GLOBAL RIGIDITY FOR SOME PARTIALLY HYPERBOLIC ABELIAN ACTIONS WITH 1-DIMENSIONAL CENTER

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ABSTRACT. We obtain a global rigidity result for abelian partially hyperbolic higher rank actions on certain $2-$step nilmanifolds X_Γ. We show that, under certain natural assumptions, all such actions are $C^\infty-$conjugated to an affine model. As a consequence, we obtain a centralizer rigidity result, classifying all possible centralizers for any C^1-small perturbation of an irreducible, affine partially hyperbolic map on X_Γ. Along the way, we also prove two results of independent interest. We describe fibered partially hyperbolic diffeomorphisms on X_Γ and we show that topological conjugacies between partially hyperbolic actions and higher rank affine actions are C^∞.

1. INTRODUCTION

Rigidity of $\mathbb{Z}^k-$actions on tori (and nilmanifolds) with some hyperbolicity have been studied extensively. The general philosophy is: large abelian actions with some hyperbolicity should be *globally rigid*, i.e. smoothly conjugated to algebraic models. A big breakthrough result in this direction was obtained by Katok and Spatzier [43] where they prove that all perturbations of certain algebraic Anosov actions are smoothly conjugated back to the corresponding algebraic models. There they also outline a rigidity program for abelian actions with hyperbolicity. Since the paper by Katok and Spatzier, a lot of results have been obtained for large abelian actions close to some algebraic model with some hyperbolicity [16, 18, 14, 59, 60, 64, 65, 63, 23, 12]. Even earlier than the result by Katok and Spatzier, Katok and Lewis [42] proved a global rigidity statement for Anosov action on tori. In [42], the authors show that a certain class of Anosov $\mathbb{Z}^d-$actions on the torus $\mathbb{T}^{d+1}$ is, necessarily, smoothly conjugated to an algebraic model, even though the action might not be close to the algebraic model. A crucial assumption in [42] is that the $\mathbb{Z}^d-$action contains *many* Anosov elements. The property of having many Anosov elements was removed by F. Rodriguez Hertz in [50], where Rodriguez Hertz only assumes that the action contains one Anosov element. Nilmanifolds are natural generalizations of tori. There has been a lot of work studying global rigidity of higher rank[1] Anosov actions on nilmanifolds

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[1]See Definition 2.3.

since the paper by Katok and Lewis [37, 38, 39, 25]. The culmination of these works is the result by F. Rodriguez Hertz, Z. Wang and Fisher, Kalinin, Spatzier [52, 26] proving that abelian higher rank Anosov actions on (infra-)nilmanifolds are smoothly conjugated to algebraic models, completely resolving the question of global rigidity of abelian higher rank Anosov actions on these manifolds. Relaxing the Anosov assumption leads to the question:

When are partially hyperbolic $\mathbb{Z}^k-$actions on nilmanifolds globally rigid?

These actions have been remarkably resistant. Even local rigidity for algebraic partially hyperbolic actions on (non-toral) nilmanifolds has been open for a long time, until recent advances by Z. J. Wang [63]. In this paper, we produce an initial result towards answering the global rigidity question.

In Theorem A we prove the first global rigidity result for higher rank abelian actions on nilmanifolds with one partially hyperbolic element.

The questions of local and global rigidity of higher rank actions can also be studied for different types of actions, either dropping the assumption that the action is abelian, or dropping the assumption that the action should have some hyperbolicity. Local rigidity has been obtained for large abelian parabolic actions, with no hyperbolicity [17, 15, 62, 13]. Removing the assumption that the action should be abelian, we can study the actions of higher rank lattices in semi-simple Lie groups, see for example [24] and the references therein. In fact, the rigidity result for Abelian actions in [42] was used to obtain local rigidity of $\mathrm{SL}(n, \mathbb{Z})-$action on tori. Considering rigidity of Anosov lattice actions on nilmanifolds, there are also global results, see for example [10]. A key point in [10] is that any conjugacy between the hyperbolic lattice action and the algebraic model also conjugates the action of a large abelian subgroup to some algebraic abelian action. So, the results of [52, 26] can be applied to improve the regularity of the conjugacy. Considering the main results of this paper, see Theorem A, a natural question is:

Are partially hyperbolic higher rank lattice actions on Heisenberg nilmanifolds globally rigid?

1.1. **Global rigidity of partially hyperbolic actions.** Let G be a simply connected $\ell-$step nilpotent Lie group. That is, the lower central series $G^{(1)} = G$, $G^{(j+1)} = [G, G^{(j)}]$, terminate at ℓ, $G^{(\ell)} = e$. Given a lattice $\Gamma \leq G$ we define the associated *compact nilmanifold* as the quotient $X_\Gamma = \Gamma \backslash G$. Compact nilmanifolds have associated groups of *automorphisms* and *affine maps*

(1.1) $\mathrm{Aut}(X_\Gamma) = \{L \in \mathrm{Aut}(G) : L\Gamma = \Gamma\},$

(1.2) $\mathrm{Aff}(X_\Gamma) = \{f_0(x) = L(x)g^{-1} : L \in \mathrm{Aut}(X_\Gamma), \ g \in G\}.$

By automorphism rigidity of nilpotent lattices [48] we can, equivalently, define $\mathrm{Aut}(X_\Gamma) = \mathrm{Aut}(\Gamma) = \mathrm{Aut}(\pi_1 X_\Gamma)$. An automorphism $\rho : \mathbb{Z}^k \to \mathrm{Aut}(X_\Gamma)$ is said to have a rank-1 factor if there is some quotient $\hat{X}_\Gamma$ of X_Γ such that ρ descends to $\hat{X}_\Gamma$ and the induced map $\hat{\rho} : \mathbb{Z}^k \to \mathrm{Aut}(\hat{X}_\Gamma)$ factor through a map $\mathbb{Z} \to \mathrm{Aut}(\hat{X}_\Gamma)$. A homomorphism $\rho : \mathbb{Z}^k \to \mathrm{Aut}(X_\Gamma)$ is *higher rank* if it has no rank-1 factor. More generally, given a smooth action $\alpha : \mathbb{Z}^k \times X_\Gamma \to X_\Gamma$ we have an induced map $\alpha_* : \mathbb{Z}^k \to \mathrm{Aut}(\pi_1 X_\Gamma)$. We say that α is higher rank if the induced map α_* is higher rank.

A diffeomorphism $f : X_\Gamma \to X_\Gamma$ is *partially hyperbolic* if there is a $Df-$invariant splitting $TX_\Gamma = E^s \oplus E^c \oplus E^u$ such that E^s is exponentially contracted, E^u is exponentially expanded and the behaviour along E^c is dominated by the behaviour of Df along E^s and E^u (for a precise definition, see Section 2.1). If the *center bundle* E^c is the trivial bundle then f is Anosov. The main result of this paper is an extension of the results of [52] to certain nilmanifolds by weakening the assumption that α is Anosov. Instead, we assume that the action α contains a partially hyperbolic element.

Theorem A. *Let G be a $2-$step nilpotent Lie group with $\dim[G,G] = 1$, $\Gamma \leq G$ a lattice and $X_\Gamma = \Gamma \backslash G$ the associated nilmanifold. Let $\alpha : \mathbb{Z}^k \times X_\Gamma \to X_\Gamma$ be a smooth higher rank action with $\mathbf{n}_0 \in \mathbb{Z}^k$ such that $f = \alpha^{\mathbf{n}_0}$ is partially hyperbolic and satisfying*

 (i) f has $1-$dimensional center,
 (ii) the stable and unstable foliations W^s, W^u are quasi-isometric in the universal cover.

Then α is $C^\infty-$conjugated to some affine action $\alpha_0 : \mathbb{Z}^k \to \mathrm{Aff}(X_\Gamma)$.

Remark 1. With G as in Theorem A the group G can be written as $G = H^n \times \mathbb{R}^m$, $n \neq 0$, where H^n is a Heisenberg group. The nilmanifold X_Γ is also a product of a Heisenberg nilmanifold and a torus. On these manifolds there exists no Anosov actions since the derived subgroup, $[G,G]$, is isometric for any automorphism. So Theorem A is the only global rigidity result on these manifolds, since [52, 26] do not apply. In fact, to the author's knowledge, Theorem A is the first global rigidity result for abelian actions assuming only one partially hyperbolic element.

Remark 2. In principle the proof of Theorem A should work for $\ell-$step G with $\ell > 2$ as long as $\dim G^{(\ell)} = 1$. In this case the quasi-isometric assumption, assumption (ii), would have to be changed. This is a work in progress.

Remark 3. If α is topologically conjugated to some affine action, then the methods from [52] generalize to partially hyperbolic systems, see Theorem 1.2. So, the main novelty of Theorem A is that we produce a topological conjugacy from α to an affine action α_0.

1.2. **Applications to centralizer classification and centralizer rigidity.**
Given a diffeomorphism $f : M \to M$ on a closed manifold we define its *smooth centralizer* as the group of diffeomorphisms that commute with f. That is, we define

$$(1.3) \qquad\qquad Z^\infty(f) = \{g \in \mathrm{Diff}^\infty(M) : fg = gf\}.$$

We are interested in two questions about the group $Z^\infty(f)$:

   (i) What are possible groups that arise as $Z^\infty(f)$ for some $f \in \mathrm{Diff}^\infty(M)$?
   (ii) If $Z^\infty(f)$ is large (compared to the conjecturally generic size $\mathbb{Z}$, [56, 57]) what can be said about f?

In this level of generality, questions (i) and (ii) are difficult (or possibly impossible) to answer. Instead, we fix $f_0 \in \mathrm{Aff}(X_\Gamma)$ for some $X_\Gamma = \Gamma \backslash G$, and consider question (i) and (ii) for those $f \in \mathrm{Diff}^\infty(X_\Gamma)$ that are C^1-close to f_0. We call (i) the question of *local centralizer classification* and (ii) the question of *local centralizer rigidity* around f_0. These questions were raised and addressed by Damjanović, Wilkinson and Xu in [20] where the authors study perturbations of time$-t_0$ map of geodesic flows on negatively curved manifolds and trivial circle extensions of hyperbolic automorphisms. In [5] the authors study local centralizer rigidity of time-1 maps of Anosov flows on $3-$manifolds, generalizing results from [20] in the context of $3-$manifolds. Another generalization of results from [20] was obtained by W. Wang in [66], where semi-simple Lie groups of higher rank were studied instead of rank-1 simple groups. For ergodic toral automorphisms, Gan, Xu, Shi and Zhang studied partially hyperbolic diffeomorphisms on $\mathbb{T}^3$ homotopic to an hyperbolic automorphism [28]. In [54] the author studies local centralizer classification and rigidity for some partially hyperbolic, irreducible[2] toral automorphisms.

If f is partially hyperbolic with (uniquely integrable) center foliation W^c, then we obtain a normal subgroup $Z_c^\infty(f) \subset Z^\infty(f)$, the *center fixing centralizer*:

$$(1.4) \qquad Z_c^\infty(f) := \{g \in \mathrm{Diff}^\infty(X_\Gamma) : gx \in W^c(x), \ x \in X_\Gamma\}.$$

From [21, Theorem 5], if $Z_c^\infty(f)$ is sufficiently big and f is fibered (see [21, Definition 1]) then f is smoothly conjugated to an isometric extension of an Anosov map. Combining this with Theorem A we completely classify the centralizers of diffeomorphisms C^1-close to affine partially hyperbolic maps.

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[2]An automorphism $A \in \mathrm{GL}(d, \mathbb{Z})$ is irreducible if the characteristic polynomial $p_A(t)$ is irreducible in $\mathbb{Q}[t]$.

Let G be the $(d+1)-$dimensional Heisenberg group and $X_\Gamma = \Gamma \setminus G$ a compact Heisenberg nilmanifold. We have a natural fibration

$$(1.5) \qquad \qquad \pi : X_\Gamma \to \mathbb{T}^d$$

such that any affine map on X_Γ descends to an affine map on $\mathbb{T}^d$. From the group relations in G, any automorphism $L \in \mathrm{Aut}(X_\Gamma)$ induce an element of $\mathrm{Sp}(d, \mathbb{Z})$ on $\mathbb{T}^d$. Any $L \in \mathrm{Sp}(d, \mathbb{Z})$ also define an element of $\mathrm{Aut}(X_\Gamma)$, see Section 2.2. Given $f_0 \in \mathrm{Aff}(X_\Gamma)$ we denote by $L_{su} \in \mathrm{Sp}(d, \mathbb{Z})$ the induced automorphism on $\mathbb{T}^d$. Before stating the theorem we define for any $f_0 \in \mathrm{Aff}(X_\Gamma)$, with irreducible, hyperbolic induced map on the base $L_{su} \in \mathrm{Sp}(d, \mathbb{Z})$, the natural number

$$(1.6) \qquad r_0(f_0) = \mathrm{rank}\left(\frac{Z^\infty(f_0)}{Z_c^\infty(f_0)} \right) = \mathrm{rank}\left(Z_{\mathrm{Sp}(d,\mathbb{Z})}(L_{su}) \right).$$

The number $r_0(f_0)$ is calculated explicitly in Lemma A.3, if $d \geq 6$ then $r_0(f_0) > 1$.

