Extracting Error Thresholds through the Framework of Approximate Quantum Error Correction Condition

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The robustness of quantum memory against physical noises is measured by two methods: the exact and approximate quantum error correction (QEC) conditions for error recoverability, and the decoder-dependent error threshold which assesses if the logical error rate diminishes with system size. Here we unravel their relations and propose a unified framework to extract an intrinsic error threshold from the approximate QEC condition, which could upper bound other decoder-dependent error thresholds. Our proof establishes that relative entropy, effectively measuring deviations from exact QEC conditions, serves as the order parameter delineating the transition from asymptotic recoverability to unrecoverability. Consequently, we establish a unified framework for determining the error threshold across both exact and approximate QEC codes, addressing errors originating from noise channels as well as those from code space imperfections. This result sharpens our comprehension of error thresholds across diverse QEC codes and error models.

Introduction. Quantum Error Correction (QEC) is crucial for fault-tolerant quantum computation [1–6], protecting quantum information from decoherence and noise. At its core lies the QEC condition, i.e. Knill-Laflamme condition [7], essential for precisely recovering quantum information from error channels. Despite its theoretical importance, perfect error correction in practice faces challenges due to various physical noises in real-world systems [8–18]. This is where the error threshold theorem [5, 19–31] becomes relevant, suggesting that if error rates are maintained below a certain critical threshold, the impact of logical errors can be substantially mitigated, a key to realizing reliable quantum computation.

A fundamental conflict exists between the precise QEC condition, aimed at exact recoverability for particular QEC code designs, and the more generalized, asymptotic perspective of the error threshold theorem [5, 19–31]. Many realistic noise models [5, 20–23, 26–29, 32–34] often fail to satisfy QEC condition, highlighting an intrinsic nonzero logical error rate in finite systems. This discrepancy has led to the adoption of error thresholds. Here we focus on the quantum memory threshold describing the robustness of quantum information under an active correction process. In fact, within the notion of mixed state phase transition, Fan and others [35] investigated a specific example of toric code with single-qubit Pauli noise, and found that the optimal error threshold is implicated in the information of encoding and noise channel without the knowledge of decoder algorithm. Relevant works on mixed state topological order can be found in Refs. [36–39].Therefore, a critical question arises: How can we reconcile the code-specific QEC condition with the decoder-dependent perspective of error thresholds in advancing practical QEC strategies against diverse noise types in quantum memory?

To resolve the discrepancy, we make use of the Approximate QEC (AQEC) condition [40–46]. Given a quantum code with encoding map $\mathcal{E}$ Suffering from physical noise channel $\mathcal{N}(\rho) = \sum_u E_u \rho E_u^\dagger$, it satisfies:

$$PE_u^\dagger E_v P = \lambda_{uv} P + PB_{uv}P. \tag{1}$$

Here P is the projection onto the code subspace, λ_{uv} is a constant satisfying $\lambda_{uv} = \lambda_{vu}^*$, B_{uv} operator term captures the deviation from exact QEC correction and can be viewed as logical error. If this logical error is sufficiently small, the recovery can be deemed high-precision, thereby enabling effective AQEC. In particular, an exact recovery channel $\mathcal{R}$ exists s.t. $\mathcal{R} \circ \mathcal{N} \circ \mathcal{E} = \mathcal{I}$, if and only if $B_{uv} = 0$ [7].

Using the framework of AQEC, we establish a direct connection between the intrinsic characteristics of a quantum code and practical error thresholds, demonstrating that the error threshold is determined by the AQEC relative entropy. This framework unifies the notion of error threshold for common QEC codes (for example stabilizer codes) as well as AQEC codes, which can only approximately preserve quantum information against local perturbations [47–51]. Remarkably, this method can assist us in characterizing the intrinsic properties of a code including AQEC codes even without an easily analyzable decoder. We also examine two examples, the ordinary qudit stabilizer codes [52] under stochastic noises, and an imperfectly prepared toric code [53–55] that is unstable under local noises.

AQEC relative entropy. We conjecture that exploring the asymptotic behaviors of AQEC conditions with increasing code size, denoted as n, is crucial for fully grasping error thresholds. Based on this assumption, we examine a series of QEC codes, $\{\mathcal{C}_n\}_n^\infty$, and their respective noise channels, $\{\mathcal{N}_n\}_n^\infty$. The dimension of the code space, K, is dependent on the specific code; for instance, $K = 4$ in toric codes [5] and $\log K = \Theta(n)$ for good LDPC codes [56–58]. In each instance of $\mathcal{C}_n$ and $\mathcal{N}_n$, Equation (1) is fulfilled. Typically, B_{uv} is nonzero for finite n. The hypothesis is that for systems below the threshold, B_{uv} diminishes as $n \to \infty$, whereas it remains significant for systems above the threshold, even as $n \to \infty$.

The subtlety arises in quantifying the magnitude of B_{uv}. Within the chosen basis $\{|q\rangle\}_q$ of the code subspace, simply examining the matrix element $\langle q_1| B_{uv} |q_2\rangle$ is insufficient to ascertain the threshold. Considering a surface code affected by single-qubit Pauli errors [5], λ_{uv} exhibits a scaling of $\mathcal{O}\left((1-p)^{\frac{n}{2}}\right)$, whereas $\langle q_1| B_{uv} |q_2\rangle$ demonstrates a scaling of $\mathcal{O}\left(p^{\frac{\delta}{2}}(1-p)^{\frac{n-\delta}{2}}\right)$, with $\delta \approx \sqrt{n}$. In the limit of $n \to \infty$, this is negligible for $p < 1/2$. However, the actual threshold, identified using a maximum likelihood decoder (MLD), is approximately $p_c \sim 0.11$, indicating that matrix elements alone are inadequate for pinpointing the criticality in QEC systems.

We utilize an entropic measure for a more precise threshold estimation. The parameters λ_{uv} and operators B_{uv} are reformulated in matrix form, pertinent to error configurations uv and code words q_1q_2, as $\Lambda_{uv,q_1q_2} = \lambda_{uv}\delta_{q_1q_2}/K$ and $B_{uv,q_1q_2} = \langle q_1| B_{uv} |q_2\rangle /K$. Here, K represents the code space dimension. To ensure $\mathrm{tr}(\Lambda + B) = 1$, we introduce a factor of $\frac{1}{K}$, with both Λ and B being Hermitian [41]. Furthermore, we can always assume $\mathrm{tr}_{\mathcal{C}}(B) = 0$, where $\mathrm{tr}_{\mathcal{C}}(*)$ is the trace over code subspace, since the trace-nonzero part can always be absorbed into the definition of Λ. Besides, both $\Lambda + B$ and Λ are positive semi-defined [59]. The *AQEC relative entropy* is defined as

$$S(\Lambda + B||\Lambda) = \mathrm{tr}\left\{(\Lambda + B)\left[\log(\Lambda + B) - \log\Lambda\right]\right\}, \quad (2)$$

which can be proved that (see Supplemental Information (SI) [59])

$$0 \leq S(\Lambda + B||\Lambda) \leq 2\log K. \quad (3)$$

It measures the magnitude of B relative to Λ, and intrinsically captures the logical error rate. Notice that in a finite system, the lower bound in Eq. (3) is saturated equality if and only if $B = 0$ [60]. In other words, the exact QEC condition is equivalent to $S(\Lambda + B||\Lambda) = 0$. As $S(\Lambda + B||\Lambda)$ is usually nonzero, the asymptotic behavior as $n \to \infty$ becomes crucial.

With the help of AQEC relative entropy, we define the *intrinsic error threshold* as follows. If the AQEC relative entropy vanishes in the thermodynamic limit $\lim_{n\to\infty} S(\Lambda + B||\Lambda) = 0$, we say that the QEC system is *below the intrinsic error threshold*. Otherwise, the QEC system is *above the intrinsic error threshold*. This intrinsic threshold demarcates different behaviors of AQEC relative entropy at large n, independent of decoder choice.

Asymptotic recoverability. Bény and Oreshkov [41] utilized worst-case entanglement fidelity to measure the deviation of AQEC from QEC condition, limited to a fixed code size. Our work examines the impact of AQEC relative entropy on recovery channels for large code sizes in the asymptotic limit.

Theorem 1. *Given a family of $\{\mathcal{C}_n\}_n^\infty$ with noise channels $\{\mathcal{N}_n\}_n^\infty$, consider the large size limit $n \to \infty$,*

(1) Below the intrinsic error threshold, i.e. $\lim_{n\to\infty} S(\Lambda + B||\Lambda) = 0$, there exists a family of recovery map $\{\mathcal{R}_n\}_n$, such that the entanglement fidelity of the whole QEC process satisfies

$$\lim_{n\to\infty} F_e(\mathcal{R}\circ\mathcal{N}\circ\mathcal{E}) = 1. \quad (4)$$

(2) Above the intrinsic error threshold, i.e. $S(\Lambda + B||\Lambda)$ does not converge to 0, if $K = \mathcal{O}(1)$, then the entanglement fidelity $F_e(\mathcal{R}\circ\mathcal{N}\circ\mathcal{E})$ cannot converges to 1 for an arbitrary family of recovery map $\{\mathcal{R}_n\}_n$.

(3) Let K be a parameter that diverges with n, such that $K = \omega(1)$. If $s(\Lambda + B||\Lambda) \equiv S(\Lambda + B||\Lambda)/\log K$ does not converge to 0, the entanglement fidelity $F_e(\mathcal{R}\circ\mathcal{N}\circ\mathcal{E})$ cannot converges to 1 for an arbitrary family of recovery map $\{\mathcal{R}_n\}_n$.

Remark. The original concept of entanglement fidelity [61–63], as reviewed in SI [59], necessitates specifying an initial state. In our case, the initial state is the maximally mixed logical state I_K/K, making $F_e(\mathcal{R}\circ\mathcal{N}\circ\mathcal{E}) = F_e(I_K/K, \mathcal{R}\circ\mathcal{N}\circ\mathcal{E})$ indicative of the QEC success in an average sense. Considering the limit $\lim_{n\to\infty} F_e(\mathcal{R}\circ\mathcal{N}\circ\mathcal{E})$ helps determine if noise is *asymptotically recovered*. The theorem's proof, found in SI [59], hinges on two finite n inequalities: a recovery channel $\mathcal{R}$ exists (not necessarily optimal) fulfilling

$$1 \geq F_e(\mathcal{R}\circ\mathcal{N}\circ\mathcal{E}) \geq 1 - \sqrt{2S(\Lambda + B||\Lambda)}, \quad (5)$$

and for any $\mathcal{R}$,

$$0 \leq S(\Lambda + B||\Lambda) \leq 2h(1 - F_e(\mathcal{R}\circ\mathcal{N}\circ\mathcal{E})), \quad (6)$$

where $h(x) = -x\log x - (1-x)\log(1-x) + x\log(K^2 - 1)$. Eqs. (5) and (6) follows from the inequalities in Refs. [40, 63, 64] by relating AQEC relative entropy to coherent information. The proof is completed by aggregating all n and considering the $n \to \infty$ limit. Notably, with K diverging with n, Eq.(6) above implies $1 - F_e(\mathcal{R}\circ\mathcal{N}\circ\mathcal{E}) = \Omega(1/\log K)$ above threshold, due to $h(x)$'s K-dependency. Therefore, the "logical qubit number" $k \propto \log K$ can be used as the denominator to evaluate the *density of AQEC relative entropy*, $s(\Lambda + B||\Lambda) = S(\Lambda + B||\Lambda)/\log K$, aiding in assessing the failure of asymptotic recoverability.

Till now, our discussion has not specified a decoder, which is essentially a method for constructing a recovery channel, denoted as $\mathcal{R}_{\mathrm{de}}$. Considering an error threshold under this decoder. In scenarios with finite K, achieving $\lim_{n\to\infty} F_e = 1$ is impossible when above the intrinsic threshold. Below this threshold, although some recovery channel $\mathcal{R}$ may achieve perfect entanglement fidelity, $\mathcal{R}_{\mathrm{de}}$ may not be equally effective. Thus, the intrinsic error threshold sets an upper bound on decoding thresholds. With K diverging, the density of AQEC relative entropy $s(\Lambda+B||\Lambda)$ serves as a metric to upper bound decoding thresholds.

The two phases of the QEC system have just been discussed. However, the critical examination of an intrinsic threshold, or a phase transition, remains. Subsequent sections address this through specific examples using our framework.

Stabilizer codes. We first consider qudit stabilizer codes [4, 52, 65] within the phase space formalism [65], where $T(v)$, $v \in \mathcal{V} = \mathbb{Z}_d^{2n}$ denotes the Heisenberg-Weyl operators, specifically for prime local dimensions d, for convenience. A stabilizer group $T(\mathcal{M})$ is associated with an isotropic subspace $\mathcal{M} \subset \mathcal{V}$ (See to SI [59] for detailed definitions). Such a stabilizer code $\mathcal{C}$ has $k = n - \dim \mathcal{M}$ logical qudits and the code space dimension $K = d^k$. We assume that the data qudits suffer from stochastic weyl errors,

$$\mathcal{N}(\rho_0) = \sum_{\eta \in \mathcal{V}} \Pr(\eta) T(\eta) \rho_0 T(\eta)^\dagger. \tag{7}$$

MLD [5, 31], the optimal decoder for this QEC system, selects recovery operators by assessing the combined likelihood of errors that produce the same syndrome. Notice that each error configuration η is uniquely decomposed as $\eta = s + l + m$, where $m \in \mathcal{M}$, $s \in \mathcal{S}$ represents a particular syndrome configuration and $l \in \mathcal{L}$ denotes logical classes. An error equivalent class is specified by syndrome s and logical class l, and its probability is $\Pr(s, l) = \sum_{m \in \mathcal{M}} \Pr(s + l + m)$. We denote the corresponding random variables as S and L. Given a syndrome s, the MLD chooses recovery operator $T(s+l)$ with the largest conditional probability $\Pr(l|s)$. Without losing generality, we redefine the representative configuration of syndrome s such that $\Pr(l = 0|s)$ maximizes the likelihood. A standard approach to the MLD threshold problem is mapping the error class probability to a statistical mechanical (SM) partition function $\Pr(s, l) = Z(\eta)$ with quenched disorder η [31], generalizing previous SM mapping constructions [5, 21, 23, 27, 28]. The order parameter $\Delta(\eta, l) = -\log(Z(\eta + l)/Z(\eta))$, i.e. the free energy cost of logical classes, relates the SM phase transition to the MLD threshold. See detailed review in SI [59].

