# Estimation Sample Complexity of a Class of Nonlinear Continuous-time Systems \*

Simon Kuang\* Xinfan Lin\*

\* University of California, Davis; Davis, CA 95616 (e-mail: slku@ucdavis.edu, lxflin@ucdavis.edu).

**Abstract:** We present a method of parameter estimation for large class of nonlinear systems, namely those in which the state consists of output derivatives and the flow is linear in the parameter. The method, which solves for the unknown parameter by directly inverting the dynamics using regularized linear regression, is based on new design and analysis ideas for differentiation filtering and regularized least squares. Combined in series, they yield a novel finite-sample bound on mean absolute error of estimation.

Keywords: Modelling, Identification and Signal Processing; Machine Learning in modeling, estimation, and control; Estimation

## 1. INTRODUCTION

Most physical laws are continuous-time. Examples include Newtonian motion  $\ddot{x} = m^{-1}F$ , Fickian diffusion  $\dot{q} = D\Delta q$ , capacitance  $\dot{v} = C^{-1}i$ . These three relationships are semilinear (in fact, linear) ordinary differential equations in the state variables x, q, and v; with vector fields that are linear in the parameters  $m^{-1}$ , D, and  $C^{-1}$ , respectively. We view the states as observable and the parameters as unknown. System identification, as a way of measurement, takes on the emphasis of parameter estimation (as opposed to model prediction); this interpretation of system identification has universal application across engineering, e.g. magnetic resonance imaging (Maidens et al., 2016) and battery diagnostics (Lin et al., 2019); and in other areas such as monetary policy econometrics (Chen and Phelan, 2023).

We focus on the case in which all of the state variables are derivatives of the observed output. In chaos-related literature, this structure is called a hyperjerk system. In the geometric control literature, the observed output can also be understood as a flat output of a differentially flat system. Relative to the econometrics and control literature, this structure looks like a nonlinear autoregressions, but adapted in continuous time, with process noise replaced by observation noise.

### 2. PROBLEM STATEMENT

Let  $\Theta = \mathbb{R}^p$  be parameter space. The output is  $y:[0,T]\times$  $\Theta \to \mathbb{R}$ , and the state  $\xi \in \mathbb{R}^m$  consists of the derivatives of y of orders 0 through m-1. The state dynamics are

$$\frac{\partial^m}{\partial t^m} y(t,\theta) = \theta^{\mathsf{T}} \phi(\xi(t,\theta)) \tag{1}$$

with a known initial condition  $\xi(0,\theta) = \xi_0$ .

Assumption 1. The feature map  $\phi: \mathbb{R}^m \to \mathbb{R}^p$  is  $\alpha$ -Hölder continuous with coefficient  $C_{\phi}$ .

Assumption 2.  $\frac{\partial^{m+1}}{\partial t^{m+1}}y$  exists and is Riemann integrable on

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space with expectation  $\mathbb{E}$ , where  $\Omega$  is the sample space,  $\sigma$ -field  $\mathcal{F}$ , and probability measure  $\mathbb{P}$ .

We may observe y at n evenly spaced observations through the random variable z with the following law:

$$z_{i,n} = y(in^{-1}T, \theta) + w(in^{-1}T),$$
 (2)  
 $i = 1, 2, \dots, n.$  (3)

$$i = 1, 2, \dots, n. \tag{3}$$

Assumption 3.  $\mathbb{E} w(t)^2$  is Riemann integrable in t.

Definition 4. A discrete estimator of  $\theta$  is a random variable  $\hat{\theta}: \Omega \to \Theta$  expressed as a function of  $\{z_{i,n}\}_{1 \leq i \leq n}$  that does not depend explicitly on  $\theta$ .

This last stipulation is simply to disallow "cheating" such as  $\hat{\theta} := \theta$ ; in a Bayesian setting where  $\theta$  is a random variable, we could make this more precise by requiring a  $\hat{\theta}$ to be measurable with respect to the  $\sigma$ -field generated by  $\{z_{i,n}\}_{1\leq i\leq n}$ .

The goal is to find an estimator design that can be made arbitrarily accurate by increasing the quantity n and bandwidth T/n of data collected.

Problem 5. Find a sequence of estimators  $\{\hat{\theta}_n\}_{n\in\mathbb{N}}$  such that  $\hat{\theta}_n \xrightarrow{\mathbb{P}} \theta$ , with quantitative guarantees for finite n.

# 3. RELATED WORK

3.1 Discretization does not solve Problem 5

The discrete-time model  $X_{i+1} \approx f(X_i, \theta)$ , where X is state, f is the dynamics, and  $\approx$  denotes process or observation

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For a system with exogeneous input, the theory also works with a non-autonomous  $\phi$ .

noise, is a mainstay in system identification due to its resemblance to nonlinear regression  $Y \approx f(X,\theta)$ . There is recent progress in finite-sample identification of discrete-time linear (Sarkar et al., 2021; Tsiamis et al., 2022; Ziemann et al., 2023; Matni and Tu, 2019) and parameter-linear (Mania et al., 2022) systems. Can discrete-time identification theory solve the continuous-time Problem 5? The answer is: not easily. An overview of continuous-to-discrete challenges in system identification is available in Ljung and Wills (2010).

One might argue that discretization converts continuoustime models to discrete-time models. But (1) cannot be translated into discrete form without compromising either accuracy or parameter-linearity. For instance, one can obtain the equivalent model

$$y((i+1)n^{-1}T) = f(y(in^{-1}T), \theta)$$
(4)

where the dynamic map f is computed by solving an initial value problem (IVP). But the right side of this equation is no longer linear in  $\theta$ .

One could take a Taylor expansion of (4) for  $n^{-1}T \approx 0$  in order to preserve the parameter-linearity. That would be equivalent to choosing a numerical scheme for integrating the IVP above, with the need to deal with the integration error. This practice would be at least as hard as estimating differentiation error (our approach).

# 3.2 Curve fitting does not solve Problem 5

In the prediction error method (PEM) for nonlinear system identification (Ljung, 1998; Ljung, 2010; Keesman, 2011, sec. 5.2), we find  $\hat{\theta}$  such that the predicted flow generated by  $\hat{\theta}^{\dagger}\phi$  best fits the measured y. Some of the PEM literature is grounded in classical theory on general nonlinear least squares estimation (Chen et al., 2020; Wu, 1981; Lai, 1994; Jennrich, 1969) In certain large-sample limits, extremum estimators such as PEM and maximum likelihood estimation can be asymptotically consistent.

Theorem 6. (Amemiya [1985, Thm. 4.1.1]). Assume:

- (1) The parameter space  $\Theta$  is a compact subset of  $\mathbb{R}^p$ .
- (2)  $Q_n(\omega, \theta)$  is measurable in the outcome  $\omega \in \Omega$  and continuous in  $\theta \in \Theta$ .
- (3)  $n^{-1}Q_n(\omega,\theta) \to Q(\theta)$  in probability uniformly <sup>3</sup> in  $\theta$  as  $n \to \infty$ , and  $Q(\theta)$  has a unique **global** maximum at  $\theta_0$ .

Let  $\hat{\theta}_n(\omega) \in \arg \max_{\theta} Q_n(\omega, \theta)$  be a measurable extremum. Then  $\hat{\theta}_n \stackrel{\mathbb{P}}{\to} \theta_0$ .

The global maximum estimator can be difficult to locate (Amemiya, 1985, p. 110), and we give such an example in Section 7 in comparison with our proposed estimator. Furthermore, the global maximum estimator does not come with even asymptotic tail guarantees such as normality. Asymptotic normality requires a number of higher regularity and rank conditions (Amemiya, 1985, Thm. 4.1.3), as well as the following result on local extrema: if  $Q_n$  has multiple local maxima, one of them converges in  $L^2(\Omega)$  to

a local maxima of Q (Amemiya, 1985, Thm. 4.1.2). On the other hand, our estimator comes with a moment guarantee valid at finite n; and it is a "plug-in" estimator that does not require solving any implicit nonlinear equations. Our estimator also works without continuity in  $\theta$ ; see Section 8.

