COARSE GEOMETRY OF QUASI-TRANSITIVE GRAPHS BEYOND PLANARITY

LOUIS ESPERET AND UGO GIOCANTI

Abstract. We study geometric and topological properties of infinite graphs that are quasi-isometric to a planar graph of bounded degree. We prove that every locally finite quasi-transitive graph excluding a minor is quasi-isometric to a planar graph of bounded degree. We use the result to give a simple proof of the result that finitely generated minor-excluded groups have Assouad-Nagata dimension at most 2 (this is known to hold in greater generality, but all known proofs use significantly deeper tools). We also prove that every locally finite quasi-transitive graph that is quasi-isometric to a planar graph is k-planar for some k (i.e. it has a planar drawing with at most k crossings per edge), and discuss a possible approach to prove the converse statement.

1. Introduction

Our work is motivated by a conjecture and a problem raised recently by Georgakopoulos and Papasoglu [GP23], lying at the intersection of metric graph theory and graph minor theory. Before we state them, we first need to introduce some terminology.

We say that a graph H is a *k-fat minor* of a graph G if there exists a family of connected subsets $(M_v)_{v\in V(H)}$ of $V(G)$ such that

(1) for each $u \neq v \in V(H)$, $d_G(M_u, M_v) \geqslant k$;
(2) for each $e = uv \in E(H)$ there is a path P_e whose two endpoints lie in M_u and M_v and internal vertices are not in $\bigcup_{v\in V(H)} M_v$, and
(3) for every $e \neq e' \in E(H)$, $d_G(P_e, P_{e'}) \geqslant k$ and for every $e = uv \in E(H)$ and $w \notin \{u, v\}$, $d_G(P_e, M_w) \geqslant k$.

A graph H is an *asymptotic minor* of G if for every $k \geqslant 0$, H is a k-fat minor of G.

Let (X, d_X) and (Y, d_Y) be two metric spaces. We say that X is *quasi-isometric* to Y if there is a map $f : X \to Y$ and constants $\varepsilon \geqslant 0$, $\lambda \geqslant 1$, and $C \geqslant 0$ such that (i) for any $y \in Y$ there is $x \in X$ such that $d_Y(y, f(x)) \leqslant C$, and (ii) for every $x_1, x_2 \in X$,

$$\frac{1}{\lambda} d_X(x_1, x_2) - \varepsilon \leqslant d_Y(f(x_1), f(x_2)) \leqslant \lambda d_X(x_1, x_2) + \varepsilon.$$

It is not difficult to check that the definition is symmetric, and we often simply say that X and Y are quasi-isometric. If condition (i) is omitted in the definition above, we say that f is a *quasi-isometric embedding of X in Y*.

We can view each graph G as a metric space, by considering the natural shortest-path metric associated to G. A graph is *locally finite* if every vertex has finite degree. Georgakopoulos and Papasoglu conjectured the following [GP23].

The authors are partially supported by the French ANR Project GrR (ANR-18-CE40-0032), TWIN-WIDTH (ANR-21-CE48-0014-01), and by LabEx PERSYVAL-lab (ANR-11-LABX-0025).

Conjecture 1.1 (Conjecture 9.3 in [GP23]). *If G is locally finite, vertex-transitive and excludes some finite graph H as an asymptotic minor, then G is quasi-isometric to a planar graph.*

It is natural to first prove this conjecture when G excludes some minor H (instead of an *asymptotic* minor, as defined above). This suggests the following:

Question 1.2. *Is it true that if G is locally finite, vertex-transitive and excludes some finite graph H as a minor, then G is quasi-isometric to a planar graph?*

Our first result is a positive answer to this question, in a slightly stronger form. An infinite graph is *quasi-transitive* if its vertex set has finitely many orbits under the action of its automorphism group. Note that any vertex-transitive graph is quasi-transitive, and that for quasi-transitive graphs, being locally finite is equivalent to having bounded degree.

The *countable clique* K_∞ is the graph with vertex set $\mathbb{N}$ in which every two vertices are adjacent (every graph which excludes a finite or countable graph H as a minor also excludes K_∞ as a minor). We prove the following.

Theorem 1.3. *Every locally finite quasi-transitive K_∞-minor free graph is quasi-isometric to a planar graph of bounded degree.*

The main technical tool that we use is a recent structural theorem on locally finite quasi-transitive graphs excluding the countable clique as a minor [EGLD23], which shows that such graphs have a canonical tree-decomposition in which all torsos are planar or finite (see the next section for the definitions). Most importantly, this result does not use the Robertson-Seymour graph minor structure theorem.

We note that the result which allows us to construct the quasi-isometry using the canonical tree-decomposition was also proved recently (and independently) by MacManus [Mac23] in a slightly different form (the "if" direction in his Corollary C). Our proof is very similar to his.

We now discuss several applications of Theorem 1.3.

Application 1. Beyond planarity. A graph is *k-planar* if it has a drawing in the plane in which each edge is involved in at most k crossings (note that with this terminology, being planar is the same as being 0-planar). The *local crossing number* of a graph G, denoted by $\mathrm{lcr}(G)$, is the infimum integer k such that G is k-planar.

Georgakopoulos and Papasoglu raised the following problem [GP23].

Problem 1.4 (Problem 9.4 in [GP23]). *For any quasi-transitive graph G of bounded degree, G is quasi-isometric to a planar graph if and only if G has finite local crossing number.*

We prove that for any integer k, every bounded degree graph which is quasi-isometric to a k-planar graph is k'-planar for some integer k'. In the particular case $k = 0$, we immediately obtain the "only if" direction of Problem 1.4 (we recently learned from Agelos Georgakopoulos that he also proved the case $k = 0$ independently). In Section 6, we raise a number of conjectures whose validity would imply a positive answer to the "if" direction of Problem 1.4. In the case $k = 0$, we also obtain our second application of Theorem 1.3:

Theorem 1.5. *Every locally finite quasi-transitive graph G which is K_∞-minor-free has finite local crossing number.*

The assumption that G is locally finite is necessary, as shown by the graph obtained from the square grid by adding a universal vertex (this graph is K_6-minor free, but is not k-planar for any $k < \infty$). The assumption that G is quasi-transitive is also crucial: consider for each integer ℓ a graph G_ℓ obtained from the square grid by adding an edge between two vertices at distance ℓ in the grid (if G_ℓ is k-planar then $k = \Omega(\ell)$), and take the disjoint union of all graphs G_ℓ, $\ell \in \mathbb{N}$.

Note that in the other direction, there exist 1-planar graphs that are vertex-transitive and locally finite, but which contain all graphs as minors (the square grid with all diagonals is such an example).

