A homomorphism from the affine Yangian $Y_{\hbar,\varepsilon}(\widehat{\mathfrak{sl}}(n))$ to the affine Yangian $Y_{\hbar,\varepsilon}(\widehat{\mathfrak{sl}}(n+1))$

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Abstract

We construct a homomorphism from the affine Yangian $Y_{\hbar,\varepsilon}(\widehat{\mathfrak{sl}}(n))$ to the standard degreewise completion of the affine Yangian $Y_{\hbar,\varepsilon}(\widehat{\mathfrak{sl}}(n+1))$. We also give the relationship between this homomorphism and the one from the affine Yangian $Y_{\hbar,\varepsilon}(\widehat{\mathfrak{sl}}(n))$ to the universal enveloping algebra of the rectangular algebra $\mathcal{W}^k(\mathfrak{gl}(2n),(2^n))$ constructed by the author [19].

1 Introduction

Drinfeld ([6], [7]) introduced the Yangian associated with a finite dimensional simple Lie algebra $\mathfrak{g}$ in order to solve the Yang-Baxter equation. The Yangian is a quantum group which is a deformation of the current algebra $\mathfrak{g}\otimes\mathbb{C}[z]$. The Yangian of type A has several presentations: the RTT presentation, the current presentation, and so on. By using the current presentation of the Yangian, we can extend the definition of the Yangian $Y_\hbar(\mathfrak{g})$ to a symmetrizable Kac-Moody Lie algebra $\mathfrak{g}$. For general symmetrizable Kac-Moody Lie algebra $\mathfrak{g}$, it has not been resolved whether the Yangian $Y_\hbar(\mathfrak{g})$ has a quantum group structure or not. However, in the case that $\mathfrak{g}$ is of affine type, this problem has been affirmatively resolved ([2], [13], and [16]).

Recently, relationships between affine Yangians and W-algebras have been studied. A W-algebra $\mathcal{W}^k(\mathfrak{g},f)$ is a vertex algebra associated with a finite dimensional reductive Lie algebra $\mathfrak{g}$ and a nilpotent element $f\in\mathfrak{g}$. It appeared in the study of two dimensional conformal field theories ([20]). We call a W-algebra associated with $\mathfrak{gl}(n)$ (resp. $\mathfrak{gl}(ln)$) and a principal nilpotent element (resp. a nilpotent element of type of (l^n)) a principal (resp. rectangular) W-algebra. The AGT (Alday-Gaiotto-Tachikawa) conjecture suggests that there exists a representation of the principal W-algebra of type A on the equivariant homology space of the moduli space of $U(r)$-instantons. Schiffmann and Vasserot [15] gave this representation by using an action of the affine Yangian associated with $\widehat{\mathfrak{gl}}(1)$ on this homology space.

In the rectangular case, the author [19] constructed a surjective homomorphism from the Guay's affine Yangian ([11] and [12]) to the universal enveloping algebra of a rectangular W-algebra of type A:

$$\Phi^n\colon Y_{\hbar,\varepsilon}(\widehat{\mathfrak{sl}}(n))\to\mathcal{U}(\mathcal{W}^k(\mathfrak{gl}(ln),(l^n))).$$

The Guay's affine Yangian $Y_{\hbar,\varepsilon}(\widehat{\mathfrak{sl}}(n))$ is a 2-parameter affine Yangian associated with $\widehat{\mathfrak{sl}}(n)$. The Guay's affine Yangian has a quantum group structure and is the deformation of the universal enveloping algebra of the central extension of $\mathfrak{sl}(n)[u^{\pm1},v]$. It is known that the Guay's affine Yangian has a representation on the equivariant homology space of affine Laumon spaces ([8] and [9]). Similarly to principal W-algebras, we expect that we can construct geometric representations of rectangular W-algebras by using the homomorphism Φ^n.

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In non-rectangular cases, it is conjectured in Creutzig-Diaconescu-Ma [4] that a geometric representation of an iterated W-algebra of type A on the equivariant homology space of the affine Laumon space will be given through an action of an affine shifted Yangian constructed in [9]. Based on this conjecture, we can expect that there exists a non-trivial homomorphism from the affine shifted Yangian to an iterated W-algebra associated with any nilpotent element. However, tackling this issue is very difficult and has not been resolved.

In [18], we constructed a homomorphism from the Guay's affine Yangian $Y_{\hbar,\varepsilon}(\widehat{\mathfrak{sl}}(n))$ to the universal enveloping algebra of a non-rectangular W-algebra. This is the affine version of De Sole-Kac-Valeri [5]. In [5], De Sole, Kac and Valeri constructed a homomorphism from the finite Yangian of type A to a finite W-algebras of type A by using the Lax operator. The homomorphism in De Sole-Kac-Valeri [5] is a restriction of the one from the shifted Yangian to a finite W-algebra given by Brundan-Kleshchev in [3]. We expect that we can extend the homomorphism given in [18] to the affine shifted Yangian and this extended homomorphism coincides with the homomorphism conjectured in Creutzig-Diaconescu-Ma [4].

For this purpose, it is helpful to give a homomorphism from the affine Yangian to the affine shifted Yangian. As a first step, in this article, we construct a homomorphism

$$\Psi\colon Y_{\hbar,\varepsilon}(\widehat{\mathfrak{sl}}(n))\to \widetilde{Y}_{\hbar,\varepsilon}(\widehat{\mathfrak{sl}}(n+1)),$$

where $\widetilde{Y}_{\hbar,\varepsilon}(\widehat{\mathfrak{sl}}(n+1))$ is the standard degreewise completion of $Y_{\hbar,\varepsilon}(\widehat{\mathfrak{sl}}(n+1))$. In finite setting, by using the RTT presentation, there exists a natural embedding from the Yangian associated with $\mathfrak{gl}(n)$ to the one associated with $\mathfrak{gl}(n+1)$. However, the Guay's affine Yangian $Y_{\hbar,\varepsilon}(\widehat{\mathfrak{sl}}(n))$ does not have the RTT presentation. For the construction of Ψ, we use the finite presentation called the *minimalistic presentation* given by Guay-Nakajima-Wendlandt [13].

As for one of the rectangular cases, we can construct a relationship with Ψ and Φ^n. There exists an embedding

$$\iota\colon \mathcal{W}^{k+1}(\mathfrak{gl}(2n),(2^n))\to(\mathcal{W}^k(\mathfrak{gl}(2n+2),(2^{n+1})).$$

In the last section of this article, we show the following relation:

$$\Phi^{n+1}\circ\Psi=\iota\circ\Phi^n.$$

We expect that the similar relation holds in the non-rectangular case.

2 Affine Yangian

Let us recall the definition of the affine Yangian of type A (Definition 3.2 in [11] and Definition 2.3 of [12]). Here after, we identify $\{0,1,2,\cdots,n-1\}$ with $\mathbb{Z}/n\mathbb{Z}$.

Definition 2.1. Suppose that $n\geq 3$. The affine Yangian $Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(n))$ is the associative algebra over generated by $x^+_{i,r},x^-_{i,r},h_{i,r}$ $(i\in\{0,1,\cdots,n-1\},r\in\mathbb{Z}_{\geq 0})$ with two parameters $\varepsilon_1,\varepsilon_2\in\mathbb{C}$ subject to the following defining relations:

$$[h_{i,r},h_{j,s}]=0,\tag{2.2}$$

$$[x^+_{i,r},x^-_{j,s}]=\delta_{i,j}h_{i,r+s},\tag{2.3}$$

$$[h_{i,0},x^\pm_{j,r}]=\pm a_{i,j}x^\pm_{j,r},\tag{2.4}$$

$$[h_{i,r+1},x^\pm_{j,s}]-[h_{i,r},x^\pm_{j,s+1}]=\pm a_{i,j}\frac{\varepsilon_1+\varepsilon_2}{2}\{h_{i,r},x^\pm_{j,s}\}-m_{i,j}\frac{\varepsilon_1-\varepsilon_2}{2}[h_{i,r},x^\pm_{j,s}],\tag{2.5}$$

$$[x^\pm_{i,r+1},x^\pm_{j,s}]-[x^\pm_{i,r},x^\pm_{j,s+1}]=\pm a_{ij}\frac{\varepsilon_1+\varepsilon_2}{2}\{x^\pm_{i,r},x^\pm_{j,s}\}-m_{ij}\frac{\varepsilon_1-\varepsilon_2}{2}[x^\pm_{i,r},x^\pm_{j,s}],\tag{2.6}$$

$$\sum_{w\in\mathfrak{S}_{1+|a_{ij}|}}[x^\pm_{i,r_{w(1)}},[x^\pm_{i,r_{w(2)}},\dots,[x^\pm_{i,r_{w(1+|a_{ij}|)}},x^\pm_{j,s}]\dots]]=0\ \text{ if }i\neq j,\tag{2.7}$$

where $\{X,Y\}=XY+YX$ and

$$a_{i,j}=\begin{cases}2 & \text{if } i=j,\\ -1 & \text{if } j=i\pm 1,\\ 0 & \text{otherwise},\end{cases}\quad m_{i,j}=\begin{cases}-1 & \text{if } i=j-1,\\ 1 & \text{if } i=j+1,\\ 0 & \text{otherwise}.\end{cases}$$