### 3.3 Linear theory does not solve Problem 5

We have finite-sample guarantees when system identification amounts to estimating an operator  $\Theta$  in the linear relationship

$$Y = \Theta X + \text{noise}, \tag{5}$$

where generally, X depends on the initial condition, system input, and observable states; Y is an output; and  $\Theta$  is an system representation such as a Hankel matrix or transfer function. After multiplying both sides by a generalized inverse of X, estimation error in  $\Theta$  is bounded in terms of X and the noise profile (Ljung, 1998; Sarkar et al., 2021; Pillonetto et al., 2022; Tsiamis et al., 2022; Pintelon and Schoukens, 2012, chap. 11; Bittanti, 2019, sec. 5.5.4.1).

We inherit an impressive legacy of theory and practice on identifying linear systems. Linear system identification, for all practical purposes, has been a solved problem for decades. But the conclusions mostly do not apply to the nonlinear system (1), which does not satisfy scaling or superposition in y. That is why we have selected a nonlinear system, (1) which does not satisfy scaling or superposition in y, and which enjoys no simplification under taking Laplace transforms.

## 4. A DIFFERENTIATION-BASED APPROACH

We attack (1) directly by solving a linear regression for predictors  $\frac{\partial^m}{\partial t^m}y(t)$  in terms of regressors  $\phi(\xi(t,\theta))$ . This rewrites (1) into the form

$$Y = \Theta\left(X + \operatorname{error}_{X}\right) - \operatorname{error}_{Y},\tag{6}$$

differing from (5) only in the errors-in-variables error  $\chi$ .

To get the regressors  $\phi(\xi(t,\theta))$  we must differentiate the measurements (Soderstrom et al., 1997; Van Breugel et al., 2020). Conventional wisdom holds that differentiation of noisy data is plagued by bias (due to aliasing high-frequency signal) and fluctuation (due to amplification of high-frequency noise). But this impression is fallible on both counts—bias and fluctuation.

Concerning bias: any derivative filter introduces bias in the form of generalized aliasing. Even without noise, it is never possible to get perfect derivatives for a general smooth y, because one cannot with certainty impute what happens in between samples. But suppose, as is often the the case, that the sampling period  $\Delta t$  is small compared to the regularity of y. When applying PEM to such a case, it would seem viable to integrate the flow using a linear multi-step scheme <sup>4</sup> with steps aligned to the measurement schedule. The quadrature error is controlled by comparing y to local polynomial approximants. Our differentiator will share the local polynomial approximation, but we manage the error systematically in Lemma 9 and beyond.

 $<sup>^2\,</sup>$  e.g. a sum-of-squares error between random observation outcome  $\omega$  and prediction at  $\theta.$ 

<sup>&</sup>lt;sup>3</sup> i.e.  $\lim_{n\to\infty} \mathbb{P}\left[\sup_{\theta\in\Theta} n^{-1}Q_n(\omega,\theta) < \epsilon\right] = 1$ .

 $<sup>^4\,</sup>$  A good exposition can be found in Dattani (2008). Butcher (2006) is a comprehensive treatment.

Concerning fluctuation: any derivative filter amplifies noise, as the ideal LTI differentiator has Laplace transfer function s. Our regressor matrix is noisy, but we manage this variability with ridge regularization in Appendix A. <sup>5</sup> To compare with PEM: consider estimation of  $\dot{y} = \theta$ . The time average of  $\hat{y}$ , which is the differentiation-based estimate, is very close to the least-squares PEM estimate.

# 4.1 Our contributions in context

An arbitrary-order finite difference filter for differentiating noisy signals: with modest regularity assumptions on trajectory and noise, we are able to bound the mean squared error of the derivative estimates. The filter resembles Savitzky-Golay (Schmid et al., 2022; Niedźwiecki et al., 2021; Ochieng et al., 2023) and orthogonal polynomial filtering (Othmane, 2022; Diekema and Koornwinder, 2012), but emphasizes the bias-variance tradeoff when filtering noisy data, following Sadeghi et al. (2020); Krishnan and Seelamantula (2013).

After differentiation, our method uses the derivative estimates to invert (1). Our use of finite-difference filters to estimate a continuous-time system resembles Soderstrom et al. (1997), which treats of a linear system with process noise, but our analysis makes the quantitative bias-variance tradeoff explicit. The concept of inverting linear equations to estimate a continuous-time dynamic parameter has a spectral interpretation by Unbehauen and Rao (1997).

A novel analysis of ridge regression subject to square-integrable error in both the regressors and predictors. Noise in both variables is called *errors-invariables* in statistics (Gleser et al., 1987; Hirshberg, 2021; Zhang et al., 2020; Pintelon and Schoukens (2012, sec 8.2)) and system identification (Söderström, 2018; Barbieri and Diversi, 2021; Khorasani and Weyer, 2021; Fosson et al., 2021). These problems are ill-behaved because they require the inversion of a random matrix which could *a priori* be singular with positive probability, leading to heavy tails. Additionally, the phenomenon of regression dilution states that the inverse of a random matrix is too small on average. We account for both of these issues in our analysis.

In brief, let  $\sigma > 0$  be an unreliable singular value in the data. Ridge regularization picks  $\lambda > 0$  to control  $(\sigma + \lambda)^{-1}$ . The naïve bound is  $(\sigma + \lambda)^{-1} \le \lambda^{-1}$  which is an  $L^{\infty}$ -type bound that is forced by the worst-case scenario  $\sigma = 0$ . Rather, we use a very detailed  $L^1$ -type pseudoinverse perturbation bound, Lemma 16 (Appendix A). This is the main idea in the proof of Thm 8.

# 5. PRELIMINARIES

Unless otherwise specified,  $\|\cdot\|$  denotes the 2-norm of a vector and the operator 2-norm of a matrix. We write  $\min(x,y)$  as  $x \wedge y$ . We write  $\max(0,x)$  as  $x^+$ . The least singular value of a matrix A is  $\sigma_{\min}(A)$ .

We say that a function is Riemann integrable if its equally-spaced Riemann sums converge.

#### 6. SOLUTION TO PROBLEM 5

This section constructs an estimator that solves Problem 5.

#### 6.1 Differentiation

Partition the n random variables  $z_j$  into to n' windows, each containing N consecutive samples, where N will be determined later. The windows can be chosen to be disjoint, n' = n/N; overlapping by n-1 points, n' = n-N+1; or anything in between.

Let  $n_i$  be the first index of the ith window. Then the start time of the ith window is  $t_i = n_i n^{-1} T$ . Augment  $\xi$  of (1) with an extra derivative to define  $x_i^d = \frac{\partial^d}{\partial t^d} y(t_i)$ ,  $d = 0, 1, \ldots, m$ . Regard  $x_i$  as an  $\mathbb{R}^{m+1}$  vector.

Theorem 7. For some choice of N, there exist statistics  $\hat{x}_i^d$  for all  $i \in [1 \dots n']$  and  $d \in [0 \dots m]$ , given as linear functions of the data  $z_j$ ,  $j \in [1 \dots n]$ , that satisfy

$$\mathbb{E}\sum_{i=1}^{n'}\sum_{d=0}^{m}\|\hat{x}_{i}^{d}-x_{i}^{d}\|^{2} \leq \epsilon n' n^{-\frac{2}{2m+3}},$$

where  $\epsilon > 0$  is independent of n.

This counterintuitive result on tuning arbitary-order finitedifference filters for noisy data may be the first of its kind suggests that noisy data can be practically differentiated for statistical use in a certain large-sample limit.

The proof is constructive and given in Appendix A. Henceforth the notation  $\hat{x}_i^d$  will refer to the estimators provided by Theorem 7.

# 6.2 Regularized least squares

Let P be the projection matrix onto the first m entries. Introduce the shorthand  $\phi_i = \phi(Px_i)$  and  $\hat{\phi}_i = \phi(P\hat{x}_i)$ .

