Thermodynamics of the five-vertex model with scalar-product boundary conditions

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Abstract. We consider the homogeneous five-vertex model on a rectangle domain of the square lattice with so-called scalar-product boundary conditions. Peculiarity of these boundary conditions is that the configurations of the model are in an one-to-one correspondence with the 3D Young diagrams limited by a box of a given size. We address the thermodynamics of the model using a connection of the partition function with the τ-function of the sixth Painlevé equation. We compute an expansion of the logarithm of the partition function to the order of a constant in the size of the system. We find that the geometry of the domain is crucial for phase transition phenomena. Two cases need to be considered separately: one is where the region has an asymptotically square shape and the second one is where it is of an arbitrary rectangle, but not square, shape. In the first case there are three regimes, which can be attributed to dominance in the configurations of a ferroelectric order, disorder, and anti-ferroelectric order. In the second case the third regime is absent.

Contents

1. Introduction

The five-vertex model had originally emerged for modeling of crystal growth and evaporation in two dimensions [1–3]. For periodic boundary conditions its thermodynamic properties, including the phase diagram, have been completely understood by Bethe ansatz methods [4, 5]. As the same time, it is known that the six-vertex model (and hence the five-vertex model as its descendant) is sensitive to boundary conditions. A paradigmatic example here is the six-vertex model with domain wall boundary conditions [6–8].

As for the five-vertex model, interesting boundary conditions are such that the configurations of the model appear to be in a one-to-one correspondence with the 3D Young diagrams limited by a box of a given size (or, "boxed" plane partitions). These boundary conditions are special fixed boundary conditions imposed to a finite-size domain of the square lattice of a rectangular shape. They can be seen as a generalization of domain wall boundary conditions and called "scalar-product" boundary conditions, as they arise when scalar products off-shell Bethe states are interpreted as partition functions of related vertex models [9–11].

Recently, a notable progress had been achieved in understanding scaling properties of the five-vertex model in a rather general setup by variational methods, with the focus on phase separation and limit shape phenomena [12–15]. On the other hand, for the case of scalar-product boundary conditions an important problem consists in constructing expansions of the partition function in the limit of large system size. In the free-fermion case, equivalent to the dimer model on a hexagonal domain (boxed 3D Young diagrams), a solution of this problem has been provided in [16].

In the present paper, we consider the five-vertex model with scalar-product boundary conditions and derive an expansion for the logarithm of the partition function for large lattice sizes. We obtain explicitly terms to the order of a constant, including the logarithmic terms. In [17], we have derived various determinant formulas for the partition function of the five-vertex model with scalar-product boundary conditions and showed that one of these representations coincides with the τ-function of the sixth Painlevé equation. To derive the asymptotic expansion, we apply here the method originally proposed in [18] which is based on use of the sixth Painlevé equation in its σ-form [19, 20]. Similarly to [18], we deal with an asymptotic expansion of the σ-function where the coefficients are large while the argument is a finite parameter.

It has to be mentioned that to address the problem of finding asymptotic expansions for solutions of Painlevé equations one can use methods such as the isomonodromy deformation techniques [21, 22] or the asymptotic analysis of the corresponding Riemann-Hilbert problem [23]. Somewhat equivalently, one can construct asymptotic expansions by relating the τ-function with a random matrix model [24] and formulating the Riemann-Hilbert problem for the orthogonal polynomials associated to the weight measure [25]. Specifically, for the present problem the matrix model appearing on this route has a discrete measure, the corresponding polynomials have been studied in [26]. The method of [18] which we exploit here can be seen as an alternative to these approaches, and it relies on the theory of asymptotic expansions for solutions of ordinary differential equations [27].

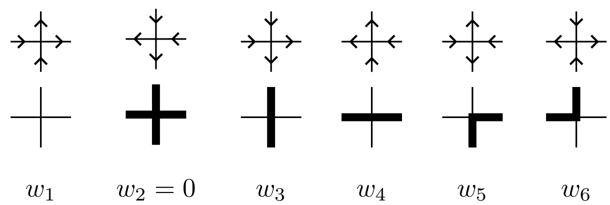

FIGURE 1. The six vertices of the six-vertex model in terms of arrows (first row) or lines (second row), and their Boltzmann weights in the five-vertex model (third row)

Our main result is collected in two theorems about the thermodynamic limit expansion for the logarithm of certain polynomial completely determining the partition function. We find that this expansion significantly depends on an asymptotic form of the domain, namely, whether the region has an asymptotically square shape, or the region has an arbitrary rectangle, but not square, shape. In the former case there are three regimes, which can be attributed to a ferroelectric order, disorder, and anti-ferroelectric order. In the latter case the third regime is absent. We also illustrate that this extra phase transition between the disorder and anti-ferroelectric order for the square-shaped domain can be seen as a "merger transition" discussed recently in [28].

1.1. The model. The five-vertex model is defined on a square lattice in terms of arrows placed on edges or, equivalently, in terms of lines "flowing" through the lattice. The standard convention [29,30] between the arrow and line pictures is that if an arrow points down or left, then this edge contains a line, otherwise the edge is empty. In the six-vertex model the admissible vertices are only those which have equal number of incoming and outgoing arrows, see Fig. 1. The five-vertex model can be obtained by requiring that only those vertices are admitted which contain non-intersecting lines, that is, the vertex of the second type is excluded.

In this paper we consider the model on a lattice obtained by intersection of L vertical and M horizontal lines (the $M \times L$ lattice). The boundary conditions are the following: the N first (last) arrows at the bottom (top) boundary point down, and the remaining arrows point up or right, see Fig. 2.

An interesting property of these boundary conditions is that there exists an one-to-one correspondence between the configurations of the five-vertex model with the 3D Young diagrams, which fit into $(L-N)\times N\times(M-N)$ box, see Fig. 3. In this correspondence, the lines of the vertex model are gradient lines; there also exists the one-to-one correspondence between vertices and flat fragments of images of 3D Young diagrams (see Fig. 3, right). In a rather general setup the boundary conditions defined above are related to the scalar products of off-shell Bethe states and their generalizations [9–11, 17]. For this reason we refer to them as scalar-product boundary conditions.

The partition function is defined as

$$Z=\sum_{\mathcal{C}}\prod_{i=1,3,\dots,6} w_i^{l_i(\mathcal{C})} \tag{1.1}$$

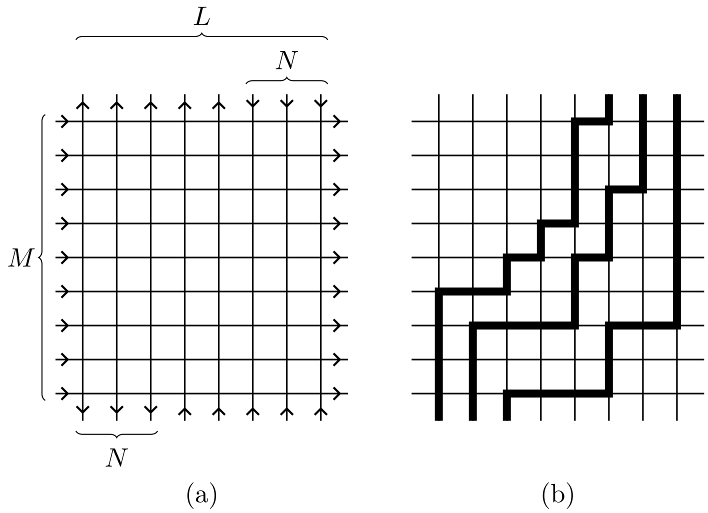

FIGURE 2. The boundary conditions (a) and one of the possible configurations (b).

where the sum is taken over all admissible configurations $\mathcal{C}$ and $l_i(\mathcal{C})$ denotes the number of vertices of the ith type in the configuration $\mathcal{C}$. Note that in all configurations the number of vertices of the first type is fixed, $l_1(\mathcal{C}) = (L-N)(M-N)$, the vertices of the third and fourth types appear in pairs, $l_3(\mathcal{C})-l_4(\mathcal{C}) = N(M+N-L)$, and the number of vertices of the fifth type is equal to the number of vertices of the sixth type, $l_5(\mathcal{C}) = l_6(\mathcal{C})$.

A standard way to parametrize the Boltzmann weights (see, e.g., [17]) is the following:

$$w_1 = \frac{\alpha}{\sqrt{x}}\frac{x-1}{\Delta}, \qquad w_3 = \frac{\sqrt{x}}{\alpha}, \qquad w_4 = \alpha\sqrt{x}, \qquad w_5 = w_6 = 1. \tag{1.2}$$

Here, $x \in (1,\infty)$ for $\Delta > 0$, and $x \in (0,1)$ for $\Delta < 0$. The parameter Δ can be defined independently of the parameterization as follows:

$$\Delta = \frac{w_3 w_4 - w_5 w_6}{w_1 w_3}. \tag{1.3}$$

The case $\Delta = 0$ can be approached in the limit $x \to 1$; this is the free-fermion point of the model (for further details, see Sect. 2.1). The parameter α is real and positive, it has the meaning of an external field.

The partition function $Z = Z(x;\Delta,\alpha)$ has the structure

$$Z = E\,\widetilde{Z}. \tag{1.4}$$

Here, $E = E(x;\Delta,\alpha)$ is a factor giving the weight of the configuration corresponding to the “empty” 3D Young diagram,

$$E = \left(\frac{x-1}{\Delta}\right)^{(L-N)(M-N)}\left(\frac{\alpha}{\sqrt{x}}\right)^{M(L-2N)} x^{N(L-N-1)}.$$

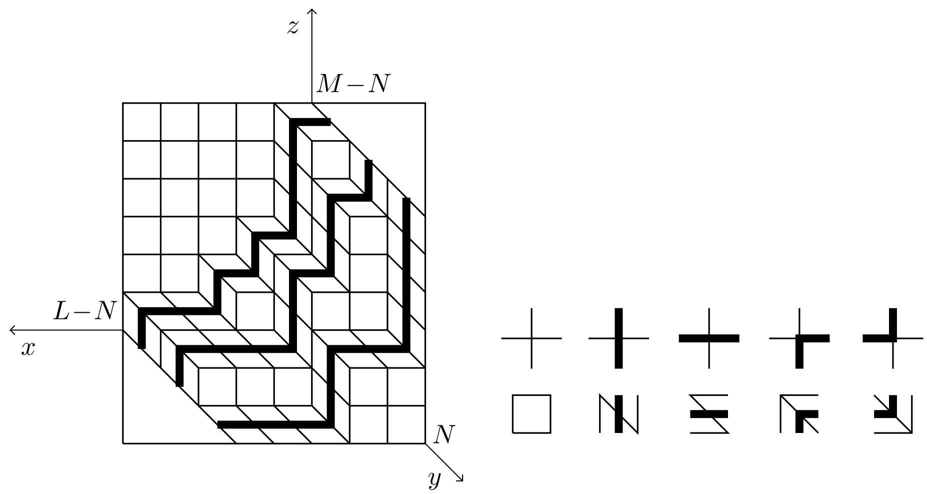

FIGURE 3. The five-vertex model configuration of Fig. 2b as a 3D Young diagram (left) and mapping of the five vertices to flat fragments of images of 3D Young diagrams (right).

The quantity $\widetilde{Z}=\widetilde{Z}(x)$ is independent of Δ and α, and it has the form

$$\widetilde{Z}=\binom{M}{N} P_{N,M,L}\left(x^{-1}\right). \tag{1.5}$$

Here, $\binom{M}{N}$ is the binomial coefficient and $P_{N,M,L}\left(x^{-1}\right)$ is a polynomial of its variable satisfying the normalization condition

$$P_{N,M,L}\left(0\right)=1.$$

The degree of $P_{N,M,L}\left(x^{-1}\right)$ is equal to the difference between the maximum and minimum number of pairs of vertices of the fifth and sixth types,

$$\deg P_{N,M,L}=N\min(M-N,L-N-1). \tag{1.6}$$

A highly nontrivial and remarkable property of this polynomial is that all its coefficients are symmetric under exchange $L\leftrightarrow M+1$, i.e.,

$$P_{N,M,L}\left(x^{-1}\right)=P_{N,L-1,M+1}\left(x^{-1}\right). \tag{1.7}$$

Though there seems no simple explanation of this property from the definition of the model, it is transparent in explicit expressions (see, e.g., representation (2.6) below) discussed in the text.

The polynomial $P_{N,M,L}\left(x^{-1}\right)$ can be seen as a generating function which counts configurations with a fixed number of turns of 'solid' lines (vertices of the fifth and sixth types). Indeed, due to the combinatorial restrictions (i.e., the fixed numbers $\ell_1(\mathcal{C})$ and $\ell_3(\mathcal{C})-\ell_4(\mathcal{C})$), one can take weights, instead of (1.2), equal to $w_1=w_3=w_4=1$ and $w_5=w_6=1/\sqrt{x}$. These are the weights which have been considered in [12] (where $1/\sqrt{x}$ has been denoted by r).

1.2. Main result. The aim of this paper is to study the thermodynamic limit of the model, i.e., the limit where the size of the domain tends to infinity with its geometry being fixed. To treat the general case one can introduce two "macroscopic" parameters $p, q \in [0, \infty)$, which will describe the side lengths of the rectangle-shaped domain in the scale of N. Specifically, for the reasons explained below, we define them as follows:

$$pN = M - N + \frac{1}{2}, \qquad qN = L - N - \frac{1}{2}. \tag{1.8}$$

We are interested in the limit $N, M, L \to \infty$ with p and q being fixed. The main thermodynamic quantity of interest is the free energy per site $F = F(x; \Delta, \alpha)$, defined as

$$F = - \lim_{N,M,L\to\infty} \frac{\log Z}{ML}.$$

It can be given in the form

$$F = -\frac{f_2(x)}{(p+1)(q+1)} - \frac{pq}{(p+1)(q+1)} \log \frac{x-1}{\Delta} + \left(\frac{1}{2} - \frac{1}{(p+1)(q+1)}\right) \log x \\ - \frac{q-1}{q+1} \log \alpha. \tag{1.9}$$

The function $f_2(x)$ describes the leading large N behavior of the nontrivial factor in (1.5),

$$f_2(x) = \lim_{N,M,L\to\infty} \frac{\log P_{N,M,L}(x^{-1})}{N^2}.$$

Our main results concern the function $f_2(x)$ and all the sub-leading corrections for $\log P_{N,M,L}(x^{-1})$ up to $O(1)$ in the limit $N, M, L \to \infty$. We often call it below simply "large N limit", assuming that L and M are connected to N via (1.8).

To treat the special case where the domain has an asymptotic square shape, we find it convenient to use a "macroscopic" parameter $r \in [0, \infty)$ and a "microscopic" parameter $\epsilon = 0, \pm 1, \pm 2, \ldots$, defined as follows:

$$rN = \frac{M+L}{2} - N, \qquad \epsilon = M - L + 1. \tag{1.10}$$

In view of the symmetry (1.7), the parameter ϵ appears below only via its absolute value, $|\epsilon|$. It describes a microscopic deformation from the "perfect" square shape which is in our problem corresponds to the relation $M - L + 1 = 0$. Our results for the square-shaped domain are obtained under the assumption that the parameter ϵ is fixed in the large N limit, i.e., that ϵ is of $O(1)$.

More broadly, the square-shaped domain asymptotically can be obtained under the assumption that $M/L \to 1$ with ϵ slowly increasing[1]. As it follows from our results, ϵ can be taken to be of $o(N)$ (see Remark 1.3 below). This agrees with the fact that in (1.10) both M and L are of the order N and the rectangle-shaped domain would correspond to ϵ to be of the order N as well.

In what follows we shall often call the square domain case simply as "symmetric case" and sometimes refer to it as "the case $p = q$", in view of (1.8). Clearly, in this case the free energy is just given by (1.9) with $p = q =: r$.

Our main finding about the thermodynamics of the model is that it depends strongly on whether the domain takes asymptotically a square or rectangular (but not square) shape. If $p = q$, then the model exhibits three different phases depending on the value of x. If $p \neq q$, then only two phases exist. All the transitions between the phases are of the third order, that are characterized by discontinuities

[1] We thank the anonymous referee for pointing this possibility to our attention.

of $f_2'''(x)$ with continuous first- and second-order derivatives at these points. We summarize the main result in two theorems.

The first theorem concerns the case of a square-shaped domain.

THEOREM 1.1. *If $M, L, N \to \infty$, such that $r = (M+L-2N)/2N$ and $\epsilon = M-L+1$ are kept fixed, there exist three asymptotic regimes which are separated by the critical values $x = x_c^{-1}$ and $x = x_c$, where*

$$x_c = (2r+1)^2, \qquad r \in (0,\infty).$$

If $x \in [x_c, \infty)$, then

$$\log P_{N,M,L}\left(x^{-1}\right) = N^2 f_2^{\mathrm{I}}(x) + N f_1^{\mathrm{I}}(x) + f_0^{\mathrm{I}}(x) + O\left(N^{-1}\right),$$

where

$$\begin{aligned}
f_2^{\mathrm{I}}(x) &= r^2 \log \frac{x}{x-1}, \\
f_1^{\mathrm{I}}(x) &= (2r+1) \log \frac{\sqrt{x_c(x-1)} + \sqrt{x-x_c}}{\left(1+\sqrt{x_c}\right)\sqrt{x}} - \log \frac{\sqrt{x-1}+\sqrt{x-x_c}}{2\sqrt{x}}, \\
f_0^{\mathrm{I}}(x) &= \frac{1}{4} \log \frac{x}{x-x_c} - \frac{\epsilon^2}{4} \log \frac{x}{x-1}.
\end{aligned}$$

If $x \in [x_c^{-1}, x_c]$, then

$$\log P_{N,M,L}\left(x^{-1}\right) = N^2 f_2^{\mathrm{II}}(x) + N f_1^{\mathrm{II}}(x) + \frac{5}{12}\log N + f_0^{\mathrm{II}}(x) + O\left(N^{-1}\right),$$

where

$$\begin{aligned}
f_2^{\mathrm{II}}(x) &= (2r+1) \log \frac{1+\sqrt{x}}{1+\sqrt{x_c}} - \left(r + \frac{1}{4}\right) \log \frac{x}{x_c} + r^2 \log \frac{x_c}{x_c - 1}, \\
f_1^{\mathrm{II}}(x) &= \log \frac{2\sqrt{x}}{\sqrt{x_c}+1} + r \log \frac{\sqrt{x_c}-1}{\sqrt{x_c}+1}, \\
f_0^{\mathrm{II}}(x) &= \frac{1}{8} \log \frac{\left(\sqrt{x_c}-\sqrt{x}\right)^3 \sqrt{x}}{\left(\sqrt{x_c x}-1\right)} - \frac{1}{12} \log\left(\sqrt{x_c}\left(x_c - 1\right)\right) \\
&\quad + \frac{\epsilon^2}{2} \log \frac{\sqrt{x_c x}-1}{\sqrt{x_c - 1}\sqrt{x}} + \zeta'(-1) + \log\sqrt{2\pi}.
\end{aligned}$$

If $x \in [0, x_c^{-1}]$, then

$$\log P_{N,M,L}\left(x^{-1}\right) = N^2 f_2^{\mathrm{III}}(x) + N f_1^{\mathrm{III}}(x) + \frac{1-\epsilon^2}{2}\log N + f_0^{\mathrm{III}}(x) + O\left(N^{-1}\right),$$

where

$$\begin{aligned}
f_2^{\mathrm{III}}(x) &= r^2 \log \frac{1}{1-x} - r \log x, \\
f_1^{\mathrm{III}}(x) &= |\epsilon|(2r+1) \log \frac{\sqrt{x_c}\sqrt{1-x} + \sqrt{1-x_c x}}{\sqrt{x_c - 1}} - |\epsilon| \log \frac{\sqrt{1-x}+\sqrt{1-x_c x}}{\sqrt{x_c-1}\sqrt{x}} \\
&\quad + \log \frac{2\sqrt{x}}{\sqrt{x_c}+1} + r \log \frac{\sqrt{x_c}-1}{\sqrt{x_c}+1}, \\
f_0^{\mathrm{III}}(x) &= \frac{1}{4}\log(1-x) - \frac{\epsilon^2}{4}\log(1-x_c x) + (1-|\epsilon|)\log\sqrt{2\pi} + \log G(1+|\epsilon|).
\end{aligned}$$

In these expressions, $\zeta(z)$ and $G(z)$ stand for the Riemann zeta-function and the Barnes G-function, respectively; $\zeta'(-1) = -0.165142...$ and $G(1) = G(2) = G(3) = 1$, $G(n+2) = 1!2!\cdots n!$.

In what follows we refer to the three intervals of values of x, namely, $[x_\mathrm{c}, \infty)$, $[x_\mathrm{c}^{-1}, x_\mathrm{c}]$, and $[0, x_\mathrm{c}^{-1}]$ as Regimes I, II, and III, respectively. We have three remarks concerning the result formulated in Thm. 1.1.