A key result of ours for stabilizer codes is the following relation.

Lemma 2. *For a qudit stabilizer code in prime d and a stochastic Weyl noise channel, the AQEC relative entropy, probability of logical classes and order parameter of the SM model are related by*

$$\begin{aligned} S(\Lambda + B||\Lambda) &= H(L|S) \\ &= \sum_{\eta \in \mathcal{V}} \Pr(\eta) \log \left\{ \sum_{l \in \mathcal{L}} \exp\left[-\Delta(\eta, l)\right] \right\}, \end{aligned} \tag{8}$$

where $H(L|S) = -\sum_{s,l} \Pr(s, l) \log \Pr(l|s)$ is the Shannon conditional entropy of logical class l given syndrome s, and the last quantity is a generalized version of the homological difference [28].

Remark. We applied replica method in the derivation of Eq. (8). The rigorousness is guaranteed by Carlson's theorem [66–68]. We calculate the AQEC relative entropy through the limit

$$S(\Lambda + B||\Lambda) = \lim_{R \to 1} \frac{1}{R - 1} \log \frac{\mathrm{tr}((\Lambda + B)^R)}{\mathrm{tr}(\Lambda^R)}. \tag{9}$$

The r.h.s. is evaluated by fixing R as integers $R > 1$,

$$\begin{aligned} \mathrm{tr}(\Lambda^R) &= \frac{1}{d^{k(R-1)}} \sum_{\eta \in \mathcal{V}} \Pr(\eta) Z(\eta)^{R-1}, \\ \mathrm{tr}((\Lambda + B)^R) &= \frac{1}{d^{k(R-1)}} \sum_{\eta \in \mathcal{V}} \Pr(\eta) \left[\sum_{l \in \mathcal{L}} Z(\eta + l) \right]^{R-1}, \end{aligned} \tag{10}$$

and then taking the $R \to 1$ limit after analytical continuation. A detailed proof is in SI [59]. Note that the generalized homological difference serves as a better-behaved order parameter compared to the simple disorder averaged $\Delta(\eta, l)$ [59].

Notice that the r.h.s. of Eq. (10) are the replica partition functions [69] of the quenched disordered SM model $Z(\eta)$ (inserted with domain wall configurations l in the second line). It suggests that the SM mapping can be derived from the intrinsic properties of the code and noise, following the spirit of Fan and others' example [35]. We also remark that our result is not a simple generalization, since their original intrinsic SM model cannot be generally applied to qudit systems due to potentially complex Boltzmann weights. For stabilizer codes and Weyl noises, the validity of SM mapping crucially relies on the Gottesman-Knill theorem [60, 65]. The code states and noises both acquire phase space descriptions, thus the error threshold problem can be mapped to probability distributions (Boltzmann weights) on the classical phase space. Therefore, $\mathrm{tr}(\Lambda^R)$ (or $Z(\eta)$) is a more suitable choice for the partition function to circumvent complex weights.

Considering Eq.(8), the first equality introduces the conditional entropy $H(L|S)$, which measures the uncertainty in MLD when deducing the logical class l from the syndrome s. This suggests a link between the intrinsic and MLD thresholds. Given that the MLD threshold is typically evaluated by the asymptotic behavior of the success probability $\Pr(l = 0) = 1$ [28, 31], we can establish a corresponding relation.

Theorem 3. *For a family of qudit stabilizer codes in prime d and stochastic Weyl noise channels, in the large size limit $n \to \infty$,*

(1) below the intrinsic error threshold, i.e. $\lim_{n\to\infty} S(\Lambda + B||\Lambda) = 0$, the success probability of MLD converges to 1,

$$\lim_{n\to\infty} \Pr(l = 0) = 1. \tag{11}$$

(2) if the logical qudit number is finite $k = \mathcal{O}(1)$, then $\lim_{n\to\infty} \Pr(l = 0) = 1$ implies the QEC system is below the intrinsic error threshold $\lim_{n\to\infty} S(\Lambda + B||\Lambda) = 0$.

(3) if k diverges with n, $k = \omega(1)$, then $\lim_{n\to\infty} \Pr(l = 0) = 1$ implies the density of AQEC relative entropy $s(\Lambda + B||\Lambda) \equiv S(\Lambda + B||\Lambda)/k$ converges to 0.

Remark. The above theorem is proved in SI [59]. Basically, it follows from the inequality

$$-\log \Pr(l=0) \le S(\Lambda+B||\Lambda) \le h(1-\Pr(l=0)), \quad (12)$$

where $h(x)$ is the same as in Eq.(6). The first two propositions in Thm. 3 suggests that the intrinsic threshold is exactly the same as the MLD threshold for those codes with finite logical qudits, if the decodable region is defined by $\lim_{n\to\infty} \Pr(l=0)=1$. For $k\to\infty$, we similarly can only conclude $1-\Pr(l=0)=\Omega(1/k)$ when above the intrinsic threshold.

The intrinsic threshold of QEC systems, as indicated by the second equality of Eq.(8), is contingent on the phase transition of the corresponding SM model. Notable instances encompass qubit or qudit toric code [5, 35], color code [21, 23, 27], qubit or qudit hypergraph-product code [28, 70, 71] and Hyperbolic surface code [26, 70, 71] under single-qubit Pauli noise, toric code under locally correlated Pauli noise [31]. Now, let us assume geometric locality for both stabilizer generators and the error channel.

In the ordered phase, nontrivial logical operators must span at least the code distance δ, incurring a free energy cost proportional to δ. This leads to the AQEC relative entropy scaling as $S(\Lambda+B \parallel \Lambda)\sim e^{-\delta/\xi}$, where ξ denotes the correlation length related to the error rate. As the code distance increases with n, the relative entropy decreases.

In the disordered phase, some logical classes l maintain a finite free energy cost, preventing $S(\Lambda+B||\Lambda)$ from reaching zero. For example, if $\Delta(\eta,l)\sim 0$ for all l, then $S(\Lambda+B||\Lambda)\sim 2k\log d$. Intuitively, when above the threshold, AQEC relative entropy characterizes how many logical qudits suffer from logical errors.

At a critical error threshold, the divergence of ξ implies a power-law behavior $S(\Lambda+B||\Lambda)\sim 1/\delta^{2h}$. For instance, in the random bond Ising model, which describes the toric code and single qubit errors, the dual relationship between an open-ended wall (disorder operator) and spin-spin correlation exhibits a power-law length dependence. This can be extended to a closed domain wall l, leading to $e^{-\Delta}\sim 1/\delta^{2h}$, where h is the Ising spin's scaling dimension.

State preparation threshold. In the preceding section, we addressed the relationship between the AQEC condition arising solely from noise channel $\mathcal{N}$ and the threshold. Yet, inevitable imperfections in the fundamental encoding channel $\mathcal{E}$ result in AQEC codes. We now illustrate how our formalism encompasses imperfect encoding or state preparation.

We consider toric code [5, 6] with single-qubit bit-flip error and assume that the logical states are prepared through measuring the plaquette stabilizers $B_p=\prod_{e\in\partial p} Z_e$ on a product state (e labels data qubit). An ancilla qubit should be brought to entangle with four data qubits and measured in order to perform B_p measurement, but coherent noises on the entanglement gates might result in a positive operator-valued measurement (POVM) rather than a projective measurement. We apply the imperfect preparation model considered in Refs. [53–55] (also see SI [59]) with the following logical basis

$$\begin{aligned}&|{+}{+}\rangle_L \propto \prod_p \exp\left[\frac{1}{2}\beta B_p\right]|+\rangle^{\otimes n}, \quad |{-}{+}\rangle_L = Z_{l_1}|{+}{+}\rangle_L,\\ &|{+}{-}\rangle_L = Z_{l_2}|{+}{+}\rangle_L, \quad |{-}{-}\rangle_L = Z_{l_1}Z_{l_2}|{+}{+}\rangle_L.\end{aligned} \quad (13)$$

Here Z_{l_1}, Z_{l_2} stands for Z logical operators. β is the error parameter and $\beta\to+\infty$ recovers projective measurement. Unlike the perfect preparation case, B_{uv,q_1q_2} is now nonzero even for certain local noises like $E_u^\dagger E_v\sim\sqrt{p}X_e$ and scales as $\mathcal{O}\left(p^{\frac{1}{2}}e^{-2\beta}\right)$ for large enough β and small enough p, which is not suppressed by code distance [59]. It suggest that as long as β is finite, B_{uv,q_1q_2} is always comparable to Λ_{uv,q_1q_2} and prevents $S(\Lambda+B||\Lambda)$ from approaching 0 in the thermodynamic limit.

We can still try to find an SM interpretation. In general, this is not always valid beyond stabilizer code states and Weyl errors due to the violation of Gottesman-Knill theorem, but luckily it works for the current simple model Eq.(13). We write $\mathrm{tr}\left((\Lambda+B)^R\right)$ as a partition function

$$\begin{aligned}&\mathrm{tr}\left((\Lambda+B)^R\right)\propto\sum_{\{\eta_e^{(\alpha)}\}}\exp[-H(\{\eta_e^{(\alpha)}\})],\\ &H(\{\eta_e^{(\alpha)}\}) = -\sum_{\alpha=1}^{R}\Bigg[h\sum_e\eta_e^{(\alpha)}\\ &\qquad+\frac{1}{2}\log\cosh\beta\sum_p U_p^{(\alpha)}U_p^{(\alpha+1)}\\ &\qquad+\log\left(1+(\tanh\beta)^{n/2}\prod_p\delta_{U_p^{(\alpha)},U_p^{(\alpha+1)}}\right)\Bigg],\end{aligned} \quad (14)$$

Where $h=\frac{1}{2}\log\frac{1-p}{p}$ and $U_p^{(\alpha)}=\prod_{e\in\partial p}\eta_e^{(\alpha)}$. The SM d.o.f. $\eta_e^{(\alpha)}=\pm1$ is defined on each data qubit e and each replica copy $\alpha=1,\cdots,R$ (we identify $R+1=1$). $\{\eta_e^{(\alpha)}\}$ correspond to the error configurations u, v in Eq. (1), thus the above partition function captures fluctuations of errors. The Rényi AQEC relative entropy is expressed as

$$\begin{aligned}S^{(R)}(\Lambda+B||\Lambda) &= \frac{1}{R-1}\log\frac{\mathrm{tr}((\Lambda+B)^R)}{\mathrm{tr}(\Lambda^R)}\\ &=\frac{1}{1-R}\log\left\langle\prod_{\alpha,l}\frac{1+\prod_{e\in l}\eta_e^{(\alpha)}\eta_e^{(\alpha+1)}}{2}\right\rangle,\end{aligned} \quad (15)$$

where $l=l_1,l_2$ labels the set of data qubits in Z logical operator Z_{l_1},Z_{l_2} and the $\langle * \rangle$ here is the SM ensemble average. The r.h.s. compares fluctuations of logical errors with trivial errors. The $R\to1$ phase transition point probed by the above quantity is the intrinsic threshold.

When $\beta\to+\infty$ the model (14) reduces to Eq. (10), or more concretely replica random bond Ising model [5, 69] with domain wall inserted. When $\beta\to0$ and keeping h finite, we obtain a trivial paramagnetic spin model and

$\lim_{n\to\infty} S^{(R)}(\Lambda+B||\Lambda) = 2\log 2$. We then focus on the case when both state preparation and Pauli error rates are small but nonzero $0 < e^{-\beta}, p \ll 1$ and perturbatively calculate $S^{(R)}(\Lambda + B||\Lambda)$ up to the first non-vanishing order [59],

$$S^{(R)}(\Lambda + B||\Lambda) \sim \frac{\sqrt{n/2}}{1-1/R} \exp\left(-2h - 4\beta\right). \tag{16}$$

Although there is a subtlety that the $R \to 1$ and $n \to \infty$ limits are not compatible with the perturbative expansion, it nonetheless suggests that the imperfect code cannot suppress $S^{(R)}(\Lambda+B||\Lambda)$ to 0 and lead to a phase transition at $1/\beta = 0$ for any $R > 1$. In the $n \to \infty$ limit where the third term of the Hamiltonian (14) is negligible, $S^{(R)}(\Lambda + B||\Lambda)$ is monotonically increasing with $1/\beta$ following from Griffiths-Kelly-Sherman inequality [70, 72, 73]. The divergent coefficient $\sqrt{n}$ in Eq. (16) indicates a sudden jump at $e^{-4\beta} \to 0$. Then we extrapolate to $R \to 1$ and conclude that the intrinsic threshold is at $e^{-4\beta} = 0$ or $\beta \to +\infty$. In other words, the parameter region below the intrinsic threshold is $1/\beta = 0$ and $p < p_c \sim 0.11$. Our theoretical framework correctly conveys the insight that information encoded in imperfectly prepared toric code (13) subjected to single-qubit Pauli errors is irrecoverable, regardless of the decoding strategy employed. This is also compatible with our previous result in Ref. [55], where we assumed a specific noisy decoding procedure.

Discussion. Note that the last example confirmed that our framework could be applied on general AQEC codes, which possess nonzero B_{uv,q_1q_2} for local error operators $E_u^\dagger E_v$. Although the intrinsic threshold in this example vanishes due to the non-vanishing B_{uv,q_1q_2} in the $n \to \infty$ limit, it may remain finite for certain AQEC codes with B_{uv,q_1q_2} suppressed by size (potentially e.g. topological ordered states [47], approximate LDPC code [48], ETH and Heisenberg chain codes [49]). Our framework also holds potential for application to codes and noises that are beyond Gottesman-Knill theorem and are intrinsically quantum. Future endeavors could include detailed examinations of threshold existence in broader contexts using this framework.