Theorem B. *Let X_Γ be a compact Heisenberg nilmanifold and let $f_0 \in \mathrm{Aff}(X_\Gamma)$ be partially hyperbolic with $1-$dimensional center and L_{su} irreducible. If $f \in \mathrm{Diff}^\infty(X_\Gamma)$ is C^1-close to f_0 then one of the following holds*

(i) either $Z^\infty(f)$ is virtually trivial,
(ii) or $Z^\infty(f)$ is virtually $\mathbb{Z} \times \mathbb{T}$ in which case f is an isometric extension of some Anosov diffeomorphism on $\mathbb{T}^d$,
(iii) or $Z^\infty(f)$ is virtually $\mathbb{Z}^{r_0} \times \mathbb{T}$ and if $r_0 > 1$ then f is $C^\infty-$conjugate to some (possibly different) affine map $\tilde{f}_0 \in \mathrm{Aff}(X_\Gamma)$.

Remark 4. All cases (i), (ii) and (iii) occur, so Theorem B completely classifies the centralizer of $f \in \mathrm{Diff}^\infty(X_\Gamma)$ close to a partially hyperbolic $f_0 \in \mathrm{Aff}(X_\Gamma)$. Case (i) holds generically [6]. Case (ii) can be produced by fixing some irreducible, hyperbolic $L \in \mathrm{Sp}(d, \mathbb{Z})$ and defining f on $G \cong \mathbb{R}^d \times \mathbb{R}$ by $f(x, t) = (Lx, t + \beta(x))$ where the second coordinate is identified with $[G, G] \cong \mathbb{R}$ and $\beta : \mathbb{T}^d \to \mathbb{R}$ is a cocycle over L that is not cohomologous to a constant. The last case holds when f is $C^\infty-$conjugate to some affine $\tilde{f}_0$, so in particular when we take the trivial perturbation $f = f_0$.

Remark 5. Similar results as Theorem B have been obtained independently by Damjanović, Wilkinson and Xu using different methods with additional assumptions [19].

1.3. **Partially hyperbolic maps on nilmanifolds.** When proving Theorems A and B, we use a description of partially hyperbolic diffeomorphisms on the nilmanifolds considered in Theorem A. The main property that we show is that, under the assumptions of Theorem A, the system f is *fibered* in the terminology of [4]:

Theorem 1.1. *Let $G = H^n \times \mathbb{R}^m$ (where we allow $n = 0$), $\Gamma \leq G$ a lattice with associated nilmanifold $X_\Gamma = \Gamma \backslash G$. Let $f \in \mathrm{Diff}^\infty(X_\Gamma)$ be partially hyperbolic and satisfy (i), (ii) from Theorem A. If G is abelian we assume, in addition, that the induced map $f_* : H_1(X_\Gamma) \to H_1(X_\Gamma)$ has at least one rational eigenvalue. The following holds*

> (i) *f is dynamically coherent with global product structure,*
> (ii) *all foliations W^σ, $\sigma = s, c, u, cs, cu$, are uniquely integrable,*
> (iii) *the center foliation W^c have compact oriented circle leaves,*
> (iv) *f is fibered over some hyperbolic $L_{su} \in \mathrm{GL}(d, \mathbb{Z})$ in the sense that there is some Hölder $\Phi : X_\Gamma \to \mathbb{T}^d$ such that $\Phi(fx) = L_{su}\Phi(x)$, $W^c(x) = \Phi^{-1}(\Phi(x))$ and Φ is homotopic to the projection $\pi : X_\Gamma \to \mathbb{T}^d$,*
> (v) *there is a finite index subgroup $Z^\infty_{\mathrm{fix}}(f) \leq Z^\infty(f)$ such that if $g \in Z^\infty_{\mathrm{fix}}(f)$ we have $\Phi(gx) = B\Phi(x)$ where $B \in \mathrm{GL}(d, \mathbb{Z})$ is defined by $B\Phi_* = \Phi_* g_*$, B is the induced map on homology if G is non-abelian,*

moreover, if G is not abelian then

> (vi) *f is accessible.*

Remark 6. The assumption that f_* has at least one rational eigenvalue is to remove derived-from-Anosov examples since these examples are not fibered.

Remark 7. Properties (i) and (ii) follow from [7].

Remark 8. This Theorem is similar to the classification of partially hyperbolic diffeomorphisms on $3-$dimensional manifolds by Hammerlindl and Hammerlindl-Potrie [33, 32, 34]. In fact, in dimension 3, using [8], Theorem 1.1 essentially reduces to the main results of [33, 32] (in [32] we must make the extra assumption that the linearization $L \in \mathrm{GL}(3, \mathbb{Z})$ has at least one rational eigenvalue).

1.4. **Improved regularity of topological conjugacies between higher rank actions.** The conjugacy in Theorems A and B, case (iii), is produced in two steps. First, we construct a topological conjugacy and second we show that the topological conjugacy is C^∞. The second step is the content of the following theorem, that may be of independent interest.

Theorem 1.2. *Let X_Γ be a nilmanifold and $\alpha_0 : \mathbb{Z}^k \to \mathrm{Aff}(X_\Gamma)$ a homomorphism. Suppose that α_0 is higher rank. If $\alpha : \mathbb{Z}^k \to \mathrm{Diff}^\infty(X_\Gamma)$ is bi-Hölder conjugate to α_0 by $H : X_\Gamma \to X_\Gamma$ and there is some $\mathbf{n}_0 \in \mathbb{Z}^k$ such that $f = \alpha^{\mathbf{n}_0}$ is partially hyperbolic and accessible with center $\dim(E^c) = \dim(E^c_{\alpha_0^{\mathbf{n}_0}})$, then H is a $C^\infty-$diffeomorphism.*

Theorem 1.2 is a generalization of the global rigidity result by F. Rodriguez Hertz and Z. Wang [52] to some higher rank partially hyperbolic actions. In fact, large parts of the results in [52] generalize immediately to partially hyperbolic actions. One exception is that Rodriguez Hertz and Wang use a characterization of Anosov diffeomorphisms due to Mañé [45], to show that many elements of the action are Anosov. This characterization can not be applied in the partially hyperbolic setting. We also change some technical aspects of the proof, removing the use of Pesin theory.

1.5. **Description of proofs.** Let G and $\alpha : \mathbb{Z}^k \times X_\Gamma \to X_\Gamma$ be as in Theorem A and $f = \alpha^{\mathbf{n}_0}$ the partially hyperbolic element. By considering the Lie algebra of G, $\mathfrak{g}$, it is immediate that G takes the form $G = H^n \times \mathbb{R}^m$, $n \neq 0$, where H^n is the $(2n+1)-$dimensional Heisenberg group (for $n = 0$ we will consider H^n as the trivial group 1). Moreover, under the assumptions of Theorem B G has to be a Heisenberg group H^n for some n (this follows since any lattice Γ in $H^n \times \mathbb{R}^m$ is, virtually, a product lattice so irreducibility of $(f_0)_*$ guarantee that either $n = 0$ or $m = 0$). The proof of Theorem A is divided into three steps, first we show that any f as in Theorem A is fibered, then we show that any action α as in A is topologically conjugated to some affine model and finally we show that the topological conjugacy can be improved to a smooth conjugacy.

1.5.1. *Step 1.* The first part of the proof of Theorem A is to show that any element f as in the theorem has to fiber over a hyperbolic automorphism of the torus and that f is accessible. Both of these properties are contents of Theorem 1.1. First, we obtain a map Φ on G, which is a contender for being the map in Theorem 1.1, this is done as in [27]. Second, we show that Φ is injective on the lifted stable and unstable leaves $\hat{W}^\sigma$, $\sigma = s, u$. By invariance of domain, this implies that the stable and unstable distributions of f satisfy $\dim(E^\sigma) \leq \dim(E_0^\sigma)$, where E_0^σ is the corresponding distribution for the linearization L of f. Since f has center E^c of dimension 1 by assumption we conclude that $\dim(E^\sigma) = \dim(E_0^\sigma)$ for $\sigma = s, c, u$. This shows, see Lemma 3.1, that Φ descends to a map $\Phi : X_\Gamma \to \mathbb{T}^d$. Showing that Φ gives a fiber bundle structure as in Theorem 1.1 is then similar to [7].

Proving accessibility uses a topological argument. Since X_Γ does not have a virtually abelian fundamental group, and since the kernel of the induced map

$$(1.7) \qquad\qquad \Phi_* = \pi_* : \pi_1 X_\Gamma = \Gamma \to \mathbb{Z}^d$$

is the center of Γ, there can not exist a connected compact set $K \subset X_\Gamma$ such that $\Phi : X_\Gamma \to \mathbb{T}^d$ is a finite covering map. This implies, in particular, that f can not have a compact $su-$leaf. So, the proof of accessibility reduces to proving that if f has a non-open accessibility class, then there is a compact $su-$leaf. Obtaining a compact $su-$leaf is done by studying the holonomies between center leaves in

the universal cover induced by the fundamental group, as in [49, map T_n defined on page 71]. If $K_x \subset G \cap \hat{W}^c(x)$ is the closed set such that $y \in K_x$ does not have open accessibility class, then there is an action of Γ on K_x defined by mapping y to the unique intersection between the accessibility class of γy and K_x. We show that the induced $\Gamma-$action on the image of K_x in $\hat{W}^c(x)/[\Gamma, \Gamma]$ has a fixed point if K_x is non-empty. This fixed point corresponds to a compact $su-$leaf, which gives a contradiction so K_x must be empty.

1.5.2. *Step 2.* The remainder of the proof of Theorem A follows an idea by Spatzier and Vinhage [58]: instead of producing the conjugacy directly, we produce a homogeneous structure on X_Γ that is compatible with α. The homogeneous structure on X_Γ is obtained as the action of a certain quotient of the *su-path group* (see Section 4). We use the map Φ to define the *su*-path group, $\mathcal{P}$, and a natural action of $\mathcal{P}$ on X_Γ. The most technical part of the paper is the following theorem from Section 6.

Theorem 1.3. *There is a unique $\alpha-$invariant measure μ such that $\Phi_*\mu = \mathrm{vol}$, moreover $\lambda_\mu^c = 0$.*

Using Theorem 1.3 we show that there is a normal subgroup, $\mathcal{N}$, of $\mathcal{P}$ such that $N = \mathcal{P}/\mathcal{N}$ is a Nilpotent Lie group that act transitively and freely on G. Moreover, the action of N is constructed such that α is compatible with the $N-$action in the sense that the joint action of $N \rtimes \mathbb{Z}^k$ is through a semi-direct product. We then use the $N-$action to produce coordinates on X_Γ, in which f is affine. These coordinates gives a bi-Hölder conjugacy H from α to some affine action α_0.

1.5.3. *Step 3.* We finish the proof of Theorem A by proving Theorem 1.2, improving the regularity of H from bi-Hölder to C^∞. The proof is similar to the proof in [52]. We begin the proof by using results from Wilkinson [67] to show that the conjugacy H is smooth along the center W^c. The proof of Theorem 1.2 then follows [52] to show that the component of the conjugacy along some coarse exponent $[\chi]$ defining a chamber wall for the chamber that contains $\mathbf{n}_0$ (see Section 2.2) is smooth. That is, we show that the $[\chi]-$component of the bi-Hölder conjugacy H restricted to $W^s(x)$ and normalized by $x \mapsto e$, denoted $H_x^{[\chi]} : W^s(x) \to G^{[\chi]}$ where $G^{[\chi]}$ is the coarse group with Lie algebra $E_0^{[\chi]}$, is uniformly C^∞. Once we know that $H_x^{[\chi]}$ is smooth, we study the map

$$(1.8) \qquad P : \mathrm{Gr}^\ell(E^s) \to \mathbb{R}, \quad P(x, V) = \det\left(D_x H_x^{[\chi]}|_V\right)$$

where $\ell = \dim(E_0^{[\chi]})$ and the determinant is calculated with respect to some background Riemannian metric. The main observation is that $P(x, V)$ can not

vanish for all $V \in \mathrm{Gr}_x^\ell(E^s)$ for any $x \in X_\Gamma$ (see Lemma 8.7). This shows that $H_x^{[\chi]}$ is a submersion for every x, so its fibers form a C^∞−foliation within W^s, denoted W^{ss}. Finally, we construct a α−invariant distribution $E^{[\chi]}$ transverse to the distribution $E^{ss} = TW^{ss}$ by using a graph transform argument. Existence of the distributions $E^{[\chi]}$ and E^{ss} allows us to produce new partially hyperbolic elements of the action α in a Weyl chamber adjacent to the Weyl chamber containing the first partially hyperbolic element, $\mathbf{n}_0$. By induction we produce a partially hyperbolic element in every Weyl chamber. Using that α contains many partially hyperbolic elements, $H_x^{[\chi]}$ is uniformly C^∞ for every coarse exponent $[\chi]$. It follows that H is uniformly C^∞ along W^s and W^u. Since H is uniformly C^∞ along W^s, W^u and W^c we can apply Journé's lemma twice to show that H is C^∞.