Guay-Nakajima-Wendland [13] gave the new presentation of the affine Yangian $Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(n))$ whose generators are

$$\{h_{i,r},x^{\pm}_{i,r}\mid 0\leq i\leq n-1,r\in\mathbb{Z}\}.$$

Proposition 2.8. *Suppose that $n\geq 3$. The affine Yangian $Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(n))$ is isomorphic to the associative algebra $Y_{\hbar,\varepsilon}(\widehat{\mathfrak{sl}}(n))$ generated by $X^{+}_{i,r},X^{-}_{i,r},H_{i,r}$ $(i\in\{0,1,\cdots,n-1\},r=0,1)$ subject to the following defining relations:*

$$[H_{i,r},H_{j,s}]=0,\tag{2.9}$$
$$[X^{+}_{i,0},X^{-}_{j,0}]=\delta_{ij}H_{i,0},\tag{2.10}$$
$$[X^{+}_{i,1},X^{-}_{j,0}]=\delta_{ij}H_{i,1}=[X^{+}_{i,0},X^{-}_{j,1}],\tag{2.11}$$
$$[H_{i,0},X^{\pm}_{j,r}]=\pm a_{ij}X^{\pm}_{j,r},\tag{2.12}$$
$$[\tilde{H}_{i,1},X^{\pm}_{j,0}]=\pm a_{ij}\left(X^{\pm}_{j,1}\right),\ \textit{if } (i,j)\neq(0,n-1),(n-1,0),\tag{2.13}$$
$$[\tilde{H}_{0,1},X^{\pm}_{n-1,0}]=\mp\left(X^{\pm}_{n-1,1}+(\varepsilon+\frac{n}{2}\hbar)X^{\pm}_{n-1,0}\right),\tag{2.14}$$
$$[\tilde{H}_{n-1,1},X^{\pm}_{0,0}]=\mp\left(X^{\pm}_{0,1}-(\varepsilon+\frac{n}{2}\hbar)X^{\pm}_{0,0}\right),\tag{2.15}$$
$$[X^{\pm}_{i,1},X^{\pm}_{j,0}]-[X^{\pm}_{i,0},X^{\pm}_{j,1}]=\pm a_{ij}\frac{\hbar}{2}\{X^{\pm}_{i,0},X^{\pm}_{j,0}\}\ \textit{if } (i,j)\neq(0,n-1),(n-1,0),\tag{2.16}$$
$$[X^{\pm}_{0,1},X^{\pm}_{n-1,0}]-[X^{\pm}_{0,0},X^{\pm}_{n-1,1}]=\mp\frac{\hbar}{2}\{X^{\pm}_{0,0},X^{\pm}_{n-1,0}\}+(\varepsilon+\frac{n}{2}\hbar)[X^{\pm}_{0,0},X^{\pm}_{n-1,0}],\tag{2.17}$$
$$(\mathrm{ad}\,X^{\pm}_{i,0})^{1+|a_{i,j}|}(X^{\pm}_{j,0})=0\quad\textit{if } i\neq j,\tag{2.18}$$

where $\tilde{H}_{i,1}=H_{i,1}-\dfrac{\hbar}{2}H^2_{i,0}$, $\hbar=\varepsilon_1+\varepsilon_2$ *and* $\varepsilon=-n\varepsilon_1$.

Proposition 2.8 is a little different from the presentation given by Guay-Nakajima-Wendland [13]. The isomorphism $\Xi\colon Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(n))\to Y_{\hbar,\varepsilon}(\widehat{\mathfrak{sl}}(n))$ is given by

$$\Xi(h_{i,0})=H_{i,0},\quad \Xi(x^{\pm}_{i,0})=X^{\pm}_{i,0},$$
$$\Xi(h_{i,1})=\begin{cases}H_{0,1} & \text{if } i=0,\\ H_{i,1}-\dfrac{i}{2}(\varepsilon_1-\varepsilon_2)H_{i,0} & \text{if } i\neq 0.\end{cases}$$

Thus, Proposition 2.8 is derived from Guay-Nakajima-Wendland [13]. By this corresponding, we find that

$$[X^{\pm}_{i,r},X^{\pm}_{j,s}]=0 \text{ if } |i-j|>1,\tag{2.19}$$
$$[X^{\pm}_{i,1},[X^{\pm}_{i,0},X^{\pm}_{j+1,r}]]+[X^{\pm}_{i,0},[X^{\pm}_{i,1},X^{\pm}_{j+1,r}]]=0.\tag{2.20}$$

Remark 2.21. In [17], the author gave the similar presentation for the affine super Yangian. We note that (2.29) in [17] contains a typo. We should replace (2.29) with

$$[X^{\pm}_{0,1},X^{\pm}_{n-1,0}]-[X^{\pm}_{0,0},X^{\pm}_{n-1,1}]=\mp(-1)^{p(m+n)}\frac{\hbar}{2}\{X^{\pm}_{0,0},X^{\pm}_{n-1,0}\}-(\varepsilon+\frac{n}{2}\hbar)[X^{\pm}_{0,0},X^{\pm}_{n-1,0}].$$

We also note that $\varepsilon=-n\varepsilon_2$ in [17]. This makes the difference between (2.14), (2.15) and (2.17) in Proposition 2.8 and (2.26), (2.27) and (2.29) in [17].

By the definition of the affine Yangian $Y_{\hbar,\varepsilon}(\widehat{\mathfrak{sl}}(n))$, we find that there exists a natural homomorphism from the universal enveloping algebra of $\widehat{\mathfrak{sl}}(n)$ to $Y_{\hbar,\varepsilon}(\widehat{\mathfrak{sl}}(n))$. In order to simplify the notation, we denote the image of $x \in U(\widehat{\mathfrak{sl}}(n))$ by x.

We take one completion of $Y_{\hbar,\varepsilon}(\widehat{\mathfrak{sl}}(n))$. We set the degree of $Y_{\hbar,\varepsilon}(\widehat{\mathfrak{sl}}(n))$ by

$$\deg(H_{i,r}) = 0,\ \deg(X^{\pm}_{i,r}) = \begin{cases} \pm 1 & \text{if } i = 0,\\ 0 & \text{if } i \neq 0.\end{cases}$$

We denote the standard degreewise completion of $Y_{\hbar,\varepsilon}(\widehat{\mathfrak{sl}}(n))$ by $\widetilde{Y}_{\hbar,\varepsilon}(\widehat{\mathfrak{sl}}(n))$. Let us set $A_i \in \widetilde{Y}_{\hbar,\varepsilon}(\widehat{\mathfrak{sl}}(n))$ as

$$\begin{aligned}
A_i &= \frac{\hbar}{2}\sum_{\substack{s\geq 0\\ u>v}} E_{u,v}t^{-s}[E_{i,i}, E_{v,u}t^{s}] + \frac{\hbar}{2}\sum_{\substack{s\geq 0\\ u<v}} E_{u,v}t^{-s-1}[E_{i,i}, E_{v,u}t^{s+1}]\\
&= \frac{\hbar}{2}\sum_{\substack{s\geq 0\\ u>i}} E_{u,i}t^{-s}E_{i,u}t^{s} - \frac{\hbar}{2}\sum_{\substack{s\geq 0\\ i>v}} E_{i,v}t^{-s}E_{v,i}t^{s}\\
&\quad + \frac{\hbar}{2}\sum_{\substack{s\geq 0\\ u>i}} E_{u,i}t^{-s-1}E_{i,u}t^{s+1} - \frac{\hbar}{2}\sum_{\substack{s\geq 0\\ i>v}} E_{i,v}t^{-s-1}E_{v,i}t^{s+1},
\end{aligned}$$

where $E_{i,j}$ is a matrix unit whose (a,b) component is $\delta_{a,i}\delta_{b,j}$ Similarly to Section 3 in [13], we define

$$J(h_i) = \widetilde{H}_{i,1} - A_i + A_{i+1} \in \widetilde{Y}_{\hbar,\varepsilon}(\widehat{\mathfrak{sl}}(n))$$

We also set $J(x_i^{\pm}) = \pm\dfrac{1}{2}[J(h_i), x_i^{\pm}]$. Guay-Nakajima-Wendland [13] defined the automorphism of $Y_{\hbar,\varepsilon}(\widehat{\mathfrak{sl}}(n))$ by

$$\tau_i = \exp(\mathrm{ad}(x^{+}_{i,0}))\exp(-\mathrm{ad}(x^{-}_{i,0}))\exp(\mathrm{ad}(x^{+}_{i,0})).$$

Let α be a positive real root. By definition of the Weyl algebra, there is an element w of the Weyl group of $\widehat{\mathfrak{sl}}(n)$ and a simple root α_j such that $\alpha = w\alpha_j$. Then we define a corresponding root vector by

$$x^{\pm}_{\alpha} = \tau_{i_1}\tau_{i_2}\cdots\tau_{i_{p-1}}(x^{\pm}_j),$$

where $w = s_{i_1}s_{i_2}\cdots s_{i_{p-1}}$ is a reduced expression of w. We can define $J(x^{\pm}_{\alpha})$ as

$$J(x^{\pm}_{\alpha}) = \tau_{i_1}\tau_{i_2}\cdots\tau_{i_{p-1}}J(x^{\pm}_j).$$

Lemma 2.22 (Proposition 3.21 in [13]). *The following relation holds:*

$$[J(h_i), x^{\pm}_{\alpha}] = \pm(\alpha_i, \alpha)J(x^{\pm}_{\alpha}) \pm c_{\alpha,i}x^{\pm}_{\alpha}$$

for some $c_{\alpha,i} \in \mathbb{C}$.

3 A homomorphism from the affine Yangian $Y_{\hbar,\varepsilon}(\widehat{\mathfrak{sl}}(n))$ to the affine Yangian $Y_{\hbar,\varepsilon}(\widehat{\mathfrak{sl}}(n+1))$

In this section, we will construct a homomorphism from the affine Yangian $Y_{\hbar,\varepsilon}(\widehat{\mathfrak{sl}}(n))$ to the degreewise completion of the affine Yangian $Y_{\hbar,\varepsilon}(\widehat{\mathfrak{sl}}(n+1))$.

