

# Perturbations of Dirac operators, spectral Einstein functionals and the Noncommutative residue

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## Abstract

In this paper, we introduce the spectral Einstein functional for perturbations of Dirac operators on manifolds with boundary. Furthermore, we provide the proof of the Dabrowski-Sitarz-Zalecki type theorems associated with the spectral Einstein functionals for perturbations of Dirac operators, particularly in the cases of on 4-dimensional manifolds with boundary.

**Keywords:** Perturbations of Dirac Operators; noncommutative residue; spectral Einstein functionals; Dabrowski-Sitarz-Zalecki type theorems.

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## 1. Introduction

An eminent spectral scheme, yielding geometric objects on manifolds, including residue, scalar curvature, and various scalar combinations of curvature tensors, is the small-time asymptotic expansion of the (localised) trace of heat kernel [4, 5]. The theory exhibits profound and rich structures, spanning both physics and mathematics. In the recent paper [2], Dabrowski etc. introduced bilinear functionals of vector fields and differential forms, the densities of which yield the metric and Einstein spectral functionals on even-dimensional Riemannian manifolds, and they obtained certain values or residues of the (localised) zeta function of the Laplacian arising from the Mellin transform and the coefficients of this expansion.

Consider  $E$  as a finite-dimensional complex vector bundle over a closed compact manifold  $M$  with dimension  $n$ , the noncommutative residue of a pseudo-differential operator  $P \in \Psi DO(E)$  can be defined as

$$\text{res}(P) := (2\pi)^{-n} \int_{S^* M} \text{trace}(\sigma_{-n}^P(x, \xi)) dx d\xi, \quad (1.1)$$

where  $S^* M \subset T^* M$  represents the co-sphere bundle on  $M$  and  $\sigma_{-n}^P$  is the component of order  $-n$  of the complete symbol

$$\sigma^P := \sum_i \sigma_i^P \quad (1.2)$$

of  $P$ , as defined in [6–9], and the linear functional  $\text{res} : \Psi DO(E) \rightarrow \mathbb{C}$  is, in fact, the unique trace (up to multiplication by constants) on the algebra of pseudo-differential operators  $\Psi DO(E)$ .

In [10], Connes used the noncommutative residue to derive a conformal 4-dimensional Polyakov action analogy. Connes proved that the noncommutative residue on a compact manifold  $M$  coincided with Dixmier's trace on pseudodifferential operators of order  $\text{-dim} M$  [11]. Furthermore, Connes claimed the noncommutative residue of the square of the inverse of the Dirac operator was proportioned to the Einstein-Hilbert action

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in [11]. Kastler provided a brute-force proof of this theorem in [13], while Kalau and Walze proved it in the normal coordinates system simultaneously in [14], which is called the Kastler-Kalau-Walze theorem now. Building upon the theory of the noncommutative residue introduced by Wodzicki, Fedosov etc. [15] constructed a noncommutative residue on the algebra of classical elements in Boutet de Monvel's calculus on a compact manifold with boundary of dimension  $n > 2$ . With elliptic pseudodifferential operators and noncommutative residue, it is natural way for investigating the Kastler-Kalau-Walze type theorem and operator-theoretic explanation of the gravitational action on manifolds with boundary. Concerning Dirac operators and signature operators, Wang performed computations of the noncommutative residue and successfully demonstrated the Kastler-Kalau-Walze type theorem for manifolds with boundaries [16–18].

Jean-Michel Bismut [19] previously established a local index theorem for Dirac operators on a Riemannian manifold  $M$ , specifically those associated with connections on  $TM$  with non zero torsion. In [20], Ackermann and Tolksdorf demonstrated a generalized version of the well-known Lichnerowicz formula. This extended formula applies to for the square of the most general Dirac operator with torsion  $D_T$  on an even-dimensional spin manifold, associated to a metric connection with torsion. In [21], Pfäffle and Stephan focused on compact Riemannian spin manifolds without boundary equipped with orthogonal connections, and delved into the induced Dirac operators. Furthermore, Pfäffle and Stephan investigated orthogonal connections with arbitrary torsion on compact Riemannian manifolds, they computed the spectral action for the induced Dirac operators, twisted Dirac operators and Dirac operators of Chamseddine-Connes type, as detailed in their work[3]. In [22], J. Wang, Y. Wang and C. Yang calculated the lower dimensional volume  $\widetilde{\text{Wres}}[\pi^+(D_T^*)^{-p_1} \circ \pi^+ D_T^{-p_2}]$  and derived a Kastler-Kalau-Walze type theorems associated with Dirac operators with torsion on compact manifolds with boundary. In [28], the authors generalized the results in [2], [17], [21] and obtained spectral functionals associated with Dirac operators with torsion on compact manifolds with boundary. In [1], Wang considered the arbitrary perturbations of Dirac operators, and established the associated Kastler-Kalau-Walze theorem.

**The aim of this paper** is to extend and generalize the results in [2], [28] and focusing on obtaining the spectral Einstein functional for perturbations of Dirac operators on compact manifolds with boundary. Specifically, for lower dimensional compact Riemannian manifolds with boundary, we compute the non-commutative residue  $\widetilde{\text{Wres}}\left[\pi^+\nabla_X^\Psi\nabla_Y^\Psi D_\Psi^{-2} \circ \pi^+ D_\Psi^{-2}\right]$  and  $\widetilde{\text{Wres}}\left[\pi^+\nabla_X^\Psi\nabla_Y^\Psi D_\Psi^{-1} \circ \pi^+ D_\Psi^{-3}\right]$  on 4 dimensional manifolds. Our main theorems are as follows.

**Theorem 1.1.** *Let  $M$  be an 4-dimensional oriented compact spin manifold with boundary  $\partial M$ , then we get the following equality:*

$$\begin{aligned} \widetilde{\text{Wres}}\left[\pi^+\nabla_X^\Psi\nabla_Y^\Psi D_\Psi^{-2} \circ \pi^+ D_\Psi^{-2}\right] &= \frac{4\pi^2}{3} \int_M EG(X, Y) vol_g + \frac{\pi^2}{2} \int_M \text{trace}\left[c(X)\nabla_Y^{S(TM)}(c(\Psi))\right. \\ &+ \left.\nabla_Y^{S(TM)}(c(\Psi))c(X) - c(Y)\nabla_X^{S(TM)}(c(\Psi)) - \nabla_X^{S(TM)}(c(\Psi))c(Y)\right] dvol_M + \frac{1}{2} \int_M \text{trace}\left[\frac{1}{4}s\right. \\ &+ \left.\sum_{j=1}^n \frac{1}{2}c(\Psi)c(e_j)c(\Psi)c(e_j) - (c(\Psi))^2\right] g(X, Y) dvol_M + \int_{\partial M} \tilde{\Phi}, \end{aligned} \quad (1.3)$$

where  $\tilde{\Phi}$  are defined in (4.48).

**Theorem 1.2.** *Let  $M$  be an 4-dimensional oriented compact spin manifold with boundary  $\partial M$ , then we get the following equality:*

$$\begin{aligned} \widetilde{\text{Wres}}\left[\pi^+\nabla_X^\Psi\nabla_Y^\Psi D_\Psi^{-1} \circ \pi^+ D_\Psi^{-3}\right] &= \frac{4\pi^2}{3} \int_M EG(X, Y) vol_g + \frac{\pi^2}{2} \int_M \text{trace}\left[c(X)\nabla_Y^{S(TM)}(c(\Psi))\right. \\ &+ \left.\nabla_Y^{S(TM)}(c(\Psi))c(X) - c(Y)\nabla_X^{S(TM)}(c(\Psi)) - \nabla_X^{S(TM)}(c(\Psi))c(Y)\right] dvol_M + \frac{1}{2} \int_M \text{trace}\left[\frac{1}{4}s\right. \\ &+ \left.\sum_{j=1}^n \frac{1}{2}c(\Psi)c(e_j)c(\Psi)c(e_j) - (c(\Psi))^2\right] g(X, Y) dvol_M + \int_{\partial M} \hat{\Phi}, \end{aligned} \quad (1.4)$$

where  $\widehat{\Phi}$  are defined in (5.48).

The paper is organized in the following way. In Section 2, we review some basic formulas related to the spectral Einstein functional for perturbations of Dirac operators. In Section 4, we prove the Dabrowski-Sitarz-Zalecki type theorem for  $\widetilde{\text{Wres}}\left[\pi^+ \nabla_X^\Psi \nabla_Y^\Psi D_\Psi^{-2} \circ \pi^+ D_\Psi^{-2}\right]$  on 4-dimensional manifolds with boundary.

In Section 5, we prove the Dabrowski-Sitarz-Zalecki type theorem  $\widetilde{\text{Wres}}\left[\pi^+ \nabla_X^\Psi \nabla_Y^\Psi D_\Psi^{-1} \circ \pi^+ D_\Psi^{-3}\right]$  on 4-dimensional manifolds with boundary.

## 2. Boutet de Monvel's calculus and the noncommutative residue

In this section, we recall some basic facts and formulas about Boutet de Monvel's calculus and the definition of the noncommutative residue for manifolds with boundary, which will be used in the following. For more details, see Section 2 in [17].

Let  $M$  be a 4-dimensional compact oriented manifold with boundary  $\partial M$ . We assume that the metric  $g^{TM}$  on  $M$  has the following form near the boundary,

$$g^M = \frac{1}{h(x_n)} g^{\partial M} + dx_n^2, \quad (2.1)$$

where  $g^{\partial M}$  is the metric on  $\partial M$  and  $h(x_n) \in C^\infty([0, 1]) := \{\widehat{h}|_{[0,1]} | \widehat{h} \in C^\infty((-\varepsilon, 1))\}$  for some  $\varepsilon > 0$  and  $h(x_n)$  satisfies  $h(x_n) > 0$ ,  $h(0) = 1$  where  $x_n$  denotes the normal directional coordinate. Let  $U \subset M$  be a collar neighborhood of  $\partial M$  which is diffeomorphic with  $\partial M \times [0, 1]$ . By the definition of  $h(x_n) \in C^\infty([0, 1])$  and  $h(x_n) > 0$ , there exists  $\widehat{h} \in C^\infty((-\varepsilon, 1))$  such that  $\widehat{h}|_{[0,1]} = h$  and  $\widehat{h} > 0$  for some sufficiently small  $\varepsilon > 0$ . Then there exists a metric  $g'$  on  $\widetilde{M} = M \cup_{\partial M} \partial M \times (-\varepsilon, 0]$  which has the form on  $U \cup_{\partial M} \partial M \times (-\varepsilon, 0]$

$$g' = \frac{1}{\widehat{h}(x_n)} g^{\partial M} + dx_n^2, \quad (2.2)$$

such that  $g'|_M = g$ . We fix a metric  $g'$  on the  $\widetilde{M}$  such that  $g'|_M = g$ .

Let the Fourier transformation  $F'$  be

$$F' : L^2(\mathbf{R}_t) \rightarrow L^2(\mathbf{R}_v); F'(u)(v) = \int_{\mathbb{R}} e^{-ivt} u(t) dt$$

and let

$$r^+ : C^\infty(\mathbf{R}) \rightarrow C^\infty(\widetilde{\mathbf{R}^+}); f \mapsto f|_{\widetilde{\mathbf{R}^+}}; \widetilde{\mathbf{R}^+} = \{x \geq 0; x \in \mathbf{R}\}.$$

We define  $H^+ = F'(\Phi(\widetilde{\mathbf{R}^+}))$ ;  $H_0^- = F'(\Phi(\widetilde{\mathbf{R}^-}))$  which satisfies  $H^+ \perp H_0^-$ , where  $\Phi(\widetilde{\mathbf{R}^+}) = r^+ \Phi(\mathbf{R})$ ,  $\Phi(\widetilde{\mathbf{R}^-}) = r^- \Phi(\mathbf{R})$  and  $\Phi(\mathbf{R})$  denotes the Schwartz space. We have the following property:  $h \in H^+$  (respectively  $H_0^-$ ) if and only if  $h \in C^\infty(\mathbf{R})$  which has an analytic extension to the lower (respectively upper) complex half-plane  $\{\text{Im}\xi < 0\}$  (respectively  $\{\text{Im}\xi > 0\}$ ) such that for all nonnegative integer  $l$ ,

$$\frac{d^l h}{d\xi^l}(\xi) \sim \sum_{k=1}^{\infty} \frac{d^l}{d\xi^l} \left( \frac{c_k}{\xi^k} \right),$$

as  $|\xi| \rightarrow +\infty, \text{Im}\xi \leq 0$  (respectively  $\text{Im}\xi \geq 0$ ) and where  $c_k \in \mathbb{C}$  are some constants.

Let  $H'$  be the space of all polynomials and  $H^- = H_0^- \oplus H'$ ;  $H = H^+ \oplus H^-$ . Denote by  $\pi^+$  (respectively  $\pi^-$ ) the projection on  $H^+$  (respectively  $H^-$ ). Let  $\tilde{H} = \{\text{rational functions having no poles on the real axis}\}$ . Then on  $\tilde{H}$ ,

$$\pi^+ h(\xi_0) = \frac{1}{2\pi i} \lim_{u \rightarrow 0^-} \int_{\Gamma^+} \frac{h(\xi)}{\xi_0 + iu - \xi} d\xi, \quad (2.3)$$

where  $\Gamma^+$  is a Jordan closed curve included  $\text{Im}(\xi) > 0$  surrounding all the singularities of  $h$  in the upper half-plane and  $\xi_0 \in \mathbf{R}$ . In our computations, we only compute  $\pi^+ h$  for  $h$  in  $\tilde{H}$ . Similarly, define  $\pi'$  on  $\tilde{H}$ ,

$$\pi' h = \frac{1}{2\pi} \int_{\Gamma^+} h(\xi) d\xi. \quad (2.4)$$

So  $\pi'(H^-) = 0$ . For  $h \in H \cap L^1(\mathbf{R})$ ,  $\pi' h = \frac{1}{2\pi} \int_{\mathbf{R}} h(v) dv$  and for  $h \in H^+ \cap L^1(\mathbf{R})$ ,  $\pi' h = 0$ .

An operator of order  $m \in \mathbf{Z}$  and type  $d$  is a matrix

$$\tilde{A} = \begin{pmatrix} \pi^+ P + G & K \\ T & \tilde{S} \end{pmatrix} : \begin{array}{c} C^\infty(M, E_1) \\ \bigoplus \\ C^\infty(\partial M, F_1) \end{array} \longrightarrow \begin{array}{c} C^\infty(M, E_2) \\ \bigoplus \\ C^\infty(\partial M, F_2) \end{array},$$

where  $M$  is a manifold with boundary  $\partial M$  and  $E_1, E_2$  (respectively  $F_1, F_2$ ) are vector bundles over  $M$  (respectively  $\partial M$ ). Here,  $P : C_0^\infty(\Omega, \overline{E_1}) \rightarrow C^\infty(\Omega, \overline{E_2})$  is a classical pseudodifferential operator of order  $m$  on  $\Omega$ , where  $\Omega$  is a collar neighborhood of  $M$  and  $\overline{E_i}|M = E_i$  ( $i = 1, 2$ ).  $P$  has an extension:  $\mathcal{E}'(\Omega, \overline{E_1}) \rightarrow \mathcal{D}'(\Omega, \overline{E_2})$ , where  $\mathcal{E}'(\Omega, \overline{E_1})$  ( $\mathcal{D}'(\Omega, \overline{E_2})$ ) is the dual space of  $C^\infty(\Omega, \overline{E_1})$  ( $C_0^\infty(\Omega, \overline{E_2})$ ). Let  $e^+ : C^\infty(M, E_1) \rightarrow \mathcal{E}'(\Omega, \overline{E_1})$  denote extension by zero from  $M$  to  $\Omega$  and  $r^+ : \mathcal{D}'(\Omega, \overline{E_2}) \rightarrow \mathcal{D}'(\Omega, E_2)$  denote the restriction from  $\Omega$  to  $X$ , then define

$$\pi^+ P = r^+ P e^+ : C^\infty(M, E_1) \rightarrow \mathcal{D}'(\Omega, E_2).$$

In addition,  $P$  is supposed to have the transmission property; this means that, for all  $j, k, \alpha$ , the homogeneous component  $p_j$  of order  $j$  in the asymptotic expansion of the symbol  $p$  of  $P$  in local coordinates near the boundary satisfies:

$$\partial_{x_n}^k \partial_\xi^\alpha p_j(x', 0, 0, +1) = (-1)^{j-|\alpha|} \partial_{x_n}^k \partial_\xi^\alpha p_j(x', 0, 0, -1),$$

then  $\pi^+ P : C^\infty(M, E_1) \rightarrow C^\infty(M, E_2)$ . Let  $G$ ,  $T$  be respectively the singular Green operator and the trace operator of order  $m$  and type  $d$ . Let  $K$  be a potential operator and  $S$  be a classical pseudodifferential operator of order  $m$  along the boundary. Denote by  $B^{m,d}$  the collection of all operators of order  $m$  and type  $d$ , and  $\mathcal{B}$  is the union over all  $m$  and  $d$ .

