# FLOWS OF PIECEWISE ANALYTIC VECTOR FIELDS IN CONVEX POLYTOPE DECOMPOSITIONS 

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#### Abstract

We prove that for a convex polytope decomposition of a domain in $\mathbb{R}^{n}$, an integral curve of a piecewise analytic vector field is chopped by the decomposition into finitely many pieces. As a consequence, we prove the finiteness of the number of the edge flips in a discrete Yamabe flow.


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## 1. Introduction

Gu et al. 4] introduced a discrete Yamabe flow with edge flips, which are combinatorial changes of triangulations. It would be theoretically interesting and practically useful to know if there are only finitely many edge flips during the flow.

In this paper we prove that for a convex polytope decomposition of a domain in $\mathbb{R}^{n}$, an integral curve of a piecewise analytic vector field is chopped by the decomposition into finitely many pieces. As a consequence, we prove the finiteness of the edge flips in a discrete Yamabe flow mentioned above.
1.1. Setup and the main theorem. (a) A subset $H$ of $\mathbb{R}^{n}$ is called a half space if

$$
H=\left\{x \in \mathbb{R}^{n}: a_{1} x_{1}+\ldots+a_{n} x_{n} \geq b\right\}
$$

for some nonzero $\left(a_{1}, \ldots, a_{n}\right) \in \mathbb{R}^{n}$ and $b \in \mathbb{R}$. A subset $D$ of $\mathbb{R}^{n}$ is called a convex polytope if $D$ has nonempty interior and is the intersection of finitely many half spaces.
(b) Given $U \subseteq \mathbb{R}^{n}$, a vector field on $U$ is a function from $U$ to $\mathbb{R}^{n}$.
(c) Given $U \subseteq \mathbb{R}^{n}$, a function $f$ on $U$ is called analytic if there exists an open superset $\tilde{U}$ of $U$ and an analytic function $\tilde{f}$ on $\tilde{U}$ such that $\tilde{f} \equiv f$ on $U$.
(d) Given $M \subseteq \mathbb{R}^{n}$, a flow on $M$ is a continuous function from $[T, \infty)$ to $M$ for some $T \in \mathbb{R}$.

Here is our main theorem.
Theorem 1.1. Let $U$ be an open domain in $\mathbb{R}^{n}$ and $U \subseteq \cup_{i} D_{i}$ be a finite closed cover such that each $D_{i}$ is a convex polytope. Suppose $M \ni 0$ is an analytic submanifold in $U$ and $V(x)$ is a $C^{1}$ vector field on $U$ such that
(i) $V(x)$ is analytic on each $D_{i} \cap U$,
(ii) $D V(x)\left(T_{x} M\right) \subseteq T_{x} M$ for all $x \in M$,
(iii) $V(0)=0$, and
(iv) $\left.D V(0)\right|_{T_{0} M}$ is diagonalizable and has only negative eigenvalues.

Suppose $x(t)$ is a flow on $M$ such that
(i) $x^{\prime}(t)=V(x(t))$ on $[0, \infty)$, and
(ii) $\lim _{t \rightarrow \infty} x(t)=0$.

Then there exists a sequence of finitely many increasing numbers $0=t_{0}<t_{1}<$ $\ldots<t_{n-1}<t_{n}=\infty$ such that for all $i \in\{1, \ldots, n\}, x\left(\left[t_{i-1}, t_{i}\right)\right) \subseteq D_{j}$ for some $j$.

To prove Theorem 1.1, by compactness we only need to prove Theorems 1.2 and 1.3 , describing the local behavior and the limiting behavior respectively.

Theorem 1.2. Let $U$ be an open domain in $\mathbb{R}^{n}$ and $U \subseteq \cup_{i} D_{i}$ be a finite closed cover such that each $D_{i}$ is a convex polytope. Suppose $V(x)$ is a $C^{1}$ vector field on $U$ and is analytic on each $D_{i} \cap U$. If $x^{\prime}(t)=V(x(t))$ near 0 , then there exists $\epsilon>0$ such that $x([0, \epsilon)) \subseteq D_{i}$ for some $i$.

Theorem 1.3. Let $U$ be an open domain in $\mathbb{R}^{n}$ and $U \subseteq \cup_{i} D_{i}$ be a finite closed cover such that each $D_{i}$ is a convex polytope. Let $M \ni 0$ be an m-dim analytic submanifold in $U$. Suppose $V(x)$ is a $C^{1}$ vector field on $U$ such that
(i) $V(x)$ is analytic on each $D_{i} \cap U$,
(ii) $D V(x)\left(T_{x} M\right) \subseteq T_{x} M$ for all $x \in M$,
(iii) $V(0)=0$, and
(iv) $\left.D V(0)\right|_{T_{0} M}$ is diagonalizable and has $m$ negative eigenvalues $\lambda_{1} \geq \ldots \geq \lambda_{m}$. Suppose $x(t)$ is a flow on $M$ such that
(i) $x^{\prime}(t)=V(x(t))$ on $[0, \infty)$, and
(ii) $\lim _{t \rightarrow \infty} x(t)=0$.

Then there exists $T>0$ such that $x([T, \infty)) \subseteq D_{j}$ for some $j$.
1.2. Finiteness of the edge flips in the discrete Yamabe flow. As a consequence of Theorem 1.1, we prove the finiteness of the number of edge flips in the discrete Yamabe flow introduced in 4]. One may refer [4] for backgrounds. Other related work on discrete conformality and geodesic triangulations could be found in $1-17$. Here we give a very brief explanation.

A discrete conformal class in [4] is parametrized by $\mathbb{R}^{n}$ and has a natural finite cell decomposition $\mathbb{R}^{n}=\cup_{i} U_{i}$. A discrete Yamabe flow $u(t)$ is an integral curve of a $C^{1}$ vector field $F(u)$ on $\mathbb{R}^{n}$, satisfying that
(a) $F(u)$ is analytic on each $U_{i}$,
(b) $F(u) \in \mathbf{1}_{n}^{\perp}=\left\{\left(x_{1}, \ldots, x_{n}\right) \in \mathbb{R}^{n}: x_{1}+\ldots+x_{n}=0\right\}$ for all $u$, and
(c) $D F(u)$ is always symmetric and negative definite as a transformation on $\mathbf{1}_{n}^{\perp}$.
(d) $F(\bar{u})=0$ for some $\bar{u} \in u(0)+\mathbf{1}_{n}^{\perp}$.

A discrete Yamabe flow $u(t)$ always exists on $[0, \infty)$ and satisfies that
(a) $u(t)-u(0) \in \mathbf{1}_{n}^{\perp}$.
(b) $u^{\prime}(t)=F(u(t))$, and
(c) $\lim _{t \rightarrow \infty} u(t)=\bar{u}$.

Furthermore, the analytic change of coordinates $\left(u_{1}, \ldots, u_{n}\right) \mapsto\left(e^{-2 u_{1}}, \ldots, e^{-2 u_{n}}\right)$ will map each cell $U_{i}$ to $D_{i} \cap \mathbb{R}_{>0}^{n}$ for some convex polytope $D_{i}$.

If we let $M$ be the image of $u(0)+\mathbf{1}_{n}^{\perp}$ under this coordinate change, then Theorem 1.1 implies the finiteness of the number of switches in the new coordinates.

### 1.3. Notations and a convex geometric estimate.

(a) Given $x \in \mathbb{R}^{n}\left(\right.$ or $\left.\mathbb{Z}^{n}\right)$, denote $x=\left(x_{1}, \ldots, x_{n}\right)$.
(b) Given $x \in \mathbb{R}^{n}$ (or $\mathbb{Z}^{n}$ ), denote $|x|=|x|_{1}=\left|x_{1}\right|+\ldots+\left|x_{n}\right|$.
(c) Given $x \in \mathbb{R}^{n}$ (or $\mathbb{Z}^{n}$ ), denote $|x|_{2}=\sqrt{x_{1}^{2}+\ldots+x_{n}^{2}}$.
(d) Given $x \in \mathbb{R}^{n}$ (or $\mathbb{Z}^{n}$ ), denote $|x|_{\infty}=\max _{i}\left|x_{i}\right|$.
(e) Given $x \in \mathbb{R}^{n}, I \in \mathbb{Z}^{n}$, denote $x^{I}=x_{1}^{I_{1}} \ldots x_{n}^{I_{n}}$.
(f) $\mathbf{1}_{n}$ denotes $(1, \ldots, 1) \in \mathbb{R}^{n}$.
(g) Given $U \subseteq \mathbb{R}^{n}$ and $p \in \mathbb{R}^{n}$, denote $d(p, U)=\inf _{q \in U}|p-q|_{2}$.
(h) Given a half space $H \subseteq \mathbb{R}^{n}, d_{s}(x, H)$ denotes the singed distance from $x$ to $H$. To be specific, $d_{s}(\cdot, H)$ is a linear function on $\mathbb{R}^{n}$ such that $d_{s}(x, H)=d(x, H)$ whenever $d(x, H)>0$.