Remark 1.1. *From (1.10) it follows that $M = (r+1)N + \frac{\epsilon-1}{2}$ and hence*

$$\log \binom{M}{N} = N\big((r+1)\log(r+1) - r\log r\big) - \frac{1}{2}\log N \\ - \log\sqrt{2\pi} - \frac{\epsilon}{2}\log\frac{r}{r+1} + O\left(N^{-1}\right),$$

that implies that the partition function $\widetilde{Z}$ defined in (1.5) in each Regime i, $i = \mathrm{I}, \mathrm{II}, \mathrm{III}$, has the form

$$\log\widetilde{Z} = N^2 F_2^i + N F_1^i + \kappa^i \log N + F_0^i + O(N^{-1}),$$

where

$$\begin{aligned} F_2^i &= f_2^i(x), \\ F_1^i &= f_1^i(x) - \log\frac{2}{\sqrt{x_\mathrm{c}}+1} - r\log\frac{\sqrt{x_\mathrm{c}}-1}{\sqrt{x_\mathrm{c}}+1}, \\ F_0^i &= f_0^i(x) - \log\sqrt{2\pi} - \frac{\epsilon}{2}\log\frac{\sqrt{x_\mathrm{c}}-1}{\sqrt{x_\mathrm{c}}+1}, \end{aligned}$$

and

$$\kappa^{\mathrm{I}} = -\frac{1}{2}, \qquad \kappa^{\mathrm{II}} = -\frac{1}{12}, \qquad \kappa^{\mathrm{III}} = -\frac{\epsilon^2}{2}.$$

Remark 1.2. *In the Regime III at $\epsilon = 0$, it can be shown that*

$$\log\widetilde{Z} = N^2 f_2^{\mathrm{III}}(x) + N\log\sqrt{x} + \frac{1}{4}\log(1-x) + O\left(N^{-\infty}\right), \tag{1.11}$$

where the symbol $O(N^{-\infty})$ denote terms decaying faster than any integer power of $1/N$.

We will explain the origin of (1.11) after the proof of Thm. 1.1 in Sect. 4. The terms which we have denoted by $O(N^{-\infty})$ are in fact exponentially small and they can also be treated by the method of [18].

Remark 1.3. *In the expressions in Thm. 1.1 the parameter ϵ can be replaced by some quantity of magnitude of $o(N)$ without altering the leading term behavior governed by the function f_2 in all the three regimes. This implies that the picture with two phase transitions obtained for the square-shaped domain is also valid in the large N limit such that $M/L \to 1$ with $M - L$ allowed to be of $o(N)$.*

The second theorem concerns the case of a rectangular, but not square, domain.

Theorem 1.2. *In the case $p \neq q$ there exist two asymptotic regimes which are separated by the critical value*

$$x_\mathrm{c} = \left(\sqrt{(p+1)(q+1)} + \sqrt{pq}\right)^2.$$

If $x \in [x_c, \infty)$, *then*

$$\log P_{N,M,L}\left(x^{-1}\right) = N^2 f_2^{\mathrm{I}}(x) + N f_1^{\mathrm{I}}(x) + f_0^{\mathrm{I}}(x) + O\left(N^{-1}\right),$$

where

$$\begin{aligned}
f_2^{\mathrm{I}}(x) &= pq \log \frac{x}{x-1}, \\
f_1^{\mathrm{I}}(x) &= \frac{2p+1}{2} \log \frac{(2p+1)x - p - q - 1 + \sqrt{s(x)}}{2(p+1)\sqrt{x(x-1)}} \\
&\quad + \frac{2q+1}{2} \log \frac{(2q+1)x - p - q - 1 + \sqrt{s(x)}}{2(q+1)\sqrt{x(x-1)}} \\
&\quad - \frac{1}{2} \log \frac{x - 2pq - p - q - 1 + \sqrt{s(x)}}{2x}, \\
f_0^{\mathrm{I}}(x) &= \frac{1}{4} \log \frac{x(x-1)}{s(x)},
\end{aligned}$$

and

$$s(x) = x^2 - 2(2pq + p + q + 1)x + (p+q+1)^2, \qquad s(x_c) = 0.$$

If $x \in [0, x_c]$, *then*

$$\log P_{N,M,L}\left(x^{-1}\right) = N^2 f_2^{\mathrm{II}}(x) + N f_1^{\mathrm{II}}(x) + \frac{5}{12} \log N + f_0^{\mathrm{II}}(x) + O\left(N^{-1}\right),$$

where

$$\begin{aligned}
f_2^{\mathrm{II}}(x) &= -\frac{(p+q)^2}{2} \log y - \frac{(p-q)^2 + 2p + 2q + 1}{2} \log(y+1) - p \log(y+p-q) \\
&\quad - q \log(y+q-p) + p(p+1) \log\big((2p+1)y + p - q\big) \\
&\quad + q(q+1) \log\big((2p+1)y + q - p\big) + (p+q+1) \log(y+p+q+2) \\
&\quad - \frac{1}{2}\Big\{(p+1)^2 \log 2(p+1) + (q+1)^2 \log 2(q+1) + p^2 \log 2p + q^2 \log 2q\Big\} \\
f_1^{\mathrm{II}}(x) &= \log \sqrt{x} - \frac{1}{2}\Big\{(p+1)\log(p+1) + (q+1)\log(q+1) - p\log p - q \log q\Big\}, \\
f_0^{\mathrm{II}}(x) &= \frac{1}{8}\Bigg\{ \log y + \log(y+1) - 2\log\big((2p+1)y + p - q\big) \\
&\quad - 2\log\big((2q+1)y + q - p\big) + 3\log\left(y^2 - 2(2pq + p + q)y + (p-q)^2\right) \\
&\quad + \frac{1}{3}\log\Big((2p+1)(2q+1)y^3 - (p-q)^2\left[3y^2 + 3y - (p+q+1)^2 + 1\right]\Big)\Bigg\} \\
&\quad - \frac{1}{24}\log\big(16p(p+1)q(q+1)\big) + \zeta'(-1) + \log\sqrt{2\pi}.
\end{aligned}$$

The function $y = y(x)$ *is the root of the quartic equation*

$$x = \frac{(y+1)^2(y-p+q)(y+p-q)}{\big((2p+1)y + p - q\big)\big((2q+1)y + q - p\big)}, \tag{1.12}$$

which takes the values $y \in [|p-q|, y_c]$ *for* $x \in [0, x_c]$, *where, moreover,* $y(0) = |p-q|$ *and*

$$y_c \equiv y(x_c) = \left(\sqrt{p(q+1)} + \sqrt{q(p+1)}\right)^2 = x_c - 1.$$

In what follows in the non-symmetric case we will refer the intervals $[x_\mathrm{c}, \infty)$ and $[0, x_\mathrm{c}]$ of values of the variable x as Regime I and Regime II, respectively. Note that there is no Regime III here.

REMARK 1.4. *In the case $q = p =: r$ the parametrization $x = x(y)$ defined by* (1.12) *becomes*

$$x = \frac{(y+1)^2}{(2r+1)^2}, \quad x : [0, y_\mathrm{c}] \mapsto \left[(2r+1)^{-2}, (2r+1)^2\right],$$

and the functions $f_2^{\mathrm{II}}(x)$ and $f_1^{\mathrm{II}}(x)$ defined in Thm. 1.1 are recovered. The function $f_0^{\mathrm{II}}(x)$ of Thm. 1.1 is recovered at $\epsilon = 0$; the ϵ^2-term of this function is recovered by setting $q = r + \epsilon/2N$ and $p = r - \epsilon/2N$ in the function $f_2^{\mathrm{II}}(x)$ and re-expanding it in $1/N$.

The purpose of the remaining part of the paper is to give proofs of Thms. 1.1 and 1.2. In brief, we apply the method of paper [18] to the results of paper [17]. We use the parameterization related to a rational elliptic curve proposed in [31] to obtain the assertion of Thm. 1.2 for Regime II. Along the proofs, we have found that the sub-leading corrections can be treated in a simplified way, in a comparison to the original approach of [18], by splitting the σ-form of the sixth Painlevé equation on two factors.

We start with exposing the main ingredients of our analysis, namely, determinant representations for the polynomial $P_{N,M,L}(x^{-1})$ and its connection with the sixth Painlevé equation in Sect. 2. In Sect. 3 we obtain expansions of $P_{N,M,L}(x^{-1})$ at the singular points of the sixth Painlevé equation for finite values of N, M, L. In Sect. 4 we show how to construct the leading order term of the asymptotic expansion in the large N limit and how to treat the sub-leading corrections in the symmetric case. In Sect. 5 we consider the solution of the same problem in the non-symmetric case. In Conclusion (Sect. 6) we briefly discuss our results and illustrate a connection with the so-called "merger transition" in the square-shaped domain case.

2. Exact results for the partition function

Here, we collect known results about the partition function of the five-vertex model with scalar-product boundary conditions, which we use below in our proofs of Thms. 1.1 and 1.2.

2.1. Basic properties of the model. We begin with commenting each factor in formulas (1.4) and (1.5) describing the partition function Z. These formulas follow from the relations satisfied by the numbers $l_i(\mathcal{C})$, discussed after (1.1). In turn, the indicated relations can also be easily understood using the outlined connection of model with the 3D Young diagrams (see Fig. 3).

The first relation $l_1(\mathcal{C}) = (M - N) \times (L - N)$ gives the number of elementary squares that obviously is conserved for the plane partitions. The factor $E = E(x; \Delta, \alpha)$ in (1.4) is the Boltzmann weight of the configuration corresponding to the "empty" partition. The binomial coefficient in (1.5) corresponds to the degeneracy of this weight as there are exactly $\binom{M}{N}$ configurations with N horizontal lines at M rows, see Fig. 4. These configurations describe ferroelectrically ordered states. Since the number of the weights w_1 is fixed, from (1.2) it follows that this is a typical form of configurations for sufficiently large values of x.

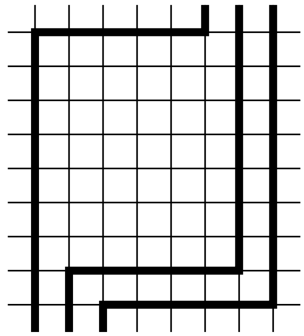

FIGURE 4. One of the $\binom{M}{N}$ ferroelectric ground states, $M = 9$, $N = 3$.

Moderate values of x correspond to a situation where vertices with turn paths (vertices of types 5 and 6) are mixed with those having straight paths (vertices of types 3 and 4). These are disordered states. This is an interesting regime because it can be characterized be appearance of nontrivial limit shapes.

An important example of such a situation is the case $x = 1$ which corresponds to the free fermion point of the model. Recall that in the general six-vertex model the free-fermion condition is $w_1 w_2 + w_3 w_4 = w_5 w_6$ which is for the five-vertex model implies $\Delta = 0$, where Δ is defined in (1.3). In the five-vertex it corresponds to the weights

$$w_1 = \lambda, \qquad w_3 = \alpha, \qquad w_4 = \alpha^{-1}, \qquad w_5 = w_6 = 1,$$

where λ is some parameter, $\lambda > 0$. This model can be obtained (recall that $x \leqslant 1$ for $\Delta \leqslant 0$) upon setting $x = \exp(\lambda\Delta)$ and taking the limit $\Delta \to 0$ in (1.2). The correspondence with the boxed plane partitions means that the partition function $Z = Z(x; \Delta, \alpha)$ at all weights equal to 1 has the following value:

$$\lim_{\Delta\to 0} Z(\mathrm{e}^{\Delta}; \Delta, 1) = \mathrm{PL}(L - N, N, M - N). \tag{2.1}$$

Here, $\mathrm{PL}(a, b, c)$ is the number of the boxed plane partitions in a box of the size $a \times b \times c$. It is well-known to be given by the famous MacMahon triple-product formula

$$\mathrm{PL}(a, b, c) = \prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{i + j + k - 1}{i + j + k - 2}.$$

We use below a less symmetric but more practical single-product expression

$$\mathrm{PL}(a, b, c) = \prod_{j=0}^{a-1} \frac{(b + c + j)!j!}{(b + j)!(c + j)!}. \tag{2.2}$$

Analogous expressions can be obtained by permuting cyclically a, b, c. Formulas (1.5) and (2.1) imply that

$$P_{N,M,L}(1) = \binom{M}{N}^{-1} \mathrm{PL}(L - N, N, M - N).$$

Using (2.2) one can check relation (1.7) at $x = 1$.

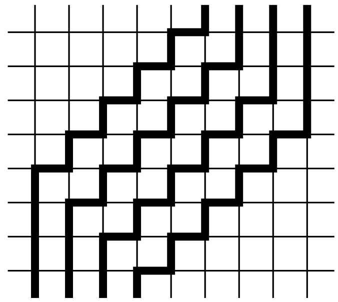

FIGURE 5. The anti-ferroelectric ground state, $M = L - 1 = 8$.

As x is small, configurations with maximally possible number of the vertices of types 5 and 6 should dominate. However, such a dominance may be affected by relations between the geometric parameters of the domain. Indeed, for M and L such that $M = L - 1$ and arbitrary N there exists an anti-ferroelectric ground state, which can be characterized by the presence of a rectangle region inside of the domain containing only vertices of types 5 and 6, see Fig. 5. For arbitrary values of M and L there exists no particular state dominating over the other states; there are many states which can contribute equally well into the partition function. This simple observation hints at the fact that in the thermodynamic limit the partition function may demonstrate a different behavior in the case of a generic rectangular, but not square, domain, in comparison with the case of the domain having a perfect square shape.

In this relation it is useful to outline some detail on the meaning of the limit $x \to 0$. It is closely related to another special case of the five-vertex model, which is known as the four-vertex model [32,33]. In this model $w_4 = 0$ that can be achieved by setting $x = \alpha^2 u^2$ in (1.2), where u is a new parameter, and taking the limit $\alpha \to 0$ [17]. Recalling that $\Delta < 0$, the nonzero weights are then

$$w_1 = \frac{1}{(-\Delta)u}, \qquad w_3 = u, \qquad w_5 = w_6 = 1.$$

As it follows from (1.4) and (1.6), the partition function $Z = Z(x; \Delta, \alpha)$ is nonzero only if $L \leqslant M + 1$. It is known (see, e.g., [34]) that for all the four weights equal to 1, it can be expressed in terms of the number of boxed plane partitions:

$$\lim_{\alpha \to 0} Z(\alpha^2; -1, \alpha) = \mathrm{PL}(N, M - L + 1, L - N).$$

This gives the leading term of the polynomial $P_{N,M,L}(x^{-1})$ in the case $L \leqslant M + 1$,

$$\lim_{x \to 0} x^{N(L-N-1)} P_{N,M,L}(x^{-1}) = \binom{M}{N}^{-1} \mathrm{PL}(N, M - L + 1, L - N).$$

As we show in the next section, it is also possible to obtain a similar formula for the leading term when $L > M + 1$ (see Prop. 3.2).

2.2. Hankel determinant representations. In what follows we need to use two statements about representations for the partition function in terms of determinants.

As it has been shown in [17], the partition function Z can be written in terms of Hankel determinants of $(L-N)\times(L-N)$ or $N\times N$ matrices. This result, rephrased for the polynomial $P_{N,M,L}(x^{-1})$, reads as follows.

THEOREM 2.1. *The polynomial $P_{N,M,L}(x^{-1})$ can be given in terms of the $(L-N)\times(L-N)$ Hankel determinant,*

$$
\begin{aligned}
P_{N,M,L}(x^{-1}) = (-1)^{\frac{(L-N)(L-N-1)}{2}} \prod_{j=0}^{L-N-1} \frac{M!(M+j)!}{(M-N)!(M+L-N-1)!(N+j)!} \\
\times \frac{1}{(x-1)^{(L-N)(M-N)} x^{\frac{(L+N)(L-N-1)}{2}}} \\
\times \det_{1\leqslant i,j\leqslant L-N} \Bigg[(x\partial_x)^{i+j-2} (x-1)^{M+L-2N-1} \\
\times {}_2F_1\left(\left. \begin{matrix} -N,\ L-N-1 \\ -M \end{matrix} \right| x \right) \Bigg],
\end{aligned}
\tag{2.3}
$$

or in terms of the $N\times N$ determinant,

$$
\begin{aligned}
P_{N,M,L}(x^{-1}) = \prod_{j=0}^{N-1} \frac{(L+M-2N)!}{(L-N+j)!(M-N+j)!} \\
\times \frac{(x-1)^{N(M+L-N)}}{x^{N(L-1)-\frac{N(N+1)}{2}}} \det_{1\leqslant i,j\leqslant N} \Bigg[(x\partial_x)^{i+j-2} \frac{1}{(x-1)^{M+L-2N+1}} \\
\times {}_2F_1\left(\left. \begin{matrix} -L+N+1,\ -L+N \\ -L-M+2N \end{matrix} \right| 1-x \right) \Bigg].
\end{aligned}
\tag{2.4}
$$

Note that because of the relation $(x\partial_x)x^a = x^a(x\partial_x + a)$, the Hankel determinants which appear here, possess the property

$$
\det_{1\leqslant i,j\leqslant N} \left[(x\partial_x)^{i+j-2} x^a f(x) \right] = x^{aN} \det_{1\leqslant i,j\leqslant N} \left[(x\partial_x)^{i+j-2} f(x) \right]. \tag{2.5}
$$

Below we often use this freedom in writing the Hankel determinants.

In addition to the two representations given above, we present here one more Hankel determinant formula for $P_{N,M,L}(x^{-1})$.

PROPOSITION 2.1. *The polynomial $P_{N,M,L}(x^{-1})$ admits the following representation*

$$
\begin{aligned}
P_{N,M,L}(x^{-1}) = N! \prod_{j=0}^{N-1} \frac{(L-N-1+j)!(M-N+j)!}{(L-2)!(M-1)!} x^{\frac{N(N-1)}{2}} \\
\times \det_{1\leqslant i,j\leqslant N} \left[(x\partial_x)^{i+j-2}\, {}_2F_1\left(\left. \begin{matrix} -L+2,\ -M+1 \\ 2 \end{matrix} \right| \frac{1}{x} \right) \right].
\end{aligned}
\tag{2.6}
$$

We prove this proposition below in two steps. At the first step we show that both representations (2.3) and (2.6) satisfy the same differential equation, which is essentially the sixth Painlevé equation in its σ-form. At the second step we show that they provide the same solution of this equation. The solution can be identified in a unique way, say, as $x\to\infty$ by the first three terms of the expansion. Specifically, as far as the equivalence of (2.3) and (2.4) is established (see [17]), we obtain such an expansion from (2.4). The first step is given in Sect. 2.3 and the second step is explained in Sect. 3.1.

As shown in Sect. 3, representation (2.6) is useful in obtaining expansions of the polynomial $P_{N,M,L}(x^{-1})$ also at the points $x=0,1$, besides the point $x=\infty$. Both (2.3) and (2.4) can hardly be used for this purpose.

2.3. Connection with the sixth Painlevé equation. We begin with recalling some useful facts from the theory of sixth Painlevé equation; for a detailed exposition, see [19, 20, 24]. An important role in the theory is played by the τ-function, $\tau=\tau(t)=\tau(t;b_1,b_2,b_3,b_4)$ where t is the time variable, and $b_1,\dots,b_4$ are parameters. It is connected to the canonical Hamiltonian by the relation $H=\partial_t\log\tau(t)$, and hence defined up to a multiplicative constant. For our purposes we need only an explicit form of the τ-function corresponding to the so-called classical solutions related to the Gauss hypergeometric function (see, e.g., [24], Sect. 2.3),

$$\begin{aligned}\tau(t)&=(t(t-1))^{-(b_3+b_4)n/2}\left(\frac{t-1}{t}\right)^{(A-B)n/2}\\&\quad\times\det_{1\leqslant i,j\leqslant n}\left[\big((t-1)t\partial_t\big)^{i+j-2}\,t^A(t-1)^B\,{}_2F_1\left(\left.\begin{matrix}b_1+b_4,\ 1-b_1+b_4\\ 1+b_2+b_4\end{matrix}\right|t\right)\right],\end{aligned}\tag{2.7}$$

where the parameters are subject to the constraints

$$b_1+b_3=n,\qquad A+B=1-b_1+b_4.$$

Note that due to a relation similar to (2.5), the expression in (2.7) is independent of $A-B$, so one can always set $A=B=\frac{1}{2}(1-b_1+b_4)$. In our cases below, we use $A=0$, $B=1-b_1+b_4$.

An important property of the τ-function is that, for generic values of the parameters $b_1,\dots,b_4$, the function

$$\sigma(t)=t(t-1)\partial_t\log\tau(t)+(b_1b_3+b_1b_4+b_3b_4)t-\frac{1}{2}\sum_{1\leqslant j<k\leqslant 4}b_jb_k\tag{2.8}$$

satisfies the equation

$$\sigma'\big(t(t-1)\sigma''\big)^2+\big(\sigma'[2\sigma+(1-2t)\sigma']+b_1b_2b_3b_4\big)^2=\prod_{j=1}^{4}\big(\sigma'+b_j^2\big).\tag{2.9}$$

The function (2.8) is called σ-function and (2.9) is usually referred to as the sixth Painleve equation in its σ-form.