Theorem B follows from Theorem A and results from [21].

1.6. **Outline of paper.** In Section 2 we go through some of the background results, and basic definitions from partially hyperbolic dynamics and higher rank actions on nilmanifolds. In Section 3 we prove Theorem 1.1. In Section 4 we introduce the su−path group, one of the main objects in this paper, and show some of its basic properties. In Section 5 we recall the suspension construction of an abelian action and use it, combined with results from [3], to derive an invariance principle for higher rank actions on nilmanifolds. Section 6 is the most technical part of the paper, here we prove Theorem 1.3. In Section 7 we prove that the action α in Theorem A is topologically conjugated to some affine action. In Section 8 we prove Theorem 1.2, showing that the topological conjugacy is C^∞. Finally, in Section 9 we complete the proofs of Theorems A and B. We also include an appendix, Appendix A, proving some basic properties of higher rank, abelian algebraic actions on nilmanifolds.

2. Background and definitions

2.1. **Partially hyperbolic diffeomorphisms.** Let M be a smooth closed manifold and $f \in \mathrm{Diff}^\infty(M)$ a diffeomorphism. We fix a smooth metric g on M inducing a norm $\|\cdot\|$. We say that f is (absolutely) *partially hyperbolic* if there is a continuous Df−invariant splitting

$$(2.1) \qquad\qquad T_x M = E^s(x) \oplus E^c(x) \oplus E^u(x)$$

and constants $\nu, \gamma, \widehat{\gamma}, \widehat{\nu} \in (0,1)$, $n_0 \in \mathbb{N}$ such that for $n \geq n_0$

$$(2.2) \qquad \|Df^n|_{E^s}\| \leq \nu^n < \gamma^n \leq \left\|(Df^n)^{-1}|_{E^c}\right\|^{-1},$$

$$(2.3) \qquad \|Df^n|_{E^c}\| \leq \widehat{\gamma}^{-n} < \widehat{\nu}^{-n} \leq \left\|(Df^n|_{E^u})^{-1}\right\|^{-1}.$$

If we can choose the constants such that

$$(2.4) \qquad \nu < \gamma\widehat{\gamma}^r, \quad \widehat{\nu} < \gamma^r\widehat{\gamma}$$

then we say that f is $r-$*bunching*. The distributions E^s, E^c and E^u are the *stable, center* and *unstable* distributions respectively.

Let $f \in \mathrm{Diff}^\infty(M)$ be partially hyperbolic. The stable and unstable distributions are always uniquely integrable to foliations W^s and W^u with uniformly C^∞ leaves, but the center distribution may fail to be integrable. A sufficient condition for E^c being integrable is *dynamical coherence*. We say that f is dynamically coherent if $E^{cs} = E^c \oplus E^s$ and $E^{cu} = E^c \oplus E^u$ are both integrable to foliations W^{cs} and W^{cu}. In this case we obtain a foliation tangent to E^c by intersecting $W^c = W^{cs} \cap W^{cu}$. We will denote the distance between two points $p, q \in W^\sigma(x)$, $\sigma = s, c, u, cs, cu$, in the leaf metric by $\mathrm{d}_\sigma(p,q)$. Denote the ball about x of radius ε in d_σ by $W_\varepsilon^\sigma(x)$. If f is $r-$bunching and dynamically coherent then W^{cs}, W^{cu} and W^c have uniformly C^r leaves [35] (or [20, Theorem 7]).

Let $\hat{M}$ be the universal cover of M. We lift $f : M \to M$ to a diffeomorphism $F : \hat{M} \to \hat{M}$. Any foliation $\mathcal{F}$ on M naturally lifts to a foliation $\hat{\mathcal{F}}$ of $\hat{M}$. In particular, if $f : M \to M$ is partially hyperbolic (and dynamically coherent) then we can lift all foliations W^σ, $\sigma = s, c, u, cs, cu$, to foliations $\hat{W}^\sigma$ on $\hat{M}$. Similarly one lifts the distributions E^σ on M to distributions on $\hat{M}$, also denoted E^σ.

Definition 2.1. *We say that a continuous foliation with C^1-leaves $\mathcal{F}$ of M have quasi isometric leaves in the universal cover if there is a constant $Q \geq 1$ such that*

$$(2.5) \qquad \mathrm{d}(x,y) \leq \mathrm{d}_\mathcal{F}(x,y) \leq Q\mathrm{d}(x,y), \quad x, y \in \hat{\mathcal{F}}(p)$$

where $\mathrm{d}_\mathcal{F}$ is the metric along $\mathcal{F}$.

Remark 9. The inequality $\mathrm{d}(x,y) \leq \mathrm{d}_\mathcal{F}(x,y)$ is immediate since any path connecting x and y along $\hat{\mathcal{F}}(p)$ also connect x and y in $\hat{M}$.

Remark 10. We could have asked $\mathrm{d}_\mathcal{F}(x,y) \leq A\mathrm{d}(x,y) + B$ in the definition, but this is equivalent to Definition 2.1 since $\mathrm{d}_\mathcal{F}$ and d are comparable in small balls in $\mathcal{F}$ if $\mathcal{F}$ have uniformly C^1-leaves.

Assume now that $f : M \to M$ is dynamically coherent. Since E^s is uniformly transverse to E^{cu}, W^s and W^{cu} have a local product structure. Similarly, W^c and W^u subfoliate W^{cu} and E^c is transverse to E^u, so the foliations W^c and W^u have a local product structure in W^{cu}.

Definition 2.2. *We say that f have global product structure* [31] *if*

(2.6) $$\#\hat{W}^{cs}(x) \cap \hat{W}^u(y) = 1, \quad x, y \in \hat{M},$$

(2.7) $$\#\hat{W}^s(x) \cap \hat{W}^{cu}(y) = 1, \quad x, y \in \hat{M},$$

(2.8) $$\#\hat{W}^s(x) \cap \hat{W}^c(y) = 1, \quad x, y \in \hat{W}^{cs}(p),$$

(2.9) $$\#\hat{W}^c(x) \cap \hat{W}^u(y) = 1, \quad x, y \in \hat{W}^{cu}(p).$$

When f has global product structure we define global holonomy maps in the universal cover $\hat{M}$. Given $x \in \hat{M}$ and $y \in \hat{W}^u(x)$ we define

(2.10) $$\pi^u_{x,y} : \hat{W}^{cs}(x) \to \hat{W}^{cs}(y), \quad \{\pi^u_{x,y}(z)\} = \hat{W}^u(z) \cap \hat{W}^{cs}(y).$$

Since $\hat{W}^u$ and $\hat{W}^c$ subfoliate $\hat{W}^{cu}$ the holonomy maps $\pi^u_{x,y}$ restricts to maps
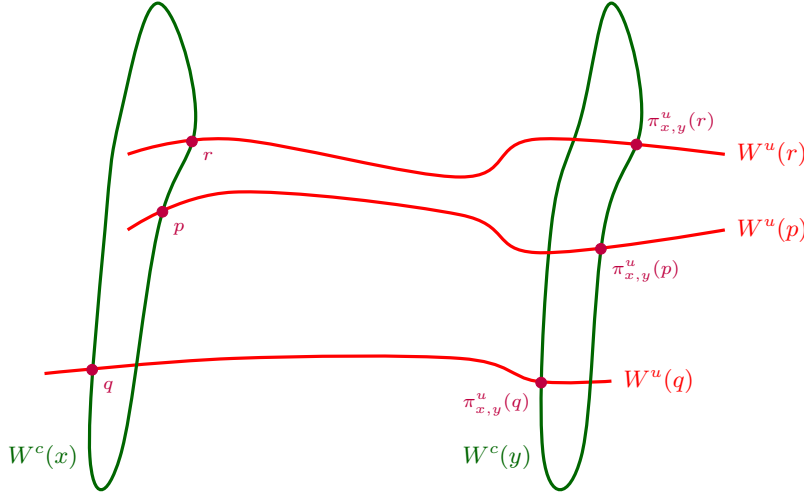

FIGURE 1. Unstable holonomy between $W^c(x)$ and $W^c(y)$, $y \in W^u(x)$.

$\pi^u_{x,y} : \hat{W}^c(x) \to \hat{W}^c(y)$. The holonomy maps $\pi^u_{x,y}$ descend to holonomy maps between center leaves on M, Figure 1. Similarly we define stable holonomies $\pi^s_{x,y} : W^c(x) \to W^c(y)$ when $y \in W^s(x)$. When we consider holonomies between center leaves then the holonomy maps are $C^{1+\alpha}$ [9]. If f is r−bunching then the holonomies $\pi^\sigma_{x,y} : W^c(x) \to W^c(y)$, $\sigma = s, u$, are C^r [47].

We say that a path $\gamma : [0,1] \to M$ is an su−path if $[0,1]$ has a subdivision $0 = t_0 < t_1 < ... < t_{N-1} < t_N = 1$ such that $\mathrm{Im}\left(\gamma|_{[t_j,t_{j+1}]}\right)$ is entirely contained in either an W^s−leaf or a W^u−leaf. If any two points $x, y \in M$ are connected by an su−path, then we say that f is *accessible*. A set $E \subset M$ is σ−saturated, $\sigma = s, u$, if $x \in E$ implies $W^\sigma(x) \subset E$, and su−saturated if it is s and u−saturated. Equivalently f is accessible if the only su−saturated sets are M and $\emptyset$. For

$x \in M$ (or $\hat{M}$) we define the *accessibility class* of x

(2.11) $\mathrm{AC}(x) = \{y \in M : \text{there is an } su-\text{path connecting } x \text{ and } y\}.$

We define a closed set $\Lambda(f)$ by $x \in \Lambda(f)$ if $\mathrm{AC}(x)$ is not open. That is

$$(2.12) \qquad \Lambda(f) = \left[\bigcup_{\mathrm{AC}(x) \text{ is open}} \mathrm{AC}(x) \right]^c = \bigcap_{\mathrm{AC}(x) \text{ is open}} \mathrm{AC}(x)^c$$

If f has $1-$dimensional center direction, then $\Lambda(f)$ is laminated by accessibility classes [51, Proposition A.3]. In particular, if f has $1-$dimensional center and $\Lambda(f) = M$ then $E^s \oplus E^u$ is jointly integrable to some continuous foliation W^{su} with smooth leaves (in fact, the foliation W^{su} will be a C^r-foliation if f is $r-$bunching). In the other extreme, f is accessible if and only if $\Lambda(f) = \emptyset$ and in this case f has a unique accessibility class.

2.2. **Nilmanifolds and higher rank actions.** Let G be a (simply connected) Lie group with Lie algebra $\mathfrak{g}$. We define the lower central series of $\mathfrak{g}$ inductively as

$$(2.13) \qquad\qquad \mathfrak{g}^{(0)} = \mathfrak{g}, \quad \mathfrak{g}^{(j+1)} = [\mathfrak{g}^{(j)}, \mathfrak{g}].$$

If there is ℓ such that $\mathfrak{g}^{(\ell)} = 0$ then we say that $\mathfrak{g}$ is nilpotent, and the minimal ℓ satisfying $\mathfrak{g}^{(\ell)} = 0$ is the step of $\mathfrak{g}$. We say that G is a $\ell-$step nilpotent Lie group if $\mathfrak{g}$ is $\ell-$step nilpotent. Given a discrete subgroup $\Gamma \leq G$, we say that Γ is a lattice if the quotient space $\Gamma \backslash G$ carries a Haar measure μ_Γ (which it always does for nilpotent groups, [11, Theorem 1.2.10]), and with respect to this measure $\mu_\Gamma(\Gamma \backslash G) < \infty$. Equivalently, for nilpotent groups [11, Corollary 5.4.6], a discrete subgroup $\Gamma \leq G$ is a lattice if the quotient $\Gamma \backslash G$ is compact.