Theorem 3.1. *There exists an algebra homomorphism*

$$\Psi\colon Y_{\hbar,\varepsilon}(\widehat{\mathfrak{sl}}(n)) \to \widetilde{Y}_{\hbar,\varepsilon}(\widehat{\mathfrak{sl}}(n+1))$$

determined by

$$\Psi(H_{i,0}) = \begin{cases} H_{0,0} + H_{n,0} & \text{if } i = 0,\\ H_{i,0} & \text{if } i \neq 0,\end{cases}$$

$$\Psi(X^+_{i,0}) = \begin{cases} E_{n,1}t & \text{if } i = 0,\\ E_{i,i+1} & \text{if } i \neq 0,\end{cases} \quad \Psi(X^-_{i,0}) = \begin{cases} E_{1,n}t^{-1} & \text{if } i = 0,\\ E_{i+1,i} & \text{if } i \neq 0,\end{cases}$$

and

$$\begin{aligned}
\Psi(H_{i,1}) &= H_{i,1} - \hbar\sum_{s\geq 0} E_{i,n+1}t^{-s-1}E_{n+1,i}t^{s+1} + \hbar\sum_{s\geq 0} E_{i+1,n+1}t^{-s-1}E_{n+1,i+1}t^{s+1},\\
\Psi(X^+_{i,1}) &= X^+_{i,1} - \hbar\sum_{s\geq 0} E_{i,n+1}t^{-s-1}E_{n+1,i+1}t^{s+1},\\
\Psi(X^-_{i,1}) &= X^-_{i,1} - \hbar\sum_{s\geq 0} E_{i+1,n+1}t^{-s-1}E_{n+1,i}t^{s+1}
\end{aligned}$$

for $i \neq 0$. In particular, we have

$$\Psi(\widetilde{H}_{i,1}) = \widetilde{H}_{i,1} - \hbar\sum_{s\geq 0} E_{i,n+1}t^{-s-1}E_{n+1,i}t^{s+1} + \hbar\sum_{s\geq 0} E_{i+1,n+1}t^{-s-1}E_{n+1,i+1}t^{s+1}$$

for $i \neq 0$.

By Theorem 3.1, we can easily compute $\Psi(X^+_{0,1})$ and $\Psi(H_{0,1})$.

Corollary 3.2. *The following equations hold:*

$$\begin{aligned}
\Psi(X^+_{0,1}) &= [X^+_{n,0}, X^+_{0,1}] - \hbar\sum_{s\geq 0} E_{n,n+1}t^{-s}E_{n+1,1}t^{s+1},\\
\Psi(X^-_{0,1}) &= [X^-_{0,1}, X^-_{n,0}] - \hbar\sum_{s\geq 0} E_{1,n+1}t^{-s-1}E_{n+1,n}t^{s},\\
\Psi(H_{0,1}) &= H_{0,1} + H_{n,1} + (\varepsilon + \frac{\hbar}{2}n)H_{n,0} + \hbar H_{n,0}H_{0,0}\\
&\quad - \hbar\sum_{s\geq 0} E_{n,n+1}t^{-s-1}E_{n+1,n}t^{s+1} + \hbar\sum_{s\geq 0} E_{1,n+1}t^{-s-1}E_{n+1,1}t^{s+1}.
\end{aligned}$$

In particular, we obtain

$$\begin{aligned}
\Psi(\widetilde{H}_{0,1}) &= \widetilde{H}_{0,1} + \widetilde{H}_{n,1} + (\varepsilon + \frac{\hbar}{2}n)H_{n,0}\\
&\quad - \hbar\sum_{s\geq 0} E_{n,n+1}t^{-s-1}E_{n+1,n}t^{s+1} + \hbar\sum_{s\geq 0} E_{1,n+1}t^{-s-1}E_{n+1,1}t^{s+1}.
\end{aligned}$$

Proof. By the definition of Ψ and (2.13), we have

$$\begin{aligned}
\Psi(X^+_{0,1}) &= -[\Psi(\widetilde{H}_{1,1}), \Psi(X^+_{0,0})]\\
&= -[\widetilde{H}_{1,1}, [X^+_{n,0}, X^+_{0,0}]] + \hbar[\sum_{s\geq 0} E_{1,n+1}t^{-s-1}E_{n+1,1}t^{s+1}, E_{n,1}t]\\
&\quad - \hbar[\sum_{s\geq 0} E_{2,n+1}t^{-s-1}E_{n+1,2}t^{s+1}, E_{n,1}t]
\end{aligned}$$

$$= [X^+_{n,0}, X^+_{0,1}] - \hbar\sum_{s\geq 0} E_{n,n+1}t^{-s}E_{n+1,1}t^{s+1},$$

where the second equality is due to $E_{n,1}t = [X^+_{n,0}, X^+_{0,0}]$. Similarly to $\Psi(X^+_{0,1})$, we can compute $\Psi(X^-_{0,1})$.

By (2.11), we obtain

$$\begin{aligned}
\Psi(H_{0,1}) &= [\Psi(X^+_{0,1}), \Psi(X^-_{0,0})] \\
&= [[X^+_{n,0}, X^+_{0,1}], [X^-_{0,0}, X^-_{n,0}]] - [\hbar\sum_{s\geq 0} E_{n,n+1}t^{-s}E_{n+1,1}t^{s+1}, E_{1,n}t^{-1}] \\
&= [[X^+_{n,0}, H_{0,1}], X^-_{n,0}] + [X^-_{0,0}, [H_{n,0}, X^+_{0,1}]] - [\hbar\sum_{s\geq 0} E_{n,n+1}t^{-s}E_{n+1,1}t^{s+1}, E_{1,n}t^{-1}], \quad (3.3)
\end{aligned}$$

where the last equality is due to (2.10) and (2.11). By (2.11) and (2.12), we have

$$[X^-_{0,0}, [H_{n,0}, X^+_{0,1}]] = -[X^-_{0,0}, X^+_{0,1}] = H_{0,1}. \quad (3.4)$$

By (2.11) and (2.13)-(2.15), we have

$$\begin{aligned}
&[[X^+_{n,0}, H_{0,1}], X^-_{n,0}] \\
&= -[[\widetilde{H}_{0,1} + \frac{\hbar}{2}H^2_{0,0}, X^+_{n,0}], X^-_{n,0}] \\
&= [X^+_{n,1} + (\varepsilon + \frac{\hbar}{2}(n+1))X^+_{n,0}, X^-_{n,0}] + \frac{\hbar}{2}[\{H_{0,0}, X^+_{n,0}\}, X^-_{n,0}] \\
&= H_{n,1} + (\varepsilon + \frac{\hbar}{2}(n+1))H_{n,0} + \frac{\hbar}{2}\{X^-_{n,0}, X^+_{n,0}\} + \frac{\hbar}{2}\{H_{0,0}, H_{n,0}\} \\
&= H_{n,1} + (\varepsilon + \frac{\hbar}{2}(n+1))H_{n,0} + \hbar X^+_{n,0}X^-_{n,0} - \frac{\hbar}{2}H_{n,0} + \hbar H_{0,0}, H_{n,0}, \quad (3.5)
\end{aligned}$$

where the second equality is due to (2.14) and (2.12), the third equality is due to (2.10) and (2.11) and the last equality is due to (2.9) and the relation

$$\frac{\hbar}{2}\{X^-_{n,0}, X^+_{n,0}\} = \hbar X^+_{n,0}X^-_{n,0} - \frac{\hbar}{2}H_{n,0}.$$

By a direct computation, we obtain

$$\begin{aligned}
&- [\hbar\sum_{s\geq 0} E_{n,n+1}t^{-s}E_{n+1,1}t^{s+1}, E_{1,n}t^{-1}] \\
&= -\hbar\sum_{s\geq 0} E_{n,n+1}t^{-s}E_{n+1,n}t^{s} + \hbar\sum_{s\geq 0} E_{1,n+1}t^{-s-1}E_{n+1,1}t^{s+1}. \quad (3.6)
\end{aligned}$$

Applying (3.4) and (3.6) to (3.3), we obtain

$$\begin{aligned}
\Psi(H_{0,1}) &= H_{0,1} + H_{n,1} + (\varepsilon + \frac{\hbar}{2}n)H_{n,0} + \hbar H_{n,0}H_{0,0} \\
&\quad - \hbar\sum_{s\geq 0} E_{n,n+1}t^{-s-1}E_{n+1,n}t^{s+1} + \hbar\sum_{s\geq 0} E_{1,n+1}t^{-s-1}E_{n+1,1}t^{s+1}.
\end{aligned}$$

We complete the proof. □

In order to prove Theorem 3.1, it is enough to show that Ψ is compatible with (2.9)-(2.18). By the definition of Ψ, Ψ is compatible with (2.10), (2.12) and (2.18). We will prove the compatibility with other relations of Proposition 2.8 in the following subsections.

3.1 Compatibility with (2.11)

The case that $i=j=0$ has already been proved in the proof of Corollary 3.2. We only show the case that $i,j\neq 0$. Other cases are proven in a similar way. We obtain

$$\begin{aligned}
&[\Psi(X^+_{i,1}),\Psi(X^-_{j,0})]\\
&=[X^+_{i,1}-\hbar\sum_{s\geq 0}E_{i,n+1}t^{-s-1}E_{n+1,i+1}t^{s+1},E_{j+1,j}]\\
&=\delta_{i,j}H_{i,1}-\delta_{i,j}\hbar\sum_{s\geq 0}E_{i,n+1}t^{-s-1}E_{n+1,i}t^{s+1}+\delta_{i,j}\hbar\sum_{s\geq 0}E_{i+1,n+1}t^{-s-1}E_{n+1,i+1}t^{s+1}\\
&=\delta_{i,j}\Psi(H_{i,1}),
\end{aligned}$$

where the first and last equalities are due to the definition of Ψ and the second equality is due to (2.11).

