A simplified Parisi Ansatz II: REM universality

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Abstract

In a previous work [*A simplified Parisi Ansatz*, Franchini, S., Commun. Theor. Phys., 73, 055601 (2021)] we introduced a simple method to compute the Random Overlap Structure of Aizenmann, Simm and Stars and the full-RSB Parisi formula for the Sherrington-Kirckpatrick Model without using replica theory. The method consists in partitioning the system into smaller sub-systems that we call layers, and iterate the Bayes rule. A central ansatz in our derivation was that these layers could be approximated by Random Energy Models of the Derrida type. In this paper we analyze the properties of the interface in detail, and show the equivalence with the Random Energy Model at any temperature.

Keywords: Sherrington-Kirkpatrick model, Cavity methods, Random Energy Model, Parisi formula, REM Universality.

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1 Introduction

The Sherrington-Kirkpatrick (SK) model is a well known toy model for complex systems, and plays a central role in the celebrated "Replica Symmetry Breaking" (RSB) theory of spin glasses (SG) by Parisi, Mezard, Virasoro [1] and many others [2]. In previous papers [3, 4] we introduced a generalized cavity method to study the SK model and many other physical systems without relying on the so-called "replica trick": as is shown in [3], this method allows a natural derivation of the Random Overlap Structure (ROSt) of Aizenmann, Simms and Starr [5], and the full-RSB Parisi functional for computing the free energy per spin [3, 4]. The main steps of our analysis where to define a sequence of SK models of increasing sizes by partitioning the vertices set into subsets, that we call layers, and then show that these layers can be approximated by a simpler noise model, that we call **interface**, where the Hamiltonian is simply the scalar product between the spin state and some external field (see Lemma 10 of [4]). In [3] a crucial claim was that the interface can be approximated by a Random Energy Model (REM) [6], the simplest toy model for disordered systems. Introduced by B. Derrida in the 1980s, this model has inspired important mathematical advances in the understanding of spin glasses, particularly trough its relation with the Poisson Point Processes (PPP). Of special interest is the **REM universality** [7]. After several precursor papers, worth to cite Ebeling and Nadler [8], Mertens [9], Borgs, Chayes and Pittel [10], etc. the REM universality has been finally recognized, by Mertens, Franz and Bauke [11], in the context of combinatorial optimization, and further investigated by other authors, see [7] for a survey. As is said in [7], the basic phenomenon is in that the micro-canonical distribution of the energies of a large class of models is close to a REM in distribution for certain energy windows. By implementing a form of REM universality, in Lemma 12 of [4] is shown that the thermal fluctuations of the interface energy near the ground state converge to a REM at near zero-temperature. In this paper we study the interface model in detail, compute the thermodynamic limit, and show the equivalence with the REM at any temperature. We remark that present paper only aims to describe the interface model, ie the one body hamiltonian in Eq. (2) below, that to best of our knowledge does not have a dedicated paper describing its properties. We do not discuss the SK model here, altough the results can be obviously applied to the SK model following the methods shown in [3, 4]

2 Summary

Let briefly introduce the basic notation. Let $V = \{1, 2, ..., N\}$ be a set of N vertices and put a spin $\sigma_i \in \Omega$ of inner states $\Omega = \{+, -\}$ on each vertex, we denote by

$$\sigma_V := \{ \sigma_i \in \Omega : i \in V \}$$
⁽¹⁾

the generic magnetization state. The support of σ_V is the product space Ω^V . We denote by $\mathbb{I}(A)$ the indicator function of the event *A*, that is $\mathbb{I}(A) = 1$ if *A* is verified and zero otherwise. Also, given two ordered sets *A* and *B* we use notation $A \otimes B$ for the tensor product and just *AB* for the Cartesian product. The scalar product is denoted by the usual \cdot symbol. The Hadamard product is denoted by the \circ symbol.

2.1 The interface model

Following ideas from Borgs [12, 13], Coja-Oghlan [14] and others, in a recent publication we showed [4] that the scalar product of a spin state with some external field, that we call **interface model**, and formally describe with the Hamiltonian

$$H(\sigma_V|h_V) := \sigma_V \cdot h_V, \tag{2}$$

can be approximated by the REM in the low temperature phase. The field components are real numbers $h_i \in \mathbb{R}$ indexed by $i \in V$ and are assumed to have been independently extracted from some probability distribution p. We will use a braket notation for the average of the test function f respect to the Gibbs measure ξ

$$\langle f(\boldsymbol{\sigma}_{V}) \rangle_{\xi} := \sum_{\boldsymbol{\sigma}_{V} \in \Omega^{V}} f(\boldsymbol{\sigma}_{V}) \exp\left[-\boldsymbol{\sigma}_{V} \cdot \boldsymbol{\beta} h_{V} + N \boldsymbol{\beta} \zeta\left(\boldsymbol{\beta} h_{V}\right)\right],$$
(3)

where N is the number of spins and ζ is the free energy density per spin:

$$-\beta \zeta \left(\beta h_V\right) := \frac{1}{N} \sum_{i \in V} \log 2 \cosh\left(\beta h_i\right).$$
(4)

We remark that the interface model is closely related to the "Number Partitioning Problem" (NPP) [11, 12, 13], and we may also refer to it as a random field model, or noise model. It correspond for example to the Random Field Ising Model (RIFM) in the limit of zero Ising interactions (or infinite field amplitude) and many other models. In general, the interface could be seen as the zero interaction limit of any lattice field theory of the kind described in [15].

2.2 Thermodynamic limit

In Section 3.1 we study the thermodynamic limit by quantile mechanics [17] and series analysis, and give explicit examples for the binary, uniform and Gaussian cases. The scaling limit of the free energy density for an infinite number of spins will converge almost surely to the following functional:

$$-\lim_{N\to\infty}\beta\zeta(\beta h_V) = \int_0^1 dq \log 2\cosh[\beta x(q)].$$
(5)

The function x(q) is called **quantile** and is found by inverting the cumulant of p,

$$q(x) := \int_0^x dz \ p(z), \tag{6}$$

by quantile mechanics [17] the quantile satisfies the differential equation

$$\partial_q^2 x(q) = \boldsymbol{\rho} \left[x(q) \right] \left[\partial_q x(q) \right]^2, \tag{7}$$

where the function ρ is defined from p according to the relation

$$\boldsymbol{\rho}\left(x\right) := -\partial_x \log p\left(x\right). \tag{8}$$

We explicitly compute the uniform and Gaussian cases. At high temperature we find that, as expected, the free energy is replica symmetric, and is therefore linear in temperature. At low temperature we find that the convergence toward the ground state energy is quadratic in the temperature, ie., the specific heat is linear like in the Dulong-Petit law. The origin of this quadratic convergence is due to vertices with small field amplitude (see Section 3), and notice that, if we restrict to linear terms, the low temperature modes can be neglected and the free energy is approximately constant in temperature, like in the REM. In Section 5 of [4] the convergence to the REM is actually shown in distribution in the near zero temperature phase.