If G is simply connected, nilpotent and $\Gamma \leq G$ is a lattice, then we define the associated *compact nilmanifold* by

$$(2.14) \qquad\qquad X_\Gamma = \Gamma \backslash G.$$

Denote by μ_Γ the normalized Haar measure on X_Γ. We write

$$(2.15) \qquad\qquad p_\Gamma : G \to X_\Gamma$$

for the natural projection map. If G is $\ell-$step then we obtain a natural sequence of (normal) subgroups

$$(2.16) \qquad G = G^{(0)} \trianglerighteq G^{(1)} \trianglerighteq ... \trianglerighteq G^{(\ell-1)} \trianglerighteq G^{(\ell)} = e, \quad G^{(j)} = \exp\left(\mathfrak{g}^{(j)}\right).$$

The intersection $\Gamma^{(j)} = G^{(j)} \cap \Gamma$ defines a lattice in $G^{(j)}$, see [11, Theorem 5.2.3]. For each $0 \leq j \leq \ell$, we define

$$(2.17) \qquad\qquad X_\Gamma^{(j)} = X_\Gamma / G^{(j)}, \quad \pi^{(j)} : X_\Gamma \to X_\Gamma^{(j)}$$

where $X_\Gamma^{(j)}$ is a compact nilmanifold and $\pi^{(j)} : X_\Gamma \to X_\Gamma^{(j)}$ is a fiber bundle. In particular, if G is 2$-$step then we get one (non-trivial) projection map $\pi^{(1)} : X_\Gamma \to X_\Gamma^{(1)}$. Since $G^{(1)} = [G, G]$, $G/G^{(1)} = G/[G, G]$ is abelian. So $X_\Gamma^{(1)}$ is a torus. In the case of 2$-$step nilpotent nilmanifolds, we will write

$$(2.18) \qquad\qquad\qquad \pi : X_\Gamma \to \mathbb{T}^d$$

for the projection. The fibers of π are $G^{(1)}/\Gamma^{(1)}$ which, since $G^{(2)} = e$, is also a torus. So $\pi : X_\Gamma \to \mathbb{T}^d$ is a fiber bundle with base and fibers both tori, but X_Γ is not a torus (if G is 2$-$step).

We define the automorphism and affine group of X_Γ by

$$(2.19) \qquad \mathrm{Aut}(X_\Gamma) = \{L \in \mathrm{Aut}(G) : L\Gamma = \Gamma\},$$

$$(2.20) \qquad \mathrm{Aff}(X_\Gamma) = \{f_0(x) = L(x)g^{-1} : L \in \mathrm{Aut}(X_\Gamma), \ g \in G\}.$$

There is a natural map $\mathrm{Aff}(X_\Gamma) \to \mathrm{Aut}(X_\Gamma)$ defined by mapping $f_0(x) = L(x)g^{-1}$ to the automorphism L. Each projection $\pi^{(j)}$ induce a map $\mathrm{Aut}(X_\Gamma) \to \mathrm{Aut}(X_\Gamma^{(j)})$.

Fix a homomorphism

$$(2.21) \qquad\qquad \rho : \mathbb{Z}^k \to \mathrm{Aut}(X_\Gamma), \quad \rho^{\mathbf{n}} : X_\Gamma \to X_\Gamma, \ \mathbf{n} \in \mathbb{Z}^k.$$

We say that ρ has a rank-1 factor if there is a nilpotent group $\hat{G}$, of positive dimension less than or equal to the dimenison of G, a homomorphism $\phi : G \to \hat{G}$ such that $\hat{\Gamma} = \phi\Gamma$ is a lattice in $\hat{G}$ and an automorphism $L \in \mathrm{Aut}(\hat{X}_{\hat{\Gamma}})$ such that for some finite index subgroup $\Lambda \subset \mathbb{Z}^k$ we have some $n : \Lambda \to \mathbb{Z}$ satisfying $\phi\rho^{\mathbf{n}} = L^{n(\mathbf{n})}\phi$, $\mathbf{n} \in \Lambda$. That is, ρ has a rank-1 factor if there is a factor of X_Γ where the projected action of $\mathbb{Z}^k$ is a $\mathbb{Z}^1-$action (up to finite index). More generally, if $\alpha_0 : \mathbb{Z}^k \to \mathrm{Aff}(X_\Gamma)$ is a homomorphism then we say that α_0 has a rank-1 factor if the induced map

$$(2.22) \qquad\qquad\qquad \mathbb{Z}^k \xrightarrow{\alpha_0} \mathrm{Aff}(X_\Gamma) \to \mathrm{Aut}(X_\Gamma)$$

has a rank-1 factor.

Definition 2.3. *A homomorphism $\alpha_0 : \mathbb{Z}^k \to \mathrm{Aff}(X_\Gamma)$ is higher rank if it has no rank-1 factor.*

Let $\alpha_0 : \mathbb{Z}^k \to \mathrm{Aff}(X_\Gamma)$ be a homomorphism. We say that $\chi : \mathbb{Z}^k \to \mathbb{R}$ is a *Lyapunov exponent* of α_0 if there is $v \in \mathfrak{g} \setminus 0$ such that

$$(2.23) \qquad\qquad\qquad \chi(\mathbf{n}) = \lim_{\ell \to \pm\infty} \frac{1}{\ell} \log \left\| \alpha_0^{\ell\mathbf{n}}(v) \right\|.$$

The Lyapunov exponents χ are linear and extends uniquely to $\mathbb{R}^k$, we will consider Lyapunov exponents as linear maps on $\mathbb{R}^k$. The *Lyapunov space* associated to χ,

$E_0^\chi \leq \mathfrak{g}$, is the subspace where Equation 2.23 hold. For the $0-$functional we write E_0^c. Note that

$$(2.24) \qquad \mathfrak{g} = E_0^c \oplus \bigoplus_\chi E_0^\chi.$$

Denote the set of Lyapunov exponents for α_0 by

$$(2.25) \qquad \mathrm{Lyap}(\alpha_0) = \{\chi \neq 0 : \chi \text{ is a Lyapunov exponent of } \alpha_0\}.$$

For $\chi \in \mathrm{Lyap}(\alpha_0)$, define the associated *coarse exponent* and *coarse space* by

$$(2.26) \qquad [\chi] = \{\chi' \in \mathrm{Lyap}(\alpha_0) : \chi' = c\chi, \text{ for some } c > 0\}, \quad E_0^{[\chi]} = \bigoplus_{\chi' \in [\chi]} E_0^{\chi'}.$$

If $\chi(\mathbf{n}) > 0$ (or $\chi(\mathbf{n}) < 0$) then $\chi'(\mathbf{n}) > 0$ (or $\chi(\mathbf{n}) < 0$) for every $\chi' \in [\chi]$, we define $[\chi](\mathbf{n})$ as the sign of $\chi(\mathbf{n})$ (or as 0 if $\chi(\mathbf{n}) = 0$). We also define $\ker[\chi] = \ker \chi$.

Definition 2.4. *Let $\{[\chi_1], ..., [\chi_N]\}$ be the coarse exponents of α_0 and*

$$(2.27) \qquad U = \left(\bigcup_{j=1}^N \ker[\chi_j] \right)^c.$$

Each connected component $\mathcal{C}$ of U is a Weyl chamber of α_0. The kernels $\ker[\chi]$ are Weyl chamber walls. A wall $\ker[\chi]$ is a wall of $\mathcal{C}$ if $\dim \overline{\mathcal{C}} \cap \ker[\chi] = k - 1$.

Two coarse exponents, $[\chi]$ and $[\eta]$, are *dependent* if $[\chi](\mathbf{n}) = -[\eta](\mathbf{n})$, otherwise the two exponents are *independent*. Given any two $\chi', \chi'' \in [\chi]$ it is immediate

$$(2.28) \qquad \left[E_0^{\chi'}, E_0^{\chi''} \right] \subset E_0^{\chi'+\chi''} \quad \left(\text{with } E_0^{\chi'+\chi''} = 0 \text{ if } \chi' + \chi'' \notin \mathrm{Lyap}(\alpha_0) \right)$$

so $E_0^{[\chi]}$ is a subalgebra of $\mathfrak{g}$. We define the associated group

$$(2.29) \qquad G^{[\chi]} \leq G, \quad G^{[\chi]} = \exp\left(E_0^{[\chi]} \right).$$

A useful fact, that we will use in Section 8, is that within stable group, G^s, any coarse group $G^{[\chi]}$ have a transverse group that is normal. More precisely, if $\alpha_0^{\mathbf{n}}$, $\mathbf{n} \in \mathbb{Z}^k$, has stable space E_0^s, $[\chi](\mathbf{n}) < 0$, and

$$(2.30) \qquad E_0^{ss} := \bigoplus_{\substack{[\eta] \neq [\chi] \\ [\eta](\mathbf{n}) < 0}} E_0^{[\eta]}$$

then $E_0^{ss} \leq E_0^s$ is an ideal in E_0^s, [52, Lemma 3.1]. Equivalently, the subgroup $G^{ss} = \exp(E_0^{ss}) \leq G^s = \exp(E_0^s)$ is a normal subgroup.

The following two lemmas are well-known, we include proofs in Appendix A.

Lemma 2.1. *If $\alpha_0 : \mathbb{Z}^k \to \mathrm{Aff}(X_\Gamma)$ is higher rank then there are at least two independent coarse exponents.*

Lemma 2.2. *If $\alpha_0 : \mathbb{Z}^k \to \mathrm{Aff}(X_\Gamma)$ is higher rank and $[\chi]$ is a coarse Lyapunov exponent then the space*

$$V = \bigoplus_{[\eta] \neq \pm[\chi]} E_0^{[\chi]}$$

defines a minimal translation action on X_Γ (the translation action by V is the translation action of the exponential of the Lie algebra generated by V).

Given a homomorphism $\alpha : \mathbb{Z}^k \to \mathrm{Diff}^\infty(X_\Gamma)$, written $\alpha(\mathbf{n}) = \alpha^\mathbf{n}$, we obtain a *linearization* $\rho : \mathbb{Z}^k \to \mathrm{Aut}(\pi_1 X_\Gamma) \cong \mathrm{Aut}(X_\Gamma)$.

Definition 2.5. *A smooth action $\alpha : \mathbb{Z}^k \to \mathrm{Diff}^\infty(X_\Gamma)$ is higher rank if the linearization $\rho : \mathbb{Z}^k \to \mathrm{Aut}(X_\Gamma)$ is higher rank.*

Fix $n \geq 1$, $d = 2n$ and define $H^n := \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$. With $g = (q, p, z) \in G$ and $h = (q', p', z') \in G$ we define a multiplication

(2.31) $\qquad gh = (q, p, z)(q', p', z') = (q + q', p + p', z + z' + q \cdot p')$.

This makes H^n into a group, the $(d+1)-$*dimensional Heisenberg group.* Denote by ω the symplectic form on $\mathbb{R}^d = \mathbb{R}^n \oplus \mathbb{R}^n$. The Lie bracket on $\mathfrak{g}$ is

(2.32) $\qquad [(X, Z), (X', Z')] = (0, \omega(X, X')), \quad X, X' \in \mathbb{R}^d \ Z, Z' \in \mathbb{R}$.

Let $\Gamma \subset H^n$ with nilmanifold $X_\Gamma = \Gamma \setminus H^n$. For $L \in \mathrm{Aut}(X_\Gamma)$ we obtain a map $L_{su} \in \mathrm{GL}(d, \mathbb{Z})$ by projecting onto the base, one checks that this element L_{su} satisfy $L_{su} \in \mathrm{Sp}(d, \mathbb{Z})$. In particular, if $[\chi]$ is a coarse exponent of $\alpha_0 : \mathbb{Z}^k \to \mathrm{Aff}(X_\Gamma)$ then $-[\chi]$ is also a coarse exponent. That is, the coarse exponents come in negatively proportional pairs.

In the remainder, we will be interested in groups G of the form $G = \mathbb{R}^\ell \times H^n$ for some $\ell \geq 0$ and $n \geq 0$. These groups constitute all abelian simply connected nilpotent groups and all $2-$step, simply connected nilpotent Lie groups with $\dim[G, G] = 1$. Recall the *Baker–Campbell–Hausdorff formula* [11]

(2.33) $\qquad\qquad e^X e^Y = e^{X+Y+[X,Y]/2}, \quad X, Y \in \mathfrak{g}$.

Fix a left invariant metric, d, on G. Using the Baker–Campbell–Hausdorff formula, it is immediate that for $e^Z = g_c \in [G, G]$, $\mathrm{d}(e, g_c) \leq 4\sqrt{\|Z\|}$.

3. Some properties of partially hyperbolic diffeomorphisms with quasi isometric leaves in the universal cover

In this section, we prove Theorem 1.1. We begin by proving that Φ from Theorem 1.1 exists in Section 3.1. In section 3.2 we show that f is accessible.

Let $G = H^n \times \mathbb{R}^\ell$ be the product of some Heisenberg group and some abelian group, $\Gamma \leq G$ a lattice and X_Γ the associated compact nilmanifold. If $n > 0$ then we write $d = 2n + \ell$ and let $\pi : G \to \mathbb{R}^d$ be the base projection. If $n = 0$ and we have an automorphism $L \in \mathrm{Aut}(X_\Gamma)$ with $1-$dimensional center, then we let $d = \ell - 1$ and $\pi : G \to \mathbb{R}^d$ be the projection from G onto G/E_0^c (note that if L has $1-$dimensional center then the center direction E_0^c is a rational line). We assume in the remainder of this section that $f \in \mathrm{Diff}^\infty(X_\Gamma)$ satisfy all the assumptions of Theorem 1.1. Denote by $L \in \mathrm{Aut}(X_\Gamma)$ the linearization of f and $L_{su} \in \mathrm{Aut}(\mathbb{T}^d)$ the induced map on the base.