3.2 Compatibility with (2.13)

We only show the case that $i=j=0$ and the sign is $+$, The other cases are proven in a similar way. By the definition of Ψ, we have

$$\begin{aligned}
&[\Psi(\widetilde{H}_{0,1}),\Psi(X^+_{0,0})]\\
&=[\widetilde{H}_{0,1}+\widetilde{H}_{n,1},E_{n,1}t]+(\varepsilon+\frac{\hbar}{2}n)[H_{n,0},E_{n,1}t]\\
&\quad-[\hbar\sum_{s\geq 0}E_{n,n+1}t^{-s-1}E_{n+1,n}t^{s+1},E_{n,1}t]+[\hbar\sum_{s\geq 0}E_{1,n+1}t^{-s-1}E_{n+1,1}t^{s+1},E_{n,1}t].
\end{aligned}\tag{3.7}$$

By a direct computation, we obtain

$$(\varepsilon+\frac{\hbar}{2}n)[H_{n,0},E_{n,1}t]=(\varepsilon+\frac{\hbar}{2}n)E_{n,1}t\tag{3.8}$$

and

$$\begin{aligned}
&-[\hbar\sum_{s\geq 0}E_{n,n+1}t^{-s-1}E_{n+1,n}t^{s+1},E_{n,1}t]+[\hbar\sum_{s\geq 0}E_{1,n+1}t^{-s-1}E_{n+1,1}t^{s+1},E_{n,1}t]\\
&=-\hbar\sum_{s\geq 0}E_{n,n+1}t^{-s-1}E_{n+1,1}t^{s+2}-\hbar\sum_{s\geq 0}E_{n,n+1}t^{-s}E_{n+1,1}t^{s+1}\\
&=-2\hbar\sum_{s\geq 0}E_{n,n+1}t^{-s}E_{n+1,1}t^{s+1}+\hbar X^+_{n,0}X^+_{0,0}.
\end{aligned}\tag{3.9}$$

By (2.13)-(2.15), we have

$$\begin{aligned}
&[\widetilde{H}_{0,1}+\widetilde{H}_{n,1},E_{n,1}t]=[\widetilde{H}_{0,1}+\widetilde{H}_{n,1},[X^+_{n,0},X^+_{0,0}]]\\
&=[X^+_{n,0},X^+_{0,1}-(\varepsilon+\frac{n+1}{2}\hbar)X^+_{0,0}]+[X^+_{n,1}+(\varepsilon+\frac{n+1}{2}\hbar)X^+_{n,0},X^+_{0,0}]\\
&=[X^+_{n,0},X^+_{0,1}]+[X^+_{n,1},X^+_{0,0}]\\
&=2[X^+_{n,0},X^+_{0,1}]-\frac{\hbar}{2}\{X^+_{0,0},X^+_{n,0}\}+(\varepsilon+\frac{n+1}{2}\hbar)[X^+_{0,0},X^+_{n,0}]\\
&=2[X^+_{n,0},X^+_{0,1}]-\frac{\hbar}{2}\{X^+_{0,0},X^+_{n,0}\}-(\varepsilon+\frac{n+1}{2}\hbar)E_{n,1}t\\
&=2[X^+_{n,0},X^+_{0,1}]-\hbar X^+_{n,0}X^+_{0,0}-\frac{\hbar}{2}[X^+_{0,0},X^+_{n,0}]-(\varepsilon+\frac{n+1}{2}\hbar)E_{n,1}t,
\end{aligned}\tag{3.10}$$

where the first equality is due to $[X^+_{n,0}, X^+_{0,0}] = E_{n,1}t$, the second equality is due to (2.13)-(2.15), the 4-th equality is due to (2.17). Applying (3.8)-(3.10) to (3.7), we obtain

$$\begin{aligned}
[\Psi(\widetilde{H}_{0,1}), \Psi(X^+_{0,0})] &= 2[X^+_{n,0}, X^+_{0,1}] - 2\hbar\sum_{s\geq 0} E_{n,n+1}t^{-s}E_{n+1,1}t^{s+1} \\
&= 2\Psi(X^+_{0,1}).
\end{aligned}$$

3.3 Compatibility with (2.14)

We only prove the + case. The + case can be shown in the same way. By the definition of Ψ, we have

$$\begin{aligned}
&[\Psi(\widetilde{H}_{0,1}), \Psi(X^+_{n-1,0})] \\
&= [\widetilde{H}_{0,1} + \widetilde{H}_{n,1}, X^+_{n-1,0}] + (\varepsilon + \frac{\hbar}{2}n)[H_{n,0}, E_{n-1,n}] \\
&\quad - [\hbar\sum_{s\geq 0} E_{n,n+1}t^{-s-1}E_{n+1,n}t^{s+1}, E_{n-1,n}] + [\hbar\sum_{s\geq 0} E_{1,n+1}t^{-s-1}E_{n+1,1}t^{s+1}, E_{n-1,n}] \\
&= -X^+_{n-1,1} - (\varepsilon + \frac{\hbar}{2}n)E_{n-1,n} + \hbar\sum_{s\geq 0} E_{n-1,n+1}t^{-s-1}E_{n+1,n}t^{s+1} \\
&= -(\Psi(X^+_{n-1,1}) + (\varepsilon + \frac{\hbar}{2}n)\Psi(X^+_{n-1,0})),
\end{aligned}$$

where the first last equalities are due to the definition of Ψ and the second equality is due to (2.13).

3.4 Compatibility with (2.15)

By the definition of Ψ, we have

$$\begin{aligned}
&[\Psi(\widetilde{H}_{n-1,1}), \Psi(X^+_{0,0})] \\
&= [\widetilde{H}_{n-1,1}, E_{n,1}t] - [\hbar\sum_{s\geq 0} E_{n-1,n+1}t^{-s-1}E_{n+1,n-1}t^{s+1}, E_{n,1}t] \\
&\quad + [\hbar\sum_{s\geq 0} E_{n,n+1}t^{-s-1}E_{n+1,n}t^{s+1}, E_{n,1}t].
\end{aligned} \tag{3.11}$$

By a direct computation, we obtain

$$\begin{aligned}
&- [\hbar\sum_{s\geq 0} E_{n-1,n+1}t^{-s-1}E_{n+1,n-1}t^{s+1}, E_{n,1}t] + [\hbar\sum_{s\geq 0} E_{n,n+1}t^{-s-1}E_{n+1,n}t^{s+1}, E_{n,1}t] \\
&= 0 + \hbar\sum_{s\geq 0} E_{n,n+1}t^{-s-1}E_{n+1,1}t^{s+2} \\
&= \hbar\sum_{s\geq 0} E_{n,n+1}t^{-s}E_{n+1,1}t^{s+1} - \hbar X^+_{n,0}X^+_{0,0}.
\end{aligned} \tag{3.12}$$

By (2.13) and (2.15), we have

$$\begin{aligned}
&[\widetilde{H}_{n-1,1}, E_{n,1}t] = [\widetilde{H}_{n-1,1}, [X^+_{n,0}, X^+_{0,0}]] = -[X^+_{n,1}, X^+_{0,0}] \\
&= -[X^+_{n,0}, X^+_{0,1}] + \frac{\hbar}{2}\{X^+_{0,0}, X^+_{n,0}\} + (\varepsilon + \frac{\hbar}{2}(n+1))[X^+_{n,0}, X^+_{0,0}] \\
&= -[X^+_{n,0}, X^+_{0,1}] + \hbar X^+_{n,0}X^+_{0,0} + \frac{\hbar}{2}[X^+_{0,0}, X^+_{n,0}] + (\varepsilon + \frac{\hbar}{2}(n+1))[X^+_{n,0}, X^+_{0,0}]
\end{aligned} \tag{3.13}$$

By applying (3.12) and (3.13) to (3.11), we have

$$\begin{aligned}
&[\Psi(\widetilde{H}_{n-1,1}),\Psi(X^+_{0,0})]\\
&=-[X^+_{n,0},X^+_{0,1}]+\hbar\sum_{s\geq 0}E_{n,n+1}t^{-s}E_{n+1,1}t^{s+1}+(\varepsilon+\frac{\hbar}{2}n)[X^+_{n,0},X^+_{0,0}]\\
&=-(\Psi(X^+_{0,1})-(\varepsilon+\frac{\hbar}{2}n)\Psi(X^+_{0,0})),
\end{aligned}$$

3.5 Compatibility with (2.16)

We only show the case that $i=0$ and the sign is $+$. The other cases are proven in a similar way.

Case 1: $i=0, j\neq 0,n-1$

By the definition of Ψ and the assumption that $j\neq 0,n-1$, we have

$$\begin{aligned}
&[\Psi(X^+_{0,0}),\Psi(X^+_{j,1})]\\
&=[[X^+_{n,0},X^+_{0,0}],X^+_{j,1}]-[E_{n,1}t,\hbar\sum_{s\geq 0}E_{j,n+1}t^{-s-1}E_{n+1,j+1}t^{s+1}]\\
&=[[X^+_{n,0},X^+_{0,0}],X^+_{j,1}]-\delta_{j,1}\hbar\sum_{s\geq 0}E_{n,n+1}t^{-s}E_{n+1,j+1}t^{s+1}.
\end{aligned}$$

and

$$\begin{aligned}
&[\Psi(X^+_{0,1}),\Psi(X^+_{j,0})]\\
&=[[X^+_{n,0},X^+_{0,1}],X^+_{j,0}]-[\hbar\sum_{s\geq 0}E_{n,n+1}t^{-s}E_{n+1,1}t^{s+1},E_{j,j+1}]\\
&=[[X^+_{n,0},X^+_{0,1}],X^+_{j,0}]-\delta_{j,1}\hbar\sum_{s\geq 0}E_{n,n+1}t^{-s}E_{n+1,2}t^{s+1}.
\end{aligned}$$

Thus, by a direct computation, we have

$$\begin{aligned}
&[\Psi(X^+_{0,1}),\Psi(X^+_{j,0})]-[\Psi(X^+_{0,0}),\Psi(X^+_{j,1})]\\
&=[[X^+_{n,0},X^+_{0,1}],X^+_{j,0}]-[[X^+_{n,0},X^+_{0,0}],X^+_{j,1}].
\end{aligned}$$

We obtain

$$\begin{aligned}
&[[X^+_{n,0},X^+_{0,1}],X^+_{j,0}]-[[X^+_{n,0},X^+_{0,0}],X^+_{j,1}]\\
&=[X^+_{n,0},([X^+_{0,1},X^+_{j,0}]-[X^+_{0,0},X^+_{j,1}])]\\
&=\frac{\hbar}{2}a_{0,j}[X^+_{n,0},\{X^+_{0,0},X^+_{j,0}\}]\\
&=\frac{\hbar}{2}a_{0,j}\{[X^+_{n,0},X^+_{0,0}],X^+_{j,0}\}\\
&=\frac{\hbar}{2}a_{0,j}\{\Psi(X^+_{0,0}),\Psi(X^+_{j,0})\},
\end{aligned}\tag{3.14}$$

where the first equality is due to (2.19) and the assumption that $j\neq 0,n-1$, the second equality is due to (2.16) and the last equlity is due to the assumption that $j\neq 0,n-1$.