Let  $\Phi \in \mathbb{R}^{n' \times p}$  be the matrix having  $\phi_i^\intercal, i = 1, 2, \dots, n'$  as rows, and let  $\hat{\Phi}$  be the same, with  $\hat{\phi}_i$  replacing  $\phi_i$ . Let  $\lambda \in (0, \sigma_{\min}(\Phi))$  be a regularization coefficient to be determined later. Define  $R, \hat{R} \in \mathbb{R}^{(p+n') \times p}$  as the block matrices

$$R = \begin{pmatrix} 0_{p \times p} \\ \Phi \end{pmatrix}, \quad \hat{R} = \begin{pmatrix} \lambda I_p \\ \hat{\Phi} \end{pmatrix}.$$
 (7)

Define  $u, \hat{u} \in \mathbb{R}^{p+n'}$  as

$$u = \begin{pmatrix} 0_{p \times 1} \\ x_1^m \\ x_2^m \\ \vdots \\ x_{n'}^m \end{pmatrix}, \quad \hat{u} = \begin{pmatrix} 0_{p \times 1} \\ \hat{x}_1^m \\ \hat{x}_2^m \\ \vdots \\ \hat{x}_{n'}^m \end{pmatrix}. \tag{8}$$

The system dynamics can be stated as  $u = R\theta$ . Assuming that R has rank m, let  $R^{\dagger}$  be the Moore-Penrose pseudoinverse of R, leading to the identity

$$\theta = R^{\dagger}u$$

which motivates the estimator

$$\hat{\theta} = \hat{R}^{\dagger} \hat{u}$$
.

Theorem 8. For some choice  $\lambda = \lambda(n)$ ,  $\mathbb{E} \|\hat{\theta} - \theta\| = O(n^{-\alpha/(2m+3)})$  as  $n \to \infty$ .

 $<sup>^5\,</sup>$  Appendices are only available in the preprint version of this article at Kuang and Lin (2023).

Constant	Meaning	Value
$\theta$	nonlinear gain	2
x(0)	initial displacement	1
$\dot{x}(0)$	initial rate	1
$\sigma^2$	observation variance	0.1
n	number of samples	10000
N	window size	50
$\lambda$	regularization coefficient	1

Table 1. Values of constants in simulating a Van der Pol oscillator and estimating its parameter.

This result is the first finite-sample frequentist guarantee for parameter estimation of a large class of nonlinear continuous-time systems from discrete data.

The proof is constructive and given in Appendix B.

### 7. NUMERICAL EXAMPLE

Consider the problem of estimating the nonlinearity coefficient  $\theta > 0$  in the Van der Pol oscillator:

$$\ddot{x}(t) = \theta(1 - x(t)^2)\dot{x}(t) - x(t)$$

The measurements are i.i.d. Normal random variables

$$z_i \sim \mathcal{N}(x(in^{-1}T), \sigma^2).$$

We constructed an estimator  $\hat{\theta}$  using the idea of this paper and used 1000 trials to capture its sampling distribution. To represent PEM, we also computed the likelihood function  $\mathcal{L}(\hat{\theta})$  based on a single random experiment. Constants pertaining to the problem and the estimator are available in Table 1.

The sampling distribution of  $\hat{\theta}$ , displayed in Fig. 1, is concentrated in the region [1.5, 1.8], close to the true value of 2. The reason for under-estimating  $\theta$  is likely regression dilution—taking the generalized inverse of a noisy matrix adds variance terms to the denominator. We will revisit this explanation when we address regression dilution in a future work.

In comparison, the log-likelihood for  $\hat{\theta}$ , displayed in Fig. 2, is challenging. While, unsurprisingly, the maximum likelihood estimator is vanishingly close to the true value of 2, there are at least four other local maxima, if one counts the boundary  $\hat{\theta} = 0$ . A priori,  $\theta$  could be any positive number, and a local optimization routine is liable to converge to a local maximum that is wrong by an order of magnitude.

Since the dimensionality of this problem is p=1, global optimization may be feasible if an upper bound on  $\theta$  is known in advance. But global optimization has computational cost exponential in p. In our method, the constant in Thm. 8 depends weakly on p.

It is a promising two-step estimation scheme to compute  $\hat{\theta}$  via our plug-in estimator, then use the rough estimate to initialize a local search algorithm for the maximum likelihood or maximum a posteriori point estimate.

# 8. THEORETICAL EXAMPLE

Let the system model be any instantiation of (1) with  $\phi$  satisfying  $\phi(0) = 0$ ,  $\nabla \phi(0) \neq 0$ :

$$\frac{\partial^m}{\partial t^m} y(t, \theta) = \theta^{\mathsf{T}} \phi(\xi(t, \theta)) \tag{9}$$

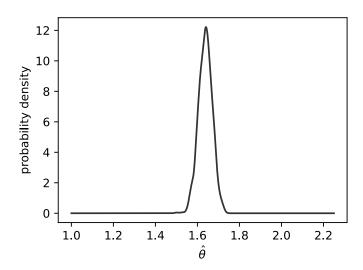


Fig. 1. Sampling distribution of  $\hat{\theta}$ .

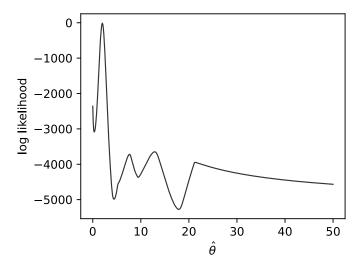


Fig. 2. Log-likelihood of  $\hat{\theta}$ . The highest peak is at  $\hat{\theta} \approx 2$ .

Let  $t_i$  be the observation times. Let the observations obey

$$z_i = y(t_i) + w_i, (10)$$

where the  $w_i$  are independently distributed as

$$w_i = \begin{cases} e + y(t_i) & \text{w.p. } 1/2\\ -e - y(t_i) & \text{w.p. } 1/2 \end{cases}$$
 (11)

for some  $e \in \mathbb{R}$ . Because the noise  $w_i$  is an atomic random variable that depends on  $\theta$ , it is essentially impossible to estimate  $\theta$  by maximum likelihood. The likelihood function is discontinuous, zero almost everywhere, and generically attains the same value on all of its support. Our method, with its accuracy guarantees, still applies, because  $\mathbb{E} \|w_i\|^2 = \|e + y(t_i)\|^2 < \infty$ .

This example also shows the need for ridge regularization. For all sufficiently small  $\epsilon$ , with probability  $2^{-n}$ , every row in the regressor matrix  $\hat{\Phi}$  will be equal to the same small vector of magnitude |e|. Accordingly, the unregularized pseudoinverse of  $\hat{\Phi}$  will have a singular value on the order of  $|e|^{-1}$ , which can cause  $\hat{\theta}$  to blow up. Such an event, in which  $\theta$  is unidentifiable, is exponentially unlikely. Ridge regularization serves as a kind of insurance that truncates this tail risk, so that the estimation error is at least finite.

#### 9. CONCLUSION

Continuous-time parameter-linear systems are widely useful, but they do not have an estimation theory to date that issues finite-sample estimation guarantees. We have provided such a theory by using the parameter-linear structure to reformulate the parameter estimation problem as linear least squares.

Theorem 8, is likely the first identification result in the literature with

- (1) quantitative, finite-sample consistency, for
- (2) a large class of nonlinear systems (1) with continuous and unbounded parameter,
- (3) observed at discrete times, with
- (4) heavy-tailed noise.

The mathematical technology underlying this result is interesting in its own right, and contains new methods for analyzing differentiation filters and regularized least squares.