Recall that  $B^{m,d}$  is a Fréchet space. The composition of the above operator matrices yields a continuous map:  $B^{m,d} \times B^{m',d'} \rightarrow B^{m+m', \max\{m'+d, d'\}}$ . Write

$$\tilde{A} = \begin{pmatrix} \pi^+ P + G & K \\ T & \tilde{S} \end{pmatrix} \in B^{m,d}, \tilde{A}' = \begin{pmatrix} \pi^+ P' + G' & K' \\ T' & \tilde{S}' \end{pmatrix} \in B^{m',d'}.$$

The composition  $\tilde{A}\tilde{A}'$  is obtained by multiplication of the matrices (For more details see [13]). For example  $\pi^+ P \circ G'$  and  $G \circ G'$  are singular Green operators of type  $d'$  and

$$\pi^+ P \circ \pi^+ P' = \pi^+(PP') + L(P, P').$$

Here  $PP'$  is the usual composition of pseudodifferential operators and  $L(P, P')$  called leftover term is a singular Green operator of type  $m' + d$ . For our case,  $P$ ,  $P'$  are classical pseudo differential operators, in other words  $\pi^+ P \in \mathcal{B}^\infty$  and  $\pi^+ P' \in \mathcal{B}^\infty$ .

Let  $M$  be a  $n$ -dimensional compact oriented manifold with boundary  $\partial M$ . Denote by  $\mathcal{B}$  the Boutet de Monvel's algebra. We recall that the main theorem in [15, 17].

**Theorem 2.1.** [15] **(Fedosov-Golse-Leichtnam-Schrohe)** *Let  $M$  and  $\partial M$  be connected,  $\dim M = n \geq 3$ , and let  $\tilde{S}$  (respectively  $\tilde{S}'$ ) be the unit sphere about  $\xi$  (respectively  $\xi'$ ) and  $\sigma(\xi)$  (respectively  $\sigma(\xi')$ ) be the corresponding canonical  $n-1$  (respectively  $(n-2)$ ) volume form. Set  $\tilde{A} = \begin{pmatrix} \pi^+ P + G & K \\ T & \tilde{S} \end{pmatrix} \in \mathcal{B}$ , and*

denote by  $p$ ,  $b$  and  $s$  the local symbols of  $P, G$  and  $\tilde{S}$  respectively. Define:

$$\begin{aligned}\widetilde{\text{Wres}}(\tilde{A}) &= \int_X \int_{\tilde{\mathbf{S}}} \text{trace}_E [p_{-n}(x, \xi)] \sigma(\xi) dx \\ &\quad + 2\pi \int_{\partial X} \int_{\tilde{\mathbf{S}'}} \{\text{trace}_E [(\text{trace} b_{-n})(x', \xi')] + \text{trace}_F [s_{1-n}(x', \xi')]\} \sigma(\xi') dx',\end{aligned}\quad (2.5)$$

where  $\widetilde{\text{Wres}}$  denotes the noncommutative residue of an operator in the Boutet de Monvel's algebra. Then a)  $\widetilde{\text{Wres}}([\tilde{A}, B]) = 0$ , for any  $\tilde{A}, B \in \mathcal{B}$ ; b) It is the unique continuous trace on  $\mathcal{B}/\mathcal{B}^{-\infty}$ .

### 3. Spectral Einstein functionals for perturbations of Dirac operators

Firstly, we recall the definition of the Dirac operator. Let  $M$  be an  $n$  dimensional oriented compact spin Riemannian manifold with a Riemannian metric  $g^M$  and let  $\nabla^L$  be the Levi-Civita connection about  $g^M$ .

In the fixed orthonormal frame  $\{e_1, \dots, e_n\}$ , the connection matrix  $(\omega_{s,t})$  is defined by

$$\nabla^L(e_1, \dots, e_n) = (e_1, \dots, e_n)(\omega_{s,t}). \quad (3.1)$$

Let  $c(X)$  be a Clifford action on  $M$ , where  $X$  is a smooth vector field on  $M$ , which satisfies

$$c(e_i)c(e_j) + c(e_j)c(e_i) = -2g^M(e_i, e_j). \quad (3.2)$$

In [25], the Dirac operator is given

$$D = \sum_{i=1}^m c(e_i) \left[ e_i - \frac{1}{4} \sum_{s,t} \omega_{s,t}(e_i) c(e_s) c(e_t) \right]. \quad (3.3)$$

Set

$$X = \sum_{\alpha=1}^n a_{\alpha} e_{\alpha} = X^T + X_n \partial_{x_n} = \sum_{j=1}^n X_j \partial_j,$$

$$L(X) := \frac{1}{4} \sum_{ij} \langle \nabla_X^L e_i, e_j \rangle c(e_i) c(e_j)$$

and

$$G(X, \Psi) := -\frac{1}{2} [c(X)c(\Psi) + c(\Psi)c(X)].$$

Let  $\nabla_X^{S(TM)}$  denotes a spin connection, defined as and

$$\nabla_X^{S(TM)} := X + L(X).$$

And let  $\nabla_X^{\Psi} = \nabla_X^{S(TM)} + G(X, \Psi)$  and  $\nabla_Y^{\Psi} = \nabla_Y^{S(TM)} + G(Y, \Psi)$ ,  $c(\Psi) = c(X_1) \cdots c(X_k)$ ,  $X_i$  ( $1 \leq i \leq k$ ) are tangent vector files,  $g^{ij} = g(dx_i, dx_j)$ ,  $\xi = \sum_k \xi_j dx_j$  and  $\nabla_{\partial_i}^L \partial_j = \sum_k \Gamma_{ij}^k \partial_k$ , we denote that

$$\sigma_i = -\frac{1}{4} \sum_{s,t} \omega_{s,t}(e_i) c(e_s) c(e_t); \quad \xi^j = g^{ij} \xi_i; \quad \Gamma^k = g^{ij} \Gamma_{ij}^k; \quad \sigma^j = g^{ij} \sigma_i. \quad (3.4)$$

Then, perturbations of Dirac Operators is defined as

$$D_{\Psi} = D + c(\Psi) = \sum_{i=1}^n c(e_i) \left[ e_i - \frac{1}{4} \sum_{s,t} \omega_{s,t}(e_i) c(e_s) c(e_t) \right] + c(\Psi), \quad (3.5)$$

and

$$\sigma_1(D_\Psi) = ic(\xi); \quad (3.6)$$

$$\sigma_0(D_\Psi) = -\frac{1}{4} \sum_{i,s,t} \omega_{s,t}(e_i)c(e_s)c(e_t) + c(\Psi), \quad (3.7)$$

where  $\sigma_l(A)$  denotes the  $l$ -th order symbol of an operator  $A$ .

In addition, we obtain

$$\begin{aligned} \nabla_X^\Psi \nabla_Y^\Psi &= \left( X + L(X) + G(X, \Psi) \right) \left( Y + L(Y) + G(Y, \Psi) \right) \\ &= XY + X[L(Y)] + L(Y)X + L(X)Y + L(X)L(Y) + G(X, \Psi)Y + G(X, \Psi)L(Y) \\ &\quad + X[G(Y, \Psi)] + G(Y, \Psi)X + L(X)G(Y, \Psi) + G(X, \Psi)G(Y, \Psi), \end{aligned} \quad (3.8)$$

where  $X = \sum_{j=1}^n X_j \partial_{x_j}$ ,  $Y = \sum_{l=1}^n Y_l \partial_{x_l}$ .

Let  $p_1, p_2$  be nonnegative integers and  $p_1 + p_2 \leq n$ , an application of (3.5) and (3.6) in [16] shows that

**Definition 3.1.** Spectral Einstein functional of spin manifolds with boundary for perturbations of the Dirac operator are defined by

$$\widetilde{Wres}[\pi^+ \nabla_X^\Psi \nabla_Y^\Psi D_\Psi^{-p_1} \circ \pi^+ D_\Psi^{-p_2}], \quad (3.9)$$

where  $\pi^+ \nabla_X^\Psi \nabla_Y^\Psi D_\Psi^{-p_1}$ ,  $\pi^+ D_\Psi^{-p_2}$  are elements in Boutet de Monvel's algebra[17].

The following lemma derived from Dabrowski et al.'s Einstein functional plays a crucial role in our demonstration of the generalized noncommutative residue for the spectral action for perturbations of Dirac operators on compact manifolds with boundary.

**Lemma 3.2.** [2] The Einstein functional equal to

$$Wres(\tilde{\nabla}_X \tilde{\nabla}_Y \Delta_\Psi^{-\frac{n}{2}}) = \frac{v_{n-1}}{6} 2^{\frac{n}{2}} \int_M EG(X, Y) vol_g + \frac{v_{n-1}}{2} \int_M F(X, Y) vol_g + \frac{1}{2} \int_M (\text{trace } E) g(X, Y) vol_g,$$

where  $EG(X, Y)$  denotes the Einstein tensor evaluated on the two vector fields,  $F(X, Y) = Tr(X_a Y_b F_{ab})$  and  $F_{ab}$  is the curvature tensor of the connection  $T$ ,  $TrE$  denotes the trace of  $E$  and  $v_{n-1} = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$ .

Consider  $X$  and  $Y$  as a pair of vector fields on a compact Riemannian manifold  $M$  with dimension  $n$ . By employing the Laplace operator  $\Delta_\Psi = D_\Psi^2 = \Delta + E$ , which acts on sections of a vector bundle  $S(TM)$  with a rank of  $2^{\frac{n}{2}}$ . The purpose of this section is to demonstrate the following.

**Theorem 3.3.** For the Laplace (type) operator with torsion  $\Delta_\Psi$ , the Einstein functional equal to

$$\begin{aligned} Wres(\nabla_X^\Psi \nabla_Y^\Psi \Delta_\Psi^{-\frac{n}{2}}) &= \frac{v_{n-1}}{6} 2^{\frac{n}{2}} \int_M EG(X, Y) vol_g + \frac{v_{n-1}}{4} \int_M \text{trace} \left[ c(X) \nabla_Y^{S(TM)} (c(\Psi)) \right. \\ &\quad \left. + \nabla_Y^{S(TM)} (c(\Psi)) c(X) - c(Y) \nabla_X^{S(TM)} (c(\Psi)) - \nabla_X^{S(TM)} (c(\Psi)) c(Y) \right] dvol_M \\ &\quad + \frac{1}{2} \int_M \text{trace} \left[ \frac{1}{4} s + \sum_{j=1}^n \frac{1}{2} c(\Psi) c(e_j) c(\Psi) c(e_j) + (1 - \frac{n}{2}) (c(\Psi))^2 \right] g(X, Y) dvol_M, \end{aligned}$$

where  $s$  is the scalar curvature.

*Proof.* Let  $X = \sum_{a=1}^n X^a e_a$ ,  $Y = \sum_{b=1}^n Y^b e_b$ . Considering that

$$F(X, Y) = \text{trace}(X_a Y_b F_{ab}) = \sum_{a,b=1}^n X^a Y^b \text{trace}^{S(TM)}(F_{e_a, e_b}). \quad (3.10)$$

Set  $\overline{A}(X) = L(X) + G(X, \Psi)$ , then we acquire

$$\begin{aligned}
F_{e_a, e_b} &= (e_a + \overline{A}(e_a))(e_b + \overline{A}(e_b)) - (e_b + \overline{A}(e_b))(e_a + \overline{A}(e_a)) - ([e_a, e_a] + \overline{A}([e_a, e_b])) \\
&= e_a \circ \overline{A}(e_b) + \overline{A}(e_a) \circ e_b + \overline{A}(e_a)A(e_b) - e_b \circ \overline{A}(e_a) - \overline{A}(e_b) \circ e_a \\
&\quad - \overline{A}(e_b)\overline{A}(e_a) - \overline{A}([e_a, e_b]) \\
&= \overline{A}(e_b)e_a + e_a(\overline{A}(e_b)) + \overline{A}(e_a) \circ e_b + \overline{A}(e_a)\overline{A}(e_b) - \overline{A}(e_a) \circ e_b - e_b(\overline{A}(e_a)) \\
&\quad - \overline{A}(e_b)e_a - \overline{A}(e_b)\overline{A}(e_a) - \overline{A}([e_a, e_b]) \\
&= e_a(\overline{A}(e_b)) - e_b(\overline{A}(e_a)) + \overline{A}(e_a)\overline{A}(e_b) - \overline{A}(e_b)\overline{A}(e_a) - \overline{A}([e_a, e_b]). \tag{3.11}
\end{aligned}$$

We note that  $\text{trace}[\overline{A}(e_a)\overline{A}(e_b) - \overline{A}(e_b)\overline{A}(e_a)] = 0$  and  $e_a(c(e_s)) = 0$ . If  $s = t$ , we have  $\omega_{s,t}(e_b) = 0$ ; if  $s \neq t$ , we have  $\text{trace}[c(e_s)c(e_t)] = 0$ . Then we obtain  $\text{trace}[e_a(\omega_{s,t}(e_b))c(e_s)c(e_t)] = 0$ , so there is the following formula

$$\begin{aligned}
\text{trace}(e_a(\overline{A}(e_b))) &= \text{trace}\left[e_a\left(-\frac{1}{4}\sum_{s,t}\omega_{s,t}(e_b)c(e_s)c(e_t) - \frac{1}{2}[c(e_b)c(\Psi) + c(\Psi)c(e_b)]\right)\right] \\
&= -\frac{1}{2}\text{trace}\left[c(e_b)e_a\left(c(\Psi)\right) + e_a\left(c(\Psi)\right)c(e_b)\right] \tag{3.12}
\end{aligned}$$

and

$$\begin{aligned}
&\text{trace}(\overline{A}([e_a, e_b])) \\
&= \text{trace}\left(-\frac{1}{4}\sum_{s,t}\omega_{s,t}([e_a, e_b])c(e_s)c(e_t) - \frac{1}{2}[c([e_a, e_b])c(\Psi) + c(\Psi)c([e_a, e_b])]\right)(x_0) \\
&= -\frac{1}{2}\text{trace}\left(c([e_a, e_b])c(\Psi) + c(\Psi)c([e_a, e_b])\right). \tag{3.13}
\end{aligned}$$

Fix a point  $x_0$ ,  $\omega_{st}(e_a)(x_0) = 0$ , then  $e_a = \nabla_{e_a}^{S(TM)}(x_0)$  and let  $e_a = \sum_j H_{aj}\partial_j$ , then we have

$$\begin{aligned}
[e_a, e_b](x_0) &= [H_{aj}\partial_j, H_{bl}\partial_l](x_0) \\
&= H_{aj}\partial_j(H_{bl})\partial_l(x_0) - H_{bl}\partial_l(H_{aj})\partial_j(x_0) \\
&= 0 \tag{3.14}
\end{aligned}$$

we therefore draw the following formula

$$\begin{aligned}
&\sum_{a,b=1}^n X^a Y^b \left\{ -\frac{1}{2}\text{trace}\left[c(e_b)e_a\left(c(\Psi)\right) + e_a\left(c(\Psi)\right)c(e_b)\right] + \frac{1}{2}\text{trace}\left[c(e_a)e_b\left(c(\Psi)\right) \right. \right. \\
&\quad \left. \left. + e_b\left(c(\Psi)\right)c(e_a)\right] + \frac{1}{2}\text{trace}\left(c([e_a, e_b])c(\Psi) + c(\Psi)c([e_a, e_b])\right) \right\} \\
&= -\frac{1}{2}\text{trace}\left(c(Y)\nabla_X^{S(TM)}(c(\Psi)) + \nabla_X^{S(TM)}(c(\Psi))c(Y)\right) \\
&\quad + \frac{1}{2}\text{trace}\left(c(X)\nabla_Y^{S(TM)}(c(\Psi)) + \nabla_Y^{S(TM)}(c(\Psi))c(X)\right) \\
&= F(X, Y). \tag{3.15}
\end{aligned}$$

Let  $\Delta_\Psi = \Delta + E$ , as the formula (2.28) in [1], we have

$$\begin{aligned}
\text{trace}(E) &= \text{trace}\left[\frac{1}{4}s + \frac{1}{2}\Psi c(e_j)\Psi c(e_j) + \left(1 - \frac{n}{2}\right)(c(\Psi))^2\right] \\
&= \text{trace}\left[\frac{1}{4}s + \sum_{j=1}^n \frac{1}{2}c(\Psi)c(e_j)c(\Psi)c(e_j) + \left(1 - \frac{n}{2}\right)(c(\Psi))^2\right]. \tag{3.16}
\end{aligned}$$

Taking all the above conclusions into consideration, we ultimately complete the proof of the Theorem.  $\square$

#### 4. The noncommutative residue $\widetilde{\text{Wres}}\left[\pi^+ \nabla_X^\Psi \nabla_Y^\Psi D_\Psi^{-2} \circ \pi^+ D_\Psi^{-2}\right]$ on manifolds with boundary

In this section, we calculate the spectral Einstein functional for 4-dimension compact manifolds with boundary and derive a Dabrowski-Sitarz-Zalecki type formula in this case.

For perturbations of Dirac Operators  $\nabla_X^\Psi \nabla_Y^\Psi D_\Psi^{-2}$  and  $D_\Psi^{-2}$ , let  $\sigma_l(A)$  denote the  $l$ -order symbol of an operator A. Then, based on  $\sigma(\partial_{x_j}) = i\xi_j$  and (3.8), we can establish the following lemmas.