Lemma 1.4. Suppose $D=\cap_{i=1}^{m} H_{i}$ is a convex polytope in $\mathbb{R}^{n}$, and $H_{i}$ 's are half spaces. Then for all $p \in \operatorname{int}(D)$ and $x \notin D$ we have that

$$
\frac{\min _{i} d\left(p, \partial H_{i}\right)}{|x-p|_{2}} \cdot d(x, D) \leq \max _{i} d\left(x, H_{i}\right) \leq d(x, D)
$$

Proof. The second part of the inequality is obvious. Denote $q$ as the intersection of $\partial D$ and the straight arc from $p$ to $x$. Assume $q \in \partial H_{j}$ and then

$$
\frac{|x-q|_{2}}{|x-p|_{2}}=\frac{d\left(x, H_{j}\right)}{d\left(p, \partial H_{j}\right)+d\left(x, H_{j}\right)} \leq \frac{d\left(x, H_{j}\right)}{d\left(p, \partial H_{j}\right)} \leq \frac{\max _{i} d\left(x, H_{i}\right)}{\min _{i} d\left(p, \partial H_{i}\right)}
$$

Notice that $|x-q|_{2} \geq d(x, D)$ and we are done.

## 2. LOCAL FINITENESS

Proof of Theorem 1.2. Without loss of generality, assume $x(0)=0$. It suffices to show that for any $D_{i}$ there exists $\epsilon>0$ such that $x([0, \epsilon]) \subseteq D_{i}$ or $x([0, \epsilon]) \cap D_{i}=$ $\{0\}$.

Fix $D=D_{i}$. Let $V_{*}$ be an analytic vector field on an open superset of $D \cap U$ such that $V_{*}(x)=V(x)$ on $D \cap U$. Let $y(t)$ be the local analytic solution to

$$
\left\{\begin{array}{l}
y(0)=0 \\
y^{\prime}(t)=V_{*}(y(t))
\end{array} .\right.
$$

If there exists $\epsilon>0$ such that $y([0, \epsilon]) \subseteq D$, then by Picard's uniqueness theorem $x(t)=y(t) \in D$ on $[0, \epsilon]$.

So we may assume that for all $\epsilon>0, y([0, \epsilon]) \nsubseteq D$. Suppose $D=\cap_{j=1}^{m} H_{j}$ where $H_{j}$ 's are half spaces. For a fixed $j, d\left(y(t), H_{j}\right)=0$ for all small $t$ or $d\left(y(t), H_{j}\right)=$ $a t^{k}+o\left(t^{k}\right)$ for some $a>0$ and $k \in \mathbb{Z}_{\geq 1}$. Since we cannot have that $d\left(y(t), H_{j}\right)=0$ for all small $t$ and for all $j$, we have that

$$
\max _{j} d\left(y(t), H_{j}\right)=a t^{k}+o\left(t^{k}\right)
$$

for some $a>0$ and $k \in \mathbb{Z}_{\geq 1}$. By Lemma 1.4, there exists constants $a_{1}, a_{2}>0$ such that $a_{1} t^{k} \leq d(y(t), D) \leq \overline{a_{2}} t^{k}$ for small $t$.

We will prove $x(t)=y(t)+o\left(t^{k}\right)$, and then $x(t) \notin D$ for small $t>0$. We will prove $x(t)=y(t)+o\left(t^{j}\right)$ inductively for all $j=0,1, \ldots, k$. Obviously $x(t)=y(t)+o(1)$. Now we assume $x(t)=y(t)+o\left(t^{j}\right)$ and $j \leq k-1$. Denote $y_{*}(t)$ as the closest point to $y(t)$ in $D$, and then $y_{*}(t)=y(t)+o\left(t^{j}\right)=x(t)+o\left(t^{j}\right)$ and

$$
\begin{gathered}
V(x(t))=V\left(y_{*}(t)\right)+o\left(t^{j}\right)=V_{*}\left(y_{*}(t)\right)+o\left(t^{j}\right) \\
=V_{*}(y(t))+o\left(t^{j}\right)=y^{\prime}(t)+o\left(t^{j}\right)
\end{gathered}
$$

and

$$
x(t)=\int_{0}^{t} V(x(s)) d s=y(t)+o\left(t^{j+1}\right)
$$

## 3. $\lambda$-SERIES AND CONVERGENCE

3.1. $n$-dim analytic functions. An $n$-dim analytic function $f(x)$ near 0 in $\mathbb{R}^{m}$ can be written as

$$
f(x)=\sum_{I \in \mathbb{Z}_{\geq 0}^{m}} b_{I} x^{I}
$$

where $b_{I} \in \mathbb{R}^{n}$ and

$$
\left|b_{I}\right| \leq M^{|I|+1}
$$

for some constant $M>0$.
Suppose $f(x)=\sum_{I} b_{I} x^{I}$ is an $n$-dim analytic function near 0 in $\mathbb{R}^{m}$, and $g(x)=$ $\sum_{J} c_{J} x^{J}$ is an $m$-dim analytic function near 0 in $\mathbb{R}^{k}$ with $c_{0}=0$. Then $(f \circ g)(x)$ is an $n$-dim function near 0 in $\mathbb{R}^{k}$, defined as

$$
(f \circ g)(x)=f(g(x))
$$

Furthermore, $f \circ g$ is analytic near 0 and

$$
(f \circ g)(x)=f(g(x))=\sum_{I} b_{I}\left(\sum_{J} c_{J} x^{J}\right)^{I}=\sum_{J} d_{J} x^{J}
$$

where

$$
\begin{equation*}
d_{J}=\sum_{I} b_{I} \sum_{J_{i, j}: \sum_{i=1}^{m} \sum_{j=1}^{I_{i}} J_{i, j}=J} \prod_{i, j}\left(c_{J_{i, j}}\right)_{i} . \tag{3.1}
\end{equation*}
$$

Here we let $d_{0}=b_{0}$ as a convention and denote $\left(c_{J_{i, j}}\right)_{i}$ as the $i$-th component of $c_{J_{i, j}}$. Notice that the summation in equation (3.1) is well-defined since it contains only finitely many nonzero terms. This is because $c_{0}=0$ and if $|I|>|J|$ then $\sum_{i=1}^{m} \sum_{j=1}^{I_{i}} J_{i, j}=J$ forces some $J_{i, j}$ to be 0 .
3.2. $\lambda$-series. $P(t)=\sum_{i=0}^{k} a_{i} t^{i}$ is called an $n$-dim polynomial if $a_{i} \in \mathbb{R}^{n}$ for all $i$. Given $\lambda \in \mathbb{R}_{<0}^{m}$, we define an $n$-dim $\lambda$-series as a formal expression

$$
x(t)=x(t ; \lambda)=\sum_{J \in \mathbb{Z}_{\geq 0}^{m}} P_{J}(t) e^{\lambda \cdot J t}
$$

where each $P_{J}(t)$ is an $n$-dim polynomial. To be rigorous, such an $n$-dim $\lambda$-series could be represented by a map $J \mapsto P_{J}$ from $\mathbb{Z}_{\geq 0}^{m}$ to the space of $n$-dim polynomials. However, we will always represent a $\lambda$-series as the above infinite summation for better intuition. The formal derivative of a $\lambda$-series is also a $\lambda$-series naturally defined by

$$
\left(\sum_{J \in \mathbb{Z}_{\geq 0}^{m}} P_{J}(t) e^{\lambda \cdot J t}\right)^{\prime}=\sum_{J \in \mathbb{Z}_{\geq 0}^{m}}\left(P_{J}^{\prime}(t)+\lambda \cdot J P_{J}(t)\right) e^{\lambda \cdot J t} .
$$