### REFERENCES

- Amemiya, T. (1985). Advanced econometrics. Harvard University Press, Cambridge, Mass.
- Barbieri, M. and Diversi, R. (2021). Recursive identification of errors-in-variables models with correlated output noise. *IFAC-PapersOnLine*, 54(7), 363–368. doi:10.1016/j.ifacol. 2021.08.386.
- Bittanti, S. (2019). Model identification and data analysis. Wiley, Hoboken, NJ.
- Butcher, J.C. (2006). General linear methods. *Acta Numerica*, 15, 157–256. doi:10.1017/S0962492906220014. Publisher: Cambridge University Press.
- Chen, P., Wen, L., and Sung, S.H. (2020). Strong and weak consistency of least squares estimators in simple linear EV regression models. *Journal of Statistical Planning* and Inference, 205, 64–73. doi:10.1016/j.jspi.2019.06.004.
- Chen, W. and Phelan, G. (2023). Should monetary policy target financial stability? *Review of Economic Dynamics*, 49, 181–200. doi:10.1016/j.red.2022.08.002.
- Dattani, N.S. (2008). Linear Multistep Numerical Methods for Ordinary Differential Equations. doi:10.48550/arXiv. 0810.4965. ArXiv:0810.4965 [math].
- Diekema, E. and Koornwinder, T.H. (2012). Differentiation by integration using orthogonal polynomials, a survey. *Journal of Approximation Theory*, 164(5), 637–667. doi: 10.1016/j.jat.2012.01.003.
- Fosson, S.M., Cerone, V., Regruto, D., and Abdalla, T. (2021). A concave approach to errors-in-variables sparse linear system identification. *IFAC-PapersOnLine*, 54(7), 298–303. doi:10.1016/i.ifacol.2021.08.375.
- Gleser, L.J., Carroll, R.J., and Gallo, P.P. (1987). The Limiting Distribution of Least Squares in an Errors-in-Variables Regression Model. *The Annals of Statistics*, 15(1), 220–233. Publisher: Institute of Mathematical Statistics.
- Hirshberg, D.A. (2021). Least Squares with Error in Variables.
- Jennrich, R.I. (1969). Asymptotic Properties of Non-Linear Least Squares Estimators. *The Annals of Mathematical Statistics*, 40(2), 633–643. doi:10.1214/aoms/1177697731. Publisher: Institute of Mathematical Statistics.

- Keesman, K.J. (2011). System identification: an introduction. Advanced textbooks in control and signal processing. Springer, London; New York. OCLC: ocn747112444.
- Khorasani, M.M. and Weyer, E. (2021). Non-asymptotic Confidence Regions for Errors-In-Variables Systems in an Extended Noise Environment. *IFAC-PapersOnLine*, 54(7), 583–588. doi:10.1016/j.ifacol.2021.08.423.
- Krishnan, S.R. and Seelamantula, C.S. (2013). On the Selection of Optimum Savitzky-Golay Filters. *IEEE Transactions on Signal Processing*, 61(2), 380–391. doi: 10.1109/TSP.2012.2225055. Conference Name: IEEE Transactions on Signal Processing.
- Kuang, S. and Lin, X. (2023). Finite-sample Identification of Continuous-time Parameter-linear Systems. doi:10. 48550/arXiv.2312.05382. ArXiv:2312.05382 [cs, eess, math, stat].
- Lai, T.L. (1994). Asymptotic Properties of Nonlinear Least Squares Estimates in Stochastic Regression Models. The Annals of Statistics, 22(4), 1917–1930. doi:10.1214/ aos/1176325764. Publisher: Institute of Mathematical Statistics.
- Lin, X., Kim, Y., Mohan, S., Siegel, J.B., and Stefanopoulou, A.G. (2019). Modeling and Estimation for Advanced Battery Management. *Annual Review of Control, Robotics, and Autonomous Systems*, 2(1), 393–426. \_eprint: https://doi.org/10.1146/annurev-control-053018-023643.
- Ljung, L. (1998). System Identification: Theory for the User, 2nd Edition.
- Ljung, L. (2010). Perspectives on system identification. Annual Reviews in Control, 34(1), 1–12. doi:10.1016/j. arcontrol.2009.12.001.
- Ljung, L. and Wills, A. (2010). Issues in sampling and estimating continuous-time models with stochastic disturbances. *Automatica*, 46(5), 925–931. doi:10.1016/j. automatica.2010.02.011.
- Maidens, J., Gordon, J.W., Arcak, M., and Larson, P.E.Z. (2016). Optimizing Flip Angles for Metabolic Rate Estimation in Hyperpolarized Carbon-13 MRI. *IEEE Transactions on Medical Imaging*, 35(11), 2403–2412. doi:10.1109/TMI.2016.2574240. Conference Name: IEEE Transactions on Medical Imaging.
- Mania, H., Jordan, M.I., and Recht, B. (2022). Active learning for nonlinear system identification with guarantees. *The Journal of Machine Learning Research*, 23(1), 32:1433–32:1462.
- Matni, N. and Tu, S. (2019). A Tutorial on Concentration Bounds for System Identification. doi:10.48550/arXiv. 1906.11395. ArXiv:1906.11395 [cs, math, stat].
- Niedźwiecki, M.J., Ciołek, M., Gańcza, A., and Kaczmarek, P. (2021). Application of regularized Savitzky–Golay filters to identification of time-varying systems. *Automatica*, 133, 109865. doi:10.1016/j.automatica.2021.109865.
- Ochieng, P.J., Maróti, Z., Dombi, J., Krész, M., Békési, J., and Kalmár, T. (2023). Adaptive Savitzky–Golay Filters for Analysis of Copy Number Variation Peaks from Whole-Exome Sequencing Data. *Information*, 14(2), 128. doi:10.3390/info14020128. Number: 2 Publisher: Multidisciplinary Digital Publishing Institute.
- Othmane, A. (2022). Contributions to numerical differentiation using orthogonal polynomials and its application to fault detection and parameter identification. doctoralThesis, Saarländische Universitäts- und

- Landesbibliothek. doi:10.22028/D291-38806. Accepted: 2023-01-30T13:05:52Z Journal Abbreviation: Sur de la différentiation numérique à l'aide de polynômes orthogonaux et son application à la détection de défauts et à l'identification de paramètres.
- Pillonetto, G., Chen, T., Chiuso, A., De Nicolao, G., and Ljung, L. (2022). Regularized System Identification: Learning Dynamic Models from Data. Communications and Control Engineering. Springer International Publishing, Cham. doi:10.1007/978-3-030-95860-2.
- Pintelon, R. and Schoukens, J. (2012). System identification: a frequency domain approach. John Wiley & Sons Inc, Hoboken, N.J, 2nd ed edition.
- Sadeghi, M., Behnia, F., and Amiri, R. (2020). Window Selection of the Savitzky–Golay Filters for Signal Recovery From Noisy Measurements. *IEEE Transactions* on *Instrumentation and Measurement*, 69(8), 5418–5427. doi:10.1109/TIM.2020.2966310. Conference Name: IEEE Transactions on Instrumentation and Measurement.
- Sarkar, T., Rakhlin, A., and Dahleh, M.A. (2021). Finite Time LTI System Identification. *Journal of Machine* Learning Research, 22(26), 1–61.
- Schmid, M., Rath, D., and Diebold, U. (2022). Why and How Savitzky–Golay Filters Should Be Replaced. ACS Measurement Science Au, 2(2), 185–196. doi:10.1021/acsmeasuresciau.1c00054.
- Soderstrom, T., Fan, H., Carlsson, B., and Bigi, S. (1997). Least squares parameter estimation of continuous-time ARX models from discrete-time data. *IEEE Transactions on Automatic Control*, 42(5), 659–673. doi:10.1109/9.580871. Conference Name: IEEE Transactions on Automatic Control.
- Söderström, T. (2018). Errors-in-variables methods in system identification. Communications and control engineering. Springer, Cham.
- Tsiamis, A., Ziemann, I., Matni, N., and Pappas, G.J. (2022). Statistical Learning Theory for Control: A Finite Sample Perspective.
- Unbehauen, H. and Rao, P. (1997). Identification of Continuous-Time Systems: A Tutorial. *IFAC Proceedings Volumes*, 30(11), 973–999. doi:10.1016/S1474-6670(17) 42970-3.
- van Breugel, F., Kutz, J.N., and Brunton, B.W. (2020). Numerical differentiation of noisy data: A unifying multi-objective optimization framework. *IEEE access: practical innovations, open solutions*, 8, 196865–196877. doi:10. 1109/access.2020.3034077.
- Wedin, P.A. (1973). Perturbation theory for pseudoinverses. BIT Numerical Mathematics, 13(2), 217–232. doi:10.1007/BF01933494.
- Wu, C.F. (1981). Asymptotic Theory of Nonlinear Least Squares Estimation. The Annals of Statistics, 9(3), 501– 513. doi:10.1214/aos/1176345455. Publisher: Institute of Mathematical Statistics.
- Zhang, Y., Liu, X., Yu, Y., and Hu, H. (2020). Asymptotic normality and mean consistency of LS estimators in the errors-in-variables model with dependent errors. *Open Mathematics*, 18(1), 930–947. doi:10.1515/math-2020-0052. Publisher: De Gruyter Open Access.
- Ziemann, I., Tsiamis, A., Lee, B., Jedra, Y., Matni, N., and Pappas, G.J. (2023). A Tutorial on the Non-Asymptotic Theory of System Identification. doi:10.48550/arXiv. 2309.03873. ArXiv:2309.03873 [cs, eess, stat].