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Supplemental Information for "Extracting Error Thresholds through the Framework of Approximate Quantum Error Correction Condition"

In this supplementary information,we provide details on

- Sec. SI: Relation with coherent information and entanglement fidelity, including the proof of Lemma 0 and Theorem 1.
- Sec. SII: Application on qudit stabilizer codes, including subsection SII A Definition of stabilizer codes, subsection SII B Proof of Lemma 2, and subsection SII C Maximum likelihood decoder and proof of Theorem 3.
- Sec. SIII: Imperfect measurement prepared toric code

SI. RELATION WITH COHERENT INFORMATION AND ENTANGLEMENT FIDELITY

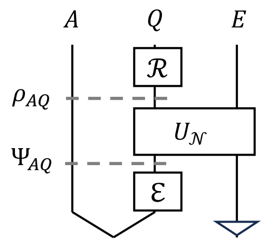

FIG. S1. The circuit that underlies the definition of coherent information and entanglement entropy. Here $\mathcal{E}$ is the encoding channel, $\mathcal{R}$ is the recovery channel, $U_{\mathcal{N}}$ comes from purifying noise channel.

Suppose that we have a state ρ_0 storing logical information and we encode it in the code subspace $\mathcal{C}$. Q labels the physical system of the code. We introduce a reference system A and purify it as $|\Psi_{AQ}\rangle$. The dimension of Hilbert space of A is the code subspace dimension K. The system Q suffers from noise channel $\mathcal{N}(\rho) = \sum_u E_u \rho E_u^\dagger$, we also purify it by introducing an environment E as in Fig. S1. The post-error pure state reads

$$|\Psi'_{AQE}\rangle = \sum_u E_u |\Psi_{AQ}\rangle \otimes |u\rangle_E \,, \tag{S1}$$

such that $\rho_{AQ} = \sum_u E_u |\Psi_{AQ}\rangle \langle \Psi_{AQ}| E_u^\dagger$ after tracing out E. The coherent information is defined as

$$I_c(\rho_0, \mathcal{N} \circ \mathcal{E}) = S(\rho_Q) - S(\rho_{AQ}), \tag{S2}$$

where $S(\rho) = -\operatorname{tr} \rho \log \rho$ is the von Neumann entropy and $\rho_Q = \operatorname{tr}_A \rho_{AQ}$. It satisfies

$$-S(\rho_A) \le I_c(\rho_0, \mathcal{N} \circ \mathcal{E}) \le S(\rho_A), \tag{S3}$$

and characterizes how much entanglement between A and Q is preserved under the noise channel. Assume that we apply a recovery $\mathcal{R}$ on Q, The entanglement fidelity [1–3] is defined as

$$F_e(\rho_0, \mathcal{R} \circ \mathcal{N} \circ \mathcal{E}) = \langle \Psi_{AQ}| \mathcal{R}(\rho_{AQ}) |\Psi_{AQ}\rangle \,, \tag{S4}$$

which quantifies how much quantum information is protected after the entire QEC process. The Nielsen-Schumacher condition for exact recovery $\mathcal{R} \circ \mathcal{N} \circ \mathcal{E} = \mathcal{I}$ is $I_c(\rho_0, \mathcal{N} \circ \mathcal{E}) = S(\rho_A)$, which is equivalent to the KL condition. Actually, it is sufficient to choose ρ_0 as the maximumly mixed state I_K/K to probe exact recovery. In the following sections, we set $\rho_0 = I_K/K$ such that the quantities (S2) and (S4) can be viewed as 'averaged' over all logical states, instead of taking a maximization or minimization as in Ref. [4]. Now the state $|\Psi_{AQ}\rangle$ can be expressed as

$$|\Psi_{AQ}\rangle = \frac{1}{\sqrt{K}} \sum_q |q\rangle_A \otimes |q\rangle_{\mathcal{C}} \,, \tag{S5}$$

where $|q\rangle_{\mathcal{C}}$'s are the logical code words in the logical space and $|q\rangle_A$'s are the corresponding ancilla states. Similarly,

$$\rho_{0Q} \equiv \mathrm{tr}_A |\Psi_{AQ}\rangle \langle\Psi_{AQ}| = \frac{1}{K} |q\rangle_{\mathcal{C}} \langle q|_{\mathcal{C}} = \frac{1}{K} P, \tag{S6}$$

where P is the projection on to the code subspace $\mathcal{C}$. We abbreviate that $I_c = I_c(I_K/K, \mathcal{N} \circ \mathcal{E})$ and $F_e(\mathcal{R} \circ \mathcal{N} \circ \mathcal{E}) = F_e(I_K/K, \mathcal{R} \circ \mathcal{N} \circ \mathcal{E})$.

Lemma 0. *For a QEC code $\mathcal{C}$ under a noise channel $\mathcal{N}$, the AQEC relative entropy is related to the coherent information through:*

$$S(\Lambda + B||\Lambda) = -I_c + \log K. \tag{S7}$$

Proof. Since $|\Psi'_{AQE}\rangle$ is a pure state, the coherent information has an alternative form

$$I_c = S(\rho_{AE}) - S(\rho_E), \tag{S8}$$

where

$$\begin{aligned}
\rho_E &= \mathrm{tr}_{AQ} |\Psi'_{AQE}\rangle \langle\Psi'_{AQE}| \\
&= \sum_{uv} \mathrm{tr}_{AQ} \left(E_u |\Psi_{AQ}\rangle \langle\Psi_{AQ}| E_v^\dagger\right) |u\rangle_E \langle v|_E \\
&= \frac{1}{K} \sum_{uv} \mathrm{tr}_Q \left(E_u P E_v^\dagger\right) |u\rangle_E \langle v|_E \\
&= \frac{1}{K} \sum_{uv} \mathrm{tr}_Q \left(P E_v^\dagger E_u P\right) |u\rangle_E \langle v|_E \\
&= \frac{1}{K} \sum_{uv} \mathrm{tr}_Q \left(\lambda_{vu} P + P B_{vu} P\right) |u\rangle_E \langle v|_E \\
&= \sum_{uv} \lambda_{vu} |u\rangle_E \langle v|_E \,.
\end{aligned} \tag{S9}$$

Notice that we have used $\mathrm{tr}_{\mathcal{C}}(B_{vu}) = 0$. Similarly,

$$\begin{aligned}
\rho_{AE} &= \mathrm{tr}_Q |\Psi'_{AQE}\rangle \langle\Psi'_{AQE}| \\
&= \sum_{uv} \mathrm{tr}_Q \left(E_u |\Psi_{AQ}\rangle \langle\Psi_{AQ}| E_v^\dagger\right) |u\rangle_E \langle v|_E \\
&= \frac{1}{K} \sum_{uv,q_1q_2} \mathrm{tr}_Q \left(E_u |q_1\rangle_{\mathcal{C}} \langle q_2|_{\mathcal{C}} E_v^\dagger\right) |u\rangle_E \langle v|_E \otimes |q_1\rangle_A \langle q_2|_A \\
&= \frac{1}{K} \sum_{uv,q_1q_2} \langle q_2|_{\mathcal{C}} E_v^\dagger E_u |q_1\rangle_{\mathcal{C}} |u\rangle_E \langle v|_E \otimes |q_1\rangle_A \langle q_2|_A \\
&= \sum_{uv} (\Lambda_{vu,q_2q_1} + B_{vu,q_2q_1}) |u\rangle_E \langle v|_E \otimes |q_1\rangle_A \langle q_2|_A \,.
\end{aligned} \tag{S10}$$

Since $\rho_A = I_K/K$, we obtain the relation between the gKL condition and the error correction circuit Fig. S1,

$$\Lambda = (\rho_E \otimes \rho_A)^T, \quad \Lambda + B = \rho_{AE}^T. \tag{S11}$$

The density matrix nature of the Λ and $\Lambda + B$ matrices tells us that they are both positive semi-defined $\Lambda \geq 0$, $\Lambda + B \geq 0$ and trace-one $\mathrm{tr}(\Lambda) = 1$, $\mathrm{tr}(\Lambda + B) = 1$. Thus the AQEC relative entropy is expressed as

$$\begin{aligned}
S(\Lambda + B||\Lambda) &= \mathrm{tr}\left\{(\Lambda + B)\left[\log(\Lambda + B) - \log \Lambda\right]\right\} \\
&= \mathrm{tr}(\Lambda + B) \log(\Lambda + B) - \mathrm{tr}\, \Lambda \log \Lambda \\
&= -S(\rho_{AE}) + S(\rho_A) + S(\rho_E) \\
&= -I_c + \log K,
\end{aligned} \tag{S12}$$

where we again used $\mathrm{tr}_{\mathcal{C}}(B) = 0$.

□

Note that the AQEC relative entropy takes value from

$$0 \leq S(\Lambda + B||\Lambda) \leq 2\log K, \tag{S13}$$

following Eq. (S3).

Theorem 1. *Given a family of $\{\mathcal{C}_n\}_n^\infty$ with noise channels $\{\mathcal{N}_n\}_n^\infty$, consider the large size limit $n \to \infty$,*

(1) Below the intrinsic error threshold, i.e. $\lim_{n\to\infty} S(\Lambda + B||\Lambda) = 0$, there exists a family of recovery map $\{\mathcal{R}_n\}_n$, such that the entanglement fidelity of the whole QEC process satisfies

$$\lim_{n\to\infty} F_e(\mathcal{R} \circ \mathcal{N} \circ \mathcal{E}) = 1. \tag{S14}$$

(2) Above the intrinsic error threshold, i.e. $S(\Lambda + B||\Lambda)$ does not converge to 0, if $K = \mathcal{O}(1)$, then the entanglement fidelity $F_e(\mathcal{R} \circ \mathcal{N} \circ \mathcal{E})$ cannot converges to 1 for an arbitrary family of recovery map $\{\mathcal{R}_n\}_n$.

(3) Let K be a parameter that diverges with n, such that $K = \omega(1)$. If $s(\Lambda + B||\Lambda) \equiv S(\Lambda + B||\Lambda)/\log K$ does not converge to 0, the entanglement fidelity $F_e(\mathcal{R} \circ \mathcal{N} \circ \mathcal{E})$ cannot converges to 1 for an arbitrary family of recovery map $\{\mathcal{R}_n\}_n$.

Proof. Given coherent information I_c, there exist a recovery channel R such that [5]

$$1 \geq F_e(\mathcal{R} \circ \mathcal{N} \circ \mathcal{E}) \geq 1 - \sqrt{2(-I_c + \log K)}, \tag{S15}$$

and for an arbitrary quantum channel $\mathcal{R}$ we have [3]

$$0 \leq -I_c + \log K \leq 2h(1 - F_e(\mathcal{R} \circ \mathcal{N} \circ \mathcal{E})), \tag{S16}$$

where $h(x) = -x\log x - (1-x)\log(1-x) + x\log(K^2 - 1)$. Now for a family of QEC code labeled by system size n and has a limit $n \to \infty$, the above two statements hold for all n, that is there exists a family of recovery channels $\{\mathcal{R}_n\}_n$ such that

$$0 \leq r(\mathcal{R}_n \circ \mathcal{N}_n \circ \mathcal{E}_n) \leq \sqrt{2S(\Lambda_n + B_n||\Lambda_n)}, \tag{S17}$$

while for every possible family of recovery channels $\{\mathcal{R}_n\}_n$,

$$0 \leq S(\Lambda_n + B_n||\Lambda_n) \leq h(r(\mathcal{R}_n \circ \mathcal{N}_n \circ \mathcal{E}_n)), \tag{S18}$$

where we have used Lemma 0. Here the subscripts label the system size, and we define infidelity $r(\mathcal{R}_n \circ \mathcal{N}_n \circ \mathcal{E}_n) = 1 - F_e(\mathcal{R}_n \circ \mathcal{N}_n \circ \mathcal{E}_n)$. Below the intrinsic threshold $\lim_{n\to\infty} S(\Lambda_n + B_n||\Lambda_n) = 0$, Eq. (S17) tells us $\lim_{n\to\infty} r(\mathcal{R}_n \circ \mathcal{N}_n \circ \mathcal{E}_n) = 0$. Above the threshold where $\lim_{n\to\infty} S(\Lambda_n + B_n||\Lambda_n) > 0$ or diverges, $\lim_{n\to\infty} r(\mathcal{R}_n \circ \mathcal{N}_n \circ \mathcal{E}_n)$ must also > 0 or diverges for the finite K case, otherwise Eq. (S18) leads to conflict. For the case K_n depend on n and $K_n \to \infty$ as $n \to \infty$. if the infidelity $r(\mathcal{R}_n \circ \mathcal{N}_n \circ \mathcal{E}_n) \to 1$, we conclude that the density of AQEC relative entropy $s(\Lambda + B||\Lambda) = \frac{1}{\log K_n} S(\Lambda + B||\Lambda)$ converges to zero rather than $S(\Lambda + B||\Lambda)$ itself.

□

Notice that when $K_n \to \infty$, following from Eq. (S13) the AQEC relative entropy might also approach infinity, but the infidelity is always bounded,

$$0 \leq r(\mathcal{R}_n \circ \mathcal{N}_n \circ \mathcal{E}_n) \leq 1. \tag{S19}$$

For large n, $h(r(\mathcal{R}_n \circ \mathcal{N}_n \circ \mathcal{E}_n)) \to 4r(\mathcal{R}_n \circ \mathcal{N}_n \circ \mathcal{E}_n)\log K_n$ and we can only conclude $r(\mathcal{R}_n \circ \mathcal{N}_n \circ \mathcal{E}_n) = \Omega(1/\log K_n)$ from Eq. (S18) if $S(\Lambda_n + B_n||\Lambda_n)$ does not approach zero. Only if $S(\Lambda_n + B_n||\Lambda_n) \to 2c\log K_n$ and $0 < c \leq 1$ is a finite constant can we get a lower bounded infidelity $r(\mathcal{R}_n \circ \mathcal{N}_n \circ \mathcal{E}_n) \geq c/2 > 0$.

We point out that there could be other quantities that measure the magnitude of B, for example using entanglement fidelity [4], operator norm [6, 7] or trace distance [8]. We might alternatively extract the intrinsic error threshold from the asymptotic behavior of these measures. To capture the intrinsic threshold, we postulate that the quantity should be nonlinear with respect to Λ and B. We choose AQEC relative entropy since it is more tractable analytically.

SII. APPLICATION ON QUDIT STABILIZER CODES

In this section, we review qudit stabilizer codes and derive the results on stabilizer codes.