2.3 **REM universality**

To this scope we introduce the eigenstates of magnetization

$$\Omega(m) := \left\{ \sigma_V \in \Omega^V : M(\sigma_V) = \lfloor mN \rfloor \right\},\tag{9}$$

where $M(\sigma_V)$ stands for the total magnetization of the state σ_V . These central objects of our analysis are studied in detail in Section 4. We also introduce the "master direction"

 ω_V , the flickering state σ_V^* and flickering function f^*

$$\omega_i := h_i / |h_i|, \quad \sigma_i^* := \sigma_i \,\omega_i, \quad f^*(\sigma_V) := f(\sigma_V \circ \omega_V), \tag{10}$$

where ω_i is the direction of the ground state in the vertex *i*, and *f* is any test function if not specified otherwise. We can now introduce a fundamental variable, that we call *J*-field: let define the following quantities

$$\psi := \frac{1}{N} \sum_{i \in V} |h_i|, \quad \delta^2 := \frac{1}{N} \sum_{i \in V} h_i^2 - \psi^2, \quad J_i := \frac{|h_i| - \psi}{\delta}, \tag{11}$$

where ψ denotes the ground state energy density and δ is the amplitude of the fluctuations of $|h_i|$ around ψ . As explained in the Sections 5 and 6, it is possible to track the fluctuations around the ground state. This is done by introducing the the vertex set *X*,

$$X(\sigma_V^*) := \{ i \in V : \sigma_i^* = -1 \},$$
(12)

that collects the vertices in which the spin σ_i is flipped with respect to the direction of the ground state ω_i , and the renormalized Gaussian field *J*:

$$J(\boldsymbol{\sigma}_{V}^{*}) := \frac{1}{\sqrt{|X(\boldsymbol{\sigma}_{V}^{*})|}} \sum_{i \in X(\boldsymbol{\sigma}_{V}^{*})} J_{i}, \tag{13}$$

that is the normalized sum of the flipped local fields. The Hamiltonian of Eq. (2) can be expressed in terms of M, J and X as follows:

$$\sigma_V \cdot h_V = \psi M(\sigma_V^*) + J(\sigma_V^*) \sqrt{4\delta^2 |X(\sigma_V^*)|}.$$
(14)

In Section 5 we show that in the thermodynamic limit the average of Eq. (3) is mostly sampled from eigenstates of magnetization with eigenvalue

$$m_0(\beta \psi) := \tanh(\beta \psi). \tag{15}$$

For any $\sigma_V \in \Omega(m_0)$ we can rewrite the Hamiltonian as:

$$\sigma_V \cdot h_V = \psi \, m_0 \left(\beta \, \psi\right) \, N + J(\sigma_V^*) \, \sqrt{K \left[m_0 \left(\beta \, \psi\right)\right]} \tag{16}$$

where we introduced the auxiliary function

$$K(m) := 2\delta^2 (1 - m)N.$$
(17)

In reference [4] we found that at low temperature the *J* field applied to $\Omega(m_0)$ actually converges in distribution to a REM. This is done by noticing that when the temperature goes to zero the state align toward the direction of the ground state almost everywhere, and only a small fraction of spins is flipped in the opposite direction. Since the flipped spins are sparse, two independent configurations will probabily have a small number of common flipped spins that can be ignored, making the corresponding *J*-fields independent. In Section 6 we show that is possible to extend the results of [4] to the full temperature range by properly renormalizing the *J* field. In particular, we will introduce the *J'* field, described in detail in the Section 6. It is shown that, if applied to the ensemble $\Omega(m)$, this field is distributed like a REM by construction. In Section 6 we show that the *J* field is distributed like *J'* up to a constant and a renormalization

$$J(\sigma_V)\sqrt{K(m_0)} = J'(\sigma_V)\sqrt{K'(m_0)} + \text{const.}$$
(18)

the renormalized amplitude is found in Section 6 to be

$$K'(m) := \delta^2 (1 - m^2) N.$$
⁽¹⁹⁾

By the averaging properties of PPP, a parameter λ exists (dependent from β , ψ and δ) such that the Gibbs average satisfy the REM-PPP average formula [3, 4, 5, 6],

$$\lim_{N \to \infty} \langle f(\sigma_V) \rangle_{\xi} = \lim_{N \to \infty} \langle f^*(\sigma_V)^{\lambda} \rangle_V^{1/\lambda},$$
(20)

that, in the subcritical region $\lambda \in [0, 1]$, interpolates between the geometric (λ is equal to zero) and the arithmetic average (λ is equal to one). We indicated with *v* the uniform distribution. The importance of the REM universality to the spin glass physics is now evident in that one could directly apply this formula to Eq. (55) of [3] (or Eq. (6.20) of [4]) and find the full-RSB Parisi functional. This complets the steps to compute the free energy of the SK model (and many other models) with methods and concepts from [3, 4, 15], see Section 6 and Section 5 of [4] for further details.

3 Thermodynamic limit

Let start by formally defining the interface [3, 4]: the Hamiltonian is that of Eq. (2), following the canonical notation we call β the inverse of the temperature. The canonical partition function is defined as follows:

$$Z(\beta h_V) := \sum_{\sigma_V \in \Omega^V} \exp\left(-\sigma_V \cdot \beta h_V\right), \qquad (21)$$

the associated Gibbs measure is given by the expression

$$\xi\left(\sigma_{V}|\beta h_{V}\right) := \frac{\exp\left(-\sigma_{V}\cdot\beta h_{V}\right)}{Z\left(\beta h_{V}\right)} = \exp\left[-\sigma_{V}\cdot\beta h_{V} + N\beta\zeta\left(\beta h_{V}\right)\right]$$
(22)

where the function ζ is the free energy density per spin

$$-\beta\zeta(\beta h_V) := \frac{1}{N}\log Z(\beta h_V) = \frac{1}{N}\sum_{i\in V}\log 2\cosh(\beta h_i).$$
(23)

Let f be a test function of σ_V the Gibbs average is as follows

$$\langle f(\boldsymbol{\sigma}_{V}) \rangle_{\xi} := \sum_{\boldsymbol{\sigma}_{V} \in \Omega^{V}} f(\boldsymbol{\sigma}_{V}) \exp\left[-\boldsymbol{\sigma}_{V} \cdot \boldsymbol{\beta} h_{V} + N \boldsymbol{\beta} \zeta\left(\boldsymbol{\beta} h_{V}\right)\right].$$
(24)

3.1 Field fluctuations

Since the free energy depends only on the absolute value of the external field and not on the direction, before proceeding with the computations it is convenient to introduce the following auxiliary variables:

$$x_i = |h_i|, \quad \omega_i = h_i/|h_i|, \quad \sigma_i^* = \sigma_i \,\omega_i$$
(25)

the flickering state σ_V^* is the Hadamard product between the initial state σ_V and the direction of the external field ω_V , that corresponds to the ground state of the system and that we call master direction. Using these variables the Hamiltonian can be rewritten as $\sigma_V^* \cdot x_V$ where x_V is a vector with all positive entries, we call it rectified field. To find the scaling limit of the free energy it is convenient to reoreder *V* according to the **order statistics**, a remapping of the index $i \in V$, usually denoted with the symbol (*i*), such that $x_{(i)+1} \ge x_{(i)}$. Then, it is easy to see that if each x_i is independently drawn according to the same probability density *p*, then the scaling limit of $x_{(i)}$ for $(i) / N \rightarrow q \in [0, 1]$ will almost surely converge to the quantile function of *p*.