3.1. **Existence of Franks-Manning coordinates.** We write $f : X_\Gamma \to X_\Gamma$

$$(3.1) \qquad\qquad\qquad fx = L(x)e^{-v(x)}$$

with $v : X_\Gamma \to \mathfrak{g}$. Fix a lift $F : G \to G$, $Fx = L(x)e^{-v(x)}$. For $x \in G$, let

$$(3.2) \qquad\qquad\qquad F^n x =: x_n, \quad n \in \mathbb{Z}.$$

The splitting $\mathfrak{g} = E_0^s \oplus E_0^c \oplus E_0^u$, with respect to L, decompose v, v_σ, $\sigma = s, c, u, cs, cu$. Denote by $\tilde{\pi} : G \to G/G^c \cong \mathbb{R}^{d'}$ the projection, where $G^c = \exp(E_0^c)$ is the center of L (we do not know, a priori, that E_0^c has dimension 1). Write $A : \mathbb{R}^{d'} \to \mathbb{R}^{d'}$ for the map induced by L, then A is hyperbolic (if $\dim E_0^c \neq 1$ then $A \neq L_{su}$). Recall the following well-known lemma.

Lemma 3.1. *There exists a unique Hölder map $\Phi : G \to G/G^c \cong \mathbb{R}^{d'}$*

$$(3.3) \qquad\qquad \Phi(x) = \tilde{\pi}(x) + \varphi(x), \quad \varphi(\gamma x) = \varphi(x), \ \gamma \in \Gamma$$

such that $\Phi(Fx) = A\Phi(x)$. If $\dim(E_0^c) = 1$ then $d' = d$, $\Phi : G \to \mathbb{R}^d$ descends to a map $\Phi : X_\Gamma \to \mathbb{T}^d$ homotopic to π and $A = L_{su}$.

Proof. The lemma follows from a calculation showing that φ satisfy $v_{su}(x) = \varphi(fx) - A(\varphi(x))$, which has a unique solution since A is hyperbolic. If $\dim(E_0^c) = 1$ then $E_0^c = [\mathfrak{g}, \mathfrak{g}]$ if G is non-abelian (since $[\mathfrak{g}, \mathfrak{g}]$ lie in the center of any automorphism) so G/G^c is the natural quotient by $[G, G]$. The lemma follows since φ is $\Gamma-$invariant. If G is abelian, then E_0^c is some $1-$dimensional rational line (since we assume that L_{su} have at least one rational eigenvalue) and the last conclusion follows. $\blacksquare$

Lemma 3.2. *If $y \in \hat{W}^\sigma(x)$, $\sigma = s, u$, then $\Phi(y) = \Phi(x)$ if and only if $x = y$. That is $\Phi : \hat{W}^\sigma(x) \to \Phi(x) + E_0^\sigma$ is injective. Moreover, $\dim E^\sigma = \dim E_0^\sigma$ for s, c, u, so Φ descends and $A = L_{su}$ is hyperbolic.*

Remark 11. We prove the lemma when G is non-abelian. The proof when G is abelian simplifies since all terms from brackets vanish.

Proof. Assume that $\sigma = u$, for the other case we reverse time. Let $y \in \hat{W}^u(x)$. Write $y_n = x_n e^{\gamma^n}$ where $\gamma^n \in \mathfrak{g}$. With respect to the decomposition $\mathfrak{g} = E_0^s \oplus E_0^c \oplus E_0^u$, decompose $\gamma^n = \gamma_s^n + \gamma_c^n + \gamma_u^n$. If $\Phi(x) = \Phi(y)$ then $\Phi(x_n) = \Phi(y_n)$ for all $n \geq 0$, so $\|\pi(x_n) - \pi(y_n)\| \leq 2\|\varphi\|_{C^0} =: C$ independently of n. On the other hand, $\|\pi(x_n) - \pi(y_n)\|^2 = \|\gamma_s^n\|^2 + \|\gamma_u^n\|^2$. So to show that Φ is injective on $\hat{W}^u(x)$, it suffices to show that $\gamma_u^n \to \infty$ as $n \to \infty$ if $x \neq y$.

Suppose for contradiction that $x \neq y$ and $\|\gamma_u^n\| \leq K$ uniformly in n. From our definitions

$$
\begin{aligned}
x_{n+1}e^{\gamma^{n+1}} &= y_{n+1} = F(y_n) = L(y_n)e^{-v(y_n)} = L(x_n)e^{L(\gamma^n)}e^{-v(y_n)} = \\
&\quad L(x_n)e^{L(\gamma^n)-v(y_n)-[L(\gamma^n),v(y_n)]/2} = \\
&\quad F(x_n)e^{v(x_n)+L(\gamma^n)-v(y_n)-([L(\gamma^n),v(y_n)]-[v(x_n),L(\gamma^n)]+[v(x_n),v(y_n)])/2}
\end{aligned}
$$

or if we take logarithms

$$\gamma^{n+1} = v(x_n) + L(\gamma^n) - v(y_n) - \tag{3.4}$$

$$\frac{[L(\gamma^n),v(y_n)] - [v(x_n),L(\gamma^n)] + [v(x_n),v(y_n)]}{2}. \tag{3.5}$$

Using 3.5 we estimate

$$\|\gamma_s^{n+1}\| \leq \|L\gamma_s^n\| + C, \tag{3.6}$$

$$\|\gamma_c^{n+1}\| \leq \|\gamma_c^n\| + C + k\left(\|\gamma_s^n\| + \|\gamma_u^n\|\right), \tag{3.7}$$

with constants C, K, k that only depend on $\|v\|_{C^0}$. Since L is contracting on E_0^s there is $\tau \in (0,1)$ such that $\|\gamma_s^{n+1}\| \leq \tau\|\gamma_s^n\| + C$, or

$$\|\gamma_s^n\| \leq \frac{C}{1-\tau} \tag{3.8}$$

uniformly in n. We have $\|\gamma_u^n\| \leq K$ by assumption, so for some possibly larger C we obtain

$$\|\gamma_c^{n+1}\| \leq \|\gamma_c^n\| + C, \quad \|\gamma_c^n\| \leq Cn + \|\gamma_c^0\|. \tag{3.9}$$

After possibly enlarging C again, we have

$$d(x_n, y_n) = d(e, e^{\gamma^n}) \leq d(e, e^{\gamma_s^n+\gamma_u^n}) + d(e, e^{\gamma_c^n}) \leq C(\sqrt{n}+1). \tag{3.10}$$

On the other hand, the assumption that $\hat{W}^u(x_n)$ is quasi isometric implies that there is some $\lambda > 1$ and $Q \geq 1$ such that

$$d(x_n, y_n) \geq \frac{1}{Q}d_u(F^n x, F^n y) \geq \frac{1}{Q}\lambda^n d_u(x, y). \tag{3.11}$$

If $d_u(x, y) \neq 0$, then Equations 3.10 and 3.11 gives a contradiction for n sufficiently big, so $x = y$.

Since $\Phi : \hat{W}^\sigma(x) \to \Phi(x) + E_0^\sigma$, $\sigma = s, u$, is injective it follows by invariance of domain that $\dim(E^\sigma) \le \dim(E_0^\sigma)$. On the other hand, we have

$$(3.12) \qquad \dim(E^s) + \dim(E^u) + 1 = d + 1 = \dim(E_0^s) + \dim(E_0^c) + \dim(E_0^u)$$

or $1 \ge \dim(E_0^c)$, so $\dim(E_0^c) = 1$. This implies $\dim(E^s) + \dim(E^u) = \dim(E_0^u) + \dim(E_0^s)$, which only hold if $\dim(E^\sigma) = \dim(E_0^\sigma)$, $\sigma = s, u$, since $\dim(E^\sigma) \le \dim(E_0^\sigma)$. That $A = L_{su}$ is hyperbolic follows by Lemma 3.1. ∎

Remark 12. We will make no notational distinction between the map $\Phi : G \to \mathbb{R}^d$ and the induced map $\Phi : X_\Gamma \to \mathbb{T}^d$.

Lemma 3.3. *For every $x \in G$ the map $\Phi_{\sigma,x} : \hat{W}^\sigma(x) \to E_0^\sigma$, defined by*

$$(3.13) \qquad\qquad \Phi_{\sigma,x}(y) = \Phi(y) - \Phi(x),$$

is a homeomorphism. For any $y \in \hat{W}^\sigma(x)$, the map $\Phi_{\sigma,x} : \hat{W}_R^\sigma(y) \to E_0^\sigma$ is uniformly bi-Hölder for fixed R.

Proof. By Lemma 3.2 $\Phi : \hat{W}^\sigma(x) \to \Phi(x) + E_0^\sigma$ is injective so $\Phi_{\sigma,x}$ is injective. Since $\hat{W}^\sigma(x)$ and E_0^σ have the same dimension, it follows by invariance of domain that $\Phi_{\sigma,x}$ has an open image and is homeomorphic onto its image. In particular, the image of $\Phi_{\sigma,x}$ contain some ball $B_{r_x}^\sigma(0)$ around 0 in E_0^σ (to make r_x well-defined we take the maximal possible r_x). Given $\gamma \in \Gamma$ we have

$$(3.14) \qquad \Phi_{\sigma,\gamma x}(\gamma y) = \Phi(\gamma y) - \Phi(\gamma x) = \Phi(y) + \pi(\gamma) - \Phi(x) - \pi(\gamma) = \Phi_{\sigma,x}(y)$$

from which it follows that $x \mapsto r_x$ is $\Gamma-$invariant. Moreover, $\Phi_{\sigma,x}$ and $\hat{W}^\sigma(x)$ vary continuously in x, so $r_x > r_0$ is open. Combined with $\Gamma-$invariance and the fact that X_Γ is compact, we find $r_0 > 0$ such that $r_x \ge r_0$ for all $x \in G$. Assume now $\sigma = u$, the other case follows by reversing time. We have

$$\Phi_{u,x}(\hat{W}^u(x)) = \Phi_{u,x}(F^n \hat{W}^u(x_{-n})) = L^n \Phi_{u,x_{-n}}(\hat{W}^u(x_{-n})) \supset L^n B_{r_0}^u(0)$$

and letting $n \to \infty$, using that L expand E_0^u, we obtain $\Phi_{u,x}(\hat{W}^u(x)) = E_0^u$.

So $\Phi_{u,x}$ is a homeomorphism. Since Φ is Hölder, $(x, y) \mapsto \Phi_{u,x}(y)$ is Hölder in x and y. The set $\Phi_{u,x}^{-1}\left(\overline{B_1^u(0)}\right)$ is compact in $\hat{W}^u(x)$, so we define K_x as the minimal radius such that the closure of $\hat{W}_{K_x}^u(x)$ contain $\Phi_{u,x}^{-1}\left(\overline{B_1^u(0)}\right)$. Since $x \mapsto \Phi_{u,x}$ vary continuously the map $x \mapsto K_x$ also vary continuosly in x. If $K = \sup_x K_x$, then K is such that if $z, w \in \hat{W}^u(x)$ satisfy $\mathrm{d}_u(z, w) \ge K$ then $\|\Phi(z) - \Phi(w)\| \ge 1$. Let $\mu = \left\|L|_{E_0^u}\right\| > 1$ and $\lambda > 1$ such that $\mathrm{d}_u(F^n w, F^n z) \ge c\lambda^n \mathrm{d}_u(z, w)$. Let $z, w \in \hat{W}^u(x)$ satisfy $\mathrm{d}_u(z, w) \le 1$. There is $n = n(z, w) \ge 0$ such that $n(z, w) \le$

$-\kappa \log \mathrm{d}_u(z, w) + C$, $\mathrm{d}_u(z_n, w_n) \geq K$ and κ only depends on λ. We have

$$\|\Phi_{u,x}(z) - \Phi_{u,x}(w)\| = \|\Phi(z) - \Phi(w)\| = \left\|L^{-n}\left(\Phi(z_n) - \Phi(w_n)\right)\right\| \geq$$
$$\mu^{-n}\|\Phi(z_n) - \Phi(w_n)\| \geq \mu^{-n} \geq \mu^{\kappa \log(\mathrm{d}_u(z,w)) - C} =$$
$$\mu^{-C}\mathrm{d}_u(z, w)^{\kappa \log(\mu)}.$$

Fix $r > 0$ such that $\Phi_{u,x}^{-1}(B_r^u(0)) \subset \hat{W}_1^u(x)$ for all x. For $v_1, v_2 \in B_r^u(0)$ we obtain

(3.15) $$\mathrm{d}_u(\Phi_{u,x}^{-1}(v_1), \Phi_{u,x}^{-1}(v_2)) \leq \mu^{C/\kappa \log(\mu)} \|v_1 - v_2\|^{1/\kappa \log(\mu)}$$

so $\Phi_{u,x}^{-1} : B_r^u(0) \to \hat{W}^u(x)$ is uniformly Hölder. Given $y \in \hat{W}^u(x)$ and $v_0 \in E_0^u$