A. Definition of Stabilizer Codes

For a n qudit system with local Hilbert space dimension $d \geq 2$, the Heisenberg-Weyl operator is defined as [9]:

$$T(v) = T(v_p, v_q) = \omega^{-\frac{1}{2} v_p^T v_q} Z^{v_p} X^{v_q}. \tag{S20}$$

Here both v_p and v_q are n dimensional $\mathbb{Z}_d$ valued vectors, and $v = (v_p, v_q) \in \mathbb{Z}_d^{2n}$. $\omega = e^{i2\pi/d}$ is a phase factor. Note that all arithmetic is done modulo d. X and Z can be viewed as a generalized version of Pauli operators, which are defined as

$$X |q\rangle = |q+1\rangle, \quad Z |q\rangle = \omega^q |q\rangle. \tag{S21}$$

Z^{v_p} and X^{v_q} denotes the Weyl strings acting on n qudits

$$Z^{v_p} = \bigotimes_{i=1}^{n} Z^{v_{pi}}, \quad X^{v_q} = \bigotimes_{i=1}^{n} X^{v_{qi}}. \tag{S22}$$

All such $T(v)$ generates the discrete Heisenberg-Weyl group. In general, v can be interpreted as a point in the classical phase space $\mathcal{V} = \mathbb{Z}_d^{2n}$. Specifically, v_q is the coordinate vector and v_p is the momentum vector. The action of $T(v)$ leads to a translation in both the coordinate and momentum space, so it is also called the "Displacement operator". The basic algebraic relation of Weyl operators is

$$T(v)T(u) = \omega^{\frac{1}{2}[v,u]} T(v+u). \tag{S23}$$

Here $[v, u] = v^T \Omega u$ is the symplectic inner product on $\mathcal{V}$, where

$$\Omega = \begin{pmatrix} 0_{n\times n} & I_{n\times n} \\ -I_{n\times n} & 0_{n\times n} \end{pmatrix}. \tag{S24}$$

We can see that T defines a projective representation of $\mathcal{V}$ on the Hilbert space.

Within the phase space formalism, let us define stabilizer codes. Given a subspace $\mathcal{M}$ of $\mathcal{V}$, $\mathcal{M}$ is called isotropic if and only if

$$[m_1, m_2] = 0, \quad \forall m_1, m_2 \in \mathcal{M}. \tag{S25}$$

If $\mathcal{M}$ is isotropic, then all corresponding Weyl operators commute with each other, $[T(m_1), T_(m_2)] = 0$. In that case, T is an isomorphism between $\mathcal{M}$ and an abelian subgroup of the discrete Heisenberg-Weyl group, which is called stabilizer group stabilizer group. Actually, the elements in stabilizer group can be redefined with some phase factor, but we will not keep track of the phases here since they are irrelevant in our discussion.

Given an isotropic subspace $\mathcal{M}$, the corresponding stabilizer group is defined as $T(\mathcal{M})$, which is the image of mapping $\mathcal{M}$ onto the operator space. The associated stabilizer code subspace $\mathcal{C}$ is defined as the maximal subspace of the Hilbert space which satisfies

$$T(m) |\psi\rangle = |\psi\rangle, \quad \forall m \in \mathcal{M}, \quad \forall |\psi\rangle \in \mathcal{C}. \tag{S26}$$

The cardinal of $\mathcal{M}$ has to be less than d^n such that the eigenspace $\mathcal{C}$ has degeneracy. In fact, the dimension of $\mathcal{C}$ (denoted as K) must satisfy $K = d^n/|\mathcal{M}|$ [10]. We specify $\mathcal{M}$ by giving a basis $\{m_1, \cdots, m_r\}$. This basis is mapped to a set of generators $\{T(m_1), \cdots, T(m_r)\}$ of stabilizer group by T.

Here a remark should be made about the local dimension d. If d is a prime number $d = 2, 3, 5, \cdots$, then $\mathbb{Z}_d$ is a field and $\mathcal{V}$ is a true vector space. Thus most of the results in the qubit case can be naturally generalized to prime dimensions. But if d is nonprime, $\mathcal{V}$ is mathematically a $\mathbb{Z}$ module, and subtleties arise in the algebraic structure [9, 10]. For example, consider a single qudit with $d = 4$. While X stabilizes a unique state $(|0\rangle + |1\rangle + |2\rangle + |3\rangle)/\sqrt{4}$, there are two states stabilized by X^2, $(|0\rangle+|2\rangle)/\sqrt{2}$ and $(|1\rangle+|3\rangle)/\sqrt{2}$. In other words, X^2 has degenerate eigenvectors.

Then we must take another operator like Z^2 into account to define a unique stabilizer state $(|0\rangle + |2\rangle)/\sqrt{2}$. Generally in prime dimensions, $\mathcal{M}$ will be a vector space with dimension $r = \log_d |\mathcal{M}|$, spanned by the basis $\{m_1, \cdots, m_r\}$. Therefore the dimension of the code subspace will be $K = d^{n-r} = d^k$, which leads to an $[[n,k]]$ code. But in nonprime dimensions, one might need $r \geq n-k$ generators to construct an $[[n,k]]$ code. Conversely given r generators, the size of $\mathcal{M}$ might be less than d^r, which leads to $K \geq d^{n-r}$. In general, K cannot be written in the form d^k.

Now we want to find the logical operators within the Heisenberg-Weyl group. Notice that any logical operator must commute with all the stabilizers so that they do not affect the error syndrome. We define the symplectic complement of M as [9]

$$\mathcal{M}^{\perp} = \{v \in \mathcal{V} | [m,v] = 0, \quad \forall m \in \mathcal{M}\}. \tag{S27}$$

The suitable logical Weyl operators must be in $\mathcal{M}^{\perp}$. Since the operators in $\mathcal{M}$ act trivially on the code subspace, the space of logical Weyl operators will be chosen as $\mathcal{L} = \mathcal{M}^{\perp}/\mathcal{M}$. For each equivalent class $[l] \in \mathcal{L}$, we choose a representative element l and define the corresponding logical operator as $[l] \mapsto T(l)$. Since the size of $\mathcal{M}^{\perp}$ satisfies $|\mathcal{M}^{\perp}| = d^{2n}/|\mathcal{M}| = d^{n+k}$ [9], we have $|\mathcal{L}| = K^2$. The code distance δ is defined as the minimal weight of a nontrivial logical operator $T(l), l \in \mathcal{M}^{\perp} - \mathcal{M}$, while 'weight' means the number of data qudits that $T(l)$ nontrivially acts on.

From now on we consider only $[[n,k,\delta]]$ code in prime local dimensions d for simplicity. It follows that $\mathcal{V}$ is a $2n$ dimensional $\mathbb{Z}_d$ vector space, and $\mathcal{M}$, $\mathcal{M}^{\perp}$ are respectively its $r = n-k$ and $n+k$ dimensional subspace. $\mathcal{L}$ is a quotient vector space with dimension $2k$. We abbreviate the logical class $[l]$ as its representative element l without leading to misunderstanding.

B. Proof of Lemma 2

Lemma 2. *For a qudit stabilizer code in prime d and a stochastic Weyl noise channel, the AQEC relative entropy, probability of logical classes and order parameter of the SM model are related by*

$$S(\Lambda + B||\Lambda) = H(L|S) = \sum_{\eta \in \mathcal{V}} \Pr(\eta) \log \left\{ \sum_{l \in \mathcal{L}} \exp\left[-\Delta(\eta, l)\right] \right\}, \tag{S28}$$

where $H(L|S) = -\sum_{s,l} \Pr(s,l) \log \Pr(l|s)$ is the Shannon conditional entropy of logical class l given syndrome s, and the last quantity is a generalized version of the homological difference [11].

Proof. First, we evaluate the expression for the matrices Λ and $\Lambda + B$. Consider the stochastic Weyl error

$$\mathcal{N}(\rho) = \sum_{u \in \mathcal{V}} \Pr(u) T(u) \rho T(u)^{\dagger}, \tag{S29}$$

where $\Pr(u)$ is a probabilistic distribution on $\mathcal{V}$, $0 \leq \Pr(u) \leq 1$, $\sum_{u \in \mathcal{V}} \Pr(u) = 1$. Now P is the projection onto the stabilizer code subspace $\mathcal{C}$ and the Kraus operator of stochastic Weyl error channel reads $E_u = \sqrt{\Pr(u)} T(u)$. Then we have

$$\begin{aligned}
PE_u^{\dagger} E_v P &= \frac{1}{d^{2r}} \sum_{m,m' \in \mathcal{M}} \sqrt{\Pr(u)\Pr(v)} T(m) T(u)^{\dagger} T(v) T(m') \\
&= \frac{1}{d^{2r}} \sum_{m,m' \in \mathcal{M}} \sqrt{\Pr(u)\Pr(v)} \omega^{-\frac{1}{2}[u,v]} \omega^{[m,v-u]} T(v-u) T(m+m') \\
&= \frac{1}{d^{r}} \sum_{m'' \in \mathcal{M}} \sqrt{\Pr(u)\Pr(v)} \omega^{-\frac{1}{2}[u,v]} \delta_{\mathcal{M}^{\perp}}(v-u) T(v-u) T(m'') \\
&= \sqrt{\Pr(u)\Pr(v)} \omega^{-\frac{1}{2}[u,v]} \delta_{\mathcal{M}^{\perp}}(v-u) T(v-u) P \\
&= \sqrt{\Pr(u)\Pr(v)} \omega^{-\frac{1}{2}[u,v]} \delta_{\mathcal{M}}(v-u) P \\
&+ \sqrt{\Pr(u)\Pr(v)} \omega^{-\frac{1}{2}[u,v]} \delta_{\mathcal{M}^{\perp} - \mathcal{M}}(v-u) P T(v-u) P.
\end{aligned} \tag{S30}$$

In the third line we have used Lemma 9 in Ref. [9]

$$\frac{1}{|\mathcal{M}|}\sum_{m\in\mathcal{M}}\omega^{[v,m]}=\delta_{\mathcal{M}^\perp}(v)=\begin{cases}1 & v\in\mathcal{M}^\perp\\ 0 & \text{else.}\end{cases} \tag{S31}$$

Here $\delta_{\mathcal{M}^\perp-\mathcal{M}}(v-u)=1$ only when $v-u\in\mathcal{M}^\perp$ but $\notin\mathcal{M}$. When $v-u\in\mathcal{M}$ the Weyl operator $T(v-u)$ is just a stabilizer and absorbed into P, which leads to the λ_{uv} term. When $v-u\in\mathcal{M}^\perp$ but $\notin\mathcal{M}$, $T(v-u)$ is a logical operator which causes logical error, hence it corresponds to the B_{uv} term. Therefore,

$$\begin{aligned}\lambda_{uv}&=\sqrt{\Pr(u)\Pr(v)}\omega^{-\frac{1}{2}[u,v]}\delta_{\mathcal{M}}(v-u),\\ B_{uv}&=\sqrt{\Pr(u)\Pr(v)}\omega^{-\frac{1}{2}[u,v]}\delta_{\mathcal{M}^\perp-\mathcal{M}}(v-u)T(v-u).\end{aligned} \tag{S32}$$

We will show that B_{uv} is suppressed in the ordered phase in an average sense.

In order to calculate the AQEC relative entropy, we apply the replica method. Owing to $\mathrm{tr}_{\mathcal{C}}(B)=0$, we have

$$S(\Lambda+B||\Lambda)=\mathrm{tr}(\Lambda+B)\log(\Lambda+B)-\mathrm{tr}\,\Lambda\log\Lambda=S(\Lambda)-S(\Lambda+B), \tag{S33}$$

which is the difference between the two von Neumann entropies $S(\Lambda)$ and $S(\Lambda+B)$. We extend both of them to Rényi entropies,

$$\begin{aligned}&S^{(R)}(\Lambda+B||\Lambda)=S^{(R)}(\Lambda)-S^{(R)}(\Lambda+B)\\ &S^{(R)}(\rho)=\frac{1}{1-R}\log\mathrm{tr}(\rho^R),\end{aligned} \tag{S34}$$

which is holomorphic in $R\in\mathbb{C}$ for $\mathrm{Re}\,R\geq 1$ [12]. The von Neumann entropy can be extracted from the Rényi entropy through the $R\to 1$ limit,

$$S(\rho)=\lim_{R\to 1}S^{(R)}(\rho)=S^{(1)}(\rho)=-\frac{\mathrm{d}}{\mathrm{d}R}\mathrm{tr}(\rho^R)\bigg|_{R\to 1}. \tag{S35}$$

Now we compute the Rényi entropies for integer $R>1$, then carefully perform analytic continuation and take the $R\to 1$ limit $\lim_{R\to 1}S^{(R)}(\Lambda+B||\Lambda)=S(\Lambda+B||\Lambda)$. We first compute $\mathrm{tr}(\Lambda^R)$,

$$\begin{aligned}\mathrm{tr}(\Lambda^R)&=\frac{1}{d^{k(R-1)}}\left(\prod_{\alpha=1}^{R}\sum_{\eta^{(\alpha)}\in\mathcal{V}}\right)\prod_{\alpha=1}^{R}\lambda_{\eta^{(\alpha)}\eta^{(\alpha+1)}}\\ &=\frac{1}{d^{k(R-1)}}\left(\prod_{\alpha=1}^{R}\sum_{\eta^{(\alpha)}\in\mathcal{V}}\right)\prod_{\alpha=1}^{R}\sqrt{\Pr(\eta^{(\alpha)})\Pr(\eta^{(\alpha+1)})}\omega^{-\frac{1}{2}[\eta^{(\alpha)},\eta^{(\alpha+1)}]}\delta_{\mathcal{M}}(\eta^{(\alpha+1)}-\eta^{(\alpha)})\\ &=\frac{1}{d^{k(R-1)}}\left(\sum_{\eta\in\mathcal{V}}\prod_{\alpha=1}^{R-1}\sum_{m^{(\alpha)}\in\mathcal{M}}\right)\Pr(\eta)\prod_{\alpha=1}^{R-1}\Pr(\eta+m^{(\alpha)})\prod_{\alpha=1}^{R}\omega^{-\frac{1}{2}[\eta+m^{(\alpha)},\eta+m^{(\alpha+1)}]}\\ &=\frac{1}{d^{k(R-1)}}\left(\sum_{\eta\in\mathcal{V}}\prod_{\alpha=1}^{R-1}\sum_{m^{(\alpha)}\in\mathcal{M}}\right)\Pr(\eta)\prod_{\alpha=1}^{R-1}\Pr(\eta+m^{(\alpha)})\prod_{\alpha=1}^{R}\omega^{-\frac{1}{2}[\eta,m^{(\alpha+1)}]+\frac{1}{2}[\eta,m^{(\alpha)}]}\\ &=\frac{1}{d^{k(R-1)}}\left(\sum_{\eta\in\mathcal{V}}\prod_{\alpha=1}^{R-1}\sum_{m^{(\alpha)}\in\mathcal{M}}\right)\Pr(\eta)\prod_{\alpha=1}^{R-1}\Pr(\eta+m^{(\alpha)})\\ &=\frac{1}{d^{k(R-1)}}\sum_{\eta\in\mathcal{V}}\Pr(\eta)\left[\sum_{m\in\mathcal{M}}\Pr(\eta+m)\right]^{R-1}.\end{aligned} \tag{S36}$$

