1/y_i(t)$, and derive the corresponding equation for $\rho_i(t)$, and translate the properties from y_i to ρ_i. We introduce the function $\rho_\ell(t,x) = \sum_{i=1}^{M_\ell} \rho_i(t)\mathbb{I}_i(t,x)$ where $\mathbb{I}_i(t,x)$ is the indicator function of the time-dependent spatial interval $[x_i(t), x_{i+1}(t))$. Since y_i is periodic, the corresponding ρ_i will be periodic in Eulerian coordinates, with some period P, see equation (2.7). We ensure that we stay away from vacuum by assuming that $\rho_0 \ge \nu > 0$. Then we can prove, cf. Lemma 3.1,

$$\begin{aligned} \inf \rho_0 \le \rho_\ell(t,x) &\le \sup \rho_0, \\ |\rho_\ell(t,\,\cdot\,)|_{BV([0,P])} &\le \frac{1}{\nu^2}\,|\rho_0|_{BV([0,P])}\,, \\ \|\rho_\ell(t,\,\cdot\,) - \rho_\ell(s,\,\cdot\,)\|_{L^1([0,P])} &\le C\,|t-s|\,. \end{aligned}$$

Observe that we get the somewhat unexpected constant $1/\nu^2$ in the estimate for bounded variation. This is the case provided $N > 1$; in the classical case $N = 1$ the constant is replaced by unity.

These estimates establish the existence of a limit $\rho \in C([0,T]; L^1([0,P])) \cap L^\infty([0,T]; BV([0,P]))$, cf. Corollary 3.4, as $\ell \to 0$. Here $L^\infty([0,T]; BV([0,P]))$ denotes the set of functions $u = u(t,x)$ with

$$\sup_{t\in[0,T]} |u(t,\,\cdot\,)|_{BV([0,P])} < \infty.$$

It remains to show that the limit equals the unique weak entropy solution of the LWR equation (1.2), which means that it satisfies, cf. Theorem 4.1,

$$\int_0^\infty \int_0^P \big(\eta(\rho)\varphi_t + q(\rho)\varphi_x\big)\,dxdt + \int_0^P \eta(\rho_0)\varphi(0,x)\,dx \ge 0$$

for any non-negative P-periodic test function $\varphi \in C_c^\infty([0,\infty) \times [0,P])$. Here η is a convex (entropy) function and q is the entropy flux, satisfying $q'(\rho) = \eta'(\rho)(\rho v(\rho))'$. This shows that ρ is indeed the unique weak entropy solution of the LWR equation.

2. The model

Consider M_ℓ identical vehicles on a unidirectional, single lane road with initial positions $x_1(0) < x_2(0) < \cdots < x_{M_\ell}(0)$ where $x_{i+1}(0) - x_i(0) > \ell$, with $\ell > 0$ being the length of each vehicle. The velocity v is assumed to be a decreasing Lipschitz function of a single variable. The "non-localness" enters the model in the following way.

Given constants c_j for $j = 0, \dots, N$

$$\sum_{j=0}^{N} c_j = 1, \quad c_0 \geq \cdots \geq c_{N-1} \geq 0, \;\; c_N = 0, \tag{2.1}$$

we define for any sequence a_i

$$\bar{a}_i = \sum_{j=0}^{N} c_j a_{i+j}. \tag{2.2}$$

Furthermore, we define the traditional shift operators as follows

$$\Delta_{\pm} a_i = \pm(a_{i\pm 1} - a_i). \tag{2.3}$$

Observe that we have, by applying summation by parts, that

$$\begin{aligned} \Delta_+ \bar{a}_i = \overline{\Delta_+ a_i} &= \sum_{j=0}^{N} c_j \left(a_{i+j+1} - a_{i+j}\right) = -c_0 a_i + \sum_{j=1}^{N} \left(c_{j-1} - c_j\right) a_{i+j} \\ &= \sum_{j=1}^{N} \left(c_j - c_{j-1}\right)\left(a_i - a_{i+j}\right) = -\sum_{j=1}^{N} \Delta_- c_j \left(a_{i+j} - a_i\right). \end{aligned} \tag{2.4}$$

Given a Lipschitz continuous non-increasing velocity function $v\colon [0,1] \to [0,1]$ with $v(0) = 1$ and $v(1) = 0$, we assume that the dynamics of the ith vehicle is given by

$$\begin{aligned} \dot{x}_i(t) &= \sum_{j=0}^{N} c_j v\big(\frac{\ell}{x_{i+j+1}(t) - x_{i+j}(t)}\big) \\ &\quad + \kappa\Big(v\big(\frac{\ell}{x_{i+1}(t) - x_i(t)}\big) - v\big(\frac{\ell}{x_i(t) - x_{i-1}(t)}\big)\Big), \quad i \in Z_\ell, \end{aligned} \tag{2.5}$$

with $Z_\ell = \{1, \dots, M_\ell\}$, and κ is a positive constant. With the introduced notation we can write (2.5) compactly as

$$\dot{x}_i(t) = \bar{v}_i + \kappa \Delta_- v_i, \tag{2.6}$$

with $v_i = v(\ell/\Delta_+ x_i(t))$. To avoid technicalities connected with boundary conditions, we assume periodicity. More concretely, we assume the existence of a positive P such that

$$x_{i+M_\ell}(t) = x_i(t) + P \text{ for all } i \in \mathbb{Z} \text{ and all } t. \tag{2.7}$$

As in (1.4) we write $y_i(t) = \Delta_+ x_i(t)/\ell$, which implies that (2.6) takes the form

$$\ell \dot{y}_i(t) = \Delta_+ \overline{V_i} + \kappa \Delta_+ \Delta_- V_i, \quad i \in Z_\ell, \tag{2.8}$$

where $V(y) = v(1/y)$, and hence $V_i = V(y_i) = v_i$. Note that $y_{i+M_\ell} = y_i$ for all i. This gives a finite-dimensional system of ordinary differential equations, and (local in time) existence of a unique solution follows from standard theory. Clearly, V is increasing and $V\colon [1,\infty) \to [0,1]$.

3. The continuum limit

Next we will study the limit when $\ell \to 0$.