(3.16) $$\Phi_{u,y}(\Phi_{u,x}^{-1}(v + v_0)) = \Phi(\Phi_{u,x}^{-1}(v + v_0)) - \Phi(y) = v + v_0 + \Phi(x) - \Phi(y)$$

or if we choose $\Phi(y) - \Phi(x) = v_0$, then

(3.17) $$\Phi_{u,x}^{-1}(v + v_0) = \Phi_{u,y}^{-1}(v)$$

which shows that $\Phi_{u,x}^{-1}$ is uniformly Hölder on any ball of radius r. By covering any ball of radius R with balls of radius r, the lemma follows. ∎

Lemma 3.4. *If $\gamma : [0, 1] \to G$ is a C^1-curve that is tangent to E^{cs} (or E^{cu}) then $\Phi(\gamma(1)) \in \Phi(\gamma(0)) + E_0^s$ (or $\Phi(\gamma(1)) \in \Phi(\gamma(0)) + E_0^u$).*

Proof. Let γ be tangent to E^{cs}. Write $H_x = \Phi^{-1}(\Phi(x))$. For $y \in H_x$

(3.18) $$\|\pi(y - x)\| \leq \|\varphi(y) - \varphi(x)\| \leq 2\|\varphi\|_{C^0} = C$$

so $H_x \subset \pi^{-1}B_C(\Phi(x))$. In particular, for $y \in H_x$ let $y = xe^{\eta(y)}$, we estimate

(3.19) $$\|\eta_s(y)\|^2 + \|\eta_u(y)\|^2 \leq C^2.$$

Let $\eta^n : \hat{W}^u(x) \to \mathfrak{g}$ be defined by $y_n = x_n e^{\eta^n(y)}$. Since

$$F^n H_x = F^n \Phi^{-1}(\Phi(x)) = \Phi^{-1}(\Phi(F^n x)) = H_{x_n}$$

we have $\|\eta_s^n(y)\|, \|\eta_u^n(y)\| \leq C$ uniformly in $n \in \mathbb{Z}$. As in the proof of Lemma 3.2 it follows that we also have $\|\eta_c^n(y)\| \leq C|n|$ for some (possibly larger) constant C. Let $\gamma : [0, 1] \to G$ be a C^1-curve tangent to E^{cs}, we denote the end-points of γ by $x = \gamma(0)$ and $y = \gamma(1)$. We find $z \in \hat{W}^s(y)$ and $w \in \hat{W}^u(z)$ such that $w \in H_x$ (this corresponds to choosing a two-legged $su-$path from $\Phi(y)$ to $\Phi(x)$ in $\mathbb{R}^d$, which always exists). By the reverse triangle inequality

(3.20) $$\mathrm{d}(x_n, w_n) \geq \mathrm{d}(w_n, z_n) - \mathrm{d}(x_n, y_n) - \mathrm{d}(y_n, z_n).$$

Using that w and z lie in the same unstable leaf and that $\hat{W}^u$ have quasi-isometric leaves, $\mathrm{d}(w_n, z_n) \geq c\lambda^n \mathrm{d}(w, z)$ for some $c > 0$ and $\lambda > 1$. Since γ is a C^1-curve along E^{cs} from x to y we have

(3.21) $$\mathrm{d}(x_n, y_n) \leq \int_0^1 \left\|D_{\gamma(t)}F^n(\dot{\gamma}(t))\right\| \mathrm{d}t \leq C\hat{\lambda}^n \cdot |\gamma|$$

where $|\gamma|$ is the length of γ and $\hat{\lambda} < \lambda$. Finally, $\mathrm{d}(y_n, z_n) \leq C$ uniformly for $n \geq 0$ since y and z lie in the same stable leaf. Equation 3.20 implies

$$(3.22) \quad c\lambda^n \mathrm{d}(w, z) - C\hat{\lambda}^n |\gamma| - C \leq \mathrm{d}(x_n, w_n) = \mathrm{d}(e, e^{\eta^n(w)}) \leq C\sqrt{n}, \quad n \geq 0$$

for some constant C. If $\mathrm{d}(w, z) > 0$, we obtain a contradiction for n large enough. We conclude that $\mathrm{d}(w, z) = 0$. That is, $w = z$ so $\hat{W}^s(y) \cap H_x = \{w\} \neq \emptyset$, which implies

$$(3.23) \qquad\qquad \Phi(x) - \Phi(y) = \Phi(w) - \Phi(y) = \Phi_{s,y}(w) \in E_0^s$$

proving the lemma for sc. The lemma is shown for cu by reversing time.   ∎

We can now prove the first five points of Theorem 1.1

Proof of Theorem 1.1. Dynamical coherence of f follows from [7]. Any curve γ tangent to E^{cs} satisfies $\Phi(\gamma(1)) \in \Phi(\gamma(0)) + E_0^s$ by Lemma 3.4, so $\hat{W}^{cs}(x) \subset \Phi^{-1}(\Phi(x) + E_0^s)$ and $\Phi^{-1}(\Phi(x) + E_0^s)$ is a union of $\hat{W}^{cs}$−leaves. We claim that $\Phi^{-1}(\Phi(x) + E_0^s)$ is path-connected, which proves that $\Phi^{-1}(\Phi(x) + E_0^s) = \hat{W}^{cs}(x)$. Given any $y \in G$ there is a unique intersection $\hat{W}^u(y) \cap \Phi^{-1}(\Phi(x) + E_0^s) = \{w\}$ (since $\hat{W}^u(y)$ maps homeomorphically onto $\Phi(y) + E_0^u$ under Φ). Since $\Phi^{-1}(\Phi(x) + E_0^s)$ is tangent to E^{cs}, $\hat{W}^u$ is tangent to E^u the map $G \to \Phi^{-1}(\Phi(x) + E_0^s)$, $y \mapsto \hat{W}^u(y) \cap \Phi^{-1}(\Phi(x) + E_0^s)$ is continuous. Since $G \to \Phi^{-1}(\Phi(x) + E_0^s)$ is surjective and G is path-connected it follows that $\Phi^{-1}(\Phi(x) + E_0^s)$ is path-connected. Properties (i) and (ii) follows. Using $\hat{W}^{cs}(x) = \Phi^{-1}(\Phi(x) + E_0^s)$, $\hat{W}^{cu}(x) = \Phi^{-1}(\Phi(x) + E_0^u)$ and $\hat{W}^c(x) = \hat{W}^{cs}(x) \cap \hat{W}^{cu}(x)$ we obtain $\hat{W}^c(x) = \Phi^{-1}(\Phi(x))$. Since Φ descends to a map $\Phi : X_\Gamma \to \mathbb{T}^d$, the fibers $W^c(x) = \Phi^{-1}(\Phi(x))$ are compact. Denote by

$$(3.24) \qquad\qquad\qquad \Gamma^c = \{\gamma \in \Gamma : L\gamma = \gamma\}$$

where $L \in \mathrm{Aut}(X_\Gamma)$ is the linearization of f. For $\gamma^c \in \Gamma^c$ we have

$$(3.25) \qquad \gamma^c \hat{W}^c(x) = \hat{W}^c(\gamma^c x) = \Phi^{-1}(\Phi(\gamma^c x)) = \Phi^{-1}(\Phi(x)) = \hat{W}^c(x)$$

where we have used $\Phi(\gamma x) = \pi(\gamma) + \Phi(x)$ and $\ker \pi = \Gamma^c$. So, if we fix a generator γ_0^c of Γ^c then we can orient $\hat{W}^c(x)$ by letting $\gamma_0^c x > x$. This is a well-defined orientation of $\hat{W}^c(x)$ since $x \mapsto \gamma_0^c x$ have no fixed points. That W^c are circles follows since they are compact 1−dimensional manifolds.

Let $\varepsilon > 0$ be small and fix $x \in X_\Gamma$. We denote by $D = W_\varepsilon^s(W_\varepsilon^u(x))$, $U = W^c(D)$ and $D' = \Phi(D) = \Phi(U)$. With ε small enough can define $\pi^{su} : U \to \hat{W}^c(x)$ by the unique holonomy first along W^s then W^u in U. We obtain a map $U \to D' \times W^c(x)$ by $y \mapsto (\Phi(y), \pi^{su}(y))$, this map is smooth along W^c since the holonomies are C^r (where r depends on the bunching, if f is 2−bunching then the holonomies are C^2 [47]). That Φ semiconjugates (a finite index subgroup of) the centralizer $Z^\infty(f)$ onto its linearization is immediate since Φ is (essentially) unique homotopic to

π (note that the uniqueness in Lemma 3.1 implies that $\Phi : X_\Gamma \to \mathbb{T}^d$ is unique modulo the fact that we could change $\Phi(x)$ to $\Phi(x) + p_0$ where $L_{su}p_0 = p_0$). Indeed, for any $g \in Z^\infty(g)$ we let $B \in \mathrm{GL}(d, \mathbb{Z})$ be the induced map on $\mathbb{T}^d$ by its linearization and let $\tilde{\Phi}(x) = B^{-1}\Phi(gx)$. Then $\tilde{\Phi}$ is homotopic to π and we still have

$$(3.26) \qquad \tilde{\Phi}(fx) = B^{-1}\Phi(gfx) = B^{-1}\Phi(fgx) = L_{su}B^{-1}\Phi(gx) = L_{su}\tilde{\Phi}(x)$$

so $\tilde{\Phi}(x) = \Phi(x) + p_0$ where p_0 is a fixed point for L_{su}. We define

$$(3.27) \qquad Z^\infty_{\mathrm{fix}}(f) = \{g \in Z^\infty(f) : gW^c(x) = W^c(x) \text{ if } fW^c(x) = W^c(x)\}.$$

Since $\Phi(fx) = L_{su}\Phi(x)$ and L_{su} has finitely many fixed points, $Z^\infty_{\mathrm{fix}}(f)$ has finite index in $Z^\infty(f)$. For $g \in Z^\infty_{\mathrm{fix}}(f)$ and $\Phi(x_0) = 0$ we have $fW^c(x_0) = W^c(x_0)$ so $gW^c(x_0) = W^c(x_0)$. It follows that

$$(3.28) \qquad\qquad p_0 = \tilde{\Phi}(0) = B^{-1}\Phi(gx_0) = B^{-1}\Phi(x_0) = 0$$

or $B^{-1}\Phi(gx) = \Phi(x)$. ∎

3.2. **Proof of accessibility.** In this section, we show the last point of Theorem 1.1. For $\gamma \in \Gamma$, define $T_\gamma : G \to G$ by

$$(3.29) \qquad T_\gamma(x) = \hat{W}^u\left[\hat{W}^s(\gamma x) \cap \hat{W}^{cu}(x)\right] \cap \hat{W}^c(x) = \hat{W}^u(\hat{W}^s(\gamma x)) \cap \hat{W}^c(x)$$

which are well-defined by point (i) of Theorem 1.1. A calculation shows

$$FT_\gamma(x) = T_{L\gamma}(Fx).$$

Recall that we denote by $\Lambda(F) \subset G$ and $\Lambda(f) \subset X_\Gamma$ the complement of open accessibility classes. The set $\Lambda(F)$ is closed and su−saturated. Since E^c is 1−dimensional, the set $\Lambda(F)$ is laminated by accessibility classes, denoted $\hat{W}^{su}(x)$ [51]. The union of open accessibility classes is Γ−invariant, so $\Lambda(F)$ is Γ−invariant. Moreover, F maps accessibility classes to accessibility classes so $F\Lambda(F) = \Lambda(F)$. Given $\gamma, \gamma' \in \Gamma$ and $x \in \Lambda(F)$ we have

$$\begin{aligned} T_\gamma(T_{\gamma'}(x)) =& \hat{W}^{su}\left(\gamma\left[\hat{W}^{su}(\gamma'x) \cap \hat{W}^c(x)\right]\right) \cap \hat{W}^c\left(\left[\hat{W}^{su}(\gamma'x) \cap \hat{W}^c(x)\right]\right) = \\ & \hat{W}^{su}(\hat{W}^{su}(\gamma\gamma'x) \cap \hat{W}^c(\gamma x)) \cap \hat{W}^c(x) = \\ & \hat{W}^{su}(\gamma\gamma'x) \cap \hat{W}^c(x) = T_{\gamma\gamma'}(x) \end{aligned}$$

so restricted to $\Lambda(F)$ the map $(\gamma, x) \mapsto T_\gamma(x)$ defines a group action of Γ, see also [49, Lemma 6.1]. Before starting the proof we will need an elementary, but important, auxiliary lemma on $\mathbb{Z}^k$−actions on the circle.

Lemma 3.5. *Let $f_1, ..., f_k, g \in \mathrm{Homeo}_+(\mathbb{T})$ be orientation preserving homeomorphisms on the circle and let $K \subset \mathbb{T}$ be a compact subset that is invariant by g*

and each f_j. Moreover, assume that $f_i f_j = f_j f_i$ on K and that there is some hyperbolic integer matrix $(A_i^j)_{1 \le i,j \le k}$ such that

$$(3.30) \qquad g f_i = f_1^{A_i^1} ... f_k^{A_i^k} g, \quad \text{on } K.$$

Then the $\mathbb{Z}^k-$action generated by $f_1, ..., f_k$ on K, $\beta : \mathbb{Z}^k \times K \to K$, has a periodic point. That is, there is a point $p \in K$ and a finite index subgroup $\Lambda \le \mathbb{Z}^k$ such that $\beta^{\mathbf{n}} p = p$ for $\mathbf{n} \in \Lambda$.