Here α denotes the replica index and we identify $R+1=1$. $\eta^{(\alpha)}$ is the error configuration for the copy α. Similarly we compute $\mathrm{tr}((\Lambda+B)^R)$,

$$
\begin{aligned}
\mathrm{tr}((\Lambda+B)^R) &= \frac{1}{d^{kR}}\left(\prod_{\alpha=1}^{R}\sum_{\eta^{(\alpha)}\in\mathcal{V}}\right)\prod_{\alpha=1}^{R}\sqrt{\Pr(\eta^{(\alpha)})\Pr(\eta^{(\alpha+1)})}\omega^{-\frac{1}{2}[\eta^{(\alpha)},\eta^{(\alpha+1)}]}\delta_{\mathcal{M}^\perp}(\eta^{(\alpha+1)}-\eta^{(\alpha)})\,\mathrm{tr}_{\mathcal{C}}\left(\prod_{\alpha=1}^{R}T(\eta^{(\alpha+1)}-\eta^{(\alpha)})\right)\\
&= \frac{1}{d^{kR}}\left(\sum_{\eta\in\mathcal{V}}\prod_{\alpha=1}^{R-1}\sum_{m^{(\alpha)}\in\mathcal{M}}\sum_{l^{(\alpha)}\in\mathcal{L}}\right)\Pr(\eta)\prod_{\alpha=1}^{R-1}\Pr(\eta+m^{(\alpha)}+l^{(\alpha)})\prod_{\alpha=1}^{R}\omega^{-\frac{1}{2}[\eta+m^{(\alpha)}+l^{(\alpha)},\eta+m^{(\alpha+1)}+l^{(\alpha+1)}]}\\
&\times\mathrm{tr}_{\mathcal{C}}\left(\prod_{\alpha=1}^{R}T(m^{(\alpha+1)}+l^{(\alpha+1)}-m^{(\alpha)}-l^{(\alpha)})\right)\\
&= \frac{1}{d^{kR}}\left(\sum_{\eta\in\mathcal{V}}\prod_{\alpha=1}^{R-1}\sum_{m^{(\alpha)}\in\mathcal{M}}\sum_{l^{(\alpha)}\in\mathcal{L}}\right)\Pr(\eta)\prod_{\alpha=1}^{R-1}\Pr(\eta+m^{(\alpha)}+l^{(\alpha)})\prod_{\alpha=1}^{R}\omega^{-\frac{1}{2}[l^{(\alpha)},l^{(\alpha+1)}]}\\
&\times\mathrm{tr}_{\mathcal{C}}\left(\prod_{\alpha=1}^{R}T(m^{(\alpha+1)}+l^{(\alpha+1)}-m^{(\alpha)}-l^{(\alpha)})\right).
\end{aligned}
\tag{S37}
$$

Here $\mathrm{tr}_{\mathcal{C}}(*)=\sum_q \langle q|_{\mathcal{C}} * |q\rangle_{\mathcal{C}}$ denotes the trace over the code subspace. Now we evaluate the $\mathrm{tr}_{\mathcal{C}}$ term. Using the fact stabilizers are symplectic orthogonal to logical operators $[m,l]=0$ and acts trivially on the logical states $T(m)\,|q\rangle_{\mathcal{C}}=|q\rangle_{\mathcal{C}}$, we obtain

$$
\mathrm{tr}_{\mathcal{C}}\left(\prod_{\alpha=1}^{R}T(m^{(\alpha+1)}+l^{(\alpha+1)}-m^{(\alpha)}-l^{(\alpha)})\right)=\mathrm{tr}_{\mathcal{C}}\left(\prod_{\alpha=1}^{R}T(l^{(\alpha+1)}-l^{(\alpha)})\right).
\tag{S38}
$$

Then use the algebraic relation Eq. (S23), we have

$$
\begin{aligned}
\mathrm{tr}_{\mathcal{C}}\left(\prod_{\alpha=1}^{R}T(l^{(\alpha+1)}-l^{(\alpha)})\right) &= \omega^{\frac{1}{2}\sum_{\alpha=2}^{R}[l^{(\alpha)}-l^{(1)},l^{(\alpha+1)}-l^{(\alpha)}]}\,\mathrm{tr}_{\mathcal{C}}\left(T\left(\sum_{\alpha=1}^{R}(l^{(\alpha+1)}-l^{(\alpha)})\right)\right)\\
&= d^k\omega^{\frac{1}{2}\sum_{\alpha=1}^{R}[l^{(\alpha)},l^{(\alpha+1)}]}.
\end{aligned}
\tag{S39}
$$

Therefore we obtain

$$
\begin{aligned}
\mathrm{tr}((\Lambda+B)^R) &= \frac{1}{d^{k(R-1)}}\left(\sum_{\eta\in\mathcal{V}}\prod_{\alpha=1}^{R-1}\sum_{m^{(\alpha)}\in\mathcal{M}}\sum_{l^{(\alpha)}\in\mathcal{L}}\right)\Pr(\eta)\prod_{\alpha=1}^{R-1}\Pr(\eta+m^{(\alpha)}+l^{(\alpha)})\\
&= \frac{1}{d^{k(R-1)}}\sum_{\eta\in\mathcal{V}}\Pr(\eta)\left[\sum_{m\in\mathcal{M}}\sum_{l\in\mathcal{L}}\Pr(\eta+m+l)\right]^{R-1}.
\end{aligned}
\tag{S40}
$$

The Equations (S36) and (S40) now only hold for integer $R\geq 1$ (the case $R=1$ is trivial). We extend these equalities to $\mathrm{Re}\,R\geq 1$ by applying Carlson's theorem [12–14]. Specifically, for Eq. (S36), the last expression can be analytically continued to a holomorphic function on $\mathrm{Re}\,R\geq 1$,

$$
f(R)=\frac{1}{d^{k(R-1)}}\sum_{\eta\in\mathcal{V}}\Pr(\eta)\left[\sum_{m\in\mathcal{M}}\Pr(\eta+m)\right]^{R-1}.
\tag{S41}
$$

For $\mathrm{Re}\,R\geq 1$, since $\Pr(\eta)$ is a probability distribution, we have $0\leq\sum_{m\in\mathcal{M}}\Pr(\eta+m)\leq 1$ and thus

$$
\left|\left[\sum_{m\in\mathcal{M}}\Pr(\eta+m)\right]^{R-1}\right|\leq 1.
\tag{S42}
$$

It follows that

$$\left|\sum_{\eta\in\mathcal{V}}\Pr(\eta)\left[\sum_{m\in\mathcal{M}}\Pr(\eta+m)\right]^{R-1}\right|\leq\sum_{\eta\in\mathcal{V}}\Pr(\eta)\left|\left[\sum_{m\in\mathcal{M}}\Pr(\eta+m)\right]^{R-1}\right|\leq 1. \tag{S43}$$

So the norm of $f(R)$ is bounded,

$$|f(R)|\leq|d^{-k(R-1)}|\leq 1, \tag{S44}$$

for constants $d\geq 2$ and $k\geq 0$. For the l.h.s., we already know that $\operatorname{tr}(\Lambda^R)$ is bounded in absolute value by 1 for $\operatorname{Re}R\geq 1$, $|\operatorname{tr}(\Lambda^R)|\leq 1$. So $|\operatorname{tr}(\Lambda^R)-f(R)|$ is bounded by 2. The difference $\operatorname{tr}(\Lambda^R)-f(R)$ satisfies the requirements of Carlson's theorem [15], thus we conclude that $\operatorname{tr}(\Lambda^R)=f(R)$ for $\operatorname{Re}R\geq 1$. A similar argument can be applied to Eq. (S40) such that it also holds for $\operatorname{Re}R\geq 1$.

The AQEC relative entropy is then obtained by taking the derivative at $R=1$,

$$\begin{aligned}
S(\Lambda+B||\Lambda)&=\frac{\mathrm{d}}{\mathrm{d}R}\operatorname{tr}((\Lambda+B)^R)\Big|_{R\to 1}-\frac{\mathrm{d}}{\mathrm{d}R}\operatorname{tr}(\Lambda^R)\Big|_{R\to 1}\\
&=\frac{\mathrm{d}}{\mathrm{d}R}\sum_{\eta\in\mathcal{V}}\Pr(\eta)\left[\sum_{m\in\mathcal{M}}\sum_{l\in\mathcal{L}}\Pr(\eta+m+l)\right]^{R-1}\Bigg|_{R\to 1}-\frac{\mathrm{d}}{\mathrm{d}R}\sum_{\eta\in\mathcal{V}}\Pr(\eta)\left[\sum_{m\in\mathcal{M}}\Pr(\eta+m)\right]^{R-1}\Bigg|_{R\to 1}\\
&=\sum_{\eta\in\mathcal{V}}\Pr(\eta)\log\frac{\sum_{m\in\mathcal{M}}\sum_{l\in\mathcal{L}}\Pr(\eta+m+l)}{\sum_{m\in\mathcal{M}}\Pr(\eta+m)}.
\end{aligned} \tag{S45}$$

The connection to the Shannon conditional entropy of MLD and generalized homological difference follows straightforwardly from the discussions in the subsequent sections. □

C. Maximum likelihood decoder

For an arbitrary Weyl error configuration $\eta\in\mathcal{V}$, it can be decomposed into the for $\eta=s+l+m$, where $s\in\mathcal{S}=\mathcal{V}/\mathcal{M}^{\perp}$ is the representative error configuration of a particular syndrome, $m\in\mathcal{M}$ and $l\in\mathcal{L}$. Note that $\mathcal{V}/\mathcal{M}^{\perp}\cong(\mathcal{V}/\mathcal{M})/\mathcal{L}$ and $\dim\mathcal{S}=n-k$. Since $\dim\mathcal{S}+\dim\mathcal{L}+\dim\mathcal{M}=\dim\mathcal{V}$, the decomposition is unique. The choice of the representative element of $s\in\mathcal{S}$ and $l\in\mathcal{L}$ can be arbitrary. Each sydrome s contains d^k logical classes l while each logical class contains d^{n-k} stabilizers m. Since stabilziers acts trivially on code subspace, we only concern about the probability of logical classes when deciding the recovery operator. We define the joint probability of two random variables syndrome S and logical class L as

$$\Pr(s,l)=\sum_{m\in\mathcal{M}}\Pr(s+l+m). \tag{S46}$$

In the process of maximum likelihood decoding [16], each syndrome s appears with probability $\Pr(s)=\sum_{l\ in\mathcal{L}}\Pr(s,l)$. After measuring The charge of stabilizer generators $i\log T(m_a)$, $a=1,\cdots,r$, we faithfully obtain the information of syndrome since $\dim\mathcal{S}=\dim\mathcal{M}$. Then what we need to do is choosing recovery operator according to the syndrome s. The maximum likelihood decoder (MLD) make the choice up to stabilizers and assign the logical class with the largest conditional probability

$$\Pr(l|s)=\Pr(s,l)/\Pr(s) \tag{S47}$$

to the required recovery operator $T(s+l)$. Without losing generality, we re-choose the representative elements of $s\in\mathcal{S}$ in the r.h.s. of Eq. (S46) such that $\Pr(l|s)$ is maximized by $l=0$. The success rate of MLD is commonly measured by the probability of $l=0$ logical class,

$$\Pr(l)=\sum_{s\in\mathcal{S}}\Pr(s,l), \tag{S48}$$

$$\Pr(\text{MLD success})=\Pr(l=0). \tag{S49}$$

We also concern about another measure, which is the Shannon entropy of L conditioned on S,

$$H(L|S) = -\sum_{s\in\mathcal{S}}\sum_{l\in\mathcal{L}} \Pr(s,l)\log\Pr(l|s) = \sum_{s\in\mathcal{S}} \Pr(s) H(L|S=s), \tag{S50}$$

where $H(L|S=s)$ is the Shannon entropy when syndrome s is specified,

$$H(L|S=s) = -\sum_{l\in\mathcal{L}} \Pr(l|s)\log\Pr(l|s). \tag{S51}$$

$H(L|S)$ measures the uncertainty of the decoder's choice.

Now we apply the above definitions to Eq. (S45) and find that

$$\begin{aligned}
S(\Lambda+B||\Lambda) &= \sum_{\eta\in\mathcal{V}} \Pr(\eta)\log\frac{\sum_{m\in\mathcal{M}}\sum_{l\in\mathcal{L}}\Pr(\eta+m+l)}{\sum_{m\in\mathcal{M}}\Pr(\eta+m)} \\
&= \sum_{s\in\mathcal{S}}\sum_{l'\in\mathcal{L}}\sum_{m'\in\mathcal{M}} \Pr(s+l'+m')\log\frac{\sum_{m\in\mathcal{M}}\sum_{l\in\mathcal{L}}\Pr(s+l'+m'+m+l)}{\sum_{m\in\mathcal{M}}\Pr(s+l'+m'+m)} \\
&= \sum_{s\in\mathcal{S}}\sum_{l'\in\mathcal{L}}\sum_{m'\in\mathcal{M}} \Pr(s+l'+m')\log\frac{\Pr(s)}{\Pr(s,l')} \\
&= -\sum_{s\in\mathcal{S}}\sum_{l'\in\mathcal{L}} \Pr(s,l')\log\Pr(l'|s) \\
&= H(L|S),
\end{aligned} \tag{S52}$$

i.e. the AQEC relative entropy is equal to the Shannon conditional entropy of the decoding process.