**Lemma 4.1.** *The following identities hold:*

$$\begin{aligned} \sigma_0(\nabla_X^\Psi \nabla_Y^\Psi) &= X[L(Y)] + L(X)L(Y) + X[G(Y, \Psi)] + G(X, \Psi)L(Y) + L(X)G(Y, \Psi) \\ &\quad + G(X, \Psi)G(Y, \Psi); \\ \sigma_1(\nabla_X^\Psi \nabla_Y^\Psi) &= i \sum_{j,l=1}^n X_j \frac{\partial Y_l}{\partial x_j} i\xi_l + i \sum_j A(Y)X_j \xi_j + i \sum_l A(Y)Y_l \xi_l + \sum_j G(X, \Psi)Y_j i\xi_j \\ &\quad + \sum_j G(X, \Psi)X_j i\xi_j; \\ \sigma_2(\nabla_X^\Psi \nabla_Y^\Psi) &= - \sum_{j,l=1}^n X_j Y_l \xi_j \xi_l. \end{aligned} \tag{4.1}$$

Next, we present the following lemmas.

**Lemma 4.2.** *The following identities hold:*

$$\sigma_{-1}(D_\Psi^{-1}) = \frac{ic(\xi)}{|\xi|^2}; \tag{4.2}$$

$$\sigma_{-2}(D_\Psi^{-1}) = \frac{c(\xi)\sigma_0(D_\Psi)c(\xi)}{|\xi|^4} + \frac{c(\xi)}{|\xi|^6} \sum_j c(dx_j) \left[ \partial_{x_j}(c(\xi))|\xi|^2 - c(\xi)\partial_{x_j}(|\xi|^2) \right]; \tag{4.3}$$

$$\sigma_{-2}(D_\Psi^{-2}) = \sigma_{-2}(D^{-2}) = |\xi|^{-2}; \tag{4.4}$$

$$\sigma_{-3}(D_\Psi^{-2}) = -i|\xi|^{-4}\xi_k(\Gamma^k - 2\delta^k) - i|\xi|^{-6}2\xi^j\xi_\alpha\xi_\beta\partial_j g^{\alpha\beta} - \left( c(\Psi)ic(\xi) + ic(\xi)c(\Psi) \right) |\xi|^{-4}. \tag{4.5}$$

According to Lemma 4.1 and Lemma 4.2, we obtain

**Lemma 4.3.** *The following identities hold:*

$$\sigma_0(\nabla_X^\Psi \nabla_Y^\Psi D_\Psi^{-2}) = - \sum_{j,l=1}^n X_j Y_l \xi_j \xi_l |\xi|^{-2}; \tag{4.6}$$

$$\begin{aligned} \sigma_{-1}(\nabla_X^\Psi \nabla_Y^\Psi D_\Psi^{-2}) &= \sigma_2(\nabla_X^\Psi \nabla_Y^\Psi) \sigma_{-3}(D_\Psi^{-2}) + \sigma_1(\nabla_X^\Psi \nabla_Y^\Psi) \sigma_{-2}(D_\Psi^{-2}) \\ &\quad + \sum_{j=1}^n \partial_{\xi_j} [\sigma_2(\nabla_X^\Psi \nabla_Y^\Psi)] D_{x_j} [\sigma_{-2}(D_\Psi^{-2})]. \end{aligned} \tag{4.7}$$

Since  $\Theta$  is a global form on  $\partial M$ , so for any fixed point  $x_0 \in \partial M$ , we can choose the normal coordinates  $U$  of  $x_0$  in  $\partial M$  (not in  $M$ ) and compute  $\Theta(x_0)$  in the coordinates  $\tilde{U} = U \times [0, 1)$  and the metric  $\frac{1}{h(x_n)}g^{\partial M} + dx_n^2$ . The dual metric of  $g^M$  on  $\tilde{U}$  is  $h(x_n)g^{\partial M} + dx_n^2$ . Write  $g_{ij}^M = g^M(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$ ;  $g_M^{ij} = g^M(dx_i, dx_j)$ , then

$$[g_{ij}^M] = \begin{bmatrix} \frac{1}{h(x_n)}[g_{ij}^{\partial M}] & 0 \\ 0 & 1 \end{bmatrix}; \quad [g_M^{ij}] = \begin{bmatrix} h(x_n)[g_{ij}^{\partial M}] & 0 \\ 0 & 1 \end{bmatrix},$$

and

$$\partial_{x_s} g_{ij}^{\partial M}(x_0) = 0, \quad 1 \leq i, j \leq n-1; \quad g_{i,j}^M(x_0) = \delta_{ij}.$$

Let  $\{e_1, \dots, e_{n-1}\}$  be an orthonormal frame field in  $U$  about  $g^{\partial M}$  which is parallel along geodesics and  $e_i = \frac{\partial}{\partial x_i}(x_0)$ , then  $\{\tilde{e}_1 = \sqrt{h(x_n)}e_1, \dots, \tilde{e}_{n-1} = \sqrt{h(x_n)}e_{n-1}, \tilde{e}_n = dx_n\}$  is the orthonormal frame field in  $\tilde{U}$  about  $g^M$ . Locally  $S(TM)|\tilde{U} \cong \tilde{U} \times \wedge_C^*(\frac{n}{2})$ . Let  $\{f_1, \dots, f_n\}$  be the orthonormal basis of  $\wedge_C^*(\frac{n}{2})$ . Take a spin frame field  $\sigma : \tilde{U} \rightarrow Spin(M)$  such that  $\pi\sigma = \{\tilde{e}_1, \dots, \tilde{e}_n\}$  where  $\pi : Spin(M) \rightarrow O(M)$  is a double covering, then  $\{[\sigma, f_i], 1 \leq i \leq n\}$  is an orthonormal frame of  $S(TM)|\tilde{U}$ . In the following, since the global form  $\Theta$  is independent of the choice of the local frame, so we can compute  $\text{trace}_{S(TM)}$  in the frame  $\{[\sigma, f_i], 1 \leq i \leq n\}$ . Let  $\{\hat{e}_1, \dots, \hat{e}_n\}$  be the canonical basis of  $\mathbb{R}^n$  and  $c(\hat{e}_i) \in cl_C(n) \cong Hom(\wedge_C^*(\frac{n}{2}), \wedge_C^*(\frac{n}{2}))$  be the Clifford action. Then

$$c(\tilde{e}_i) = [(\sigma, c(\hat{e}_i))]; \quad c(\tilde{e}_i)[(\sigma, f_i)] = [\sigma, (c(\hat{e}_i))f_i]; \quad \frac{\partial}{\partial x_i} = [(\sigma, \frac{\partial}{\partial x_i})],$$

then we have  $\frac{\partial}{\partial x_i}c(\tilde{e}_i) = 0$  in the above frame. In accordance with Lemma 2.2 in [17], we obtain

**Lemma 4.4.** *With the metric  $g^M$  on  $M$  near the boundary*

$$\partial_{x_j}(|\xi|_{g^M}^2)(x_0) = \begin{cases} 0, & \text{if } j < n; \\ h'(0)|\xi'|_{g^{\partial M}}^2, & \text{if } j = n. \end{cases} \quad (4.8)$$

$$\partial_{x_j}[c(\xi)](x_0) = \begin{cases} 0, & \text{if } j < n; \\ \partial_{x_n}(c(\xi'))(x_0), & \text{if } j = n, \end{cases} \quad (4.9)$$

where  $\xi = \xi' + \xi_n dx_n$ .

An application of (2.1.4) in [16] shows that

$$\begin{aligned} & \widetilde{\text{Wres}} \left[ \pi^+ \nabla_X^\Psi \nabla_Y^\Psi D_\Psi^{-p_1} \circ \pi^+ D_\Psi^{-p_2} \right] \\ &= \int_M \int_{|\xi|=1} \text{trace}_{S(TM)} [\sigma_{-n} (\nabla_X^\Psi \nabla_Y^\Psi D_\Psi^{-p_1} \circ D_\Psi^{-p_2}) \sigma(\xi)] dx + \int_{\partial M} \Phi, \end{aligned} \quad (4.10)$$

where

$$\begin{aligned} \Phi = & \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{j,k=0}^{\infty} \sum_{\alpha} \frac{(-i)^{|\alpha|+j+k+\ell}}{\alpha!(j+k+1)!} \text{trace}_{S(TM)} [\partial_{x_n}^j \partial_{\xi'}^\alpha \partial_{\xi_n}^k \sigma_r^+ ((\nabla_X^\Psi \nabla_Y^\Psi D_\Psi^{-p_1})(x', 0, \xi', \xi_n) \\ & \times \partial_{x_n}^\alpha \partial_{\xi_n}^{j+1} \partial_{x_n}^k \sigma_l (D_\Psi^{-p_2})(x', 0, \xi', \xi_n)] d\xi_n \sigma(\xi') dx', \end{aligned} \quad (4.11)$$

and the sum is taken over  $r - k + |\alpha| + \ell - j - 1 = -n, r \leq -p_1, \ell \leq -p_2$ .

In the following, we will compute the residue  $\widetilde{\text{Wres}} \left[ \pi^+ \nabla_X^\Psi \nabla_Y^\Psi D_\Psi^{-2} \circ \pi^+ D_\Psi^{-2} \right]$  on 4-dimensional oriented compact spin manifolds with boundary and get a Dabrowski-Sitarz-Zalecki type theorem in this case. By (4.10) and (4.11), we have

$$\begin{aligned} & \widetilde{\text{Wres}} \left[ \pi^+ \nabla_X^\Psi \nabla_Y^\Psi D_\Psi^{-2} \circ \pi^+ D_\Psi^{-2} \right] \\ &= \int_M \int_{|\xi|=1} \text{trace}_{S(TM)} [\sigma_{-4} (\nabla_X^\Psi \nabla_Y^\Psi D_\Psi^{-2} \circ D_\Psi^{-2}) \sigma(\xi)] dx + \int_{\partial M} \widetilde{\Phi}, \end{aligned} \quad (4.12)$$

where

$$\begin{aligned} \widetilde{\Phi} = & \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{j,k=0}^{\infty} \sum_{\alpha} \frac{(-i)^{|\alpha|+j+k+\ell}}{\alpha!(j+k+1)!} \text{trace}_{S(TM)} [\partial_{x_n}^j \partial_{\xi'}^\alpha \partial_{\xi_n}^k \sigma_r^+ (\nabla_X^\Psi \nabla_Y^\Psi D_\Psi^{-2})(x', 0, \xi', \xi_n) \\ & \times \partial_{x_n}^\alpha \partial_{\xi_n}^{j+1} \partial_{x_n}^k \sigma_l (D_\Psi^{-2})(x', 0, \xi', \xi_n)] d\xi_n \sigma(\xi') dx', \end{aligned} \quad (4.13)$$

and the sum is taken over  $r - k + |\alpha| + \ell - j - 1 = -4, r \leq 0, \ell \leq -2$ .

Locally we can use Theorem 3.3 to compute the interior term of (4.12), then

$$\begin{aligned} & \int_M \int_{|\xi|=1} \text{trace}_{S(TM)} [\sigma_{-4}(\nabla_X^\Psi \nabla_Y^\Psi D_\Psi^{-2} \circ D_\Psi^{-2}) \sigma(\xi)] dx \\ &= \frac{4\pi^2}{3} \int_M EG(X, Y) vol_g + \frac{\pi^2}{2} \int_M \text{trace} [c(X) \nabla_Y^{S(TM)}(c(\Psi)) \\ & \quad + \nabla_Y^{S(TM)}(c(\Psi)) c(X) - c(Y) \nabla_X^{S(TM)}(c(\Psi)) - \nabla_X^{S(TM)}(c(\Psi)) c(Y)] dvol_M \\ & \quad + \frac{1}{2} \int_M \text{trace} [\frac{1}{4}s + \sum_{j=1}^n \frac{1}{2} c(\Psi) c(e_j) c(\Psi) c(e_j) - (c(\Psi))^2] g(X, Y) dvol_M, \end{aligned} \quad (4.14)$$

so we only need to compute  $\int_{\partial M} \tilde{\Phi}$ .

When  $n = 4$ , then  $\text{trace}_{S(TM)}[\text{id}] = 4$ , the sum is taken over  $r - k + |\alpha| + \ell - j = -3, r \leq 0, \ell \leq -2$ , then we have the  $\int_{\partial M} \tilde{\Phi}$  is the sum of the following five cases:

**case (a) (I)**  $r = 0, l = -2, k = j = 0, |\alpha| = 1$

By (4.13), we get

$$\tilde{\Phi}_1 = - \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{|\alpha|=1} \text{trace} [\partial_{\xi'}^\alpha \pi_{\xi_n}^+ \sigma_0(\nabla_X^\Psi \nabla_Y^\Psi D_\Psi^{-2}) \times \partial_{x'}^\alpha \partial_{\xi_n} \sigma_{-2}(D_\Psi^{-2})](x_0) d\xi_n \sigma(\xi') dx'. \quad (4.15)$$

By Lemma 2.2 in [17], for  $i < n$ , then

$$\partial_{x_i} \sigma_{-2}(D_\Psi^{-2})(x_0) = \partial_{x_i}(|\xi|^{-2})(x_0) = -\frac{\partial_{x_i}(|\xi|^2)(x_0)}{|\xi|^4} = 0, \quad (4.16)$$

so  $\tilde{\Phi}_1 = 0$  and **case (a) (I)** vanishes.

**case (a) (II)**  $r = 0, l = -2, k = |\alpha| = 0, j = 1$ .

By (4.13), we get

$$\tilde{\Phi}_2 = -\frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace} [\partial_{x_n} \pi_{\xi_n}^+ \sigma_0(\nabla_X^\Psi \nabla_Y^\Psi D_\Psi^{-2}) \times \partial_{\xi_n}^2 \sigma_{-2}(D_\Psi^{-2})](x_0) d\xi_n \sigma(\xi') dx'. \quad (4.17)$$

By Lemma 4.2, we have

$$\partial_{\xi_n}^2 \sigma_{-2}(D_\Psi^{-2})(x_0) = \partial_{\xi_n}^2(|\xi|^{-2})(x_0) = \frac{6\xi_n^2 - 2}{(1 + \xi_n^2)^3}. \quad (4.18)$$

It follows that

$$\partial_{x_n} \sigma_0(\nabla_X^\Psi \nabla_Y^\Psi D_\Psi^{-2})(x_0) = \partial_{x_n} \left( - \sum_{j,l=1}^n X_j Y_l \xi_j \xi_l |\xi|^{-2} \right) = \frac{\sum_{j,l=1}^n X_j Y_l \xi_j \xi_l h'(0) |\xi'|^2}{(1 + \xi_n^2)^2}. \quad (4.19)$$

By integrating formula, we obtain

$$\begin{aligned}
& \pi_{\xi_n}^+ \partial_{x_n} \sigma_0(\nabla_X^\Psi \nabla_Y^\Psi D_\Psi^{-2})(x_0) \\
&= \partial_{x_n} \pi_{\xi_n}^+ \sigma_0(\nabla_X^\Psi \nabla_Y^\Psi D_\Psi^{-2}) \\
&= -\frac{i\xi_n}{4(\xi_n - i)^2} \sum_{j,l=1}^{n-1} X_j Y_l \xi_j \xi_l h'(0) + \frac{2 - i\xi_n}{4(\xi_n - i)^2} X_n Y_n h'(0) - \frac{i}{4(\xi_n - i)^2} \sum_{j=1}^{n-1} X_j Y_n \xi_j \\
&\quad - \frac{i}{4(\xi_n - i)^2} \sum_{l=1}^{n-1} X_n Y_l \xi_l.
\end{aligned} \tag{4.20}$$

We note that  $i < n$ ,  $\int_{|\xi'|=1} \xi_{i_1} \xi_{i_2} \cdots \xi_{i_{2d+1}} \sigma(\xi') = 0$ , so we omit some items that have no contribution for computing **case (a) (III)**. From (4.18) and (4.20), we obtain

$$\begin{aligned}
& \text{trace}[\partial_{x_n} \pi_{\xi_n}^+ \sigma_0(\nabla_X^\Psi \nabla_Y^\Psi D_\Psi^{-2}) \times \partial_{\xi_n}^2 \sigma_{-2}(D_\Psi^{-2})](x_0) \\
&= \frac{2 + 2\xi_n i - 6\xi_n^3 i - 2i}{(\xi_n - i)^5 (\xi_n + i)^3} \sum_{j,l=1}^{n-1} X_j Y_l \xi_j \xi_l h'(0) + \frac{2 + 2\xi_n i - 6\xi_n^3 i - 2i}{(\xi_n - i)^5 (\xi_n + i)^3} X_n Y_n h'(0) \\
&\quad + \frac{2(1 - 3\xi_n^2)i}{(\xi_n - i)^5 (\xi_n + i)^3} \sum_{j=1}^{n-1} X_j Y_n \xi_j + \frac{2(1 - 3\xi_n^2)i}{(\xi_n - i)^5 (\xi_n + i)^3} \sum_{l=1}^{n-1} X_n Y_l \xi_l.
\end{aligned} \tag{4.21}$$

Therefore, we get

$$\begin{aligned}
\tilde{\Phi}_2 &= -\frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \left\{ \frac{2 + 2\xi_n i - 6\xi_n^3 i - 2i}{(\xi_n - i)^5 (\xi_n + i)^3} \sum_{j,l=1}^{n-1} X_j Y_l \xi_j \xi_l h'(0) + \frac{2 + 2\xi_n i - 6\xi_n^3 i - 2i}{(\xi_n - i)^5 (\xi_n + i)^3} X_n Y_n h'(0) \right\} \\
&\quad \times d\xi_n \sigma(\xi') dx' \\
&= -\sum_{j,l=1}^{n-1} X_j Y_l h'(0) \Omega_3 \int_{\Gamma^+} \frac{1 + \xi_n i - 3\xi_n^3 i - i}{(\xi_n - i)^5 (\xi_n + i)^3} \xi_j \xi_l d\xi_n dx' - X_n Y_n h'(0) \Omega_3 \int_{\Gamma^+} \frac{1 + \xi_n i - 3\xi_n^3 i - i}{(\xi_n - i)^5 (\xi_n + i)^3} \\
&\quad \times d\xi_n dx' \\
&= -\frac{2\pi i}{4!} h'(0) \left( \sum_{j,l=1}^{n-1} X_j Y_l \left[ \frac{1 + \xi_n i - 3\xi_n^3 i - i}{(\xi_n + i)^3} \right]^{(4)} \Big|_{\xi_n=i} + X_n Y_n \left[ \frac{1 + \xi_n i - 3\xi_n^3 i - i}{(\xi_n + i)^3} \right]^{(4)} \Big|_{\xi_n=i} \right) \Omega_3 dx' \\
&= \frac{13}{8} h'(0) \pi \Omega_3 dx' \left( \frac{\pi}{3} \sum_{j=1}^{n-1} X_j Y_j + \frac{1}{4} X_n Y_n \right),
\end{aligned} \tag{4.22}$$

where  $\Omega_3$  is the canonical volume of  $S^2$ .