If $f(x)=\sum_{I \in \mathbb{Z}_{\geq 0}^{n}} b_{I} x^{I}$ is an analytic function in $n$ variables and

$$
x(t)=\sum_{J \in \mathbb{Z}_{\geq 0}^{m}} P_{J}(t) e^{\lambda \cdot J t}
$$

is an $n$-dim $\lambda$-series with $P_{0}(t)=0$, we can heuristically formally expand $f(x(t))$ as

$$
f(x(t))=\sum_{I \in \mathbb{Z}_{\geq 0}^{n}} b_{I}\left(\sum_{J \in \mathbb{Z}_{\geq 0}^{m}} P_{J}(t) e^{\lambda \cdot J t}\right)^{I}=\sum_{J \in \mathbb{Z}_{\geq 0}^{m}} Q_{J}(t) e^{\lambda \cdot J t}
$$

where

$$
\begin{equation*}
Q_{J}(t)=\sum_{I \in \mathbb{Z}_{\geq 0}^{n}} b_{I} \sum_{J_{i, j}: \sum_{i=1}^{n} \sum_{j=1}^{I_{i}} J_{i, j}=J} \prod_{i, j}\left(P_{J_{i, j}}\right)_{i}(t) \tag{3.2}
\end{equation*}
$$

Here we let $Q_{0}=b_{0}$ as a convention and denote $\left(P_{J_{i, j}}\right)_{i}(t)$ as the $i$-th component of the $n$-dim polynomial $P_{J_{i, j}}(t)$. So given such analytic function $f(x)$ and $\lambda$-series $x(t)$, we define $(f \circ x)(t)$ as a formal $\lambda$-series

$$
(f \circ x)(t)=\sum_{J} Q_{J}(t) e^{\lambda \cdot J t}
$$

where $Q(t)$ is defined as in equation (3.2). Notice that the summation in equation (3.2) is well-defined since it contains only finitely many nonzero terms. This is because $P_{0}(t)=0$ and if $|I|>|J|$ then $\sum_{i=1}^{n} \sum_{j=1}^{I_{i}} J_{i, j}=J$ forces some $J_{i, j}$ to be 0 . Also notice that equation (3.2) is similar to equation (3.1). This is because a $\lambda$-series could be viewed as a power series with $m$ variables $e^{\lambda_{1} t}, \ldots, e^{\lambda_{m} t}$ and polynomial-valued coefficients.

Proposition 3.1. Suppose $f(x)=\sum_{I} b_{I} x^{I}$ is a $k$-dim analytic function in $n$ variables, $g(x)=\sum_{I} c_{I} x^{I}$ is an n-dim analytic function in $m$ variables with $c_{0}=0$, and $x(t)=\sum_{I} P_{I}(t) e^{\lambda \cdot J t}$ is an m-dim $\lambda$-series with $P_{0}=0$. Then

$$
(f \circ(g \circ x))(t)=((f \circ g) \circ x)(t)
$$

Proof. This can be shown by a routine but tedious computation, using equations (3.1) and (3.2).
3.3. Convergence of a $\lambda$-series. We have a simple sufficient condition for a $\lambda$ series to converge.
Proposition 3.2. Given $\lambda \in \mathbb{R}_{<0}^{m}$ and a $\lambda$-series $x(t)=\sum_{J} P_{J}(t) e^{\lambda \cdot J t}$, suppose there exists $T>0$ and $q \in \mathbb{R}_{>0}$ such that $\left|P_{J}(t)\right| \leq t^{q|J|}$ for all $J \in \mathbb{Z}_{\geq 0}^{m}$ and $t \geq T$. Then
(a) the series $x(t)$ converges absolutely and uniformly for sufficiently large $t$, and
(b) for any $a \in \mathbb{R}$, as a real-valued function

$$
x(t)=\sum_{J \in \mathbb{Z}_{\geq 0}^{m}: \lambda \cdot J \geq a} P_{J}(t) e^{\lambda \cdot J t}+o\left(e^{a t}\right)
$$

as $t \rightarrow \infty$.
Proof. (a) Without loss of generality, we may assume that $T$ is so large that $t^{q} e^{\lambda_{i} t}$ is decreasing and less than 1 on $[T, \infty)$ for all $i=1, \ldots, m$.

For $t \in[T, \infty)$

$$
\begin{gathered}
\sum_{|J|_{\infty}>n}\left|P_{J}(t) e^{\lambda \cdot J t}\right| \leq \sum_{|J|_{\infty}>n} t^{q|J|} e^{\lambda \cdot J t} \leq \sum_{|J|_{\infty}>n} T^{q|J|} e^{\lambda \cdot J T} \\
=\sum_{J} T^{q|J|} e^{\lambda \cdot J T}-\sum_{|J|_{\infty} \leq n} T^{q|J|} e^{\lambda \cdot J T} \\
=\prod_{i=1}^{m}\left(\sum_{j=0}^{\infty}\left(T^{q} e^{\lambda_{i} T}\right)^{j}\right)-\prod_{i=1}^{m}\left(\sum_{j=0}^{n}\left(T^{q} e^{\lambda_{i} T}\right)^{j}\right)
\end{gathered}
$$

which is independent on $t$ and goes to 0 as $n$ goes to infinity, since for all $i$

$$
\sum_{j=0}^{n}\left(T^{k} e^{\lambda_{i} T}\right)^{j} \rightarrow \sum_{j=0}^{\infty}\left(T^{q} e^{\lambda_{i} T}\right)^{j}=\frac{1}{1-T^{q} e^{\lambda_{i} T}}<\infty
$$

as $n \rightarrow \infty$.
(b) Pick a large $n$ such that $\lambda \cdot J<a$ for all $J$ with $|J|_{\infty}>n$. Then

$$
\sum_{\lambda \cdot J<a} P_{J}(t) e^{\lambda \cdot J t}=\sum_{|J|_{\infty}>n} P_{J}(t) e^{\lambda \cdot J t}+o\left(e^{a t}\right)
$$

Then we finish the proof by showing that for sufficiently large $t$

$$
\begin{aligned}
& \sum_{|J|_{\infty}>n} P_{J}(t) e^{\lambda \cdot J t} \leq \sum_{|J|_{\infty}>n} t^{q|J|} e^{\lambda \cdot J t} \\
&=\sum_{J} t^{q|J|} e^{\lambda \cdot J t}-\sum_{|J|_{\infty} \leq n} t^{q|J|} e^{\lambda \cdot J t} \\
&=\prod_{i=1}^{m}\left(\sum_{j=0}^{\infty}\left(t^{q} e^{\lambda_{i} t}\right)^{j}\right)-\prod_{i=1}^{m}\left(\sum_{j=0}^{n}\left(t^{q} e^{\lambda_{i} t}\right)^{j}\right)
\end{aligned}
$$

$$
=\sum_{i=1}^{m} O\left(\left(t^{q} e^{\lambda_{i} t}\right)^{n+1}\right)=o\left(e^{a t}\right)
$$

Proposition 3.3. Given $\lambda \in \mathbb{R}_{<0}^{m}$ and an $n$-dim $\lambda$-series $x(t)=\sum_{J} P_{J}(t) e^{\lambda \cdot J t}$ with $P_{0}(t)=0$, suppose there exists $T>1$ and $q \in \mathbb{R}_{>0}$ such that $\left|P_{J}(t)\right| \leq t^{q|J|}$ for all $J \in \mathbb{Z}_{\geq 0}^{m}$ and $t \geq T$. If $f(x)=\sum_{I} b_{I} x^{I}$ is a 1-dim analytic function on $a$ neighborhood of 0 in $\mathbb{R}^{n}$, then there exists $r>0$ such that the $\lambda$-series

$$
(f \circ x)(t)=\sum_{J} Q_{J}(t) e^{\lambda \cdot J t}
$$

satisfies that $\left|Q_{J}(t)\right| \leq t^{r|J|}$ for all $J$ and $t \geq T$. Furthermore, for sufficiently large $t$,

$$
\begin{equation*}
(f \circ x)(t)=f(x(t)) \tag{3.3}
\end{equation*}
$$

Here $(f \circ x)(t)$ and $x(t)$ are viewed as two functions induced by taking the limit of the corresponding $\lambda$-series. Equation (3.3) means that the limit of the $\lambda$-series $(f \circ x)(t)$ is equal to the analytic function $f$ evaluated at the limit of the $\lambda$-series $x(t)$.