Theorem 3. *For a family of qudit stabilizer codes in prime d and stochastic Weyl noise channels, in the large size limit $n\to\infty$,*

(1) below the intrinsic error threshold, i.e. $\lim_{n\to\infty} S(\Lambda+B||\Lambda)=0$, the success probability of MLD converges to 1*,*

$$\lim_{n\to\infty}\Pr(l=0)=1. \tag{S53}$$

(2) if the logical qudit number is finite $k=\mathcal{O}(1)$, then $\lim_{n\to\infty}\Pr(l=0)=1$ implies the QEC system is below the intrinsic error threshold $\lim_{n\to\infty} S(\Lambda+B||\Lambda)=0$.

(3) if k diverges with n, $k=\omega(1)$, then $\lim_{n\to\infty}\Pr(l=0)=1$ implies the density of AQEC relative entropy $s(\Lambda+B||\Lambda)\equiv S(\Lambda+B||\Lambda)/k$ converges to 0*.*

Proof. Assume $\lim_{n\to\infty} S(\Lambda+B||\Lambda)=0$, the first proposition follows from the concavity of logarithmic function,

$$\begin{aligned}
1 \ge \Pr(l=0) &= \sum_{s\in\mathcal{S}} \Pr(s)\Pr(l=0|s) \ge \exp\left[\sum_{s\in\mathcal{S}}\Pr(s)\log\Pr(l=0|s)\right] \\
&= \exp\left[\sum_{s\in\mathcal{S}}\sum_{l\in\mathcal{L}}\Pr(s,l)\log\Pr(0|s)\right] \ge \exp\left[\sum_{s\in\mathcal{S}}\sum_{l\in\mathcal{L}}\Pr(s,l)\log\Pr(l|s)\right] = \exp\left[-H(L|S)\right].
\end{aligned} \tag{S54}$$

Note that we have used $\Pr(0|s)\ge\Pr(l|s)$, $\forall l\in\mathcal{L}$. So $\lim_{n\to\infty}\Pr(l=0)=1$.

The second and third propositions follow that conditioning reduces entropy,

$$0 \le H(L|S) \le H(L), \tag{S55}$$

where $H(L) = -\sum_{l\in\mathcal{L}}\Pr(l)\log\Pr(l)$. Fixing the success probability $\Pr(l=0)$, the Shannon entropy $H(L)$ is maximized when the remaining logical classes are uniformly distributed,

$$0 \le H(L|S) \le H(L) \le -\Pr(l=0)\log\Pr(l=0) - (K^2-1)\frac{1-\Pr(l=0)}{K^2-1}\log\frac{1-\Pr(l=0)}{K^2-1} = h(1-\Pr(l=0)). \tag{S56}$$

The above inequality is called the classical Fano's inequality [17]. For finite $k = \log K/\log d$, $\Pr(l=0) \to 1$ implies $\lim_{n\to\infty} S(\Lambda+B||\Lambda) = \lim_{n\to\infty} H(L|S) = 0$. For $k \to \infty$, $\Pr(l=0) \to 1$ in turn implies $\lim_{n\to\infty} s(\Lambda+B||\Lambda) = 0$.
□

D. Statistical Mechanical Mapping

We then reveal the relation between Eq. (S45) to statistical mechanical (SM) mapping proposed in Ref. [16]. They write the probability of error equivalent classes $[\eta] \in \mathcal{V}/\mathcal{M}$ into classical partition functions,

$$\begin{aligned} &Z(\eta) = \Pr([\eta]) = \Pr(s,l) = \sum_{m\in\mathcal{M}} \Pr(\eta+m) = \sum_{c\in\mathbb{Z}_d^r} \exp\left(-H_\eta(c)\right), \\ &H_\eta(c) = -\log \Pr(\eta + Mc), \quad c \in \mathbb{Z}_d^r, \end{aligned} \tag{S57}$$

were $M = (m_1, \cdots, m_r)$ is the $n \times r$ matrix where each column is a basis vector of $\mathcal{M}$, $c \in \mathbb{Z}_d^r$ serves as the SM d.o.f. and η inherits a quenched disorder configuration from error probability $\Pr(\eta)$. For example, for single-qudit noise channels $\mathcal{N} = \otimes_{i=1}^n \mathcal{N}_i$ where $\mathcal{N}_i(\rho_0) = \sum_{\eta_i\in\mathbb{Z}_d^2} p_i(\eta_i) T(\eta_i)\rho_0 T(\eta_i)^\dagger$, the Hamiltonian H_η has a more explicit form

$$\begin{aligned} H_\eta(c) &= -\sum_{i=1}^n \sum_{v_i\in\mathbb{Z}_d^2} J_i(v_i)\omega^{[v_i,\eta_i+M_ic]}, \\ J_i(v_i) &= \frac{1}{d^2}\sum_{u_i\in\mathbb{Z}_d^2} \omega^{-[v_i,u_i]}\log p_i(u_i), \end{aligned} \tag{S58}$$

Where M_i is the i-th row of M. Or suppose that the noises are locally correlated and factored into the form

$$\Pr(\eta) = \prod_R p_R(\eta_R), \tag{S59}$$

where R labels possibly overlapped regions on the lattice and η_R is the restriction of η on the subsystem R. In this case the SM model becomes

$$\begin{aligned} H_\eta(c) &= -\sum_R \sum_{v_R\in\mathbb{Z}_d^2} J_R(v_R)\omega^{[v_R,\eta_R+M_Rc]}, \\ J_R(v_R) &= \frac{1}{d^{2|R|}}\sum_{u_R\in\mathbb{Z}_d^2} \omega^{-[v_R,u_R]}\log p_R(u_R), \end{aligned} \tag{S60}$$

where M_R is now a $|R|\times r$ matrix constituted by the $i \in R$ rows of M. Suppose that the error rate of $\mathcal{N}$ can be tuned, then the error threshold of MLD is identified with the order-disorder phase transition of the SM model. The order parameter of such a phase transition can be chosen as the average free energy cost of a nontrivial logical operator $l \neq 0$,

$$\overline{\Delta_l} = \sum_{\eta\in\mathcal{V}} \Pr(\eta)\Delta(\eta,l), \quad \Delta(\eta,l) = -\log\frac{Z(\eta+l)}{Z(\eta)}, \tag{S61}$$

with $\Delta(\eta,l)$ the free energy cost under a particular disorder configuration η. It diverges below the decoding threshold and in most cases finite above the decoding threshold. Notice that for $l=0$, $\Delta(\eta,l)$ is trivially 0.

Now we substitute (S57) into (S36) and (S40) and find that

$$\mathrm{tr}(\Lambda^R) = \frac{1}{d^{k(R-1)}}\sum_{\eta\in\mathcal{V}} \Pr(\eta)Z(\eta)^{R-1}, \tag{S62}$$

$$\mathrm{tr}((\Lambda+B)^R) = \frac{1}{d^{k(R-1)}}\sum_{\eta\in\mathcal{V}} \Pr(\eta)\left[\sum_{l\in\mathcal{L}} Z(\eta+l)\right]^{R-1}. \tag{S63}$$

The r.h.s. of Eq. (S62) is the classical replica partition function of the disorder SM model. In classical spin glass

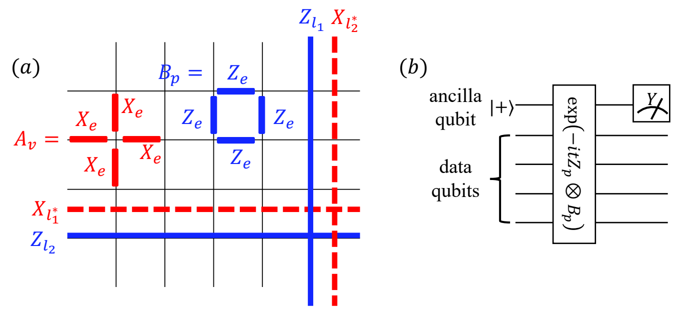

FIG. S2. (a) Toric code defined on 2D periodic lattice. Physical qubits stay on the edges of the lattice. The two kinds of stabilizers are A_v defined on each vertex and B_p defined on each plaquette as shown in the figure. The logical Pauli Z operators Z_{l_1} and Z_{l_2} are product of Z's along non-contractible loops. Correspondingly the logical Pauli X operators $X_{l_1^*}$ and $X_{l_2^*}$ are defined as X's along non-contractible loops on the dual lattice. (b) A circuit model of B_{p_0} measurement circuit.

theories, in order to compute the disorder averaged free energy $F = -\sum_\eta \Pr(\eta) \log Z(\eta)$ in an accessible way, one often averages the R-th power of partition function first and then take the $R \to 1$ limit after analytical continuation,

$$F = \lim_{R\to 1} \frac{1 - \sum_\eta \Pr(\eta) Z(\eta)^{R-1}}{R-1}. \tag{S64}$$

The replica partition function is defined as $Z^{(R-1)} = \sum_\eta \Pr(\eta) Z(\eta)^{R-1}$. The r.h.s. of Eq. (S63) can also be viewed as the replica partition function with additional SM d.o.f. labeled by logical classes l.

Similarly, the AQEC relative entropy Eq. (S45) can be expressed as

$$S(\Lambda + B||\Lambda) = \sum_{\eta\in\mathcal{V}} \Pr(\eta) \log \frac{\sum_{l\in\mathcal{L}} Z(\eta + l)}{Z(\eta)} = \sum_{\eta\in\mathcal{V}} \Pr(\eta) \log \left\{ \sum_{l\in\mathcal{L}} \exp\left[-\Delta(\eta, l)\right] \right\}. \tag{S65}$$

This expression measures the free energy difference between the SM models with or without including the logical class d.o.f. l, and is a generalized version of homological difference defined in [11].

Now assume we are dealing with a qubit LDPC code with $K = 2^k$ and infinitely many logical qubits $k \to \infty$. Intuitively, AQEC relative entropy characterizes how many logical qudits suffer from logical errors. Suppose that $\Delta(\eta, l) \sim 0$ for all l, we have a divergent $S(\Lambda + B||\Lambda) \sim 2k \log 2$ and $1 - F_e \geq 1/2$, $1 - \Pr(l = 0) \geq 1/2$ from Eq. (S14) and (S56). But if $\Delta(\eta, l) \sim 0$ only for a finite number of logical classes l and otherwise $\Delta(\eta, l) \to +\infty$, we get $S(\Lambda + B||\Lambda) \sim c \log 2$ where c is the number of failed logical qudits. So despite there being an incompatibility between AQEC relative entropy and fidelity or probability measures in Thm. 1 and Thm. 3 above the threshold, the AQEC relative entropy can still be a proper measure, while it is stronger in determining the recoverable region.

SIII. IMPERFECT MEASUREMENT PREPARED TORIC CODE

Now we consider the case of toric code on a 2D periodic lattice. The data qubits are located at edges e, and the stabilizer generators are

$$A_v = \prod_{e|v\in\partial e} X_e, \quad B_p = \prod_{e\in\partial p} Z_e, \tag{S66}$$

as in Fig. S2 (a). Normally the initial code states are prepared by measuring the stabilizer generators on a product state and projecting it onto the $+1$ subspace, for example,

$$|{+}{+}\rangle_L = \prod_p \frac{I + B_p}{2} |+\rangle^{\otimes n}. \tag{S67}$$

However, the measurement procedure might suffer from imperfection in reality. We need to entangle the data qubits with an ancilla qubit and then measure the ancilla in order to perform a four-qubit measurement, but the entanglement gate might acquire coherent error, leading to general positive operator-valued measurement (POVM) rather than projective measurement. We consider the following model for imperfect measurement [18–20] as in Fig. S2 (b),

1. prepare the ancilla qubit in $|+\rangle$ state for each plaquette p;

2. apply a joint time evolution involving each ancilla and its four neighboring data qubits $\exp[-itZ_p \otimes B_p]$ where Z_p is the Pauli Z acting on ancilla at p and we assume $0 \le t \le \pi/4$;

3. perform projective measurement on ancilla in Y basis.

The resulting measurement operator on data qubits is no longer a projection operator but instead

$$M_{\{s_p\}} = \frac{1}{(\sqrt{2}\cosh\beta)^{\frac{n}{2}}} \exp\left[\frac{1}{2}\beta \sum_p s_p B_p\right], \tag{S68}$$

up to an irrelevant phase factor. Here $s_p = \pm$ denotes the binary outcomes for every plaquette p and $\tanh\beta/2 = \tan t$. The projective measurement is recovered when $t \to \pi/4$ or $\beta \to +\infty$. We therefore assume the imperfect code subspace is spanned by (fixing the position of logical operators)

$$\begin{aligned} |{+}{+}\rangle_L &= \frac{M_{\{s_p=+\}}\,|+\rangle^{\otimes n}}{\sqrt{\langle+|^{\otimes n} M^\dagger_{\{s_p=+\}} M_{\{s_p=+\}}\,|+\rangle^{\otimes n}}} \propto \exp\left[\frac{1}{2}\beta B_p\right]|+\rangle^{\otimes n}, \\ |{-}{+}\rangle_L &= Z_{l_1}\,|{+}{+}\rangle_L, \quad |{+}{-}\rangle_L = Z_{l_2}\,|{+}{+}\rangle_L, \quad |{-}{-}\rangle_L = Z_{l_1}Z_{l_2}\,|{+}{+}\rangle_L. \end{aligned} \tag{S69}$$

It is able to verify that the above four states are orthogonal to each other. This model is a rather simplified one which is easier to study analytically, but it can capture the fundamental influence of imperfect stabilizer measurement on QEC. In Ref. [18, 19] it is shown that these states lose long-range entanglement or topological order, and in Ref. [20] we showed that it is undecodable under single-qubit Pauli X noises through a common multi-round syndrome measurement protocol. Here we analyze this model using AQEC condition and AQEC relative entropy to extract an optimal threshold.