3.1. **Entropy estimates.** Inspired by conservation laws, let (η, Q) be an entropy/entropy flux pair, i.e., η is twice continuously differentiable and convex, and Q is defined by $Q' = \eta' V'$. We multiply (2.8) with $\eta'(y_i)$ and use (2.4) to obtain

$$\begin{aligned}
\ell \frac{d}{dt}\eta_i &= \eta'(y_i)\Delta_+\overline{V_i} + \kappa\eta'(y_i)\Delta_+\Delta_- V_i \\
&= \Delta_+\overline{Q_i} + \eta'(y_i)\Delta_+\overline{V_i} - \Delta_+\overline{Q_i} \\
&\quad + \kappa\Delta_+\left(\eta'(y_i)\Delta_- V_i\right) - \kappa\left(\Delta_+\eta'(y_i)\right)\left(\Delta_+ V_i\right) \\
&= \Delta_+\overline{Q_i} + \sum_{j=1}^{N}\left(c_j - c_{j-1}\right)\left[\left(\eta'(y_i)V_i - Q_i\right) - \left(\eta'(y_i)V_{i+j} - Q_{i+j}\right)\right] \\
&\quad + \kappa\Delta_+\left(\eta'(y_i)\Delta_- V_i\right) - \kappa\left(\Delta_+\eta'(y_i)\right)\left(\Delta_+ V_i\right) \\
&= \Delta_+\overline{Q_i} + \sum_{j=1}^{N}\left(c_j - c_{j-1}\right) H(y_i, y_{i+j}) \\
&\quad + \kappa\Delta_+\left(\eta'(y_i)\Delta_- V_i\right) - \kappa\left(\Delta_+\eta'(y_i)\right)\left(\Delta_+ V_i\right),
\end{aligned}$$

where

$$\begin{aligned}
H(a,b) &= \left[\left(\eta'(a)V(a) - Q(a)\right) - \left(\eta'(a)V(b) - Q(b)\right)\right] \\
&= \int_0^a \left(\eta'(a) - \eta'(\sigma)\right)V'(\sigma)\,d\sigma - \int_0^b \left(\eta'(a) - \eta'(\sigma)\right)V'(\sigma)\,d\sigma \\
&= \int_a^b \left(\eta'(\sigma) - \eta'(a)\right)V'(\sigma)\,d\sigma = \int_a^b \int_a^\sigma \eta''(\mu)\,d\mu V'(\sigma)\,d\sigma \ge 0.
\end{aligned}$$

Here we have written, in obvious notation, $\eta_i = \eta(y_i)$, $Q_i = Q(y_i)$. Furthermore, we get

$$\begin{aligned}
\ell \frac{d}{dt}\eta_i - \sum_{j=1}^{N}\left(\Delta_- c_j\right) H(y_i, y_{i+j}) &+ \kappa\left(\Delta_+\eta'(y_i)\right)\left(\Delta_+ V_i\right) \\
&= \Delta_+\overline{Q_i} + \kappa\Delta_+\left(\eta'(y_i)\Delta_- V_i\right),
\end{aligned} \tag{3.1}$$

and since $c_j \le c_{j-1}$, we see that the second term is non-negative. Furthermore, since η' and V are both increasing functions, also the third term is non-negative. This entropy equality immediately implies the entropy inequality

$$\ell \frac{d}{dt}\eta_i \le \Delta_+\overline{Q_i} + \kappa\Delta_+\left(\eta'(y_i)\Delta_- V_i\right), \tag{3.2}$$

and by an approximation argument, this is valid for any Lipschitz continuous convex entropy η. Hence

$$\frac{d}{dt}\ell \sum_{i\in Z_\ell} \eta(y_i(t)) \le 0. \tag{3.3}$$

Choosing

$$\eta(y) = \left(y - \inf_i y_i(0)\right)^- \quad \text{and} \quad \eta(y) = \left(y - \sup_i y_i(0)\right)^+,$$

where $a^\pm = (|a| \pm a)/2$, implies that

$$\inf_{i\in Z_\ell} y_i(0) \le y_i(t) \le \sup_{i\in Z_\ell} y_i(0),$$

for any positive t. Incidentally, this shows that the systems of ordinary differential equations, (2.6) and (2.8), both have unique global solutions for $t \in (0, \infty)$. Furthermore, it shows that the model does not allow for collisions.

Consider next another solution $z_i(t)$ of (2.8) with initial data $z_i(0)$. Subtract the equation for z_i from the corresponding equation for y_i, and multiply by $\operatorname{sign}(y_i - z_i)$ to get

$$
\begin{aligned}
\ell \frac{d}{dt} |y_i - z_i| &= \ell \operatorname{sign}(y_i - z_i) \frac{d}{dt} (y_i - z_i) \\
&= \operatorname{sign}(y_i - z_i) \Delta_+ \big(\overline{V(y_i) - V(z_i)}\big) \\
&\quad + \kappa \operatorname{sign}(y_i - z_i) \Delta_- \Delta_+ (V(y_i) - V(z_i)) \\
&= \operatorname{sign}(y_i - z_i) \sum_{j=1}^{N} (c_j - c_{j-1}) \left[(V(y_i) - V(z_i)) - (V(y_{i+j}) - V(z_{i+j}))\right] \\
&\quad + \kappa \Delta_+ (\operatorname{sign}(y_i - z_i) \Delta_- (V(y_i) - V(z_i))) \\
&\quad - \kappa (\Delta_+ \operatorname{sign}(y_i - z_i)) (\Delta_+ (V(y_i) - V(z_i))) \\
&\leq \sum_{j=1}^{N} (c_j - c_{j-1}) \left[|V(y_i) - V(z_i)| - |V(y_{i+j}) - V(z_{i+j})|\right] \\
&\quad + \kappa \Delta_+ (\operatorname{sign}(y_i - z_i) \Delta_- (V(y_i) - V(z_i))) \\
&= \Delta_+ \overline{|V(y_i) - V(z_i)|} + \kappa \Delta_+ (\operatorname{sign}(y_i - z_i) \Delta_- (V(y_i) - V(z_i))),
\end{aligned} \tag{3.4}
$$

where we used the fact that $c_j \leq c_{j-1}$ to get the inequality. In addition, we used that

$$
\begin{aligned}
&(\Delta_+ \operatorname{sign}(y_i - z_i)) (\Delta_+ (V(y_i) - V(z_i)) \\
&\qquad = (\Delta_+ \operatorname{sign}(V(y_i) - V(z_i))) (\Delta_+ (V(y_i) - V(z_i))) \geq 0,
\end{aligned}
$$

as both the signum function and V are increasing functions. This immediately gives the stability estimate

$$
\ell \sum_{i \in Z_\ell} |y_i(t) - z_i(t)| \leq \ell \sum_{i \in Z_\ell} |y_i(0) - z_i(0)|, \tag{3.5}
$$

for any $t > 0$.

Choosing $z_i = y_{i-1}$ yields a bound on the variation of y_i, viz.

$$
\sum_{i \in Z_\ell} |y_i(t) - y_{i-1}(t)| \leq \sum_{i \in Z_\ell} |y_i(0) - y_{i-1}(0)|. \tag{3.6}
$$

It is convenient to view y_i as a spatially 1-periodic function. Define for $i = 0, \dots, M_\ell$ the quantities $\zeta_i = i/M_\ell$ and $I_i = [\zeta_i, \zeta_{i+1})$. Then we let

$$
y_\ell(t, \zeta) = \sum_{i \in Z_\ell} y_i(t) \mathbb{I}_{I_i}(\zeta), \tag{3.7}
$$

where $\mathbb{I}_A$ is the indicator function of the set A. Using this, (3.5) and (3.6) read $\|y_\ell(t, \cdot) - z_\ell(t, \cdot)\|_{L^1([0,1])} \leq \|y_\ell(0, \cdot) - z_\ell(0, \cdot)\|_{L^1([0,1])}$ and $|y_\ell(t, \cdot)|_{BV([0,1])} \leq |y_\ell(0, \cdot)|_{BV([0,1])}$, respectively.