To show that the polynomial $P_{N,M,L}(x^{-1})$ is nothing but the τ-function, let us consider the representation (2.3). We apply the Euler transformation of the Gauss hypergeometric function,

$${}_2F_1\left(\left.\begin{matrix}a,\ b\\ c\end{matrix}\right|x\right)=(1-x)^{-a}\,{}_2F_1\left(\left.\begin{matrix}a,\ c-b\\ c\end{matrix}\right|\frac{x}{x-1}\right),$$

to the function standing in the determinant in (2.3), that yields

$${}_2F_1\left(\left.\begin{matrix}-N,\ L-N-1\\ -M\end{matrix}\right|x\right)=(1-x)^N\,{}_2F_1\left(\left.\begin{matrix}-N,\ -M-L+N+1\\ -M\end{matrix}\right|\frac{x}{x-1}\right).$$

Ignoring an overall constant factor, we then get[2]

$$P_{N,M,L}(x^{-1}) \propto \frac{1}{(x-1)^{(L-N)(M-N)} x^{\frac{(L+N)(L-N-1)}{2}}} \times \det_{1\leqslant i,j\leqslant L-N} \Bigg[(x\partial_x)^{i+j-2} (x-1)^{M+L-N-1} \times {}_2F_1\left(\begin{matrix} -N,\ -M-L+N+1 \\ -M \end{matrix} \middle| \frac{x}{x-1} \right) \Bigg].$$

If we make change of the variable

$$x = \frac{t}{t-1}, \tag{2.10}$$

then a comparison with (2.7) shows that

$$P_{N,M,L}(x^{-1}) \propto \frac{1}{t^{N(L-N-1)}(t-1)^N} \tau(t), \tag{2.11}$$

where the parameters of the τ-function are

$$b_1 = \frac{L+M}{2} - N, \quad b_2 = \frac{L-M}{2} - 1, \quad b_3 = \frac{L-M}{2}, \quad b_4 = -\frac{L+M}{2}. \tag{2.12}$$

Note that $n = L-N$, $A = 0$ and $B = -L-M+N+1$.

Let us now consider the representation (2.6). Recall that at the moment it is unproven and we need to show that it is identical to (2.3). We make the change of the variable

$$x = \frac{t-1}{t}, \tag{2.13}$$

so that (2.6) takes the following form:

$$P_{N,M,L}(x^{-1}) \propto \left(\frac{t-1}{t}\right)^{\frac{N(N-1)}{2}} \times \det_{1\leqslant i,j\leqslant N} \left[\big((t-1)t\partial_t\big)^{i+j-2} {}_2F_1\left(\begin{matrix} -L+2,\ -M+1 \\ 2 \end{matrix} \middle| \frac{t}{t-1} \right) \right].$$

After the Euler transformation we get

$$P_{N,M,L}(x^{-1}) \propto \left(\frac{t-1}{t}\right)^{\frac{N(N-1)}{2}} \times \det_{1\leqslant i,j\leqslant N} \left[\big((t-1)t\partial_t\big)^{i+j-2} (t-1)^{-L+2} {}_2F_1\left(\begin{matrix} M+1,\ -L+2 \\ 2 \end{matrix} \middle| t \right) \right].$$

A comparison with (2.7) shows that

$$P_{N,M,L}(x^{-1}) \propto \frac{1}{(t-1)^{N(L-N-1)}} \tau(t), \tag{2.14}$$

where the parameters of the τ-function are

$$b_1 = \frac{L+M}{2}, \quad b_2 = \frac{L-M}{2}, \quad b_3 = N - \frac{L+M}{2}, \quad b_4 = \frac{M-L}{2} + 1. \tag{2.15}$$

In this case $n = N$, $A = 0$, and $B = -L+2$.

[2]Recall that $f(x) \propto g(x)$ means that $f(x) = Cg(x)$ for some constant C.

A crucial observation which can be made by inspecting (2.12) and (2.15) is that these two sets of the parameters can be obtained one from another, modulo signs of the elements. To get more insight on the relation between these τ-functions, it is useful to consider the corresponding σ-functions appearing in both cases. In the first case, described by (2.10), (2.11) and (2.12), the σ-function constructed by (2.8) reads

$$\sigma(t) = \frac{t-1}{t}\frac{P'_{N,M,L}(x^{-1})}{P_{N,M,L}(x^{-1})}\bigg|_{x=\frac{t}{t-1}} - \left(N - \frac{L+M}{2}\right)^2 t + N^2 - \frac{3N(M+L)}{4} + \frac{N+ML}{2} + \frac{L-M}{4}. \quad (2.16)$$

In the second case, described by (2.13), (2.14) and (2.15), the σ-function constructed by (2.8) reads

$$\sigma(t) = \frac{t}{t-1}\frac{P'_{N,M,L}(x^{-1})}{P_{N,M,L}(x^{-1})}\bigg|_{x=\frac{1-t}{t}} - \left(N - \frac{L+M}{2}\right)^2 t - \frac{N(M+L)}{4} - \frac{N}{2} + \frac{L^2+M^2-L+M}{4}. \quad (2.17)$$

It is easy to see that the two σ-functions (2.16) and (2.17) are related by the map

$$\sigma(t) \mapsto -\sigma(1-t), \qquad b_1b_2b_3b_4 \mapsto -b_1b_2b_3b_4. \quad (2.18)$$

The map (2.18) leaves the σ-form (2.9) intact and it is an example of symmetry transformations of the sixth Painleve equation [20].

Furthermore, using these transformations (for further details, see [20], Sect. 4) one can obtain the σ-function directly in terms of our initial variable x. This can be done by making the corresponding change of the variables $t \mapsto x$ in each of the two cases (2.10) and (2.13). In each case the set of parameters $\{b_1, b_2, b_3, b_4\}$ is mapped into another set of parameters $\{\nu_1, \nu_2, \nu_3, \nu_4\}$. In our construction the map (2.18) guarantees that the resulting σ-form appears to be the same in both cases, that is the two expressions for the polynomial $P_{N,M,L}(x^{-1})$ provided by (2.3) and (2.6) satisfy the same equation.

This result in terms of the variable x can be formulated as follows.

Proposition 2.2. *The σ-function*

$$\sigma(x) = x(x-1)\partial_x \log P_{N,M,L}\left(x^{-1}\right) - \widetilde{A}\,x + \widetilde{B}, \quad (2.19)$$

with

$$\widetilde{A} = \frac{(N+1)^2}{4}, \qquad \widetilde{B} = \frac{L(M+1)}{2} - \frac{(L+M)(3N+1)}{4} + \frac{N}{2} + N^2. \quad (2.20)$$

satisfies the sixth Painlevé equation in its σ-form

$$\sigma'\left(x(x-1)\sigma''\right)^2 + \left(\sigma'[2\sigma + (1-2x)\sigma'] + \nu_1\nu_2\nu_3\nu_4\right)^2 = \prod_{j=1}^{4}\left(\sigma' + \nu_j^2\right), \quad (2.21)$$

where $\sigma' \equiv \partial_x\sigma(x)$, $\sigma'' \equiv \partial_x^2\sigma(x)$, and the parameters $\nu_1, \ldots, \nu_4$ can be chosen to be

$$\nu_1 = M - \frac{N-1}{2}, \quad \nu_2 = -L + \frac{N+1}{2}, \quad \nu_3 = \frac{N+1}{2}, \quad \nu_4 = \frac{N-1}{2}. \quad (2.22)$$

In [17] (see Prop. 9 therein) this proposition have been formulated for the partition function Z, related to the σ-function as

$$\sigma(x) = x(x-1)\partial_x \log Z - A\,x + B,$$

with

$$A = \frac{LM}{2} + \frac{(N-1)^2}{4}, \qquad B = \frac{(N+1)(L+M-2N)}{4} + \frac{N^2-M}{2}.$$

For a later use we also note that instead of the parameters M and L, one can use the parameters ν_1 and ν_2 given in (2.22) together with N (which, in turn, is related to ν_3 and ν_4). For example, the constant $\widetilde{B}$ in (2.20) can be written as

$$\widetilde{B} = -\frac{\nu_1\nu_2}{2} - \frac{N(\nu_1-\nu_2)}{2} + \frac{3N^2+1}{8} + \frac{N}{2}. \tag{2.23}$$

Below we will often use this way of writing for various expressions in addressing their behavior as $N, M, L \to \infty$.

Given function (2.19) one can reconstruct the polynomial $P_{N,M,L}(x^{-1})$ by integrating the σ-function,

$$\log P_{N,M,L}(x^{-1}) = \int \Big(\sigma(x) + \widetilde{A}x - \widetilde{B}\Big) \frac{\mathrm{d}x}{x(x-1)} + \widetilde{C}, \tag{2.24}$$

where $\widetilde{C}$ is some integration constant.

3. Asymptotic expansions at the singular points

To uniquely identify the solution of the sixth Painlevé equation (2.21) as being governed by the polynomial $P_{N,M,L}(x^{-1})$, we use the asymptotic expansions of this polynomial at the singular points of the sixth Painleve equation, namely, at the points $x = 0, 1, \infty$. It is known from the general theory (see, e.g., [21]) that a solution is uniquely determined by at least first three terms of the asymptotic expansion. In this section we construct these expansions of $P_{N,M,L}(x^{-1})$ and obtain those for the σ-function.

3.1. Expansion as $x \to \infty$. Here we have the following result.

PROPOSITION 3.1. *As $x \to \infty$,*

$$P_{N,M,L}(x^{-1}) = 1 + \frac{\kappa_1}{x} + \frac{\kappa_2}{x^2} + O\left(x^{-3}\right), \tag{3.1}$$

where the coefficients are

$$\kappa_1 = \frac{abc}{a+1}, \qquad \kappa_2 = \frac{bc\,[a(a+1)(bc+1) - (b+1)(c+1)]}{2(a+1)(a+2)}, \tag{3.2}$$

with

$$a = N, \qquad b = L - N - 1, \qquad c = M - N.$$

We first show how this result follows from the representation (2.4). Next, we will show that the same result follows from (2.6), that completes the proof of Prop. 2.1.

We start with a standard calculation from the random matrix theory. Let $\mu(m)$ denote an arbitrary measure on $\mathbb{Z}_{\geqslant 0}$. We represent the Hankel determinant as a multiple sum

$$\begin{aligned}\det_{1\leqslant i,j\leqslant N}\left[\sum_{m=0}^{\infty} m^{i+j-2}\frac{\mu(m)}{x^m}\right] &= \sum_{m_1,\ldots,m_N=0}^{\infty}\det_{1\leqslant i,j\leqslant N}\left[m_j^{i+j-2}\right]\prod_{j=1}^{N}\frac{\mu(m_j)}{x^{m_j}}\\ &= \sum_{0\leqslant m_1<\ldots<m_N\leqslant\infty}\prod_{1\leqslant i<j\leqslant N}(m_j-m_i)^2\prod_{j=1}^{N}\frac{\mu(m_j)}{x^{m_j}}.\end{aligned} \tag{3.3}$$

From this expression it is clear that the leading term of the $x\to\infty$ expansion of the determinant corresponds to the values $m_i = i-1$, $i=1,\ldots,N$. The first order correction to the leading term comes from the values

$$m_i = i-1, \quad i=1,\ldots,N-1, \qquad m_N = N.$$

The second-order correction is the sum of two contributions, which corresponds to the values:

$$m_i = i-1, \quad i=1,\ldots,N-1, \qquad m_N = N+1,$$

and

$$m_i = i-1, \quad i=1,\ldots,N-2, \qquad m_{N-1}=N, \qquad m_N = N+1.$$

Hence, as $x\to\infty$, we have

$$\det_{1\leqslant i,j\leqslant N}\left[\sum_{m=0}^{\infty} m^{i+j-2}\frac{\mu(m)}{x^m}\right] = \frac{C}{x^{N(N-1)/2}}\left(1+\frac{\gamma_1}{x}+\frac{\gamma_2}{x^2}+O\left(\frac{1}{x^3}\right)\right), \tag{3.4}$$

where

$$C = \prod_{1\leqslant i<j\leqslant N}(j-i)^2\prod_{j=0}^{N-1}\mu(j) = \prod_{j=0}^{N-1}(j!)^2\mu(j).$$

The coefficients γ_1 and γ_2 can be readily computed to be

$$\begin{aligned}\gamma_1 &= \frac{\mu(N)}{\mu(N-1)}N^2,\\ \gamma_2 &= \frac{\mu(N+1)}{\mu(N-1)}\left(\frac{N(N+1)}{2}\right)^2+\frac{\mu(N)}{\mu(N-2)}\left(\frac{N(N-1)}{2}\right)^2.\end{aligned} \tag{3.5}$$

Let us consider the determinant in (2.4), namely, we focus on the Gauss hypergeometric function determining the elements of the matrix. Using the identity

$${}_2F_1\left(\left.\begin{matrix}a,\, b\\ c\end{matrix}\,\right|z\right) = (1-z)^{-a}\,{}_2F_1\left(\left.\begin{matrix}a,\, c-b\\ c\end{matrix}\,\right|\frac{z}{z-1}\right), \tag{3.6}$$

we first rewrite it in the form

$${}_2F_1\left(\left.\begin{matrix}-L+N+1,\, -L+N\\ -L-M+2N\end{matrix}\,\right|1-x\right) = x^{L-N-1}\,{}_2F_1\left(\left.\begin{matrix}-L+N+1,\, -M+N\\ -L-M+2N\end{matrix}\,\right|1-\frac{1}{x}\right).$$

The ${}_2F_1$-function in the right-hand side in the above relation is a polynomial in x^{-1} of the degree $\min(L-N-1, M-N)$. It can further rewritten in the form

$$
{}_2F_1\left(\begin{matrix} -L+N+1,\ -M+N \\ -L-M+2N \end{matrix}\middle| 1-\frac{1}{x}\right)
= \frac{(L-N)!(M-N+1)!}{(L+M-2N)!}\, {}_2F_1\left(\begin{matrix} -L+N+1,\ -M+N \\ 2 \end{matrix}\middle| \frac{1}{x}\right).
$$

Let us now take into account the $(1-x)$-factor standing in the determinant in (2.4). Applying (3.6) twice, one has the relation

$$
{}_2F_1\left(\begin{matrix} a,\ b \\ c \end{matrix}\middle| z\right) = (1-z)^{c-a-b}\, {}_2F_1\left(\begin{matrix} c-a,\ c-b \\ c \end{matrix}\middle| z\right).
$$

This relation implies that

$$
\frac{1}{(1-x^{-1})^{M+L-2N+1}}\, {}_2F_1\left(\begin{matrix} -L+N+1,\ -M+N \\ 2 \end{matrix}\middle| \frac{1}{x}\right)
= {}_2F_1\left(\begin{matrix} L-N+1,\ M-N+2 \\ 2 \end{matrix}\middle| \frac{1}{x}\right).
$$

In total, we have thus obtained the identity

$$
\begin{aligned}
\frac{1}{(1-x)^{M+L-2N+1}}\, {}_2F_1\left(\begin{matrix} -L+N+1,\ -L+N \\ -L-M+2N \end{matrix}\middle| 1-x\right)
&= (-1)^{M+L+1} \frac{(L-N)!(M-N+1)!}{(L+M-2N)!} \\
&\quad \times \frac{1}{x^{M-N+2}}\, {}_2F_1\left(\begin{matrix} L-N+1,\ M-N+2 \\ 2 \end{matrix}\middle| \frac{1}{x}\right).
\end{aligned}
$$

As a result, using also identity (2.5) to move the x-factor from the determinant in (2.4), we find that the polynomial $P_{N,M,L}(x^{-1})$ admits the following representation:

$$
\begin{aligned}
P_{N,M,L}(x^{-1}) = N! \prod_{i=0}^{N-1} \frac{(L-N)!(M-N+1)!}{(L-N+i)!(M-N+1+i)!} \left(1-\frac{1}{x}\right)^{N(M+L-N)} \\
\times x^{\frac{N(N-1)}{2}} \det_{1\leqslant i,j\leqslant N} \left[(x\partial_x)^{i+j-2}\, {}_2F_1\left(\begin{matrix} L-N+1,\ M-N+2 \\ 2 \end{matrix}\middle| \frac{1}{x}\right)\right].
\end{aligned} \tag{3.7}
$$

The determinant here is of the form (3.3), with

$$
\mu(m) = \frac{(L-N+1)_m (M-N+2)_m}{(m+1)!m!},
$$

where the standard notation for the Pochhammer symbol have been used,

$$
(z)_m := z(z+1)\cdots(z+m-1).
$$

Hence, as $x \to \infty$,

$$\det_{1\leqslant i,j\leqslant N}\left[(x\partial_x)^{i+j-2}\,{}_2F_1\left(\begin{matrix}L-N+1,\ M-N+2\\ 2\end{matrix}\,\middle|\,\frac{1}{x}\right)\right]$$
$$=\frac{\prod_{j=0}^{N-1}(L-N+1)_j(M-N+2)_j}{N!\,x^{N(N-1)/2}}\left\{1+\frac{L(M+1)N}{N+1}\frac{1}{x}\right.$$
$$\left.+\frac{L(M+1)}{4}\left(\frac{(L+1)(M+2)}{N+2}+\frac{(L-1)M}{N+1}\right)\frac{1}{x^2}+O(x^{-3})\right\}. \quad (3.8)$$

Clearly, the leading term here cancels the prefactor in (3.7), so that $P_{N,M,L}(0)=1$. Furthermore, expanding the factor $(1-x^{-1})^{N(M+L-N)}$ in (3.7) in the Taylor series, from (3.8) one can easily obtain the coefficients κ_1 and κ_2 in (3.1). This finalizes the proof of Prop. 3.1.

Let us now comment that exactly the same result follows from the new representation (2.6). Indeed, for the determinant in (2.6) we have the expansion (3.4), with

$$\mu(m)=\frac{(-L+2)_m(-M+1)_m}{(m+1)!m!}. \quad (3.9)$$

More explicitly, we have

$$\det_{1\leqslant i,j\leqslant N}\left[(x\partial_x)^{i+j-2}\,{}_2F_1\left(\begin{matrix}-L+2,\ -M+1\\ 2\end{matrix}\,\middle|\,\frac{1}{x}\right)\right]$$
$$=\frac{\prod_{j=0}^{N-1}(-L+2)_j(-M+1)_j}{N!\,x^{N(N-1)/2}}\left\{1+\frac{\kappa_1}{x}+\frac{\kappa_2}{x^2}+O\left(x^{-3}\right)\right\},$$

where κ_1 and κ_2 are exactly those given by (3.2). Clearly, the overall constant here cancels the prefactor in (2.6), and the property $P_{N,M,L}(0)=1$ is recovered. In total, Prop. 3.1 follows from (2.6), as it should. This finalizes the proof of Prop. 2.1.

Using Prop. 3.1 together with Prop. 2.2 one can compute an expansion of the corresponding σ-function near the point $x=\infty$, namely

$$\sigma(x)=-\widetilde{A}x+\widetilde{B}-\kappa_1+\frac{\kappa_1+\kappa_1^2-2\kappa_2}{x}+O\big(x^{-2}\big),$$

where κ_1 and κ_2 are to be taken from (3.2) and $\widetilde{A}$ and $\widetilde{B}$ are given by (2.20). More explicitly, we have the following.

Corollary 3.1.1. *The σ-function* (2.19) *has the following $x\to\infty$ behavior*

$$\sigma(x)=-\frac{(N+1)^2}{4}x+\frac{N-1}{2(N+1)}\nu_1\nu_2+\frac{(N+1)^2}{8}$$
$$+\frac{\left[\nu_1^2-\left(\frac{N+1}{2}\right)^2\right]\left[\nu_2^2-\left(\frac{N+1}{2}\right)^2\right]}{(N+1)^2(N+2)}x^{-1}+O\big(x^{-2}\big), \quad (3.10)$$

where ν_1 and ν_2 are given by (2.22).

3.2. Expansion as $x\to 0$. Next we consider asymptotic behavior of the polynomial $P_{N,M,L}(x^{-1})$ near the point $x=0$. We have the following result.

Proposition 3.2. *As $x\to 0$,*

$$P_{N,M,L}(x^{-1})=\frac{C}{x^{ac}}\left\{1+\kappa_1 x+\kappa_2 x^2+O(x^{-3})\right\}, \quad (3.11)$$

where

$$C=\binom{a+c}{a}^{-1}\mathrm{PL}(a,b,c)=\binom{a+b+c}{a}^{-1}\mathrm{PL}(a,b,c+1)$$

and the coefficients are

$$\kappa_1=\frac{ac(c+1)}{a+b},\qquad \kappa_2=\frac{ac(c+1)}{a+b}\,\frac{(c^2+c+1)(a^2+ab-1)-b-2bc}{2(a+b-1)(a+b+1)},\tag{3.12}$$

with

$$a=N,\qquad b=|M-L+1|,\qquad c=\min(L-N-1,M-N).\tag{3.13}$$

We will prove Prop. 3.2 using representation (2.6). Constructively, the proof will be based again on formula (3.4) in which we make the change $x^{-1}\mapsto x$. The details however depend on whether $M\geqslant L-1$ or $M\leqslant L-1$.