**case (a) (III)**  $r = 0$ ,  $l = -2$ ,  $j = |\alpha| = 0$ ,  $k = 1$ .

By (4.13), we get

$$\begin{aligned}
\tilde{\Phi}_3 &= -\frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\partial_{\xi_n} \pi_{\xi_n}^+ \sigma_0(\nabla_X^\Psi \nabla_Y^\Psi D_\Psi^{-2}) \times \partial_{\xi_n} \partial_{x_n} \sigma_{-2}(D_\Psi^{-2})](x_0) d\xi_n \sigma(\xi') dx' \\
&= \frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\partial_{\xi_n}^2 \pi_{\xi_n}^+ \sigma_0(\nabla_X^\Psi \nabla_Y^\Psi D_\Psi^{-2}) \times \partial_{x_n} \sigma_{-2}(D_\Psi^{-2})](x_0) d\xi_n \sigma(\xi') dx'.
\end{aligned} \tag{4.23}$$

Based on Lemma 4.2, we can express the equation as follows:

$$\partial_{x_n} \sigma_{-2}(D_\Psi^{-2})(x_0)|_{|\xi'|=1} = -\frac{h'(0)}{(1 + \xi_n^2)^2}. \tag{4.24}$$

A straightforward computation yields

$$\begin{aligned} \pi_{\xi_n}^+ \sigma_0(\nabla_X^\Psi \nabla_Y^\Psi D_\Psi^{-2})(x_0)|_{|\xi'|=1} &= \frac{i}{2(\xi_n - i)} \sum_{j,l=1}^{n-1} X_j Y_l \xi_j \xi_l - \frac{1}{2(\xi_n - i)} X_n Y_n \\ &\quad - \frac{1}{2(\xi_n - i)} \sum_{j=1}^{n-1} X_j Y_n \xi_j - \frac{1}{2(\xi_n - i)} \sum_{l=1}^{n-1} X_n Y_l \xi_l. \end{aligned} \quad (4.25)$$

Additionally, simple calculations provide

$$\partial_{\xi_n}^2 \pi_{\xi_n}^+ \sigma_0(\nabla_X^\Psi \nabla_Y^\Psi D_\Psi^{-2})(x_0)|_{|\xi'|=1} = \frac{i}{(\xi_n - i)^3} \sum_{j,l=1}^{n-1} X_j Y_l \xi_j \xi_l - \frac{1}{(\xi_n - i)^3} X_n Y_n. \quad (4.26)$$

From (4.24) and (4.26), we obtain

$$\begin{aligned} &\text{trace}[\partial_{\xi_n} \pi_{\xi_n}^+ \sigma_0(\nabla_X^\Psi \nabla_Y^\Psi D_\Psi^{-2}) \times \partial_{\xi_n} \partial_{x_n} \sigma_{-2}(D_\Psi^{-2})](x_0) \\ &= -4 \frac{h'(0)i}{(\xi_n - i)^5 (\xi_n + i)^2} \sum_{j,l=1}^{n-1} X_j Y_l \xi_j \xi_l + 4 \frac{h'(0)}{(\xi_n - i)^5 (\xi_n + i)^2} X_n Y_n. \end{aligned} \quad (4.27)$$

Hence, we obtain

$$\begin{aligned} \tilde{\Phi}_3 &= \frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \left( -4 \frac{h'(0)i}{(\xi_n - i)^5 (\xi_n + i)^2} \sum_{j,l=1}^{n-1} X_j Y_l \xi_j \xi_l + 4 \frac{h'(0)}{(\xi_n - i)^5 (\xi_n + i)^2} X_n Y_n \right) d\xi_n \sigma(\xi') dx' \\ &= -2 \sum_{j,l=1}^{n-1} X_j Y_l h'(0) \Omega_3 \int_{\Gamma^+} \frac{i}{(\xi_n - i)^5 (\xi_n + i)^2} \xi_j \xi_l d\xi_n dx' + 2 X_n Y_n h'(0) \Omega_3 \int_{\Gamma^+} \frac{1}{(\xi_n - i)^5 (\xi_n + i)^2} d\xi_n dx' \\ &= -2 \sum_{j,l=1}^{n-1} X_j Y_l h'(0) \Omega_3 \frac{2\pi i}{4!} \left[ \frac{i}{(\xi_n + i)^2} \right]^{(4)} \Big|_{\xi_n=i} dx' + 2 X_n Y_n h'(0) \Omega_3 \frac{2\pi i}{4!} \left[ \frac{1}{(\xi_n + i)^2} \right]^{(4)} \Big|_{\xi_n=i} dx' \\ &= \frac{5}{4} h'(0) \pi \Omega_3 dx' \left( \frac{\pi}{3} \sum_{j=1}^{n-1} X_j Y_j + \frac{i}{4} X_n Y_n \right). \end{aligned} \quad (4.28)$$

**case (b)**  $r = 0, l = -3, k = j = |\alpha| = 0$ .

According to equation (4.13), we get

$$\begin{aligned} \tilde{\Phi}_4 &= -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\pi_{\xi_n}^+ \sigma_0(\nabla_X^\Psi \nabla_Y^\Psi D_\Psi^{-2}) \times \partial_{\xi_n} \sigma_{-3}(D_\Psi^{-2})](x_0) d\xi_n \sigma(\xi') dx' \\ &= i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\partial_{\xi_n} \pi_{\xi_n}^+ \sigma_0(\nabla_X^\Psi \nabla_Y^\Psi D_\Psi^{-2}) \times \sigma_{-3}(D_\Psi^{-2})](x_0) d\xi_n \sigma(\xi') dx'. \end{aligned} \quad (4.29)$$

According to Lemma 4.2, we have

$$\begin{aligned} \sigma_{-3}(D_\Psi^{-2})(x_0)|_{|\xi'|=1} &= -\frac{i}{(1 + \xi_n^2)^2} \left( -\frac{1}{2} h'(0) \sum_{k < n} \xi_n c(e_k) c(e_n) + \frac{5}{2} h'(0) \xi_n \right) - \frac{2ih'(0)\xi_n}{(1 + \xi_n^2)^3} \\ &\quad - \left[ c(\Psi) ic(\xi) + ic(\xi) c(\Psi) \right] |\xi|^{-4}. \end{aligned} \quad (4.30)$$

In the (4.30), we set

$$B_0 = -\frac{i}{(1+\xi_n^2)^2} \left( -\frac{1}{2} h'(0) \sum_{k < n} \xi_n c(e_k) c(e_n) + \frac{5}{2} h'(0) \xi_n \right) - \frac{2ih'(0)\xi_n}{(1+\xi_n^2)^3},$$

and

$$B_1 = - \left[ c(\Psi) ic(\xi) + ic(\xi) c(\Psi) \right] |\xi|^{-4},$$

thus we obtain

$$\sigma_{-3}(D_\Psi^{-2})(x_0)|_{|\xi'|=1} := B_0 + B_1.$$

By (4.29), we have

$$\text{trace}[\partial_{\xi_n} \pi_{\xi_n}^+ \sigma_0(\nabla_X^\Psi \nabla_Y^\Psi D_\Psi^{-2}) \times \sigma_{-3}(D_\Psi^{-2})] = \text{trace} \left[ \partial_{\xi_n} \pi_{\xi_n}^+ \sigma_0(\nabla_X^\Psi \nabla_Y^\Psi D_\Psi^{-2}) \times (B_0 + B_1) \right] (x_0). \quad (4.31)$$

By Lemma 4.3, we have

$$\begin{aligned} \partial_{\xi_n} \pi_{\xi_n}^+ \sigma_0(\nabla_X^\Psi \nabla_Y^\Psi D_\Psi^{-2})(x_0)|_{|\xi'|=1} &= -\frac{i}{2(\xi_n - i)^2} \sum_{j,l=1}^{n-1} X_j Y_l \xi_j \xi_l - \frac{1}{2(\xi_n - i)^2} X_n Y_n \\ &\quad + \frac{1}{2(\xi_n - i)^2} \sum_{j=1}^{n-1} X_j Y_n \xi_j + \frac{1}{2(\xi_n - i)^2} \sum_{l=1}^{n-1} X_n Y_l \xi_l. \end{aligned} \quad (4.32)$$

We observe that  $i < n$ ,  $\int_{|\xi'|=1} \xi_{i_1} \xi_{i_2} \cdots \xi_{i_{2d+1}} \sigma(\xi') = 0$  and

$$\text{trace}[c(\Psi)c(\xi)] = \text{trace}[c(dx_n)c(\Psi)] \xi_n + \sum_{j=1}^{n-1} \text{trace}[c(dx_j)c(\Psi)] \xi'_j,$$

so we omit some items that have no contribution for computing **case (b)**. Then, we have

$$\begin{aligned} &\text{trace} \left[ \partial_{\xi_n} \pi_{\xi_n}^+ \sigma_0(\nabla_X^\Psi \nabla_Y^\Psi D_\Psi^{-2}) \times B_0 \right] (x_0) \\ &= -\frac{h'(0)(5\xi_n^2 - 5 + 4\xi_n)}{(\xi_n - i)^5 (\xi_n + i)^3} \sum_{j,l=1}^{n-1} X_j Y_l \xi_j \xi_l + \frac{h'(0)i(5\xi_n^3 - \xi_n)}{(\xi_n - i)^5 (\xi_n + i)^3} X_n Y_n. \end{aligned} \quad (4.33)$$

Therefore, we get

$$\begin{aligned} &i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace} \left[ \partial_{\xi_n} \pi_{\xi_n}^+ \sigma_0(\nabla_X^\Psi \nabla_Y^\Psi D_\Psi^{-2}) \times B_0 \right] (x_0) d\xi_n \sigma(\xi') dx' \\ &= i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \left( -\frac{h'(0)(5\xi_n^2 - 5 + 4\xi_n)}{(\xi_n - i)^5 (\xi_n + i)^3} \sum_{j,l=1}^{n-1} X_j Y_l \xi_j \xi_l + \frac{h'(0)i(5\xi_n^3 - \xi_n)}{(\xi_n - i)^5 (\xi_n + i)^3} X_n Y_n \right) d\xi_n \sigma(\xi') dx' \\ &= -i \sum_{j,l=1}^{n-1} X_j Y_l h'(0) \Omega_3 \int_{\Gamma^+} \frac{5\xi_n^2 - 5 + 4\xi_n}{(\xi_n - i)^5 (\xi_n + i)^2} \xi_j \xi_l d\xi_n dx' + i X_n Y_n h'(0) \Omega_3 \int_{\Gamma^+} \frac{5\xi_n^3 - \xi_n}{(\xi_n - i)^5 (\xi_n + i)^2} d\xi_n dx' \\ &= -i \sum_{j,l=1}^{n-1} X_j Y_l h'(0) \Omega_3 \frac{2\pi i}{4!} \left[ \frac{5\xi_n^2 - 5 + 4\xi_n}{(\xi_n + i)^2} \right]^{(4)} \Big|_{\xi_n=i} dx' + i X_n Y_n h'(0) \Omega_3 \frac{2\pi i}{4!} \left[ \frac{5\xi_n^3 - \xi_n}{(\xi_n + i)^2} \right]^{(4)} \Big|_{\xi_n=i} dx' \\ &= \frac{1}{4} h'(0) \pi \Omega_3 dx' \left( \frac{(1-5i)\pi}{3} \sum_{j=1}^{n-1} X_j Y_j + \frac{11i}{4} X_n Y_n \right). \end{aligned} \quad (4.34)$$

Similar to (4.33) and (4.34), we have

$$\begin{aligned}
& i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace} \left[ \partial_{\xi_n} \pi_{\xi_n}^+ \sigma_0 (\nabla_X^\Psi \nabla_Y^\Psi D_\Psi^{-2}) \times B_1 \right] (x_0) d\xi_n \sigma(\xi') dx' \\
& = \left( -i \sum_{j,l=1}^{n-1} X_j Y_l - X_n Y_n \Omega_3 dx' \right) \frac{2\pi i}{3!} \left[ \frac{\xi_n}{2(\xi_n + i)^2} \right]^{(3)} \Big|_{\xi_n=i} \cdot 2\text{trace}[c(\Psi)c(dx_n)] \\
& = \left( \frac{\pi^2}{12} \sum_{j=1}^{n-1} X_j Y_j - \frac{\pi i}{8} X_n Y_n \right) \cdot \text{trace}[c(\Psi)c(dx_n)] \Omega_3 dx'.
\end{aligned} \tag{4.35}$$

From all of this, we get

$$\begin{aligned}
\tilde{\Phi}_4 = & \left\{ \left( \frac{\pi^2 - 5i\pi^2}{12} \sum_{j=1}^{n-1} X_j Y_j + \frac{11\pi i}{16} X_n Y_n \right) h'(0) + \left( \frac{\pi^2}{12} \sum_{j=1}^{n-1} X_j Y_j - \frac{\pi i}{8} X_n Y_n \right) \right. \\
& \left. \times \text{trace}[c(\Psi)c(dx_n)] \right\} \Omega_3 dx'.
\end{aligned} \tag{4.36}$$

**case (c)**  $r = -1$ ,  $\ell = -2$ ,  $k = j = |\alpha| = 0$ .

By (4.13), we get

$$\tilde{\Phi}_5 = -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\pi_{\xi_n}^+ \sigma_{-1} (\nabla_X^\Psi \nabla_Y^\Psi D_\Psi^{-2}) \times \partial_{\xi_n} \sigma_{-2} (D_\Psi^{-2})] (x_0) d\xi_n \sigma(\xi') dx'. \tag{4.37}$$

By Lemma 4.2, we have

$$\partial_{\xi_n} \sigma_{-2} (D_\Psi^{-2})|_{|\xi'|=1} = -\frac{2\xi_n}{(\xi_n^2 + 1)^2}. \tag{4.38}$$

By Lemma 4.3, we get

$$\begin{aligned}
\sigma_{-1} (\nabla_X^\Psi \nabla_Y^\Psi D_\Psi^{-2}) = & \sigma_2 (\nabla_X^\Psi \nabla_Y^\Psi) \sigma_{-3} (D_\Psi^{-2}) + \sigma_1 (\nabla_X^\Psi \nabla_Y^\Psi) \sigma_{-2} (D_\Psi^{-2}) \\
& + \sum_{j=1}^n \partial_{\xi_j} [\sigma_2 (\nabla_X^\Psi \nabla_Y^\Psi)] D_{x_j} [\sigma_{-2} (D_\Psi^{-2})].
\end{aligned} \tag{4.39}$$

First, we present the explicit expression for the first term in equation (4.39)

$$\begin{aligned}
& \sigma_2 (\nabla_X^\Psi \nabla_Y^\Psi) \sigma_{-3} (D_\Psi^{-2}) (x_0)|_{|\xi'|=1} \\
& = - \sum_{j,l=1}^n X_j Y_l \xi_j \xi_l \left[ -i|\xi|^{-4} \xi_k (\Gamma^k - 2\delta^k) - i|\xi|^{-6} 2\xi^j \xi_\alpha \xi_\beta \partial_j g^{\alpha\beta} - \left( c(\Psi)ic(\xi) + ic(\xi)c(\Psi) \right) |\xi|^{-4} \right] \\
& = - \sum_{j,l=1}^n X_j Y_l \xi_j \xi_l \left[ \frac{h'(0)i \sum_{k<n} \xi_n c(e_k) c(e_n) - 5ih'(0)\xi_n}{2|\xi|^4} - \frac{2ih'(0)\xi_n}{(1+\xi_n^2)^3} - \frac{c(\Psi)ic(\xi) + ic(\xi)c(\Psi)}{|\xi|^4} \right]. \tag{4.40}
\end{aligned}$$