Application 2. Assouad-Nagata dimension. Our second application of Theorem 1.3 requires the notions of asymptotic dimension and Assouad-Nagata dimension of metric spaces, which we introduce now. Let (X, d) be a metric space, and let $\mathcal{U}$ be a family of subsets of X. We say that $\mathcal{U}$ is *D-bounded* if each set $U \in \mathcal{U}$ has diameter at most D. We say that $\mathcal{U}$ is *r-disjoint* if for any a, b belonging to different elements of $\mathcal{U}$ we have $d(a, b) > r$.

We say that $D_X : \mathbb{R}^+ \to \mathbb{R}^+$ is an *n-dimensional control function* for (X, d) if for any $r > 0$, (X, d) has a cover $\mathcal{U} = \bigcup_{i=1}^{n+1} \mathcal{U}_i$, such that each $\mathcal{U}_i$ is r-disjoint and each element of $\mathcal{U}$ is $D_X(r)$-bounded. A control function D_X for a metric space X is said to be a *dilation* if there is a constant $c > 0$ such that $D_X(r) \leqslant cr$, for any $r > 0$.

The *asymptotic dimension* of (X, d), introduced by Gromov in [Gro93], is the least integer n such that (X, d) has an n-dimensional control function. If no such integer n exists, then the asymptotic dimension is infinite. The *Assouad-Nagata dimension* of (X, d), introduced by Assouad in [Ass82], is the least n such that (X, d) has an n-dimensional control function which is a dilation. Clearly the asymptotic dimension is at most the Assouad-Nagata dimension.

It was proved in [BBE+20] that every *bounded degree* graph excluding a minor has asymptotic dimension at most 2, and that any planar graph has asymptotic dimension at most 2. This was improved in [BBE+23], where it was shown that any graph excluding a minor has asymptotic dimension at most 2, and any planar graph has Assouad-Nagata dimension at most 2. This was finally extended by Liu in [Liu23], who proved that any graph avoiding a minor has Assouad-Nagata dimension at most 2 (a different proof was then given by Distel in [Dis23]). All the results on graphs excluding a minor mentioned above (even for bounded degree graphs) crucially rely on the Graph minor structure theorem by Robertson and Seymour [RS03], a deep result proved in a series of 16 papers.

Using the invariance of Assouad-Nagata dimension under bilipschitz embedding [LS05], we will give a short proof of the fact that minor-excluded quasi-transitive graphs have Assouad-Nagata dimension at most 2. A finitely generated group is said to be *minor-excluded* if it has a Cayley graph which excludes a minor. Our result directly implies that finitely generated minor-excluded groups have Assouad-Nagata dimension at most 2, and thus asymptotic dimension at most 2 (which was originally conjectured by Ostrovskii and Rosenthal in [OR15]). We will only use Theorem 1.3 and a few simple tools from [BBE+20, BBE+23] based on the work of Brodskiy, Dydak, Levin and Mitra [BDLM08]. In particular we will give a short proof of the fact that planar graphs of bounded degree have Assouad-Nagata dimension at most

2. Crucially, our proof for quasi-transitive graphs excluding a minor *does not rely* on the Graph minor structure theorem of Robertson and Seymour.

2. Preliminaries

All graphs in this paper are assumed to be infinite, unless stated otherwise.

A *tree-decomposition* of a graph G is a pair $(T, \mathcal{V})$ such that T is a tree and $\mathcal{V}$ is a collection $(V_t : t \in V(T))$ of subsets of $V(G)$, called the *bags*, such that

- $\bigcup_{t\in V(T)} V_t = V(G)$,
- for every $uv \in E(G)$, there exists $t \in V(T)$ such that $u, v \in V_t$, and
- for every $v \in V(G)$, the set $\{t \in V(T) : v \in V_t\}$ induces a connected subgraph of T.

For a tree-decomposition $(T, \mathcal{V})$, the *width* of $(T, \mathcal{V})$ is $\sup_{t\in V(T)}|V_t|-1 \in \mathbb{N}\cup\{\infty\}$. The *treewidth* of G is the minimum width of a tree-decomposition of G.

The sets $V_t \cap V_{t'}$ for every $tt' \in E(T)$ are called the *adhesion sets* of $(T, \mathcal{V})$ and the *adhesion* of $(T, \mathcal{V})$ is the supremum of the sizes of its adhesion sets (possibly infinite). For $t \in V(T)$, the *torso* $G[\![V_t]\!]$ is the graph obtained from $G[V_t]$ (the subgraph of G induced by the bag V_t) by adding all edges uv with $u, v \in V_t$ for which there exist t' such that u and v lie in the adhesion set $V_t \cap V_{t'}$.

We say that a tree-decomposition $(T, \mathcal{V})$ is *canonical*, if for every automorphism γ of G, γ sends bags of $(T, \mathcal{V})$ to bags, and adhesion sets to adhesion sets. In other words, the automorphism group of G induces a group action on T.

A *separation* in a graph $G = (V, E)$ is a triple (Y, S, Z) such that Y, S, Z are pairwise disjoint, $V = Y \cup S \cup Z$ and there is no edge between vertices of Y and Z. The separation (Y, S, Z) is said to be *tight* if there are some components C_Y, C_Z respectively of $G[Y], G[Z]$ such that $N_G(C_Y) = N_G(C_Z) = S$.

Consider a tree-decomposition $(T, \mathcal{V})$ of a graph G, with $\mathcal{V} = (V_t)_{t\in V(T)}$. Let A be an orientation of the edges of $E(T)$, i.e. a choice of either (t_1, t_2) or (t_2, t_1) for every edge t_1t_2 of T. For a pair $(t_1, t_2) \in A$, and for each $i \in \{1, 2\}$, let T_i denote the component of $T - \{t_1t_2\}$ containing t_i. Then the *edge-separation* of G associated to (t_1, t_2) is (Y_1, S, Y_2) with $S := V_{t_1} \cap V_{t_2}$ and $Y_i := \bigcup_{s\in V(T_i)} V_s \setminus S$ for $i \in \{1, 2\}$.

We will need the main result of [EGLD23], which gives the structure of locally finite quasi-transitive graph excluding a minor (note that this result does not use the Graph minor structure theorem by Robertson and Seymour [RS03]).

Theorem 2.1 ([EGLD23])**.** *Let G be a locally finite quasi-transitive graph excluding K_∞ as a minor. Then there is an integer k such that G admits a canonical tree-decomposition $(T, \mathcal{V})$ of adhesion at most k, whose torsos have size at most k or are planar. Moreover, the edge-separations of $(T, \mathcal{V})$ are tight.*

The property that the edge-separations $(T, \mathcal{V})$ are tight in the statement of Theorem 2.1 will be particularly useful in combination with the following result of Thomassen and Woess [TW93] (which was explicitly proved for transitive graphs, but the same proof also holds for quasi-transitive graphs).