Next we consider the time continuity of the approximate solution. Let now $t \geq s \geq 0$, and calculate

$$
\begin{aligned}
\|y_\ell(t, \cdot) - y_\ell(s, \cdot)\|_{L^1([0,1])} &= \ell \sum_{i \in Z_\ell} |y_i(t) - y_i(s)| \\
&= \ell \sum_{i \in Z_\ell} \left| \int_s^t \frac{d}{d\tau} y_i(\tau)\, d\tau \right| \\
&\leq \sum_{i \in Z_\ell} \int_s^t \left| \ell \frac{d}{d\tau} y_i(\tau) \right| d\tau
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i\in Z_\ell}\int_s^t \left|\Delta_+\overline{V(y_i(\tau))}+\kappa\Delta_+\Delta_- V(y_i(\tau))\right| d\tau\\
&\le \|V'\|_\infty \sum_{i\in Z_\ell}\int_s^t \left|\Delta_+\overline{y_i(\tau)}\right| + 2\kappa\left|\Delta_+ y_i(\tau)\right| d\tau\\
&\le \|V'\|_\infty (1+2\kappa)\left|y_\ell(0,\,\cdot\,)\right|_{BV}(t-s),
\end{aligned}
\tag{3.8}
$$

so that the map $t\mapsto y_\ell(t,\,\cdot\,)$ is Lipschitz continuous in L^1. Here we used that

$$
\begin{aligned}
\left|\overline{a}\right|_{BV} = \sum_{i\in Z_\ell}\left|\overline{\Delta_+ a_i}\right| &= \sum_{i\in Z_\ell}\left|\sum_{j=0}^N c_j\Delta_+ a_{i+j}\right| \le \sum_{i\in Z_\ell}\sum_{j=0}^N c_j\left|\Delta_+ a_{i+j}\right|\\
&= \sum_{j=1}^N c_j \sum_{i\in Z_\ell}\left|\Delta_+ a_{i+j}\right| = \sum_{j=0}^N c_j\left|a\right|_{BV} = \left|a\right|_{BV}.
\end{aligned}
\tag{3.9}
$$

Note that the total variation is computed on D_ℓ.

Recall that $\ell M_\ell = 1$, from the estimates (3.5), (3.6), and (3.8) we can conclude the strong convergence

$$
y_\ell \;\to\; y \quad \text{in } C([0,T];L^1([0,1])) \text{ as } \ell\to 0,
$$

as $\ell\to 0$, see [13, Thm. A.8].

At this point we could have shown that the limit y is the unique entropy solution of the conservation law $y_t - V_z = 0$, subsequently transfer the result to Eulerian coordinates, and finally show that the corresponding density, or rather saturation, is an entropy solution of the LWR model.

However, we shall do this directly for ρ in the Eulerian setting. To this end, we define $\rho_i(t) = 1/y_i(t)$. Applying equation (2.8) we find

$$
\dot\rho_i = -\rho_i\Big(\frac{\Delta_+\overline{v}_i}{\Delta_+ x_i} + \kappa\frac{\Delta_+\Delta_- v_i}{\Delta_+ x_i}\Big),
\tag{3.10}
$$

for $t>0$, using $\rho_i(t) = \ell/\Delta_+ x_i(t)$. Observe the straightforward transition between the Lagrangian formulation (2.8) and the Eulerian formulation (3.10), sharply contrasting the cumbersome transition in the continuum case.

Next we define the appropriate initial data. Assume that we are given a non-negative P-periodic function $\rho_0\in L^1([0,P])\cap BV([0,P])$, such that $\rho_0(x)\in[\nu,1]$ for all x, where ν is some small positive number. This last assumption excludes vacuum.

Choose

$$
M_\ell\in\mathbb{N}, \quad \text{and define } \; \ell = \frac{1}{M_\ell}\int_0^P \rho_0(x)\,dx.
$$

Let $x_0(0)=0$ and define $x_{i+1}(0)$ inductively by

$$
\int_{x_i(0)}^{x_{i+1}(0)} \rho_0(\xi)\,d\xi = \ell, \quad i=0,\dots,M_\ell-1.
\tag{3.11}
$$

Next, we set

$$
\rho_i(0) = \frac{\ell}{x_{i+1}(0)-x_i(0)} =: \frac{1}{y_i(0)}, \quad i=1,\dots,M_\ell,
\tag{3.12}
$$

extending it periodically by $\rho_{i+M_\ell}(0) = \rho_i(0)$ for $i\in\mathbb{Z}$. Finally, we define $\rho_i(t)$ as the solution of (3.10) with initial data $\rho_i(0)$ defined by (3.12). We can then introduce

$$
\rho_\ell(t,x) = \sum_{i\in\mathbb{Z}}\rho_i(t)\mathbb{I}_i(t,x),
\tag{3.13}
$$

where $\mathbb{I}_i(t,x) = \mathbb{I}_{[x_i(t),x_{i+1}(t))}(x)$. Note that $\mathbb{I}_i$ and $\mathbb{I}_{I_i}$ are distinct. Observe that at this point the variable y_i is superfluous; it is used only as a tool to obtain the convergence. Converting the above calculations we get the following result.

Lemma 3.1. *Let ρ_i for $i \in Z_\ell$ be defined by (3.10), and $\rho_\ell(t,x)$ by (3.13). Assume also that $\rho_\ell(0,x) = \rho_{\ell,0}(x) \geq \nu > 0$ for all $\ell > 0$ and x. Then*

$$\inf \rho_0 \leq \rho_\ell(t,x) \leq \sup \rho_0, \tag{3.14}$$

$$\sum_{i\in Z_\ell} |\rho_i(t) - \tilde{\rho}_i(t)| \leq \frac{1}{\nu^2} \sum_{i\in Z_\ell} |\rho_{i,0} - \tilde{\rho}_{i,0}|, \tag{3.15}$$

$$|\rho_\ell(t,\,\cdot\,)|_{BV([0,P])} \leq \frac{1}{\nu^2} |\rho_0|_{BV([0,P])}, \tag{3.16}$$

$$\|\rho_\ell(t,\,\cdot\,) - \rho_\ell(s,\,\cdot\,)\|_{L^1([0,P])} \leq 2(1+2\kappa)\,\|v'\|_\infty\,(t-s), \tag{3.17}$$

for all $0 \leq s \leq t$. Here ρ_ℓ and $\tilde{\rho}_\ell$ are two solutions with P-periodic initial data $\rho_{\ell,0}$ and $\tilde{\rho}_{\ell,0}$, respectively.

Remark 3.2. *Estimate (3.15) follows directly from the corresponding estimate (3.5). However, this estimate cannot directly be expressed in terms of the L^1-norm in Eulerian coordinates.*[1]

Proof. The inequalities (3.14), (3.15), and (3.16) all follow from the corresponding inequalities for y_ℓ, using that $\rho_i \geq \nu$.