Next, we present the explicit expression for the second term in equation (4.39),

$$\begin{aligned}
& \sigma_1 (\nabla_X^\Psi \nabla_Y^\Psi) \sigma_{-2} (D_\Psi^{-2}) (x_0)|_{|\xi'|=1} \\
& = \frac{i \sum_{j,l=1}^n X_j \frac{\partial Y_l}{\partial x_j} i\xi_l + i \sum_j A(Y) X_j \xi_j + i \sum_l A(Y) Y_l \xi_l + \sum_j G(X, \Psi) Y_j i\xi_j + \sum_j G(Y, \Psi) X_j i\xi_j}{|\xi|^2}.
\end{aligned} \tag{4.41}$$

Finally, we present the explicit expression for the third item of (4.39) in equation (4.39),

$$\begin{aligned} & \sum_{j=1}^n \sum_{\alpha} \frac{1}{\alpha!} \partial_{\xi}^{\alpha} [\sigma_2(\nabla_X^{\Psi} \nabla_Y^{\Psi})] D_x^{\alpha} [\sigma_{-2}(D_{\Psi}^{-2})](x_0)|_{|\xi'|=1} = \sum_{j=1}^n \partial_{\xi_j} \left[ - \sum_{j,l=1}^n X_j Y_l \xi_j \xi_l \right] (-i) \partial_{x_j} [|\xi|^{-2}] \\ & = \sum_{j=1}^n \sum_{l=1}^n i(x_j Y_l + x_l Y_j) \xi_l \partial_{x_j} (|\xi|^{-2}). \end{aligned} \quad (4.42)$$

We observe that  $i < n$ ,  $\int_{|\xi'|=1} \xi_{i_1} \xi_{i_2} \cdots \xi_{i_{2d+1}} \sigma(\xi') = 0$ , so we omit some items that have no contribution for computing **case (c)**. In addition, we have trace  $L(Y) = 0$  and trace  $[G(X, \Psi)] = -\text{trace}[c(X)c(\Psi)]$ . A straightforward computation yields

$$\begin{aligned} & \text{trace}[\pi_{\xi_n}^+ (\sigma_2(\nabla_X^{\Psi} \nabla_Y^{\Psi}) \sigma_{-3}(D_{\Psi}^{-2})) \times \partial_{\xi_n} \sigma_{-2}(D_{\Psi}^{-2})](x_0)|_{|\xi'|=1} \\ & = \frac{h'(0)(2\xi_n^2 - \xi_n - 2\xi_n i)}{(\xi_n - i)^4(\xi_n + i)^2} \sum_{j,l=1}^{n-1} X_j Y_l \xi_j \xi_l \sum_{k < n} \xi_k c(e_k) c(e_n) + \frac{h'(0)(17\xi_n i - \xi_n^2 + 4\xi_n^3 i)}{(\xi_n - i)^5(\xi_n + i)^2} \sum_{j,l=1}^{n-1} X_j Y_l \xi_j \xi_l \\ & \quad + \frac{-2i\xi_n^2}{(\xi_n - i)^4(\xi_n + i)^4} \sum_{j,l=1}^n X_j Y_l \xi_j \xi_l \cdot 2\text{trace}[c(\Psi)c(dx_n)] \end{aligned} \quad (4.43)$$

Additionally, simple calculations provide

$$\begin{aligned} & \text{trace}[\pi_{\xi_n}^+ (\sigma_1(\nabla_X^{\Psi} \nabla_Y^{\Psi}) \sigma_{-2}(D_{\Psi}^{-2})) \times \partial_{\xi_n} \sigma_{-2}(D_{\Psi}^{-2})](x_0)|_{|\xi'|=1} \\ & = \frac{-\xi_n}{(\xi_n + i)^2(\xi_n - i)^3} \cdot \text{trace}\left(iX_n \frac{\partial Y_n}{\partial x_n} + L(Y)X_n + L(X)Y_n + G(X, \Psi)Y_n + G(Y, \Psi)X_n\right), \end{aligned} \quad (4.44)$$

and

$$\begin{aligned} & -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\pi_{\xi_n}^+ (\sigma_1(\nabla_X^{\Psi} \nabla_Y^{\Psi}) \sigma_{-2}(D_{\Psi}^{-2})) \times \partial_{\xi_n} \sigma_{-2}(D_{\Psi}^{-2})](x_0) d\xi_n \sigma(\xi') dx' \\ & = \frac{i\pi}{8} \Omega_3 dx' \cdot \text{trace}\left(iX_n \frac{\partial Y_n}{\partial x_n} + L(Y)X_n + L(X)Y_n + G(X, \Psi)Y_n + G(Y, \Psi)X_n\right) \\ & = \frac{i\pi}{8} \Omega_3 dx' \cdot \left( \text{trace}(iX_n \frac{\partial Y_n}{\partial x_n}) - \text{trace}[c(X)c(\Psi)]Y_n - \text{trace}[c(Y)c(\Psi)]X_n \right). \end{aligned} \quad (4.45)$$

In addition, we have

$$\begin{aligned} & \text{trace}[\pi_{\xi_n}^+ (\sum_{j=1}^n \sum_{\alpha} \frac{1}{\alpha!} \partial_{\xi}^{\alpha} [\sigma_2(\nabla_X^{\Psi} \nabla_Y^{\Psi})] D_x^{\alpha} [\sigma_{-2}(D_{\Psi}^{-2})]) \times \partial_{\xi_n} \sigma_{-2}(D_{\Psi}^{-2})](x_0)|_{|\xi'|=1} \\ & = 2iX_n Y_n h'(0)\xi_n \cdot \frac{-2\xi_n}{(\xi_n^2 + 1)^2}. \end{aligned} \quad (4.46)$$

Substituting (4.43), (4.44) and (4.46) into (4.37) yields

$$\begin{aligned} \widetilde{\Phi}_5 = & \left( \frac{5i - 13}{6} \sum_{j=1}^{n-1} X_j Y_j + \frac{3 - 96i}{8} X_n Y_n \right) h'(0) \pi^2 \Omega_3 dx' - \left( \frac{\pi^2}{6} \sum_{j=1}^{n-1} X_j Y_j \Omega_3 dx' + \frac{\pi}{8} X_n Y_n \Omega_3 dx' \right) \\ & \times \text{trace}[c(\Psi)c(dx_n)] + \frac{i\pi}{8} \Omega_3 dx' \cdot \left( \text{trace}(iX_n \frac{\partial Y_n}{\partial x_n}) - \text{trace}[c(X)c(\Psi)]Y_n - \text{trace}[c(Y)c(\Psi)]X_n \right). \end{aligned} \quad (4.47)$$

Let  $X = X^T + X_n \partial_n$ ,  $Y = Y^T + Y_n \partial_n$ , then we have  $\sum_{j=1}^{n-1} X_j Y_j = g(X^T, Y^T)$ . Now  $\tilde{\Phi}$  is the sum of the cases (a), (b) and (c). Therefore, we get

$$\begin{aligned}\tilde{\Phi} = \sum_{i=1}^5 \tilde{\Phi}_i &= \left( \frac{15 - 362i}{32} X_n Y_n + \frac{(10i - 27)\pi}{24} g(X^T, Y^T) \right) h'(0) \pi \Omega_3 dx' \\ &\quad + \left( \frac{\pi^2}{6} \sum_{j=1}^{n-1} X_j Y_j + \frac{\pi - \pi i}{8} X_n Y_n \right) \cdot \text{trace}[c(\Psi)c(dx_n)] \Omega_3 dx' \\ &\quad + \frac{i\pi}{8} \left( \text{trace}(iX_n \frac{\partial Y_n}{\partial x_n}) - \text{trace}[c(X)c(\Psi)] Y_n - \text{trace}[c(Y)c(\Psi)] X_n \right) \cdot \Omega_3 dx'\end{aligned}\quad (4.48)$$

Combining (4.14) and (4.48), we obtain Theorem 1.1.

When  $c(\Psi) = f$ , we can calculate  $\text{trace}[c(dx_n)f] = 0$  and  $\text{trace}[c(X)f] = 0$ , then we can straightforwardly assert the following corollary:

**Corollary 4.5.** *Let  $M$  be a 4-dimensional oriented compact manifolds with the boundary  $\partial M$  and the metric  $g^M$  as above, and let  $c(\Psi) = f$ , then*

$$\begin{aligned}&\widetilde{\text{Wres}}[\pi^+ \nabla_X^\Psi \nabla_Y^\Psi D_\Psi^{-2} \circ \pi^+ D_\Psi^{-2}] \\ &= \frac{4\pi^2}{3} \int_M EG(X, Y) vol_g + \frac{1}{2} \int_M \text{trace}[\frac{1}{4}s - 3f^2] g(X, Y) dvol_M \\ &\quad + \int_{\partial M} \left\{ \left( \frac{15 - 362i}{32} X_n Y_n + \frac{(10i - 27)\pi}{24} g(X^T, Y^T) \right) h'(0) \pi \Omega_3 + \frac{i\pi}{8} \text{trace}\left(iX_n \frac{\partial Y_n}{\partial x_n}\right) \cdot \Omega_3 \right\} dVol_{\partial M},\end{aligned}$$

where  $s$  is the scalar curvature.

When  $c(\Psi) = c(\overline{X})$ , we can calculate

$$\text{trace}[c(dx_n)c(\Psi)] = -g(\partial x_n, \overline{X}) \text{trace}[id], \quad (4.49)$$

then we can straightforwardly assert the following corollary:

**Corollary 4.6.** *Let  $M$  be a 4-dimensional oriented compact manifolds with the boundary  $\partial M$  and the metric  $g^M$  as above, and let  $c(\Psi) = c(\overline{X})$ , then*

$$\begin{aligned}&\widetilde{\text{Wres}}[\pi^+ \nabla_X^\Psi \nabla_Y^\Psi D_\Psi^{-2} \circ \pi^+ D_\Psi^{-2}] = \frac{4\pi^2}{3} \int_M EG(X, Y) vol_g + \frac{\pi^2}{2} \int_M \left[ -8g(X, \nabla_Y^{TM} \overline{X}) - 8g(Y, \nabla_X^{TM} \overline{X}) \right] dvol_M \\ &\quad + \frac{1}{2} \int_M \text{trace}\left[\frac{1}{4}s + \sum_{j=1}^n \frac{1}{2} c(\overline{X}) c(e_j) c(\overline{X}) c(e_j) + |\overline{X}|^2\right] g(X, Y) dvol_M + \int_{\partial M} \left\{ \left( \frac{15 - 362i}{32} X_n Y_n \right. \right. \\ &\quad \left. \left. + \frac{(10i - 27)\pi}{24} g(X^T, Y^T) \right) h'(0) \pi \Omega_3 - \left( \frac{2\pi^2}{3} \sum_{j=1}^{n-1} X_j Y_j + \frac{\pi - \pi i}{2} X_n Y_n \right) g(\partial x_n, \overline{X}) \Omega_3 \right. \\ &\quad \left. + \frac{i\pi}{8} \text{trace}\left(iX_n \frac{\partial Y_n}{\partial x_n} + 4g(X, \overline{X}) Y_n + 4g(Y, \overline{X}) X_n\right) \cdot \Omega_3 \right\} dVol_{\partial M},\end{aligned}\quad (4.50)$$

where  $s$  is the scalar curvature.

When  $c(\Psi) = c(X_1)c(X_2)$ , we have

$$\text{trace}[c(dx_n)c(\Psi)] = 0, \quad (4.51)$$

and we can compute  $\widetilde{\text{Wres}}[\pi^+\nabla_X^\Psi\nabla_Y^\Psi D_\Psi^{-2} \circ \pi^+D_\Psi^{-2}]$ .

**Corollary 4.7.** *Let  $M$  be a 4-dimensional oriented compact manifolds with the boundary  $\partial M$  and the metric  $g^M$  as above, and let  $c(\Psi) = c(X_1)c(X_2)$ , then*

$$\begin{aligned} \widetilde{\text{Wres}}[\pi^+\nabla_X^\Psi\nabla_Y^\Psi D_\Psi^{-2} \circ \pi^+D_\Psi^{-2}] &= \frac{4\pi^2}{3} \int_M EG(X, Y) \text{vol}_g \\ &+ \frac{1}{2} \int_M \text{trace}\left[\frac{1}{4}s + \sum_{j=1}^n \frac{1}{2}c(\Psi)c(e_j)c(\Psi)c(e_j) - (c(\Psi))^2\right] g(X, Y) d\text{vol}_M \\ &+ \int_{\partial M} \left\{ \left( \frac{15-362i}{32} X_n Y_n + \frac{(10i-27)\pi}{24} g(X^T, Y^T) \right) h'(0) \pi \Omega_3 \right. \\ &\left. + \frac{i\pi}{8} \text{trace}(iX_n \frac{\partial Y_n}{\partial x_n}) \Omega_3 \right\} d\text{Vol}_{\partial M}, \end{aligned} \quad (4.52)$$

where  $s$  is the scalar curvature.

When  $c(\Psi) = c(X_1)c(X_2)c(X_3)$ , then we have

$$\begin{aligned} \text{trace}[c(dx_n)c(\Psi)] &= g(\partial x_n, X_1)g(X_2, X_3)\text{trace}[\text{id}] - g(\partial x_n, X_2)g(X_1, X_3)\text{trace}[\text{id}] \\ &+ g(\partial x_n, X_3)g(X_1, X_2)\text{trace}[\text{id}], \end{aligned} \quad (4.53)$$

and we compute that:

**Corollary 4.8.** *Let  $M$  be a 4-dimensional oriented compact manifolds with the boundary  $\partial M$  and the metric  $g^M$  as above, and let  $c(\Psi) = c(X_1)c(X_2)c(X_3)$ , then*

$$\begin{aligned} \widetilde{\text{Wres}}[\pi^+\nabla_X^\Psi\nabla_Y^\Psi D_\Psi^{-2} \circ \pi^+D_\Psi^{-2}] &= \frac{4\pi^2}{3} \int_M EG(X, Y) \text{vol}_g + \frac{\pi^2}{2} \int_M \left\{ 8 \left[ g(X, X_1) \left( g(\nabla_Y^{TM} X_2, X_3) \right. \right. \right. \\ &+ g(X_2, \nabla_Y^{TM} X_3) \left. \right) - g(X, X_2) \left( g(\nabla_Y^{TM} X_1, X_3) + g(X_1, \nabla_Y^{TM} X_3) \right) + g(X, X_3) \left( g(\nabla_Y^{TM} X_1, X_2) \right. \\ &+ g(X_1, \nabla_Y^{TM} X_2) \left. \right) - g(X, \nabla_Y^{TM} X_1)g(X_2, X_3) - g(X, \nabla_Y^{TM} X_2)g(X_1, X_3) + g(X, \nabla_Y^{TM} X_3)g(X_1, X_2) \left. \right] \\ &- 8 \left[ g(Y, X_1) \left( g(\nabla_X^{TM} X_2, X_3) + g(X_2, \nabla_X^{TM} X_3) \right) - g(Y, X_2) \left( g(\nabla_X^{TM} X_1, X_3) + g(X_1, \nabla_X^{TM} X_3) \right) \right. \\ &+ g(Y, X_3) \left( g(\nabla_X^{TM} X_1, X_2) + g(X_1, \nabla_X^{TM} X_2) \right) - g(Y, \nabla_Y^{TM} X_1)g(X_2, X_3) - g(Y, \nabla_Y^{TM} X_2)g(X_1, X_3) \\ &+ g(Y, \nabla_Y^{TM} X_3)g(X_1, X_2) \left. \right] \left. \right\} d\text{vol}_M + \frac{1}{2} \int_M \text{trace}\left[\frac{1}{4}s + \sum_{j=1}^n \frac{1}{2}c(\Psi)c(e_j)c(\Psi)c(e_j) - (c(\Psi))^2\right] g(X, Y) d\text{vol}_M \\ &+ \int_{\partial M} \left\{ \left( \frac{15-362i}{32} X_n Y_n + \frac{(10i-27)\pi}{24} g(X^T, Y^T) \right) h'(0) \pi \Omega_3 + \left( \frac{\pi^2}{6} \sum_{j=1}^{n-1} X_j Y_j + \frac{\pi - \pi i}{8} X_n Y_n \right) \right. \\ &\times \left( 4g(\partial x_n, X_1)g(X_2, X_3) - 4g(\partial x_n, X_2)g(X_1, X_3) + 4g(\partial x_n, X_3)g(X_1, X_2) \right) \Omega_3 \\ &+ \frac{i\pi}{8} \left( \text{trace}(iX_n \frac{\partial Y_n}{\partial x_n}) - 4 \left( g(X, X_1)g(X_2, X_3) - g(X, X_2)g(X_1, X_3) + g(X, X_3)g(X_1, X_2) \right) Y_n \right. \\ &\left. \left. - 4 \left( g(Y, X_1)g(X_2, X_3) - g(Y, X_2)g(X_1, X_3) + g(Y, X_3)g(X_1, X_2) \right) X_n \right) \cdot \Omega_3 \right\} d\text{Vol}_{\partial M}, \end{aligned} \quad (4.54)$$

where  $s$  is the scalar curvature.