3.2 Uniform distribution

There are several important cases that can be treated exactly, one could consider the uniform distribution $p(x) = \mathbb{I}(x \in [0, 1])$: here the cumulant is q(x) = x and the quantile is therefore x(q) = q, then, the free energy density converges to the following integral:

$$-\lim_{N\to\infty}\beta\zeta(\beta x_V) = \int_0^1 dq \log 2\cosh(\beta q) = \frac{1}{\beta}\int_0^\beta d\tau \log 2\cosh(\tau)$$
(26)

the primitive of this integral is easily found via computer algebra,

$$\int d\tau \log 2 \cosh\left(\tau\right) = \frac{\operatorname{Li}_{2}\left[-\exp\left(-2\tau\right)\right] + \tau^{2}}{2}$$
(27)

where Li₂ is the dilogarithm [16], or Spence's function, that is often encountered in particle physics: for $z \in [0, 1]$ the following relations holds

$$\operatorname{Li}_{2}(-z) = \sum_{k \ge 1} \frac{(-z)^{k}}{k^{2}}, \quad \operatorname{Li}_{2}(-1) = -\frac{\pi^{2}}{12}, \quad \operatorname{Li}_{2}(0) = 0,$$
(28)

the first derivative obeys the following formula

$$\partial_z \operatorname{Li}_2(-z) = \sum_{k \ge 1} \frac{(-z)^{k-1} (-1)}{k} = z^{-1} \sum_{k \ge 1} \frac{(-z)^k}{k} = \frac{\log(1-z)}{z}$$
(29)

and notice that at -1 the derivative converges to the nontrivial value

$$\partial_z \operatorname{Li}_2(-1) = -\log 2 \tag{30}$$

Then, the scaling limit of the free energy density converges to the expression

$$-\lim_{N \to \infty} \beta \zeta \left(\beta x_V\right) = \frac{\beta}{2} + \frac{\pi^2}{24\beta} + \frac{\text{Li}_2\left[-\exp\left(-2\beta\right)\right]}{2\beta},\tag{31}$$

where the last term is negative in the whole temperature range. Notice that the convergence to the ground state energy is quadratic in temperature: the specific heat is linear like in the Dulong-Petit law.

3.3 Half-Gaussian distribution

We could also consider more complex shapes, like the half-normal distribution,

$$p(x) = \sqrt{2/\pi} \exp(-x^2/2).$$
 (32)

From quantile mechanics [17] one finds

$$\rho(x) = -\partial_x(-x^2/2 + \log\sqrt{2/\pi}) = x,$$
 (33)

then, the quantile equation and its boundary conditions is as follows:

$$\partial_q^2 x(q) = x(q) [\partial_q x(q)]^2, \quad x(0) = 0, \quad x(1/2) = \sqrt{2\pi} \operatorname{erf}^{-1}(1/2), \quad (34)$$

solving the equation with these boundary give us

$$x(q) = \sqrt{2\pi} \operatorname{erf}^{-1}(q).$$
 (35)

Then, for the half-normal (and normal) noise model we expect the free energy to converge toward the following limit expression

$$-\lim_{N\to\infty}\beta\zeta(\beta x_V) = \int_0^1 dq \log 2\cosh\left[\beta\sqrt{2\pi}\operatorname{erf}^{-1}(q)\right].$$
(36)

We notice that the differential equation in Eq. (34) before is remarkably similar to the term in parentesis in Eq. (10) of [18], that is the nonlinear antiparabolic equation from which we obtain the Parisi functional. Further investigation on the relation between the Guerra interpolation theory and quantile mechanics would be certainly interesting, we hope to explore this in a future work.

3.4 Series analysis of the Half-Gaussian

To highligt the low temperature features it will be more instructive to rather perform a series analysis. We start from the expression

$$-\beta\zeta(\beta x_V) = \lim_{N \to \infty} \frac{1}{N} \sum_{i \in V} \log 2 \cosh(\beta x_i), \qquad (37)$$

the ground state in the thermodynamic limit (TL) is

$$-\zeta(\infty) = \lim_{N \to \infty} \lim_{\beta \to \infty} \frac{1}{N} \sum_{i \in V} x_i \tanh(\beta x_i) = \lim_{N \to \infty} \frac{1}{N} \sum_{i \in V} x_i = \sqrt{\frac{2}{\pi}},$$
(38)

where the numeric value is obtained by computing the average of x with x Gaussian variable of zero average and unitary variance:

$$\lim_{N \to \infty} \frac{1}{N} \sum_{i \in V} x_i = \sqrt{\frac{2}{\pi}} \int_0^\infty x \exp\left(-x^2/2\right) dx = \sqrt{\frac{2}{\pi}} \int_0^\infty \exp\left(-t\right) dt = \sqrt{\frac{2}{\pi}}, \quad (39)$$

in the last step we applied the substitution $t = x^2/2$. Now, let consider the equivalent expression

$$\lim_{N \to \infty} \frac{1}{N} \sum_{i \in V} \log 2 \cosh\left(\beta x_i\right) = \beta \sqrt{\frac{2}{\pi}} + \lim_{N \to \infty} \frac{1}{N} \sum_{i \in V} \log\left[1 + \exp\left(-2\beta x_i\right)\right], \tag{40}$$

the logarithm can be expanded in the limit of large β ,

$$\log\left[1 + \exp\left(-2k\beta x_{i}\right)\right] = \sum_{k \ge 1} \frac{\left(-1\right)^{k-1}}{k} \exp\left(-2k\beta x_{i}\right),$$
(41)

the average of the exponential term can be computed by Gaussian integration

$$\lim_{N \to \infty} \frac{1}{N} \sum_{i \in V} \exp\left(-2k\beta x_i\right) = \sqrt{\frac{2}{\pi}} \int_0^\infty dx \exp\left(-2k\beta x - x^2/2\right) = \\ = \exp\left(2k^2\beta^2\right) \sqrt{\frac{2}{\pi}} \int_0^\infty dx \exp\left[-(\sqrt{2}k\beta + x/\sqrt{2})^2\right] = \\ = \exp\left(2k^2\beta^2\right) \sqrt{\frac{4}{\pi}} \int_0^\infty d(\sqrt{2}k\beta + x/\sqrt{2}) \exp\left[-(\sqrt{2}k\beta + x/\sqrt{2})^2\right] = \\ = \exp\left(2k^2\beta^2\right) \sqrt{\frac{4}{\pi}} \int_{\sqrt{2}k\beta}^\infty dt \exp\left(-t^2\right) = \exp\left(2k^2\beta^2\right) \operatorname{erfc}\left(\sqrt{2}k\beta\right), \quad (42)$$

the last substitution is $t = x/\sqrt{2} + \sqrt{2}k\beta$. We found a series representation for the free energy of the gaussian noise model

$$-\beta\zeta(\beta x_V) = \beta\sqrt{\frac{2}{\pi}} + \sum_{k\geq 1} \frac{(-1)^{k-1}}{k} \exp(2k^2\beta^2)\operatorname{erfc}(\sqrt{2}k\beta).$$
(43)

For large β one can use the following approximation for the Error Function

$$\exp\left(2k^{2}\beta^{2}\right)\operatorname{erfc}(\sqrt{2}k\beta) = \frac{1}{\sqrt{2\pi}k\beta} + O\left(\beta^{-3}\right). \tag{44}$$

Put the last expression back into the logarithm expansion before and one finds:

$$\lim_{N \to \infty} \frac{1}{N} \sum_{i \in V} \log\left[1 + \exp\left(-2\beta x_i\right)\right] = \frac{K_0}{\sqrt{2\pi\beta}} + O\left(\beta^{-3}\right),\tag{45}$$

where the constant K_0 is the convergent sum

$$K_0 := \sum_{k \ge 1} \frac{(-1)^{k-1}}{k^2} = \frac{\pi^2}{12}$$
(46)

The asymptotic expansion of ζ near zero temperature is then found to be

$$-\beta\zeta\left(\beta x_{V}\right) = \beta\sqrt{\frac{2}{\pi}} + \frac{1}{\beta}\sqrt{\frac{\pi^{3}}{48}} + O\left(\beta^{-3}\right),\tag{47}$$

also in this case the convergence to the ground state is quadratic in temperature.