To prove (3.17), let ω_ε be a standard mollifier and define

$$\chi_i^\varepsilon(t,x) = \int_{-\infty}^{x} \big(\omega_\varepsilon(\sigma - x_i(t)) - \omega_\varepsilon(\sigma - x_{i+1}(t))\big)\,d\sigma, \tag{3.18}$$

and

$$\rho_\ell^\varepsilon(t,x) = \sum_i \rho_i(t)\chi_i^\varepsilon(t,x).$$

Using that

$$\frac{\partial}{\partial t}\chi_i^\varepsilon(t,x) = \Delta_+\left[\omega_\varepsilon(x-x_i)\,(\overline{v}_i + \kappa\Delta_- v_i)\right],$$

we get

$$\begin{aligned}
\partial_t\rho_\ell^\varepsilon(x,t) &= \sum_i \partial_t\left(\rho_i\chi_i^\varepsilon(t,x)\right)\\
&= \sum_i \dot{\rho}_i\chi_i^\varepsilon(t,x) + \rho_i\frac{\partial}{\partial t}\chi_i^\varepsilon(t,x)\\
&= \sum_i -\rho_i\Big(\frac{\Delta_+\overline{v}_i + \kappa\Delta_+\Delta_- v_i}{\Delta_+ x_i}\Big)\chi_i^\varepsilon(t,x) + \rho_i\Delta_+[\omega_\varepsilon(x-x_i)(\overline{v}_i + \kappa\Delta_- v_i)].
\end{aligned}$$

Consequently

$$\begin{aligned}
\|\rho_\ell^\varepsilon(t,\,\cdot\,) - \rho_\ell^\varepsilon(s,\,\cdot\,)\|_{L^1([0,P])} &= \Big\|\int_s^t \partial_t\rho_\ell^\varepsilon(\sigma,\,\cdot\,)\,d\sigma\Big\|_{L^1([0,P])}\\
&\leq \int_s^t\int_0^P \Big|\partial_t\rho_\ell^\varepsilon(\sigma,\,\cdot\,)\Big|\,dx d\sigma\\
&\leq \int_s^t\sum_i\int_0^P \Big[\rho_i\frac{|\Delta_+\overline{v}_i + \kappa\Delta_+\Delta_- v_i|}{\Delta_+ x_i}\chi_i^\varepsilon\\
&\qquad + \rho_i\left|\Delta_+\left[\omega_\varepsilon(x-x_i)(\overline{v}_i + \kappa\Delta_- v_i)\right]\right|\Big]\,dx d\sigma.
\end{aligned}$$

[1]We are grateful to Halvard O. Storbugt for pointing this out.

Now we can send ε to zero to obtain

$$\begin{aligned}\|\rho_\ell(t,\,\cdot\,)-\rho_\ell(s,\,\cdot\,)\|_{L^1([0,P])} &\le \int_s^t \sum_i \Big[\rho_i\left(|\Delta_+\overline{v}_i| + \kappa\,|\Delta_+\Delta_- v_i|\right) \\ &\qquad + \rho_i\left(|\Delta_+\overline{v}_i| + \kappa\,|\Delta_+\Delta_- v_i|\right)\Big]\,d\sigma \\ &\le 2\,(1+2\kappa)\,\|v'\|_\infty\,(t-s),\end{aligned}$$

where we have used that $\rho_i \le 1$ and (3.9). □

Remark 3.3. *It is natural to ask whether (3.14), (3.15), and (3.16) can be proved directly from the scheme for ρ_i, i.e., from (3.10). Using elementary techniques, this is easily accomplished for (3.14). For (3.15) and (3.16) one would hope to eliminate the constant $1/\nu^2$ (in fact, one would surmise that the constant would equal unity). However, we only managed to prove (3.15) and (3.16) with constant unity if $N=1$. To do this we rewrite the equation (3.10) for $\dot\rho_i$ as*

$$\ell\dot\rho_i = -\rho_i^2\Delta_+(v_i+\kappa\Delta_- v_i),$$

where we have used $\rho_i = \ell/\Delta_+ x_i$. Then we have

$$\begin{aligned}\ell\frac{d}{dt}\,|\rho_i-\rho_{i-1}| &= \operatorname{sign}(\Delta_-\rho_i)\,\Delta_-\dot\rho_i \\ &= \operatorname{sign}(\Delta_- v_i)\,\Delta_-\left(\rho_i^2\Delta_+ v_i\right) + \kappa\operatorname{sign}(\Delta_- v_i)\,\Delta_-\left(\rho_i^2\Delta_+\Delta_- v_i\right).\end{aligned}$$

We sum over $i\in Z_\ell$, and consider each term on the right separately. For the first term we get

$$\begin{aligned}\sum_i \operatorname{sign}(\Delta_- v_i)\,\Delta_-\left(\rho_i^2\Delta_+ v_i\right) &= \sum_i \operatorname{sign}(\Delta_- v_i)\left(\rho_i^2\Delta_+ v_i - \rho_{i-1}^2\Delta_- v_i\right) \\ &= \sum_i \operatorname{sign}(\Delta_- v_i)\,\rho_i^2\Delta_+ v_i - \rho_i^2\,|\Delta_+ v_i| \le 0.\end{aligned}$$

As to the second term

$$\begin{aligned}\sum_i \operatorname{sign}(\Delta_- v_i)\,\Delta_-\left(\rho_i^2\Delta_+\Delta_- v_i\right) &= \sum_i \operatorname{sign}(\Delta_- v_i)\left(\rho_i^2\Delta_+\Delta_- v_i - \rho_{i-1}^2\Delta_-\Delta_- v_i\right) \\ &= \sum_i \operatorname{sign}(\Delta_- v_i)\Big(\rho_i^2\left(\Delta_+ v_i - \Delta_- v_i\right) \\ &\qquad - \rho_{i-1}^2\left(\Delta_- v_i - \Delta_- v_{i-1}\right)\Big) \\ &= \sum_i \rho_i^2\left[\Delta_+ v_i - \Delta_- v_i\right]\left(\operatorname{sign}(\Delta_- v_i) - \operatorname{sign}\Delta_+ v_i\right) \le 0. \qquad (3.19)\end{aligned}$$

Thus

$$\frac{d}{dt}\,|\rho_\ell(t,\,\cdot\,)|_{BV} \le 0,$$

and (3.16) holds with constant equal to 1. We investigate the general case $N>1$ numerically. A natural numerical scheme consists in solving (2.6) by the Euler method, i.e., replacing (2.6) by

$$\frac{1}{\Delta t}\big(x_i(t+\Delta t) - x_i(t)\big) = \overline{v_i(t)} + \kappa\Delta_- v_i(t), \qquad (3.20)$$

where we must choose $\Delta t \le \ell$ in order to avoid collisions. We choose $\kappa = 0$, $N=10$, $c_j = 1/10$ for $j=0,\dots,4$, $\ell = 1/45$, $\Delta t = \ell$ and initial data given by

$$\rho_0(x) = \begin{cases} 1.0 & |x| < 0.5, \\ 0.05 & \textit{otherwise}, \end{cases}$$

for x in the interval $[-2,2]$, and extended periodically. From Figure 1 we see that