## 5. The noncommutative residue $\widetilde{\text{Wres}}\left[\pi^+(\nabla_X^\Psi \nabla_Y^\Psi D_\Psi^{-1}) \circ \pi^+(D_\Psi^{-3})\right]$ on manifolds with boundary

In the following, we will compute the residue  $\widetilde{\text{Wres}}\left[\pi^+(\nabla_X^\Psi \nabla_Y^\Psi D_\Psi^{-1}) \circ \pi^+(D_\Psi^{-3})\right]$  on 4 dimensional oriented compact spin manifolds with boundary and establish a Kastler-Kalau-Walze type theorem in this case. By (4.10), we have

$$\begin{aligned} & \widetilde{\text{Wres}}\left[\pi^+(\nabla_X^\Psi \nabla_Y^\Psi D_\Psi^{-1}) \circ \pi^+(D_\Psi^{-3})\right] \\ &= \int_M \int_{|\xi|=1} \text{trace}_{S(TM)} [\sigma_{-4}((\nabla_X^\Psi \nabla_Y^\Psi D_\Psi^{-1}) \circ (D_\Psi^{-3}))] \sigma(\xi) dx + \int_{\partial M} \widehat{\Phi}, \end{aligned} \quad (5.1)$$

where

$$\begin{aligned} \widehat{\Phi} &= \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{j,k=0}^{\infty} \sum_{\alpha} \frac{(-i)^{|\alpha|+j+k+\ell}}{\alpha!(j+k+1)!} \text{trace}_{S(TM)} \left[ \partial_{x_n}^j \partial_{\xi'}^\alpha \partial_{\xi_n}^k \sigma_r^+ (\nabla_X^\Psi \nabla_Y^\Psi D_\Psi^{-1})(x', 0, \xi', \xi_n) \right. \\ &\quad \times \left. \partial_{x_n}^\alpha \partial_{\xi_n}^{j+1} \partial_{x_n}^k \sigma_l (D_\Psi^{-3})(x', 0, \xi', \xi_n) \right] d\xi_n \sigma(\xi') dx', \end{aligned} \quad (5.2)$$

and the sum is taken over  $r - k + |\alpha| + \ell - j - 1 = -4, r \leq 0, \ell \leq -2$ .

Locally we can use Theorem 3.3 to compute the interior term of (5.1), then

$$\begin{aligned} & \int_M \int_{|\xi|=1} \text{trace}_{S(TM)} [\sigma_{-4}((\nabla_X^\Psi \nabla_Y^\Psi D_\Psi^{-1}) \circ (D_\Psi^{-3}))] \sigma(\xi) dx \\ &= \frac{4\pi^2}{3} \int_M EG(X, Y) vol_g + \frac{\pi^2}{2} \int_M \text{trace} [c(X) \nabla_Y^{S(TM)}(c(\Psi)) \\ &+ \nabla_Y^{S(TM)}(c(\Psi)) c(X) - c(Y) \nabla_X^{S(TM)}(c(\Psi)) - \nabla_X^{S(TM)}(c(\Psi)) c(Y)] dvol_M \\ &+ \frac{1}{2} \int_M \text{trace} [\frac{1}{4} s + \sum_{j=1}^n \frac{1}{2} c(\Psi) c(e_j) c(\Psi) c(e_j) - (c(\Psi))^2] g(X, Y) dvol_M \end{aligned} \quad (5.3)$$

From Lemma 4.1 and Lemma 4.2, we have

**Lemma 5.1.** *The following identities hold:*

$$\sigma_1(\nabla_X^\Psi \nabla_Y^\Psi D_\Psi^{-1}) = -i \sum_{j,l=1}^n X_j Y_l \xi_j \xi_l c(\xi) |\xi|^{-2}; \quad (5.4)$$

$$\begin{aligned} \sigma_0(\nabla_X^\Psi \nabla_Y^\Psi D_\Psi^{-1}) &= \sigma_2(\nabla_X^\Psi \nabla_Y^\Psi) \sigma_{-2}(D_\Psi^{-1}) + \sigma_1(\nabla_X^\Psi \nabla_Y^\Psi) \sigma_{-1}(D_\Psi^{-1}) \\ &+ \sum_{j=1}^n \partial_{\xi_j} [\sigma_2(\nabla_X^\Psi \nabla_Y^\Psi)] D_{x_j} [\sigma_{-1}(D_\Psi^{-1})]. \end{aligned} \quad (5.5)$$

Write

$$D_x^\alpha = (-i)^{|\alpha|} \partial_x^\alpha; \quad \sigma(A^3) = p_3 + p_2 + p_1 + p_0; \quad \sigma(A^{-3}) = \sum_{j=3}^{\infty} q_{-j}. \quad (5.6)$$

By the composition formula of pseudodifferential operators, we have

$$\begin{aligned}
1 = \sigma(A^3 \circ A^{-3}) &= \sum_{\alpha} \frac{1}{\alpha!} \partial_{\xi}^{\alpha} [\sigma(A^3)] A_x^{\alpha} [\sigma(A^{-3})] \\
&= (p_3 + p_2 + p_1 + p_0)(q_{-3} + q_{-4} + q_{-5} + \dots) \\
&\quad + \sum_j (\partial_{\xi_j} p_3 + \partial_{\xi_j} p_2 + \partial_{\xi_j} p_1 + \partial_{\xi_j} p_0)(A_{x_j} q_{-3} + A_{x_j} q_{-4} + A_{x_j} q_{-5} + \dots) \\
&= p_3 q_{-3} + (p_3 q_{-4} + p_2 q_{-3} + \sum_j \partial_{\xi_j} p_3 D_{x_j} q_{-3}) + \dots,
\end{aligned} \tag{5.7}$$

so

$$q_{-3} = p_3^{-1}; \quad q_{-4} = -p_3^{-1}[p_2 p_3^{-1} + \sum_j \partial_{\xi_j} p_3 D_{x_j}(p_3^{-1})]. \tag{5.8}$$

Then it is easy to check that

**Lemma 5.2.** *The following identities hold:*

$$\begin{aligned}
\sigma_{-3}(D_{\Psi}^{-3}) &= ic(\xi)|\xi|^{-4}; \\
\sigma_{-4}(D_{\Psi}^{-3}) &= \frac{c(\xi)\sigma_2(D_{\Psi}^3)c(\xi)}{|\xi|^8} + \frac{ic(\xi)}{|\xi|^8} \left( |\xi|^4 c(dx_n) \partial_{x_n} c(\xi') - 2h'(0)c(dx_n)c(\xi) + 2\xi_n c(\xi) \partial_{x_n} c(\xi') \right. \\
&\quad \left. + 4\xi_n h'(0) \right),
\end{aligned} \tag{5.9}$$

where

$$\sigma_2(D_{\Psi}^3) = c(\xi)(4\sigma^k - 2\Gamma^k)\xi_k - \frac{1}{4}|\xi|^2 \sum_{s,t} \omega_{s,t}(\tilde{e}_l)c(e_l)c(\tilde{e}_s)c(\tilde{e}_t) - 2|\xi|^2 c(\Psi) - 2c(\xi)c(\Psi)c(\xi). \tag{5.10}$$

Next, we need to compute  $\int_{\partial M} \widehat{\Psi}$ . When  $n = 4$ , then  $\text{trace}_{S(TM)}[\text{id}] = \dim(\wedge^*(\mathbb{R}^2)) = 4$ , the sum is taken over  $r + l - k - j - |\alpha| = -3$ ,  $r \leq 0$ ,  $l \leq -2$ , then we have the following five cases:

**case (1)**  $r = 1$ ,  $l = -2$ ,  $k = j = 0$ ,  $|\alpha| = 1$ .

By (5.2), we get

$$\widehat{\Phi}_1 = - \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{|\alpha|=1} \text{trace}[\partial_{\xi'}^{\alpha} \pi_{\xi_n}^+ \sigma_1(\nabla_X^{\Psi} \nabla_Y^{\Psi} D_{\Psi}^{-1}) \times \partial_{x'}^{\alpha} \partial_{\xi_n} \sigma_{-3}(D_{\Psi}^{-3})](x_0) d\xi_n \sigma(\xi') dx'. \tag{5.11}$$

By Lemma 2.2 in [17], for  $i < n$ , then

$$\partial_{x_i} \sigma_{-3}(D_{\Psi}^{-3})(x_0) = \partial_{x_i}(ic(\xi)|\xi|^{-4})(x_0) = i \frac{\partial_{x_i} c(\xi)}{|\xi|^4}(x_0) + i \frac{c(\xi) \partial_{x_i}(|\xi|^4)}{|\xi|^8}(x_0) = 0, \tag{5.12}$$

so  $\widehat{\Phi}_1 = 0$ .

**case (2)**  $r = 1$ ,  $l = -3$ ,  $k = |\alpha| = 0$ ,  $j = 1$ .

By (5.2), we get

$$\widehat{\Phi}_2 = - \frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\partial_{x_n} \pi_{\xi_n}^+ \sigma_1(\nabla_X^{\Psi} \nabla_Y^{\Psi} D_{\Psi}^{-1}) \times \partial_{\xi_n}^2 \sigma_{-3}(D_{\Psi}^{-3})](x_0) d\xi_n \sigma(\xi') dx'. \tag{5.13}$$

By Lemma 5.2, we have

$$\partial_{\xi_n}^2 \sigma_{-3}(D_{\Psi}^{-3})(x_0) = \partial_{\xi_n}^2 (c(\xi)|\xi|^{-4})(x_0) = \frac{(20\xi_n^2 - 4)ic(\xi') + 12(\xi^3 - \xi)ic(dx_n)}{(1 + \xi_n^2)^4}, \quad (5.14)$$

In addition, by Lemma 5.1, we get

$$\partial_{x_n} \sigma_1(\nabla_X^\Psi \nabla_Y^\Psi D_{\Psi}^{-1})(x_0) = \sum_{j,l=1}^n X_j Y_l \xi_j \xi_l \left[ \frac{\partial_{x_n} c(\xi')}{1 + \xi_n^2} + \frac{c(\xi) h'(0) |\xi'|^2}{(1 + \xi_n^2)^2} \right]. \quad (5.15)$$

Then, we have

$$\begin{aligned} & \pi_{\xi_n}^+ \partial_{x_n} \sigma_1(\nabla_X^\Psi \nabla_Y^\Psi D_{\Psi}^{-1})(x_0) = \partial_{x_n} \pi_{\xi_n}^+ \sigma_1(\nabla_X^\Psi \nabla_Y^\Psi D_{\Psi}^{-1})(x_0) \\ &= i \sum_{j,l=1}^{n-1} X_j Y_l \xi_j \xi_l h'(0) |\xi'|^2 \left[ \frac{ic(\xi')}{4(\xi_n - i)} + \frac{c(\xi') + ic(dx_n)}{4(\xi_n - i)^2} \right] - \sum_{j,l=1}^{n-1} X_j Y_l \xi_j \xi_l \frac{\partial_{x_n} c(\xi')}{2(\xi_n - i)} - i X_n Y_n \left\{ \frac{\partial_{x_n} c(\xi')}{2(\xi_n - i)} \right. \\ & \quad \left. + h'(0) |\xi'| \left[ -\frac{2ic(\xi') - 3c(dx_n)}{4(\xi_n - i)} \frac{[c(\xi') + ic(dx_n)][i(\xi_n - i) + 1]}{4(\xi_n - i)^2} \right] \right\} - \sum_{j=1}^{n-1} X_j Y_n \xi_j \left[ i \frac{\partial_{x_n} c(\xi')}{2(\xi_n - i)} \right. \\ & \quad \left. - \frac{ih'(0) |\xi'| [c(\xi') + 2ic(dx_n)]}{4(\xi_n - i)} - \frac{[ic(\xi') - c(dx_n)][i(\xi_n - i) + 1]}{(\xi_n - i)^2} \right] - \sum_{l=1}^{n-1} X_n Y_l \xi_l \left[ i \frac{\partial_{x_n} c(\xi')}{2(\xi_n - i)} \right. \\ & \quad \left. - \frac{ih'(0) |\xi'| [c(\xi') + 2ic(dx_n)]}{4(\xi_n - i)} - \frac{[ic(\xi') - c(dx_n)][i(\xi_n - i) + 1]}{(\xi_n - i)^2} \right]. \end{aligned} \quad (5.16)$$

We note that  $i < n$ ,  $\int_{|\xi'|=1} \xi_{i_1} \xi_{i_2} \cdots \xi_{i_{2r+1}} \sigma(\xi') = 0$ , so we omit some items that have no contribution for computing **case (2)**. Then there is the following formula

$$\begin{aligned} & \text{trace}[\partial_{x_n} \pi_{\xi_n}^+ \sigma_1(\nabla_X^\Psi \nabla_Y^\Psi D_{\Psi}^{-1}) \times \partial_{\xi_n}^2 \sigma_{-3}(D_{\Psi}^{-3})](x_0) \\ &= \sum_{j,l=1}^{n-1} X_j Y_l \xi_j \xi_l h'(0) \left[ 8i \frac{5\xi_n^2 - 1}{(\xi_n - i)^5 (\xi_n + i)^4} + \frac{4(5\xi_n^2 - 1) + 12i(\xi_n^3 - \xi_n)}{(\xi_n - i)^6 (\xi_n + i)^4} \right] \\ & \quad + X_n Y_n h'(0) \left[ \frac{(4i - 4)(\xi^2 - 1) + 48(\xi_n^3 - \xi_n)}{(\xi_n - i)^5 (\xi_n + i)^4} - \frac{4(5\xi_n^2 - 1) + 12i(\xi_n^3 - \xi_n)}{(\xi_n - i)^6 (\xi_n + i)^4} \right] \\ & \quad + 8 \sum_{j=1}^{n-1} X_j Y_n \xi_j \left[ \frac{(6 - 3ih'(0))(\xi_n^3 - \xi_n) - 2i(5\xi_n^2 - 1)}{(\xi_n - i)^5 (\xi_n + i)^4} + \frac{2(5\xi_n^2 - 1) + 6i(\xi_n^3 - \xi_n)}{(\xi_n - i)^6 (\xi_n + i)^4} \right] \\ & \quad + 8 \sum_{l=1}^{n-1} X_n Y_l \xi_l \left[ \frac{(6 - 3ih'(0))(\xi_n^3 - \xi_n) - 2i(5\xi_n^2 - 1)}{(\xi_n - i)^5 (\xi_n + i)^4} + \frac{2(5\xi_n^2 - 1) + 6i(\xi_n^3 - \xi_n)}{(\xi_n - i)^6 (\xi_n + i)^4} \right]. \end{aligned} \quad (5.17)$$

Therefore, we get

$$\begin{aligned}
\widehat{\Phi}_2 &= \frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \left\{ \sum_{j,l=1}^{n-1} X_j Y_l \xi_j \xi_l h'(0) \left[ 8i \frac{5\xi_n^2 - 1}{(\xi_n - i)^5 (\xi_n + i)^4} + \frac{4(5\xi_n^2 - 1) + 12i(\xi_n^3 - \xi_n)}{(\xi_n - i)^6 (\xi_n + i)^4} \right] \right. \\
&\quad \left. + X_n Y_n h'(0) \left[ \frac{(4i - 4)(\xi^2 - 1) + 48(\xi_n^3 - \xi_n)}{(\xi_n - i)^5 (\xi_n + i)^4} - \frac{4(5\xi_n^2 - 1) + 12i(\xi_n^3 - \xi_n)}{(\xi_n - i)^6 (\xi_n + i)^4} \right] \right\} d\xi_n \sigma(\xi') dx' \\
&= \sum_{j,l=1}^{n-1} X_j Y_l h'(0) \Omega_3 \int_{\Gamma^+} \left[ 8i \frac{5\xi_n^2 - 1}{(\xi_n - i)^5 (\xi_n + i)^4} + \frac{4(5\xi_n^2 - 1) + 12i(\xi_n^3 - \xi_n)}{(\xi_n - i)^6 (\xi_n + i)^4} \right] \xi_j \xi_l d\xi_n dx' \\
&\quad + X_n Y_n h'(0) \Omega_3 \int_{\Gamma^+} \left[ \frac{(4i - 4)(\xi^2 - 1) + 48(\xi_n^3 - \xi_n)}{(\xi_n - i)^5 (\xi_n + i)^4} - \frac{4(5\xi_n^2 - 1) + 12i(\xi_n^3 - \xi_n)}{(\xi_n - i)^6 (\xi_n + i)^4} \right] d\xi_n dx' \\
&= - \left[ \frac{592}{3} \pi \sum_{j=1}^{n-1} X_j Y_j + \left( \frac{461}{4} + \frac{23}{4}i \right) X_n Y_n \right] h'(0) \pi \Omega_3 dx', \tag{5.18}
\end{aligned}$$

where  $\Omega_3$  is the canonical volume of  $S^2$ .