Case 2: $i=j=0$

In this case, (2.16) is equivalent to

$$[X^+_{0,1},X^+_{0,0}]=\hbar(X^+_{0,0})^2.\tag{3.15}$$

We will prove the compatibility with (3.15). By the definition of Ψ, we have

$$[\Psi(X^+_{0,1}),\Psi(X^+_{0,0})]$$

$$
\begin{aligned}
&= [[X^+_{n,0}, X^+_{0,1}], [X^+_{n,0}, X^+_{0,0}]] - [\hbar \sum_{s\geq 0} E_{n,n+1} t^{-s} E_{n+1,1} t^{s+1}, E_{n,1} t] \\
&= [[X^+_{n,0}, X^+_{0,1}], [X^+_{n,0}, X^+_{0,0}]] - 0. \qquad (3.16)
\end{aligned}
$$

By (2.18), we obtain

$$
\begin{aligned}
&[[X^+_{n,0}, X^+_{0,1}], [X^+_{n,0}, X^+_{0,0}]] \\
&= [X^+_{n,0}, [X^+_{0,1}, [X^+_{n,0}, X^+_{0,0}]]] + [[X^+_{n,0}, [X^+_{n,0}, X^+_{0,0}]], X^+_{0,1}] \\
&= [X^+_{n,0}, [X^+_{0,1}, [X^+_{n,0}, X^+_{0,0}]]] + 0, \qquad (3.17)
\end{aligned}
$$

where the last equality is due to (2.18). We obtain

$$
\begin{aligned}
&[X^+_{0,1}, [X^+_{n,0}, X^+_{0,0}]] \\
&= \frac{1}{2}[X^+_{0,1}, [X^+_{n,0}, X^+_{0,0}]] + \frac{1}{2}([[X^+_{0,1}, X^+_{n,0}], X^+_{0,0}] + [X^+_{n,0}, [X^+_{0,1}, X^+_{0,0}]]) \\
&= \frac{1}{2}([X^+_{0,1}, [X^+_{n,0}, X^+_{0,0}]] + [[X^+_{0,1}, X^+_{n,0}], X^+_{0,0}]) + \frac{1}{2}[X^+_{n,0}, [X^+_{0,1}, X^+_{0,0}]] \\
&= 0 + \frac{\hbar}{2}[X^+_{n,0}, (X^+_{0,0})^2] \\
&= \frac{\hbar}{2}\{[X^+_{n,0}, X^+_{0,0}], X^+_{0,0}\}, \qquad (3.18)
\end{aligned}
$$

where the third equality is due to (2.20) and (3.15). By applying (3.17) and (3.18) to (3.16), we obtain

$$
\begin{aligned}
&[\Psi(X^+_{0,1}), \Psi(X^+_{0,0})] \\
&= [X^+_{n,0}, \frac{\hbar}{2}\{[X^+_{n,0}, X^+_{0,0}], X^+_{0,0}\}] \\
&= \frac{\hbar}{2}\{[X^+_{n,0}, X^+_{0,0}], [X^+_{n,0}, X^+_{0,0}]\} = \hbar(\Psi(X^+_{0,0}))^2
\end{aligned}
$$

by (2.18).

3.6 Compatibility with (2.17)

We only show the $+$ case. The $-$ case can be proven in a similar way. By the definition of Ψ, we have

$$
\begin{aligned}
&[\Psi(X^+_{0,0}), \Psi(X^+_{n-1,1})] \\
&= [[X^+_{n,0}, X^+_{0,0}], X^+_{n-1,1}] - [E_{n,1} t, \hbar \sum_{s\geq 0} E_{n-1,n+1} t^{-s-1} E_{n+1,n} t^{s+1}] \\
&= [[X^+_{n,0}, X^+_{0,0}], X^+_{n-1,1}] + \hbar \sum_{s\geq 0} E_{n-1,n+1} t^{-s-1} E_{n+1,1} t^{s+2} \qquad (3.19)
\end{aligned}
$$

and

$$
\begin{aligned}
&[\Psi(X^+_{0,1}), \Psi(X^+_{n-1,0})] \qquad (3.20) \\
&= [[X^+_{n,0}, X^+_{0,1}], X^+_{n-1,0}] - [\hbar \sum_{s\geq 0} E_{n,n+1} t^{-s} E_{n+1,1} t^{s+1}, E_{n-1,n}] \qquad (3.21) \\
&= [[X^+_{n,0}, X^+_{0,1}], X^+_{n-1,0}] + \hbar \sum_{s\geq 0} E_{n-1,n+1} t^{-s} E_{n+1,1} t^{s+1}. \qquad (3.22)
\end{aligned}
$$

By comparing the right hand sides of (3.19) and (3.22), we have

$$
[\Psi(X^+_{0,1}), \Psi(X^+_{n-1,0})] - [\Psi(X^+_{0,0}), \Psi(X^+_{n-1,1})]
$$

$$= [[X^+_{n,0}, X^+_{0,1}], X^+_{n-1,0}] + \hbar E_{n-1,n+1}E_{n+1,1}t - [[X^+_{n,0}, X^+_{0,0}], X^+_{n-1,1}]. \tag{3.23}$$

We obtain

$$\begin{aligned}
&[[X^+_{n,0}, X^+_{0,1}], X^+_{n-1,0}]\\
&= [[X^+_{n,1}, X^+_{0,0}] + \frac{\hbar}{2}\{X^+_{0,0}, X^+_{n,0}\} - (\varepsilon + \frac{\hbar}{2}(n+1))[X^+_{0,0}, X^+_{n-1,0}], X^+_{n-1,0}]\\
&= [[X^+_{n,1}, X^+_{0,0}], X^+_{n-1,0}] + \frac{\hbar}{2}\{X^+_{0,0}, [X^+_{n,0}, X^+_{n-1,0}]\} - (\varepsilon + \frac{\hbar}{2}(n+1))[[X^+_{0,0}, X^+_{n,0}], X^+_{n-1,0}]\\
&= [[X^+_{n,1}, X^+_{0,0}], X^+_{n-1,0}] - \frac{\hbar}{2}\{E_{n+1,1}t, E_{n-1,n+1}\} - (\varepsilon + \frac{\hbar}{2}(n+1))E_{n-1,1}t,
\end{aligned} \tag{3.24}$$

where the second equality is due to (2.17). By the similar way to (3.14), we have

$$\begin{aligned}
&[[X^+_{n,1}, X^+_{0,0}], X^+_{n-1,0}] - [[X^+_{n,0}, X^+_{0,0}], X^+_{n-1,1}]\\
&= -\frac{\hbar}{2}\{[X^+_{n,0}, X^+_{0,0}], X^+_{n-1,0}\}.
\end{aligned} \tag{3.25}$$

By applying (3.25) to (3.23), we have

$$\begin{aligned}
&[\Psi(X^+_{0,1}), \Psi(X^+_{n-1,0})] - [\Psi(X^+_{0,0}), \Psi(X^+_{n-1,1})]\\
&= -\frac{\hbar}{2}\{[X^+_{n,0}, X^+_{0,0}], X^+_{n-1,0}\} - \frac{\hbar}{2}\{E_{n+1,1}t, E_{n-1,n+1}\}\\
&\quad - (\varepsilon + \frac{\hbar}{2}(n+1))E_{n-1,1}t + \hbar E_{n-1,n+1}E_{n+1,1}t\\
&= -\frac{\hbar}{2}\{[X^+_{n,0}, X^+_{0,0}], X^+_{n-1,0}\} - (\varepsilon + \frac{\hbar}{2}n)E_{n-1,1}t\\
&= -\frac{\hbar}{2}\{\Psi(X^+_{0,0}), \Psi(X^+_{n-1,0})\} + (\varepsilon + \frac{\hbar}{2}n)[\Psi(X^+_{0,0}), \Psi(X^+_{n-1,0})].
\end{aligned}$$

3.7 Compatibility with (2.9)

By the definition of $\Psi(H_{i,1})$, Ψ is compatible with (2.9) in the case that $r+s \leq 1$. Thus, it is enough to prove $[\Psi(\widetilde{H}_{i,1}), \Psi(\widetilde{H}_{j,1})] = 0$. We only show the case that $i, j \neq 0$. The case that $i = 0$ or $j = 0$ can be proven in a similar way. By the definition of Ψ, we have

$$[\Psi(\widetilde{H}_{i,1}), \Psi(\widetilde{H}_{j,1})] = [\widetilde{H}_{i,1}, \widetilde{H}_{j,1}] - [\widetilde{H}_{i,1}, P_j - P_{j+1}] + [\widetilde{H}_{j,1}, P_i - P_{i+1}] + [P_i - P_{i+1}, P_j - P_{j+1}],$$

where $P_i = \hbar \sum_{s\geq 0} E_{i,n+1}t^{-s-1}E_{n+1,i}t^{s+1}$. By the definition of $J(h_i)$, we have

$$\begin{aligned}
&-[\widetilde{H}_{i,1}, P_j - P_{j+1}] + [\widetilde{H}_{j,1}, P_i - P_{i+1}]\\
&= -[J(h_i), P_j - P_{j+1}] + [J(h_j), P_i - P_{i+1}] + [A_i - A_{i+1}, P_j - P_{j+1}] - [A_j - A_{j+1}, P_j - P_{j+1}].
\end{aligned} \tag{3.26}$$