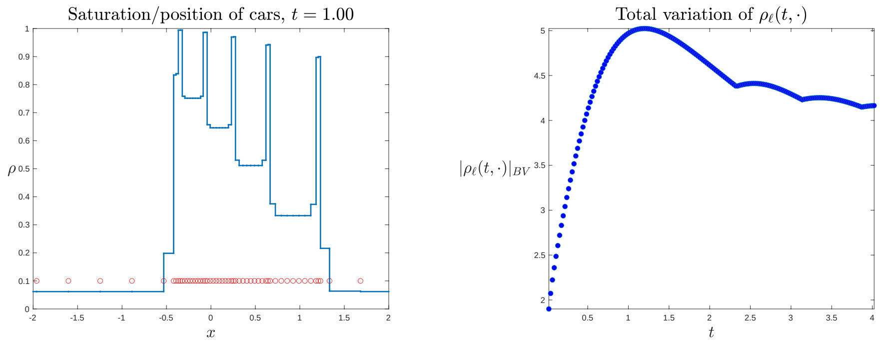

FIGURE 1. Left: $\rho_\ell(1,x)$, the red circles at the bottom indicate the position of the vehicles. Right: The total variation of ρ_ℓ as a function of $t \in [0,4]$.

ρ_ℓ develops large oscillations, and that it most likely is not true that $|\rho_\ell(t,\cdot)|_{BV} \le |\rho_0|_{BV}$ for the model specified by (2.6) when $N > 1$.

Corollary 3.4. *Under the same assumptions as in Lemma 3.1, for any $T > 0$ there exists a function $\rho \in C([0,T]; L^1([0,P])) \cap L^\infty([0,T]; BV([0,P]))$ such that (up to a subsequence)*

$$\lim_{\ell\to 0} \rho_\ell = \rho.$$

4. THE LIMIT

Having established the existence of the limit of ρ_ℓ to a function ρ as $\ell \to 0$ in Corollary 3.4, we now need to show that the limit is the unique weak entropy solution of the LWR equation, that is,

$$\int_0^\infty \int_0^P \big(\eta(\rho)\varphi_t + q(\rho)\varphi_x\big)\,dxdt + \int_0^P \eta(\rho_0)\varphi(0,x)\,dx \ge 0 \tag{4.1}$$

for all non-negative P-periodic test functions $\varphi \in C_c^\infty([0,\infty)\times[0,P])$. Here η is a convex (entropy) function and q is the entropy flux, satisfying $q'(\rho) = \eta'(\rho)(\rho v(\rho))'$.

Theorem 4.1. *Let v be a Lipschitz continuous non-increasing velocity function $v\colon [0,1] \to [0,1]$, with $v(0) = 1$ and $v(1) = 0$. Let $P > 0$ and $\rho_0 \in L^1([0,P]) \cap BV([0,P])$ be P-periodic such that $\rho_0(x) \in [\nu, 1]$ for all x for some $\nu > 0$. Define $x_i(0)$ by (3.11) for $i \in Z_\ell$, and let $x_i(t)$ be defined by (2.6).*

Define ρ_i for $i \in Z_\ell$ by (3.10), and $\rho_\ell(t,x)$ by (3.13) for $\ell > 0$. Denote the limit as $\ell \to 0$ of $\rho_\ell(t,x)$, granted by Corollary 3.4, by ρ with $\rho \in C([0,T]; L^1([0,P])) \cap L^\infty([0,T]; BV([0,P]))$. Then ρ is the unique weak entropy solution of the scalar conservation law

$$\rho_t + (\rho v(\rho))_x = 0, \quad \rho|_{t=0} = \rho_0. \tag{4.2}$$

Proof. Let (η, q) be an entropy/entropy flux pair. We define

$$\eta_\ell(t,x) = \sum_i \eta_i(t)\mathbb{I}_i(t,x),$$
$$\eta_\ell^\varepsilon(t,x) = \sum_i \eta_i(t)\chi_i^\varepsilon(t,x),$$

where, in the obvious notation, $\eta_i = \eta(\rho_i)$, and χ_i^ε is given by (3.18). The dynamics of η_i is given by

$$\dot\eta_i = -\eta'(\rho_i)\rho_i \frac{\Delta_+\overline{v}_i + \kappa\Delta_+\Delta_- v_i}{\Delta_+ x_i}.$$

Fix a nonnegative test function φ and calculate

$$\begin{aligned}
&\int_0^\infty \int_0^P \eta_\ell^\varepsilon \varphi_t \, dxdt \\
&\quad = -\int_0^P \eta_\ell^\varepsilon(0,x)\varphi(0,x)\, dx - \int_0^\infty \int_0^P \partial_t \eta_\ell^\varepsilon \varphi \, dxdt \\
&\quad = -\int_0^P \eta_\ell^\varepsilon(0,x)\varphi(0,x)\, dx - \int_0^\infty \int_0^P \sum_i \Big(\dot\eta_i \chi_i^\varepsilon + \eta_i \frac{\partial}{\partial t}\chi_i^\varepsilon\Big)\varphi \, dxdt \\
&\quad = -\int_0^P \eta_\ell^\varepsilon(0,x)\varphi(0,x)\, dx \\
&\qquad - \int_0^\infty \int_0^P \sum_i \Big[\Big(-\eta'(\rho_i)\rho_i \frac{\Delta_+\overline{v}_i + \kappa\Delta_+\Delta_- v_i}{\Delta_+ x_i}\Big)\chi_i^\varepsilon \\
&\qquad\qquad + \eta_i \Delta_+\left[\omega_\varepsilon(x-x_i)\left(\overline{v}_i + \kappa\Delta_- v_i\right)\right]\Big]\varphi \, dxdt. \qquad (4.3)
\end{aligned}$$