Proof. Without loss of generality, assume $f(0)=0$. Suppose $t \geq T$ and by equation (3.2),

$$
\begin{gathered}
\left|Q_{J}(t)\right| \leq \tilde{Q}_{J}(t):=\sum_{I \in \mathbb{Z}_{\geq 0}^{n}}\left|b_{I}\right| \sum_{J_{i, j}: \sum_{i=1}^{n} \sum_{j=1}^{I_{i}} J_{i, j}=J} \prod_{i, j}\left|\left(P_{J_{i, j}}\right)_{i}(t)\right| \\
=\sum_{I \in \mathbb{Z}_{\geq 0}^{n}}\left|b_{I}\right| \sum_{J_{i, j} \neq 0: \sum_{i=1}^{n} \sum_{j=1}^{I_{i}}} \prod_{J_{i, j}=J}\left|\left(P_{J_{i, j}}\right)_{i}(t)\right| \\
\leq \sum_{I \in \mathbb{Z}_{\geq 0}^{n}}\left|b_{I}\right| \sum_{J_{i, j} \neq 0: \sum_{i=1}^{n} \sum_{j=1}^{I_{i}} J_{i, j}=J} \prod_{i, j} t^{q\left|J_{i, j}\right|} \\
=t^{q|J|} \sum_{I \in \mathbb{Z}_{\geq 0}^{n}}\left|b_{I}\right| \sum_{J_{i, j} \neq 0: \sum_{i=1}^{n} \sum_{j=1}^{I_{i}} J_{i, j}=J} 1
\end{gathered}
$$

where

$$
c_{J}:=\sum_{I \in \mathbb{Z}_{\geq 0}^{n}}\left|b_{I}\right| \sum_{J_{i, j} \neq 0: \sum_{i=1}^{n} \sum_{j=1}^{I_{i}} J_{i, j}=J} 1
$$

is the coefficient of $x^{J}$ in the local analytic function

$$
\sum_{I}\left|b_{I}\right|\left(\mathbf{1}_{n} \sum_{i=1}^{\infty} x^{i}\right)^{I}=f_{*}\left(\frac{x}{1-x} \mathbf{1}_{n}\right)
$$

where $f_{*}(x)=\sum_{I}\left|b_{I}\right| x^{I}$ is analytic near 0 .
So $c_{J} \leq R^{|J|}$ for some $R>0$, and for all $J \in \mathbb{Z}_{\geq 0}^{m}$ and $t \geq T$

$$
\left|Q_{J}(t)\right| \leq \tilde{Q}_{J}(t) \leq R^{|J|} t^{q|J|} \leq t^{\left(q+\log _{T} R\right)|J|}
$$

Suppose $t$ is sufficiently large so that $x(t)$ and $(f \circ x)(t)$ and $\sum_{J} \tilde{Q}_{J} e^{\lambda \cdot J t}$ all converge absolutely and $f$ is analytic near $x(t)$. To prove $(f \circ x)(t)=f(x(t))$ we need to show that

$$
\sum_{J} Q_{J}(t) e^{\lambda \cdot J t}=\sum_{I} b_{I}\left(\sum_{J} P_{J}(t) e^{\lambda \cdot J t}\right)^{I}
$$

Let

$$
A_{N, M}=\sum_{|I| \leq N} b_{I}\left(\sum_{J:|J| \leq M} P_{J}(t) e^{\lambda \cdot J t}\right)^{I}
$$

and

$$
B_{N}=\sum_{J:|J| \leq N} Q_{J}(t) e^{\lambda \cdot J t}
$$

If $N \leq M$, then

$$
\begin{aligned}
\left|A_{N, M}-B_{N}\right|=\mid \sum_{J:|J|>N} e^{\lambda \cdot J t} & \sum_{I:|I| \leq N} b_{I} \sum_{J_{i, j} \neq 0: \sum_{i=1}^{n} \sum_{j=1}^{I_{i}} J_{i, j}=J,\left|J_{i, j}\right| \leq M} \prod_{i, j}\left(P_{J_{i, j}}\right)_{i}(t) \mid \\
& \leq \sum_{J:|J|>N} \tilde{Q}_{J}(t) e^{\lambda \cdot J t}
\end{aligned}
$$

which is independent on $M$ and goes to 0 as $N$ goes to infinity. Let $M \rightarrow \infty$ and then $N \rightarrow \infty$, and then we get

$$
\left|\sum_{I} b_{I}\left(\sum_{J} P_{J}(t) e^{\lambda \cdot J t}\right)^{I}-\sum_{J} Q_{J}(t) e^{\lambda \cdot J t}\right|=0
$$

## 4. Limiting behavior for analytic vector fields

4.1. Formal $\lambda$-series solutions to ODEs. Let $V(x)=\sum_{I} b_{I} x^{I}$ be an analytic vector field on a neighborhood of 0 in $\mathbb{R}^{m}$ such that $V(0)=0$ and

$$
\Lambda:=D V(0)=\operatorname{diag}\left(\lambda_{1}, \ldots, \lambda_{m}\right)
$$

for some $\lambda=\left(\lambda_{1}, \ldots, \lambda_{m}\right) \in \mathbb{R}_{<0}^{m}$. A $\lambda$-series

$$
x(t)=\sum_{J} P_{J}(t) e^{\lambda \cdot J t}
$$

with $P_{0}(t)=0$ formally satisfies $x^{\prime}=V(x)$ if and only if for all $J \neq 0$,

$$
P_{J}^{\prime}+\lambda \cdot J P_{J}=\sum_{I} b_{I} \sum_{J_{i, j}: \sum_{i=1}^{n} \sum_{j=1}^{I_{i}} J_{i, j}=J} \prod_{i, j}\left(P_{J_{i, j}}\right)_{i}
$$

which is equivalent to that

$$
\begin{equation*}
\left(\frac{d}{d t}-(\Lambda-\lambda \cdot J)\right) P_{J}=Q_{J} \tag{4.1}
\end{equation*}
$$

where

$$
\begin{equation*}
Q_{J}=\sum_{I:|I| \geq 2} b_{I} \sum_{J_{i, j}: \sum_{i=1}^{m} \sum_{j=1}^{I_{i}} J_{i, j}=J} \prod_{i, j}\left(P_{J_{i, j}}\right)_{i} \tag{4.2}
\end{equation*}
$$

4.2. Construction of $\lambda$-series solutions. Given $u \in \mathbb{R}$ and a 1-dim polynomial $Q(t)$, define

$$
\left(\frac{d}{d t}-u\right)^{-1} Q(t)=-u^{-1}\left(1+u^{-1} \frac{d}{d t}+u^{-2} \frac{d^{2}}{d t^{2}}+\ldots\right) Q(t)
$$

if $u \neq 0$ and

$$
\left(\frac{d}{d t}-u\right)^{-1} Q(t)=\int_{0}^{t} Q(s) d s
$$

if $u=0$. It is straightforward to verify that

$$
\left(\frac{d}{d t}-u\right)\left(\frac{d}{d t}-u\right)^{-1} Q(t)=Q(t)
$$

Given an $m$-dim diagonal matrix $U=\operatorname{diag}\left(u_{1}, \ldots, u_{m}\right)$ and an $m$-dim polynomial $Q(t)$,

$$
P(t)=\left(\frac{d}{d t}-U\right)^{-1} Q(t)
$$

is defined to be such that

$$
P_{i}(t)=\left(\frac{d}{d t}-u_{i}\right)^{-1} Q_{i}(t)
$$

for all $i=1, \ldots, m$. Clearly we have that

$$
\left(\frac{d}{d t}-U\right)\left(\frac{d}{d t}-U\right)^{-1} Q(t)=Q(t)
$$

Denote

$$
\begin{gathered}
\vec{e}_{1}=(1,0, \ldots, 0) \in \mathbb{R}^{m} \\
\vdots \\
\vec{e}_{m}=(0, \ldots, 0,1) \in \mathbb{R}^{m}
\end{gathered}
$$

Given any $c \in \mathbb{R}^{m}$, we construct a formal $\lambda$-series solution

$$
x(t ; c)=\sum_{J} P_{J}(t ; c) e^{\lambda \cdot J t}
$$

to $x^{\prime}=V(x)$ as follows.

$$
\begin{gather*}
P_{0}(t ; c)=0  \tag{4.3}\\
P_{\vec{e}_{i}}(t ; c)=c_{i} \vec{e}_{i}  \tag{4.4}\\
P_{J}(t ; c)=\left(\frac{d}{d t}-(\Lambda-\lambda \cdot J)\right)^{-1} Q_{J}(t ; c), \tag{4.5}
\end{gather*}
$$

if $|J| \geq 2$ where

$$
\begin{equation*}
Q_{J}(t ; c)=\sum_{I \in \mathbb{Z}_{\geq 0}^{m}:|I| \geq 2} b_{I} \sum_{J_{k, l}: \sum_{i=1}^{m} \sum_{j=1}^{I_{i}} J_{i, j}=J} \prod_{i, j}\left(P_{J_{i, j}}\right)_{i}(t ; c) . \tag{4.6}
\end{equation*}
$$