3.5 Convergence to the ground state

The origin of the quadratic difference in the low temperature behavior is in that for both uniform and Gaussian distributions some couplings may be close to zero for a fraction of spins. To see this, let analyze the energy contribution from the subset of spins

$$V(\delta) = \{i \in V : x_i > \delta\}, \tag{48}$$

following the steps before we find

$$\int_{\delta}^{\infty} dx \exp\left(-2k\beta x - x^2/2\right) =$$
$$= \exp\left(2k^2\beta^2\right) \operatorname{erfc}\left(\delta/\sqrt{2} + \sqrt{2}k\beta\right) = O\left[\exp\left(-2\delta k\beta\right)\right]. \quad (49)$$

and then the limit in Eq. (45) restricted to $V(\delta)$ is

$$\lim_{N \to \infty} \frac{1}{|V(\delta)|} \sum_{i \in V(\delta)} \log \left[1 + \exp\left(-2\beta x_i\right)\right] = O\left[\exp\left(-2\delta\beta\right)\right],\tag{50}$$

we can see if we exclude the spins with small coupling the temperature dependence is again exponentially suppressed in β .

4 Eigenstates of magnetization

We introduce a central element in our analysis: the kernel of the eigenstates of magnetization. Let define the microcanonical set

$$\Omega(M) := \left\{ \sigma_V \in \Omega^V : M(\sigma_V) = M \right\},\tag{51}$$

that is equivalent to the kernel of all magnetization states with a given magnetization M, or a lattice gas with exactly E particles. Hereafter we denote by $|\Omega(M)|$ the number of such eigenstates, that we will call cardinality of $\Omega(M)$, or "complexity", as is sometimes found in the spin glass literature. Also, define a notation for the average,

$$\langle f(\sigma_V) \rangle_{\Omega(M)} := \frac{1}{|\Omega(M)|} \sum_{\sigma_V \in \Omega(M)} f(\sigma_V).$$
 (52)

The eigenstates of magnetization $\Omega(M)$ can be studied in detail using Large Deviation Theory (LDT) [19], even at the "sample-path" LDT level. For example, by a simple applications of the Varadhan Integral Lemma, the Mogulskii theorem, and other standard LDT methods [19, 20, 21, 22, 23], one can compute the number of the eigenstates with given magnetization $\lfloor mN \rfloor$, ie the integer part of mN, with $m \in [-1, 1]$: to simplify the notation, hereafter

$$\Omega(\lfloor mN \rfloor) =: \Omega(m), \quad \langle f(\sigma_V) \rangle_{\Omega(\lfloor mN \rfloor)} =: \langle f(\sigma_V) \rangle_m.$$
(53)

With some algebraic effort is possible to show that the cardinality of $\Omega(m)$ is proportional to $\exp[N\phi(m)]$, with rate function ϕ equal to

$$\phi(m) = \log 2 - \frac{1}{2}\log(1 - m^2) + \frac{m}{2}\log\left(\frac{1+m}{1-m}\right),$$
(54)

this result can be obtained by applying the inverse Legendre transform to the free energy of the binary noise model, studied in Section 5. In Ref. [20] a detailed description of the magnetization eigenstates is achieved by adapting methods from the large deviations theory, in particular, the Varadhan lemma and the Mogulskii theorem. Further details can be found in [20, 21], where a full mathematical derivation is shown for the more generel HLS model (for example, the binary noise model is recovered in the most

trivial case of constant urn function).

4.1 Lattice gas

Notice that the set $\Omega(M)$ is equivalent to a self-avoiding lattice gas of E = N/2 - M/2 particles on a lattice of size *N*. Let define the particle displacements

$$X(\sigma_V) := \{i \in V : \sigma_i = -1\} \subset V$$
(55)

in the magnetic representation would be the subsets of *V* where the flickering state σ_V^* is flipped with respect to the master direction 1_V (that indicates a vector with all 1 entries). Inside $\Omega(M)$ the size of *X* is fixed, and related to the magnetization by

$$|X(\sigma_V)| = (N - M)/2 =: E, \quad \forall \sigma_V \in \Omega(M).$$
(56)

We interpret *E* as the number of particles in our self-avoiding gas. Then, we introduce the set of all possible displacements of $E = |\varepsilon N|$ particles

$$\mathscr{V}(E) := \{ X \subset V : |X| = E \}, \tag{57}$$

as for the magnetic rapresentation before we use the notation

$$\mathscr{V}(\lfloor \varepsilon N \rfloor) =: \mathscr{V}(\varepsilon), \quad \langle f(\sigma_V) \rangle_{\mathscr{V}(\lfloor \varepsilon N \rfloor)} =: \langle f(\sigma_V) \rangle_{\varepsilon}.$$
(58)

that is in fact the exact image of $\Omega(m)$ if one takes $\varepsilon = (1-m)/2$. We represent the spin states in terms of *X* as follows: let $\sigma_V = \sigma_{V\setminus X} \cup \sigma_X$, then for the flipped spins we have $\sigma_X = -1_X$, vector with all negative entries, for the others $\sigma_{V\setminus X} = 1_{V\setminus X}$ with all positive entries. The spin state σ_V can be reconstructed from the flipped vertices

$$\sigma_V = \mathbf{1}_{V \setminus X} \cup (-\mathbf{1}_X),\tag{59}$$

this representation in terms of particle displacements allows to easily explore the overlap structure. Consider two configurations $X, Y \in \mathcal{V}(\varepsilon)$, corresponding to

$$\sigma_V = \mathbf{1}_{V \setminus X} \cup (-\mathbf{1}_X), \ \tau_V = \mathbf{1}_{V \setminus Y} \cup (-\mathbf{1}_Y).$$
(60)

Within $\mathscr{V}(\varepsilon)$ the number of particles is fixed $|X| = |Y| = \varepsilon N$. Now, let $X \cap Y$ be the set of points of *V* at which the two particle configurations *X* and *Y* overlap, and define the

non-overlapping components

$$X' := X \setminus (X \cap Y), \quad Y' := Y \setminus (X \cap Y), \tag{61}$$

that correspond to the non overlapping points of the particle displacements. By definition, their intersection is void, ie $X' \cap Y' = \emptyset$, moreover, the following equalities hold for of the union of *X* and *Y*

$$X \cup Y = X' \cup Y' \cup (X \cap Y), \ X' \cup Y' = (X \cup Y) \setminus (X \cap Y).$$
(62)

It follows that $|X \cup Y|$ total fraction of *V* occupied by the particles is

$$|X \cup Y| = |X| + |Y| - |X \cap Y|,$$
(63)

while the total non-overlapping volume is

$$|X' \cup Y'| = |X \cup Y| - |X \cap Y| = |X| + |Y| - 2|X \cap Y|.$$
(64)

The overlap between the corresponding spin states can be expressed as

$$\sigma_{V} \cdot \tau_{V} = \left[\mathbf{1}_{V \setminus X} \cup (-\mathbf{1}_{X}) \right] \cdot \left[\mathbf{1}_{V \setminus Y} \cup (-\mathbf{1}_{Y}) \right] = N - 2|X'| - 2|Y'| = N - 2|X| - 2|Y| + 4|X \cap Y|, \quad (65)$$

since we are considering magnetizations eigenstates with fixed eigenvalue, the volumes of |X| and |Y| are also fixed at εN , and the overlap of the spin states can be expressed in terms of the overlap between the particle configurations:

$$\sigma_V \cdot \tau_V = (1 - 4\varepsilon)N + 4|X \cap Y|. \tag{66}$$

The overlap size is $0 \le |X \cap Y| \le \varepsilon N$, but it can be shown (e.g., see the next section) that for large *N* the overlap concentrates on $\varepsilon^2 N$, with fluctuations of order \sqrt{N} (it converges to a gaussian), then from $m = 1 - 2\varepsilon$ and the Eq. (65) follows that the spin overlap concentrates almost surely on the mean value $1 - 4\varepsilon + 4\varepsilon^2 = m^2$.