Next we want to take the $\varepsilon \to 0$ limit. To that end, we consider some of the terms separately. We have

$$\begin{aligned}
&\int_0^P \sum_i \Big(-\eta'(\rho_i)\rho_i \frac{\Delta_+\overline{v}_i + \kappa\Delta_+\Delta_- v_i}{\Delta_+ x_i}\Big)\chi_i^\varepsilon \varphi \, dx \\
&\quad \xrightarrow{\varepsilon\to 0} \sum_i \int_{x_i}^{x_{i+1}} \Big(-\eta'(\rho_i)\rho_i \frac{\Delta_+\overline{v}_i + \kappa\Delta_+\Delta_- v_i}{\Delta_+ x_i}\Big)\varphi \, dx \\
&\qquad = \sum_i \Big(-\eta'(\rho_i)\rho_i(\Delta_+\overline{v}_i + \kappa\Delta_+\Delta_- v_i)\Big)\varphi_{i+1/2}
\end{aligned}$$

where

$$\varphi_{i+1/2} = \frac{1}{\Delta_+ x_i}\int_{x_i}^{x_{i+1}} \varphi \, dx.$$

Furthermore, we will use

$$\begin{aligned}
&\int_0^P \eta_i \Delta_+\left[\omega_\varepsilon(x-x_i)\left(\overline{v}_i + \kappa\Delta_- v_i\right)\right]\varphi \, dx \\
&\quad = \int_0^P \eta_i \Delta_+\left[\omega_\varepsilon(x-x_i)\left(\overline{v}_i + \kappa\Delta_- v_i\right)\varphi\right] dx \\
&\quad \xrightarrow{\varepsilon\to 0} \eta_i \Delta_+\left[\left(\overline{v}_i + \kappa\Delta_- v_i\right)\varphi_i\right],
\end{aligned}$$

with $\varphi_i = \varphi(t, x_i(t))$. Note that while $\varphi_{i+1/2}$ is defined by a spatial average, φ_i is given as a pointwise value. This yields that the limit as $\varepsilon \to 0$ in (4.3) satisfies

$$\begin{aligned}
&\int_0^\infty \int_0^P \eta_\ell \varphi_t \, dxdt \\
&\quad = -\int_0^P \eta_\ell(0,x)\varphi(0,x)\, dx \\
&\qquad - \int_0^\infty \Big(\sum_i \Big(-\eta'(\rho_i)\rho_i(\Delta_+\overline{v}_i + \kappa\Delta_+\Delta_- v_i)\Big)\varphi_{i+1/2} \\
&\qquad\qquad + \eta_i \Delta_+\left[\left(\overline{v}_i + \kappa\Delta_- v_i\right)\phi_i\right]\Big)\, dt \\
&\quad = -\int_0^P \eta_\ell(0,x)\varphi(0,x)\, dx \\
&\qquad + \int_0^\infty \sum_i \Big(\rho_i \eta'(\rho_i)\Delta_+\overline{v}_i\varphi_{i+1/2} - \eta_i\Delta_+\overline{v}_i\varphi_{i+1} - \eta_i\overline{v}_i\Delta_+\varphi_i
\end{aligned}$$

$$+\kappa\left(\Delta_+\Delta_- v_i\right)\left(\rho_i\eta'(\rho_i)\varphi_{i+1/2}-\eta_i\varphi_i\right)\Big)\,dt.$$

Next we want to replace $\varphi_{i+1/2}$ and φ_{i+1} with φ_i, and by doing so, we will introduce an error term of order $\mathcal{O}(\ell)$. First observe that

$$\left|\varphi_{i+1/2}-\varphi_i\right| \le \frac{1}{2}\left\|\varphi_x\right\|_\infty \Delta_+ x_i \le \frac{\ell}{2\nu}\left\|\varphi_x\right\|_\infty,$$

$$\left|\Delta_+\varphi_i\right| = \left|\varphi_i-\varphi_{i+1}\right| \le \left\|\varphi_x\right\|_\infty \Delta_+ x_i \le \frac{\ell}{\nu}\left\|\varphi_x\right\|_\infty,$$

since

$$\Delta_+ x_i(t) \le \ell \sup_i y_i(t) \le \frac{\ell}{\nu},$$

where $\left\|\varphi_x\right\|_\infty = \left\|\varphi_x\right\|_{L^\infty([0,\infty)\times[0,P])}$.

For the relevant terms we add and subtract φ_i. Thus

$$\begin{aligned}
\Big|\int_0^\infty \sum_i \rho_i\eta'(\rho_i)\Delta_+\overline{v}_i(\varphi_{i+1/2}-\varphi_i)\,dt\Big| &\le \sup_{\rho\in[0,1]}\left|\eta'(\rho)\right| T\sum_i \left|\Delta_+\overline{v}_i\right|\left\|\varphi_x\right\|_\infty \frac{\ell}{\nu}\\
&\le \sup_{\rho\in[0,1]}\left|\eta'(\rho)\right| \frac{T\ell}{\nu}\left\|\varphi_x\right\|_\infty \left|\overline{v}\right|_{BV}\\
&\le \sup_{\rho\in[0,1]}\left|\eta'(\rho)\right| \frac{T\ell}{\nu}\left\|\varphi_x\right\|_\infty \left\|v'\right\|_\infty \left|\rho_\ell(t,\,\cdot\,)\right|_{BV}\\
&\le \sup_{\rho\in[0,1]}\left|\eta'(\rho)\right| \frac{T\ell}{\nu^3}\left\|\varphi_x\right\|_\infty \left\|v'\right\|_\infty \left|\rho_0\right|_{BV} = \mathcal{O}(\ell),
\end{aligned}$$

where we used that $\rho_\ell \in [0,1]$, (3.9), (3.16), and $T>0$ is such that $\varphi(t,x)=0$ for $t>T$ and all x. Next, we find

$$\Big|\int_0^\infty \sum_i \eta_i\Delta_+\overline{v}_i(\varphi_{i+1}-\varphi_i)\,dt\Big| \le \sup_{\rho\in[0,1]}\left|\eta(\rho)\right|\frac{T\ell}{\nu^3}\left\|\varphi_x\right\|_\infty \left\|v'\right\|_\infty \left|\rho_0\right|_{BV} = \mathcal{O}(\ell),$$

by the same estimates as above. Finally, we consider

$$\begin{aligned}
\Big|\int_0^\infty \sum_i \left(\Delta_+\Delta_- v_i\right)\rho_i\eta'(\rho_i)(\varphi_{i+1/2}-\varphi_i)\,dt\Big| &\le \sup_{\rho\in[0,1]}\left|\eta'(\rho)\right|\frac{T\ell}{2\nu}\left\|\varphi_x\right\|_\infty 2\left|v\right|_{BV}\\
&\le \sup_{\rho\in[0,1]}\left|\eta'(\rho)\right|\left\|v'\right\|_\infty \left|\rho_0\right|_{BV}\frac{T\ell}{\nu} = \mathcal{O}(\ell),
\end{aligned}$$

as in the estimates above. Thus

$$\begin{aligned}
\int_0^\infty\int_0^P \eta_\ell\varphi_t\,dxdt = &-\int_0^P \eta_\ell(0,x)\varphi(0,x)\,dx\\
&+\int_0^\infty\sum_i\Big(\rho_i\eta'(\rho_i)\Delta_+\overline{v}_i\varphi_i-\eta_i\Delta_+\overline{v}_i\varphi_i-\eta_i\overline{v}_i\Delta_+\varphi_i\\
&\qquad+\kappa\left(\Delta_+\Delta_- v_i\right)\left(\rho_i\eta'(\rho_i)-\eta_i\right)\varphi_i\Big)\,dt+\mathcal{O}(\ell).
\end{aligned}\tag{4.4}$$

Define $h(\rho)=\rho\eta'(\rho)-\eta(\rho)$ and note that $h'(\rho)=\rho\eta''(\rho)\ge 0$. Using this, the last term above equals

$$\begin{aligned}
\sum_i\left(\Delta_+\Delta_- v_i\right)\varphi_i h_i &= -\sum_i \Delta_- v_i\Delta_-(h_i\varphi_i)\\
&= -\sum_i\Big(\Delta_- v_i\Delta_- h_i\,\varphi_{i-1}+h_i\Delta_- v_i\Delta_-\varphi_i\Big)\\
&= -\sum_i \Delta_- v_i\Delta_-\rho_i h'(\hat\rho_i)\,\varphi_{i-1}+\mathcal{O}(\ell)\\
&\ge \mathcal{O}(\ell),
\end{aligned}$$

for some $\hat{\rho}_i$ between ρ_i and ρ_{i-1}. We used the nonnegativity of the test function and that $\Delta_- v_i \Delta_- \rho_i \le 0$ since v is non-increasing in ρ.