**case (3)**  $r = 1, l = -3, j = |\alpha| = 0, k = 1$ .

By (5.2), we get

$$\begin{aligned}
\widehat{\Phi}_3 &= -\frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\partial_{\xi_n} \pi_{\xi_n}^+ \sigma_1(\nabla_X^\Psi \nabla_Y^\Psi D_\Psi^{-1}) \times \partial_{\xi_n} \partial_{x_n} \sigma_{-3}(D_\Psi^{-3})](x_0) d\xi_n \sigma(\xi') dx' \\
&= \frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\partial_{\xi_n}^2 \pi_{\xi_n}^+ \sigma_1(\nabla_X^\Psi \nabla_Y^\Psi D_\Psi^{-1}) \times \partial_{x_n} \sigma_{-3}(D_\Psi^{-3})](x_0) d\xi_n \sigma(\xi') dx'. \tag{5.19}
\end{aligned}$$

By Lemma 5.3, we have

$$\partial_{x_n} \sigma_{-3}(D_\Psi^{-3})(x_0)|_{|\xi'|=1} = \frac{i \partial_{x_n} [c(\xi')]}{(1 + \xi_n^2)^4} - \frac{2ih'(0)c(\xi)|\xi'|^2_{g_{\partial M}}}{(1 + \xi_n^2)^6}. \tag{5.20}$$

By integrating formula we obtain

$$\begin{aligned}
\pi_{\xi_n}^+ \sigma_1(\nabla_X^\Psi \nabla_Y^\Psi D_\Psi^{-1}) &= - \frac{c(\xi') + ic(dx_n)}{2(\xi_n - i)} \sum_{j,l=1}^{n-1} X_j Y_l \xi_j \xi_l - \frac{c(\xi') + ic(dx_n)}{2(\xi_n - i)} X_n Y_n \\
&\quad - \frac{ic(\xi') - c(dx_n)}{2(\xi_n - i)} \sum_{j=1}^{n-1} X_j Y_n \xi_j - \frac{ic(\xi') - c(dx_n)}{2(\xi_n - i)} \sum_{l=1}^{n-1} X_n Y_l \xi_l. \tag{5.21}
\end{aligned}$$

Then, we have

$$\partial_{\xi_n}^2 \pi_{\xi_n}^+ \sigma_1(\nabla_X^\Psi \nabla_Y^\Psi D_\Psi^{-1}) = - \frac{c(\xi') + ic(dx_n)}{(\xi_n - i)^3} \sum_{j,l=1}^{n-1} X_j Y_l \xi_j \xi_l - \frac{c(\xi') + ic(dx_n)}{(\xi_n - i)^3} X_n Y_n. \tag{5.22}$$

We note that  $i < n, \int_{|\xi'|=1} \xi_{i_1} \xi_{i_2} \cdots \xi_{i_{2r+1}} \sigma(\xi') = 0$ , so we omit some items that have no contribution for computing **case (3)**, then

$$\text{trace}[\partial_{\xi_n} \pi_{\xi_n}^+ \sigma_1(\nabla_X^\Psi \nabla_Y^\Psi D_\Psi^{-1}) \times \partial_{\xi_n} \partial_{x_n} \sigma_{-3}(D_\Psi^{-3})](x_0) = \frac{-2h'(0)}{(\xi_n - i)^5 (\xi_n + i)^2} \left( \sum_{j,l=1}^{n-1} X_j Y_l \xi_j \xi_l + X_n Y_n \right). \tag{5.23}$$

Therefore, we get

$$\begin{aligned}
\widehat{\Phi}_3 &= \frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \left( \frac{-2h'(0)}{(\xi_n - i)^5 (\xi_n + i)^2} \left( \sum_{j,l=1}^{n-1} X_j Y_l \xi_j \xi_l + X_n Y_n \right) d\xi_n \sigma(\xi') dx' \right. \\
&= h'(0) \Omega_3 \left( - \sum_{j,l=1}^{n-1} X_j Y_l \int_{\Gamma^+} \frac{1}{(\xi_n - i)^5 (\xi_n + i)^2} \xi_j \xi_l d\xi_n dx' - X_n Y_n \int_{\Gamma^+} \frac{1}{(\xi_n - i)^5 (\xi_n + i)^2} d\xi_n dx' \right) \\
&= \left( - \sum_{j,l=1}^{n-1} X_j Y_l + 2X_n Y_n \right) h'(0) \Omega_3 \frac{2\pi i}{4!} \left[ \frac{1}{(\xi_n + i)^2} \right]^{(4)} \Big|_{\xi_n=i} dx' \\
&= \left( \frac{5\pi i}{6} \sum_{j=1}^{n-1} X_j Y_j + \frac{5i}{8} X_n Y_n \right) h'(0) \pi \Omega_3 dx'. \tag{5.24}
\end{aligned}$$

**case (4)**  $r = 0$ ,  $l = -3$ ,  $k = j = |\alpha| = 0$ .

By (5.2), we get

$$\widehat{\Phi}_4 = -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\pi_{\xi_n}^+ \sigma_0(\nabla_X^\Psi \nabla_Y^\Psi D_\Psi^{-1}) \times \partial_{\xi_n} \sigma_{-3}(D_\Psi^{-3})](x_0) d\xi_n \sigma(\xi') dx'. \tag{5.25}$$

By Lemma 5.3, we obtain

$$\partial_{\xi_n} \sigma_{-3}(D_\Psi^{-3})(x_0)|_{|\xi'|=1} = \frac{ic(dx_n)}{(1 + \xi_n^2)^2} - \frac{4i\xi_n c(\xi)}{(1 + \xi_n^2)^3}. \tag{5.26}$$

By Lemma 5.2, we have

$$\begin{aligned}
\sigma_0(\nabla_X^\Psi \nabla_Y^\Psi D_\Psi^{-1}) &= \sigma_2(\nabla_X^\Psi \nabla_Y^\Psi) \sigma_{-2}(D_\Psi^{-1}) + \sigma_1(\nabla_X^\Psi \nabla_Y^\Psi) \sigma_{-1}(D_\Psi^{-1}) \\
&\quad + \sum_{j=1}^n \partial_{\xi_j} [\sigma_2(\nabla_X^\Psi \nabla_Y^\Psi)] D_{x_j} [\sigma_{-1}(D_\Psi^{-1})]. \tag{5.27}
\end{aligned}$$

(1) First, we present the explicit expression for the first term in equation (5.27)

$$\begin{aligned}
&\sigma_2(\nabla_X^\Psi \nabla_Y^\Psi) \sigma_{-2}(D_\Psi^{-1})(x_0)|_{|\xi'|=1} \\
&= - \sum_{j,l=1}^n X_j Y_l \xi_j \xi_l \left[ \frac{c(\xi) \sigma_0(D_\Psi) c(\xi)}{|\xi|^4} + \frac{c(\xi)}{|\xi|^6} \sum_j c(dx_j) \left( \partial_{x_j} [c(\xi)] |\xi|^2 - c(\xi) \partial_{x_j} (|\xi|^2) \right) \right], \tag{5.28}
\end{aligned}$$

By integrating formula, we obtain

$$\begin{aligned}
&\pi_{\xi_n}^+ \left\{ - \sum_{j,l=1}^n X_j Y_l \xi_j \xi_l \left[ \frac{c(\xi) \sigma_0(D_\Psi) c(\xi) + c(\xi) c(dx_n) \partial_{x_n} [c(\xi')](x_0)}{(1 + \xi_n^2)^2} \right] \right\} \\
&= - \frac{\sum_{j,l=1}^{n-1} X_j Y_l \xi_j \xi_l \cdot K_1}{4(\xi_n - i)} - \frac{\sum_{j,l=1}^{n-1} X_j Y_l \xi_j \xi_l \cdot K_2}{4(\xi_n - i)^2} + \frac{K_3}{4(\xi_n - i)^2}, \tag{5.29}
\end{aligned}$$

where

$$K_1 = ic(\xi')\sigma_0(D)c(\xi') + ic(dx_n)(-\frac{3}{4}h'(0)c(dx_n))c(dx_n) + ic(\xi')c(dx_n)\partial_{x_n}[c(\xi')], \quad (5.30)$$

$$K_2 = [c(\xi') + ic(dx_n)]\sigma_0(D)[c(\xi') + ic(dx_n)] + c(\xi')c(dx_n)\partial_{x_n}c(\xi') - i\partial_{x_n}[c(\xi')], \quad (5.31)$$

$$\begin{aligned} K_3 = & \sum_{j,l=1}^{n-1} X_j Y_l \xi_j \xi_l \left( (-2 - i\xi_n)c(\xi')c(\Psi)c(\xi') - ic(dx_n)c(\Psi)c(\xi') - ic(\xi')c(\Psi)c(dx_n) \right. \\ & \left. - i\xi_n c(dx_n)c(\Psi)c(dx_n) \right) + X_n Y_n \left( -i\xi_n c(\xi')c(\Psi)c(\xi') - ic(dx_n)c(\Psi)c(\xi') \right. \\ & \left. - ic(\xi')c(\Psi)c(dx_n) + c(dx_n)c(\Psi)c(dx_n) \right). \end{aligned} \quad (5.32)$$

Using a similar approach, we derive

$$\pi_{\xi_n}^+ \left[ \frac{c(\xi)c(dx_n)c(\xi)}{(1+\xi_n)^3}(x_0)|_{|\xi'|=1} \right] = \frac{1}{2} \left[ \frac{c(dx_n)}{4i(\xi_n-i)} + \frac{c(dx_n)-ic(\xi')}{8(\xi_n-i)^2} + \frac{3\xi_n-7i}{8(\xi_n-i)^3}[ic(\xi')-c(dx_n)] \right]. \quad (5.33)$$

(2) Next, we present the explicit expression for the second term in equation (5.28)

$$\begin{aligned} & \sigma_1(\nabla_X^\Psi \nabla_Y^\Psi) \sigma_{-1}(D_\Psi^{-1})(x_0)|_{|\xi'|=1} \\ &= \left( i \sum_{j,l=1}^n X_j \frac{\partial Y_l}{\partial x_j} \partial_{x_l} + i \sum_j L(Y) X_j \xi_j + i \sum_l L(Y) Y_l \xi_l + \sum_j iG(X, \Psi) Y_j \xi_j \right. \\ & \quad \left. + \sum_j iG(Y, \Psi) X_j \xi_j \right) \frac{ic(\xi)}{|\xi|^2}; \end{aligned} \quad (5.34)$$

By integrating formula we get

$$\begin{aligned} & \pi_{\xi_n}^+ \left( \left( \sum_j^n iG(X, \Psi) Y_j \xi_j + \sum_j^n iG(Y, \Psi) X_j \xi_j \right) \frac{ic(\xi)}{|\xi|^2} \right) \\ &= \pi_{\xi_n}^+ \left( \left( \sum_j^{n-1} iG(X, \Psi) Y_j \xi_j + \sum_j^{n-1} iG(Y, \Psi) X_j \xi_j \right) \frac{ic(\xi)}{|\xi|^2} \right) \\ & \quad + \pi_{\xi_n}^+ \left( \left( iG(X, \Psi) Y_n \xi_n + iG(Y, \Psi) X_n \xi_n \right) \frac{ic(\xi)}{|\xi|^2} \right) \\ &= \left( \sum_j^{n-1} iG(X, \Psi) Y_j \xi_j + \sum_j^{n-1} iG(Y, \Psi) X_j \xi_j \right) \frac{ic(\xi') - c(dx_n)}{2(\xi_n - i)} \\ & \quad + \left( iG(X, \Psi) Y_n \xi_n + iG(Y, \Psi) X_n \xi_n \right) \frac{-c(\xi') - ic(dx_n)}{2(\xi_n - i)} \end{aligned} \quad (5.35)$$

We note that  $i < n$ ,  $\int_{|\xi'|=1} \xi_{i_1} \xi_{i_2} \cdots \xi_{i_{2d+1}} \sigma(\xi') = 0$ , and  $\text{trace}[c(\xi')] = \text{trace}[c(dx_n)] = -4$ , then

$$\begin{aligned} & -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace} \left( \pi_{\xi_n}^+ \left( \sigma_1(\nabla_X^\Psi \nabla_Y^\Psi) \sigma_{-1}(D_\Psi^{-1}) \right) \times \partial_{\xi_n} \sigma_{-3}(D_\Psi^{-3}) \right) (x_0)(x_0) d\xi_n \sigma(\xi') dx' \\ &= \frac{3\pi}{4} \left[ iX_n \cdot \frac{\partial Y_n}{\partial x_n} + L(Y) X_n + L(X) Y_n + \text{trace}[G(X, \Psi)] Y_n + \text{trace}[G(Y, \Psi)] X_n \right]. \end{aligned} \quad (5.36)$$

(3) Finally, we present the explicit expression for the second term in equation (5.28)

$$\begin{aligned}
& \sum_{j=1}^n \sum_{\alpha} \frac{1}{\alpha!} \partial_{\xi}^{\alpha} [\sigma_2(\nabla_X^{\Psi} \nabla_Y^{\Psi})] D_{x_j} [\sigma_{-1}(D_{\Psi}^{-1})](x_0)|_{|\xi'|=1} \\
&= \sum_{j=1}^n \partial_{\xi_j} [\sigma_2(\nabla_X^{\Psi} \nabla_Y^{\Psi})](-i) \partial_{x_j} [\sigma_{-1}(D_{\Psi}^{-1})] \\
&= \sum_{j=1}^n \partial_{\xi_j} \left[ - \sum_{j,l=1}^n X_j Y_l \xi_j \xi_l \right] (-i) \partial_{x_j} \left[ \frac{i c(\xi)}{|\xi|^2} \right] \\
&= \sum_{j=1}^n \sum_{l=1}^n i(x_j Y_l + x_l Y_j) \xi_l \partial_{x_j} \left( \frac{i c(\xi)}{|\xi|^2} \right).
\end{aligned} \tag{5.37}$$

By integrating formula we obtain

$$\begin{aligned}
& \pi_{\xi_n}^+ \left( \sum_{j=1}^n \sum_{\alpha} \frac{1}{\alpha!} \partial_{\xi}^{\alpha} [\sigma_2(\nabla_X^{\Psi} \nabla_Y^{\Psi})] D_{x_j} [\sigma_{-1}(D_{\Psi}^{-1})] \right) \\
&= \pi_{\xi_n}^+ \left( \sum_{l=1}^{n-1} i(x_n Y_l + x_l Y_n) \xi_l \partial_{x_n} \left( \frac{i c(\xi)}{|\xi|^2} \right) \right) + \pi_{\xi_n}^+ \left( i(x_n Y_n + x_n Y_n) \xi_n \partial_{x_n} \left( \frac{i c(\xi)}{|\xi|^2} \right) \right) \\
&= \sum_{l=1}^{n-1} (x_n Y_l + x_l Y_n) \xi_l \left( \frac{i \partial_{x_n}(c(\xi'))}{2(\xi_n - i)} + h'(0) \frac{(-2 - i \xi_n) c(\xi')}{4(\xi_n - i)^2} - h'(0) \frac{i c(dx_n)}{4(\xi_n - i)^2} \right) \\
&\quad + x_n Y_n \left( \frac{-\partial_{x_n}(c(\xi'))}{(\xi_n - i)} + h'(0) \frac{(-i) c(\xi')}{2(\xi_n - i)^2} - h'(0) \frac{-i \xi_n c(dx_n)}{2(\xi_n - i)^2} \right)
\end{aligned} \tag{5.38}$$

Substituting (5.26) and (5.38) into (5.25) yields

$$\begin{aligned}
& -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace} \left[ \pi_{\xi_n}^+ \left( \sum_{j=1}^n \sum_{\alpha} \frac{1}{\alpha!} \partial_{\xi}^{\alpha} [\sigma_2(\nabla_X^{\Psi} \nabla_Y^{\Psi})] D_{x_j} [\sigma_{-1}(D_{\Psi}^{-1})] \right) \partial_{\xi_n} \sigma_{-3}(D_{\Psi}^{-3}) \right] (x_0) \\
&\quad \times d\xi_n \sigma(\xi') dx' \\
&= \frac{7 - 15i}{8} X_n Y_n \pi h'(0) \Omega_3 dx'.
\end{aligned} \tag{5.39}$$

Combining (1), (2), and (3) yields the desired equality

$$\begin{aligned}
\widehat{\Phi}_4 &= \left( \frac{55\pi}{24} \sum_{j=1}^{n-1} X_j Y_j + \frac{15i - 60 + \pi}{8} X_n Y_n \right) h'(0) \pi \Omega_3 dx' - \left( \frac{\pi^2}{3} \sum_{j,l=1}^{n-1} X_j Y_l + \frac{3\pi}{4} X_n Y_n \right) \text{trace}[c(dx_n)c(\Psi)] \Omega_3 dx' \\
&\quad - \frac{3\pi}{4} X_n Y_n \text{trace}[c(dx_n)\sigma_0(D)] \Omega_3 dx' + \frac{3\pi i}{16} \text{trace}[-c(X)c(\Psi)X_n - c(Y)c(\Psi)Y_n] \Omega_3 dx'.
\end{aligned} \tag{5.40}$$