4.2 Entropy of the overlap

We compute the probability that two configuration randomly extracted from $\mathscr{V}(\varepsilon)$ have an intersection of size $\lfloor xN \rfloor$, with $x \in [0, \varepsilon]$. The limit entropy density (rate function) of such event is defined as follows

$$\eta(x|\varepsilon) := -\lim_{N \to \infty} \frac{1}{N} \log P(|X \cap Y| = \lfloor xN \rfloor | X, Y \in \mathscr{V}(\varepsilon))$$
(67)

that gives the shape of the distribution for large number of spins

$$P(|X \cap Y| = \lfloor xN \rfloor | X, Y \in \mathscr{V}(\varepsilon)) \sim \exp\left[-N\eta(x|\varepsilon)\right].$$
(68)

The first step is to notice that due to the uniformity of the distribution of the flipped spins the intersection size does not depend on the special realization of both states, then we can fix one of the two states: let's fix Y and call it 'target' set, then

$$P(|X \cap Y| = \lfloor xN \rfloor | X, Y \in \mathscr{V}(\varepsilon)) =$$
$$= P(|X \cap Y| = \lfloor xN \rfloor | X \in \mathscr{V}(\varepsilon)), \ \forall Y \in \mathscr{V}(\varepsilon).$$
(69)

Since only the size of the target set actually matters, to highligh its internal components it will be convenient to chose a special configuration of the target (see Figure 1)

$$Y_0 := \{ 1 \le k \le \lfloor \varepsilon N \rfloor \}$$

$$\tag{70}$$

where the vertices of the flipped spins are placed at the beginning of the set *V* (ie, the labels $k \in V \setminus Y_0$ are all larger than $|\mathcal{E}N|$), formally holds that

$$P(|X \cap Y| = \lfloor xN \rfloor | X, Y \in \mathscr{V}(\varepsilon)) = P(|X \cap Y_0| = \lfloor xN \rfloor | X \in \mathscr{V}(\varepsilon)), \quad (71)$$

The entropy density is given by the limit

$$\eta(x|\varepsilon) := -\lim_{N \to \infty} \frac{1}{N} \log P(|X \cap Y_0| = \lfloor xN \rfloor | X \in \mathscr{V}(\varepsilon)),$$
(72)

and it can be computed in many ways by Varadhan Lemma, the Mogulskii theorem and other large deviations techniques.

4.3 Urn methods

We can adapt methods from the urn process theory (see [20, 21]) to compute the shape of the overlap entropy density. The method consists in defining a nested set sequence



Figure 1: Two configurations X (first row) and Y_0 (second row) extracted from $\mathscr{V}(\varepsilon)$, reordered in such way that both X and $X \cap Y_0$ are compact sets. The last row shows the partition into the disjoint non-overlapping components X', Y'_0 and the common component $X \cap Y_0$ projected on V (last row).

that start from the null set and converges to X in exactly E steps,

$$X_n := \bigcup_{n \le E} \{i_n\},\tag{73}$$

that is a markov chain with transition matrix

$$P(i_{n+1} = k) = \frac{\mathbb{I}(k \in V \setminus X_n)}{|V \setminus X_n|}.$$
(74)

We indicate the overlap between X_n and the target set Y_0 with

$$R_n := |X_n \cap Y_0|, \tag{75}$$

the final conditions of the processes are fixed at $X_E = X$ and $R_E = |X \cap Y_0|$ respectively, reached in $E = \lfloor \varepsilon N \rfloor$ steps. It can be shown that the overlap between X_n and the target set follows a urn process [20, 21, 22, 23] in the step variable *n*

$$R_{n+1} = \begin{cases} R_n + 1 & \pi_n(R_n) \\ R_n & 1 - \pi_n(R_n) \end{cases}$$
(76)

the urn function π_n at step *n* is the ratio between the number of vertices in Y_0 that have not been occupied in the preceeding *n* extractions (that are $E - R_n$) and the number of vertices of *V* that have not been occupied (ie, N - n),

$$\pi_n(R_n) := \frac{E - R_n}{N - n}.$$
(77)

adapting large-deviation methods shown in [20, 21, 22, 23] for generalized urn models it is possible to show that the distribution of the overlap is aproximately Gaussian. It is also possible to compute the parameters by solving the difference equation

$$\mathbb{E}(R_{n+1}) = \mathbb{E}\left[\left(R_n + 1\right)\pi_n(R_n)\right] + \mathbb{E}\left\{R_n\left[1 - \pi_n(R_n)\right]\right\}$$
(78)

where $\mathbb{E}\left(\,\cdot\,\right)$ indicates the average respect to the urn process. Substituting the expression of the urn function we find

$$\mathbb{E}(R_{n+1}) - \mathbb{E}(R_n) = -\frac{1}{N-n}\mathbb{E}(R_n) + \frac{E}{N-n},$$
(79)

solving with null initial condition brings to the linear average solution $\mathbb{E}(R_n) = nE/N$ from which follows that the average overlap converges to

$$\lim_{N \to \infty} \frac{\langle |X \cap Y_0| \rangle_{\varepsilon}}{N} = \lim_{N \to \infty} \frac{\mathbb{E}(R_E)}{N} = \varepsilon^2.$$
(80)

We can show that the fluctuations are small: consider

$$\mathbb{E}\left(R_{n+1}^{2}\right) = \mathbb{E}\left[\left(R_{n}+1\right)^{2}\pi_{n}\left(R_{n}\right)\right] + \mathbb{E}\left\{R_{n}^{2}\left[1-\pi_{n}\left(R_{n}\right)\right]\right\},\tag{81}$$

substituting the urn function and the formula for the linear we find

$$\mathbb{E}\left(R_{n+1}^{2}\right) - \mathbb{E}\left(R_{n}^{2}\right) = -\frac{2}{N-n}\mathbb{E}\left(R_{n}^{2}\right) + \frac{E}{N-n}\left[1 + n\left(\frac{2E-1}{N}\right)\right], \quad (82)$$

solving again for null initial condition gives another linear solution

$$\mathbb{E}\left(R_{n}^{2}\right) = \frac{nE\left[E\left(n-1\right)-n+N\right]}{N\left(N-1\right)}$$
(83)

Let now compute the variance of R_n : the variance is defined by

$$\mathbb{E}(R_n^2) - \mathbb{E}(R_n)^2 = \frac{nE[E(n-1) - n + N]}{N(N-1)} - \frac{n^2 E^2}{N^2}$$
(84)

and after some algebra it can be shown that for the variance holds

$$\lim_{N \to \infty} \frac{\langle |X \cap Y_0|^2 \rangle_{\varepsilon} - \langle |X \cap Y_0| \rangle_{\varepsilon}^2}{N} = \lim_{N \to \infty} \frac{\mathbb{E} \left(R_E^2 \right) - \mathbb{E} \left(R_E \right)^2}{N} = \varepsilon^2 \left(1 - \varepsilon \right)^2.$$
(85)

The entropy density η can thus be expanded at second order in the variable

$$\Delta(x|\varepsilon) := \frac{x - \varepsilon^2}{\varepsilon (1 - \varepsilon)},\tag{86}$$

in the limit of large N it can be shown that

$$\eta(x|\varepsilon) = \frac{1}{2}\Delta(x|\varepsilon)^2 + O[\Delta(x|\varepsilon)^3], \qquad (87)$$

and since $m = 1 - 2\varepsilon$, according to Eq. (65), the corresponding spin overlap concentrates almost surely on the average value, i.e., $1 - 4\varepsilon + 4\varepsilon^2 = m^2$. Notice that the spin overlap concentrates on the same value of the correlation matrix $\langle \sigma_i \sigma_j \rangle_m = m^2$. This means that the kernel of the magnetization eigenstates commutes in distribution, i.e., that the correlation matrix converges to the overlap matrix, see in Section 2 of [4] for further details on kernel commutation and its implications.