Remark 13. The condition in Equation 3.30 says that $\beta : \mathbb{Z}^k \times K \to K$ joint with g form an Abelian-by-Cyclic (AbC) action on K.

In the proof, we will use the following two lemmas.

Lemma 3.6. If $f \in \mathrm{Homeo}_+(\mathbb{T})$ is an orientation preserving homeomorphism on the circle with zero rotation number, $\omega(f) = 0$, and $K \subset \mathbb{T}$ is a compact $f-$invariant set, then f have a fixed point in K.

Proof. If f has zero rotation number then for any $p \in \mathbb{T}$ the sequence $f^n p$ converges to a fixed point of f as $n \to \infty$. For any $x \in K$ the sequence $f^n x$ lie in K since K is $f-$invariant. By compactness of K any limit point of $f^n x$ also lies in K, so f has a fixed point in K. $\blacksquare$

Lemma 3.7. Let $f, g \in \mathrm{Homeo}_+(\mathbb{T})$ be orientation preserving homeomorphisms of the circle and assume that there is a f and $g-$invariant probability measure ν. Then the rotation numbers satisfies $\omega(fg) = \omega(f) + \omega(g)$.

Proof. The proof of this lemma is standard. We include it for completeness.

We write $f(x) = x + u(x)$, $g(x) = x + v(x)$ where $u, v : \mathbb{T} \to \mathbb{R}$. Recall that if μ is a $f-$invariant measure then we obtain the rotation number of f as

$$(3.31) \qquad \omega(f) = \int_{\mathbb{T}} u(x) \mathrm{d}\mu(x) + \mathbb{Z}.$$

Similarly, we can obtain the rotation number for g. Noting that the measure ν is $fg-$invariant and that $f(gx) = x + u(gx) + v(x)$ we can write the rotation number of fg as

$$\omega(fg) = \int_{\mathbb{T}} [u(gx) + v(x)] \, \mathrm{d}\nu(x) + \mathbb{Z} =$$
$$\int_{\mathbb{T}} u(gx) \mathrm{d}\nu(x) + \int_{\mathbb{T}} v(x) \mathrm{d}\nu(x) + \mathbb{Z} =$$
$$\int_{\mathbb{T}} u(x) \mathrm{d}\nu(x) + \int_{\mathbb{T}} v(x) \mathrm{d}\nu(x) + \mathbb{Z} =$$
$$\omega(f) + \omega(g)$$

where the second to last equality use that ν is $g-$invariant. ∎

Proof of Lemma 3.5. Since the $\beta-$action is abelian on K it has an invariant measure on K. By Lemma 3.7 the rotation numbers satisfy

$$(3.32) \qquad \omega(f_1^{n_1}...f_k^{n_k}) = n_1\omega(f_1) + ... + n_k\omega(f_k)$$

for all integers $n_1, ..., n_k \in \mathbb{Z}$. Since g preserves orientation, conjugacy invariance of rotation number and Equation 3.30 implies

$$(3.33) \qquad \omega(f_i) = \omega(f_1^{A_i^1}) + ... + \omega(f_k^{A_i^k}).$$

Or if we denote the map $f_1^{n_1}...f_k^{n_k}$ by $\beta^{\mathbf{n}}$, then we can write $\omega(\beta^{\mathbf{n}}) = \omega(\beta^{A\mathbf{n}})$. Using Equation 3.32 we obtain

$$(3.34) \qquad \omega\left(\beta^{(A-I)\mathbf{n}}\right) = 0$$

for all $\mathbf{n} \in \mathbb{Z}^k$. Since A is hyperbolic $A - I$ is invertible over the rationals. So there is a finite index subgroup $\Lambda \subset \mathbb{Z}^k$ such that $\omega(\beta^{\mathbf{n}}) = 0$ for all $\mathbf{n} \in \Lambda$. Let $e_1, ..., e_k \in \Lambda$ be generators. By Lemma 3.6 the map β^{e_1} has a fixed point in K. Since β^{e_2} commute with β^{e_1} within K it follows that β^{e_2} preserve the compact set $\text{Fix}(\beta^{e_1}) \cap K \neq \emptyset$. So, if we apply Lemma 3.6 once more we see that the set

$$(3.35) \qquad K \cap \text{Fix}(\beta^{e_1}) \cap \text{Fix}(\beta^{e_2})$$

is non-empty. Proceeding by induction, we find a point $p \in K$ that is fixed by $\beta^{e_1}, ..., \beta^{e_k}$, and therefore by Λ. Since Λ has a finite index in $\mathbb{Z}^k$ the lemma follows. ∎

Recall that $L \in \text{Aut}(X_\Gamma)$ is the linearization of f and we define the $L-$fixed part of Γ by Γ^c, see Equation 3.24.

Lemma 3.8. *Assume that $\Lambda(f) \neq \emptyset$. There is a finite index subgroup $\Gamma' \leq \Gamma$ and $x \in \Lambda(F)$ such that $T_\gamma(x) \in \Gamma^c x$ for $\gamma \in \Gamma'$.*

Proof. Fix generators $\gamma_1, \gamma_2, ..., \gamma_d \in \Gamma$. Let $x_0 \in G$ be such that $\Phi(x_0) = e$. We identify $\hat{W}^c(x_0)/\Gamma^c \cong W^c(p_\Gamma(x_0))$ with $\mathbb{T}$. Since Γ^c is central in Γ we can identify $T_{\gamma_j} : \hat{W}^c(x_0)/\Gamma^c \to \hat{W}^c(x_0)/\Gamma^c$ with circle diffeomorphisms. We also identify F with a circle diffeomorphism (by our choice of x_0 and the fact that Γ^c is $L-$fixed we have $F\hat{W}^c(x_0)/\Gamma^c = \hat{W}^c(x_0)/\Gamma^c$). Let $K = (\Lambda(F) \cap \hat{W}^c(x_0))/\Gamma^c$, which is compact, $T_{\gamma_j}-$invariant, $F-$invariant and non-empty. Since any $\gamma^c \in \Gamma^c$ act trivially on $\mathbb{T}$ under T_{γ^c} (and therefore K), the action T_γ on K factor through $\Gamma/\Gamma^c \cong \mathbb{Z}^d$. Moreover, F satisfy $FT_\gamma = T_{L\gamma}F$ so the assumptions of Lemma 3.5 are satisfied with $T_{\gamma_j} = f_j$ and $g = F$. So, there is a finite index subgroup of Γ/Γ^c that admits a fixed point on $\mathbb{T}$, which implies the lemma. ∎

An immediate corollary of Lemma 3.8 is that there is a compact $su-$leaf.

Lemma 3.9. *If $\Lambda(f) \neq \emptyset$, or equivalently if f is not accessible, then there is a compact $su-$leaf intersecting each center leaves $q < \infty$ times.*

Proof. For any $x \in \Lambda(F)$, the map

$$(3.36) \qquad \hat{W}^{su}(x) \xrightarrow{\Phi} \mathbb{R}^d$$

is a homeomorphism by points (i) and (iv) in Theorem 1.1. We choose x_0 as in Lemma 3.8 and obtain $\Gamma' \leq \Gamma$ of finite index such that $\gamma \hat{W}^{su}(x_0) \subset \Gamma^c \hat{W}^{su}(x_0)$ for all $\gamma \in \Gamma'$. Note that $\Phi_* \Gamma' = \pi_* \Gamma' = \Lambda \subset \mathbb{Z}^d$ has finite index in $\mathbb{Z}^d$. We define

$$(3.37) \qquad \Lambda_* = \{\gamma \in \Gamma : \gamma \hat{W}^{su}(x_0) = \hat{W}^{su}(x_0)\} \subset \Gamma'.$$

Since $\gamma \hat{W}^{su}(x_0) \subset \Gamma^c \hat{W}^{su}(x_0)$ for all $\gamma \in \Gamma'$ there is for each $\gamma \in \Gamma'$ some $\gamma^c \in \Gamma^c$ such that $\gamma \gamma^c \in \Lambda_*$. In particular, $\Phi_* \Lambda_* = \Lambda$. Since Λ is a lattice in $\mathbb{R}^d$, and we obtain a homeomorphism $\hat{W}^{su}(x_0)/\Lambda_* \cong \mathbb{R}^d/\Lambda$ it follows that $\hat{W}^{su}(x_0)/\Lambda_*$ is compact. The image $p_\Gamma \hat{W}^{su}(x_0) = W^{su}(p_\Gamma(x_0))$ is homeomorphic to $\hat{W}^{su}(x_0)/\Lambda_*$, proving the lemma. ∎

We can now prove the last claim of Theorem 1.1

Proof of (vi) in Theorem 1.1. If f is not accessible then we construct $\Lambda_* \subset \Gamma$ as in Equation 3.37. Since $\pi_* \Lambda_*$ has finite index in $\mathbb{Z}^d$ the group $\Lambda_* \times \ker \pi_*$ has finite index in Γ. Since $\Lambda_* \times \ker \pi_*$ is abelian this implies that Γ is virtually abelian, which is a contradiction if G is a non-abelian nilpotent Lie group. ∎

4. Action of the su-path group

In this section, we introduce and prove the basic properties of the $su-$path group, $\mathcal{P}$. The $su-$path group naturally acts on X_Γ (Definition 4.1). The group $\mathcal{P}$, its various subgroups and its action on X_Γ will be the key object in the proofs in the following sections.

4.1. **The su-path group.** Let $\alpha : \mathbb{Z}^k \times X_\Gamma \to X_\Gamma$ be some smooth action and let $\rho : \mathbb{Z}^k \to \mathrm{GL}(d, \mathbb{Z})$ be the action defined by $\rho^{\mathbf{n}} \Phi = \Phi \alpha^{\mathbf{n}}$ from Theorem 1.1. We will assume that we have some $f = \alpha^{\mathbf{n}_0}$ satisfying the assumptions of Theorem A. Let L be the linearization of f and let E_0^s, E_0^u be the stable and unstable distributions for L.

Definition 4.1. *We define the $su-$path group $\mathcal{P}$ as the free product*

$$(4.1) \qquad \mathcal{P} = E_0^s * E_0^u.$$

If $w \in \mathcal{P}$ is a word in $\mathcal{P}$ then we define $\Pi(w) = w_1 + ... + w_N$ to be the sum of all factors in w. We define the normal subgroup $\mathcal{P}^c = \Pi^{-1}(0)$.

Given any pair of negatively proportional course Lyapunov spaces, $E_0^{-[\chi]}$, $E_0^{[\chi]}$, of ρ, we define the $[\chi]-$*path group*

$$(4.2) \qquad \mathcal{P}_{[\chi]} = E_0^{-[\chi]} * E_0^{[\chi]}, \quad \mathcal{P}_{[\chi]}^c = \mathcal{P}_{[\chi]} \cap \Pi^{-1}(0)$$

with $E_0^{-[\chi]} = 0$ of $-[\chi]$ is not a coarse exponent. We also define the *complementary* $[\chi]-$*path group*

$$(4.3) \qquad \mathcal{Q}_{[\chi]} = \left(\bigoplus_{\substack{[\eta] \neq \pm[\chi] \\ [\eta](\mathbf{n}_0) > 0}} E^{-[\eta]} \right) * \left(\bigoplus_{\substack{[\eta] \neq \pm[\chi] \\ [\eta](\mathbf{n}_0) > 0}} E^{[\eta]} \right), \quad \mathcal{Q}_{[\chi]}^c = \mathcal{Q}_{[\chi]} \cap \mathcal{P}^c.$$

It is immediate that $\mathcal{P}_{[\chi]}, \mathcal{Q}_{[\chi]} \subset \mathcal{P}$, $\mathcal{P}_{[\chi]}^c, \mathcal{Q}_{[\chi]}^c \subset \mathcal{P}^c$. The following well-known lemma on free products will be useful.

Lemma 4.1. *Let V, W be vector spaces, $\mathcal{G} = V * U$ and $\Pi : G \to V \oplus W$ the map defined by*

$$(4.4) \qquad \Pi(v_1 u_1 ... v_n u_n) = v_1 + ... + v_n + u_1 + ... + u_n.$$

Then $\Pi^{-1}(0) = \ker \Pi = [\mathcal{G}, \mathcal{G}]$ and any $w \in \mathcal{G}$ can be written $w = \tilde{w} v u$ with $v \in V$, $u \in U$ and $\tilde{w} \in [\mathcal{G}, \mathcal{G}]$.