# Appendix A. PROOF OF THEOREM 7

# A.1 Building blocks of the filter

Fix  $\rho:[0,1]\to (0,\infty)$ , a Riemann integrable function. Define an inner product on the polynomials of degree at most N-1:

$$\langle f, g \rangle_N = \frac{1}{N} \sum_{j=1}^N \rho\left(\frac{j}{N}\right) f\left(\frac{j}{N}\right) g\left(\frac{j}{N}\right).$$

Let the polynomials  $\{p_N^d: 0 \le d \le N-1\}$ ,  $\deg p_N^d = d$  be an orthogonal family for each N:

$$\left\langle p_N^d, p_N^{d'} \right\rangle_N = 0 \quad \text{when} \quad d \neq d'.$$

As  $\langle f,g\rangle_N$  converges to a Riemann integral as  $N\to\infty,$  so  $p_N^d$  converges too. Define the filter-specific constants

$$h_N^d = \frac{1}{N} \sum_{j=1}^N \rho\left(\frac{j}{N}\right) p_N^d \left(\frac{j}{N}\right) \left(\frac{j}{N}\right)^a$$

and

$$g_N^d = \frac{1}{N} \sum_{j=1}^N \rho\left(\frac{j}{N}\right) \left| p_N^d \left(\frac{j}{N}\right) \right|.$$

Define the problem-specific constants

$$M_i^d = \sup_{t_i \le t \le t_i + n^{-1}NT} \left| \frac{\partial^{d+1}}{\partial t^{d+1}} y^{(d+1)}(t) \right|$$

and

$$(s_i^d)^2 = \frac{1}{N} \sum_{i=1}^N \rho \left(\frac{j}{N}\right)^2 p_N^d \left(\frac{j}{N}\right)^2 \text{var } w((n_i + j)n^{-1}T).$$

# A.2 Choice of filter coefficients

The filter is given by

$$\hat{x}_{i}^{d} = C \sum_{i=1}^{N} \rho\left(\frac{j}{N}\right) p_{N}^{d} \left(\frac{j}{N}\right) z_{n_{i}+j,n}, \quad (A.1)$$

where

$$C = \frac{d!n^d}{N^{d+1}T^dh_N^d}.$$

Let  $\Delta x_i = \hat{x}_i - x(t_i)$ .

Lemma 9. (Bias of the estimated derivative). Let  $\hat{x}_i^d$  be as in (A.1). Then

$$\left| \mathbb{E} \, \Delta x_i^d \right| \leq \left( \frac{M_i^d T}{d+1} \right) \left( \frac{N}{n} \right) \frac{g_N^d}{h_N^d}.$$

**Proof.** By Taylor's theorem,

$$\mathbb{E} z_{n_i+j,n} = y \left( \frac{(n_i+j)T}{n} \right)$$

$$= \sum_{k=0}^d \frac{y^{(k)}(t_i)}{k!} \left( \frac{NT}{n} \right)^k \left( \frac{j}{N} \right)^k + R \qquad (A.2)$$

where the remainder R = R(i, d, n, N) satisfies

$$|R| \le R_{\text{max}} = \left(\frac{NT}{n}\right)^{d+1} \frac{M_i^d}{(d+1)!}.$$
 (A.3)

We now insert this into (A.1). By the orthogonality property of  $p_N^d$ , all of the lower-order Taylor expansion terms of  $\mathbb{E} z_{n_i+j,n}$  vanish.

$$\mathbb{E}\,\hat{x}_i^d = C\sum_{j=1}^N \rho\left(\frac{j}{N}\right) p_N^d \left(\frac{j}{N}\right) \frac{x_i^d}{d!} \left(\frac{NT}{n}\right)^d \left(\frac{j}{N}\right)^d + C\sum_{j=1}^N \rho\left(\frac{j}{N}\right) p_N^d \left(\frac{j}{N}\right) R(j, d, n, N) \tag{A.4}$$

By our choice of C, the first sum evaluates to  $x_i^d$ .

$$\begin{split} \left| \mathbb{E} \, \hat{\boldsymbol{x}}_i^d - \boldsymbol{x}_i^d \right| &\leq CNR_{\max} g_N^d \\ &= \left( \frac{M_i^d T}{d+1} \right) \left( \frac{N}{n} \right) \frac{g_N^d}{h_N^d}. \end{split}$$

Lemma 10. (Variance of the estimated derivative). Let  $\hat{x}_i^d$  be as in (A.1). Then

$$\operatorname{var} \hat{x}_i^d \le \frac{n^{2d}}{N^{2d+1}} \left( \frac{d! s_{i,N}^d}{T^d h_N^d} \right)^2.$$

**Proof.** Direct calculation beginning with

$$\operatorname{var} \hat{x}_{i}^{d} = C^{2} \sum_{j=1}^{N} \left[ \rho \left( \frac{j}{N} \right)^{2} p_{N}^{d} \left( \frac{j}{N} \right)^{2} \cdot \operatorname{var} w((n_{i} + j) n^{-1} T) \right].$$

Theorem 11. (MSE of the estimated derivative). The mean squared error (MSE) of  $\hat{x}_i^d$  satisfies

$$\mathbb{E} \left\| \Delta x_i^d \right\|^2 \le A_{i,N}^d \frac{N^2}{n^2} + B_{i,N}^d \frac{n^{2d}}{N^{2d+1}},$$

with constants

$$\begin{split} A_{i,N}^d &= \left(\frac{M_i^d T}{d+1}\right)^2 \left(\frac{g_N^d}{h_N^d}\right)^2 \quad \text{and} \\ B_{i,N}^d &= \left(\frac{d! s_{i,N}^d}{T^d h_N^d}\right)^2. \end{split}$$

**Proof.** Follows from applying the identity  $MSE = Bias^2 + Variance to Lemmas 9 and 10.$ 

# A.3 Choice of N

To pick the filter window size N, add up Thm. 11 across all  $d=0,1,\ldots,m$ .

$$\mathbb{E} \|\Delta x_i\|^2 \le \sum_{d=0}^m A_{i,N}^d \frac{N^2}{n^2} + \sum_{d=0}^m B_{i,N}^d \frac{n^{2d}}{N^{2d+1}}$$
 (A.5)

We can then bound the total MSE across all windows as

$$\mathbb{E}\sum_{i=1}^{n'} \|\Delta x_i\|^2 \le \tilde{A}_N \frac{N^2}{n^2} + \tilde{B}_N \frac{n^{2m}}{N^{2m+1}},\tag{A.6}$$

$$\tilde{A}_{N} = \sum_{i=1}^{n'} \sum_{d=0}^{m} A_{i,N}^{d}, \tag{A.7}$$

$$\tilde{B}_N = \sum_{i=1}^{n'} \sum_{d=0}^{m} B_{i,N}^d. \tag{A.8}$$

 $<sup>^6</sup>$  This is a rank-N quadratic form and therefore doesn't extend to an inner product on general f and g.

Optimizing in N, we pick

$$N = \left[ \frac{(2m+1)\hat{B}_N}{2\hat{A}_N} \right]^{\frac{1}{2m+3}} n^{\frac{2m+2}{2m+3}},$$

so that the total MSE obeys

$$\mathbb{E}\sum_{i=1}^{n'} \|\Delta x_i\|^2 \le \epsilon_N n' n^{-\frac{2}{2m+3}},\tag{A.9}$$

where

$$\epsilon_N = 2(2m+1)^{\frac{1}{2m+3}} \hat{A}_N^{\frac{2m+1}{2m+3}} \hat{B}_N^{\frac{2}{2m+3}} (n')^{-1}$$
 (A.10)

Likewise, let  $\epsilon_{N,m} \leq \epsilon_N$  be a constant such that

$$\mathbb{E}\sum_{i=1}^{n'} \|\Delta x_i^m\|^2 \le \epsilon_{N,m} n' n^{-\frac{2}{2m+3}}.$$
 (A.11)

# Appendix B. PROOF OF THEOREM 8

Let  $\Delta \phi_i = \hat{\phi}_i - \phi_i$ . By Hölder continuity,  $\|\Delta \phi_i\| \le C_{\phi} \|\Delta x_i\|^{\alpha}$ . Let  $\delta = \|\Delta \Phi\|$ .