5 Binary noise model

Let now consider the simplest situation where the field has two states only (binary noise model), the absolute value is a delta function centered on one, that is $p(x) = \delta(x-1)$. We study the Hamiltonian $\sigma_V \cdot \omega_V$, scalar product between σ_V and the input ω_V ,

$$H(\sigma_V|\omega_V) = \sum_{i \in V} \sigma_i \omega_i.$$
(88)

Since $|\omega_i| = 1$ the canonical analysis here is very simple: notice that due to parity of the cosh function the partition function does not depend on the input state ω_V ,

$$Z(\boldsymbol{\beta}) = \sum_{\boldsymbol{\sigma}_{V} \in \Omega^{V}} \exp\left(-\boldsymbol{\beta} \boldsymbol{\sigma}_{V} \cdot \boldsymbol{\omega}_{V}\right) = \left[2\cosh\left(\boldsymbol{\beta}\right)\right]^{N},\tag{89}$$

the free energy per spin and the ground state energy are

$$-\beta \zeta \left(\beta\right) = \log 2 \cosh\left(\beta\right), \quad \psi = -\lim_{\beta \to \infty} \tanh\left(\beta\right) = -1. \tag{90}$$

5.1 Free energy phases

In the low temperature limit the free energy is

$$-\beta\zeta(\beta) = \beta + \exp(-2\beta) + O[\exp(-4\beta)], \qquad (91)$$

then, the free energy per spin converges to the ground state energy $\zeta(\infty)$ exponentially fast in β . Moreover, we find at high temperature the free energy converges to the replica symmetric (RS) free energy of the spin glass theory: Taylor expansion of the log cosh function for small β gives

$$-\beta\zeta(\beta) = \log 2 + \frac{\beta^2}{2} + O\left(\beta^4\right). \tag{92}$$

It can be shown that in the zero temperature limit the Gibbs measure can be approximated by a random energy model: this will be discussed later.

5.2 Flickering states and thermal average

Let study the formula for the average:

$$\langle f(\sigma_V) \rangle_{\xi} = \frac{1}{Z(\beta)} \sum_{\sigma_V \in \Omega^V} f(\sigma_V) \exp(-\beta \sigma_V \cdot \omega_V).$$
 (93)

Given the independence of the partition function from ω_V it will be convenient to introduce some notation. Define the flickering state $\sigma_V^* := \sigma_V \circ \omega_V$ such that the resulting vector has the following components $\sigma_i^* := \sigma_i \,\omega_i \in \Omega$. Notice that since $\omega_i^2 = 1$ a further multiplication of σ_V^* by ω_V gives back the original vector σ_V , i.e., $\sigma_V^* \circ \omega_V = \sigma_V$. Then we introduce the flickering function

$$f^{*}(\boldsymbol{\sigma}_{V}) := f(\boldsymbol{\sigma}_{V} \circ \boldsymbol{\omega}_{V}), \quad f^{*}(\boldsymbol{\sigma}_{V}^{*}) = f(\boldsymbol{\sigma}_{V}^{*} \circ \boldsymbol{\omega}_{V}) = f(\boldsymbol{\sigma}_{V}).$$
(94)

Finally, we consider the scalar product (overlap) of σ_V with the input state ω_V , that is equivalent to the total magnetization of σ_V^* ,

$$\sigma_V \cdot \omega_V = (\sigma_V \circ \omega_V) \cdot 1_V = \sigma_V^* \cdot 1_V =: M(\sigma_V^*).$$
(95)

Putting together, the sum of f weighted with the Gibbs weights satisfies the following chain of equivalences

$$\sum_{\sigma_{V}\in\Omega^{V}} f(\sigma_{V}) \exp(-\beta \sigma_{V} \cdot \omega_{V}) =$$

$$= \sum_{\sigma_{V}\in\Omega^{V}} f(\sigma_{V}^{*} \circ \omega_{V}) \exp[-\beta M(\sigma_{V}^{*})] = \sum_{\sigma_{V}\in\Omega^{V}} f^{*}(\sigma_{V}^{*}) \exp[-\beta M(\sigma_{V}^{*})] =$$

$$= \sum_{\sigma_{V}\in\Omega^{V}} f^{*}(\sigma_{V}) \exp[-\beta M(\sigma_{V})], \quad (96)$$

where in the last step we used that σ_V and σ_V^* are in a bijective relation, this implies that we can change the sum index to σ_V as the dependence on the input state affects only f^* . Then, the formula for the average is as follows:

$$\langle f(\sigma_V) \rangle_{\xi} = \frac{1}{Z(\beta)} \sum_{\sigma_V \in \Omega^V} \exp\left[-\beta M(\sigma_V^*)\right] f^*(\sigma_V) =$$

$$= \frac{1}{Z(\beta)} \sum_M \exp\left(-\beta M\right) \sum_{\sigma_V \in \Omega(M)} f^*(\sigma_V) =$$

$$= \frac{1}{Z(\beta)} \sum_M |\Omega(M)| \exp\left(-\beta M\right) \langle f^*(\sigma_V) \rangle_{\Omega(M)}.$$
(97)

5.3 Average in thermodynamic limit

Assuming that *f* exists in the thermodynamic limit $N \rightarrow \infty$, we can write also a continuous representation. From [4, 20, 21, 22, 23] it can be shown that

$$\lim_{N \to \infty} \langle f(\sigma_V) \rangle_{\xi} = \lim_{N \to \infty} \frac{\int_{-1}^{1} dm \exp\left\{-N\left[\phi\left(m\right) + \beta m\right]\right\} \langle f^*(\sigma_V) \rangle_m}{\int_{-1}^{1} dm \exp\left\{-N\left[\phi\left(m\right) + \beta m\right]\right\}}$$
(98)

and it can be also shown that the probability mass concentrates on the value $m_0(\beta)$ that maximize $\beta m + \phi(m)$. Putting together

$$\lim_{N \to \infty} \langle f(\sigma_V) \rangle_{\xi} = \lim_{N \to \infty} \langle f^*(\sigma_V) \rangle_{m_0(\beta)}$$
(99)

and after some manipulations one can prove that $m_0(\beta) = \tanh(\beta)$. Then, it is possible to compute the average in terms of the eigenstates of magnetization and their effects on the flickering function f^* .