Lemma 2.2 (Corollary 4.3 in [TW93])**.** *Let G be a locally finite graph. Then for every $v \in V(G)$ and $k \geqslant 1$, there is only a finite number of tight separations (Y, S, Z) of order k in G such that $v \in S$. Moreover, if G is quasi-transitive then for any $k \geqslant 1$,* *there is only a finite number of orbits of tight separations of order at most k in G under the action of the automorphism group of G.*

3. Proof of Theorem 1.3

We assume that G is connected, since otherwise we can consider each connected component separately. By Theorem 2.1, G has a canonical tree-decomposition $(T, \mathcal{V})$ whose torsos $G[\![V_t]\!]$, $t \in V(T)$, are either planar or finite and whose adhesion sets have bounded size. For each $u \in V(G)$, we let T_u be the subtree of T with vertex set $\{t \in V(T), u \in V_t\}$. Note that as the edge-separations of $(T, \mathcal{V})$ are tight, Lemma 2.2 implies that T_u is finite for each u.

We let G' be the graph constructed as follow: for each $t \in V(T)$, we let V'_t be a copy of V_t and G'_t (with vertex set V'_t) be a copy of $G[\![V_t]\!]$ if V_t is infinite, or a spanning tree of $G[\![V_t]\!]$ if V_t is finite. For each $u \in V(G)$ and $t \in V(T_u)$, we let $u^{(t)}$ denote the copy of u in V'_t. We let $V(G') := \biguplus_{t \in V(T)} V'_t$. Now for every edge $st \in E(T)$, we choose an arbitrary vertex $u_{st} \in V_t \cap V_s$ (such a vertex exists, since G is assumed to be connected). We let:

$$E(G') := \left(\biguplus_{t \in V(T)} E(G'_t) \right) \uplus \{u^{(s)}_{st} u^{(t)}_{st}, st \in E(T)\}.$$

We also let T' be the 1-subdivision of T (the graph obtained from T by replacing each edge $e = st$ by a two-edge path s, t_e, t). Finally, for each $e = st \in E(T)$, we set $V'_{t_e} := \{u^{(s)}_{st}, u^{(t)}_{st}\}$ and $\mathcal{V}' := (V'_t)_{t \in V(T')}$. We observe that by definition, $(T', \mathcal{V}')$ is a tree-decomposition of G' whose adhesion sets all have size 1. In particular, for every $t \in V(T')$, $G'[V'_t] = G'[\![V'_t]\!]$. We also note that by the definition of G' and Lemma 2.2, G' has bounded degree.

Claim 3.1. *For every graph G, if G has a tree-decomposition $(T, \mathcal{V})$ such that every torso is planar and adhesion sets have size at most 1, then G is planar.*

Proof of the Claim: If G contains K_5 or $K_{3,3}$ as a minor, then some torso of $(T, \mathcal{V})$ must also contain K_5 or $K_{3,3}$ as a minor, which is a contradiction. The result then follows from Wagner's theorem [Wag37] stating that any graph excluding K_5 and $K_{3,3}$ is planar. $\Diamond$

We now construct a quasi-isometry f from G to G'. For each $u \in V(G)$, we choose some $t_u \in V(T_u)$ and set $f(u) := u^{(t_u)}$. We also let $A_1 := \max\{\mathrm{diam}_G(V_t), V_t \text{ is finite}\}$, $A_2 := \max(1, \max\{|V_t|, V_t \text{ is finite}\})$ and $B := \max\{\mathrm{diam}_T(T_u), u \in V(T)\}$, which all exist by Lemma 2.2, as the edge-separations of $(T, \mathcal{V})$ are tight. We note that for each $t \in V(T)$ such that V_t is finite, since $G'[V'_t] = G'[\![V'_t]\!]$ is connected, its diameter is at most A_2.

We first show the following:

Claim 3.2. *There exists a constant $C \geqslant 0$ such that for each $t \in V(T), u, v \in V_t$:*

$$d_G(u, v) \leqslant C \cdot d_{G[\![V_t]\!]}(u, v).$$

Proof of the Claim: By Lemma 2.2, $E(T)$ has finitely many orbits under the action of the automorphism group of G. Hence, up to automorphism there are only finitely many pairs $\{u, v\}$ such that u, v lie in a common adhesion set of $(T, \mathcal{V})$. In particular, as G is connected this means that the set of values $\{d_G(u,v), \exists st \in E(T), u,v \in V_s \cap V_t\}$ admits a maximum C. The claim follows from this observation. $\Diamond$

We now show that there is a constant $\alpha > 0$ such that for every $u, v \in V(G)$ and every $f(u)f(v)$-path P' in G', there exists a uv-path P of size at most $\alpha \cdot |P'|$ in G. By taking P' to be a shortest path from $f(u)$ to $f(v)$ in G', this will imply in particular that $d_G(u,v) \leqslant \alpha \cdot d_{G'}(f(u), f(v))$.

Claim 3.3. *For every $u, v \in V(G)$ and $t, s \in V(T)$ such that $u^{(t)}v^{(s)} \in E(G')$ we have*

$$d_G(u,v) \leqslant \alpha := \max(A_1, C).$$

Proof of the Claim: Assume first that $s = t$. If V_t is finite, then $d_G(u,v) \leqslant A_1$. If V_t is infinite we must have $uv \in E(G[\![V_t]\!])$, and thus $d_G(u,v) \leqslant C$ by Claim 3.2.

Assume now that $s \neq t$. Then by definition of G', we must have $st \in E(T)$ and $u = v$, and thus $d_G(u,v) = 0$. $\Diamond$

We now show that there exists a constant $\beta > 0$ such that for every $u, v \in V(G)$ and every uv-path P in G, there exists a $f(u)f(v)$-path P' of size at most $\beta|P|$ in G'. This directly implies that $d_{G'}(f(u), f(v)) \leqslant \beta \cdot d_G(u,v)$.