# B.1 Error analysis

To analyze the error  $\Delta \theta = \hat{\theta} - \theta$ , let  $v = \hat{u} - u$  and  $S = \hat{R}^{\dagger} - R^{\dagger}.$ 

$$\Delta\theta = (R^{\dagger} + S)(u + v) - R^{\dagger}u \tag{B.1}$$

$$= R^{\dagger} v + S \left( u + v \right) \tag{B.2}$$

$$\|\Delta\theta\| \le \frac{\|v\|}{\sigma_{\min}(R)} + \|S\| \|u\| + \|S\| \|v\|$$
 (B.3)

To extract the dependence on n', we introduce the variables

$$\bar{\sigma} = (n')^{-1/2} \sigma_{\min}(\Phi), \tag{B.4}$$

$$\bar{U} = (n')^{-1/2} \|u\| \tag{B.5}$$

These are scaled so that they approach limits as  $n \to \infty$ . Lemma 12. (Moment of v). For all  $r \in (0, 2]$ , the random variable v satisfies

$$\mathbb{E} \|v\|^r \le \frac{\epsilon_{N,m}^{r/2} (n')^{r/2}}{n^{\frac{r}{2m+3}}}.$$

**Proof.** By (A.11),

$$\mathbb{E} \|v\|^2 = \mathbb{E} \sum_{i=1}^{n'} \|\Delta x_i^m\|^2$$

$$\leq \epsilon_{N,m} n' n^{-\frac{2}{2m+3}}.$$
(B.6)

For all  $0 < r \le 2$ , Jensen's gives

$$\mathbb{E} \|v\|^r \le \frac{\epsilon_{N,m}^{r/2} (n')^{r/2}}{n^{\frac{r}{2m+3}}}.$$
 (B.7)

Lemma 13. (Moment of  $\delta$ ). The random variable  $\delta$  satis-

$$\mathbb{E}\,\delta^{2/\alpha} \le \frac{C_{\phi}^{2/\alpha} \epsilon_N(n')^{1/\alpha}}{n^{\frac{2}{2m+3}}}.$$

**Proof.** Bounding  $\delta$  by the Frobenius norm of  $\Delta\Phi$ , we have

$$\delta^2 \le \sum_{i=1}^{n'} \|\Delta \phi_i\|^2 \tag{B.8}$$

$$\leq C_{\phi}^{2} \sum_{i=1}^{n'} \|\Delta x_{i}\|^{2\alpha}$$
 (B.9)

By Hölder's inequality with exponents  $1/\alpha$  and  $1/(1-\alpha)$ ,

$$\leq C_{\phi}^{2} \left( \sum_{i=1}^{n'} \left( \|\Delta x_{i}\|^{2\alpha} \right)^{1/\alpha} \right)^{\alpha} \left( \sum_{i=1}^{n'} 1^{\frac{1}{1-\alpha}} \right)^{1-\alpha}$$
(B.10)

Raising both sides to the power  $1/\alpha$  and taking expecta-

$$\delta^{2/\alpha} \le C_{\phi}^{2/\alpha} (n')^{1/\alpha - 1} \sum_{i=1}^{n'} \|\Delta x_i\|^2$$
 (B.11)

$$\mathbb{E}\,\delta^{2/\alpha} \le C_{\phi}^{2/\alpha}(n')^{1/\alpha - 1} \sum_{i=1}^{n'} \mathbb{E}\,\|\Delta x_i\|^2 \tag{B.12}$$

Applying (A.9),

$$\leq \frac{C_{\phi}^{2/\alpha} \epsilon_N(n')^{1/\alpha}}{n^{\frac{2}{2m+3}}}.$$
(B.13)

To control S, we state and then apply a self-contained technical lemma, proven in Appendix C.

Lemma 14. Let  $A \in \mathbb{R}^{m \times n}$  have full column rank and minimal singular value  $\sigma$ , and let D be a real matrix of the same dimensions as A, having operator norm  $\delta$ . For any

$$B = \begin{pmatrix} 0_{n \times n} \\ A \end{pmatrix}^{\dagger}, \quad \hat{B} = \begin{pmatrix} \lambda I_n \\ A + D \end{pmatrix}^{\dagger}, \quad \text{and} \quad S = B - \hat{B}$$

Then

$$||S|| \le \frac{2(\lambda + \delta)}{\sigma(\lambda + (\sigma - \delta)^+)}.$$

Lemma 15. (High moment of S). The random matrix S = $\hat{R}^{\dagger} - R^{\dagger}$  satisfies

$$\mathbb{E} \left\| S \right\|^{2/\alpha} \leq \frac{2^{4/\alpha - 1} \mathbb{E} \, \delta^{2/\alpha}}{\lambda^{2/\alpha} \overline{\sigma}^{2/\alpha} (n')^{1/\alpha}} + \frac{2^{4/\alpha - 1}}{\overline{\sigma}^{2/\alpha} (n')^{1/\alpha}}.$$

**Proof.** Applying Lemma 14 to the facts of the case,

$$||S|| \le \frac{2(\delta + \lambda)}{\overline{\sigma}\sqrt{n'}\left((\overline{\sigma}\sqrt{n'} - \delta)^{+} + \lambda\right)}$$
(B.14)

As we will see that  $\delta \stackrel{\mathbb{P}}{\longrightarrow} 0$  as  $n \to \infty$ , the following bound is very conservative for large n, but it is difficult to do better when we need a high moment of S.

$$\leq \frac{2\left(\delta + \lambda\right)}{\lambda \overline{\sigma} \sqrt{n'}} \tag{B.15}$$

Using the fact that  $(a+b)^p \le 2^{p-1}(a^p+b^p)$  for  $a,b \ge 0$ 

$$||S||^{2/\alpha} \le 2^{4/\alpha - 1} \frac{\delta^{2/\alpha} + \lambda^{2/\alpha}}{\lambda^{2/\alpha} \sigma_{\min}^{2/\alpha}(\Phi)}$$

$$\mathbb{E} ||S||^{2/\alpha} \le \frac{2^{4/\alpha - 1} \mathbb{E} \delta^{2/\alpha}}{\lambda^{2/\alpha} \overline{\sigma}^{2/\alpha}(n')^{1/\alpha}} + \frac{2^{4/\alpha - 1}}{\overline{\sigma}^{2/\alpha}(n')^{1/\alpha}}$$
(B.16)

$$\mathbb{E} \|S\|^{2/\alpha} \le \frac{2^{4/\alpha - 1} \mathbb{E} \delta^{2/\alpha}}{\lambda^{2/\alpha} \overline{\sigma}^{2/\alpha} (n')^{1/\alpha}} + \frac{2^{4/\alpha - 1}}{\overline{\sigma}^{2/\alpha} (n')^{1/\alpha}}$$
(B.17)

This bound is not sharp enough to control the Su error term by Hölder's inequality. Some improvement is possible by dividing and conquering  $S = S\mathbf{1}_{\delta \leq \lambda} + S\mathbf{1}_{\delta > \lambda}$ , head and tail. Restricted to the head, S satisfies, in the terms of Lemma 14,  $||S|| \lesssim \frac{\lambda}{\sigma^2}$ ; restricted to the tail S satisfies the a priori  $S \leq \frac{1}{\sigma}$ , which can be combined by Hölder's inequality with the decaying  $\mathbb{E}_{\delta > \lambda}$ . However since  $\lambda < \sigma$  by hypothesis and we will later prefer  $\lambda \ll \sigma$ , it behooves to get tighter control of S on the event  $\{\lambda < \delta \leq \sigma\}$ .

We first state a self-contained technical lemma, proved in Appendix D, concerning Tikhonov pseudoinverses of perturbed matrices.