6 Relation with the Random Energy Model

It can be shown that at low temperature the Gibbs measure converges in distribution to a Random Energy Model (REM) of the Derrida type [4]. Define

$$\boldsymbol{\psi} = \frac{1}{N} \sum_{i \in V} x_i, \quad \boldsymbol{\varphi}_i := x_i - \boldsymbol{\psi}, \tag{100}$$

where ψ is the ground state energy and φ_i describes the field fluctuations. The Hamiltonian can be rewritten once again as follows

$$\sigma_V \cdot h_V = \psi M(\sigma_V^*) + \sigma_V^* \cdot \varphi_V. \tag{101}$$

As in previous section, we recall the special notation for the composition between the master direction and the test function, we called it flickering function

$$f^*(\boldsymbol{\sigma}_V) = f(\boldsymbol{\sigma}_V \circ \boldsymbol{\omega}_V), \qquad (102)$$

and notice that it does not depend on the external field x_V . Then, the average is rewritten in terms of the flickering variables only

$$\langle f(\boldsymbol{\sigma}_{V}) \rangle_{\xi} = \sum_{\boldsymbol{\sigma}_{V} \in \Omega^{V}} f^{*}(\boldsymbol{\sigma}_{V}) \exp\left[-\beta \psi M(\boldsymbol{\sigma}_{V}) - \beta \boldsymbol{\sigma}_{V} \cdot \boldsymbol{\varphi}_{V} + N\beta \zeta \left(\beta x_{V}\right)\right]$$
(103)

so that the dependence on ω_V is all inside the flickering function f^* and both the free energy density and the Gibbs measure depend only on the rectified field x_V .

6.1 Field fluctuations revisited

By the lattice gas representation described before, the following holds:

$$\frac{\sigma_V \cdot \varphi_V}{2} = \mathbf{1}_{X(\sigma_V)} \cdot \varphi_{X(\sigma_V)},\tag{104}$$

in fact, consider the chain of identities

$$\sum_{i \in V} \sigma_i \varphi_i = \sum_{i \in V} \varphi_i - \sum_{i \in V} (1 - \sigma_i) \varphi_i = \sum_{i \in V} \varphi_i - \sum_{i \in X(\sigma_V)} 2\varphi_i,$$
(105)

by definition we have that the first sum is zero,

$$\sum_{i\in V} \varphi_i = \sum_{i\in V} x_i - \sum_{i\in V} \psi = 0.$$
(106)

Let now introduce a notation for the variance inside φ_V , that we denote by δ^2 , and the variance over the vertex set γ^2

$$\delta^{2} = \frac{1}{N} \sum_{i \in V} \varphi_{i}^{2}, \quad \gamma^{2} = \frac{1}{N} \sum_{i \in V} x_{i}^{2}$$
(107)

this quantity is related to ground state and variance of x_i^2 by the relation $\delta^2 = \gamma^2 - \psi^2$ where γ is the average variance over the vertex set.

6.2 The *J*-field

We can now introduce a fundamental variable, that we call J-field

$$J_{X(\sigma_V)} := \{J_i := \varphi_i / \delta : i \in X(\sigma_V)\},$$
(108)

from which we define the normalized field amplitude

$$J(\sigma_V) := \frac{\mathbf{1}_{X(\sigma_V)} \cdot J_{X(\sigma_V)}}{\sqrt{|X(\sigma_V)|}}.$$
(109)

This variable converges to a Gaussian with zero mean and unitary variance in the thermodynamic limit, moreover, given two states σ_V and τ_V independently extracted from $\Omega(m)$ the average overlap converges to (1-m)/2. From previous considerations the Hamiltonian can be rewritten as follows

$$\sigma_V \cdot h_V = \psi M(\sigma_V^*) + J(\sigma_V^*) \sqrt{K[M(\sigma_V)]}, \qquad (110)$$

where we introduced a notation for the normalization of the J-amplitude

$$K[M(\sigma_V)] := 2\delta^2 N[1 - M(\sigma_V)/N].$$
(111)

The formula for the average is rewritten in terms of the new variables

$$\langle f(\sigma_V) \rangle_{\xi} = \frac{\sum_M |\Omega(M)| \exp(-\beta \psi M) \langle f^*(\sigma_V) \exp[-\beta J(\sigma_V) \sqrt{K(M)}] \rangle_{\Omega(M)}}{\sum_M |\Omega(M)| \exp(-\beta \psi M) \langle \exp[-\beta J(\sigma_V) \sqrt{K(M)}] \rangle_{\Omega(M)}}.$$
 (112)

Now, let take the themodynamic limit: it can be shown by simple saddle point methods [20, 21] that the average admit the following integral representation

$$\lim_{N \to \infty} \langle f(\sigma_V) \rangle_{\xi} = \\ = \lim_{N \to \infty} \frac{\int_{-1}^{1} dm \exp\left\{-N[\phi(m) + \beta \psi m]\right\} \langle f^*(\sigma_V) \exp\left[-\beta J(\sigma_V) \sqrt{K(m)}\right] \rangle_m}{\int_{-1}^{1} dm \exp\left\{-N[\phi(m) + \beta \psi m]\right\} \langle \exp\left[-\beta J(\sigma_V) \sqrt{K(m)}\right] \rangle_m}$$
(113)

introducing the auxiliary functions

$$m_0(\boldsymbol{\beta}\boldsymbol{\psi}) := \tanh\left(\boldsymbol{\beta}\boldsymbol{\psi}\right), \quad K(m) := 2\delta^2\left(1-m\right)N, \tag{114}$$

we arrive to the final form for our average formula, that is

$$\lim_{N \to \infty} \langle f(\sigma_V) \rangle_{\xi} = \lim_{N \to \infty} \frac{\langle f^*(\sigma_V) \exp\left[-\beta J(\sigma_V) \sqrt{K(m_0)}\right] \rangle_{m_0}}{\langle \exp\left[-\beta J(\sigma_V) \sqrt{K(m_0)}\right] \rangle_{m_0}} = \\ = \lim_{N \to \infty} \frac{\sum_{\sigma_V \in \Omega(m_0)} f^*(\sigma_V) \exp\left[-\beta J(\sigma_V) \sqrt{K(m_0)}\right]}{\sum_{\sigma_V \in \Omega(m_0)} \exp\left[-\beta J(\sigma_V) \sqrt{K(m_0)}\right]}.$$
(115)

In [4] is shown that in the low temperature limit the Gaussian amplitude J converges in the bulk to a random energy model of the Derrida type [6] (ie., with Gaussian energies). This is done by noticing that when the temperature goes to zero the state aligns toward the direction of the ground state almost everywhere, and only a small fraction of spins get flipped in the opposite direction. Since the flipped spins are sparse, any two independent configurations will most probabily have a negligible number of common flips. The number of this common flips (see Figure 1) converges to zero faster than the size of the whole flipped set when the temperature is lowered to near zero (ie., net of quadratic terms) and can be therefore ignored in that limit: see Section 5 of [4] for further details. The crucial fact is in that the field J_i is sampled independently for each vertex *i*, then for any two disjoint subsets of *V* the corresponding J-fields are independent like in a REM. The argument works also for multiple replicas if temperature is low enough.

6.3 **REM at all temperatures**

In this last sub-section we show how is possible to correct the fromulas of [4] in order to make it valid also at higher temperatures. Let consider two subsets of $X, Y \subset V$ of same size E and their non-overlapping components X' and Y' as defined in Eq. (60) of Section 4. Now notice that the following holds:

$$X = X' \cup \{X \cap Y\}, \quad Y = Y' \cup \{X \cap Y\}.$$
(116)

The the REM contribution comes only from the non-overlapping components, then we would like to get rid of the overlapping component (ie, the energy of the spins placed on the vertices in $X \cap Y$) and write everything in terms of the sets X' and Y'. This is made possible by considering the difference between the corresponding *J*-fields

$$1_X \cdot J_X - 1_Y \cdot J_Y = (1_{X'} \cdot J_{X'} + 1_{X \cap Y} \cdot J_{X \cap Y}) - (1_{Y'} \cdot J_{Y'} + 1_{X \cap Y} \cdot J_{X \cap Y}) =$$

= $1_{X'} \cdot J_{X'} - 1_{Y'} \cdot J_{Y'}$ (117)