**case (5)**  $r = 1$ ,  $\ell = -4$ ,  $k = j = |\alpha| = 0$ .

By (5.2), we get

$$\begin{aligned}
\widehat{\Phi}_5 &= - \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\pi_{\xi_n}^+ \sigma_1(\nabla_X^{\Psi} \nabla_Y^{\Psi} D_{\Psi}^{-1}) \times \partial_{\xi_n} \sigma_{-4}(D_{\Psi}^{-3})](x_0) d\xi_n \sigma(\xi') dx' \\
&= \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\partial_{\xi_n} \pi_{\xi_n}^+ \sigma_1(\nabla_X^{\Psi} \nabla_Y^{\Psi} D_{\Psi}^{-1}) \times \sigma_{-4}(D_{\Psi}^{-3})](x_0) d\xi_n \sigma(\xi') dx'.
\end{aligned} \tag{5.41}$$

By Lemma 5.2, we have

$$\begin{aligned} \sigma_{-4}(D_\Psi^{-3})(x_0)|_{|\xi'|=1} &= \frac{1}{(\xi_n^2 + 1)^4} \left[ \left( \frac{11}{2} \xi_n (1 + \xi_n^2) + 8i \xi_n \right) h'(0) c(\xi') \right. \\ &\quad + \left[ -2i + 6i \xi_n^2 - \frac{7}{4} (1 + \xi_n^2) + \frac{15}{4} \xi_n^2 (1 + \xi_n^2) \right] h'(0) c(dx_n) \\ &\quad - 3i \xi_n (1 + \xi_n^2) \partial_{x_n} c(\xi') + i (1 + \xi_n^2) c(\xi') c(dx_n) \partial_{x_n} c(\xi') \\ &\quad \left. + 2 \left[ \frac{c(\xi) c(\Psi) c(\xi)}{|\xi|^6} - \frac{c(\Psi)}{|\xi|^4} \right] \right], \end{aligned} \quad (5.42)$$

and

$$\begin{aligned} \partial_{\xi_n} \pi_{\xi_n}^+ \sigma_1 (\nabla_X^\Psi \nabla_Y^\Psi D_\Psi^{-1})(x_0)|_{|\xi'|=1} &= \frac{c(\xi') + ic(dx_n)}{2(\xi_n - i)^2} \sum_{j,l=1}^{n-1} X_j Y_l \xi_j \xi_l - \frac{c(\xi') + ic(dx_n)}{2(\xi_n - i)^2} X_n Y_n \\ &\quad + \frac{ic(\xi') - c(dx_n)}{2(\xi_n - i)^2} \sum_{j=1}^n X_j Y_n \xi_j + \frac{ic(\xi') - c(dx_n)}{2(\xi_n - i)^2} \sum_{l=1}^n X_n Y_l \xi_l. \end{aligned} \quad (5.43)$$

We note that  $i < n$ ,  $\int_{|\xi'|=1} \xi_{i_1} \xi_{i_2} \cdots \xi_{i_{2d+1}} \sigma(\xi') = 0$ , so we omit some items that have no contribution for computing **case (5)**. Here

$$\text{trace}[c(\xi') c(\xi') c(dx_n) \partial_{x_n} c(\xi')] = 0; \quad \text{trace}[c(dx_n) c(\xi') c(dx_n) \partial_{x_n} c(\xi')] = -2h'(0). \quad (5.44)$$

Also, straightforward computations yield

$$\begin{aligned} &\text{trace} \left[ \partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-1} (\nabla_X^\Psi \nabla_Y^\Psi D_\Psi^{-1}) \times \frac{1}{(\xi_n^2 + 1)^4} \left( \left( \frac{11}{2} \xi_n (1 + \xi_n^2) + 8i \xi_n \right) h'(0) c(\xi') \right. \right. \\ &\quad + \left( -2i + 6i \xi_n^2 - \frac{7}{4} (1 + \xi_n^2) + \frac{15}{4} \xi_n^2 (1 + \xi_n^2) \right) h'(0) c(dx_n) \\ &\quad \left. \left. - 3i \xi_n (1 + \xi_n^2) \partial_{x_n} c(\xi') + i (1 + \xi_n^2) c(\xi') c(dx_n) \partial_{x_n} c(\xi') \right) \right] (x_0)|_{|\xi'|=1} \\ &= \sum_{j,l=1}^{n-1} X_j Y_l \xi_j \xi_l \frac{h'(0)(7 + 6i - (20 - 15i)\xi_n - (7 - 6i)\xi_n^2 + 15i\xi_n^3)}{(\xi_n - i)^5 (\xi_n + i)^4} \\ &\quad + X_n Y_n \frac{(3i - 11)\xi_n (1 - \xi_n^2) - 16i\xi_n + (13 + \frac{7}{2}i)(1 + \xi_n^2) - 16 - \frac{15}{2}\xi_n^2 (1 + \xi_n^2)}{(\xi_n - i)^2 (\xi_n + i)^4}, \end{aligned} \quad (5.45)$$

and

$$\begin{aligned} &\text{trace} \left[ \partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-1} (\nabla_X^\Psi \nabla_Y^\Psi D_\Psi^{-1}) \times 2 \left( \frac{c(\xi) c(\Psi) c(\xi)}{|\xi|^6} - \frac{c(\Psi)}{|\xi|^4} \right) \right] \\ &= \sum_{j,l=1}^{n-1} X_j Y_l \xi_j \xi_l \frac{-i\pi}{2} \text{trace}[c(dx_n) c(\Psi)] + X_n Y_n \frac{i\pi}{2} \text{trace}[c(dx_n) c(\Psi)]. \end{aligned} \quad (5.46)$$

From (5.41), (5.45) and (5.46), we get

$$\begin{aligned} \widetilde{\Phi}_5 &= \sum_{j,l=1}^{n-1} X_j Y_l \frac{-2\pi^2}{3} \text{trace}[c(dx_n) c(\Psi)] \Omega_3 dx' + X_n Y_n \frac{i\pi}{2} \text{trace}[c(dx_n) c(\Psi)] \Omega_3 dx' \\ &\quad + \sum_{j,l=1}^{n-1} X_j Y_l h'(0) \left( \frac{55}{26} + \frac{85i}{24} \right) \pi^2 \Omega_3 dx' + X_n Y_n \left( -\frac{50 + 7i}{16} \right) \pi \Omega_3 dx' \end{aligned} \quad (5.47)$$

Let  $X = X^T + X_n \partial_n$ ,  $Y = Y^T + Y_n \partial_n$ , then we have  $\sum_{j=1}^{n-1} X_j Y_j = g(X^T, Y^T)$ . Now  $\widehat{\Phi}$  is the sum of the cases (a), (b) and (c). Combining with the five cases, this yields

$$\begin{aligned} \widetilde{\Phi} = \sum_{i=1}^5 \widetilde{\Phi}_i &= \left[ \left( \frac{-4681}{24} + \frac{5i}{6} \right) \pi^2 + \left( \frac{55}{26} + \frac{85i}{24} \right) \right] g(X^T, Y^T) h'(0) \Omega_3 dx' + \left[ \left( \frac{451}{4} + 7i + \frac{\pi}{8} \right) h'(0) \right. \\ &\quad \left. - \frac{50+7i}{16} \right] X_n Y_n \pi \Omega_3 dx' - \left[ \frac{3\pi + 2\pi i}{4} X_n Y_n + \pi^2 g(X^T, Y^T) \right] \text{trace}[c(dx_n) c(\Psi)] \Omega_3 dx' \\ &\quad - \frac{9}{16} h'(0) \pi \Omega_3 dx' + \frac{3\pi i}{16} \text{trace}[-c(X) c(\Psi) X_n - c(Y) c(\Psi) Y_n] \Omega_3 dx'. \end{aligned} \quad (5.48)$$

So, we are reduced to prove the following.

Combining (5.3) and (5.48), we obtain Theorem 1.2.

When  $c(\Psi) = f$ , we can calculate  $\text{trace}[c(dx_n)f] = 0$  and  $\text{trace}[c(X)f] = 0$ , then we can straightforwardly assert the following corollary:

**Corollary 5.3.** *Let  $M$  be a 4-dimensional oriented compact manifolds with the boundary  $\partial M$  and the metric  $g^M$  as above, and let  $c(\Psi) = f$ , then*

$$\begin{aligned} \widetilde{\text{Wres}}[\pi^+(\nabla_X^\Psi \nabla_Y^\Psi D_\Psi^{-1}) \circ \pi^+(D_\Psi^{-3})] &= \frac{4\pi^2}{3} \int_M EG(X, Y) vol_g + \frac{1}{2} \int_M \text{trace}\left[\frac{1}{4}s - 3f^2\right] g(X, Y) dvol_M \\ &\quad + \int_{\partial M} \left\{ \left[ \left( \frac{-4681}{24} + \frac{5i}{6} \right) \pi^2 + \left( \frac{55}{26} + \frac{85i}{24} \right) \right] g(X^T, Y^T) h'(0) \Omega_3 + \left[ \left( \frac{451}{4} + 7i + \frac{\pi}{8} \right) h'(0) \right. \right. \\ &\quad \left. \left. - \frac{50+7i}{16} \right] X_n Y_n \pi \Omega_3 - \frac{9}{16} h'(0) \pi \Omega_3 \right\} dVol_{\partial M}, \end{aligned} \quad (5.49)$$

where  $s$  is the scalar curvature.

By (4.49), when  $c(\Psi) = c(\overline{X})$ , then we can straightforwardly assert the following corollary:

**Corollary 5.4.** *Let  $M$  be a 4-dimensional oriented compact manifolds with the boundary  $\partial M$  and the metric  $g^M$  as above, and let  $c(\Psi) = c(\overline{X})$ , then*

$$\begin{aligned} \widetilde{\text{Wres}}[\pi^+(\nabla_X^\Psi \nabla_Y^\Psi D_\Psi^{-1}) \circ \pi^+(D_\Psi^{-3})] &= \frac{4\pi^2}{3} \int_M EG(X, Y) vol_g - 4\pi^2 \int_M \left[ g(X, \nabla_Y^{TM} \overline{X}) + g(Y, \nabla_X^{TM} \overline{X}) \right] dvol_M \\ &\quad + \frac{1}{2} \int_M \text{trace}\left[\frac{1}{4}s + \sum_{j=1}^n \frac{1}{2} c(\overline{X}) c(e_j) c(\overline{X}) c(e_j) + |\overline{X}|^2\right] g(X, Y) dvol_M + \int_{\partial M} \left\{ \left[ \left( \frac{-4681}{24} + \frac{5i}{6} \right) \pi^2 + \left( \frac{55}{26} + \frac{85i}{24} \right) \right] \right. \\ &\quad \times g(X^T, Y^T) h'(0) \Omega_3 + \left[ \left( \frac{451}{4} + 7i + \frac{\pi}{8} \right) h'(0) - \frac{50+7i}{16} \right] X_n Y_n \pi \Omega_3 + \left[ (3\pi + 2\pi i) X_n Y_n + \pi^2 g(X^T, Y^T) \right] \\ &\quad \times g(\partial x_n, X) \Omega_3 - \frac{9}{16} h'(0) \pi \Omega_3 dx' + \left. \frac{3\pi i}{16} \text{trace}[4g(X, \overline{X}) X_n + 4g(Y, \overline{X}) Y_n] \Omega_3 \right\} dVol_{\partial M}, \end{aligned} \quad (5.50)$$

where  $s$  is the scalar curvature.

By (4.51), when  $c(\Psi) = c(X_1)c(X_2)$ , we compute  $\widetilde{\text{Wres}}[\pi^+(\nabla_X^\Psi \nabla_Y^\Psi D_\Psi^{-1}) \circ \pi^+(D_\Psi^{-3})]$ .

**Corollary 5.5.** *Let  $M$  be a 4-dimensional oriented compact manifolds with the boundary  $\partial M$  and the metric*

$g^M$  as above, and let  $c(\Psi) = c(X_1)c(X_2)$ , then

$$\begin{aligned} & \widetilde{\text{Wres}}[\pi^+(\nabla_X^\Psi \nabla_Y^\Psi D_\Psi^{-1}) \circ \pi^+(D_\Psi^{-3})] \\ &= \frac{4\pi^2}{3} \int_M EG(X, Y) vol_g + \frac{1}{2} \int_M \text{trace} \left[ \frac{1}{4}s + \sum_{j=1}^n \frac{1}{2}c(\Psi)c(e_j)c(\Psi)c(e_j) - (c(\Psi))^2 \right] g(X, Y) dvol_M \\ &+ \int_{\partial M} \left\{ \left[ \left( \frac{-4681}{24} + \frac{5i}{6} \right) \pi^2 + \left( \frac{55}{26} + \frac{85i}{24} \right) \right] g(X^T, Y^T) h'(0) \Omega_3 + \left[ \left( \frac{451}{4} + 7i + \frac{\pi}{8} \right) h'(0) \right. \right. \\ &\left. \left. - \frac{50+7i}{16} \right] X_n Y_n \pi \Omega_3 - \frac{9}{16} h'(0) \pi \Omega_3 dx' \right\} d\text{Vol}_{\partial M}, \end{aligned} \quad (5.51)$$

where  $s$  is the scalar curvature.

By (4.53), when  $c(\Psi) = c(X_1)c(X_2)c(X_3)$ , we compute that:

**Corollary 5.6.** Let  $M$  be a 4-dimensional oriented compact manifolds with the boundary  $\partial M$  and the metric  $g^M$  as above, and let  $c(\Psi) = c(X_1)c(X_2)c(X_3)$ , then

$$\begin{aligned} & \widetilde{\text{Wres}}[\pi^+(\nabla_X^\Psi \nabla_Y^\Psi D_\Psi^{-1}) \circ \pi^+(D_\Psi^{-3})] = \frac{4\pi^2}{3} \int_M EG(X, Y) vol_g + \frac{\pi^2}{2} \int_M \left\{ 8 \left[ g(X, X_1) \left( g(\nabla_Y^{TM} X_2, X_3) \right. \right. \right. \\ & \left. \left. \left. + g(X_2, \nabla_Y^{TM} X_3) \right) - g(X, X_2) \left( g(\nabla_Y^{TM} X_1, X_3) + g(X_1, \nabla_Y^{TM} X_3) \right) + g(X, X_3) \left( g(\nabla_Y^{TM} X_1, X_2) \right. \right. \\ & \left. \left. + g(X_1, \nabla_Y^{TM} X_2) \right) - g(X, \nabla_Y^{TM} X_1) g(X_2, X_3) - g(X, \nabla_Y^{TM} X_2) g(X_1, X_3) + g(X, \nabla_Y^{TM} X_3) g(X_1, X_2) \right] \right. \\ & \left. - 8 \left[ g(Y, X_1) \left( g(\nabla_X^{TM} X_2, X_3) + g(X_2, \nabla_X^{TM} X_3) \right) - g(Y, X_2) \left( g(\nabla_X^{TM} X_1, X_3) + g(X_1, \nabla_X^{TM} X_3) \right) \right. \right. \\ & \left. \left. + g(Y, X_3) \left( g(\nabla_X^{TM} X_1, X_2) + g(X_1, \nabla_X^{TM} X_2) \right) - g(Y, \nabla_Y^{TM} X_1) g(X_2, X_3) - g(Y, \nabla_Y^{TM} X_2) g(X_1, X_3) \right. \right. \\ & \left. \left. + g(Y, \nabla_Y^{TM} X_3) g(X_1, X_2) \right] dvol_M + \int_{\partial M} \left\{ \left[ \left( \frac{-4681}{24} + \frac{5i}{6} \right) \pi^2 + \left( \frac{55}{26} + \frac{85i}{24} \right) \right] g(X^T, Y^T) h'(0) \Omega_3 \right. \right. \\ & \left. \left. + \left[ \left( \frac{451}{4} + 7i + \frac{\pi}{8} \right) h'(0) - \frac{50+7i}{16} \right] X_n Y_n \pi \Omega_3 - \left[ \frac{3\pi+2\pi i}{4} X_n Y_n + \pi^2 g(X^T, Y^T) \right] \cdot \left( 4g(\partial x_n, X_1) g(X_2, X_3) \right. \right. \right. \\ & \left. \left. \left. - 4g(\partial x_n, X_2) g(X_1, X_3) + 4g(\partial x_n, X_3) g(X_1, X_2) \right) \Omega_3 - \frac{9}{16} h'(0) \pi \Omega_3 dx' + \frac{3\pi i}{4} \left[ \left( g(X, X_1) g(X_2, X_3) \right. \right. \right. \\ & \left. \left. \left. - g(X, X_2) g(X_1, X_3) + g(X, X_3) g(X_1, X_2) \right) X_n + \left( g(Y, X_1) g(X_2, X_3) - g(Y, X_2) g(X_1, X_3) \right. \right. \right. \\ & \left. \left. \left. + g(Y, X_3) g(X_1, X_2) \right) Y_n \right] \Omega_3 \right\} d\text{Vol}_{\partial M}, \end{aligned} \quad (5.52)$$

where  $s$  is the scalar curvature.

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