Let us perform calculations assuming that $M\geqslant L-1$. We first transform the ${}_2F_1$-function in the determinant in (2.6) such that it will become a polynomial in x. For this end one can use the following relation valid for m and n positive integers, $n\leqslant m$, and c real (not to be confused with that in Prop. 3.2), $c\geqslant 1$:

$${}_2F_1\left(\left.\begin{matrix}-n,\,-m\\ c\end{matrix}\right|x\right)=\frac{m!}{(m-n)!\,(c)_n}x^n\,{}_2F_1\left(\left.\begin{matrix}-n,\,-n-c+1\\ m-n+1\end{matrix}\right|\frac{1}{x}\right).$$

For $M\geqslant L-1$, one has

$$\begin{aligned}{}_2F_1\left(\left.\begin{matrix}-L+2,\,-M+1\\ 2\end{matrix}\right|\frac{1}{x}\right)&=\frac{(M-1)!}{(M-L+1)!(L-1)!}\\&\quad\times x^{-L+2}\,{}_2F_1\left(\left.\begin{matrix}-L+2,\,-L+1\\ M-L+2\end{matrix}\right|x\right).\end{aligned}$$

Removing the factor x^{-L+2} from determinant by relation (2.5), we then have

$$\begin{aligned}P_{N,M,L}(x^{-1})&=N!\prod_{j=0}^{N-1}\frac{(L-N-1+j)!(M-N+j)!}{(L-2)!(M-L+1)!(L-1)!}\\&\quad\times x^{\frac{N(N-1)}{2}-(L-2)N}\det_{1\leqslant i,j\leqslant N}\left[(x\partial_x)^{i+j-2}\,{}_2F_1\left(\left.\begin{matrix}-L+2,\,-L+1\\ M-L+2\end{matrix}\right|x\right)\right].\end{aligned}$$

The determinant here can be written, up to the change $x\mapsto x^{-1}$, in the form (3.4), with

$$\mu(m)=\frac{(-L+2)_m(-L+1)_m}{(M-L+2)_m m!}.\tag{3.14}$$

Hence, for $M\geqslant L-1$, as $x\to 0$,

$$\begin{aligned}\det_{1\leqslant i,j\leqslant N}\left[(x\partial_x)^{i+j-2}\,{}_2F_1\left(\left.\begin{matrix}-L+2,\,-L+1\\ M-L+2\end{matrix}\right|x\right)\right]&=\prod_{j=0}^{N-1}\frac{(-L+2)_j(-L+1)_j j!}{(M-L+2)_j}\\&\quad\times x^{\frac{N(N-1)}{2}}\left\{1+\kappa_1x+\kappa_2x^2+O(x^3)\right\},\end{aligned}$$

where the constants κ_1 and κ_2 can be computed by the formulas (3.5) for γ_1 and γ_2, respectively, with $\mu(m)$ in (3.14), and they are are given by the expressions in (3.12) with

$$a=N,\qquad b=M-L+1,\qquad c=L-N-1.$$

As a result, we obtain that for $M \geqslant L-1$, as $x \to 0$,

$$P_{N,M,L}(x^{-1}) = \frac{C}{x^{(L-N-1)N}} \left\{1 + \kappa_1 x + \kappa_2 x^2 + O(x^3)\right\},$$

where

$$C = N! \prod_{j=0}^{N-1} \frac{(M-N+j)!j!}{(L-N+j)!(M-L+1+j)!}.$$

This constant can also be written as follows:

$$C = \frac{N!(L-N-1)!}{(L-1)!} \prod_{j=0}^{N-1} \frac{(M-N+j)!j!}{(L-N-1+j)!(M-L+1+j)!}.$$

Here the product can be recognized as the numbers of the boxed plane partitions $\mathrm{PL}(N, M-L+1, L-N-1)$, see (2.2). Equivalently, one can write

$$C = \frac{N!(M-N)!}{M!} \prod_{j=0}^{N-1} \frac{(M-N+1+j)!j!}{(L-N+j)!(M-L+1+j)!},$$

where the product equals $\mathrm{PL}(N, M-L+1, L-N)$, which is essentially the partition function of the four-vertex model, see (1.5) and the discussion in Sect. 2.1.

In the case $M \leqslant L-1$, the calculations are essentially similar. In fact all formulas in this case can be obtained from those given above by formal substitution $M \leftrightarrow L-1$. As a result, this lead to the expansion (3.11), where the parameters a, b, c are given by (3.13). This finalize the proof of Prop. 3.2.

From Props. 3.2 and 2.2 it follows that the σ-function as $x \to 0$ is given by

$$\sigma(x) = ac + \widetilde{B} - \big(ac + \widetilde{A} + \kappa_1\big)x + \big(\kappa_1 + \kappa_1^2 - 2\kappa_2\big)x^2 + O\big(x^3\big),$$

where κ_1 and κ_2 are given in (3.12) and a and c are defined in (3.13). More explicitly, we have the following.

Corollary 3.2.1. *The σ-function* (2.19) *has the following $x \to 0$ behavior:*

$$\sigma(x) = -\frac{\nu_1\nu_2}{2} - \frac{N|\nu_1+\nu_2|}{2} - \frac{N^2-1}{8} + \frac{N\nu_1\nu_2 + \frac{N^2-1}{4}|\nu_1+\nu_2|}{|\nu_1+\nu_2|+N}x \\ + \frac{N|\nu_1+\nu_2|\Big[\big(\nu_1\nu_2 + \frac{N}{2}|\nu_1+\nu_2| + \frac{N^2+1}{4}\big)^2 - \frac{1}{4}(|\nu_1+\nu_2|+N)^2\Big]}{(|\nu_1+\nu_2|+N)^2\big[(|\nu_1+\nu_2|+N)^2-1\big]}x^2 \\ + O\big(x^3\big). \tag{3.15}$$

It is interesting to note, that if $\nu_1+\nu_2 > 0$, that is $M > L-1$, this expression can be conveniently written in the form

$$\sigma(x) = -\frac{S_2}{2} + \frac{S_3}{S_1}x + \frac{\prod_{i<j}(\nu_i+\nu_j)}{S_1^2\big[S_1^2-1\big]}x^2 + O\big(x^3\big), \tag{3.16}$$

where

$$S_1 = \sum_i \nu_i, \qquad S_2 = \sum_{i<j} \nu_i\nu_j, \qquad S_3 = \sum_{i<j<k} \nu_i\nu_j\nu_k.$$

If $\nu_1+\nu_2 < 0$, then one has to make the replacement $\nu_{1,2} \mapsto -\nu_{2,1}$. Finally, if $\nu_1+\nu_2 = 0$, we just have

$$\sigma(x) = \frac{\nu_1^2}{2} - \frac{N^2-1}{8} - \nu_1^2 x + O\big(x^3\big). \tag{3.17}$$

3.3. Expansion as $x \to 1$. We conclude this section by considering the technically most difficult case of an expansion at the point $x = 1$.

PROPOSITION 3.3. *As $x \to 1$,*

$$P_{N,M,L}(x^{-1}) = C\left\{1 + \kappa_1(x-1) + \kappa_2(x-1)^2 + O\left((x-1)^3\right)\right\}, \tag{3.18}$$

where

$$C = \binom{a+c}{a}^{-1} \mathrm{PL}(a, b+1, c) \tag{3.19}$$

and the coefficients are

$$\kappa_1 = -\frac{abc}{b+c+1}, \quad \kappa_2 = abc\frac{abc(b+c+1) + b^2 + c^2 + 3bc + 3c + 3b + a + 1}{2(b+c)(b+c+1)(b+c+2)}, \tag{3.20}$$

with

$$a = N, \qquad b = L - N - 1, \qquad c = M - N. \tag{3.21}$$

We will prove this proposition using the connection of the Hankel matrix in the representation (2.6) at $x = 1$ with the ensemble of the Hahn polynomials, for a list of properties of the Hahn polynomials, see, e.g., [35], Sect. 9.5. We will use the Hahn polynomials in the normalization with the highest coefficient being equal to one,

$$p_i(m) = \frac{(a+1)_i(-n)_i}{(i+\alpha+\beta+1)_i}\, {}_3F_2\left(\left.\begin{matrix} -i,\ i+\alpha+\beta+1,\ -x \\ \alpha+1,\ -n \end{matrix}\right| 1\right).$$

For $\alpha, \beta > -1$ or $\alpha, \beta < -n$ (see [35], formula (9.5.2)) these polynomials satisfy the orthogonality condition

$$\sum_{m=0}^{n} \binom{\alpha+m}{m}\binom{\beta+n-m}{n-m} p_i(m)p_j(m) = \delta_{ij}h_i, \tag{3.22}$$

with

$$h_i = \frac{i!}{(n-i)!}\frac{(i+\alpha+\beta+1)_{n+1}(\alpha+1)_i(\beta+1)_i}{(i+\alpha+\beta+1)_i(i+\alpha+\beta+1)_{i+1}}. \tag{3.23}$$

To see the connection of the Hankel matrix in (2.6) at $x = 1$ with the ensemble of the Hahn polynomials, we multiply the entries of the matrix by the factor $(-1)^{i+j-2}$ that obviously does not alter the value of the determinant, and denote the resulting matrix by $H(x)$. Its entries are

$$(H(x))_{ij} = \sum_{m\geqslant 0} \mu(m)\frac{m^{i+j-2}}{x^m} \qquad (i, j = 1, \ldots, N),$$

where the measure $\mu(m)$ is given by (3.9). This measure is essentially that of the Hahn polynomials, due to the identity

$$\frac{(-L+2)_m(-M+1)_m}{(m+1)!m!} = \frac{(-1)^n}{n+1}\binom{\alpha+m}{m}\binom{\beta+n-m}{n-m},$$

where one should set

$$\begin{aligned} \alpha &= \min(-M, -L+1), \\ \beta &= \max(-L, -M-1), \\ n &= \min(L-2, M-1). \end{aligned} \tag{3.24}$$

Note that we deal with the case $\alpha, \beta < -n$; we recall that $\binom{-a+m}{m} = (-1)^m \binom{a-1}{m}$. The three parameters α, β and n are constrained by the condition

$$n = -\beta - 2. \tag{3.25}$$

To simplify writing below, we denote $H \equiv H(1)$. Since the entries of H are independent of N, the size of the matrix, the orthogonality condition (3.22) yields

$$\det H = \prod_{i=0}^{N-1} \tilde{h}_i, \qquad \tilde{h}_i \equiv \frac{(-1)^n}{n+1} h_i.$$

Plugging (3.24) into (3.23) gives

$$\tilde{h}_i = \frac{i!(L-2)!(M-1)!(M+L-1-2i)!(M+L-2-2i)!}{(L-1-i)!(L-2-i)!(M-i)!(M-1-i)!(M+L-1-i)!},$$

and rearranging the factors in the product, using

$$\prod_{i=0}^{N-1} (a+2i)!(a+1+2i)! = \prod_{i=0}^{2N-2} (a+i)! = \prod_{i=0}^{N-1} (a+i)!(a+N+i)!,$$

one can readily find

$$\det H = \prod_{i=0}^{N-1} \frac{(L-2)!(M-1)!(M+L-2N+i)!i!}{(L-1-i)!(L-2-i)!(M-i)!(M-1-i)!}.$$

As a result, from (2.6) it follows that the constant C in (3.18) reads

$$\begin{aligned} C &= N! \prod_{i=0}^{N-1} \frac{(M+L-2N+i)!i!}{(L-N+i)!(M-N+1+i)!} \\ &= \frac{N!(M-N)!}{M!} \prod_{i=0}^{N-1} \frac{(M+L-2N+i)!i!}{(L-N+i)!(M-N+i)!}, \end{aligned}$$

where the product in the second equality can easily recognized as the number of the boxed plane partitions $\mathrm{PL}(N, L-N, M-N)$, see (2.2).

To compute the coefficients κ_1 and κ_2 in (3.18) we first compute the coefficients in the Taylor-series expansion

$$\frac{\det H(x)}{\det H} = 1 + \gamma_1 (x-1) + \gamma_2 (x-1)^2 + O\left((x-1)^3\right)$$

from the relation $\det H(x) = \exp\{\operatorname{tr} \log H(x)\}$, that gives

$$\gamma_1 = \operatorname{tr}(H^{-1}H'), \qquad \gamma_2 = \frac{1}{2}\left\{\left(\operatorname{tr} H^{-1}H'\right)^2 + \operatorname{tr} H^{-1}H'' - \operatorname{tr}(H^{-1}H')^2\right\}, \tag{3.26}$$

where $H' \equiv H'(x)|_{x=1}$ and $H'' \equiv H''(x)|_{x=1}$. The entries of the matrix H^{-1} can be expressed (see, e.g., [36], Thm. 1.1, and also [37], Thm. 9) in terms of the coefficients of the polynomials $p_i(m)$,

$$\left(H^{-1}\right)_{jk} = \sum_{i=0}^{N-1} \tilde{h}_i^{-1} p_{i,j-1} p_{i,k-1}, \qquad p_i(m) = \sum_{k=0}^{i} p_{i,k} m^k.$$

The traces in (3.26) can be evaluated with the help of the recurrence relation (see [35], formula (9.5.4))

$$m p_i(m) = p_{i+1} + B_i p_i(m) + C_i p_{i-1}(m).$$

Here,

$$C_i = \frac{h_i}{h_{i-1}} \tag{3.27}$$

and the coefficient B_i in the case of condition (3.25) can be written in the form

$$B_i = \frac{(\alpha+\beta)(\alpha-\beta)(\alpha-\beta-2)}{4(2i+\alpha+\beta)(2i+\alpha+\beta+2)} - \frac{\alpha+\beta}{4} - 1. \tag{3.28}$$

The first trace in (3.26) can be computed as follows:

$$\operatorname{tr} H^{-1}H' = -\sum_{m=0}^{r} m\mu(m) \sum_{i=0}^{N-1} \tilde{h}_i^{-1} p_i^2(m) = -\sum_{i=0}^{N-1} B_i.$$

Essentially similarly, but slightly more involved calculation gives

$$\operatorname{tr} H^{-1}H'' = \sum_{i=0}^{N-1} \left(B_i + B_i^2\right) + 2\sum_{i=1}^{N-1} C_i + C_N,$$

$$\operatorname{tr}(H^{-1}H')^2 = \sum_{i=0}^{N-1} B_i^2 + 2\sum_{i=1}^{N-1} C_i.$$

Hence,

$$\gamma_1 = -\sum_{i=0}^{N-1} B_i, \qquad \gamma_2 = \frac{1}{2}\left(\gamma_1^2 - \gamma_1 + C_N\right).$$

To compute the sum of B_i's we expand the rational part in (3.28) in elementary ratios, that gives

$$\sum_{i=0}^{N-1} B_i = \frac{N}{4}\left(\frac{(\alpha-\beta)(\alpha-\beta-2)}{2N+\alpha+\beta} - \frac{\alpha+\beta}{4} - 1\right).$$

We also have (see (3.23), (3.25) and (3.27))

$$C_N = -\frac{N(N+\alpha-1)(N+\alpha)(N+\beta+1)(N+\beta)(N+\alpha+\beta)}{(2N+\alpha+\beta-1)(2N+\alpha+\beta)^2(2N+\alpha+\beta+1)}.$$

Expanding the factor $x^{\frac{N(N-1)}{2}}$ in (2.6) at $x=1$, for the coefficients κ_1 and κ_2 in (3.18) we obtain

$$\kappa_1 = \gamma_1 + \frac{N(N-1)}{2}, \qquad \kappa_2 = \frac{1}{2}\left(\kappa_1^2 - \kappa_1 + C_N\right).$$

Finally, plugging α and β from (3.24) into the above expressions and simplifying, we arrive at (3.20). This completes the proof of Prop. 3.3.

From Props. 2.2 and 3.3 it follows that the σ-function has the following expression near the point $x=1$:

$$\sigma(x) = \widetilde{B} - \widetilde{A} + \left(\kappa_1 - \widetilde{A}\right)(x-1) + \left(\kappa_1 + 2\kappa_2 - \kappa_1^2\right)(x-1)^2 + O\big((x-1)^3\big).$$

Here κ_1 and κ_2 are given by (3.20), and $\widetilde{A}$ and $\widetilde{B}$ are given in (2.20); note that the coefficient of the second-order term is equal to the constant C_N. In terms of ν_1 and ν_2 the result reads as follows.

Corollary 3.3.1. *The σ-function* (2.19) *has the following $x \to 1$ behavior:*

$$\sigma(x) = -\frac{\nu_1\nu_2}{2} - \frac{N(\nu_1-\nu_2)}{2} + \frac{N^2-1}{8} + \frac{N\nu_1\nu_2 + \frac{N^2-1}{4}(\nu_1-\nu_2)}{\nu_1-\nu_2-N}(x-1)$$
$$+ \frac{N(\nu_1-\nu_2)\Big[\big(\nu_1-\frac{N}{2}\big)^2-\frac{1}{4}\Big]\Big[\big(\nu_2+\frac{N}{2}\big)^2-\frac{1}{4}\Big]}{(\nu_1-\nu_2-N)^2\big[(\nu_1-\nu_2-N)^2-1\big]}(x-1)^2 + O\big((x-1)^3\big). \quad (3.29)$$

Note that the coefficients in this expansion can also be written in the form analogous to (3.16) up to the change $\nu_1 \mapsto -\nu_1$.

4. Thermodynamic limit in the symmetric case

In this section we focus on construction of the asymptotic expansions for the σ-function and the corresponding polynomial $P_{N,M,L}(x^{-1})$ in the limit $N, M, L \to \infty$ such the two parameters p and q defined by (1.8) are finite and equal to each other, $p = q$. In this case it is suitable to use the parameters $r \equiv p = q$ and ϵ defined in (1.10). This will provide a proof of Thm. 1.1. The non-symmetric case, $p \neq q$, is considered in the next section.

4.1. Preliminaries. To derive expansions of the function $\log P_{N,M,L}(x^{-1})$ in the thermodynamic limit, we start with analyzing the σ-form of the sixth Painlevé equation in the large N limit. We recall that the σ-function is given in terms of $\log P_{N,M,L}(x^{-1})$ by (2.19). Expressions (2.22) for the parameters $\nu_1, \dots, \nu_4$ and the expansions of the σ-function at the singular points given by Cor. 3.1.1, 3.2.1, and 3.3.1, suggest that the σ-function may be searched in the form of the following asymptotic ansatz in the decaying powers of N:

$$\sigma(x) = \sum_{k\geqslant 0} N^{2-k}\sigma_{2-k}(x). \quad (4.1)$$

Following [18], we note that the justification of ansatz (4.1) is based on the Wasow theorem, see [27], Chap. IX, Thm. 36.1. This theorem implies that if one succeeds in the construction of the expansion (4.1) with the functions $\sigma_i(x)$, which are piecewise *analytic* functions of x, then there exists a genuine solution of equation (2.21) with the asymptotic expansion (4.1). To justify that this solution indeed coincides with the solution given by (2.19), one has to verify that they have the same behaviors at the singular points, $x = 0, 1, \infty$, given by Cors. 3.1.1, 3.2.1, and 3.3.1. We recall that for the Painlevé equations expansions of solutions at the singular points fix the solutions [21].

The Wasow theorem is applicable to the sixth Painlevé equation in its Hamiltonian formulation. Indeed, the Wasow theorem deals with the first-order vector ordinary differential equations resolved with respect to the derivatives. The σ-function is intimately related to the Hamiltonian and there exists a one-to-one correspondence between the canonical variables (the coordinate and momentum) and the σ-function [20]. The conditions of the Wasow theorem can be verified by writing the Hamiltonian equations of motion of the sixth Painlevé system in a vector form. Details of this calculation can be found in [18], App. A. For further comments concerning the method of obtaining asymptotic expansions for the σ-function, see [18], Sect. 4.2, Rem. 4.3 and the discussion thereafter.

The expansion (4.1) can be constructed in a standard way by substituting it in the sixth Painlevé equation (2.21). On this way, we first obtain the leading term,

$\sigma_2(x)$, by requiring that it reproduces the conditions at the points $x = 0, 1, \infty$. Next we derive the further terms, and show that they can be obtained recursively. For these terms we also have to obtain that they reproduce the conditions at the points $x = 0, 1, \infty$.

Having the outlined strategy in mind, we now turn to equation (2.21) and consider the construction of the leading term of the large N expansion of the σ-function. Our first aim here is to expose how the function $\sigma_2(x)$ can be found under assumption that, as $N \to \infty$,

$$\nu_i = v_i N + O(1), \qquad i = 1, \ldots, 4, \tag{4.2}$$

where v_i are some parameters to be specified later. Below we drop the dependence on x in the notation for functions to simplify writing.