By Lemma 2.22 and the definition of P_i, we find that the sum of the first two terms of the right hand side of (3.26) are equal to zero. Thus, it is enough to show that

$$[A_i, P_j] - [A_j, P_i] + [P_i, P_j] = 0. \tag{3.27}$$

By a direct computation, we obtain

$$\begin{aligned}
[P_i, P_j] &= \hbar^2 \sum_{s,v\geq 0} E_{j,n+1}t^{-v-1}E_{i,j}t^{v-s}E_{n+1,i}t^{s+1}\\
&\quad - \hbar^2 \sum_{s,v\geq 0} E_{i,n+1}t^{-s-1}E_{j,i}t^{s-v}E_{n+1,j}t^{v+1}.
\end{aligned} \tag{3.28}$$

By the definition of A_i, we can divide $[A_i, P_j]$ into four pieces:

$$\begin{aligned}
[A_i, P_j] &= [\frac{\hbar}{2}\sum_{\substack{s\geq 0\\ u>i}} E_{u,i}t^{-s}E_{i,u}t^{s}, P_j] - [\frac{\hbar}{2}\sum_{\substack{s\geq 0\\ i>u}} E_{i,u}t^{-s}E_{u,i}t^{s}, P_j]\\
&\quad + [\frac{\hbar}{2}\sum_{\substack{s\geq 0\\ u<i}} E_{u,i}t^{-s-1}E_{i,u}t^{s+1}, P_j] - [\frac{\hbar}{2}\sum_{\substack{s\geq 0\\ i<u}} E_{i,u}t^{-s-1}E_{u,i}t^{s+1}, P_j]. \qquad (3.29)
\end{aligned}$$

We compute the right hand side of (3.29). By a direct computation, we obtain

$$\begin{aligned}
&[\frac{\hbar}{2}\sum_{\substack{s\geq 0\\ u>i}} E_{u,i}t^{-s}E_{i,u}t^{s}, P_j]\\
&= \delta(j>i)\frac{\hbar^2}{2}\sum_{s,v\geq 0} E_{j,i}t^{-s}E_{i,n+1}t^{s-v-1}E_{n+1,j}t^{v+1}\\
&\quad + \frac{\hbar^2}{2}\sum_{s,v\geq 0} E_{n+1,i}t^{-s}E_{j,n+1}t^{-v-1}E_{i,j}t^{s+v+1} - \frac{\hbar^2}{2}\sum_{\substack{s,v\geq 0\\ u>i}} \delta_{i,j}E_{u,i}t^{-s}E_{j,n+1}t^{-v-1}E_{n+1,u}t^{s+v+1}\\
&\quad + \frac{\hbar^2}{2}\sum_{s,v\geq 0} \delta_{i,j}E_{u,n+1}t^{-s-v-1}E_{n+1,j}t^{v+1}E_{i,u}t^{s} - \frac{\hbar^2}{2}\sum_{s,v\geq 0} E_{j,i}t^{-s-v-1}E_{n+1,j}t^{v+1}E_{i,n+1}t^{s}\\
&\quad - \delta(j>i)\frac{\hbar^2}{2}\sum_{s,v\geq 0} E_{j,n+1}t^{-v-1}E_{n+1,i}t^{v+1-s}E_{i,j}t^{s}, \qquad (3.30)\\
&\quad - [\frac{\hbar}{2}\sum_{\substack{s\geq 0\\ i>u}} E_{i,u}t^{-s}E_{u,i}t^{s}, P_j]\\
&= -\frac{\hbar^2}{2}\sum_{s,v\geq 0} \delta_{i,j}E_{i,u}t^{-s}E_{u,n+1}t^{s-v-1}E_{n+1,j}t^{v+1}\\
&\quad + \delta(i>j)\frac{\hbar^2}{2}\sum_{s,v\geq 0} E_{i,j}t^{-s}E_{j,n+1}t^{-v-1}E_{n+1,i}t^{s+v+1}\\
&\quad - \delta(i>j)\frac{\hbar^2}{2}\sum_{s,v\geq 0} E_{i,n+1}t^{-s-v-1}E_{n+1,j}t^{v+1}E_{j,i}t^{s}\\
&\quad + \frac{\hbar^2}{2}\sum_{s,v\geq 0} \delta_{i,j}E_{j,n+1}t^{-v-1}E_{n+1,u}t^{v+1-s}E_{u,i}t^{s}, \qquad (3.31)\\
&[\frac{\hbar}{2}\sum_{\substack{s\geq 0\\ u<i}} E_{u,i}t^{-s-1}E_{i,u}t^{s+1}, P_j]\\
&= \delta(j<i)\frac{\hbar^2}{2}\sum_{s,v\geq 0} E_{j,i}t^{-s-1}E_{i,n+1}t^{s-v}E_{n+1,j}t^{v+1}\\
&\quad + \frac{\hbar^2}{2}\sum_{s,v\geq 0} \delta_{i,j}E_{u,i}t^{-s-1}E_{j,n+1}t^{-v-1}E_{n+1,u}t^{s+v+1}\\
&\quad + \frac{\hbar^2}{2}\sum_{\substack{s,v\geq 0\\ u<i}} \delta_{i,j}E_{u,n+1}t^{-s-v-2}E_{n+1,j}t^{v+1}E_{i,u}t^{s+1}\\
&\quad - \delta(j<i)\frac{\hbar^2}{2}\sum_{s,v\geq 0} E_{j,n+1}t^{-v-1}E_{n+1,i}t^{v-s}E_{i,j}t^{s+1}, \qquad (3.32)
\end{aligned}$$

$$
\begin{aligned}
&-[\frac{\hbar}{2}\sum_{\substack{s\geq 0\\ i<u}}E_{i,u}t^{-s-1}E_{u,i}t^{s+1},P_j]\\
=&-\frac{\hbar^2}{2}\sum_{s,v\geq 0}\delta_{i,j}E_{i,u}t^{-s-1}E_{u,n+1}t^{s-v}E_{n+1,j}t^{v+1}+\frac{\hbar^2}{2}\sum_{s,v\geq 0}E_{i,n+1}t^{-s}E_{j,i}t^{s-v-1}E_{n+1,j}t^{v+1}\\
&+\delta(i<j)\frac{\hbar^2}{2}\sum_{s,v\geq 0}E_{i,j}t^{-s-1}E_{j,n+1}t^{-v-1}E_{n+1,i}t^{s+v+2}\\
&-\delta(i<j)\frac{\hbar^2}{2}\sum_{s\geq 0}E_{i,n+1}t^{-s-v-2}E_{n+1,j}t^{v+1}E_{j,i}t^{s+1}\\
&-\frac{\hbar^2}{2}\sum_{s,v\geq 0}E_{j,n+1}t^{-v-1}E_{i,j}t^{v-s}E_{n+1,i}t^{s+1}+\frac{\hbar^2}{2}\sum_{s,v\geq 0}\delta_{i,j}E_{j,n+1}t^{-v-1}E_{n+1,u}t^{v-s}E_{u,i}t^{s+1}.
\end{aligned}\tag{3.33}
$$

Here after, we denote (equation number)$_{a,b}$ means that the value of (equation number) at $i=a,j=b$. Moreover, we denote the r-th term of the right hand side of (equation number) by (equation number)$_r$.

By the definition of A_i, we have

$$
\begin{aligned}
[A_i,P_j]-[A_j,P_i]=&(3.30)_{i,j}+(3.31)_{i,j}+(3.32)_{i,j}+(3.33)_{i,j}\\
&-(3.30)_{j,i}-(3.31)_{j,i}-(3.32)_{j,i}-(3.33)_{j,i}.
\end{aligned}\tag{3.34}
$$

By the definition, we find that the terms containing $\delta_{i,j}$ in the right hand side of (3.34) vanish. By a direct computation, we can compute the sum of the terms containing $\delta(j>i)$ in the right hand side of (3.34):

$$
\begin{aligned}
&(3.30)_{i,j,1}+(3.30)_{i,j,6}-(3.31)_{j,i,2}-(3.31)_{j,i,,3}\\
&\qquad-(3.32)_{j,i,1}-(3.32)_{j,i,4}+(3.33)_{i,j,3}+(3.33)_{i,j,4}\\
=&\,\delta(j>i)\frac{\hbar^2}{2}\sum_{s,v\geq 0}E_{j,i}t^{-s-v-1}E_{i,n+1}t^{s}E_{n+1,j}t^{v+1}\\
&-\delta(j>i)\frac{\hbar^2}{2}\sum_{s,v\geq 0}E_{j,n+1}t^{-v-1}E_{n+1,i}t^{-s}E_{i,j}t^{s+v+1}\\
&-\delta(i<j)\frac{\hbar^2}{2}\sum_{s,v\geq 0}E_{i,j}t^{-s-v-1}E_{j,n+1}t^{s}E_{n+1,i}t^{v+1}\\
&+\delta(i<j)\frac{\hbar^2}{2}\sum_{s,v\geq 0}E_{i,n+1}t^{-v-1}E_{n+1,j}t^{-s}E_{j,i}t^{s+v+1}.
\end{aligned}\tag{3.35}
$$

Similarly, we can compute the sum of the terms containing $\delta(j<i)$ in the right hand side of (3.34):

$$
\begin{aligned}
&-(3.30)_{j,i,1}-(3.30)_{j,i,6}+(3.31)_{i,j,2}+(3.31)_{i,j,3}\\
&\qquad+(3.32)_{i,j,1}+(3.32)_{i,j,4}-(3.33)_{j,i3}-(3.33)_{j,i,4}\\
=&-\delta(i>j)\frac{\hbar^2}{2}\sum_{s,v\geq 0}E_{i,j}t^{-s-v-1}E_{j,n+1}t^{s}E_{n+1,i}t^{v+1}\\
&+\delta(i<j)\frac{\hbar^2}{2}\sum_{s,v\geq 0}E_{i,n+1}t^{-v-1}E_{n+1,j}t^{-s}E_{j,i}t^{s+v+1}\\
&+\delta(j<i)\frac{\hbar^2}{2}\sum_{s,v\geq 0}E_{j,i}t^{-s-v-1}E_{i,n+1}t^{s}E_{n+1,j}t^{v+1}
\end{aligned}
$$

$$-\delta(j<i)\frac{\hbar^2}{2}\sum_{s,v\geq 0}E_{j,n+1}t^{-v-1}E_{n+1,i}t^{-s}E_{i,j}t^{s+v+1}. \tag{3.36}$$

By a direct computation, we obtain

$$\begin{aligned}&(3.35)_2+(3.36)_4+(3.30)_{i,j,2}\\&=\delta(i\neq j)\frac{\hbar^2}{2}\sum_{s,v\geq 0}[E_{n+1,i}t^{-s},E_{j,n+1}t^{-v-1}]E_{i,j}t^{s+v+1}=0.\end{aligned}$$

Similarly, we have

$$(3.35)_3+(3.36)_1-(3.30)_{j,i,5}=0.$$

Then, we find that $[A_i,P_j]-[A_j,P_i]+[P_i,P_j]$ is equal to the sum of the following four terms:

$$\begin{gathered}(3.35)_1+(3.36)_3+(3.30)_{i,j,5},\\(3.35)_4+(3.36)_2-(3.30)_{j,i,2},\\(3.28)_1-(3.33)_{j,i,2}+(3.33)_{i,j,5},\\(3.28)_2+(3.33)_{i,j,2}-(3.33)_{j,i,5}.\end{gathered}$$

By a direct computation, these four sums are equal to zero. We complete the proof of the compatibility with (2.9).