We want to replace the term $\eta_i \overline{v}_i$ by $(\overline{\eta v})_i$, and this invokes an error of $\mathcal{O}(\ell)$, as the following computation reveals.

$$\begin{aligned}
\Big|\int_0^\infty \sum_i \big(\eta_i \overline{v}_i - \overline{\eta v}_i\big)\Delta_+\varphi_i \,dt\Big| &= \Big|\int_0^\infty \sum_i \sum_{j=0}^N c_j\big(\eta_i - \eta_{i+j}\big)v_{i+j}\Delta_+\varphi_i \,dt\Big| \\
&= \Big|\int_0^\infty \sum_i \sum_{j=0}^N \sum_{k=0}^{j-1} c_j\big(\eta_{i+k} - \eta_{i+k+1}\big)v_{i+j}\Delta_+\varphi_i \,dt\Big| \\
&\le \|v\|_\infty \int_0^T \sum_{j=0}^N c_j \sum_{k=0}^{j-1}\sum_i \left|\eta_{i+k} - \eta_{i+k+1}\right| \left|\Delta_+\varphi_i\right| \,dt \\
&\le \|v\|_\infty \|\varphi_x\|_\infty \frac{\ell}{\nu}\int_0^T |\eta(t)|_{BV} \sum_{j=0}^N jc_j \,dt \\
&\le \|v\|_\infty \|\varphi_x\|_\infty \frac{\ell}{\nu}\|\eta'\|_\infty \int_0^T |\rho(t)|_{BV} \sum_{j=0}^N jc_j \,dt \\
&\le T\|v\|_\infty \|\varphi_x\|_\infty \frac{\ell}{\nu}\|\eta'\|_\infty |\rho_0|_{BV} \sum_{j=0}^N jc_j \\
&= \mathcal{O}(\ell). \qquad (4.5)
\end{aligned}$$

Thus we find that (4.4) can be re-written as

$$\begin{aligned}
\int_0^\infty \int_0^P \eta_\ell \varphi_t \,dxdt \ge &- \int_0^P \eta_\ell(0,x)\varphi(0,x)\,dx \\
&+ \int_0^\infty \sum_i \left(\rho_i \eta'(\rho_i) - \eta_i\right)\Delta_+\overline{v}_i\varphi_i \,dt - \int_0^\infty\int_0^P \left(\overline{\eta v}\right)_\ell \varphi_x \,dxdt + \mathcal{O}(\ell). \qquad (4.6)
\end{aligned}$$

In order to understand the convective term we compare this with the term involving the entropy flux. We define

$$q_\ell(t,x) = \sum_i q_i(t)\mathbb{I}_i(t,x), \quad q_i(t) = q(\rho_i(t)),$$

which implies that

$$\begin{aligned}
\int_0^\infty \int_0^P q_\ell \varphi_x \,dxdt = \int_0^\infty \sum_i q_i \Delta_+\varphi_i \,dt &= \int_0^\infty \sum_i \overline{q}_i \Delta_+\varphi_i \,dt + \mathcal{O}(\ell) \\
&= -\int_0^\infty \sum_i \Delta_+\overline{q}_i\varphi_i \,dt + \mathcal{O}(\ell), \qquad (4.7)
\end{aligned}$$

where we have replaced q_i by $\overline{q}_i$, resulting in an error of order $\mathcal{O}(\ell)$, from the following computation

$$\begin{aligned}
\Big|\int_0^\infty \sum_i \big(q_i - \overline{q}_i\big)\Delta_+\varphi_i \,dt\Big| &= \Big|\int_0^\infty \sum_i \sum_{j=0}^N c_j\big(q_i - q_{i+j}\big)\Delta_+\varphi_i \,dt\Big| \\
&\le \int_0^T \sum_i \sum_{j=0}^N c_j \left|q_i - q_{i+j}\right| \left|\Delta_+\varphi_i\right| \,dt \\
&\le T\|\varphi_x\|_\infty \frac{\ell}{\nu}\|q'\|_\infty |\rho_0|_{BV} \sum_{j=0}^N jc_j
\end{aligned}$$

$$= \mathcal{O}(\ell),$$

similarly to (4.5). The observant reader will also have noticed that we have replaced φ_{i+1} by φ_i in (4.7), which also yields another $\mathcal{O}(\ell)$ error.

Adding (4.6) and (4.7) we get

$$\int_0^\infty \int_0^P (\eta_\ell \varphi_t + q_\ell \varphi_x)\,dxdt + \int_0^P \eta_\ell(0,x)\varphi(0,x)\,dx$$
$$\geq \int_0^\infty \sum_i \Big[\underbrace{(\rho_i \eta'(\rho_i) - \eta_i)\,\Delta_+ \overline{v}_i - \Delta_+ \overline{q}_i}_{e_i}\Big]\varphi_i\,dt - \int_0^\infty \int_0^P (\overline{\eta v})_\ell\,\varphi_x\,dxdt + \mathcal{O}(\ell),$$

and it remains to estimate the term e_i. To that end we find, using the formula (2.4), that

$$\begin{aligned}
e_i &= -\sum_{j=1}^N \Delta_- c_j \left[(\rho_i \eta'(\rho_i) - \eta_i)\,(v_{i+j} - v_i) - (q_{i+j} - q_i)\right] \\
&= -\sum_{j=1}^N \Delta_- c_j \int_{\rho_i}^{\rho_{i+j}} \Big((\rho_i \eta'(\rho_i) - \eta_i)\, v'(s) - \eta'(s) s v'(s) - \eta'(s) v(s)\Big)\,ds \\
&= -\sum_{j=1}^N \Delta_- c_j \int_{\rho_i}^{\rho_{i+j}} \Big(v'(s)\,(\rho_i \eta'(\rho_i) - s\eta'(s)) - \eta_i' v'(s) - \eta'(s) v(s)\Big)\,ds \\
&= -\sum_{j=1}^N \Delta_- c_j \int_{\rho_i}^{\rho_{i+j}} \Big(v'(s) \int_s^{\rho_i} \frac{d}{d\sigma}(\sigma\eta'(\sigma))\,d\sigma - \eta'(\rho_i) v'(s) - \eta'(s) v(s)\Big)\,ds \\
&= -\sum_{j=1}^N \Delta_- c_j \int_{\rho_i}^{\rho_{i+j}} \Big(v'(s) \int_s^{\rho_i} \sigma\eta''(\sigma) + \eta'(\sigma)\,d\sigma - \eta'(\rho_i) v'(s) - \eta'(s) v(s)\Big)\,ds \\
&= \sum_{j=1}^N \Delta_- c_j \int_{\rho_i}^{\rho_{i+j}} \int_{\rho_i}^{s} v'(s)\sigma\eta''(\sigma)\,d\sigma ds \\
&\quad - \sum_{j=1}^N \Delta_- c_j \int_{\rho_i}^{\rho_{i+j}} \big(v'(s)(\eta_i - \eta(s)) - \eta_i v'(s) - \eta'(s) v(s)\big)\,ds \\
&\geq \sum_{j=1}^N \Delta_- c_j \int_{\rho_i}^{\rho_{i+j}} \big(v'(s)\eta(s) + v(s)\eta'(s)\big)\,ds \\
&= \sum_{j=1}^N \Delta_- c_j \,(v_{i+j}\eta_{i+j} - v_i \eta_i) \\
&= -\Delta_+ \left(\overline{v\eta}\right)_i.
\end{aligned}$$