It is easy to verify such $x(t ; c)$ 's are formal $\lambda$-series solutions to $x^{\prime}=V(x)$ parameterized by $c \in \mathbb{R}^{m}$.
4.3. Dominating functions. Given a 1-dim polynomial in one variable

$$
P(t)=\sum_{i=0}^{q} a_{i} t^{i}
$$

we denote

$$
P^{*}(t)=\left(\sum_{i=0}^{q} a_{i} t^{i}\right)^{*}=\sum_{i=0}^{q}\left|a_{i} t^{i}\right|
$$

as a 1-dim function in one variable. Given an $n$-dim polynomial in one variable $P(t)=\left(P_{1}(t), \ldots, P_{n}(t)\right)$, then denote

$$
\operatorname{deg}(P)=\max _{i} \operatorname{deg}\left(P_{i}\right)
$$

and

$$
P^{*}(t)=\max _{i} P_{i}^{*}(t)
$$

It is routine to verify the following property.
Proposition 4.1. Suppose $t \in \mathbb{R}$ and $a \in \mathbb{R}$ and $P, \tilde{P}$ are two n-dim polynomials in one variable. Then
(a) $(P+\tilde{P})^{*}(t) \leq P^{*}(t)+\tilde{P}^{*}(t)$,
(b) $(a P(t))^{*}=|a| P^{*}(t)$, and
(c) $(P(t) \tilde{P}(t))^{*} \leq P^{*}(t) \tilde{P}^{*}(t)$ if $n=1$.

Lemma 4.2. Suppose $u \geq 2$ and $Q(t)$ are 1-dim polynomial and

$$
P(t)=\left(\frac{d}{d t}-u\right)^{-1} Q(t)
$$

Then

$$
P^{*}(t) \leq Q^{*}(t) \quad \text { for all } \quad t \geq \frac{2 \operatorname{deg}(Q)}{u}
$$

Proof. Assume $Q(t)=a_{q} t^{q}+\ldots+a_{1} t+a_{0}$ and denote

$$
\mathcal{A}=\left(\frac{d}{d t}-u\right)^{-1}=-u^{-1}\left(1+u^{-1} \frac{d}{d t}+u^{-2} \frac{d^{2}}{d t^{2}}+\ldots\right)
$$

If $0 \leq i \leq q$ and $t \geq 2 q / u \geq 2 i / u$,

$$
\left(\mathcal{A}\left(t^{i}\right)\right)^{*}=u^{-1} t^{i}\left(1+\frac{i}{u t}+\frac{i(i-1)}{(u t)^{2}}+\ldots\right) \leq \frac{1}{2} t^{i}\left(1+\frac{1}{2}+\frac{1}{2^{2}}+\ldots\right) \leq t^{i}
$$

and then

$$
P^{*}(t)=\left(\sum_{i=0}^{q} a_{i} \mathcal{A}\left(t^{i}\right)\right)^{*} \leq \sum_{i=0}^{q}\left|a_{i}\right|\left(\mathcal{A}\left(t^{i}\right)\right)^{*} \leq \sum_{i=1}^{q}\left|a_{i}\right| t^{i}=Q^{*}(t) .
$$

We have a similar result for $n$-dim polynomials as a direct consequence.
Corollary 4.3. Suppose $U=\operatorname{diag}\left(u_{1}, \ldots, u_{n}\right)$ is a diagonal matrix such that $u_{i} \geq 2$ for all $i$. If $P(t), Q(t)$ are $n$-dim polynomials such that

$$
\left(\frac{d}{d t}-U\right) P(t)=Q(t)
$$

then

$$
P^{*}(t) \leq Q^{*}(t) \quad \text { for all } \quad t \geq \frac{2 \operatorname{deg}(Q)}{\min _{i} u_{i}}
$$

### 4.4. Convergence of $\lambda$-series solutions.

Lemma 4.4. Let $n \in \mathbb{Z}_{>0}$ and $M>0$ and $a_{J} \in \mathbb{R}$ for all $J \in \mathbb{Z}_{\geq 0}^{m}$. $a_{J}$ is defined inductively as follows.

$$
\begin{array}{lr}
a_{0}=0 \\
a_{J}=1 \\
a_{J}=\sum_{I \in \mathbb{Z}_{\geq 0}^{n}:|I| \geq 2} M^{|I|} \sum_{J_{i, j}: \sum_{i=1}^{n} \sum_{j=1}^{I_{i}} J_{i, j}=J} \prod_{i, j} a_{J_{i, j}} & \text { if }|J|=1 \\
\text { if }|J| \geq 2 . \tag{4.9}
\end{array}
$$

Then there exists $R>0$ such that $\left|a_{J}\right| \leq R^{|J|}$.
Proof. Consider the following analytic function $F\left(x_{1}, \ldots, x_{m}, y\right)$ of $(m+1)$ variables.

$$
F\left(x_{1}, \ldots, x_{m}, y\right)=\frac{1}{(1-M y)^{n}}-(M n+1) y+\left(x_{1}+\ldots+x_{m}\right)
$$

We have that $F(0)=1$ and $F_{y}^{\prime}(0)=-1 \neq 0$. Then by the analytic implicit function theorem, there exists an analytic function $f(x)=\sum_{J} b_{J} x^{J}$ near $0 \in \mathbb{R}^{m}$ such that $f(0)=0$ and

$$
F\left(x_{1}, \ldots, x_{m}, f\left(x_{1}, \ldots, x_{m}\right)\right)=1
$$

i.e.,

$$
\begin{equation*}
1-\left(x_{1}+\ldots+x_{m}\right)+(M n+1) f=\frac{1}{(1-M f)^{n}}=\left(1+M f+(M f)^{2}+\ldots\right)^{n} \tag{4.10}
\end{equation*}
$$

It suffices to show that $b_{J}=a_{J}$ for all $J$. Clearly we have that $b_{0}=0=a_{0}$ and $b_{J}=1=a_{J}$ for all $J$ with $|J|=1$. Then it suffices to show that

$$
b_{J}=\sum_{I \in \mathbb{Z}_{\geq 0}^{n}:|I| \geq 2} M^{|I|} \sum_{J_{i, j}: \sum_{i=1}^{n} \sum_{j=1}^{I_{i}} J_{i, j}=J} \prod_{i, j} b_{J_{i, j}}
$$

for all $J$ with $|J| \geq 2$. Now we fix $J \in \mathbb{Z}_{\geq 0}^{m}$ with $|J| \geq 2$. The coefficient of $x^{J}$ on the left-hand side of equation (4.10) is

$$
(M n+1) b_{J}
$$

Denote $C_{J}(P)$ as the coefficient of $x^{J}$ in the power series $P$. Then the coefficient of $x^{J}$ on the right-hand side of equation (4.10) is

$$
\sum_{I \in \mathbb{Z}_{\geq 1}^{n}} C_{J}\left(\prod_{i=1}^{n}(M f)^{I_{i}}\right)=\sum_{I \in \mathbb{Z}_{\geq 1}^{n}} M^{|I|} C_{J}\left(\prod_{i=1}^{n} f^{I_{i}}\right)
$$

where

$$
C_{J}\left(\prod_{i=1}^{n} f^{I_{i}}\right)=C_{J}\left(\prod_{i=1}^{n}\left(\sum_{J} b_{J} x^{J}\right)^{I_{i}}\right)=\sum_{J_{i, j}: \sum_{i=1}^{n} \sum_{j=1}^{I_{i} J_{i, j}=J}} \prod_{i, j} b_{J_{i, j}}
$$

It remains to show that

$$
M n b_{J}=\sum_{I \in \mathbb{Z}_{\geq 0}^{n}:|I|=1} M^{|I|} \sum_{J_{i, j}: \sum_{i=1}^{n} \sum_{j=1}^{I_{i}} J_{i, j}=J} \prod_{i, j} b_{J_{i, j}}
$$

This is a consequence of that for all $I$ with $|I|=1$

$$
\sum_{J_{i, j}: \sum_{i=1}^{n} \sum_{j=1}^{I_{i}} J_{i, j}=J} \prod_{i, j} b_{J_{i, j}}=b_{J} .
$$