4 The rectangular W-algebra $\mathcal{W}^k(\mathfrak{gl}(2n),(2^n))$

Let us set some notations of a vertex algebra. For a vertex algebra V, we denote the generating field associated with $v\in V$ by $v(z)=\sum_{n\in\mathbb{Z}}v_{(n)}z^{-n-1}$. We also denote the OPE of V by

$$u(z)v(w)\sim\sum_{s\geq 0}\frac{(u_{(s)}v)(w)}{(z-w)^{s+1}}$$

for all $u,v\in V$. We denote the vacuum vector (resp. the translation operator) by $|0\rangle$ (resp. ∂).

We denote the universal affine vertex algebra associated with a finite dimensional Lie algebra $\mathfrak{g}$ and its inner product κ by $V^\kappa(\mathfrak{g})$. By the PBW theorem, we can identify $V^\kappa(\mathfrak{g})$ with $U(t^{-1}\mathfrak{g}[t^{-1}])$. In order to simplify the notation, here after, we denote the generating field $(ut^{-1})(z)$ as $u(z)$. By the definition of $V^\kappa(\mathfrak{g})$, the generating fields $u(z)$ and $v(z)$ satisfy the OPE

$$u(z)v(w)\sim\frac{[u,v](w)}{z-w}+\frac{\kappa(u,v)}{(z-w)^2} \tag{4.1}$$

for all $u,v\in\mathfrak{g}$. For a matrix unit $e_{i,j}$, we denote $e_{i,j}t^{-m}\in U(t^{-1}\mathfrak{g}[t^{-1}])=V^\kappa(\mathfrak{g})$ by $e_{i,j}[-m]$.

The W-algebra $\mathcal{W}^k(\mathfrak{g},f)$ is a vertex algebra associated with the reductive Lie algebra $\mathfrak{g}$ and a nilpotent element f. We call the W-algebra associated with $\mathfrak{gl}(ln)$ and a nilpotent element of type (l^n) the rectangular W-algebra and denote it by $\mathcal{W}^k(\mathfrak{gl}(ln),(l^n))$. In this article, we only consider the case that $l=2$. The nilpotent element is

$$f=\sum_{u=1}^n e_{n+u,u}.$$

By Theorem 3.1 and Corollary 3.2 in [1], we obtain the following theorem.

Theorem 4.2 (Theorem 3.1 and Corollary 3.2 in [1]). *1. We define the inner product on $\mathfrak{gl}(n)$ by*

$$\kappa(e_{i,j},e_{p,q})=\delta_{j,p}\delta_{i,q}\alpha+\delta_{i,j}\delta_{p,q},$$

where $\alpha=k+n$. Then, the rectangular W-algebra $\mathcal{W}^k(\mathfrak{gl}(2n),(2^n))$ can be realized as a vertex subalgebra of $V^\kappa(\mathfrak{gl}(n))^{\otimes 2}$.

2. The W-algebra $\mathcal{W}^k(\mathfrak{g},f)$ has the following strong generators:

$$\begin{aligned}
W^{(1)}_{i,j}&=e^{(1)}_{i,j}[-1]+e^{(2)}_{i,j}[-1],\\
W^{(2)}_{i,j}&=\sum_{1\leq u\leq n}e^{(1)}_{u,j}[-1]e^{(2)}_{i,u}[-1]-\alpha e^{(1)}_{i,j}[-1]
\end{aligned}$$

for $1\leq i,j\leq n$, where $e^{(1)}_{i,j}[-1]=e_{i,j}[-1]\otimes 1\in V^\kappa(\mathfrak{gl}(n))^{\otimes 2}$ and $e^{(2)}_{i,j}[-1]=1\otimes e_{i,j}[-1]\in V^\kappa(\mathfrak{gl}(n))^{\otimes 2}$.

Remark 4.3. We note that $W^{(2)}_{i,j}$ in this article is different from the one in [1]. We shift $W^{(2)}_{i,j}$ in this article is corresponding to $W^{(2)}_{j,i}-\alpha\partial W^{(1)}_{j,i}$ in [1].

We can compute all OPEs of these strong generators. The computation can be done by using the computation process in the appendix of [17].

Theorem 4.4. *1. The following equations hold:*

$$\begin{aligned}
(W^{(1)}_{p,q})_{(0)}W^{(1)}_{i,j}&=\delta_{q,i}W^{(1)}_{p,j}-\delta_{p,j}W^{(1)}_{i,q},\\
(W^{(1)}_{p,q})_{(1)}W^{(1)}_{i,j}&=2\delta_{q,i}\delta_{p,j}\alpha|0\rangle+\delta_{p,q}\delta_{i,j}(1+1)|0\rangle,\\
(W^{(1)}_{p,q})_{(s)}W^{(1)}_{i,j}&=0 \text{ for all } s>1.
\end{aligned}$$

2. The following four equations hold:

$$\begin{aligned}
(W^{(1)}_{p,q})_{(0)}W^{(2)}_{i,j}&=-\delta_{p,j}W^{(2)}_{i,q}+\delta_{i,q}W^{(2)}_{p,j},\\
(W^{(1)}_{p,q})_{(1)}W^{(2)}_{i,j}&=\delta_{p,j}\alpha W^{(1)}_{i,q}+\delta_{p,q}W^{(1)}_{i,j},\\
(W^{(1)}_{p,q})_{(2)}W^{(2)}_{i,j}&=-2\delta_{q,i}\delta_{p,j}\alpha^2|0\rangle-2\delta_{p,q}\delta_{i,j}\alpha|0\rangle,\\
(W^{(1)}_{p,q})_{(s)}W^{(2)}_{i,j}&=0 \text{ for all } s>2.
\end{aligned}$$

3. The following relations hold:

$$\begin{aligned}
&(W^{(2)}_{p,q})_{(0)}W^{(2)}_{i,j}\\
&=(W^{(2)}_{p,j})_{(-1)}W^{(1)}_{i,q}-(W^{(1)}_{p,j})_{(-1)}W^{(2)}_{i,q}+\alpha(\partial W^{(1)}_{p,j})_{(-1)}W^{(1)}_{i,q}+(\partial W^{(1)}_{p,q})_{(-1)}W^{(1)}_{i,j}\\
&\quad-\delta_{q,i}\alpha\partial W^{(2)}_{p,j}-\delta_{i,q}\frac{2\alpha^2+1}{2}\partial^2W^{(1)}_{p,j}-\delta_{i,j}\partial W^{(2)}_{p,q}-\frac{3}{2}\delta_{i,j}\alpha\partial^2W^{(1)}_{p,q},
\end{aligned}\tag{4.5}$$

$$\begin{aligned}
&(W^{(2)}_{p,q})_{(1)}W^{(2)}_{i,j}\\
&=\alpha(W^{(1)}_{p,j})_{(-1)}W^{(1)}_{i,q}+(W^{(1)}_{p,q})_{(-1)}W^{(1)}_{i,j}\\
&\quad-\delta_{q,i}\alpha W^{(2)}_{p,j}-2\delta_{q,i}\alpha^2\partial W^{(1)}_{p,j}-\delta_{p,j}\alpha W^{(2)}_{i,q}-\delta_{i,j}(1)W^{(2)}_{p,q}-2\delta_{i,j}\alpha\partial W^{(1)}_{p,q}-\delta_{p,q}W^{(2)}_{i,j},
\end{aligned}\tag{4.6}$$

$$\begin{aligned}
&(W^{(2)}_{p,q})_{(2)}W^{(2)}_{i,j}\\
&=\delta_{p,j}\alpha(2\alpha-1)W^{(1)}_{i,q}-\delta_{i,j}\alpha W^{(1)}_{p,q}-\delta_{i,q}\alpha(2\alpha-1)W^{(1)}_{i,q}+\delta_{p,q}\alpha W^{(1)}_{i,j},
\end{aligned}\tag{4.7}$$

$$\begin{aligned}
&(W^{(2)}_{p,q})_{(3)}W^{(2)}_{i,j}\\
&=(1+\alpha^2-6\alpha^2)\delta_{p,q}\delta_{i,j}|0\rangle+(2\alpha-6\alpha^3)\delta_{p,j}\delta_{i,q}|0\rangle,
\end{aligned}\tag{4.8}$$

$$(W^{(2)}_{p,q})_{(s)}W^{(2)}_{i,j}=0 \text{ for all } s>0.\tag{4.9}$$

We note that these OPEs are only dependent on α and independent of n. Thus, by Theorem 4.4, we find the following embedding:

$$\mathcal{W}^{k+1}(\mathfrak{gl}(2n),(2^n)) \to \mathcal{W}^k(\mathfrak{gl}(2(n+1)),(2^{n+1})),\ W^{(u)}_{i,j} \mapsto W^{(u)}_{i,j}.$$

5 The relationship between homomorphism Ψ and the rectangular W-algebra $\mathcal{W}^k(\mathfrak{gl}(2n),(2^n))$

Let us recall the definition of a universal enveloping algebra of a vertex algebra in the sense of [10] and [14]. For any vertex algebra V, let $L(V)$ be the Borchards Lie algebra, that is,

$$L(V)=V\otimes\mathbb{C}[t,t^{-1}]/\mathrm{Im}(\partial\otimes\mathrm{id}+\mathrm{id}\otimes\frac{d}{dt}), \tag{5.1}$$

where the commutation relation is given by

$$[ut^a,vt^b]=\sum_{r\geq 0}\binom{a}{r}(u_{(r)}v)t^{a+b-r}$$

for all $u,v\in V$ and $a,b\in\mathbb{Z}$. Now, we define the universal enveloping algebra of V.