We assume a single-qubit Pauli X noise channel,

$$\mathcal{N}(\rho) = \prod_e \mathcal{N}_e(\rho), \quad \mathcal{N}_e(\rho) = (1-p)\rho + pX_e\rho X_e. \tag{S70}$$

The corresponding Kraus operator takes the form

$$E_{c^*} = \sqrt{\Pr(c^*)}X_{c^*} = \sqrt{p^{|c^*|}(1-p)^{n-|c^*|}}X_{c^*}, \tag{S71}$$

where we view Pauli X error strings as cochain on the lattice [21], and $X_{c^*} = \prod_{e\in c^*} X_e$. In order to compute the AQEC relative entropy, we need to know about the quantity

$$\langle q_1|_L\, X_{c^*}\, |q_2\rangle_L\,, \quad q_1, q_2 = {+}{+}, {+}{-}, {-}{+}, {-}{-}\,. \tag{S72}$$

Here we show the derivation. When X_{c^*} is open-ended $\partial^* c^* \neq 0$, the action of X_{c^*} on the post-measurement state is:

$$\begin{aligned}
X_{c^*} |{++}\rangle_L &= \frac{X_{c^*} M_+ \bigotimes_e |+\rangle_e}{\sqrt{\langle +|^{\otimes n} M_+^\dagger M_+ |+\rangle^{\otimes n}}} \\
&= \frac{\left(\frac{1}{\sqrt{2\cosh\beta}}\right)^{n/2}}{\sqrt{\langle +|^{\otimes n} M_+^\dagger M_+ |+\rangle^{\otimes n}}} \exp\left[\frac{1}{2}\beta\left(\sum_{p\notin\partial^* c^*} B_p - \sum_{p\in\partial^* c^*} B_p\right)\right] X_{c^*} \bigotimes_e |+\rangle_e \\
&= \frac{\left(\frac{1}{\sqrt{2\cosh\beta}}\right)^{n/2}}{\sqrt{\langle +|^{\otimes n} M_+^\dagger M_+ |+\rangle^{\otimes n}}} \exp\left[\frac{1}{2}\beta\left(\sum_{p\notin\partial^* c^*} B_p - \sum_{p\in\partial^* c^*} B_p\right)\right] \bigotimes_e |+\rangle_e \\
&= \frac{1}{\sqrt{\langle +|^{\otimes n} M_+^\dagger M_+ |+\rangle^{\otimes n}}} \exp\left[-\beta \sum_{p\in\partial^* c^*} B_p\right] M_+ \bigotimes_e |+\rangle_e \\
&= \exp\left[-\beta \sum_{p\in\partial^* c^*} B_p\right] |{++}\rangle_L .
\end{aligned} \tag{S73}$$

It changes the sign of B_p operators that belong to the endpoints of c^*, $\partial^* c^*$, and we rewrite its action as an operator $\exp\left[-\beta \sum_{p\in\partial^* c^*} B_p\right]$ considering the fact that B_p commute with each other. Notice that when $\partial^* c^* = 0$, X_{c^*} factories in to A_v stabilizers and X logical operators and trivially $X_{c^*} |{++}\rangle_L = |{++}\rangle_L$. The expectation value of X_{c^*} can be computed through:

$$\begin{aligned}
\langle {++}|_L X_{c^*} |{++}\rangle_L &= \langle {++}|_L \exp\left[-\beta \sum_{p\in\partial^* c^*} B_p\right] |{++}\rangle_L \\
&= \frac{1}{Z_+} \sum_{\{\sigma_e=\pm\}} \exp\left[\beta \sum_{p\notin\partial^* c^*} U_p\right] = \frac{2^{n/2+1}}{Z_+} \sum_{\{U_p=\pm\}} \frac{1+\prod_p U_p}{2} \exp\left[\beta \sum_{p\notin\partial^* c^*} U_p\right] \\
&= \frac{2^{n/2}}{Z_+}\left(\prod_{p\notin\partial^* c^*} \sum_{U_p} \exp\left[\beta U_p\right] \prod_{p\in\partial^* c^*} \sum_{U_p} 1\right) + \frac{2^{n/2}}{Z_+}\left(\prod_{p\notin\partial^* c^*} \sum_{U_p} U_p \exp\left[\beta U_p\right] \prod_{p\in\partial^* c^*} \sum_{U_p} U_p\right) \\
&= \frac{2^{n/2}}{Z_+}(2^{n/2}(\cosh\beta)^{n/2-|\partial^* c^*|}) \\
&= \frac{1}{(\cosh\beta)^{|\partial^* c^*|}} \frac{1}{1+(\tanh\beta)^{n/2}},
\end{aligned} \tag{S74}$$

when $\partial^* c^* \neq 0$. Here we have expanded the post-measurement state under computational (Pauli Z) basis,

$$\exp\left[\frac{1}{2}\beta B_p\right] |+\rangle^{\otimes n} = \frac{1}{2^{n/2}} \sum_{\{\sigma_e=\pm\}} \exp\left[\frac{1}{2}\beta U_p\right] |\{\sigma_e\}\rangle \tag{S75}$$

σ_e is the eigenvalue of Z_e, $U_p = \prod_{e\in\partial p} \sigma_e$ is the eigenvalue of B_p and $Z_+ = \sum_{\{\sigma_e\}} \exp\left[\beta U_p\right]$ is the partition function of of $\mathbb{Z}_2$ gauge theory. As for other matrix elements,

$$\langle q_1|_L X_{c^*} |q_2\rangle_L = \langle {++}|_L Z_{q_1} X_{c^*} Z_{q_2} |{++}\rangle_L = \chi(X_{c^*}, Z_{q_2}) \langle {++}|_L Z_{q_1+q_2} \exp\left[-\beta \sum_{p\in\partial^* c^*} B_p\right] |{++}\rangle_L , \tag{S76}$$

where Z_q denotes the logical Z operator that send $|{++}\rangle_L$ to $|q\rangle_L$, for example $Z_{--} = Z_{l_1} Z_{l_2}$, $\chi(X_{c^*}, Z_{q_2}) = \pm 1$ is the constant commutation factor between X_{c^*} and Z_{q_2}. In the above expression, the off-diagonal terms $q_1 \neq q_2$ vanishes since then $Z_{q_1+q_2}$ changes since under 1-form symmetry operation $X_{l_1^*}$ or $X_{l_2^*}$. The diagonal value follows directly from Eq. (S74) with an additional factor of the commutator,

$$\langle q_1|_L \, X_{c^*} \, |q_2\rangle_L = \frac{1}{(\cosh\beta)^{|\partial^* c^*|}} \frac{1+(\tanh\beta)^{n/2}\delta(\partial^* c^* = 0)}{1+(\tanh\beta)^{n/2}} \chi(X_{c^*}, Z_{q_2})\delta_{q_1,q_2}. \tag{S77}$$

Therefore, we obtain the matrix element of $\Lambda + B$,

$$\begin{aligned}(\Lambda+B)_{c_1^* c_2^*, q_1 q_2} &= \sqrt{\Pr(c_1^*)\Pr(c_2^*)}\, \langle q_1|_L \, X_{c_1^*} X_{c_2^*} \, |q_2\rangle_L \\ &= \frac{1}{4}\sqrt{\Pr(c_1^*)\Pr(c_2^*)} \frac{1+(\tanh\beta)^{n/2}\delta(\partial^* c_1^* + \partial^* c_2^* = 0)}{(\cosh\beta)^{|\partial^* c_1^* + \partial^* c_2^*|}[1+(\tanh\beta)^{n/2}]} \chi(X_{c_1^* + c_2^*}, Z_{q_2})\delta_{q_1,q_2}\end{aligned} \tag{S78}$$

We assign a logical X operation $l^*(c^*)$ to each c^*, such that $[X_{l^*(c^*)}, Z_q] = [X_{c^*}, Z_q]$, $\forall q$. $l^*(c^*)$ represents the logical class of c^*. The Eq. (S78) is equivalent to

$$(\Lambda+B)_{c_1^* c_2^*, q_1 q_2} = \frac{1}{4}\sqrt{\Pr(c_1^*)\Pr(c_2^*)} \frac{1+(\tanh\beta)^{n/2}\delta(\partial^* c_1^* + \partial^* c_2^* = 0)}{(\cosh\beta)^{|\partial^* c_1^* + \partial^* c_2^*|}[1+(\tanh\beta)^{n/2}]} \langle q_1| \, X_{l^*(c_1^*)} X_{l^*(c_2^*)} \, |q_2\rangle \, \delta_{q_1,q_2}. \tag{S79}$$

The matrix elements of Λ does not depend on the code work q, thus

$$\Lambda_{c_1^* c_2^*, q_1 q_2} = \frac{1}{4}\sqrt{\Pr(c_1^*)\Pr(c_2^*)} \frac{1+(\tanh\beta)^{n/2}\delta(\partial^* c_1^* + \partial^* c_2^* = 0)}{(\cosh\beta)^{|\partial^* c_1^* + \partial^* c_2^*|}[1+(\tanh\beta)^{n/2}]} \delta_{l^*(c_1^*), l^*(c_2^*)}\delta_{q_1,q_2}. \tag{S80}$$

We compute AQEC relative entropy through the replica method. To do so we need to introduce a replica index $\alpha = 1, \cdots, R \in \mathbb{Z}_R$ to the error chain $c^{*(\alpha)}$. We alternatively represent the error configuration $c^{*(\alpha)}$ with $\mathbb{Z}_2$ spin variables $\{\eta_e^{(\alpha)} = \pm 1\}$ assigned with each edge e and replica copy α. $\eta_e^{(\alpha)} = -1$ suggests $e \in c^{*(\alpha)}$ and $\eta_e^{(\alpha)} = 1$ otherwise. The coboundary of $c^{*(\alpha)}$, $\partial^* c^{*(\alpha)}$,is marked by $U_p^{(\alpha)} = \prod_{e\in\partial p} \eta_e^{(\alpha)}$, while $U_p^{(\alpha)} = -1$ means $p \in \partial^* c^{*(\alpha)}$. Written in the spin variables, we have

$$\begin{aligned} &\Pr(c^{*(\alpha)}) = \frac{\exp\left(h\sum_e \eta_e^{(\alpha)}\right)}{(2\cosh h)^n}, \quad h = \frac{1}{2}\log\frac{1-p}{p}, \\ &|\partial^* c^{*(\alpha)} + \partial^* c^{*(\alpha+1)}| = \frac{n/2 - \sum_p U_p^{(\alpha)} U_p^{(\alpha+1)}}{2}, \\ &\delta(\partial^* c^{*(\alpha)} + \partial^* c^{*(\alpha+1)} = 0) = \delta_{U_p^{(\alpha)}, U_p^{(\alpha+1)}}. \end{aligned} \tag{S81}$$

Using the above substitutions, we have

$$\begin{aligned} \operatorname{tr}\left((\Lambda+B)^R\right) &= \frac{1}{4^{R-1}} \left(\prod_{\alpha=1}^R \sum_{c^{*(\alpha)}}\right) \left(\prod_{\alpha=1}^R \Pr(c^{*(\alpha)})\right) \prod_{\alpha=1}^R \frac{1+(\tanh\beta)^{n/2}\delta(\partial^* c^{*(\alpha)} + \partial^* c^{*(\alpha+1)} = 0)}{(\cosh\beta)^{|\partial^* c^{*(\alpha)} + \partial^* c^{*(\alpha+1)}|}[1+(\tanh\beta)^{n/2}]} \\ &= \frac{1}{4^{R-1}(2\cosh h)^{nR}(\cosh\beta)^{nR/4}(1+(\tanh\beta)^{n/2})^R} \\ &\times \sum_{\{\eta_e^{(\alpha)}\}} \exp\left\{\sum_{\alpha=1}^R \left[h\sum_e \eta_e^{(\alpha)} + \frac{1}{2}\log\cosh\beta \sum_p U_p^{(\alpha)} U_p^{(\alpha+1)} + \log\left(1 + (\tanh\beta)^{n/2}\prod_p \delta_{U_p^{(\alpha)}, U_p^{(\alpha+1)}}\right)\right]\right\}. \end{aligned} \tag{S82}$$

This is a classical partition function $Z(\beta, h) = \sum_{\{\eta_e^{(\alpha)}\}} \exp[-H(\{\eta_e^{(\alpha)}\})]$ for the Hamiltonian

$$H(\{\eta_e^{(\alpha)}\}) = -\sum_{\alpha=1}^R \left[h\sum_e \eta_e^{(\alpha)} + \frac{1}{2}(\log\cosh\beta)\sum_p U_p^{(\alpha)} U_p^{(\alpha+1)} + \log\left(1 + (\tanh\beta)^{n/2}\prod_p \delta_{U_p^{(\alpha)}, U_p^{(\alpha+1)}}\right)\right]. \tag{S83}$$

In computing $\operatorname{tr}(\Lambda^R)$, we need to insert $\prod_\alpha \delta_{l^*(c^{*(\alpha)}), l^*(c^{*(\alpha+1)})}$ in the partition function. It forces the error configuration of each replica copy to be in the same logical class, in other words they have the same commutation relation with the

Z logical operators. We notice that

$$\prod_{\alpha=1}^{R}\delta_{l^*(c^{*(\alpha)}),l^*(c^{*(\alpha+1)})}=\prod_{\alpha=1}^{R}\prod_{l\in\{l_1,l_2\}}\frac{1+\prod_{e\in l}\eta_e^{(\alpha)}\eta_e^{(\alpha+1)}}{2}, \tag{S84}$$

where $l=l_1,l_2$ labels the set of data qubits in Z logical operator Z_{l_1},Z_{l_2}. The Rényi version of AQEC relative entropy can be represented by the classical expectation value of the above expression,

$$S^{(R)}(\Lambda+B||\Lambda)=\frac{1}{R-1}\log\frac{\mathrm{tr}((\Lambda+B)^R)}{\mathrm{tr}(\Lambda^R)}=\frac{1}{1-R}\log\left\langle\prod_{\alpha,l}\frac{1+\prod_{e\in l}\eta_e^{(\alpha)}\eta_e^{(\alpha+1)}}{2}\right\rangle, \tag{S85}$$

where we have used $\mathrm{tr}_{\mathcal{C}}(B)=0$. The SM model Eq. (S83) captures the fluctuation of error configurations $\{\eta_e^{(\alpha)}\}$. The decodable order corresponds to that all spins are aligned up, $eta_e^{(\alpha)}=+$. The first term tends to point the spins up, while the second and third terms tend to pin the endpoints of the error chains on different replica copies together at the same plaquette. Large temperatures or small Interaction parameters β, h introduce disorder to the SM system.