In the inequality we used that $\Delta_- c_j v'(s) \geq 0$, as both c_j and v are non-increasing, ρ_i and ρ_{i+j} are positive, and finally $\eta''(\sigma) \geq 0$.

Therefore

$$\int_0^\infty \int_0^P (\eta_\ell \varphi_t + q_\ell \varphi_x)\,dxdt + \int_0^P \eta_\ell(0,x)\varphi(0,x)\,dx$$
$$\geq \int_0^\infty \sum_i e_i \varphi_i\,dt - \int_0^\infty \int_0^P (\overline{\eta v})_\ell\,\varphi_x\,dxdt + \mathcal{O}(\ell)$$
$$\geq -\int_0^\infty \sum_i \Delta_+ \left(\overline{v\eta}\right)_i \varphi_i\,dt - \int_0^\infty \int_0^P (\overline{\eta v})_\ell\,\varphi_x\,dxdt + \mathcal{O}(\ell)$$

$$
\begin{aligned}
&= \int_0^\infty \sum_i \left(\overline{v\eta}\right)_i \Delta_- \varphi_i \, dt \quad - \int_0^\infty \int_0^P \left(\overline{\eta v}\right)_\ell \varphi_x \, dx dt + \mathcal{O}(\ell) \\
&= \mathcal{O}(\ell),
\end{aligned}
$$

and by taking $\ell \to 0$, we obtain (4.1). By standard theory, see, e.g., [13, Thm. 2.14], it follows that ρ is the unique weak entropy solution. □

References

[1] B. Argall, E. Cheleshkin, J. M. Greenberg, C. Hinde, and P.-J. Lin. A rigorous treatment of a follow-the-leader traffic model with traffic lights present. *SIAM J. Appl. Math.* 63(19): 149–168, 2002.

[2] A. Aw, A. Klar, T. Materne, and M. Rascle. Derivation of continuum traffic flow models from microscopic follow-the-leader models. *SIAM J. Appl. Math.*, 63(1): 259–278, 2002.

[3] M. Brackstone and M. McDonald. Car-following: a historical review. *Transportation Research* Part F 2, pp. 181–196, 1999.

[4] G.M. Coclite, J.-M. Coron, N. De Nitti, A. Keimer, and L. Pflug. A general result on the approximation of local conservation laws by nonlocal conservation laws: the singular limit problem for exponential kernels. *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, 40(5):1205–1223, 2023.

[5] M. Colombo, G. Crippa, M. Graff, and L. V. Spinola. Recent results on the singular local limit for nonlocal conservation laws. In *Hyperbolic Problems: Theory, Numerics, Applications*, Am. Inst. Math. Sci. (AIMS), Springfield, MO, pp. 369–376, 2020.

[6] R. M. Colombo, H. Holden, and F. Marcellini. On the microscopic modeling of vehicular traffic on general networks. *SIAM Journal of Applied Mathematics* 80 (3): 1377–1391, 2020.

[7] R. M. Colombo and E. Rossi. On the micro-macro limit in traffic flow. *Rend. Sem. Math. Univ. Padova* 131:217–235, 2014.

[8] E. Cristiani and S. Sahu. On the micro-to-macro limit for first-order traffic flow models on networks. *Networks and Heterogeneous Media* 11(3):395–413, 2016.

[9] M. Di Francesco, S. Fagioli, and M. D. Rosini. Deterministic particle approximation of scalar conservation laws. *Boll. Unione Mat. Ital.* 10:487–501, 2017.

[10] M. Di Francesco and M. D. Rosini. Rigorous derivation of nonlinear scalar conservation laws from follow-the-leader type models via many particle limit. *Arch. Ration. Mech. Anal.*, 217(3):831–871, 2015.

[11] M. Di Francesco and G. Stivaletta. The one-sided lipschitz condition in the follow-the-leader approximation of scalar conservation laws. *J. Hyp. Diff. Equations* 19(4):775–807, 2022.

[12] P. Goatin and F. Rossi. A traffic flow model with non-smooth metric interaction: well-posedness and micro-macro limit. *Commun. Math. Sci.* 15(1):261–287, 2017.

[13] H. Holden and N. H. Risebro. *Front Tracking for Hyperbolic Conservation Laws.* Springer-Verlag, New York, 2015, Second edition.

[14] H. Holden and N. H. Risebro. Follow-the-leader models can be viewed as a numerical approximation to the Lighthill–Whitham–Richards model for traffic flow. *Networks & Heterogeneous Media* 13(3):409–421, 2018.

[15] H. Holden and N. H. Risebro. Continuum limit of Follow-the-Leader models — a short proof. *Discrete and Continuous Dynamical Systems* 38(2):715–722, 2018.

[16] M. J. Lighthill and G. B. Whitham. Kinematic waves. II. A theory of traffic flow on long crowded roads. *Proc. Roy. Soc. (London), Series A*, 229(1178):317–345, 1955.

[17] P. I. Richards. Shockwaves on the highway. *Operations Research*, 4(1): 42–51, 1956.

[18] M. D. Rosini. *Macroscopic Models for Vehicular Flows and Crowd Dynamics: Theory and Applications. Understanding Complex Systems.* Springer, 2013.

[19] E. Rossi. A justification of a LWR model based on a follow the leader description. *Discrete Cont. Dyn. Syst. Series S* 7(3): 579–591, 2014.

(Holden)
Department of Mathematical Sciences, NTNU Norwegian University of Science and Technology, NO–7491 Trondheim, Norway
Email address: helge.holden@ntnu.no
URL: https://www.ntnu.edu/employees/holden

(Risebro)
Department of Mathematics, University of Oslo, P.O. Box 1053, Blindern, NO–0316 Oslo, Norway
Email address: nilshr@math.uio.no
URL: https://www.mn.uio.no/math/english/people/aca/nilshr/index.html