Claim 3.4. *For every $u, v \in V(G)$ and $t, s \in V(T)$ such that $uv \in E(G)$, $u \in V_t$ and $v \in V_s$ we have:*

$$d_{G'}(u^{(t)}, v^{(s)}) \leqslant \beta := (4A_2 + 2)B + A_2.$$

Proof of the Claim: First note that if V_t is finite, then for each $u, v \in V_t$ we have:

$$d_{G'}(u^{(t)}, v^{(t)}) = d_{G'[V_t']}(u^{(t)}, v^{(t)}) \leqslant A_2.$$

If V_t is infinite, then for each $u, v \in V_t$ such that $uv \in E(G)$, we have $u^{(t)}v^{(t)} \in E(G')$ and thus $d_{G'}(u^{(t)}, v^{(t)}) \leqslant 1$. Since $A_2 \geqslant 1$, it follows that for each $t \in V(T)$ and $u, v \in V_t$ such that $uv \in E(G)$, we have

$$d_{G'}(u^{(t)}, v^{(t)}) \leqslant A_2. \tag{1}$$

Now let $u \in V(G)$ and $s, t \in V(T_u)$. We let $(s = t_0, t_1, \ldots, t = t_\ell)$ be the shortest st-path in T. Note that it is also a path in T_u, hence $\ell \leqslant B$. Recall that in the construction of G', we have chosen for each edge $st \in E(T)$ a vertex $u_{st} \in V_s \cap V_t$ and we have added an edge in G' between $u_{st}^{(s)} \in V_s'$ and $u_{st}^{(t)} \in V_t'$. For each $i \in [\ell]$, we write $x_i := u_{t_{i-1}t_i} \in V_{t_{i-1}} \cap V_{t_i}$ for the sake of readability. Note that for each $i \in [\ell]$, x_i might be equal to u and that both u and x_i lie in the adhesion set $V_{t_{i-1}} \cap V_{t_i}$. This implies that $u^{(t_i)}$ and $x_i^{(t_i)}$ are adjacent in G' if V_{t_i} is infinite, and $d_{G'}(u^{(t_i)}, x_i^{(t_i)}) \leqslant A_2$ otherwise. So $d_{G'}(u^{(t_i)}, x_i^{(t_i)}) \leqslant A_2$ in both cases, and similarly $d_{G'}(u^{(t_{i-1})}, x_i^{(t_{i-1})}) \leqslant A_2$. It follows that for each $i \in [\ell]$, we have

$$d_{G'}(u^{(t_{i-1})}, u^{(t_i)}) \leqslant d_{G'}(u^{(t_{i-1})}, x_i^{(t_{i-1})}) + d_{G'}(x_i^{(t_{i-1})}, x_i^{(t_i)}) + d_{G'}(x_i^{(t_i)}, u^{(t_i)}) \leqslant 2A_2 + 1.$$

This implies that for every $u \in V(G)$ and $s, t \in V(T_u)$

$$d_{G'}(u^{(s)}, u^{(t)}) \leqslant (2A_2 + 1)B. \tag{2}$$

To conclude the proof of the claim, let $uv \in E(G)$. As $(T, \mathcal{V})$ is a tree-decomposition, there exists some $t \in V(T)$ such that $u, v \in V_t$. Then:

$$d_{G'}(f(u), f(v)) \leqslant d_{G'}(u^{(t_u)}, u^{(t)}) + d_{G'}(u^{(t)}, v^{(t)}) + d_{G'}(v^{(t)}, v^{(t_v)}),$$

thus by inequalities (1) and (2) we obtain $d_{G'}(f(u), f(v)) \leqslant (4A_2 + 2)B + A_2$. $\Diamond$

To prove that f is a quasi-isometry, it remains to prove that each $y \in V(G')$ is at bounded distance in G' from $f(V(G))$. For this, let $y \in V(G')$ and $t \in V(T), u \in V(G)$ be such that $y = u^{(t)}$. Then by inequality (2), $d_{G'}(y, f(u)) \leqslant (2A_2+1)B$ so f is indeed a quasi-isometry. This concludes the proof of Theorem 1.3. □

4. Beyond planarity

In this section we prove Theorem 1.5. We first show that for graphs of bounded degree, having finite local crossing number is preserved under quasi-isometry.

Theorem 4.1. *Let G be a graph of bounded degree which is quasi-isometric to a graph H of finite local crossing number. Then G also has finite local crossing number.*

Note that in general, the property of being locally finite, or even of having countably many vertices is not preserved under quasi-isometry. The next lemma will be useful to make sure that we can restrict ourselves to locally finite graphs in the remainder of the proof.

Lemma 4.2. *Let G be a graph of bounded degree which is quasi-isometric to a graph H. Then G is quasi-isometric to a subgraph H' of H of bounded degree.*

Proof. We let $f : V(G) \to V(H)$ and $A \geqslant 1$ be such that for each $x, x' \in V(G)$:

$$\frac{1}{A} \cdot d_G(x, x') - A \leqslant d_H(f(x), f(x')) \leqslant A \cdot d_G(x, x') + A,$$

and such that the A-neighborhood of $f(V(G))$ covers H. We also let $\Delta \in \mathbb{N}$ denote the maximum degree of G. Note that for each $xy \in E(G)$, there exists a $f(x)f(y)$-path P_{xy} in H such that $|P_{xy}| \leqslant A \cdot 1 + 1 = 2A$. We let H' be the subgraph of H given by the union of all such paths P_{xy}.

We first observe that H' is quasi-isometric to G, and that f gives the corresponding quasi-isometric embedding. Note that for every $z \in V(H')$, by construction there must be some edge $xy \in E(G)$ such that $z \in P_{xy}$. In particular, $d_{H'}(z, f(x)) \leqslant 2A$. Note that by construction we clearly have $d_{H'}(f(x), f(y)) \leqslant 2A d_G(x, y)$ for each $x, y \in V(G)$, and as $d_{H'}(f(x), f(y)) \geqslant d_H(f(x), f(y))$, f indeed induces a quasi-isometric embedding between G and H'.

Now we show that H' has bounded degree. Let $z \in V(H')$ and $xy \in E(G)$ such that $z \in V(P_{xy})$. Then $d_{H'}(z, f(x)) \leqslant 2A$ so $X := \{x \in V(G), z \in V(P_{xy})\}$ has diameter at most $4A$ in G. Note that as G degree at most Δ, we have $|X| \leqslant \Delta^{4A}$. In particular it implies that H' has degree at most Δ^{4A}. □

Given a graph G and an integer $k \geqslant 1$, the *k-th power* of G, denoted by G^k, is the graph with the same vertex set as G in which two vertices are adjacent if and only if they are at distance at most k in G. The *k-blow-up* of G, denoted by $G \boxtimes K_k$, is the graph obtained from G by replacing each vertex u by a copy C_u of the complete graph K_k, and by adding all edges between pairs C_u, C_v if and only if u and v are adjacent in G (so that each edge of G is replaced by a complete bipartite graph $K_{k,k}$ in $G \boxtimes K_k$). Quasi-isometries of bounded degree graphs are related to graph powers and blow-ups by the following lemma.