Clearly, with (4.1) and (4.2) the right-hand side of (2.21) is of $O(N^8)$ and the same is valid for the second term in the left-hand side. The first term in the left-hand side, with the second-order derivative, is just of $O(N^6)$. Excluding the trivial root of the constant solution, $\sigma' = 0$ (and hence assuming that $\sigma_2' \neq 0$), we thus find that the equation for the σ-function splits into two first-order equations for the σ_2-functions:

$$\sigma_2 = x\sigma_2' - \frac{\sigma_2'}{2} - \frac{v_1 v_2 v_3 v_4}{2\sigma_2'} \pm \frac{\sqrt{\prod_i \left(\sigma_2' + v_i^2\right)}}{2\sigma_2'}. \tag{4.3}$$

For later use we introduce two functions $f_\pm(\sigma_2')$ by rewriting these equations in the form

$$\sigma_2 = x\sigma_2' + f_\pm(\sigma_2'), \tag{4.4}$$

where the plus and minus signs match those in (4.3).

Equations (4.4) are the Clairaut equations (see, e.g., [38]), i.e., they are of the form

$$y = xy' + \Phi(y'), \qquad y' = y'(x).$$

Differentiation with respect to x gives

$$\big(x + \Phi'(y')\big)y'' = 0. \tag{4.5}$$

If $y'' = 0$, then y is a linear function,

$$y = Cx + \Phi(C), \tag{4.6}$$

where C is a constant. This is the so-called general solution of the Clairaut equation. If instead the first factor in (4.5) vanishes, then the corresponding solution is called singular solution and it is of the form

$$y = \big(xy' + \Phi(y')\big)\big|_{y'=(\Phi')^{-1}(-x)}.$$

Note that there could be many such solutions or none.

To study our problem, let us first consider the situation where the parameters $v_1, \ldots, v_4$ are related by

$$v_4 = v_3, \qquad v_2 = -v_1. \tag{4.7}$$

We also assume that $v_1 \geqslant v_3$. Then,

$$f_\pm(\sigma_2') = \begin{cases} g_\pm(\sigma_2') & \sigma_2' \in (-\infty, -v_1^2] \bigcup [-v_3^2, \infty) \\ g_\mp(\sigma_2') & \sigma_2' \in [-v_1^2, -v_3^2], \end{cases}$$

where

$$g_+(\sigma_2') = \frac{v_1^2 v_3^2}{\sigma_2'} + \frac{v_1^2 + v_3^2}{2}, \qquad g_-(\sigma_2') = -\sigma_2' - \frac{v_1^2 + v_3^2}{2}.$$

In the case of the function $g_+(\sigma_2')$ we have $g_+'(\sigma_2') = -(v_1 v_3/\sigma_2')^2$ and hence there are two singular solutions corresponding to $\sigma_2' = \pm v_1 v_3/\sqrt{x}$; in the case of the function $g_-(\sigma_2')$ we have $g_-'(\sigma_2') = -1$ and there are no singular solutions. Thus equations (4.3) have two general solutions

$$(\sigma_2)_{\mathrm{g},+} = Cx + \frac{v_1^2 v_3^2}{C} + \frac{v_1^2 + v_3^2}{2}, \qquad (\sigma_2)_{\mathrm{g},-} = C(x-1) - \frac{v_1^2 + v_3^2}{2}, \tag{4.8}$$

and two singular ones,

$$(\sigma_2)_{\mathrm{s},\pm} = \pm 2 v_1 v_3 \sqrt{x} + \frac{v_1^2 + v_3^2}{2}. \tag{4.9}$$

Note that the two general solutions may correspond to the same linear function if the integration constant C has the same value in both of them and satisfies $(C + v_1^2)(C + v_3^2) = 0$. We meet exactly this situation in our considerations below.

To proceed, we fix now values of the parameters. Recalling (2.22) and (4.7), we set

$$v_1 = -v_2 = r + \frac{1}{2} \equiv w, \qquad v_3 = v_4 = \frac{1}{2},$$

where we have introduced a new parameter w.

Consider now the function $\log P_{N,M,L}(x^{-1})$. In the leading order,

$$\log P_{N,M,L}(x^{-1}) = f_2 N^2 + O(N).$$

From (2.24) we have

$$f_2 = \int \left(\sigma_2 + \widetilde{A}_2 x - \widetilde{B}_2\right) \frac{\mathrm{d}x}{x(x-1)} + \widetilde{C}_2, \tag{4.10}$$

where $\widetilde{A}_2$, $\widetilde{B}_2$, $\widetilde{C}_2$ are $O(N^2)$ terms of the constants $\widetilde{A}$, $\widetilde{B}$, $\widetilde{C}$, respectively, e.g., $\widetilde{A} = N^2 \widetilde{A}_2 + O(N)$. The constants $\widetilde{A}$ and $\widetilde{B}$ are defined in (2.20); the constant $\widetilde{C}$ fixes the normalization, $P_{N,M,L}(0) = 1$.

We obtain $\widetilde{A}_2$ and $\widetilde{B}_2$ from $\widetilde{A} = (N+1)^2/4$ (see (2.20)) and the expression (2.23) for $\widetilde{B}$, respectively,

$$\widetilde{A}_2 = \frac{1}{4}, \qquad \widetilde{B}_2 = \frac{w^2}{2} - w + \frac{3}{8}. \tag{4.11}$$

As for the constant $\widetilde{C}_2$, it can fixed by requiring that the function f_2 attains its values at the points $x = \infty, 0, 1$, as prescribed by the statements of Props. 3.1, 3.2, and 3.3, respectively. As we see below, there exists one and only one such a function.

To obtain these values of the function f_2, we have to rely on some auxiliary asymptotic result. As far as the number of boxed plane partitions is involved in Props. 3.3 and 3.2, we represent this number in the form

$$\mathrm{PL}(a,b,c) = \frac{G(a+1)G(b+1)G(c+1)G(a+b+c+1)}{G(a+b+1)G(a+c+1)G(b+c+1)}, \tag{4.12}$$

where $G(z)$ is the Barnes G-function defined by the relations

$$G(z+1) = G(z)\Gamma(z), \qquad G(2) = G(1) = 1. \tag{4.13}$$

It is well known that [39], as $z\to\infty$,

$$\log G(z+1)=\frac{z^2}{2}\log z-\frac{3}{4}z^2+\frac{\ln 2\pi}{2}z-\frac{1}{12}\log z+\zeta'(-1)+O\left(z^{-2}\right), \qquad (4.14)$$

where $\zeta'(-1) = -0.165142...$ is the derivative of the Riemann function $\zeta(z)$ at $z=-1$.

Now we ready to compute values of the function f_2. First, from Prop. 3.1 it follows that $P_{N,M,L}(0)=1$ and hence, for arbitrary values of the parameters (i.e., not just limited to the symmetric case):

$$\lim_{x\to\infty} f_2(x)=0. \qquad (4.15)$$

Next, we find the value of f_2 at the point $x=1$ using Prop. 3.3. From (3.19) and (4.13), we find

$$P_{N,M,L}(1)=\frac{G(a+2)G(b+2)G(c+2)G(a+b+c+2)}{G(a+b+2)G(a+c+2)G(b+c+2)}, \qquad (4.16)$$

where $a=N$, $b=L-N-1$, and $c=M-N$ (see (3.21)). In the symmetric case $a=N$, $b=rN+O(1)$, and $c=rN+O(1)$, so from (4.16) and (4.14) we find that

$$f_2(1)=\frac{(2r+1)^2}{2}\log(2r+1)-(r+1)^2\log(r+1)-r^2\log 4r. \qquad (4.17)$$

Finally, let us consider the behavior of f_2 near the point $x=0$. From Prop. 3.2 it follows that

$$x^{ac}P_{N,M,L}(x^{-1})\big|_{x=0}=\frac{G(a+2)G(b+1)G(c+2)G(a+b+c+1)}{G(a+b+1)G(a+c+2)G(b+c+1)}, \qquad (4.18)$$

where $a=N$, $b=|M-L+1|$, $c=\min(L-N-1,M-N)$ (see (3.13)). In the symmetric case $a=N$, $c=rN+O(1)$, but $b=O(1)$ as $N\to\infty$, hence (4.14) yields

$$\big(r\log x+f_2(x)\big)\big|_{x=0}=0. \qquad (4.19)$$

The values of the function f_2 at the points $x=1$ and $x=0$ in the non-symmetric case ($p\neq q$) which follow from (4.16) and (4.18), respectively, are computed in Sect. 5.1 (see (5.10) and (5.11)).

4.2. Construction of the leading term. Now we ready to address the problem of construction of the function f_2 describing the leading term of the function $\log P_{N,M,L}(x^{-1})$ in the thermodynamic limit. The function f_2 should satisfy the following properties. First, the corresponding σ_2-function should be given in terms of the solutions (4.8) and (4.9) of the associated Clairaut equation. Second, the function f_2 obtained from the function σ_2 by (4.10) should be consistent with the statements of Props. 3.1–3.3. In particular, it should satisfy the conditions (4.15), (4.17), and (4.19). Third, it should be a *continuous* function of x, or, more exactly, piece-wise continuous, in case if several solutions from the first property are involved.

Let us consider the first property, namely, we intend to identify the function σ_2 by requiring that its expansions at the points $x=\infty,1,0$ are consistent with (3.10), (3.29), and (3.15), respectively, specified to the symmetric case. Note that in doing so, we also involve partially the second property, because these expansions follow from the Props. 3.1–3.3. We recall that we deal with the following values of the parameters:

$$v_1 = -v_2 = w, \qquad v_3 = v_4 = \frac{1}{2}.$$

We start with considering the vicinity of the point $x = \infty$, where, as it follows from (3.10), we should have

$$\sigma_2 = -\frac{x}{4} - \frac{w^2}{2} + \frac{1}{8} + O(x^{-2}), \qquad x \to \infty.$$

Clearly, the solution of the Clairaut equation which fulfills the required $x \to \infty$ behavior is any of the two general solutions $(\sigma_2)_{g,\pm}$ with $C = -v_3^2 = -1/4$, see (4.8).

Next, in the vicinity of the point $x = 1$, as it follows from (3.29), we should have

$$\sigma_2 = \frac{w^2}{2} - w + \frac{1}{8} - \frac{w}{2}(x-1) + \frac{w}{8}(x-1)^2 + O\left((x-1)^3\right), \qquad x \to 1.$$

Apparently, the solution which has such an expansion is the singular solution $(\sigma_2)_{\mathrm{s},-}$, see (4.9), with $C = w^2/2 + 1/8$.

Finally, in the vicinity of the point $x = 0$ from (3.17) it follows that

$$\sigma_2 = \frac{w^2}{2} - \frac{1}{8} - w^2 x + O\big(x^3\big), \qquad x \to 0.$$

The solution which fulfills the required $x \to 0$ behavior is any of the two general solutions in (4.8) with $C = -w^2$.

Let us denote the obtained expressions for the σ_2-function by σ_2^{I}, σ_2^{II}, and σ_2^{III}, respectively. Summarizing, we thus have obtained

$$\begin{aligned} \sigma_2^{\mathrm{I}} &= -\frac{x}{4} - \frac{w^2}{2} + \frac{1}{8}, \\ \sigma_2^{\mathrm{II}} &= -w\sqrt{x} + \frac{w^2}{2} + \frac{1}{8}, \\ \sigma_2^{\mathrm{III}} &= -w^2 x + \frac{w^2}{2} - \frac{1}{8}. \end{aligned} \tag{4.20}$$

We recall that these expressions are valid near the points $x = \infty, 1, 0$, respectively.

Let us now consider the function $f_2 = f_2(x)$. We denote by f_2^{I}, f_2^{II}, and f_2^{III} the functions which are related to σ_2^{I}, σ_2^{II}, and σ_2^{III}, respectively, via (4.10). Taking into account (4.11), we obtain the following expressions

$$\begin{aligned} f_2^{\mathrm{I}} &= \left(w - \frac{1}{2}\right)^2 \log\frac{x}{x-1} + \widetilde{C}_2^{\mathrm{I}}. \\ f_2^{\mathrm{II}} &= 2w\log\left(1+\sqrt{x}\right) - \left(w - \frac{1}{4}\right)\log x + \widetilde{C}_2^{\mathrm{II}}. \\ f_2^{\mathrm{III}} &= -\left(w - \frac{1}{2}\right)^2 \log(1-x) - \left(w - \frac{1}{2}\right)\log x + \widetilde{C}_2^{\mathrm{III}}. \end{aligned}$$

Note that just like for the functions σ_2^{I}, σ_2^{II}, and σ_2^{III}, the obtained expressions for f_2^{I}, f_2^{II}, and f_2^{III} are valid near the points $x = \infty, 1, 0$, respectively. This finishes consideration of the first property of the function f_2.

The second property of the function f_2 in question concerns the values of the integration constants $\widetilde{C}_2^{\mathrm{I}}$, $\widetilde{C}_2^{\mathrm{II}}$, and $\widetilde{C}_2^{\mathrm{III}}$. The conditions (4.15) and (4.19) are fulfilled with

$$\widetilde{C}_2^{\mathrm{I}} = 0, \qquad \widetilde{C}_2^{\mathrm{III}} = 0. \tag{4.21}$$

The condition (4.17) means that

$$\widetilde{C}_2^{\mathrm{II}} = 2w^2 \log 2w - \left(w + \frac{1}{2}\right)^2 \log(2w+1) - \left(w - \frac{1}{2}\right)^2 \log(2w-1), \tag{4.22}$$

or

$$\widetilde{C}_2^{\mathrm{II}} = \frac{1}{2} \log \frac{1}{2w} - \left(w + \frac{1}{2}\right)^2 \log\left(1 + \frac{1}{2w}\right) - \left(w - \frac{1}{2}\right)^2 \log\left(1 - \frac{1}{2w}\right). \tag{4.23}$$

As a result, we have fixed all the three functions f_2^{I}, f_2^{II}, f_2^{III} completely.

Now we address the third property, namely, that the function f_2 must be piece-wise continuous. We consider the simplest possible ansatz that each of these three expressions is valid in some interval which contains the corresponding point. Specifically, we require that f_2 is a piece-wise continuous function of $x \in [0, \infty)$ and there exist two critical points $x_{\mathrm{c}}^{\pm} \gtrless 1$ such that

$$f_2 = \begin{cases} f_2^{\mathrm{I}} & x \in [x_{\mathrm{c}}^+, \infty) \\ f_2^{\mathrm{II}} & x \in [x_{\mathrm{c}}^-, x_{\mathrm{c}}^+] \\ f_2^{\mathrm{III}} & x \in [0, x_{\mathrm{c}}^-]. \end{cases}$$

The points $x_{\mathrm{c}}^{\pm}$ must obey the equations

$$f_2^{\mathrm{I}}(x_{\mathrm{c}}^+) = f_2^{\mathrm{II}}(x_{\mathrm{c}}^+), \qquad f_2^{\mathrm{II}}(x_{\mathrm{c}}^-) = f_2^{\mathrm{III}}(x_{\mathrm{c}}^-). \tag{4.24}$$

It turns out that despite the fact that these equations are in general transcendent, they can be solved, and, furthermore, uniqueness of their solutions can be proven.

Let us consider the first equation in (4.24). Introduce the function

$$\rho_+(x) = f_2^{\mathrm{II}}(x) - f_2^{\mathrm{I}}(x).$$

Using the first relation in (4.21) and (4.22), we get

$$\rho_+(x) = \left(w + \frac{1}{2}\right)^2 \log \frac{\sqrt{x}+1}{2w+1} + \left(w - \frac{1}{2}\right)^2 \log \frac{\sqrt{x}-1}{2w-1} - w^2 \log \frac{x}{4w^2}.$$

Apparently, the equation $\rho_+(x) = 0$ has the root $x = 4w^2$, and so we conclude that

$$x_{\mathrm{c}}^+ = 4w^2 =: x_{\mathrm{c}}.$$

To show that there are no other roots on the interval $(1, \infty)$, we evaluate the derivative of the function $\rho_+(x)$ and find

$$\rho_+'(x) = \frac{(\sqrt{x} - 2w)^2}{4x(x-1)}.$$

Thus, the function $\rho_+(x)$ is a monotonously growing function for $x \in (1, \infty)$, except the point $x = 4w^2$. This point is exactly the root we have obtained, and so there are no other roots on the interval $(1, \infty)$. Note that the second derivative of $\rho_+(x)$ also vanishes at this point but the third one does not, that implies that this is a point of the third-order phase transition.

Let us consider the second equation in (4.24). Introduce the function

$$\rho_-(x) = f_2^{\mathrm{II}}(x) - f_2^{\mathrm{III}}(x).$$

Using now the second relation in (4.21) and (4.23), we get

$$\rho_-(x) = \left(w + \frac{1}{2}\right)^2 \log \frac{1+\sqrt{x}}{1+(2w)^{-1}} + \left(w - \frac{1}{2}\right)^2 \log \frac{1-\sqrt{x}}{1-(2w)^{-1}} - \frac{1}{4}\log 4w^2 x.$$

Obviously, $\rho_-(x) = 0$ for $x = 1/4w^2$, and so

$$x_c^- = \frac{1}{4w^2} = x_c^{-1}.$$

We also have

$$\rho_-'(x) = -\frac{(2w\sqrt{x}-1)^2}{4x(1-x)},$$

so the function $\rho_-(x)$ is a monotonously decreasing function for $x \in (0,1)$, except the point $x = (4w^2)^{-1}$ where it vanishes together with its first and second derivatives. Thus, $x = 1/4w^2$ is the only root of $\rho_-(x)$ for $x \in (0,1)$. This is another point of the third-order phase transition.

As a comment to this calculation, it is useful to note that equations (4.24) appear to be elementary if we would assume that the function f_2 is continuous together with its first derivative. In other words, the assumption that the system demonstrates no first-order phase transitions can be very handy. Indeed, it means that the function σ_2 is required to be continuous and therefore (4.24) are replaced by the similar equations for the values of the function σ_2 at these points. From (4.20) then one immediately obtains that $x_c^{\pm} = (4w^2)^{\pm 1}$.

The assumption that there are no first-order phase transitions is in a complete agreement with general properties of discrete random matrix models, which are known to exhibit phase transitions not harder than third-order ones [40]. We recall that $P_{N,M,L}(x^{-1})$ can be regarded as such a model, see (3.3) and (2.6).

4.3. Sub-leading corrections. We now address the problem of computing the sub-leading corrections. For the σ-function it means construction of other terms in the $1/N$ expansion (4.1); we limit ourselves here by obtaining the functions σ_1 and σ_0 though the procedure admits derivation of all the terms recursively [18]. The corresponding expansion for the function $\log P_{N,M,L}(x^{-1})$, as we see below, may additionally contain a $\log N$ term with the constant coefficient.

To fix the structure of $1/N$ corrections in a unique way, we first consider the $O(1)$ terms in (4.2). Indeed, for ν_3 and ν_4, see (2.22), the $O(1)$ terms are equal to $1/2$ and $-1/2$, respectively, and there are no further $1/N$ corrections. The bulk system parameters (besides N) are contained in ν_1 and ν_2. These bulk system parameters can be identified in such a way that ν_1 and ν_2 have no $O(1)$ terms, that is, v_1 and v_2 are defined such that following relations hold exactly:

$$\nu_1 = v_1 N, \qquad \nu_2 = v_2 N. \tag{4.25}$$

If we further set $v_1 = 1/2 + p$ and $v_2 = -1/2 - q$, then we arrive at p and q defined in (1.8). In the symmetric case one cannot however require absence of $O(1)$ terms, but can require, for example, that v_1 and v_2 have these terms equal,

$$\nu_1 = wN + \frac{\epsilon}{2}, \qquad \nu_2 = -wN + \frac{\epsilon}{2}. \tag{4.26}$$

An advantage of the choice (4.26) is that the σ-form of the sixth Painleve equation, (2.21), then contains only even powers of N in its coefficients, just like in the non-symmetric case (4.25). This property of the coefficients is very useful and can imply, under further conditions to be met, that all terms σ_{1-2k}, $k = 0, 1, \ldots$, in (4.1) vanish.

The expansion (4.1) can be constructed by plugging it in (2.21) and matching powers in N. Instead of operating with (2.21) directly, one can simplify calculations by noting that since $\sigma_2' \neq 0$ for all values of $x \in [0, \infty)$, a systematic treatment of $1/N$ corrections can be done by considering the factorization of (2.21) on two equations

$$\sigma = x\sigma' + F_{\pm}(\sigma', \sigma''), \tag{4.27}$$

where the functions $F_{\pm}(\sigma', \sigma'') = F_{\pm}(\sigma', \sigma''; \nu_1, \nu_2, \nu_3, \nu_4)$ are

$$F_{\pm}(\sigma', \sigma'') = -\frac{\sigma'}{2} - \frac{\prod_i \nu_i}{2\sigma'} \pm \frac{\sqrt{\prod_i \left(\sigma' + \nu_i^2\right) - \sigma'\big(x(x-1)\sigma''\big)^2}}{2\sigma'}. \tag{4.28}$$

In the large N limit the functions $F_{\pm}(\sigma', \sigma'')$ in the leading order turn into the functions $f_{\pm}(\sigma_2')$ appearing in the Clairaut equations (4.4),

$$F_{\pm}(N^2\sigma_2', 0; v_1N, v_2N, v_3N, v_4N) = N^2 f_{\pm}(\sigma_2'). \tag{4.29}$$

Thus, the expansion (4.1) can be constructed by identifying which one of the two equations in (4.27) actually is satisfied in all orders in N.