Theorem 4.5. Let $U \ni 0$ be an open set in $\mathbb{R}^{m}$ and $V$ be an analytic vector field on $U$ such that $V(0)=0$ and

$$
D V(0)=\operatorname{diag}\left(\lambda_{1}, \ldots, \lambda_{m}\right)
$$

for some $\lambda=\left(\lambda_{1}, \ldots, \lambda_{m}\right) \in \mathbb{R}_{<0}^{m}$. Suppose the $n$-dim $\lambda$-series

$$
x(t)=\sum_{J \in \mathbb{Z}_{\geq 0}^{m}} P_{J}(t) e^{\lambda \cdot J t}
$$

formally satisfies that $P_{0}=0$ and $x^{\prime}(t)=V(x(t))$ as $\lambda$-series. Then we have the following.
(a) There exists $T>0$ and $r>0$ such that $\left|P_{J}(t)\right| \leq t^{r|J|}$ for all $J \in \mathbb{Z}_{\geq 0}^{m}$ and $t \geq T$.
(b) $x(t), x^{\prime}(t)$ converge and satisfy $x^{\prime}(t)=V(x(t))$ as functions for sufficiently large $t$.
(c) If $\lambda_{0}>\max _{i} \lambda_{i}$ and $C \in \mathbb{R}^{m}$ is nonzero and $x(t ; c)$ denotes the formal solution defined as in Section 4.2, as $t \rightarrow \infty$ we have

$$
x(t ; c+C)-x(t ; c)=\sum_{i=1}^{m} C_{i} e^{\lambda_{i} t} \vec{e}_{i}+o\left(e^{\left(\lambda_{C}+\lambda_{0}\right) t}\right)
$$

where $\lambda_{C}=\max \left\{\lambda_{i}: i \in\{1, \ldots, m\}, C_{i} \neq 0\right\}$.
Proof. (a)
Without loss of generality, we may assume $0>\lambda_{1} \geq \ldots \geq \lambda_{m}$. Denote $\Lambda=$ $\operatorname{diag}\left(\lambda_{1}, \ldots, \lambda_{m}\right)=D V(0)$ and suppose

$$
V(x)=\Lambda x+\sum_{I \in \mathbb{Z}_{\geq 0}^{m}:|I| \geq 2} b_{I} x^{I} .
$$

Since formally $x^{\prime}(t)=V(x(t))$ we have that

$$
P_{J}^{\prime}(t)+\lambda \cdot J P_{J}(t)=\Lambda P_{J}(t)+Q_{J}(t)
$$

where

$$
Q_{J}(t)=\sum_{I \in \mathbb{Z}_{\geq 0}^{m}:|I| \geq 2} b_{I} \sum_{J_{k, l}: \sum_{k=1}^{m} \sum_{l=1}^{I_{k}} J_{k, l}=J} \prod_{k, l}\left(P_{J_{k, l}}\right)_{k}(t) .
$$

So

$$
Q_{J}^{*}(t) \leq \sum_{I \in \mathbb{Z}_{\geq 0}^{m}:|I| \geq 2}\left|b_{I}\right| \sum_{J_{k, l}: \sum_{k=1}^{m} \sum_{l=1}^{I_{k}} J_{k, l}=J} \prod_{k, l}\left(P_{J_{k, l}}^{*}\right)(t) .
$$

and

$$
\begin{equation*}
\left(\frac{d}{d t}-(\Lambda-\lambda \cdot J)\right) P_{J}=Q_{J} \tag{4.11}
\end{equation*}
$$

Here $\Lambda-\lambda \cdot J$ is a diagonal matrix whose smallest diagonal entry is

$$
\left(\lambda_{m}-\lambda \cdot J\right)
$$

Pick $s \in \mathbb{Z}_{\geq 0}$ sufficiently large such that for all $J$ with $|J| \geq s$,

$$
\lambda_{m}-\lambda \cdot J \geq \frac{-\lambda_{1}|J|}{2} \geq 2
$$

Then for any $J$ with $|J| \geq s, \operatorname{deg}\left(P_{J}\right)=\operatorname{deg}\left(Q_{J}\right)$.

Let $p \in \mathbb{R}_{\geq 0}$ be a constant such that

$$
\operatorname{deg}\left(P_{J}\right) \leq p|J| \quad \text { and } \quad \operatorname{deg}\left(Q_{J}\right) \leq p|J|
$$

for all $J \in \mathbb{Z}_{\geq 0}^{m}$ with $|J|<s$. Then by induction it is straightforward to see that

$$
\operatorname{deg}\left(P_{J}\right) \leq p|J| \quad \text { and } \quad \operatorname{deg}\left(Q_{J}\right) \leq p|J|
$$

for all $J$. By Corollary 4.3, for $J \in \mathbb{Z}_{\geq 0}^{m}$ with $|J| \geq s$ and

$$
t \geq \frac{4 p}{-\lambda_{1}}=\frac{2 p|J|}{-\lambda_{1}|J| / 2} \geq \frac{2 \operatorname{deg}\left(Q_{J}\right)}{\lambda_{m}-\lambda \cdot J}
$$

we have $P_{J}^{*}(t) \leq Q_{J}^{*}(t)$.
Pick a sufficiently large $q$ such that

$$
P_{J}^{*}(t) \leq t^{q|J|}
$$

for all $t \geq 2$ and $J \in \mathbb{Z}_{\geq 0}^{m}$ with $|J| \leq s$. Suppose $\left|b_{I}\right| \leq M^{|I|}$ for some $M>0$ and all $I \in \mathbb{Z}_{\geq 0}^{m}$. Let $a_{J}$ be defined as in Lemma 4.4. By induction it is straightforward to see that $P_{J}^{*}(t) \leq a_{J} t^{q|J|}$ for all

$$
t \geq \max \left\{\frac{4 p}{-\lambda_{1}}, 2\right\}
$$

and $J \in \mathbb{Z}_{\geq 0}^{m}$. By Lemma 4.4 there exists $R>0$ such that $a_{J} \leq R^{|J|}$. So

$$
\left|P_{J}(t)\right|_{\infty} \leq P_{J}^{*}(t) \leq R^{|J|} t^{q|J|} \leq 2^{|J| \log _{2} R} \cdot t^{q|J|} \leq t^{\left(q+\log _{2} R\right)|J|}
$$

for all $J$ and

$$
t \geq \max \left\{\frac{4 p}{-\lambda_{1}}, 2\right\}
$$

(b) By Propositions 3.2 (a) and 3.3, $x(t), V(x(t))$ converge uniformly. Since any partial sum of $x^{\prime}(t)=V(x(t))=(V \circ x)(t)$ is the derivative of the corresponding partial sum of $x(t)$ as functions, $x^{\prime}(t)=V(x(t))$ is the derivative of $x(t)$ as functions.
(c) By Proposition 3.2 (b), we only need to compare the coefficients of $e^{\lambda \cdot J t}$ for nonzero $J$ satisfying $\lambda \cdot J \geq \lambda_{C}+\lambda_{0}$. Such $J$ could either be $\vec{e}_{i}$ or satisfy that $J_{i}=0$ if $C_{i} \neq 0$. The first case is immediate from the definition. In the second case of $J$, we can straightforwardly show that $P_{J}(t ; c)=P_{J}(t, c+C)$ by induction.

## 5. Limiting behavior for piecewise analytic vector fields

### 5.1. Preliminary estimates.

Lemma 5.1. Let $U \ni 0$ be an open domain in $\mathbb{R}^{n}$ and $U \subset \cup_{i} D_{i}$ be a closed finite cover. Suppose $V(x)$ is a $C^{1}$ vector field on $U$ such that $V(x)$ is analytic on each $D_{i} \cap U$. Then

$$
V(x)-V(y)=D V(0)(x-y)+O((|x|+|y|)|x-y|)
$$

Proof. On a small neighborhood $U_{0}$ of $0, D V(x)$ is Lipschitz continuous on each $D_{i} \cap U_{0}$, since $V(x)$ is analytic on $D_{i} \cap U_{0}$. Since $D V(x)$ is continuous on $U_{0}$, it is easy to see that $D V(x)$ is Lipschitz continuous on $U_{0}$. So on $U_{0}$ we may assume that

$$
|D V(x)-D V(0)| \leq C|x|
$$

for some constant $C>0$. Then

$$
V(x)-V(y)-D V(0)(x-y)
$$

$$
\begin{gathered}
=\left(\int_{0}^{1} D V(t x+(1-t) y)(x-y) d t\right)-D V(0)(x-y) \\
=\int_{0}^{1}(D V(t x+(1-t) y)-D V(0))(x-y) d t \\
\leq\left(\int_{0}^{1} C|(t x+(1-t) y)|_{2} d t\right)|x-y|_{2} \\
=O((|x|+|y|)|x-y|)
\end{gathered}
$$

Lemma 5.2. Suppose $V(x)$ is a $C^{1}$ vector field on a neighborhood $U$ of 0 in $\mathbb{R}^{m}$, such that

$$
V(x)=\Lambda x+O\left(|x|^{2}\right)
$$

where

$$
\Lambda=\operatorname{diag}\left(\lambda_{1}, \ldots, \lambda_{m}\right)
$$

and

$$
0>\lambda_{1} \geq \ldots \geq \lambda_{m}
$$

If $x^{\prime}(t)=V(x(t))$ on $[0, \infty)$ and $x(t)=o(1)$ as $t \rightarrow \infty$, then $x(t)=O\left(e^{\lambda_{1} t}\right)$ as $t \rightarrow \infty$.