Lemma 4.3. *Let H be a graph, and let G be a graph of degree at most $\Delta \in \mathbb{N}$ which is quasi-isometric to H. Then there is an integer k such that G is a subgraph of $H^k \boxtimes K_k$.*

Proof. We let $A \geqslant 1$ and $f : V(G) \to V(H)$ be such that for each $x, x' \in V(G)$:

$$\frac{1}{A} \cdot d_G(x, x') - A \leqslant d_H(f(x), f(x')) \leqslant A \cdot d_G(x, x') + A,$$

and such that the A-neighborhood of $f(V(G))$ covers H. Note that for each $x, x' \in V(G)$ such that $f(x) = f(x') = y$ we must have $d_G(x, x') \leqslant A^2$, hence

$$|f^{-1}(y)| \leqslant B := \Delta^{A^2}$$

for every $y \in V(H)$. We now show that G is a subgraph of $H' := H^{2A} \boxtimes K_B$, which implies the lemma for $k := \max(2A, B)$.

As in the definition of a blow-up, for each $v \in V(H)$ we denote by C_v the associated clique of size B in H'. For every $v \in V(H)$ we fix an arbitrary injection $g_v : f^{-1}(v) \to C_v$, and define an injective mapping $g : V(G) \to V(H')$ by letting $g(x) := g_{f(x)}(x)$ for each $x \in V(G)$. In other words every two vertices of G having the same image v by f are sent by g in the same clique C_v in H'. By construction g is injective, so we just need to check that it defines a graph homomorphism to conclude that G is a subgraph of H'. Let $xy \in E(G)$. Then $d_H(f(x), f(y)) \leqslant 2A$ so in particular every vertex in $V_{f(x)}$ is at distance at most $2A$ to every vertex in $V_{f(y)}$ in H'. In particular, this means that $g(x)g(y) \in E(H')$, as desired. $\square$

The next observation will allow us to slightly simplify the statement of Lemma 4.3.

Observation 4.4. *For every graph H of bounded maximum degree, and every integer k, there exists a graph G of bounded maximum degree such that* $\mathrm{lcr}(G) = \mathrm{lcr}(H)$ *and $H^k \boxtimes K_k$ is a subgraph of G^{k+2}.*

Proof. Let G be the graph obtained from H by attaching to each vertex k pendant vertices of degree 1. Note that the graph $H^k \boxtimes K_k$ is a subgraph of G^{k+2}. To see this, one can bijectively map each clique C_v of H for $v \in V(H)$ to the k pendant vertices we attached to v in G, and observe that it gives an isomorphism between the graph induced by these vertices in G^3, and $H \boxtimes K_k$. Since adding pendant vertices does not change the local crossing number, $\mathrm{lcr}(G) = \mathrm{lcr}(H)$. Moreover G has bounded maximum degree if and only if H has bounded maximum degree. $\square$

We can now combine the results above to deduce the following corollary.

Corollary 4.5. *If a graph G with bounded degree is quasi-isometric to a graph of bounded local crossing number, then there exists a planar graph H of bounded maximum degree and an integer k, such that G is a subgraph of H^k.*

Proof. By Lemmas 4.2 and 4.3, there is a graph H_1 of bounded local crossing number and maximum degree and an integer ℓ such that G is a subgraph of $H_1^\ell \boxtimes K_\ell$. By Observation 4.4, there is a graph H_2 of bounded local crossing number and maximum degree such that G is a subgraph of $H_2^{\ell+2}$. Observe that every s-planar graph F_1 is a subgraph of F_2^{s+1}, where F_2 is the planar graph obtained from F_1 by placing a new vertex at each crossing (and note that if F_1 has bounded degree, then F_2 also has bounded degree). It follows that there is a planar graph H of bounded degree and an integer k, such that G is a subgraph of H^k. $\square$

We now prove that bounded powers of planar graphs of bounded degree are ℓ-planar for some ℓ. This was proved for finite graphs in [DMW23, Lemma 12].

Lemma 4.6 ([DMW23])**.** *Let H be a finite planar graph of maximum degree at most Δ and let G be a subgraph of H^k, for some integer k. Then G is ℓ-planar, for $\ell := 2k(k+1)\Delta^k$.*

However, we need a version of Lemma 4.6 for infinite locally finite graphs. The first option is to simply follow the proof of [DMW23], which starts with a planar drawing of G, and adds for any path P of length at most k, an edge between the endpoints of P, drawn in a close neighborhood around P. However in the locally finite case this approach requires that the original planar drawing has the property that every edge has a small neighborhood which does not intersect any other vertices or edges of the graph. Such a drawing always exists but it requires a little bit of work. So instead, we chose to extend Lemma 4.6 to infinite locally finite graphs using a simple compactness argument.

Lemma 4.7. *Let G be a locally finite graph. If there is an integer ℓ such that all finite induced subgraphs of G are ℓ-planar, then G is also ℓ-planar.*

Proof. We first observe that any ℓ-planar embedding of a graph H can be described combinatorially, by considering the planar graph H^+ obtained from H by replacing all crossings by new vertices. The corresponding planar embedding of H^+ can be completely described (up to homeomorphism) by its rotation system (the clockwise cyclic ordering of the neighbors around each vertex), and there are only finitely many such rotation systems if H (and thus H^+) is finite.

We are now ready to prove the lemma. We can assume that G is connected, since otherwise we can consider each connected component independently. Since G is locally finite and connected, it is countable and we can write $V(G) = \{v_1, v_2, \ldots\}$. We define a rooted tree T as follows. The root of T is the unique ℓ-planar embedding of $G[\{v_1\}]$, up to homeomorphism. For every $k \geqslant 1$, and any ℓ-planar embedding of $G_k := G[\{v_1, \ldots, v_k\}]$ we add a node in the tree and connect it to the node corresponding to the resulting ℓ-planar embedding of $G_{k-1} = G_k - v_k$ (the embedding obtained by deleting v_k in the embedding of G_k). The resulting tree T is infinite (since every graph G_k is ℓ-planar by assumption), and locally finite (since every graph G_k has only finitely many different ℓ-planar embeddings, up to homeomorphism). By König's infinity lemma [Kön27] (or by repeated applications of the pigeonhole principle), T has an infinite path starting at the root. This infinite path corresponds to a sequence of ℓ-planar embeddings of G_k, $k \geqslant 0$, with the property that for every $k \geqslant 0$, the ℓ-planar embedding of G_k can be obtained from the ℓ-planar embedding of G_{k+1} by deleting v_{k+1} (and all edges incident to v_{k+1}). By taking the union of all the ℓ-planar embeddings of G_k, $k \geqslant 0$, we thus obtain an ℓ-planar embedding of G, as desired. □

We obtain the following as a direct consequence.