This appears to be straightforward in Regime II where the leading term, σ_2, is given by a singular solution of $\sigma_2 = x\sigma_2' + f_-(\sigma_2')$ thus identifying the equation which contains the function $F_-(\sigma', \sigma'')$. Furthermore, it can also be easily shown that in this case the first sub-leading correction vanishes, $\sigma_1 = 0$. Indeed, recalling that the σ''-term and the $O(1)$ corrections of the parameters $\nu_1, \ldots, \nu_4$ can contribute only to the order $1/N^2$ with respect to the leading term, the substitution $\sigma = N^2\sigma_2 + N\sigma_1 + O(1)$ yields

$$\begin{aligned} F_-\big(\sigma', \sigma''\big) &= F_-(N^2\sigma_2' + N\sigma_1', 0; wN, -wN, N/2, N/2)\left(1 + O(N^{-2})\right) \\ &= N^2 f_-(\sigma_2' + \sigma_1'/N) + O(1) \\ &= N^2 f_-(\sigma_2') + N f_-'(\sigma_2')\sigma_1' + O(1), \end{aligned} \tag{4.30}$$

where at the second step we have used (4.29) specified to the symmetric case for a concreteness. Taking into account that for the singular solution $x + f_-'(\sigma_2') = 0$, for the function σ_1 we obtain

$$\sigma_1 = \big(x + f_-'(\sigma_2')\big)\sigma_1' = 0. \tag{4.31}$$

Note that $\sigma_1 = 0$ for Regime II implies that in the expansion (4.1) all the terms of odd powers in $1/N$ also vanish, and (4.1) becomes an expansion in $1/N^2$ (see also the discussion in [18], Sect. 4, where the similar phenomenon have been argued differently).

As for Regime I and Regime III, we have obtained that σ_2-function in these cases is given by regular solutions of the Clairaut equation. These solutions are such that $f_-(\sigma_2') = f_+(\sigma_2')$, and so both equations in (4.27) vanish in the leading order. To identify which one of the two equations responsible for the $1/N$ expansion, below we expand functions $F_{\pm}(\sigma', \sigma'')$ to find equations for the σ_1-function, and choose the solutions which possess the required $x \to \infty$ (for Regime I) and $x \to 0$ (for Regime III) expansions as prescribed by Cor. 3.1.1 and Cor 3.2.1, respectively. The σ_1-functions in both cases appear to be given by singular solutions of some other Clairaut equations. This allows us to identify the relevant equation among the two in (4.27) just like it has been done above for the Regime II. It turns out that the equation with the function $F_-(\sigma',\sigma'')$ in (4.27) is the relevant one for all the three regimes.

We now turn to giving details of calculations for each of the regimes separately.

4.3.1. *Regime I.* We start with noting that plugging (4.26) in (3.10) gives the following expression for the function σ_1 as $x\to\infty$:

$$\sigma_1 = -\frac{x}{2} + w^2 + \frac{1}{4} + \left(w^2 - \frac{1}{4}\right)^2 \frac{1}{x} + O\left(x^{-2}\right). \tag{4.32}$$

To obtain the σ_1-function it is sufficient to consider large N expansion of the functions $F_\pm(\sigma',\sigma'')$ to order N; we expand them to order N^0, so that we can obtain next the function σ_0. We recall that in Regime I $\sigma_2' = -1/4$ and hence the square root term in (4.28) is not contributing to the leading, N^2, order. More exactly, the term $\prod_i(\sigma'+\nu_i^2)$ is of $O(N^6)$ (instead of $O(N^8)$); explicitly

$$\begin{aligned}\prod_i \left(\sigma' + \nu_i^2\right) &= N^6 \left(w^2 - \frac{1}{4}\right)^2 \left[(\sigma_1')^2 - \frac{1}{4}\right] \\ &+ N^5 \left\{\left(w^2 - \frac{1}{4}\right)\left[(\sigma_1')^2 - \frac{1}{4}\right] + \left(w^2 - \frac{1}{4}\right)^2 \left(\sigma_0' + \frac{1}{4}\right)\right\} 2\sigma_1' + O(N^4).\end{aligned}$$

Furthermore, since $\sigma_2''=0$, the term $\sigma'(\sigma'')^2 \sim N^4\sigma_2'(\sigma_1'')^2$ contributes to the large N expansion of $F_\pm(\sigma',\sigma'')$ starting from order N^{-1}. As a result, we obtain

$$\begin{aligned}F_\pm(\sigma',\sigma'') &= N^2\left(\frac{1}{8} - \frac{w^2}{2}\right) + N\left(-\frac{\sigma_1'}{2} - 2w^2\sigma_1' \mp 2\left(w^2 - \frac{1}{4}\right)\sqrt{(\sigma_1')^2 - \frac{1}{4}}\right) \\ &+ \left(-\frac{1}{2} - 2w^2 \mp \frac{2\left(w^2 - \frac{1}{4}\right)\sigma_1'}{\sqrt{(\sigma_1')^2 - \frac{1}{4}}}\right)\sigma_0' - 8w^2(\sigma_1')^2 + \frac{w^2}{2} + \frac{\epsilon^2}{8} \\ &\mp \left(8w^2\sqrt{(\sigma_1')^2 - \frac{1}{4}} + \frac{w^2 - \frac{1}{4}}{2\sqrt{(\sigma_1')^2 - \frac{1}{4}}}\right)\sigma_1' + O(N^{-1}).\end{aligned} \tag{4.33}$$

Hence, σ_1 must satisfy one of the two equations

$$\sigma_1 = \left(x - \frac{1}{2} - 2w^2\right)\sigma_1' \mp 2\left(w^2 - \frac{1}{4}\right)\sqrt{(\sigma_1')^2 - \frac{1}{4}}, \tag{4.34}$$

where the signs correspond to $F_\pm(\sigma',\sigma'')$. These equations are the Clairaut equations. The presence of the $1/x$ term in (4.32) indicates that we deal here with a singular solution. All such solutions of (4.34) satisfy

$$(\sigma_1')^2 = \frac{(2x - 4w^2 - 1)^2}{16(x-1)(x-4w^2)}.$$

Specifically, the solution which obeys (4.32) is

$$\sigma_1 = -\frac{1}{2}\sqrt{(x-1)(x-4w^2)},$$

and it can be easily checked that it corresponds to the plus sign in (4.34), that is, to the function $F_-(\sigma',\sigma'')$.

As far as the function in (4.27) is determined, the function σ_0 can be computed. One can easily see a remarkable property of the expansion (4.33): the coefficient of the σ_0' term is exactly the derivative with respect to σ_1' of the N-order term. Since σ_1 is given by the singular solution of (4.34), the σ_0' term exactly vanishes at order N^0 in (4.27), that yields

$$\sigma_0 = -8w^2(\sigma_1')^2 + \frac{w^2}{2} + \frac{\epsilon^2}{8} + 8w^2\sigma_1'\sqrt{(\sigma_1')^2 - \frac{1}{4}} + \frac{\left(w^2 - \frac{1}{4}\right)\sigma_1'}{2\sqrt{(\sigma_1')^2 - \frac{1}{4}}}.$$

Explicitly, the result reads

$$\sigma_0 = -\frac{w^2(x-1)}{x-4w^2} - \frac{x}{4} + \frac{1+\epsilon^2}{8}.$$

Let us now consider the function $\log P_{N,M,L}(x^{-1})$. From Props. 3.1 and 2.2, it follows that

$$\log P_{N,M,L}(x^{-1}) = N^2 f_2 + N f_1 + f_0 + \dots, \tag{4.35}$$

where, since $P_{N,M,L}(0) = 1$, all the terms must vanish as $x \to \infty$. In particular,

$$\lim_{x\to\infty} f_1(x) = 0, \qquad \lim_{x\to\infty} f_0(x) = 0.$$

We compute f_1 by

$$f_1 = \int \left(\sigma_1 + \widetilde{A}_1 x - \widetilde{B}_1\right) \frac{\mathrm{d}x}{x(x-1)} + \widetilde{C}_1, \tag{4.36}$$

where, $\widetilde{A}_1$ and $\widetilde{B}_1$ are $O(N)$ terms of the large N expansion of $\widetilde{A}$ and $\widetilde{B}$ in (2.20), respectively. Using (2.23) for $\widetilde{B}$ and taking into account (4.26), we find

$$\widetilde{A}_1 = \frac{1}{2}, \qquad \widetilde{B}_1 = \frac{1}{2}.$$

Choosing $\widetilde{C}_1$ to ensure that $\lim_{x\to\infty} f_1(x) = 0$, we get

$$f_1 = 2w \log \frac{2w\sqrt{x-1} + \sqrt{x-4w^2}}{(2w+1)\sqrt{x}} - \log \frac{\sqrt{x-1} + \sqrt{x-4w^2}}{2\sqrt{x}}.$$

Essentially similarly, for f_0, using

$$f_0 = \int \left(\sigma_0 + \widetilde{A}_0 x - \widetilde{B}_0\right) \frac{\mathrm{d}x}{x(x-1)} + \widetilde{C}_0, \tag{4.37}$$

where

$$\widetilde{A}_0 = \frac{1}{4}, \qquad \widetilde{B}_0 = \frac{1-\epsilon^2}{8},$$

and choosing $\widetilde{C}_0$ such that $\lim_{x\to\infty} f_0(x) = 0$, we obtain

$$f_0 = -\frac{1}{4}\log\left(1 - \frac{4w^2}{x}\right) + \frac{\epsilon^2}{4}\log\left(1 - \frac{1}{x}\right).$$

Finally, rewriting these formulas in terms of $x_{\mathrm{c}} = 4w^2$ we arrive at the expressions for the functions f_1^{I} and f_0^{I} appearing in Thm. 1.1.

4.3.2. *Regime II.* Given that $\sigma_1 = 0$, we are left with the task of obtaining the σ_0-function. This can be done directly by expanding the function $F_-(\sigma', \sigma'')$ around the leading term by setting $\sigma = N^2\sigma_2 + \sigma_0$ and taking into account that $\sigma_2' \in (-w^2, -1/4)$. As it can be anticipated from the considerations above for the σ_1-function, see (4.30), all terms at N^0 order in (4.27) depending on σ_0' vanish, just like it takes place in (4.31) due to the overall factor $x + f_-'(\sigma_2') = 0$. As a result, we get the following expression the function σ_0:

$$\sigma_0 = -\frac{[x(x-1)\sigma_2'']^2}{\left(4\sigma_2'+1\right)\left(\sigma_2'+w^2\right)} - \frac{w^2}{8\sigma_2'} + \frac{\left(\sigma_2'+w^2\right)\left(4\sigma_2'-1\right)}{8\sigma_2'\left(4\sigma_2'+1\right)} - \frac{\epsilon^2}{32\sigma_2'} + \frac{\epsilon^2\left(4\sigma_2'+1\right)\left(\sigma_2'-w^2\right)}{32\sigma_2'\left(\sigma_2'+w^2\right)}.$$

Using $\sigma_2' = -w/2\sqrt{x}$ and $\sigma_2'' = w/4x^{3/2}$, we get

$$\sigma_0 = -\frac{w(x-1)}{8}\left\{\frac{3}{2w-\sqrt{x}} + \frac{\sqrt{x}}{2w\sqrt{x}-1}\right\} + \frac{\epsilon^2(x-1)}{4(2w\sqrt{x}-1)} - \frac{1+\epsilon^2}{8}.$$

Let us now consider the function $\log P_{N,M,L}(x^{-1})$. We first note that a more detailed calculation with the help of (4.14) applied to (4.16) with $a = N$, $b = rN - \frac{\epsilon+1}{2}$, and $c = rN + \frac{\epsilon-1}{2}$ yields

$$\log P_{N,M,L}(1) = N^2 f_2(1) + N f_1(1) + \frac{5}{12}\log N + f_0(1) + O(N^{-1}), \tag{4.38}$$

where $f_2(1)$ is given by (4.17), and the values $f_1(1)$ and $f_0(1)$ are

$$f_1(1) = -(r+1)\log(r+1) + r\log r \tag{4.39}$$

and

$$f_0(1) = \frac{1}{12}\log\frac{2r^2}{(r+1)(2r+1)} + \frac{\epsilon^2}{4}\log\frac{r}{r+1} + \zeta'(-1) + \log\sqrt{2\pi}, \tag{4.40}$$

respectively. From Prop. 3.3 and expansion (4.38) we conclude that for the values of x corresponding to Regime II the following expansion is valid:

$$\log P_{N,M,L}(x^{-1}) = N^2 f_2(x) + N f_1(x) + \frac{5}{12}\log N + f_0(x) + O(N^{-1}). \tag{4.41}$$

Clearly, the functions f_1 and f_0 can be found from the functions σ_1 and σ_0 by (4.36) and (4.37), respectively.

Computing f_1 by (4.36), where $\widetilde{A}_1 = \widetilde{B}_1 = 1/2$, we get

$$f_1 = \frac{1}{2}\log x + \widetilde{C}_1.$$

Computing f_0 by (4.37), where $\widetilde{A}_0 = 1/4$ and $\widetilde{B}_0 = (1-\epsilon^2)/8$, we get

$$f_0 = \frac{1}{8}\left\{3\log\left(2w-\sqrt{x}\right) - \log\left(2w\sqrt{x}-1\right) + \log\sqrt{x}\right\} + \frac{\epsilon^2}{2}\log\frac{2w\sqrt{x}-1}{\sqrt{x}} + \widetilde{C}_0.$$

The integration constants can be fixed by using (4.39) and (4.40). We obtain

$$\widetilde{C}_1 = -(1+r)\log(1+r) + r\log r$$

and

$$\widetilde{C}_0 = -\frac{1}{12}\log\left(4r(r+1)(2r+1)\right) - \frac{\epsilon^2}{4}\log\left(4r(r+1)\right) + \zeta'(-1) + \log\sqrt{2\pi}.$$

As a result, using $\sqrt{x_c} = 2w = 2r + 1$ we arrive at the expressions for the functions f_1^{II} and f_0^{II} given in Thm. 1.1.

4.3.3. *Regime III.* Considerations in this regime in general are very similar to those in Regime I. We start with expansion (3.15), which with (4.26) gives the following expression for the function σ_1 as $x \to 0$,

$$\sigma_1 = -\frac{|\epsilon|}{2} + |\epsilon| \left(w^2 + \frac{1}{4}\right) x - |\epsilon| \left(w^2 - \frac{1}{4}\right)^2 x^2 + O(x^3). \tag{4.42}$$

To find the function σ_1 and which one of the two equations in (4.27) is relevant for $1/N$ expansion, and next obtain the function σ_0, we expand the functions $F_\pm(\sigma', \sigma'')$ to order N^0:

$$\begin{aligned} F_\pm(\sigma', \sigma'') = N^2 \left(\frac{w^2}{2} - \frac{1}{8}\right) + N \left(-\frac{\sigma_1'}{2} - \frac{\sigma_1'}{8w^2} \mp \frac{(4w^2-1)\sqrt{(\sigma_1')^2 - \epsilon^2 w^2}}{8w^2}\right) \\ + \left(-\frac{1}{2} - \frac{1}{8w^2} \mp \frac{(4w^2-1)\sigma_1'}{8w^2\sqrt{(\sigma_1')^2 - \epsilon^2 w^2}}\right) \sigma_0' - \frac{(\sigma_1')^2}{8w^4} + \frac{1}{8} + \frac{\epsilon^2}{32w^2} \\ \pm \left(\frac{\sqrt{(\sigma_1')^2 - \epsilon^2 w^2}}{8w^4} - \frac{\epsilon^2(4w^2-1)}{32w^2\sqrt{(\sigma_1')^2 - \epsilon^2 w^2}}\right) \sigma_1' + O(N^{-1}). \end{aligned}$$

Hence, the function σ_1 must be a singular solution (as far as (4.42) contains x^2 term) of one of the following two Clairaut equations:

$$\sigma_1 = x\sigma_1' - \frac{\sigma_1'}{2} - \frac{\sigma_1'}{8w^2} \mp \frac{(4w^2-1)\sqrt{(\sigma_1')^2 - \epsilon^2 w^2}}{8w^2}. \tag{4.43}$$

The solution which obeys (4.42) is

$$\sigma_1 = -\frac{|\epsilon|}{2}\sqrt{(1 - 4w^2 x)(1 - x)},$$

and it corresponds to the plus sign in (4.43), that is, to the function $F_-(\sigma', \sigma'')$ in (4.27).

As a result, for σ_0 we get

$$\sigma_0 = -\frac{(\sigma_1')^2}{8w^4} + \frac{1}{8} + \frac{\epsilon^2}{32w^2} - \frac{\sqrt{(\sigma_1')^2 - \epsilon^2 w^2}}{8w^4} + \frac{\epsilon^2(4w^2-1)\sigma_1'}{32w^2\sqrt{(\sigma_1')^2 - \epsilon^2 w^2}}$$

and substitution of the function σ_1' gives

$$\sigma_0 = -\frac{\epsilon^2(1-x)}{4(1 - 4w^2 x)} - \frac{\epsilon^2 x}{4} + \frac{1 + \epsilon^2}{8}.$$

Let us now turn to the function $\log P_{N,M,L}(x^{-1})$. We begin with addressing its $x \to 0$ behavior, using (4.18). We have $a = N$, $b = |\epsilon|$, and $c = rN - \frac{1+|\epsilon|}{2}$, so that, as $N \to \infty$, (4.14) yields

$$\begin{aligned} x^{ac} P_{N,M,L}(x^{-1})\big|_{x=0} = N(1 - |\epsilon|)\big(r \log r - (r+1)\log(r+1)\big) + \frac{1-\epsilon^2}{2}\log N \\ + (1 - |\epsilon|)\log\sqrt{2\pi} + \log G(1 + |\epsilon|) + O(N^{-1}). \end{aligned} \tag{4.44}$$

This formula implies that in the Regime III the following expansion is valid:

$$\log P_{N,M,L}(x^{-1}) = N^2 f_2(x) + N f_1(x) + \frac{1-\epsilon^2}{2}\log N + f_0(x) + O(N^{-1}).$$

The $x \to 0$ behavior of the function $f_2(x)$ is given by (4.19), and from (4.44) we also get

$$\big(-(1+|\epsilon|)\log\sqrt{x} + f_1(x)\big)\Big|_{x=0} = (1-|\epsilon|)\big(r\log r - (r+1)\log(r+1)\big) \qquad (4.45)$$

and

$$f_0(0) = (1-|\epsilon|)\log\sqrt{2\pi} + \log G(1+|\epsilon|). \qquad (4.46)$$

Computing f_1 by (4.36), where $\widetilde{A}_1 = \widetilde{B}_1 = 1/2$, we get

$$f_1 = 2|\epsilon| w \log\left(2w\sqrt{1-x} + \sqrt{1-4w^2x}\right) - |\epsilon|\log\left(\sqrt{1-x} + \sqrt{1+4w^2x}\right) + \frac{1+|\epsilon|}{2}\log x + \widetilde{C}_1.$$

Computing f_0 by (4.37), where $\widetilde{A}_0 = 1/4$ and $\widetilde{B}_0 = (1-\epsilon^2)/8$, we get

$$f_0 = -\frac{\epsilon^2}{4}\log(1-4w^2x) + \frac{1}{4}\log(1-x) + \widetilde{C}_0.$$

From (4.45) we find

$$\widetilde{C}_1 = r\log r - (r+1)\log(r+1) - |\epsilon| r\log\big(4r(r+1)\big)$$

and from (4.46) we find

$$\widetilde{C}_0 = (1-|\epsilon|)\log\sqrt{2\pi} + \log G(1+|\epsilon|).$$

Finally, rewriting the arguments of the logarithms in terms of $\sqrt{x_{\mathrm{c}}} = 2w = 2r+1$, we arrive at the expressions for the functions f_1^{III} and f_0^{III} provided in Thm. 1.1.

This finalizes the proof of Thm. 1.1.

Let us now consider the special case of $\epsilon = 0$ of Regime III, and show how the expansion (1.11) follows from our considerations above. Indeed, in this case $\sigma_1 = 0$ and $\sigma_0 = 1/8$, so (4.1) reads

$$\sigma = N^2\left(-w^2x + \frac{w^2}{2} - \frac{1}{8}\right) + \frac{1}{8} + O(N^{-1}). \qquad (4.47)$$

This expression has to be compared with a trivial solution of (2.21) valid for $\nu_1 = -\nu_2$, and ν_1, ν_3, ν_4 arbitrary, of the form

$$\sigma_{\mathrm{triv}} = -\nu_1^2 x + \frac{\nu_1^2 - \nu_3\nu_4}{2}. \qquad (4.48)$$

Setting $\nu_1 = wN$, $\nu_3 = (N+1)/2$, and $\nu_4 = (N-1)/2$ in (4.48) one can reproduce the terms shown in (4.47). This means that in Regime III at $\epsilon = 0$ all terms in the expansion (4.1) beyond σ_0 vanish. Note that does not mean that $\sigma = \sigma_{\mathrm{triv}}$ but it implies that

$$\sigma = \sigma_{\mathrm{triv}} + O(N^{-\infty}),$$

where $O(N^{-\infty})$ stands for terms which are less than any given degree in $1/N$. In fact, this term corresponds to exponentially small corrections. These corrections can also be tackled though some additional information is necessary. The situation here is similar to that considered in [18] for the so-called ordered regime (Sect. 5 therein).