Definition 5.2 (Section 6 in [14])**.** We set $\mathcal{U}(V)$ as the quotient algebra of the standard degreewise completion of the universal enveloping algebra of $L(V)$ by the completion of the two-sided ideal generated by

$$(u_{(a)}v)t^b-\sum_{i\geq 0}\binom{a}{i}(-1)^i(ut^{a-i}vt^{b+i}-(-1)^a vt^{a+b-i}ut^i), \tag{5.3}$$

$$|0\rangle t^{-1}-1. \tag{5.4}$$

We call $\mathcal{U}(V)$ the universal enveloping algebra of V.

In [19] Theorem 5.1, the author constructed a surjective homomorphism from the affine super Yangian to the universal enveloping algebra of a rectangular W-superalgebra. Setting $m=n$, $n=0$ and $l=2$, we obtain the following theorem.

Theorem 5.5. *Suppose that $\hbar=-1$ and $\varepsilon=-\alpha$. There exists an algebra homomorphism*

$$\Phi^n\colon Y_{\hbar,\varepsilon}(\widehat{\mathfrak{sl}}(n))\to\mathcal{U}(\mathcal{W}^k(\mathfrak{gl}(2n),(2^n)))$$

determined by

$$\Phi^n(H_{i,0})=\begin{cases}W^{(1)}_{n,n}-W^{(1)}_{1,1}+2\alpha & \text{if } i=0,\\ W^{(1)}_{i,i}-W^{(1)}_{i+1,i+1} & \text{if } i\neq 0,\end{cases}$$

$$\Phi^n(X^+_{i,0})=\begin{cases}W^{(1)}_{n,1}t & \text{if } i=0,\\ W^{(1)}_{i,i+1} & \text{if } i\neq 0,\end{cases}\quad \Phi^n(X^-_{i,0})=\begin{cases}W^{(1)}_{1,n}t^{-1} & \text{if } i=0,\\ W^{(1)}_{i+1,i} & \text{if } i\neq 0,\end{cases}$$

$$\Phi^n(H_{i,1})=\begin{cases}W^{(2)}_{n,n}t-W^{(2)}_{1,1}t+\alpha W^{(1)}_{n,n}-2\alpha\Phi^n(H_{0,0})+W^{(1)}_{n,n}(W^{(1)}_{1,1}-2\alpha)\\ \quad -\displaystyle\sum_{s\geq 0}\sum_{u=1}^{n}W^{(1)}_{n,u}t^{-s}W^{(1)}_{u,n}t^s+\sum_{s\geq 0}\sum_{u=1}^{n}W^{(1)}_{1,u}t^{-s-1}W^{(1)}_{u,1}t^{s+1}, & \\ & \text{if } i=0,\\ W^{(2)}_{i,i}t-W^{(2)}_{i+1,i+1}t+\dfrac{i}{2}\Phi^n(H_{i,0})+W^{(1)}_{i,i}W^{(1)}_{i+1,i+1}\\ \quad -\displaystyle\sum_{s\geq 0}\sum_{u=1}^{i}W^{(1)}_{i,u}t^{-s}W^{(1)}_{u,i}t^s-\sum_{s\geq 0}\sum_{u=i+1}^{n}W^{(1)}_{i,u}t^{-s-1}W^{(1)}_{u,i}t^{s+1}\\ \quad +\displaystyle\sum_{s\geq 0}\sum_{u=1}^{i}W^{(1)}_{i+1,u}t^{-s}W^{(1)}_{u,i+1}t^s+\sum_{s\geq 0}\sum_{u=i+1}^{n}W^{(1)}_{i+1,u}t^{-s-1}W^{(1)}_{u,i+1}t^{s+1} & \\ & i\neq 0,\end{cases}$$

$$\Phi^n(X^+_{i,1})=\begin{cases}W^{(2)}_{n,1}t^2+\alpha W^{(1)}_{n,1}t-2\alpha\Phi^n(X^+_{0,0})-\displaystyle\sum_{s\geq 0}\sum_{u=1}^{n}W^{(1)}_{n,u}t^{-s}W^{(1)}_{u,1}t^{s+1} & \\ & \text{if } i=0,\\ W^{(2)}_{i,i+1}t+\dfrac{i}{2}\Phi^n(X^+_{i,0})\\ \quad -\displaystyle\sum_{s\geq 0}\sum_{u=1}^{i}W^{(1)}_{i,u}t^{-s}W^{(1)}_{u,i+1}t^s-\sum_{s\geq 0}\sum_{u=i+1}^{n}W^{(1)}_{i,u}t^{-s-1}W^{(1)}_{u,i+1}t^{s+1} & \\ & \text{if } i\neq 0,\end{cases}$$

$$\Phi^n(X^-_{i,1})=\begin{cases}W^{(2)}_{1,n}-2\alpha\Phi^n(X^-_{0,0})-\displaystyle\sum_{s\geq 0}\sum_{u=1}^{n}W^{(1)}_{1,u}t^{-s-1}W^{(1)}_{u,n}t^s, & \\ & \text{if } i=0,\\ W^{(2)}_{i+1,i}t+\dfrac{i}{2}\Phi^n(X^-_{i,0})\\ \quad -\displaystyle\sum_{s\geq 0}\sum_{u=1}^{i}W^{(1)}_{i+1,u}t^{-s}W^{(1)}_{u,i}t^s-\sum_{s\geq 0}\sum_{u=i+1}^{n}W^{(1)}_{i+1,u}t^{-s-1}W^{(1)}_{u,i}t^{s+1} & \\ & \text{if } i\neq 0.\end{cases}$$

By the definition of Ψ^n, we obtain the following theorem.

Theorem 5.6. *Suppose that $\hbar=-1$ and $\varepsilon=-k-(n+1)$. We obtain the following commutative diagram:*

$$\Phi^{n+1}\circ\Psi=\iota\circ\Phi^n.$$

Proof. The affine Yangian is generated by $X^{\pm}_{i,0}$ for $0\leq i\leq n-1$ and $X^+_{j,1}$ for $1\leq j\leq n-1$ by the defining relations (2.9)-(2.18). Thus, it is enough to show that

$$\Phi^{n+1}\circ\Psi(X^{\pm}_{i,0})=\iota\circ\Phi^n(X^{\pm}_{i,0}), \tag{5.7}$$
$$\Phi^{n+1}\circ\Psi(X^{+}_{j,1})=\iota\circ\Phi^n(X^{+}_{j,1}). \tag{5.8}$$

By the definition of Φ^n, ι and Ψ, (5.7) holds. By the definition of Φ^n and Ψ, we have

$$\begin{aligned}&\Phi^{n+1}\circ\Psi(X^+_{j,1})\\ &=\Phi^{n+1}(X^+_{j,1}+\sum_{s\geq 0}E_{j,n+1}t^{-s-1}E_{n+1,j+1}t^{s+1})\\ &=W^{(2)}_{j,j+1}t+\frac{i}{2}\Phi^{n+1}(X^+_{j,0})\end{aligned}$$

$$
\begin{aligned}
&-\sum_{s\geq 0}\sum_{u=1}^{j} W^{(1)}_{j,u}t^{-s}W^{(1)}_{u,j+1}t^{s}-\sum_{s\geq 0}\sum_{u=j+1}^{n+1} W^{(1)}_{j,u}t^{-s-1}W^{(1)}_{u,j+1}t^{s+1}\\
&+\sum_{s\geq 0} W^{(1)}_{j,n+1}t^{-s-1}W^{(1)}_{n+1,j+1}t^{s+1}\\
=\ & W^{(2)}_{j,j+1}t+\frac{i}{2}\Phi^{n+1}(X^+_{j,0})\\
&-\sum_{s\geq 0}\sum_{u=1}^{j} W^{(1)}_{j,u}t^{-s}W^{(1)}_{u,j+1}t^{s}-\sum_{s\geq 0}\sum_{u=j+1}^{n} W^{(1)}_{j,u}t^{-s-1}W^{(1)}_{u,j+1}t^{s+1}.
\end{aligned}
$$

On the other hand, by the definition of Φ^n and ι, we obtain

$$
\begin{aligned}
&\iota\circ\Phi^n(X^+_{j,1})\\
=\ & W^{(2)}_{j,j+1}t+\frac{i}{2}\Phi^{n+1}(X^+_{j,0})\\
&-\sum_{s\geq 0}\sum_{u=1}^{j} W^{(1)}_{j,u}t^{-s}W^{(1)}_{u,j+1}t^{s}-\sum_{s\geq 0}\sum_{u=j+1}^{n} W^{(1)}_{j,u}t^{-s-1}W^{(1)}_{u,j+1}t^{s+1}.
\end{aligned}
$$

Thus, the relation (5.8) holds. □

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Data Availability

The authors confirm that the data supporting the findings of this study are available within the article and its supplementary materials.

Declarations

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Conflicts of interests/Competing interests

The authors have no competing interests to declare that are relevant to the content of this article.

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