Corollary 4.8. *Let H be a locally finite planar graph of maximum degree at most Δ and let G be a subgraph of H^k, for some integer k. Then G is ℓ-planar, for $\ell := 2k(k+1)\Delta^k$.*

Proof. Let X be a finite subset of $V(G) \subseteq V(H)$ and for any pair $x, x' \in X$ with $d_H(x, x') \leqslant k$, consider a path $P_{x,x'}$ of length at most k between x and x' in H. Let Y be the union of X and the vertex sets of all the paths $P_{x,x'}$ defined above. Then $H[Y]$ is a finite planar graph, and $G[X]$, the finite subgraph of G induced by X, is a subgraph of $H[Y]^k$. By Lemma 4.6, $G[X]$ is ℓ-planar with $\ell := 2k(k+1)\Delta^k$. Since this holds for any finite set X, it follows from Lemma 4.7 that G itself is ℓ-planar, as desired. □

Theorem 4.1 is now a direct consequence of Corollary 4.5 and Corollary 4.8. By combining Theorems 1.3 and 4.1, we then immediately deduce Theorem 1.5.

5. Assouad-Nagata dimension of minor-excluded groups

For two graphs G and H, we say that a function $f : V(G) \to V(H)$ is a *bilipschitz* mapping from G to H if there are constants $c_1, c_2 > 0$ such that for any $x, y \in V(G)$,

$$c_1 \cdot d_G(x, y) \leqslant d_H(f(x), f(y)) \leqslant c_2 \cdot d_G(x, y).$$

When such a function f exists we also say that the graph G has a *bilipschitz embedding in* H. Since graphs are uniformly discrete metric spaces (in the sense that any two distinct elements lie at distance at least 1 apart), any quasi-isometric embedding of a graph G in a graph H is also a bilipschitz embedding of G in H. Hence, Theorem 1.3 has the following immediate consequence.

Corollary 5.1. *Every locally finite quasi-transitive K_∞-minor free graph has a bilipschitz embedding in a planar graph of bounded degree.*

We first prove that planar graphs of bounded degree have Assouad-Nagata dimension at most 2 (a stronger version of this result, without the bounded degree assumption, was proved in [BBE+23]).

We need the following result, first proved in [BST12] in a slightly different form. Another proof can be found in [BBE+20, BBE+23] based on a result of Ding and Oporowski [DO95] which states that every graph of treewidth at most t and maximum degree at most Δ is a subgraph of the strong product of a tree with a complete graph on $24t\Delta$ vertices. The proofs of all these results are fairly short.

Theorem 5.2 ([BST12])**.** *If a graph G has bounded degree and bounded treewidth, then G has Assouad-Nagata dimension at most 1.*

A *layering* of a graph G is a partition of the vertex set of G into sets $L_0, L_1, \ldots$, called *layers*, so that any pair of adjacent vertices in G either lies in the same layer or in consecutive layers (i.e. layers L_i, L_{i+1} for some $i \geqslant 1$). A simple example of layering is given by a *BFS-layering* of G, obtained by choosing one root vertex v_C in each connected component C of G, and then defining L_i (for all $i \geqslant 0$) as the set of vertices of G at distance exactly i from one of the vertices v_C.

It was proved by Bodlaender [Bod88] that planar graphs of bounded diameter have bounded treewidth. This directly implies the following.

Lemma 5.3. *For any BFS-layering of a planar graph G, and any integer k, the subgraph of G induced by k consecutive layers of $\mathcal{L}$ has bounded treewidth.*

Proof. Let $L_0, L_1, \ldots$ be a BFS-layering of G. Consider the planar subgraph H of G induced by k consecutive layers $L_i, L_{i+1}, \ldots, L_{i+k-1}$. If $i = 0$, then H is a disjoint union of graphs of radius at most k, and thus has bounded treewidth by Bodlaender's result [Bod88]. Assume now that $i \geqslant 1$, and let H^+ be the supergraph of H obtained by adding a vertex r that dominates all the vertices of L_i. Note that H^+ can be obtained from the subgraph of G induced by the layers $L_0, L_1, \ldots, L_{i+k-1}$, by contracting all layers $L_0, L_1, \ldots, L_{i-1}$ (which induce a connected subgraph of G, by definition of a BFS-layering) into a single vertex. Thus H^+ is a minor of G, which implies that H^+ is planar. Moreover, it follows from the definition of a BFS-layering that each vertex of H^+ lies at distance at most k from r, and thus H^+ has diameter at most $2k$. By Bodlaender's result [Bod88], H^+ has bounded treewidth, and thus H (as a subgraph of H^+) also has bounded treewidth. □

The next result appears as Theorem 4.3 in [BBE+23], and is a simple application of the main result in [BDLM08] (which has a nice and short combinatorial proof).

Theorem 5.4 ([BBE+23]). *If a graph G has a layering $\mathcal{L} = (L_0, L_1, \ldots)$ such that for any integer k, the disjoint union of all subgraphs of G induced by k consecutive layers of $\mathcal{L}$ has Assouad-Nagata dimension at most n, then G has Assouad-Nagata dimension at most $n + 1$.*

We immediately deduce the following.

Corollary 5.5. *Every planar graph G of bounded degree has Assouad-Nagata dimension at most 2.*

Proof. Consider a BFS-layering $\mathcal{L}$ of G. By Lemma 5.3, for any $k \geqslant 1$, the disjoint union of all subgraphs of G induced by k consecutive layers of $\mathcal{L}$ has bounded treewidth. As G has bounded degree, this disjoint union of subgraphs of G also has bounded degree, and thus by Theorem 5.2 it has Assouad-Nagata dimension at most 1. Hence, it follows from Theorem 5.4 that G itself has Assouad-Nagata dimension at most 2. □

We are now ready to prove the main result of this section. Recall that a stronger version (without the quasi-transitivity assumption) was proved in [Liu23, Dis23], but the version below has a reasonably simple proof that does not rely on the Robertson-Seymour graph minor structure theorem.

Theorem 5.6. *Every locally finite quasi-transitive K_∞-minor free graph has Assouad-Nagata dimension at most 2.*

Proof. By Corollary 5.1, every locally finite quasi-transitive K_∞-minor free graph G has a bilipschitz embedding in some planar graph H of bounded degree, which has Assouad-Nagata dimension at most 2. Since Assouad-Nagata dimension is invariant under bilipschitz embedding [LS05], G has Assouad-Nagata dimension at most 2. □

6. Open problems

We now discuss possible extensions or variants of Theorem 1.3.