It is worth mentioning that the phenomenon of absence of the $1/N$ corrections in (1.11) can already be anticipated from the $x \to 0$ expansion for the polynomial

$P_{N,M,L}(x^{-1})$ given in Prop. 3.2. The case of $\epsilon = 0$ corresponds to $b = 0$ in (3.12), and (3.11) says that, as $x \to 0$,

$$P_{N,M,L}(x^{-1}) = \frac{1}{\binom{a+c}{a} x^{ac}} \left\{ 1 + c(c+1)x + \frac{c(c+1)(c^2+c+1)}{2} x^2 + O(x^3) \right\}, \tag{4.49}$$

or

$$\log\left(\binom{a+c}{a} P_{N,M,L}(x^{-1}) \right) = ac \log x + c(c+1)\left(x + \frac{x^2}{2} \right) + O(x^3). \tag{4.50}$$

Recalling that $a = N$ and $c = rN - 1/2$, we thus see that the right-hand side of (4.50) in the large N limit contains no $1/N$ corrections; there are only terms of orders N^2, N and N^0. Furthermore, the expression in the braces in (4.49) is nothing but a truncated expansion of $(1-x)^{-c(c+1)}$ (one can check by expanding further in x) and so the corresponding $c(c+1)$ term in (4.50) is $\log(1-x)$. One can therefore expect that for x taking positive values at some interval attached to the origin, the following must hold:

$$\log\left(\binom{a+c}{a} P_{N,M,L}(x^{-1}) \right) = ac \log x + c(c+1)\log(1-x) + O(N^{-\infty}). \tag{4.51}$$

As we have established, Regime III corresponds to $x \in [0, x_{\mathrm{c}}^{-1}) = [0, (2r+1)^{-2})$ and so (4.51) holds for these values of x.

As a final comment here, we mention that (4.51) admits a simple interpretation when translated back to the partition function using (1.4) and (1.5). Reverting to the weights (1.2), one can write

$$Z = \frac{w_1^{(M-N)(L-N)} w_3^{(M-L+N)N} (w_5 w_6)^{M(L-N)}}{(w_5 w_6 - w_3 w_4)^{(M-N)(L-N)}} \big(1 + O(N^{-\infty})\big) \tag{4.52}$$

where $M - L + 1 = 0$. The weights are subject to the constraint that they must obey the condition of the Regime III, which now reads

$$\frac{w_3 w_4}{w_5 w_6} \in \big[0, N^2/(M+L-N)^2\big).$$

If one expands the denominator in (4.52) in Taylor series in $w_3 w_4 / w_5 w_6$, then the leading term gives the weight of the anti-ferroelectric ground state configuration shown in Fig. 5. Thus, formula (4.52) can be interpreted as the result of summation over relevant perturbations from this ground state, valid up to exponentially small corrections in the large N limit.

5. Thermodynamic limit in the non-symmetric case

In this section we focus on construction of the asymptotic expansions for the σ-function and the corresponding polynomial $P_{N,M,L}(x^{-1})$ in the limit $N, M, L \to \infty$ such that the parameters p and q defined by (1.8) are finite and *not* equal to each other, $p \neq q$. This will provide a proof of Thm. 1.2.

5.1. Preliminaries. In Sect. 4.1 it is shown that the leading term of the σ-function in the thermodynamic limit, the function σ_2, see (4.1), can be found as a solution of the Clairaut equations (4.4). The non-symmetric case mean that the two parameters v_1 and v_2 are unrelated, thought the relation $v_4 = v_3$ holds. Henceforth we set

$$v_1 := v, \qquad v_2 := -u, \qquad v_3 = v_4 = \frac{1}{2}. \tag{5.1}$$

Note that $v = p + 1/2$ and $u = q + 1/2$ where p and q are defined in (1.8). Since $p, q > 0$, we have $v, u > 1/2$. We focus our attention on the case where the function σ_2' satisfies

$$\sigma_2' \in (-\min(v^2, u^2), -1/4]. \tag{5.2}$$

For the functions $f_\pm(\sigma_2')$ in (4.4) we then have

$$f_\pm(\sigma_2') = -\frac{\sigma_2'}{2} + \frac{vu}{8\sigma_2'} \mp \frac{\left(\sigma_2' + 1/4\right)\sqrt{\left(\sigma_2' + v^2\right)\left(\sigma_2' + u^2\right)}}{2\sigma_2'}. \tag{5.3}$$

As we see below, the condition (5.2) is indeed always fulfilled in our problem.

As usual, we have general solutions given by linear functions (4.6). In the non-symmetric case (5.1) some concerns may arise in dealing with the singular solutions. Recall that these are the solutions which correspond to vanishing of the first factor in (4.5). In the case of the functions (5.3) one has to find roots of *quartic* equations.

Instead of dealing with these roots explicitly, which are given by bulky expressions, one can search the solutions in a parametric form [31]. To solve the equations $x + f_\pm'(\sigma_2') = 0$ for the function σ_2' in terms of x, we introduce function $y = y(x)$ by defining it such that

$$\sqrt{\frac{\sigma_2' + u^2}{\sigma_2' + v^2}} = \frac{\alpha y + \beta}{\gamma y + \delta}, \tag{5.4}$$

where $\alpha, \ldots, \delta$ are some functions of v and u only. One can set $\alpha(v,u) = \gamma(u,v)$ and $\beta(v,u) = \delta(u,v)$, so that (5.4) holds identically at $u = v$. In our calculations below we make a particular choice of these functions, though this choice is not essential for obtaining a solution in the parametric form.

To proceed, we introduce the notation

$$X_\pm = \alpha y + \beta \pm (\gamma y + \delta), \qquad Y_\pm = v(\alpha y + \beta) \pm u(\gamma y + \delta).$$

From (5.4) we get

$$\sigma_2' = -\frac{Y_+ Y_-}{X_+ X_-}. \tag{5.5}$$

Substituting (5.5) into the relation $x + f_\pm'(\sigma_2') = 0$, and using (5.4), we obtain

$$x = \frac{X_\pm^2 \left(X_\mp + 2Y_\pm\right)\left(X_\mp - 2Y_\pm\right)}{16 Y_\pm^2 (\alpha y + \beta)(\gamma y + \delta)}. \tag{5.6}$$

Expression (5.6) together with (5.4) and (5.5) substituted in (4.4) yields

$$\sigma_2 = -\frac{v}{4u} + \frac{(v^2 - u^2)(\alpha y + \beta)}{4u Y_\pm} \mp \frac{(4u^2 - 1)(\gamma y + \delta)}{16(\alpha y + \beta)} \mp \frac{(4v^2 - 1)(\alpha y + \beta)}{16(\gamma y + \delta)}. \tag{5.7}$$

In these expressions the $\pm$-signs corresponds to the functions $f_\pm(\sigma_2')$. We also note that $x - 1$ has a factorized form as well:

$$x - 1 = \frac{X_\mp^2 \left(X_\pm + 2Y_\pm\right)\left(X_\pm - 2Y_\pm\right)}{16 Y_\pm^2 (\alpha y + \beta)(\gamma y + \delta)}. \tag{5.8}$$

This is a remarkable property of the parametrization (5.4) because we need to integrate the function σ_2 to obtain the corresponding function f_2.

Indeed, according to (2.24), we have

$$f_2 = \int \left(\sigma_2 + \widetilde{A}_2 x - \widetilde{B}_2\right) \frac{\mathrm{d}x}{x(x-1)} + \widetilde{C}_2, \tag{5.9}$$

where (see (2.20), (2.23), and (5.1))

$$\widetilde{A}_2 = \frac{1}{4}, \qquad \widetilde{B}_2 = \frac{vu - u - v}{2} + \frac{3}{8}.$$

From expressions (5.6), (5.7), and (5.8) it is clear that if we change of the integration variable $x \mapsto y$ and take into account that $\mathrm{d}x = (\partial_y x)\mathrm{d}y$, then the integrand in (5.9) appears to be an algebraic function of y. Hence, we can compute f_2 explicitly in terms of y.

To construct the function f_2 as a piece-wise continuous function one has to take into account its values at the points $x = \infty, 1, 0$, for generic values of p and q. We recall that at the point $x = \infty$ the function f_2 vanishes, see (4.15). The value $f_2(1)$ can be found from (4.16) and (4.14),

$$\begin{aligned} f_2(1) = \frac{1}{2}\Big\{ & p^2 \log p + q^2 \log q + (p+q+1)^2 \log(p+q+1) \\ & - (p+1)^2 \log(p+1) - (q+1)^2 \log(q+1) - (p+q)^2 \log(p+q)\Big\}. \end{aligned} \tag{5.10}$$

Concerning the point $x = 0$, from Prop. 3.2 one can find

$$\begin{aligned} \big[\min(p,q)\log x + f_2(x)\big]\big|_{x=0} &= \frac{(p-q)^2}{2} \log|p-q| - \frac{(|p-q|+1)^2}{2} \log(|p-q|+1) \\ &+ \frac{\operatorname{sgn}(p-q)}{2} \Big\{ (p+1)^2 \log(p+1) - p^2 \log p \\ &\qquad - (q+1)^2 \log(q+1) + q^2 \log q \Big\}, \end{aligned} \tag{5.11}$$

where formulas (4.12) and (4.14) have been used.

5.2. Construction of the leading term. We start with listing properties of the function σ_2 near the points $x = \infty, 1, 0$ assuming that $p \neq q$.

Behavior of the σ-function at the point $x = \infty$ is established in Cor. 3.1.1. From (3.10) in the parameterization (5.1) we have

$$\sigma_2 = -\frac{x}{4} - \frac{uv}{2} + \frac{1}{8} + O\left(x^{-2}\right), \qquad x \to \infty. \tag{5.12}$$

The case of the point $x = 1$ is considered in Cor. 3.3.1. From (3.29) it follows that

$$\begin{aligned} \sigma_2 = \frac{vu - v - u}{2} &+ \frac{1}{8} - \frac{4vu - v - u}{4(v+u-1)}(x-1) \\ &+ \frac{\left(v - \frac{1}{2}\right)^2 \left(u - \frac{1}{2}\right)^2 (v+u)}{(v+u-1)^4}(x-1)^2 + O\left((x-1)^3\right), \qquad x \to 1. \end{aligned} \tag{5.13}$$

The case of the point $x = 0$ is considered in Cor. 3.2.1. From (3.15) it follows that

$$\begin{aligned} \sigma_2 = \frac{vu - |v-u|}{2} &- \frac{1}{8} - \frac{4vu - |v-u|}{4(|v-u|+1)} x \\ &+ \frac{(4vu - 2|v-u| + 1)^2 |v-u|}{16(|v-u|+1)^4} x^2 + O\left(x^3\right), \qquad x \to 0. \end{aligned} \tag{5.14}$$

We now construct the function σ_2 satisfying all these properties.

The linear growth at infinity of the function σ_2 and the absence of an $1/x$ term in (5.12) imply that it is given by the general solution (4.6) where we have to choose $C = -1/4$. Denoting this function by σ_2^{I}, we conclude that the function σ_2 for sufficiently large values of x is given by σ_2^{I}, which reads

$$\sigma_2^{\mathrm{I}} = -\frac{x}{4} - \frac{uv}{2} + \frac{1}{8}. \tag{5.15}$$

More exactly, we have $\sigma_2 = \sigma_2^{\mathrm{I}}$ for $x \in [x_{\mathrm{c}}, \infty)$, where the value of the critical point x_{c} needs to be determined. The interval $[x_{\mathrm{c}}, \infty)$ corresponds to Regime I.

Let us now consider the case of the vicinity of the point $x = 1$. Expression (5.13) imply a non-linear behavior of the function σ_2 near $x = 1$ and hence we have to search this function among the singular solutions of the Clairaut equations (4.4) where the functions $f_\pm(\sigma_2')$ are given by (5.3). We first identify which equation, with $f_+(\sigma_2')$ or $f_-(\sigma_2')$, may possess the required asymptotics (5.13). Substituting the values of σ_2 and σ_2' in (4.4) at $x = 1$ we find that it is the equation with the function $f_-(\sigma_2')$.

Next we pass to the solution in the parametric form defined by the relation (5.4), which we choose in the following form:

$$\sqrt{\frac{\sigma_2' + u^2}{\sigma_2' + v^2}} = \frac{2uy + u - v}{2vy + v - u}.$$

Hence,

$$\sigma_2' = \frac{(u-v)^2 - 4uvy}{4y(1+y)} \tag{5.16}$$

and therefore

$$x = \frac{(y+1)^2(y+u-v)(y+v-u)}{(2vy+v-u)(2uy+u-v)}. \tag{5.17}$$

To indicate that in the vicinity of the point $x = 1$ the function σ_2 belongs to Regime II, we denote it σ_2^{II}. We have

$$\sigma_2^{\mathrm{II}} = \frac{vu}{2} - \frac{v^2+u^2}{16vu} - \frac{1}{4} - \frac{y}{2} + \frac{\left(4v^2-1\right)\left(u^2-v^2\right)}{16v(2vy+v-u)} + \frac{\left(4u^2-1\right)\left(v^2-u^2\right)}{16u(2uy+u-v)}. \tag{5.18}$$

Now the crucial step in the whole procedure is to identify which one of the four roots of the quartic equation (5.17) corresponds to the asymptotic expansion (5.13).

To do this, we use (5.8) to obtain

$$x - 1 = \frac{y^2(y-v-u+1)(y+v+u+1)}{(2vy+v-u)(2uy+u-v)}. \tag{5.19}$$

Computing the values of the expression in (5.18) at $y = -v-u-1, 0, v+u-1$ we find that the required value $\sigma_2(1) = (uv-u-v+4)/2$, see (5.13), is attained at $y = v+u-1$. Furthermore, representing y near this value as

$$y = v + u - 1 + \gamma_1\varepsilon + \gamma_2\varepsilon^2 + O(\varepsilon^3) \tag{5.20}$$

and choosing the coefficients γ_1 and γ_2 such that (5.19) becomes

$$x - 1 = \varepsilon + O(\varepsilon^3),$$

we find

$$\gamma_1 = \frac{(2v-1)\,(2u-1)\,(v+u)}{2(v+u-1)^2},$$
$$\gamma_2 = \frac{(2v-1)\,(2u-1)\,(v+u)\,\big(1+6(v^2+u^2)-(4uv+3)(v+u)\big)}{8(v+u-1)^5}.$$

Clearly, expression (5.20) provides an expansion of the function $y = y(x)$ near the point $x = 1$. This function is uniquely defined as the root of the quartic equation (5.17) by its value $y(1) = u + v - 1$. Substitution of (5.20) into (5.18) exactly reproduces the required expansion (5.13). This means that we have constructed the function σ_2^{II} which corresponds to the function σ_2 in Regime II.

Let us now address the position of the critical value of $x = x_{\mathrm{c}}$ at which Regime I changes into Regime II. At the moment, we do this under the assumption that the function σ_2 is continuous at this point,

$$\sigma_2^{\mathrm{I}}(x_{\mathrm{c}}) = \sigma_2^{\mathrm{II}}(x_{\mathrm{c}}).$$

Below we lift this assumption and derive the result for x_{c} from the similar equation for the function f_2, that proves absence of first-order phase transitions. Denoting the corresponding value of y at the critical point by $y_{\mathrm{c}} \equiv y(x_{\mathrm{c}})$ we find from (5.15), (5.18), and (5.17) that y_{c} is one of the two roots of the equation

$$y^2 + (1-4vu)y + (v-u)^2 = 0. \tag{5.21}$$

Choosing the root which lies on the right from the value $y(1) = u + v - 1$ (note that $\gamma_1 > 0$ in (5.20), so $y(x)$ is expected to be an increasing function), we get

$$y_{\mathrm{c}} = \frac{4vu - 1 + \sqrt{(4v^2-1)(4u^2-1)}}{2}. \tag{5.22}$$

The corresponding value of x_{c} is

$$\begin{aligned} x_{\mathrm{c}} &= \frac{4vu + 1 + \sqrt{(4v^2-1)(4u^2-1)}}{2} \\ &= \frac{1}{4}\Big(\sqrt{(2v-1)(2u-1)} + \sqrt{(2v+1)(2u+1)}\Big)^2. \end{aligned} \tag{5.23}$$

Thus, for the values $x \in [1, x_{\mathrm{c}}]$ the function $y(x)$ monotonously increases from the value $u + v - 1$ up to the value y_{c}, given in (5.22). Note also that $x_{\mathrm{c}} \to 4w^2$ as $v, u \to w$, in agreement with the symmetric case.

Let us now consider the values of y on the left from the point $x = 1$. As it follows from (5.17), as y decreases from the value $u+v-1$ down to the value $|u-v|$, the variable x runs its values from 1 to 0. We thus conclude that our function y is the monotonous bijective map on the interval $[0, x_{\mathrm{c}}]$:

$$y(x) : [0, x_{\mathrm{c}}] \mapsto [|v-u|, y_{\mathrm{c}}],$$

where $y(0) = |v-u|$ and $y(x_{\mathrm{c}}) = y_{\mathrm{c}}$. Furthermore, since $x \to \infty$ as $y \to \infty$ in (5.17), this map extends to the whole domain $x \in [0, \infty)$ and it corresponds to $y \in [|v-u|, \infty)$.

The obtained property of the function y makes it possible to study the function σ_2^{II} near the point $x = 0$. Essentially similarly, as we have found the expansion near the point $x = 1$ above, we find that, as $x \to 0$,

$$
\begin{aligned}
y &= |v-u| + \frac{|v-u|(4vu - 2|v-u| + 1)}{2(|v-u|+1)^2} x \\
&\quad + \frac{|v-u|(4uv - 2|v-u| + 1)\big(1 + 6(v^2+u^2) - (4vu-3)|v-u|\big)}{8(|p-q|+1)^5} x^2 + O(x^3).
\end{aligned}
$$

Substituting this expansion into (5.18), we immediately arrive at the expression given in (5.14). This means that our solution obtained for Regime II also satisfies the conditions near the point $x = 0$ and no analogue of Regime III arises in the non-symmetric case, $p \neq q$ (or $u \neq v$).

Now turning to the function f_2, we conclude that all these considerations imply that

$$
f_2 = \begin{cases} f_2^{\mathrm{I}} & x \in [x_{\mathrm{c}}, \infty) \\ f_2^{\mathrm{II}} & x \in [0, x_{\mathrm{c}}]. \end{cases} \tag{5.24}
$$

As for the function f_2^{I}, by substituting (5.9) into (5.15) and fixing the constant of integration to match the condition (4.15), we obtain

$$
f_2^{\mathrm{I}} = \frac{(2v-1)(2u-1)}{4} \log \frac{x}{x-1}. \tag{5.25}
$$

As for the function f_2^{II}, we make the change of the integration variable $x \mapsto y$ in (5.9), where, due to (5.18), we have

$$
\frac{\mathrm{d}x}{x} = \left\{ \frac{2}{y+1} + \frac{1}{y+u-v} + \frac{1}{y+v-u} - \frac{2v}{2vy+v-u} - \frac{2u}{2uy+u-v} \right\} \mathrm{d}y. \tag{5.26}
$$

Taking into account (5.19), from (5.17) we obtain

$$
\begin{aligned}
f_2^{\mathrm{II}} = &-\frac{(u+v-1)^2}{2} \log y - \frac{(u-v)^2 + 2v + 2u - 1}{2} \log(y+1) \\
&+ \frac{4u^2-1}{4} \log(2uy+u-v) + \frac{4v^2-1}{4} \log(2vy+v-u) \\
&- \frac{2u-1}{2} \log(y+u-v) - \frac{2v-1}{2} \log(y+v-u) \\
&+ (u+v) \log(y+u+v+1) + \widetilde{C}_2.
\end{aligned} \tag{5.27}
$$

The constant of integration $\widetilde{C}_2$ can be fixed by imposing the condition (5.10). Using that $y(1) = u + v - 1$ from (5.27) we obtain

$$
\begin{aligned}
f_2^{\mathrm{II}}(1) = &-\frac{(u+v-1)^2}{2} \log(u+v-1) + \frac{(u+v)^2}{2} \log(u+v) \\
&+ \frac{(2u-1)^2}{4} \log(2u-1) + \frac{(2v-1)^2}{4} \log(2v-1) + (u+v)\log 2 + \widetilde{C}_2,
\end{aligned}
$$

and a comparison (recall that $v = p + 1/2$ and $u = q + 1/2$) with (5.10) yields

$$
\begin{aligned}
\widetilde{C}_2 = &-\frac{(2v+1)^2}{8} \log(2v+1) - \frac{(2v-1)^2}{8} \log(2v-1) \\
&- \frac{(2u+1)^2}{8} \log(2u+1) - \frac{(2u-1)^2}{8} \log(2u-1).
\end{aligned} \tag{5.28}
$$

This fixes the function f_2^{II}.