We now discuss the property of Eq. (S85). We first analyze the low-temperature limit. Taking the $\beta\to+\infty$ limit and keeps h finite, the partition function becomes

$$Z(\beta,h)\propto\sum_{\{\eta_e^{(\alpha)}\}}\left(\prod_{\alpha,p}\delta_{U_p^{(\alpha)},U_p^{(\alpha+1)}}\right)\exp\left[h\sum_{\alpha,e}\eta_e^{(\alpha)}\right], \tag{S86}$$

with the constraint that all endpoints of the replica copies are exactly at the same plaquettes. Following the same spirit in deriving Eq. (S40), $\partial^*c^{*(\alpha)}=\partial^*c^{*(\alpha+1)}$ for all α suggests that we can rewrite $c^{*(\alpha)}=c^{*(R)}+b^{*(\alpha)}+l^{*(\alpha)}$ where $b^{*(\alpha)}$ is a coboundary (contractible loops) and $l^{*(\alpha)}$ is a cohomological class (non-contractible loops or logical class). We can further rewrite the summation of coboundaries as summations of spin variables on vertices $\sigma_v^{(\alpha)}$ by viewing $b^{*(\alpha)}$ as domain walls. Sign difference of σ spins at two ends of a edges $\prod_{v\in\partial e}\sigma_v^{(\alpha)}=-1$ corresponds to $\{\eta_e^{(\alpha)}\}=-1$. We also denote $\eta_e^{l^{*(\alpha)}}=-1$ when $e\in l^{*(\alpha)}$ and $\eta_e^{l^{*(\alpha)}}=+1$ otherwise. Thus

$$Z(\beta,h)\propto\sum_{\{\eta_e^{(R)}\}}\sum_{\{\sigma_v^{(\alpha)}\}}\sum_{\{l^{*(\alpha)}\}}\exp\left[h\sum_e\eta_e^{(R)}+h\sum_{\alpha=1}^{R-1}\sum_e\eta_e^{(R)}\eta_e^{l^{*(\alpha)}}\prod_{v\in\partial e}\sigma_v^{(\alpha)}\right], \tag{S87}$$

and self-consistently arrive at the replica partition function of the random bond Ising model (RBIM) [22, 23] with additional fluctuation of non-contractible domain walls. When computing $\mathrm{tr}\left(\Lambda^R\right)$, the non-contractible fluctuations are forbidden and we obtain

$$\mathrm{tr}\left(\Lambda^R\right)\propto\sum_{\{\eta_e^{(R)}\}}\sum_{\{\sigma_v^{(\alpha)}\}}\exp\left[h\sum_e\eta_e^{(R)}+h\sum_{\alpha=1}^{R-1}\sum_e\eta_e^{(R)}\prod_{v\in\partial e}\sigma_v^{(\alpha)}\right]\propto\sum_{\{\eta_e^{(R)}\}}\Pr(\{\eta_e^{(R)}\})Z_{RBIM}(\{\eta_e^{(R)}\})^{R-1}, \tag{S88}$$

It means that at the axis $\mathcal{T}=1/\beta=0$, the intrinsic threshold is located at $p_{th}\simeq 0.11$ [23]. Note that the phase transition point might be different for different R [24] and the correct value of threshold is obtained by taking the $R\to 1$ limit. When $h\to+\infty$ and fixing β, all η spins are forced to point up $\{\eta_e^{(\alpha)}\}=+1$, and the AQEC relative entropy is trivially 0.

Then we analyze the high-temperature limit. Take $\beta\to 0$, the SM model becomes a trivial paramagnetic model,

$$Z(\beta,h)=\sum_{\{\eta_e^{(\alpha)}\}}\exp\left[h\sum_{\alpha,e}\eta_e^{(\alpha)}\right]=2^n\cosh^n h, \tag{S89}$$

Assume the lattice is square with linear size $L=\sqrt{n/2}$, which is also the weight of logical operators. Through a straightforward calculation of the expectation value in Eq. (S85) with the above paramagnetic model, we get

$$S^{(R)}(\Lambda+B||\Lambda)=\left[\left(\frac{1+\tanh^L h}{2}\right)^R+\left(\frac{1-\tanh^L h}{2}\right)^R\right]^2. \tag{S90}$$

Take the thermodynamic limit $n\to\infty$ and the replica limit $R\to 1$, we have

$$\lim_{R\to 1}\lim_{n\to\infty}S^{(R)}(\Lambda+B||\Lambda)=2\log 2. \tag{S91}$$

The reason why it does not saturate the upper bound $4\log 2$ is that we only considered Pauli X noises and ignored Pauli Z noises. In the $h\to 0$ limit with finite β, the Hamiltonian has the form

$$H(\{\eta_e^{(\alpha)}\})=-\sum_{\alpha=1}^{R}\left[\frac{1}{2}(\log\cosh\beta)\sum_p U_p^{(\alpha)}U_p^{(\alpha+1)}+\log\left(1+(\tanh\beta)^{n/2}\prod_p\delta_{U_p^{(\alpha)},U_p^{(\alpha+1)}}\right)\right]. \tag{S92}$$

Ignoring the last term, the model is basically a union of one-dimensional Ising models of the $U_p^{(\alpha)}=\pm 1$ variables along the replica direction. Since the 1-D Ising models are disordered at finite temperatures, we anticipate that they lead to undecodable phase.

We then consider the monotonicity of $S^{(R)}(\Lambda+B||\Lambda)$ varying interaction parameters h, β. For a finite $\beta\geq 0$, the third term of Eq. (S83) is exponentially suppressed and negligible in the thermodynamic limit $n\to\infty$, and we approximate the Hamiltonian by

$$H(\{\eta_e^{(\alpha)}\})=-\sum_{\alpha=1}^{R}\left[h\sum_e\eta_e^{(\alpha)}+J\sum_p U_p^{(\alpha)}U_p^{(\alpha+1)}\right],\quad J=\frac{1}{2}\log\cosh\beta \tag{S93}$$

For a ferromagnetic spin model $J\geq 0$, the Griffiths-Kelly-Sherman (GKS) inequalities holds [25–27],

$$\langle\Gamma_{\mathcal{A}}\rangle\geq 0,\quad\langle\Gamma_{\mathcal{A}}\Gamma_{\mathcal{B}}\rangle\geq\langle\Gamma_{\mathcal{A}}\rangle\langle\Gamma_{\mathcal{B}}\rangle, \tag{S94}$$

where $\mathcal{A}$, $\mathcal{B}$ denotes sets containing pairs of edge and replica copy, (e,α), and $\Gamma_{\mathcal{A}}$, $\Gamma_{\mathcal{B}}$ is the product of corresponding spin variables,

$$\Gamma_X=\prod_{(e,\alpha)\in\mathcal{X}}\eta_e^{(\alpha)},\quad X=\mathcal{A},\mathcal{B}. \tag{S95}$$

In particular, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}J}\langle\Gamma_{\mathcal{A}}\rangle&=\sum_p\left\{\left\langle\Gamma_{\mathcal{A}}U_p^{(\alpha)}U_p^{(\alpha+1)}\right\rangle-\langle\Gamma_{\mathcal{A}}\rangle\left\langle U_p^{(\alpha)}U_p^{(\alpha+1)}\right\rangle\right\}\geq 0,\\ \frac{\mathrm{d}}{\mathrm{d}h}\langle\Gamma_{\mathcal{A}}\rangle&=\sum_e\left\{\left\langle\Gamma_{\mathcal{A}}\eta_e^{(\alpha)}\right\rangle-\langle\Gamma_{\mathcal{A}}\rangle\left\langle\eta_e^{(\alpha)}\right\rangle\right\}\geq 0,\end{aligned} \tag{S96}$$

Notice that the expectation value in Eq. (S85) can be expanded into summation of terms like $\Gamma_{\mathcal{A}}$ for different sets of spin variables $\mathcal{A}$, so we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}J}\left\langle\prod_{\alpha,l}\frac{1+\prod_{e\in l}\eta_e^{(\alpha)}\eta_e^{(\alpha+1)}}{2}\right\rangle\geq 0,\quad\frac{\mathrm{d}}{\mathrm{d}h}\left\langle\prod_{\alpha,l}\frac{1+\prod_{e\in l}\eta_e^{(\alpha)}\eta_e^{(\alpha+1)}}{2}\right\rangle\geq 0. \tag{S97}$$

In the thermodynamic limit, the AQEC relative entropy should be monotonically increasing with β and h following the above discussions,

$$\left.\frac{\mathrm{d}}{\mathrm{d}\beta}S^{(R)}(\Lambda+B||\Lambda)\right|_{n\to\infty}\geq 0,\quad\left.\frac{\mathrm{d}}{\mathrm{d}h}S^{(R)}(\Lambda+B||\Lambda)\right|_{n\to\infty}\geq 0. \tag{S98}$$

Note that $S^{(R)}(\Lambda+B||\Lambda)$ might be discontinuous at the phase transition point in the $n\to\infty$ limit, and we keep in

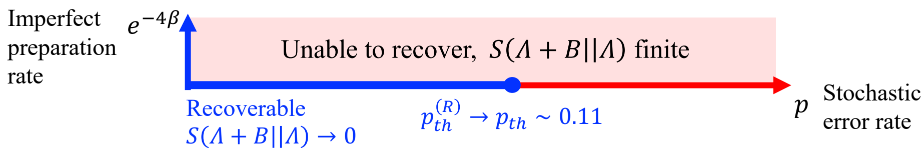

FIG. S3. The phase diagram. The undecodable phase stays at any finite preparation and Pauli error rates, as well as the region $e^{-4\beta} = 0$ and $p > p_{th}$. Notice that the RBIM phase transition point will be different for different replica R, and p_{th} is obtained in $R \to 1$.

mind that the above inequality includes the case that the derivative behaves like a delta function.

Now in order to study the properties of AQEC relative entropy at low but finite temperatures, we assume $e^{-\beta} \sim e^{-h} \ll 1$ and expand Eq. (S85) to the first non-vanishing order. Recall that the quantity $\prod_{\alpha,l} \frac{1+\prod_{e\in l} \eta_e^{(\alpha)}\eta_e^{(\alpha+1)}}{2}$ measures the difference of logical classes of the replica copies. It is 1 when all copies have the same logical class and 0 when there is a difference. In low temperature, the ground state is that all spins stay in $+1$, and the lowest order excitation is a single spin flip $\eta_e^{(\alpha)} = -1$ which costs energy $2h + 8J + 2\log(1 + \tanh^{n/2}\beta)$. There are nR possible configurations of such excitation, so the partition function reads

$$Z(\beta, h) \simeq 1 + nR\exp[-2h - 8J - 2\log(1 + \tanh^{n/2}\beta)] \simeq 1 + 4nRe^{-2h-4\beta}, \tag{S99}$$

normalized by the ground state Boltzmann weight. Now we insert $\prod_{\alpha,l} \frac{1+\prod_{e\in l} \eta_e^{(\alpha)}\eta_e^{(\alpha+1)}}{2}$ in the summation, and it yields 0 When the flipped spin $\eta_e^{(\alpha)} = -1$ is located on Z logical operators l_1, l_2. There are $2LR$ such configurations, thus

$$\sum_{\{\eta_e^{(\alpha)}\}} \prod_{\alpha,l} \frac{1 + \prod_{e\in l} \eta_e^{(\alpha)}\eta_e^{(\alpha+1)}}{2} \exp[-H] \simeq 1 + (nR - 2LR)\exp[-2h - 8J - 2\log(1 + \tanh^{n/2}\beta)] \simeq 1 + (4n - 8L)Re^{-2h-4\beta}, \tag{S100}$$

So the expectation value is

$$\left\langle \prod_{\alpha,l} \frac{1 + \prod_{e\in l} \eta_e^{(\alpha)}\eta_e^{(\alpha+1)}}{2} \right\rangle \simeq \frac{1 + (4n - 8L)Re^{-2h-4\beta}}{1 + 4nRe^{-2h-4\beta}} \simeq 1 - 8LRe^{-2h-4\beta}. \tag{S101}$$

Substitute in Eq. (S85), we can approximate the Rényi AQEC relative entropy by

$$S^{(R)}(\Lambda + B||\Lambda) \simeq \frac{8LR}{R-1} e^{-2h-4\beta}. \tag{S102}$$

This expression indicates a sudden jump at the zero error rate point. Keep e^{-h} small but finite, we have

$$\frac{\mathrm{d}}{\mathrm{d}e^{-4\beta}} S^{(R)}(\Lambda + B||\Lambda)\bigg|_{e^{-4\beta}\to 0+} = \frac{8LR}{R-1} e^{-2h}. \tag{S103}$$

It diverges in the thermodynamic limit,

$$\lim_{n\to\infty} \frac{\mathrm{d}}{\mathrm{d}e^{-4\beta}} S^{(R)}(\Lambda + B||\Lambda)\bigg|_{e^{-4\beta}\to 0+} \to +\infty. \tag{S104}$$

Similarly keep $e^{-\beta}$ small but finite, we have

$$\frac{\mathrm{d}}{\mathrm{d}e^{-2h}} S^{(R)}(\Lambda + B||\Lambda)\bigg|_{e^{-2h}\to 0+} = \frac{8LR}{R-1} e^{-4\beta}, \quad \frac{\mathrm{d}}{\mathrm{d}e^{-2h}} S^{(R)}(\Lambda + B||\Lambda) \lim_{n\to\infty}\bigg|_{e^{-2h}\to 0+} \to +\infty. \tag{S105}$$

So we conclude that the phase diagram has the form in Fig. S3 for any $R > 1$ as well as $R \to 1$. The system stays in undecodable phase as long as the preparation and Pauli error rates are both finite. The intuition for this phenomenon is gained from the SM model. When the preparation is perfect $\beta \to +\infty$, then endpoints of different replica copies are pinned together and it is hard to create logical difference between replica copies,

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