Proof. As $t \rightarrow \infty$,

$$
\begin{gathered}
\left(|x|_{2}^{2}\right)^{\prime}=2 x \cdot x^{\prime}=2 x \cdot V(x) \\
=2 x \cdot\left(\Lambda x+O\left(|x|^{2}\right)\right) \leq 2 \lambda_{1}|x|_{2}^{2}+O\left(|x|^{3}\right)
\end{gathered}
$$

If $\lambda_{0}>\lambda_{1}$,

$$
\begin{gathered}
\left(e^{-2 \lambda_{0} t}|x|_{2}^{2}\right)^{\prime}=-2 \lambda_{0} e^{-2 \lambda_{0} t}|x|_{2}^{2}+\left(2 \lambda_{1}|x|_{2}^{2}+O\left(|x|^{3}\right)\right) e^{-2 \lambda_{0} t} \\
\leq\left(2 \lambda_{1}-2 \lambda_{0}\right) e^{-2 \lambda_{0} t}|x|_{2}^{2}+e^{-2 \lambda_{0} t} O(|x|)^{3}
\end{gathered}
$$

So $\left(e^{-2 \lambda_{0} t}|x|_{2}^{2}\right)^{\prime}<0$ if $t$ is sufficiently large, and then

$$
e^{-2 \lambda_{0} t}|x|_{2}^{2}=O(1)
$$

and

$$
|x|=O\left(e^{\lambda_{0} t}\right)
$$

So

$$
\begin{aligned}
\left(e^{-2 \lambda_{1} t}|x|_{2}^{2}\right)^{\prime} & =-2 \lambda_{1} e^{-2 \lambda_{1} t}|x|_{2}^{2}+\left(2 \lambda_{1}|x|_{2}^{2}+O\left(|x|^{3}\right)\right) e^{-2 \lambda_{1} t} \\
& =e^{-2 \lambda_{1} t} O(|x|)^{3}=O\left(e^{\left(3 \lambda_{0}-2 \lambda_{1}\right) t}\right)
\end{aligned}
$$

So if $\lambda_{0}=3 \lambda_{1} / 4>\lambda_{1}$

$$
e^{-2 \lambda_{1} t}|x|_{2}^{2}=|x(0)|_{2}^{2}+\int_{0}^{t} O\left(e^{\lambda_{1} s / 4}\right) d s=|x(0)|_{2}^{2}+O(1)=O(1)
$$

and $|x|=O\left(e^{\lambda_{1} t}\right)$.

Lemma 5.3. Suppose $V(x)$ is a $C^{1}$ vector field on a neighborhood $U$ of 0 in $\mathbb{R}^{m}$, such that

$$
D V(0)=\operatorname{diag}\left(\lambda_{1}, \ldots, \lambda_{m}\right)
$$

where

$$
0>\lambda_{1} \geq \ldots \geq \lambda_{m}
$$

Suppose

$$
x^{\prime}(t)=V(x(t)), \quad y^{\prime}(t)=V(x(t))
$$

on $[0, \infty)$ and

$$
x(t)=o(1), \quad y(t)=o(1)
$$

as $t \rightarrow \infty$. If $y(t)-x(t)=O\left(e^{\lambda_{0} t}\right)$ for some $\lambda_{0}<\lambda_{m}$, then $x(t)=y(t)$.
Proof. Let $f(s)=s y+(1-s) x$. Then

$$
V(y)-V(x)=V(f(1))-V(f(0))=\int_{0}^{1}(V \circ f)^{\prime}(s) d s
$$

and

$$
\begin{gathered}
|V(y)-V(x)|_{2} \leq \int_{0}^{1}\left|(V \circ f)^{\prime}(s)\right|_{2} d s \\
=\int_{0}^{1}|D V(f(s))|_{2} \cdot\left|f^{\prime}(s)\right|_{2} d s=\max _{0 \leq s \leq 1}|D V(f(s))|_{2} \cdot|y-x|_{2} \leq \frac{\lambda_{0}+\lambda_{m}}{2}|y-x|_{2}
\end{gathered}
$$

if $|x|,|y|$ are sufficiently small. So for $t$ sufficiently large,
$\left(|y-x|_{2}^{2}\right)^{\prime}=2(y-x) \cdot(V(y)-V(x)) \geq-2|y-x|_{2} \cdot|V(y)-V(x)|_{2} \geq\left(\lambda_{0}+\lambda_{m}\right)|y-x|_{2}^{2}$.
So $|y-x|_{2}^{2} \geq e^{\left(\lambda_{0}+\lambda_{m}\right) t}|y(0)-x(0)|_{2}^{2}$. So we must have $y(0)=x(0)$ if $y(t)-x(t)=$ $O\left(e^{\lambda_{0} t}\right)$.

Lemma 5.4. Suppose $a<0, b<0, a \neq b$, and $x(t), h(t)$ are two functions on $[T, \infty)$ for some $T \in \mathbb{R}$, such that

$$
x^{\prime}(t)=a x(t)+h(t)
$$

and

$$
h(t)=o\left(e^{b t}\right)
$$

as $t \rightarrow \infty$. Then

$$
x(t)=C e^{a t}+o\left(e^{b t}\right)
$$

as $t \rightarrow \infty$ for some constant $C$.
Proof. If $a<b$,

$$
\begin{gathered}
x(t)=e^{a t} \int_{T}^{t} e^{-a s} h(s) d s+C e^{a t} \\
=e^{a t} \int_{T}^{t} e^{-a s} \cdot o\left(e^{b s}\right) d s+C e^{a t} \\
=e^{a t} \int_{T}^{t} o\left(e^{(b-a) s}\right) d s+C e^{a t} \\
=e^{a t} \cdot o\left(e^{(b-a) t}\right)+C e^{a t} \\
=o\left(e^{b t}\right)+C e^{a t}
\end{gathered}
$$

for some constant $C$.

If $a>b$,

$$
\begin{gathered}
x(t)=e^{a t} \int_{\infty}^{t} e^{-a s} h(s) d s+C e^{a t} \\
=e^{a t} \int_{\infty}^{t} e^{-a s} \cdot o\left(e^{b s}\right) d s+C e^{a t} \\
=e^{a t} \int_{\infty}^{t} o\left(e^{(b-a) s}\right) d s+C e^{a t} \\
=e^{a t} \cdot o\left(e^{(b-a) t}\right)+C e^{a t} \\
=o\left(e^{b t}\right)+C e^{a t}
\end{gathered}
$$

for some constant $C$.

### 5.2. Proof of Theorem 1.3.

Proof of Theorem 1.3. It suffices to prove that for any cell $D$ in the finite cover, $x(t) \in D$ for sufficiently large $t$ or $x(t) \notin D$ for sufficiently large $t$. Let us fix a cell $D$ and assume $0 \in D$ without loss of generality.