Therefore, let consider two independent replicas σ_V and τ_V and let indicate with $X(\sigma_V)$ and $Y(\tau_V)$ the associated flipped components. Let introduce the auxiliary Δ -field, that is the difference between the *J*-fields of the two replicas

$$\Delta(\sigma_V | \tau_V) := J(\sigma_V) - J(\tau_V), \qquad (118)$$

by multiplying both numerator and denominator of the average formula in Eq. (114) by the proper τ_V -dependent amplitude: we find

$$\frac{\sum_{\sigma_{V}\in\Omega(m_{0})}f^{*}(\sigma_{V})\exp\left[-\beta J(\sigma_{V})\sqrt{K(m_{0})}\right]}{\sum_{\sigma_{V}\in\Omega(m_{0})}\exp\left[-\beta J(\sigma_{V})\sqrt{K(m_{0})}\right]} = \frac{\sum_{\sigma_{V}\in\Omega(m_{0})}f^{*}(\sigma_{V})\exp\left\{-\beta\left[J(\sigma_{V})-J(\tau_{V})\right]\sqrt{K(m_{0})}\right\}}{\sum_{\sigma_{V}\in\Omega(m_{0})}\exp\{-\beta\left[J(\sigma_{V})-J(\tau_{V})\right]\sqrt{K(m_{0})}\}} = \frac{\sum_{\sigma_{V}\in\Omega(m_{0})}f^{*}(\sigma_{V})\exp\left[-\beta\Delta(\sigma_{V}|\tau_{V})\sqrt{K(m_{0})}\right]}{\sum_{\sigma_{V}\in\Omega(m_{0})}\exp\left[-\beta\Delta(\sigma_{V}|\tau_{V})\sqrt{K(m_{0})}\right]} \quad (119)$$

from previous considerations and Eq. (116) is easy to verify that the overlapping component cancels out and

$$\Delta(\sigma_{V}|\tau_{V}) = \frac{1}{\sqrt{E}} [\mathbf{1}_{X(\sigma_{V})} \cdot J_{X(\sigma_{V})} - \mathbf{1}_{Y(\tau_{V})} \cdot J_{Y(\tau_{V})}] =$$

$$= \frac{1}{\sqrt{E}} [\mathbf{1}_{X'(\sigma_{V})} \cdot J_{X'(\sigma_{V})} - \mathbf{1}_{Y'(\tau_{V})} \cdot J_{Y'(\tau_{V})}] =$$

$$= \sqrt{\frac{E'}{E}} [J'(\sigma_{V}) - J'(\tau_{V})] = \sqrt{\frac{E'}{E}} \Delta'(\sigma_{V}|\tau_{V}), \quad (120)$$

Now, since E'/E converges to $1 - \varepsilon_0$ in the thermodynamic limit, and $\Delta' = J'$ net of τ_V – dependent terms that we can ignore, then we have

$$J(\sigma_V) = \sqrt{1 - \varepsilon_0} J'(\sigma_V) + \text{const.}$$
(121)

Most important: notice that the J'-amplitude is distributed like a REM by construction, as we obtained it by removing the "non-REM" component from J. Then, let define one last auxiliary function

$$K'(m_0) := (1 - \varepsilon_0) K(m_0) = [1 - (1 - m_0)/2] [2\delta^2 (1 - m_0)N] =$$

= $\delta^2 (1 + m_0) (1 - m_0) N = \delta^2 (1 - m_0^2) N$ (122)

and put everything together, the average formula can be transformed into

$$\lim_{N \to \infty} \langle f(\sigma_V) \rangle_{\xi} = \frac{\sum_{\sigma_V \in \Omega(m_0)} f^*(\sigma_V) \exp\left[-\beta J'(\sigma_V) \sqrt{K'(m_0)}\right]}{\sum_{\sigma_V \in \Omega(m_0)} \exp\left[-\beta J'(\sigma_V) \sqrt{K'(m_0)}\right]} = \lim_{N \to \infty} \frac{\langle f^*(\sigma_V) \exp\left[-\beta J'(\sigma_V) \sqrt{K'(m_0)}\right] \rangle_{m_0}}{\langle \exp\left[-\beta J'(\sigma_V) \sqrt{K'(m_0)}\right] \rangle_{m_0}}.$$
 (123)

We can immediately verify that after this change of variable the average is done with respect to a REM of some type at any temperature.

6.4 **REM-PPP** average

We could now integrate the REM variable J' by applying the well known REM-PPP average formula [3, 4, 5, 6]. The final result is the relation in Eq. (20)

$$\lim_{N \to \infty} \langle f(\sigma_V) \rangle_{\xi} = \lim_{N \to \infty} \langle f^*(\sigma_V)^{\lambda} \rangle_V^{1/\lambda}$$
(124)

with v uniform measure and λ depending on β , ψ and δ . Notice that the REM-PPP average formula interpolates between arithmetic and geometric average, in fact,

$$\lim_{\lambda \to 1} \langle f(\sigma_V)^{\lambda} \rangle_{\nu}^{1/\lambda} = \langle f(\sigma_V) \rangle_{\nu} = \frac{1}{2^N} \sum_{\sigma_V \in \Omega^V} f(\sigma_V), \qquad (125)$$

and with little more work it is possible to show that

$$\lim_{\lambda \to 0} \langle f(\sigma_V)^{\lambda} \rangle_{V}^{1/\lambda} = \lim_{\lambda \to 0} \left[\frac{1}{2^{N}} \sum_{\sigma_V \in \Omega^V} f(\sigma_V)^{\lambda} \right]^{1/\lambda} = \\ = \lim_{\lambda \to 0} \left[\frac{1}{2^{N}} \sum_{\sigma_V \in \Omega^V} \exp \lambda \log f(\sigma_V) \right]^{1/\lambda} = \\ = \lim_{\lambda \to 0} \left[1 + \frac{\lambda}{2^{N}} \sum_{\sigma_V \in \Omega^V} \log f(\sigma_V) \right]^{1/\lambda} = \lim_{\lambda \to 0} \left[1 + \lambda \log \prod_{\sigma_V \in \Omega^V} f(\sigma_V)^{\frac{1}{2^{N}}} \right]^{1/\lambda} = \\ = \lim_{\lambda \to 0} \left[\exp \lambda \log \prod_{\sigma_V \in \Omega^V} f(\sigma_V)^{\frac{1}{2^{N}}} \right]^{1/\lambda} = \prod_{\sigma_V \in \Omega^V} f(\sigma_V)^{\frac{1}{2^{N}}}, \quad (126)$$

that is the geometric average. These formulas allows the computation of the average with respect to the thermal fluctuations, although notice the dependence of f^* from the ground state still remains. Following Lemma 13 of [4] at low temperature holds

$$\lambda = C\beta^{-1/2} + O\left(\beta^{-1}\right), \quad C := \delta^{-1}\sqrt{\psi}. \tag{127}$$

See Section 5 of [4] for further details on how to obtain the exact epression. Anyway, notice that the J'-field is only approximately Gaussian, and that its rate function [19] may be different from that of a Gaussian (quadratic form) when field fluctuations are of the order of the ground state energy. The reason why at low temperatures we can actually consider the bulk (wich makes the arguments relatively elementary) is in that the contribution from spins with near zero external field is only quadratic in temperature, as shown in Section 3 for the Gaussian and uniform cases. This remarkable fact guarantees that the approximate gaussianity of J' works up to the linear order in temperature, and so J' is properly approximated by a REM of the Derrida type in that limit. More general forms of REM should be considered if we are interested to extend this argument to the whole temperature range, like those studied by Jana in his PhD thesis [24] that consider random energies with an arbitrary large deviation profile. This will be addressed elsewhere.

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