Quasi-isometries to quasi-transitive planar graphs. It was proved by MacManus [Mac23] that if a finitely generated group has a Cayley graph which is quasi-isometric to a planar graph, then it is quasi-isometric to a planar Cayley graph. In the same spirit, it is natural to ask whether we can require the planar graph of bounded degree in Theorem 1.3 to be quasi-transitive, or even a Cayley graph. Our current proof does not preserve symmetries, as we do a number of non-canonical choices for the images of the vertices. As remarked by a referee, the stronger question above, whether we can require the planar graph in Theorem 1.3 to be a Cayley graph, is a special case of a Problem of Woess [Woe91, Problem 1], which asked whether every transitive graph is quasi-isometric to a Cayley graph. This turned out to have a negative answer [EFW12] in general, but the question restricted to (quasi-)transitive graphs excluding a minor might still have a positive answer.

A finite list of obstructions. MacManus [Mac23] recently gave a precise characterization of quasi-transitive graphs which are quasi-isometric to a planar graph, in terms of the existence of a canonical tree-decomposition sharing similarities with that of Theorem 2.1. It would be interesting to also find a characterization in terms of obstructions. For instance, it was conjectured in [GP23] that a graph is quasi-isometric to a planar graph if and only if it does not contain K_5 or $K_{3,3}$ as an asymptotic minor (see also Question 1.1 for a version of this problem in the particular case of transitive graphs).

Examples of quasi-transitive graphs that are not quasi-isometric to any planar graph include Cayley graphs of a group of Assouad-Nagata dimension at least 3, for instance any grid in dimension 3. This rules out any generalization of Theorem 1.3 using classes of polynomial growth or expansion. This example also shows that we cannot extend Theorem 1.3 to all families of bounded queue-number or stack-number.

Here is perhaps a more interesting example. The *strong product* $G \boxtimes H$ of two graphs G and H has vertex set $V(G) \times V(H)$, and two distinct vertices (u, x) and (v, y) are adjacent if and only if ($u = v$ or $uv \in E(G)$) and ($x = y$ or $xy \in E(H)$). Consider the strong product $T \boxtimes P$ of the infinite binary tree T and the infinite 2-way path P. Using Theorems 5.2 and Lemma 5.4, this graph has Assouad-Nagata dimension at most 2. As it contains a quasi-isometric copy of a 2-dimensional grid, the Assouad-Nagata dimension of $T \boxtimes P$ is indeed equal to 2. On the other hand, we observe that for any integer k, $T \boxtimes P$ contains the complete bipartite graph $K_{k,k}$ as an asymptotic minor. To see this, remark that T contains an infinite k-claw (the graph obtained by gluing k infinite 1-way paths at their starting vertex) as an asymptotic minor (obtained by contracting a subtree of T with k leaves into a single vertex, and pruning the additional branches). The strong product of this infinite k-claw with P consists of k copies of an infinite 2-dimensional grid (restricted to the upper half-plane, say), glued on a common infinite path π. On this path we can select k vertices, arbitrarily far apart, and on each infinite grid we can select a single vertex, arbitrarily far from π, and connect it to the k vertices of π via disjoint paths. By taking a ball of sufficiently large radius around each of the $2k$ vertices, we obtain $K_{k,k}$ as an r-fat minor for arbitrarily large r, and thus $K_{k,k}$ as an asymptotic minor. This is illustrated for $k = 3$ in Figure 1. Since containing a graph H as an asymptotic minor is invariant under quasi-isometry, any graph G which is quasi-isometric to $T \boxtimes P$ also contains every $K_{k,k}$ as an asymptotic minor, and thus every finite graph as a minor (in particular, G cannot be planar).

Recall that any graph excluding a minor has Assouad-Nagata dimension at most 2 [Liu23, Dis23]. It is natural to wonder whether some sort of converse holds, that is whether any graph of Assoud-Nagata dimension at most 2 is quasi-isometric to a graph excluding a minor (this would be a natural extension of Theorem 1.3). The example above shows that this is false, even for vertex-transitive graphs.

As explained above, it was conjectured in [GP23] that graphs that are quasi-isometric to a planar graph can be characterized by a finite list of forbidden asymptotic minors. A natural question is whether this can be replaced by a finite list of forbidden quasi-isometric graphs, at least in the case of quasi-transitive graphs. We

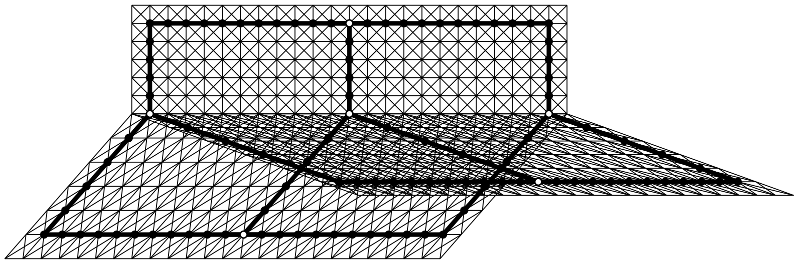

FIGURE 1. A fat $K_{3,3}$-minor in $T \boxtimes P$.

do not have a good candidate for such a finite list, but it should contain at least the two examples mentioned above: 3-dimensional grids and the product of the binary tree with a path. One difficulty is that no such list is even known (or conjectured to exist) for Assouad-Nagata dimension at most 2, or asymptotic dimension at most 2.

k-planar graphs. We conjecture the following variant of Theorem 1.3 for k-planar graphs.

Conjecture 6.1. *Every quasi-transitive graph of bounded local crossing number and degree is quasi-isometric to a planar graph.*

Using Theorem 4.1, Conjecture 6.1 would imply a positive answer to Problem 1.4. We observe that it is enough to prove Conjecture 6.1 for 1-planar graphs.

Conjecture 6.2. *Every quasi-transitive 1-planar graph of bounded degree is quasi-isometric to a planar graph.*

To see that the case $\mathrm{lcr} \geqslant 2$ reduces to the case $\mathrm{lcr} = 1$, observe that for every k-planar graph G with $k \geqslant 2$, its $(k-1)$-subdivision $G^{(k-1)}$ (the graph obtained from G by replacing every edge by a path on k edges) is locally finite, quasi-transitive, quasi-isometric to G, and 1-planar. To see the last point, consider any embedding of G in $\mathbb{R}^2$ in which every edge is involved in at most k crossings and assume without loss of generality that the crossing points between every two edges are all pairwise distinct. Then for every edge $e \in E(G)$, one can add the $k-1$ corresponding vertices of $G^{(k-1)}$ subdividing e in the drawing by putting at least one vertex on each of the curves connecting two consecutive crossing points of e with other edges.

We note that Conjecture 6.2 (and thus Conjecture 6.1) would be a direct consequence of the following.