By a linear transformation, we may assume that $T_{0} M$ is tangent to the $\left(x_{1} \ldots x_{m}\right)$ plane and

$$
\Lambda:=\left.D V(0)\right|_{T_{0} M}=\operatorname{diag}\left(\lambda_{1}, \ldots, \lambda_{m}\right)
$$

Without loss of generality, we may assume that

$$
M=\left\{\left(x_{1}, \ldots, x_{n}\right):\left(x_{1}, \ldots, x_{m}\right) \in U_{0},\left(x_{m+1}, \ldots, x_{n}\right)=f\left(x_{1}, \ldots, x_{m}\right)\right\}
$$

for some open set $U_{0} \subset \mathbb{R}^{m}$ and analytic function $f$ on $U_{0}$. Then $f(0)=0$ and $D f(0)=0$. Denote

$$
\begin{gathered}
\bar{x}(t)=\left(x_{1}(t), \ldots, x_{m}(t)\right), \text { and } \\
\bar{V}(x)=\left(V_{1}(x), \ldots, V_{m}(x)\right)
\end{gathered}
$$

Then

$$
\bar{x}^{\prime}(t)=\bar{V}(\bar{x}, f(\bar{x}))=: \tilde{V}(\bar{x})
$$

where $\tilde{V}(\bar{x})$ is $C^{1}$ in a neighborhood of 0 in $\mathbb{R}^{m}$. Furthermore,

$$
D \tilde{V}(0)=\left(\Lambda, \frac{\partial \bar{V}}{\partial\left(x_{m+1}, \ldots, x_{n}\right)}(0)\right)\left(I d_{m}, D f(0)\right)^{T}=\Lambda
$$

So by Lemma $5.2 \bar{x}(t)=O\left(e^{\lambda_{1} t}\right)$ and then $x(t)=O\left(e^{\lambda_{1} t}\right)$.
Suppose

$$
p_{m}\left(x_{1}, \ldots, x_{n}\right)=\left(x_{1}, \ldots, x_{m}\right)
$$

is the projection from $\mathbb{R}^{n}$ to $\mathbb{R}^{m}$. Then $\tilde{V}(\bar{x})$ is analytic on $p_{m}\left(M \cap D_{i}\right)$ for each cell $D_{i}$ in the finite cover. Then by Lemma 5.1,

$$
\begin{equation*}
\tilde{V}(\bar{x})-\tilde{V}(\bar{y})=\Lambda(\bar{x}-\bar{y})+O((|x|+|y|)|x-y|) \tag{5.1}
\end{equation*}
$$

Let $V_{*}$ be the analytic vector field on an open superset of $D \cap U$ such that $V_{*}(x)=V(x)$ on $D \cap U$. Denote

$$
\bar{V}_{*}(x)=\left(V_{*, 1}(x), \ldots, V_{*, m}(x)\right)
$$

and

$$
\tilde{V}_{*}(\bar{x})=\bar{V}_{*}(\bar{x}, f(\bar{x}))
$$

Given $c \in \mathbb{R}^{m}$, let $\bar{y}(t ; c)$ be the formal $\lambda$-series solution to $\bar{y}^{\prime}=\tilde{V}_{*}(\bar{y})$ as constructed in Section 4.2.

Pick $\lambda_{0} \in\left(\lambda_{1}, 0\right)$ such that $k \lambda_{0} \neq \lambda_{i}$ for all $k \in \mathbb{Z}_{\geq 0}$ and $i=1, \ldots, m$. Let
$A=\left\{k \in \mathbb{Z}_{\geq 0}\right.$ : there exists $c \in \mathbb{R}^{m}$ such that $\bar{x}(t)-\bar{y}(t ; c)=o\left(e^{k \lambda_{0} t}\right)$ as $\left.t \rightarrow \infty\right\}$
$A$ is nonempty since $0 \in A$. Suppose $D=\cap_{i} H_{i}$ where $H_{i}$ 's are half spaces, and denote $y(t ; c)=(\bar{y}(t ; c), f(\bar{y}(t ; c)))$ as a $n$-dim $\lambda$-series.
(a) If $A$ has no maximum, then we can find $k \in A$ and $c \in \mathbb{R}^{m}$ such that $k \lambda_{0}<\lambda_{m}$ and $\bar{x}(t)-\bar{y}(t ; c)=o\left(e^{k \lambda_{0} t}\right)$. Then by Lemma $5.3 \bar{x}(t)=\bar{y}(t ; c)$ as $\mathbb{R}^{m}$-valued functions. Then by Proposition $3.3 x(t)=y(t ; c)$ as $\mathbb{R}^{n}$-valued functions. Then for each $H_{i}$, the signed distance $d_{s}\left(y(t ; c), H_{i}\right)$ is a $\lambda$-series and converge for sufficiently large $t$. Furthermore, by Proposition 3.3 the limit of the $\lambda$-series $d_{s}\left(y(t ; c), H_{i}\right)$ is indeed the signed distance from $y(t ; c)$ to $H_{i}$. For each $H_{i}, d\left(y(t ; c), H_{i}\right) \equiv 0$ for sufficiently large $t$ or $d\left(y(t ; c), H_{i}\right)=a t^{q} e^{r t}+o\left(a t^{q} e^{r t}\right)$ for some $a>0, q \in \mathbb{Z}_{\geq 0}$ and $r<0$. So $x(t)=y(t ; c) \notin D$ for sufficiently large $t$ or $x(t)=y(t ; c) \in D$ for sufficiently large $t$.
(b) If $k=\max A$, let $c$ be such that

$$
\bar{x}(t)=\bar{y}(t ; c)+o\left(e^{k \lambda_{0} t}\right)
$$

If

$$
d\left(y(t ; c), H_{i}\right)=a t^{q} e^{r t}+o\left(a t^{q} e^{r t}\right)
$$

for some $H_{i}$ and $a>0$ and $q \in \mathbb{Z}_{\geq 0}$ and $r \geq k \lambda_{0}$, then

$$
d\left(x(t), H_{i}\right)=a t^{q} e^{r t}+o\left(a t^{q} e^{r t}\right)
$$

If not, $d\left(y(t ; c), H_{i}\right)=o\left(e^{k \lambda_{0} t}\right)$ for all $H_{i}$, and then

$$
d(y(t ; c), D) \leq d(y(t ; c), 0)=O\left(e^{\lambda_{1} t}\right)
$$

and by Lemma 1.4

$$
d(y(t ; c), D)=o\left(e^{k \lambda_{0} t}\right)
$$

Denote $y_{*}(t)$ as the closest point to $y(t ; c)$ in $D$. Then

$$
y_{*}(t)=O(y(t))=O\left(e^{\lambda_{1} t}\right)
$$

and

$$
x(t)=y(t ; c)+o\left(e^{k \lambda_{0} t}\right)=y_{*}(t)+o\left(e^{k \lambda_{0} t}\right)
$$

and by Lemmas 5.1 and 5.2

$$
\begin{gathered}
(\bar{x}-\bar{y})^{\prime}=\tilde{V}(\bar{x})-\tilde{V}_{*}(\bar{y}) \\
=\left(\tilde{V}(\bar{x})-\tilde{V}\left(\bar{y}_{*}\right)\right)+\left(\tilde{V}_{*}\left(\bar{y}_{*}\right)-\tilde{V}_{*}(\bar{y})\right) \\
=\Lambda\left(\bar{x}-\bar{y}_{*}\right)+O\left(\left(|\bar{x}|+\left|\bar{y}_{*}\right|\right)\left|\bar{x}-\bar{y}_{*}\right|\right)+\Lambda\left(\bar{y}_{*}-\bar{y}\right)+O\left(\left(\left|\bar{y}_{*}\right|+|\bar{y}|\right)\left|\bar{y}_{*}-\bar{y}\right|\right) \\
=\Lambda(\bar{x}-\bar{y})+O\left(e^{\lambda_{1} t} \cdot e^{k \lambda_{0} t}\right)=\Lambda(\bar{x}-\bar{y})+o\left(e^{(k+1) \lambda_{0} t}\right)
\end{gathered}
$$

Then by Lemma 5.4 for all $i=1, \ldots, m$,

$$
\bar{x}_{i}(t)-\bar{y}_{i}(t ; c)=C_{i} e^{\lambda_{i} t}+o\left(e^{(k+1) \lambda_{0} t}\right)
$$

for some constant $C_{i}$. Let $C=\left(C_{1}, \ldots, C_{m}\right)$, and then $C$ is nonzero by the maximality of $k$. Since $\bar{x}(t)-\bar{y}(t ; c)=o\left(e^{k \lambda_{0} t}\right), C_{i}=0$ if $\lambda_{i}>k \lambda_{0}$. So

$$
\lambda_{C}:=\max \left\{\lambda_{i}: i \in\{1, \ldots, m\}, C_{i} \neq 0\right\}<k \lambda_{0}
$$

By Theorem 4.5 (c),

$$
\bar{y}(t ; c+C)=\bar{y}(t ; c)+\sum_{i=1}^{m} C_{i} e^{\lambda_{i} t} \vec{e}_{i}+o\left(e^{\left(\lambda_{C}+\lambda_{0}\right) t}\right)
$$

$$
=\bar{x}(t ; c)+o\left(e^{\left(\lambda_{C}+\lambda_{0}\right) t}\right)=\bar{x}(t ; c)+o\left(e^{(k+1) \lambda_{0} t}\right)
$$

This contradicts the maximality of $k$.

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