Lemma 16. (Head-body-tail). Let  $A \in \mathbb{R}^{m \times n}$  have full column rank and minimal singular value  $\sigma$ , and let D be a real random matrix of the same dimensions as A, having operator norm  $\delta$ . For  $\lambda \in (0, \sigma)$ , let

$$B = \begin{pmatrix} 0_{m \times m} \\ A \end{pmatrix}^{\dagger}, \quad \hat{B} = \begin{pmatrix} \lambda I_m \\ A + D \end{pmatrix}^{\dagger}, \quad \text{and} \quad S = B - \hat{B}$$

Then

$$\mathbb{E} \|S\| \le s_{\text{head}} + s_{\text{body}} + s_{\text{tail}},$$

where

$$s_{\text{head}} = \frac{4\lambda}{\sigma^2},$$

$$s_{\text{body}} = 3 \int_{\lambda}^{\sigma} \frac{(\lambda + \sigma) \mathbb{P}(\delta > v)}{\sigma(\lambda + \sigma - v)^2} \, dv,$$

and

$$s_{\text{tail}} = \mathbb{E} \frac{4\delta}{\sigma \lambda} \mathbf{1}_{\delta > \sigma}.$$

Now we return the problem at hand by going through the steps of Lemma 16, in which we have an integrability condition on  $\delta$ .

Lemma 17. The random matrix ||S|| satisfies

$$\mathbb{E} \|S\| \le s_{\text{head}} + s_{\text{body}} + s_{\text{tail}}$$

where

$$\begin{split} s_{\rm head} &= \frac{4\lambda}{\overline{\sigma}^2 n'}, \\ s_{\rm body} &= \frac{6C_\phi^2 \epsilon_N^\alpha}{n^{\frac{2\alpha}{2m+3}}} \left[ \frac{2\log(\overline{\sigma}\sqrt{n'}/\lambda)}{\overline{\sigma}^3 \sqrt{n'}} + \frac{1}{\lambda \overline{\sigma}^2} \right], \end{split}$$

and

$$s_{\text{tail}} = \frac{4C_{\phi}^{2/\alpha} \epsilon_{N}}{\lambda \overline{\sigma}^{2/\alpha} n^{\frac{2}{2m+3}}}.$$

**Proof.** We can immediately transcribe  $s_{\text{head}}$  from the statement of Lemma 16.

We proceed with  $s_{\text{body}}$ , using Chebyshev's inequality to bound  $\mathbb{P}(\delta > v)$ . Using  $\sigma = \overline{\sigma} \sqrt{n'}$  for temporary brevity,

$$3 \int_{\lambda}^{\sigma} \frac{(\lambda + \sigma) \mathbb{P}(\delta > v)}{\sigma(\lambda + \sigma - v)^{2}} dv$$

$$\leq 3 \mathbb{E} \delta^{2} \int_{\lambda}^{\sigma} \frac{\lambda + \sigma}{(\lambda + \sigma - v)^{2} v^{2} dv}$$

$$= \frac{6 \mathbb{E} \delta^{2}}{\sigma} \left[ \frac{2 \log(\sigma/\lambda)}{(\sigma + \lambda)^{2}} + \frac{\sigma - \lambda}{(\sigma + \lambda)\sigma\lambda} \right]$$

"Rounding up" for simplicity.

$$\leq \frac{6 \, \mathbb{E} \, \delta^2}{\sigma} \left\lceil \frac{2 \log(\sigma/\lambda)}{\sigma^2} + \frac{1}{\sigma \lambda} \right\rceil$$

Applying Lemma 13 we get this lemma's  $s_{\text{body}}$ .

$$\leq \frac{6C_{\phi}^2\epsilon_N^{\alpha}}{n^{\frac{2\alpha}{2m+3}}} \left\lceil \frac{2\log(\overline{\sigma}\sqrt{n'}/\lambda)}{\overline{\sigma}^3\sqrt{n'}} + \frac{1}{\lambda\overline{\sigma}^2} \right\rceil.$$

As for  $s_{\text{tail}}$ , by Hölder's inequality,

$$\mathbb{E} \frac{4\delta}{\sigma \lambda} \mathbf{1}_{\delta > \sigma} 
\leq \frac{4}{\lambda \overline{\sigma} \sqrt{n'}} \left( \mathbb{E} \, \delta^{2/\alpha} \right)^{\alpha/2} \left[ \mathbb{P} \left( \delta > \overline{\sigma} \sqrt{n'} \right) \right]^{(2-\alpha)/2} 
\leq \frac{4}{\lambda \overline{\sigma} \sqrt{n'}} \left( \mathbb{E} \, \delta^{2/\alpha} \right)^{\alpha/2} \left[ \frac{\mathbb{E} \, \delta^{2/\alpha}}{(\overline{\sigma}^2 n')^{1/\alpha}} \right]^{(2-\alpha)/2} 
= \frac{4}{\lambda \overline{\sigma} \sqrt{n'}} \frac{\mathbb{E} \, \delta^{2/\alpha}}{(\overline{\sigma}^2 n')^{\frac{2-\alpha}{2\alpha}}} 
= \frac{4 \mathbb{E} \, \delta^{2/\alpha}}{\lambda \overline{\sigma}^{2/\alpha} (n')^{1/\alpha}} 
\leq \frac{4 C_{\phi}^{2/\alpha} \epsilon_N}{\lambda \overline{\sigma}^{2/\alpha} n^{\frac{2}{2m+3}}}.$$

#### B.2 Bound on $\Delta\theta$

We have now controlled each term in (B.3),

$$\|\Delta\theta\| \le \frac{\|v\|}{\sigma_{\min}(R)} + \|S\| \|u\| + \|S\| \|v\|.$$

Theorem 18. (First moment of  $\Delta\theta$ ). The random vector  $\Delta\theta$  satisfies

$$\begin{split} \mathbb{E} \left\| \Delta \theta \right\| & \leq \frac{\epsilon_{N,m}^{1/2}}{\overline{\sigma} n^{\frac{1}{2m+3}}} + \frac{4 \overline{U} \lambda}{\overline{\sigma}^2 \sqrt{n'}} \\ & + \frac{6 C_{\phi}^{2\alpha} \overline{U} \epsilon_N^{\alpha}}{n^{\frac{2\alpha}{2m+3}}} \left[ \frac{2 \log(\overline{\sigma} \sqrt{n'}/\lambda)}{\overline{\sigma}^3} + \frac{\sqrt{n'}}{\lambda \overline{\sigma}^2} \right] \\ & + \frac{4 C_{\phi}^2 \overline{U} \epsilon_N \sqrt{n'}}{\lambda \overline{\sigma}^{2/\alpha} n^{\frac{2}{2m+3}}} \\ & + \frac{2^{2-\frac{\alpha}{2}} \epsilon_{N,m}^{1/2}}{n^{\frac{1}{2m+3}}} \left( \frac{C_{\phi}^2 \epsilon_N (n')^{1/\alpha}}{\overline{\sigma}^{2/\alpha} n^{\frac{2}{2m+3}} \lambda^{2/\alpha}} + \frac{1}{\overline{\sigma}^{2/\alpha}} \right)^{\alpha/2}. \end{split}$$

**Proof.** According to Lemma 12,

$$\mathbb{E}\,\frac{\|v\|}{\sigma_{\min}(R)} \leq \frac{\epsilon_{N,m}^{1/2}}{\overline{\sigma}n^{\frac{1}{2m+3}}}$$

According to Lemma 17, we can bound  $||u|| \mathbb{E} ||S||$  as the sum of

$$\begin{aligned} & \|u\| \, s_{\text{head}} = \frac{4 \bar{U} \lambda}{\bar{\sigma}^2 \sqrt{n'}}, \\ & \|u\| \, s_{\text{body}} = \frac{6 C_\phi^{2\alpha} \bar{U} \epsilon_N^\alpha}{n^{\frac{2\alpha}{2m+3}}} \left[ \frac{2 \log(\bar{\sigma} \sqrt{n'}/\lambda)}{\bar{\sigma}^3} + \frac{\sqrt{n'}}{\lambda \bar{\sigma}^2} \right], \end{aligned}$$

and

$$\left\|u\right\|s_{\mathrm{tail}} = \frac{4C_{\phi}^2 \bar{U} \epsilon_N \sqrt{n'}}{\lambda \overline{\sigma}^{2/\alpha} n^{\frac{2}{2m+3}}}.$$