Using (5.25), (5.27), and (5.28), one can now show directly that (5.24) indeed holds. Namely, we prove that there exists one and only one solution $x_{\mathrm{c}} \in (1, \infty)$ of the equation $f_2^{\mathrm{II}}(x_{\mathrm{c}}) = f_2^{\mathrm{I}}(x_{\mathrm{c}})$. Calculation goes along the same lines as in the symmetric case (see end of the Sect. 4.2). Introduce the function

$$\rho(x) = f_2^{\mathrm{II}}(x) - f_2^{\mathrm{I}}(x).$$

Substituting (5.17) and (5.19) into (5.25), we get

$$\begin{aligned}
\rho(x) = \left(\frac{v^2}{2} + \frac{1}{8}\right) \log \frac{(2vy+v-u)^2}{(4v^2-1)\, y(y+1)} + \left(\frac{u^2}{2} + \frac{1}{8}\right) \log \frac{(2uy+u-v)^2}{(4u^2-1)\, y(y+1)} \\
+ \left(uv + \frac{1}{4}\right) \log \frac{y(y+u+v+1)(y-v-u+1)}{(y+1)(y+u-v)(y+v-u)} \\
+ \frac{1}{2} v \log \frac{(2v-1)(y+u-v)(y+v+u+1)}{(2v+1)(y+v-u)(y-v-u+1)} \\
+ \frac{1}{2} u \log \frac{(2u-1)(y+v-u)(y+v+u+1)}{(2u+1)(y+u-v)(y-v-u+1)} \\
+ \frac{1}{2} \log \frac{(y+u-v)(y+v-u)(y+1)}{(2uy+u-v)(2vy+v-u)}.
\end{aligned} \tag{5.29}$$

It is not difficult to see, that all the six logarithms in (5.29) vanish as soon as (5.21) holds, that leads us to (5.23). Hence, our result for x_{c} obtained above under the assumption of absence of the first order transition is recovered. Let us now show that there are no other roots of the equation $\rho(x) = 0$ on the interval $(1, \infty)$. Evaluating the derivative of the function $\rho(x)$, from (5.29) and (5.17) we get

$$\begin{aligned}
\rho'(x) &= \left(\frac{\partial x(y)}{\partial y}\right)^{-1} \frac{\partial \rho(x(y))}{\partial y} \\
&= \frac{(2vy+v-u)(2uy+u-v)\left[y^2 + (1-4vu)y + (v-u)^2\right]^2}{4y^2(y+1)^2(y+v+u+1)(y-v-u+1)(y+v-u)(y+u-v)},
\end{aligned}$$

and, again using (5.17),

$$\rho'(x) = \frac{\left[y^2 + (1-4vu)y + (v-u)^2\right]^2}{4xy^2(y+v+u+1)(y-v-u+1)}. \tag{5.30}$$

Recalling that $x \to 1$ as $y \to v+u-1$ (see (5.19)) and $x \to \infty$ as $y \to \infty$ (see (5.17)), we conclude from (5.30) that the function $\rho(x)$ is an increasing function on the interval $(1, \infty)$ except the point $x = x_{\mathrm{c}}$ where it has a simple zero, and where its first and second derivatives vanish, but the third one does not. This means that, besides that the point $x = x_{\mathrm{c}}$ is the only possible point of the phase transition, our system undergoes a third-order phase transition at this point.

It can also be directly checked that the resulting function f_2, given by (5.24), has the expected $x \to 0$ behavior as prescribed by (5.11).

5.3. Sub-leading corrections. Now we address calculation of the corrections to the leading term. Recall that we fix our parameters such that ν_1 and ν_2 has no $O(1)$ terms as N is large,

$$\nu_1 = vN, \qquad \nu_2 = -uN,$$

and v and u are related to the parameters p and q defined in (1.8) by $v = p + 1/2$ and $u = q + 1/2$.

Just like in the symmetric case, the $1/N$ expansion (4.1) in the non-symmetric case can be constructed by using a relevant equation among the two in (4.27). Again, such an equation can be easily identified provided the leading term, the function σ_2, is given by a singular solution of the Clairaut equation. We meet such a situation in our problem in Regime II, and it is the equation containing the function $F_(\sigma', \sigma'')$. Calculation (4.30) can be repeated without modifications for the non-symmetric case, again providing the result $\sigma_1 = 0$, see (4.31). In fact, all the terms of odd powers in $1/N$ also vanish, and (4.1) is an expansion in $1/N^2$ in Regime II.

In Regime I the situation in the non-symmetric case repeats that in the symmetric case, since the leading term σ_2 is given by a regular solution of the Clairaut equation. This term solves both equations in (4.27) in the leading order, since $f_+(\sigma_2) = f_-(\sigma_2)$. To identify which one of the two equations in (4.27) is actually responsible for the $1/N$ expansion, we should search for a suitable solution for the function σ_1. In total, the recipe of derivation of sub-leading corrections in the non-symmetric case goes along the same lines as in the symmetric one.

We turn to the details of calculation for Regime I and Regime II separately.

5.3.1. *Regime I.* We start with $x \to \infty$ asymptotic expansion (4.32) yielding

$$\sigma_1 = -\frac{x}{2} + uv + \frac{1}{4} + \frac{(4v^2-1)(4u^2-1)}{16x} + O\left(x^{-2}\right), \qquad x \to \infty. \tag{5.31}$$

We recall that in this regime $\sigma_2' = -1/4$ and so both equations in (4.27) vanish in the leading order. To identify which one of the two equations in (4.27) is responsible to $1/N$ expansion and find the functions σ_1 and σ_0, we expand the functions $F_\pm(\sigma', \sigma'')$ to order N^0:

$$\begin{aligned}
F_\pm(\sigma', \sigma'') &= N^2\left(\frac{1}{8} - \frac{vu}{2}\right) \\
&+ N\left(-\frac{\sigma_1'}{2} - 2vu\sigma_1' \mp 2\sqrt{\left(v^2-\frac{1}{4}\right)\left(u^2-\frac{1}{4}\right)\left((\sigma_1')^2-\frac{1}{4}\right)}\right) \\
&+ \left(-\frac{1}{2} - 2uv \mp 2\sigma_1' \frac{\sqrt{\left(v^2-\frac{1}{4}\right)\left(u^2-\frac{1}{4}\right)}}{\sqrt{(\sigma_1')^2-\frac{1}{4}}}\right)\sigma_0' - 8vu(\sigma_1')^2 + \frac{vu}{2} \\
&\mp \left[\frac{\left(8v^2u^2-v^2-u^2\right)\sqrt{(\sigma_1')^2-\frac{1}{4}}}{\sqrt{\left(v^2-\frac{1}{4}\right)\left(u^2-\frac{1}{4}\right)}} + \frac{\sqrt{\left(v^2-\frac{1}{4}\right)\left(u^2-\frac{1}{4}\right)}}{2\sqrt{(\sigma_1')^2-\frac{1}{4}}}\right]\sigma_1' + O(N^{-1}).
\end{aligned}$$

Hence, σ_1 must be a solution of one the following two equations:

$$\sigma_1 = x\sigma_1' - \frac{\sigma_1'}{2} - 2vu\sigma_1' \mp 2\sqrt{\left(v^2-\frac{1}{4}\right)\left(u^2-\frac{1}{4}\right)\left((\sigma_1')^2-\frac{1}{4}\right)}. \tag{5.32}$$

These are the Clairaut equations and we have to search for a singular solution that matches the asymptotic expansion (5.31). The proper solution reads

$$\sigma_1 = -\frac{1}{2}\sqrt{s(x)}, \qquad s(x) = x^2 - (1+4vu)x + (v+u)^2. \tag{5.33}$$

Note that the critical value $x = x_\mathrm{c}$ given by (5.23) is one of the roots of the polynomial $s(x)$, the second root is always smaller than x_c. The solution (5.33) corresponds to the plus sign in (5.32) and hence the equation in (4.27) relevant for the $1/N$ expansion is the one involving the function $F_-(\sigma', \sigma'')$.

The function σ_0 can be directly computed. Since σ_1 is given by a singular solution, the σ_0' term vanishes in the equation in the order N^0, that yields

$$\sigma_0 = -8vu(\sigma_1')^2 + \frac{vu}{2} + \frac{\left(8v^2u^2 - v^2 - u^2\right)\sigma_1'\sqrt{(\sigma_1')^2 - \frac{1}{4}}}{\sqrt{\left(v^2 - \frac{1}{4}\right)\left(u^2 - \frac{1}{4}\right)}} + \frac{\sqrt{\left(v^2 - \frac{1}{4}\right)\left(u^2 - \frac{1}{4}\right)}}{2\sqrt{(\sigma_1')^2 - \frac{1}{4}}}\sigma_1'.$$

Substituting (5.33), we get

$$\sigma_0 = -\frac{x}{4} + \frac{1}{8} - \frac{4vux^2 - 2(v+u)^2x + (v+u)^2}{4s(x)}.$$

Let us now consider the function $\log P_{N,M,L}(x^{-1})$. For this quantity we have the expansion (4.35). Note that all the terms must vanish at infinity, $\lim_{x\to\infty} f_1 = 0$, $\lim_{x\to\infty} f_0 = 0$, etc. Computing f_1 by (4.36) with $\widetilde{A}_1 = \widetilde{B}_1 = 1/2$ and choosing $\widetilde{C}_1$ such that $f_1(\infty) = 0$, we get

$$f_1 = v\log\left(\frac{2vx - v - u + \sqrt{s(x)}}{(2v+1)\sqrt{x(x-1)}}\right) + u\log\left(\frac{2ux - v - u + \sqrt{s(x)}}{(2u+1)\sqrt{x(x-1)}}\right) \\ - \frac{1}{2}\log\frac{2x - 4uv - 1 + 2\sqrt{s(x)}}{4x}.$$

Essentially similarly, for f_0, using (4.37) with $\widetilde{A}_0 = 1/4$ and $\widetilde{B}_0 = 1/8$ and choosing $\widetilde{C}_0$ such that $f_0(\infty) = 0$, we obtain

$$f_0 = \frac{1}{4}\log\frac{x(x-1)}{s(x)}.$$

Finally, rewriting these formulas in terms of p and q we arrive at the expressions for the functions f_1^{I} and f_0^{I} appearing in Thm. 1.2.

5.3.2. *Regime II.* We recall that here we have $\sigma_1 = 0$. The function σ_0 can be found by expanding the function $F_-(\sigma', \sigma'')$ to order N^0, that yields

$$\sigma_0 = -\frac{vu}{8\sigma_2'} + \frac{\left(\sigma_2' - \frac{1}{4}\right)\sqrt{\left(\sigma_2' + v^2\right)\left(\sigma_2' + u^2\right)}}{8\sigma_2'\left(\sigma_2' + \frac{1}{4}\right)} - \frac{[x(x-1)\sigma_2'']^2}{4\left(\sigma_2' + \frac{1}{4}\right)\sqrt{\left(\sigma_2' + v^2\right)\left(\sigma_2' + u^2\right)}}.$$

To obtain an explicit expression for σ_0 in terms of $y = y(x)$, one can use the expression (5.16) for σ_2'. Taking into account (see (5.26)) that

$$\partial_y x = \frac{2y(y+1)\Big(4vuy^3 - (v-u)^2\left[3y^2 + 3y - (v+u)^2 + 1\right]\Big)}{(2uy + u - v)^2(2vy + v - u)^2}$$

one can also express σ_2'' in terms of y and hence find σ_0 as a function of y.

Let us now consider the function $\log P_{N,M,L}(x^{-1})$. As in the symmetric case, for $\log P_{N,M,L}(1)$ we have the expansion (4.38), where $f_2(1)$ is given by (5.10), and the values $f_1(1)$ and $f_0(1)$ are given by

$$f_1(1) = -\frac{1}{2}\Big\{(p+1)\log(p+1) + (q+1)\log(q+1) - p\log p - q\log q\Big\}, \tag{5.34}$$

and

$$f_0(1) = -\frac{1}{24}\log\frac{(p+1)(q+1)}{pq} - \frac{1}{12}\log\frac{p+q+1}{p+q} + \zeta'(-1) + \log\sqrt{2\pi}, \tag{5.35}$$

respectively. These values can be used to construct the expansion (4.41) where the functions f_1 and f_0 can be found from σ_1 and σ_0 by the usual formulas (4.36) and (4.37). As for the function f_1, since $\sigma_1 = 0$, we just repeat the calculation from the symmetric case, now using (5.34) to fix the integration constant, that yields

$$f_1 = \frac{1}{2}\left\{\log x + p\log p - (p+1)\log(p+1) + q\log q - (q+1)\log(q+1)\right\}.$$

As for function f_0, one can perform the integration (4.37) by changing the integration variable $x \mapsto y$, similarly to the case of the function f_2. The calculation of f_0 appears to be notably involved, the final result reads

$$\begin{aligned} f_0 = \frac{1}{8}\Bigg\{ & \log y + \log(y+1) - 2\log(2vy + v - u) \\ & - 2\log(2uy + u - v) + 3\log\left(y^2 + (1-4vu)y + (v-u)^2\right) \\ & + \frac{1}{3}\log\Big(4vuy^3 - (v-u)^2\left[3y^2 + 3y - (v+u)^2 + 1\right]\Big)\Bigg\} + \widetilde{C}_0. \end{aligned}$$

The constant of integration can be fixed by computing the value at $x = 1$, or $y = u + v - 1$, and we get

$$f_0\big|_{y=u+v-1} = \frac{1}{12}\log\frac{(2v-1)(2u-1)(v+u-1)}{v+u} + \widetilde{C}_0.$$

Comparison with (5.35) gives

$$\widetilde{C}_0 = -\frac{1}{24}\log\left(16p(1+p)q(1+q)\right) + \zeta'(-1) + \log\sqrt{2\pi}.$$

In total, we arrive at the functions f_1^{II} and f_0^{II} appearing in Thm. 1.2, which is now finally proven.

6. Conclusion

In this paper, we have studied the five-vertex model on a rectangular domain with scalar-product boundary conditions. Relying on the connection between the partition function of the model and the sixth Painlevé equation, we have derived the expansion of the free energy in the limit where the size of the domain tends to infinity. The key advantage of this approach lies in its capability to provide not only the leading term of such an expansion, but also sub-leading corrections. All terms of the expansion can be computed recursively. Here, we limit ourselves by explicit expressions to the order of a constant (see Thms. 1.1 and 1.2).

Our results reveal an interesting feature: in the case of a rectangular domain, there is no Regime III, in other words, one phase transition disappears. To gain a better understanding of this phenomenon, we have generated several configurations of the model numerically. In order to ensure sampling from the correct probability distribution, we resorted to the Coupling From the Past Algorithm [41,42]. For the simulation we use $w_1 = w_3 = w_4 = 1$ and $w_5 = w_6 = 1/\sqrt{x}$ (see the comment at the end of Sect. 1.1 and the discussion in Sect. 2.1). Examples of configurations are presented in Fig. 6 (square domain) and Fig. 7 (rectangular domain). Numerical

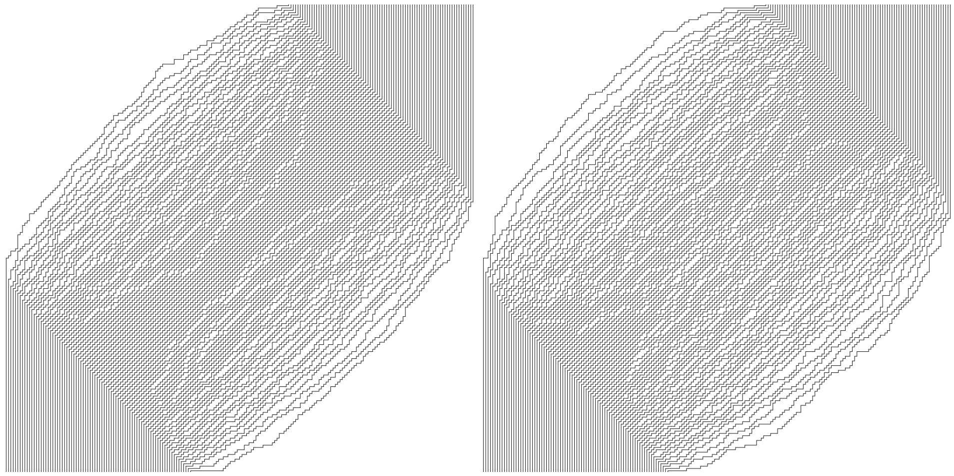

FIGURE 6. Configurations of the five-vertex model at $\sqrt{x} = 0.24$ (left) and $\sqrt{x} = 0.3$ (right) on a 'square' domain with $N = 80$, $M = 200$, and $L = 201$. On the left picture the two disordered regions are separated by a region of the anti-ferroelectric order. On the right picture these two disordered regions merge with each other.

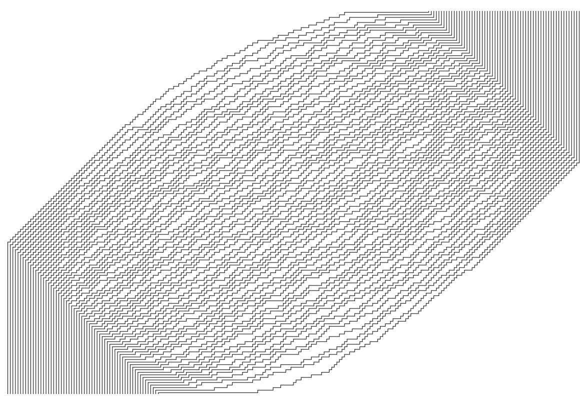

FIGURE 7. A configuration of the five-vertex model at $\sqrt{x} = 0.01$ on a rectangular domain with $N = 60$, $M = 150$, and $L = 225$. The disordered region does not split, and there is no region of the anti-ferroelectric order.

simulation clearly shows the distinction in the behavior of the model for the square and rectangular domains.

Specifically, for a square domain (where $\epsilon = M - L + 1$ is of $O(1)$ as $N, M, L \to \infty$) if $x > x_{\mathrm{c}}^{-1}$, then there exists a single disordered region. For $x = 1$, which is known as the free-fermion point of the five-vertex model, the configurations are described by random boxed plane partitions and the disordered region has a form of the ellipse [43]. In numerical simulations, we have been interested in the values of x slightly above and below the critical value $x = x_c^{-1}$ separating Regime II and Regime III, according to Thm. 1.1. Pictures of Fig. 6 show typical configurations for a 'square' domain with $N = 80$, $M = 200$, and $L = 201$. In this geometry the phase transition between Regime III and Regime II occurs at $1/\sqrt{x_c} \approx 0.25$. On the left picture where $\sqrt{x} = 0.24$, which corresponds to the Regime III, one can see two disordered regions separated by a region with the anti-ferroelectric order. On the right picture $\sqrt{x} = 0.3$, which corresponds to the Regime II, these two disordered regions merge with each other and we observe just a single disordered region.

Based on this simple illustration one can conclude, that the phase transition occurring at $x = x_c^{-1}$ corresponds to the split of the disordered region into two distinct parts and hence it resembles the "merger transition" studied in [28]. This interpretation also aligns perfectly with the results of [12]. It is useful also to mention that as $x \to 0$ both disordered regions shrink down and the model falls into the anti-ferroelectric ground state (see Fig. 5).

On the other hand, in the case of a rectangular domain, the disordered region remains connected and the split does not occur. The picture of Fig. 7 shows an example of configuration for a very small value (as small as the algorithm has allowed us to produce the picture for a reasonable amount of computing time) of the parameter $\sqrt{x} = 0.01$ for the rectangle domain with $N = 60$, $M = 150$, and $L = 225$. The disordered region changes its shape from the ellipse (which occurs at $x = 1$) but one can observe no signal of splitting it on two (or whatever) regions. Thus, one can conclude that the disordered region remains connected as x decreases. On the contrary to the case of the square domain, there is no analogue of the anti-ferroelectric ground state, and as a consequence, the disordered region does not disappear as $x \to 0$ [34].

We end up by a brief discussion of the phase transition between Regime I and Regime II at $x = x_c$, which take place for both rectangular and square shaped domains. This transition can be characterized by disappearance of the disordered region in the center of the domain as the parameter x increases from $x < x_c$ (Regime II) to $x > x_c$ (Regime I). It is clear, that the dominance of the b-weight vertices (see also discussion in Sect. 2.1) that occurs for large x cannot be spoiled by the geometry of the domain. One could expect to see this in numerical simulations, but unfortunately for large x, especially for $x > x_c$, the methods such as used above for small x are not able to produce a configuration for sufficiently large domains in a reasonable time. It definitely deserves further study how various algorithms can be adapted to produce meaningful pictures at large sizes of the domain.

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