Conjecture 6.3. *Let G be a quasi-transitive 1-planar graph of bounded degree. Then there is an integer k and an embedding of G in the plane with at most 1 crossing per edge such that for every pair of crossing edges uv, xy in G, we have $d_G(u, x) \leqslant k$.*

In a previous version of this manuscript we were conjecturing something stronger, namely that for any embedding of G with at most 1 crossing per edge, there is an integer k such that all pairs of crossing edges lie at distance at most k in G. But this is false (as shown by the two-way infinite path, drawn in such a way that it self-intersects at more and more distant points).

In this paper we have mainly considered graphs with finite local crossing number. A natural generalization is the following: a graph is $(<\omega)$-*planar* if it has a drawing in the plane in which each edge is involved in finitely many crossings. This raises the following question.

Question 6.4. *Let G be a quasi-transitive graph of bounded degree which is $(<\omega)$-planar. Is it true that G has finite local crossing number?*

It was observed by Kolja Knauer (personal communication) that Question 6.4 has a negative answer, as indeed *any* infinite locally finite graph G is $(<\omega)$-planar. To see this, consider an ordering $v_1, v_2, \ldots$ of $V(G)$, map each vertex v_i to the point with coordinates (i, i^2) in the plane, and each edge v_iv_j as a segment joining v_i and v_j. Note that by convexity of the function $x \mapsto x^2$, every edge crossing an edge v_iv_j must have an endpoint v_k with $i < k < j$. As for every pair $i < j$ there are only finitely many such vertices v_k and each of them has finite degree, the edge v_iv_j is crossed by only finitely many other edges.

As there exist quasi-transitive graphs of bounded degree that have unbounded local crossing number (the 3-dimensional grid for instance, see [DEW17]), the paragraph above implies that Question 6.4 has a negative answer.

Acknowledgements. We thank Agelos Georgakopoulos and Joseph MacManus for their helpful comments and Kolja Knauer for allowing us to mention his negative answer to Question 6.4. We also thank a referee for mentioning references [EFW12] and [Woe91] and for the helpful suggestions.

References

[Ass82] Patrice Assouad. Sur la distance de Nagata. *Comptes Rendus de l'Académie des Sciences, Paris, Série I*, 294:31–34, 1982.

[BBE+20] Marthe Bonamy, Nicolas Bousquet, Louis Esperet, Carla Groenland, François Pirot, and Alex Scott. Surfaces have (asymptotic) dimension 2, 2020.

[BBE+23] Marthe Bonamy, Nicolas Bousquet, Louis Esperet, Carla Groenland, Chun-Hung Liu, François Pirot, and Alex Scott. Asymptotic dimension of minor-closed families and Assouad–Nagata dimension of surfaces. *Journal of the European Mathematical Society*, 2023.

[BDLM08] Nikolay Brodskiy, Jerzy Dydak, Michael Levin, and Atish J. Mitra. A Hurewicz theorem for the Assouad–Nagata dimension. *Journal of the London Mathematical Society*, 77(3):741–756, 2008.

[Bod88] Hans L. Bodlaender. Planar graphs with bounded treewidth. Technical Report RUU-CS-88-14, Department of Computer Science, University of Utrecht, 1988.

[BST12] Itai Benjamini, Oded Schramm, and Ádám Timár. On the separation profile of infinite graphs. *Groups Geom. Dyn.*, 6(4):639–658, 2012.

[DEW17] Vida Dujmović, David Eppstein, and David R. Wood. Structure of graphs with locally restricted crossings. *SIAM J. Discrete Math.*, 31(2):805–824, 2017.

[Dis23] Marc Distel. Proper minor-closed classes of graphs have Assouad-Nagata dimension 2. arXiv e-print 2308.10377, 2023.

[DMW23] Vida Dujmović, Pat Morin, and David R. Wood. Graph product structure for non-minor-closed classes. *Journal of Combinatorial Theory, Series B*, 162:34–67, 2023.

[DO95] Guoli Ding and Bogdan Oporowski. Some results on tree decomposition of graphs. *Journal of Graph Theory*, 20(4):481–499, 1995.

[EFW12] Alex Eskin, David Fisher, and Kevin Whyte. Coarse differentiation of quasi-isometries. I: Spaces not quasi-isometric to Cayley graphs. *Ann. Math. (2)*, 176(1):221–260, 2012.

[EGLD23] Louis Esperet, Ugo Giocanti, and Clément Legrand-Duchesne. The structure of quasi-transitive graphs avoiding a minor with applications to the domino problem. arXiv e-print 2304.01823, 2023.

[GP23] Agelos Georgakopoulos and Panos Papasoglu. Graph minors and metric spaces. arXiv e-print 2305.07456, 2023.

[Gro93] Mikhael Gromov. *Geometric group theory. Volume 2: Asymptotic invariants of infinite groups. Proceedings of the symposium held at the Sussex University, Brighton, July*

14-19, 1991, volume 182 of *Lond. Math. Soc. Lect. Note Ser.* Cambridge: Cambridge University Press, 1993.

[Kön27] Dénes König. Über eine Schlußweise aus dem Endlichen ins Unendliche. *Acta Litt. Sci. Szeged*, 3:121–130, 1927.

[Liu23] Chun-Hung Liu. Assouad-Nagata dimension of minor-closed metrics. arXiv e-print 2308.12273, 2023.

[LS05] Urs Lang and Thilo Schlichenmaier. Nagata dimension, quasisymmetric embeddings, and Lipschitz extensions. *Int. Math. Res. Not.*, 2005(58):3625–3655, 2005.

[Mac23] Joseph MacManus. Accessibility, planar graphs, and quasi-isometries. arXiv e-print 2310.15242, 2023.

[OR15] Mikhail Ostrovskii and David Rosenthal. Metric dimensions of minor excluded graphs and minor exclusion in groups. *International Journal of Algebra and Computation*, 25(04):541–554, 2015.

[RS03] Neil Robertson and Paul Seymour. Graph minors. XVI. Excluding a non-planar graph. *Journal of Combinatorial Theory, Series B*, 89(1):43–76, 2003.

[TW93] Carsten Thomassen and Wolfgang Woess. Vertex-transitive graphs and accessibility. *Journal of Combinatorial Theory, Series B*, 58(2):248–268, 1993.

[Wag37] Klaus Wagner. Über eine Eigenschaft der ebenen Komplexe. *Math. Ann.*, 114:570–590, 1937.

[Woe91] Wolfgang Woess. Topological groups and infinite graphs. *Discrete Math.*, 95(1-3):373–384, 1991.

(L. Esperet) Univ. Grenoble Alpes, CNRS, Laboratoire G-SCOP, Grenoble, France
Email address: `louis.esperet@grenoble-inp.fr`

(U. Giocanti) Univ. Grenoble Alpes, CNRS, Laboratoire G-SCOP, Grenoble, France
Email address: `ugo.giocanti@grenoble-inp.fr`