Using Lemmas 15 and 12 in Hölder's inequality,

$$\begin{split} \mathbb{E} \, \|S\| \, \|v\| & \leq \left( \mathbb{E} \, \|S\|^{2/\alpha} \right)^{\alpha/2} \left( \mathbb{E} \, \|v\|^{2/(2-\alpha)} \right)^{(2-\alpha)/2} \\ & \leq \left( \frac{2^{4/\alpha - 1} \, \mathbb{E} \, \delta^{2/\alpha}}{\lambda^{2/\alpha} \overline{\sigma}^{2/\alpha} (n')^{1/\alpha}} + \frac{2^{4/\alpha - 1}}{\overline{\sigma}^{2/\alpha} (n')^{1/\alpha}} \right)^{\alpha/2} \\ & \cdot \left( \frac{\epsilon_{N,m}^{1/2} (n')^{1/2}}{n^{\frac{1}{2m+3}}} \right) \\ & \leq \frac{2^{2 - \frac{\alpha}{2}} \epsilon_{N,m}^{1/2}}{n^{\frac{1}{2m+3}}} \left( \frac{\mathbb{E} \, \delta^{2/\alpha}}{\lambda^{2/\alpha} \overline{\sigma}^{2/\alpha}} + \frac{1}{\overline{\sigma}^{2/\alpha}} \right)^{\alpha/2} \\ & \leq \frac{2^{2 - \frac{\alpha}{2}} \epsilon_{N,m}^{1/2}}{n^{\frac{1}{2m+3}}} \left( \frac{C_{\phi}^{2/\alpha} \epsilon_{N} (n')^{1/\alpha}}{\lambda^{2/\alpha} \overline{\sigma}^{2/\alpha} n^{\frac{2}{2m+3}}} + \frac{1}{\overline{\sigma}^{2/\alpha}} \right)^{\alpha/2}. \end{split}$$

The terms in this bound go both ways with respect to n'. On one hand, there is  $(n')^{-1/2}$  scaling reminiscent of the concentration of measure in conventional linear regression. On the other hand, pseudo-inverting a random matrix with n' rows offers n' ways to go wrong, and we get a  $(n')^{1/2}$  scaling in those terms.

A parsimonious choice of  $\lambda$  is

$$\frac{\lambda^*}{\sqrt{n'}} = \frac{C_{\phi} \epsilon_N^{\alpha/2}}{n^{\alpha/(2m+3)}},\tag{B.18}$$

which consolidates the two terms of Lemma 15, making the ||S|| ||v|| estimate proportional to the  $||v|| / \sigma_{\min}(R)$  estimate. Applying the choice  $\lambda = \lambda^*$  of (B.18) to Thm. 18 proves Theorem 8.

# Appendix C. PROOF OF LEMMA 14

Using Thm. 4.1 of Wedin (1973) for the full-rank perturbation of a full-rank rectangular matrix,

$$||S|| \le \sqrt{2} \left\| \begin{pmatrix} 0_{n \times n} \\ A \end{pmatrix}^{\dagger} \right\| \left\| \begin{pmatrix} \lambda I_n \\ A + D \end{pmatrix}^{\dagger} \right\| \left\| \begin{pmatrix} \lambda I_n \\ D \end{pmatrix} \right\|. \quad (C.1)$$

Recall that the operator norm of a matrix's pseudoinverse is the inverse of its least singular value. Thus,

$$\left\| \begin{pmatrix} 0_{n \times n} \\ A \end{pmatrix}^{\dagger} \right\| = \frac{1}{\sigma} \tag{C.2}$$

and

$$\left\| \begin{pmatrix} \lambda I_n \\ A+D \end{pmatrix}^{\dagger} \right\|^{-2} = \lambda^2 + \sigma_{\min}^2(A+D).$$

By Jensen's and Weyl's inequalities,

$$\left\| \begin{pmatrix} \lambda I_n \\ A+D \end{pmatrix}^{\dagger} \right\|^{-1} \ge \frac{\lambda + \sigma_{\min}(A+D)}{\sqrt{2}}.$$

$$\ge \frac{\lambda + (\sigma - \delta)^+}{\sqrt{2}} \tag{C.3}$$

where  $(\cdot)^+ = \max(0, \cdot)$ . We bound the last factor of (C.1) using the triangle inequality.

# Appendix D. PROOF OF LEMMA 16

Recall the a priori bound from Lemma 14,

$$||S|| \le \frac{2(\delta + \lambda)}{\sigma(\lambda + (\sigma - \delta)^+)}.$$
 (D.1)

We will integrate  $\mathbb{E} \|S\|$  over three zones of  $\delta$ : "head,"  $[0, \lambda]$ ; "body,"  $(\lambda, \sigma]$ ; and "tail,"  $(\sigma, \infty)$ .

First, we have the deterministic

$$||S|| \mathbf{1}_{\delta \le \lambda} \le \frac{4\lambda}{\sigma^2} \mathbf{1}_{\delta \le \lambda} \tag{D.2}$$

Second, define the function  $F(x) = \frac{2(\lambda+x)}{\sigma(\lambda+\sigma-x)}$ . We will execute a "layer cake" rearrangement of  $F(\delta)$ .

$$||S|| \mathbf{1}_{\lambda < \delta \le \sigma} \le F(\delta) \mathbf{1}_{\lambda < \delta \le \sigma}$$

By the Fundamental Theorem of Calculus,

$$= \left( F(\lambda) + \int_{\lambda}^{\delta} F'(v) \, dv \right) \mathbf{1}_{\lambda < \delta \le \sigma} \quad (D.3)$$
$$= F(\lambda) \mathbf{1}_{\lambda < \delta} + \left( \int_{\lambda}^{\delta} F'(v) \, dv \right) \mathbf{1}_{\lambda < \delta \le \sigma} \quad (D.4)$$

Let us combine the former term of (D.4) with (D.2), and call it  $s_{\text{head}}$ . The latter term we will call  $s_{\text{body},0}$ . We get the bound

$$\begin{split} \|S\| \, \mathbf{1}_{\delta \le \sigma} & \le s_{\text{head}} + s_{\text{body},0}, \\ s_{\text{body},0} & = \left( \int_{\lambda}^{\delta} F'(v) \, dv \right) \mathbf{1}_{\lambda < \delta \le \sigma} \\ & = \left( \int_{\lambda}^{\infty} F'(v) \mathbf{1}_{\lambda < v < \delta} \, dv \right) \mathbf{1}_{\lambda < \delta \le \sigma} \end{split}$$

We conservatively (giving up the predicate  $\delta \leq \sigma$ ) rework the logic so that the bounds of integration do not depend on  $\delta$ .

$$\leq \int_0^\infty F'(v) \mathbf{1}_{\lambda < v < \sigma} \mathbf{1}_{v < \delta} \, dv$$
$$= \int_{\lambda}^{\sigma} F'(v) \mathbf{1}_{v < \delta} \, dv$$

Taking the expectation of both sides and applying Tonelli's theorem,

$$\mathbb{E} \, s_{\text{body},0} \le \int_{\lambda}^{\sigma} F'(v) \, \mathbb{P}(\delta > v) \, \, \mathrm{d} \, v$$

By Chebyshev's inequality,

$$= \int_{\lambda}^{\sigma} \frac{2(2\lambda + \sigma) \mathbb{P}(\delta > v)}{\sigma(\lambda + \sigma - v)^2} dv$$

To balance  $\lambda$  and  $\sigma$ , we trade in  $\lambda/2$  for  $\sigma/2$ .

$$\leq 3 \int_{\lambda}^{\sigma} \frac{(\lambda + \sigma) \mathbb{P}(\delta > v)}{\sigma(\lambda + \sigma - v)^2} dv$$

We call this last quantity  $s_{\text{body}}$ . When  $\delta$  is large, (D.1) becomes

$$||S|| \mathbf{1}_{\delta > \sigma} \le \frac{3(\delta + \lambda)}{\sigma \lambda} \mathbf{1}_{\delta > \sigma}.$$

We do not lose much by simplifying  $\lambda < \sigma < \delta$ .

$$\leq \frac{4\delta}{\sigma\lambda} \mathbf{1}_{\delta > \sigma}.$$

The expectation of this quantity constitutes  $s_{\text{tail}}$ .