Proof. For $w = v_1 u_1 ... v_n u_n \in \mathcal{G}$ we have

$$v_1 u_1 ... v_n u_n = [v_1 u_1 (-v_1)(-u_1)] u_1 (v_1 + v_2) u_2 ... v_n u_n =$$
$$[v_1 u_1 (-v_1)(-u_1)] \cdot [u_1 (v_1 + v_2)(-u_1)(-v_1 - v_2)] \cdot$$
$$(v_1 + v_2)(u_1 + u_2) v_3 u_3 ... v_n u_n.$$

Since $v_1 u_1 (-v_1)(-u_1), u_1 (v_1 + v_2)(-u_1)(-v_1 - v_2) \in [\mathcal{G}, \mathcal{G}]$ and $(v_1 + v_2)(u_1 + u_2) v_3 u_3 ... v_n u_n$ consists of $n - 1$ pairs, we find $w_1, ..., w_\ell \in [\mathcal{G}, \mathcal{G}]$ such that

$$(4.5) \qquad w = w_1 ... w_\ell \cdot (v_1 + ... + v_n)(u_1 + ... + u_n).$$

This proves the last part. Since $\Pi(w_j) = 0$ for each j, we have $\Pi(w) = v_1 + ... + v_n + u_1 + ... + u_n$. In particular, if $w \in \ker \Pi$ then $v_1 + ... + v_n = 0$ and $u_1 + ... + u_n = 0$, so $w = w_1 ... w_\ell \in [\mathcal{G}, \mathcal{G}]$. $\blacksquare$

Definition 4.2. *Given $t \in E_0^\sigma$ we define $\eta_\sigma^t : G \to G$ (or $\eta_\sigma^t : X_\Gamma \to X_\Gamma$) by $\eta_\sigma^t x = \Phi_{\sigma,x}^{-1}(t)$, where $\Phi_{\sigma,x}$ is defined in Lemma 3.3.*

Remark 14. By the conclusion of Lemma 3.3, the map η_σ^t is well-defined.

Lemma 4.2. *The map $\eta_\sigma : E_0^\sigma \times G \to G$, $(t, x) \mapsto \eta_\sigma^t x$, is a Hölder E_0^σ−action that satisfies $F\eta_\sigma^t x = \eta_\sigma^{L_{su} t} F x$. The action η_σ^t naturally descend to X_Γ, and if $g \in Z^\infty(f)$, B is the automorphism defined by $\Phi(gx) = B\Phi(x)$, then $g\eta_\sigma^t x = \eta_\sigma^{Bt} g x$. Finally, $\Phi(\eta_\sigma^t x) = t + \Phi(x)$, so Φ semi-conjugates η_σ^t to the standard translation action along E_0^σ on the base.*

Proof. That η_σ^t defines an action is immediate from the definition. Indeed, given $t, s \in E_0^\sigma$ we have

$$\Phi(\eta_\sigma^s \eta_\sigma^t x) = \Phi(\Phi_{\sigma, \eta_\sigma^t x}^{-1}(s)) = s + \Phi(\eta_\sigma^t x) = s + \Phi_{\sigma, x}(\eta_\sigma^t x) + \Phi(x) =$$
$$s + t + \Phi(x)$$

so if we subtract $\Phi(x)$ then $\Phi_{\sigma, x}(\eta_\sigma^s \eta_\sigma^t x) = s + t$. Applying $\Phi_{\sigma, x}^{-1}$ on both sides of the equality we have $\eta_\sigma^s \eta_\sigma^t x = \eta_\sigma^{s+t} x$. This also shows that $\Phi(\eta_u^t x) = \Phi(x) + t$. Given $\gamma \in \Gamma$ we have $\Phi_{\sigma, \gamma x}(\gamma y) = \Phi(\gamma y) - \Phi(\gamma x) = \Phi(y) - \Phi(x) = \Phi_{\sigma, x}(y)$ which implies $\eta_\sigma^t(\gamma x) = \gamma \eta_\sigma^t x$, so η_σ^t descend to X_Γ. For $g \in Z^\infty(f)$ we have $\Phi(gx) = B\Phi(x) + p_0$ for some $p_0 \in \mathbb{T}^d$ (that is fixed by L_{su}). It follows that

$$(4.6) \qquad \Phi_{\sigma, gx}(gy) = \Phi(gy) - \Phi(gx) = B\left(\Phi(x) - \Phi(y)\right) = B\Phi_{\sigma, x}(y)$$

or with $\Phi_{\sigma, x}(y) = t$

$$(4.7) \qquad\qquad\qquad \eta_\sigma^{Bt}(gx) = gy = g\eta_\sigma^t(x).$$

Next, we show that η_σ is Hölder. Let $u = \sigma$, the other case is similar. Since $\Phi_{u, x}$ is a bi-Hölder homeomorphism, Lemma 3.3, it is immediate that $(t, x) \mapsto \eta_u^t x$ is Hölder in t. The foliations $\hat{W}^u$ and $\hat{W}^{cs}$ are uniformly transverse, so we find $\varepsilon_0 > 0$ and K such that for $x, y \in X_\Gamma$ with $d(x, y) \leq \varepsilon_0$ we have $\hat{W}_{\mathrm{loc}}^u(x) \cap \hat{W}_{\mathrm{loc}}^{cs}(y) = \{z\}$,

$$(4.8) \qquad\qquad d_u(x, z) \leq K d(x, y), \quad d_{cs}(z, y) \leq K d(x, y).$$

If $y \in \hat{W}^{cs}(x)$ then the proof of Theorem 1.1 shows $\Phi(y) \in \Phi(x) + E_0^s$, so

$$\Phi(\eta_u^t y) = \Phi(y) + t = \Phi(x) + t + [\Phi(y) - \Phi(x)] =$$
$$\Phi(\eta_u^t x) + [\Phi(y) - \Phi(x)] \in \Phi(\eta_u^t x) + E_0^s,$$

or $\eta_u^t y \in \hat{W}^{cs}(\Phi(\eta_u^t x))$. That is, η_u^t preserve the foliation $\hat{W}^{cs}$. Given $t \in E_0^u$

$$(4.9) \qquad\qquad d(\eta_u^t x, \eta_u^t y) \leq d_u(\eta_u^t x, \eta_u^t z) + d_{cs}(\eta_u^t z, \eta_u^t y)$$

so it suffices to show that η_u^t is Hölder along $\hat{W}^u$ and $\hat{W}^{cs}$. For $y \in \hat{W}^u(x)$

$$\Phi_{u, x}(\eta_u^t y) = \Phi(\eta_u^t y) - \Phi(x) = \Phi_{u, y}(\eta_u^t y) + \Phi(y) - \Phi(x) =$$
$$t + \Phi(y) - \Phi(x)$$

or $\eta_u^t y = \Phi_{u, x}^{-1}(t + \Phi(y) - \Phi(x))$. Since Φ is Hölder and $\Phi_{u, x}^{-1}$ is uniformly Hölder

$$(4.10) \qquad\qquad d(\eta_u^t x, \eta_u^t y) \leq C \left\| \Phi(y) - \Phi(x) \right\|^\theta \leq C' d(x, y)^{\theta'}$$

so η_u^t is Hölder along $\hat{W}^u$. Given $y \in \hat{W}^{cs}(x)$ we have $\eta_u^t y \in \hat{W}^{cs}(\eta_u^t x)$ since η_u^t preserve $\hat{W}^{cs}$. On the other hand, $\eta_u^t y \in \hat{W}^u(y)$ by the definition of $\Phi_{u,y}$. So $\eta_u^t y = \pi_{x,\eta_u^t x}^u(y)$. The unstable Holonomy is (uniformly) Hölder [47], so η_u^t is Hölder along $\hat{W}^{cs}$. ∎

Definition 4.3. *We define an action of* $\mathcal{P}$ *(or* $\mathcal{P}^c, \mathcal{P}_{[\chi]}, \mathcal{Q}_{[\chi]}, \mathcal{P}_{[\chi]}^c, \mathcal{Q}_{[\chi]}^c$*) on* X_Γ *(and on* G*) by*

$$(4.11) \qquad \eta_{\mathcal{P}}^w x = \eta_{\mathcal{P}}^{w_1^s w_1^u \dots w_N^s w_N^u} x = \eta_s^{w_1^s} \eta_u^{w_1^u} \dots \eta_s^{w_N^s} \eta_u^{w_N^u} x$$

where η_σ *is defined in Definition 4.2 and shown to be an action in Lemma 4.2.*

Remark 15. We make no notational distinction between the action on X_Γ and G. It is clear that the action on G covers the action on X_Γ in the sense that the projection $p_\Gamma : G \to X_\Gamma$ intertwines the two actions.

Lemma 4.3. *We have the relationship* $\Phi(\eta_{\mathcal{P}}^w x) = \Phi(x) + \Pi(w)$. *That is* Φ *semi-conjugates the* $\mathcal{P}-$*action onto the translation action on* $\mathbb{T}^d$ *(and* $\mathbb{R}^d$*).*

Proof. By induction, it suffices to consider $v \in E_0^\sigma$. The Lemma follows from Lemma 4.2. ∎

Lemma 4.4. *For any* $w \in \mathcal{P}$ *the homeomorphism* $\eta_{\mathcal{P}}^w$ *preserve the center foliation,* W^c. *Moreover, if* $w = v_N^s v_N^u \dots v_1^s v_1^u$ *with* $v_j^\sigma \in E_0^\sigma$, $x_0 = x$ *and*

$$x_1^u = \eta_u^{v_1^u} x_0, \quad x_1^s = \eta_s^{v_1^s} x_1^u,$$

$$\vdots$$

$$x_N^u = \eta_u^{v_N^u} x_{N-1}^s, \quad x_N^s = \eta_s^{v_N^s} x_N^u$$

then $\eta_{\mathcal{P}}^w x = x_N^s$ *and the map* $\eta_{\mathcal{P}}^w : W^c(x) \to W^c(\eta_{\mathcal{P}}^w x)$ *coincide with the composition*

$$(4.12) \qquad W^c(x_0) \xrightarrow{\pi_{x,x_1^u}^u} W^c(x_1^u) \xrightarrow{\pi_{x_1^u,x_1^s}^s} W^c(x_1^s) \xrightarrow{\pi_{x_1^s,x_2^u}^u} \dots \xrightarrow{\pi_{x_N^u,x_N^s}^s} W^c(x_N^s).$$

That is, we have

$$(4.13) \qquad \eta_{\mathcal{P}}^w|_{W^c(x)} = \pi_{x_N^u,x_N^s}^s \circ \pi_{x_{N-1}^s,x_N^u}^u \circ \pi_{x_{N-1}^s,x_{N-1}^u}^s \circ \dots \circ \pi_{x_1^u,x_1^s}^s \circ \pi_{x_0,x_1^u}^u.$$

If f *is* $r-$*bunching then holonomies between center manifolds are* C^r-*smooth, so* $\eta_{\mathcal{P}}^w$ *is* C^r *along* W^c *for all* $w \in \mathcal{P}$.

Proof. By induction it suffices to consider $w = v \in E_0^\sigma$ for $\sigma = s, u$. This was shown in the proof of Lemma 4.2. The regularity follows from [47]. ∎

Lemma 4.5. *We have* $w \in \mathcal{P}^c$ *if and only if* $\eta_{\mathcal{P}}^w x \in \hat{W}^c(x)$ *for every* $x \in G$.

Remark 16. When $w \in \mathcal{P}^c$ then Lemma 4.5 shows that $\eta_{\mathcal{P}}^w W^c(x) = W^c(x)$ for every x, so $\mathcal{P}^c$ is the (homotopically trivial) *center fixing* part of $\mathcal{P}$.

Proof. It is immediate from the definition of $\mathcal{P}^c$ and Lemma 4.3 that if $w \in \mathcal{P}^c$ then $\eta_{\mathcal{P}}^w x \in \Phi^{-1}(\Phi(x)) = \hat{W}^c(x)$ for every $x \in G$. Conversely, if $\eta_{\mathcal{P}}^w x \in \hat{W}^c(x)$ then $\Phi(x) + \Pi(w) = \Phi(\eta_{\mathcal{P}}^w x) = \Phi(x)$ so $\Pi(w) = 0$. $\blacksquare$

Lemma 4.6. *If f is accessible, then the $\mathcal{P}^c-$action is transitive on $\hat{W}^c(x)$ for all $x \in G$, and the $\mathcal{P}^c-$action is transitive on $W^c(x)$ for all $x \in X_\Gamma$.*

Proof. Since f is accessible $\eta_{\mathcal{P}}^{\mathcal{P}} x \supset W^c(x)$ for every $x \in X_\Gamma$, and $\eta_{\mathcal{P}}^w x \in W^c(x)$ if and only if $\Pi(w) \in \mathbb{Z}^d$, Lemma 4.3. Given $\mathbf{n} \in \mathbb{Z}^d$ we write $\mathcal{P}_{\mathbf{n}} = \{w \in \mathcal{P} : \Pi(w) = \mathbf{n}\}$. Then $\mathcal{P}^c = \mathcal{P}_0$. It follows

$$W^c(x) = \bigcup_{\mathbf{n} \in \mathbb{Z}^d} \eta_{\mathcal{P}}^{\mathcal{P}_{\mathbf{n}}} x$$

and since $W^c(x)$ is uncountable and $\mathbb{Z}^d$ is countable there is at least one $\mathbf{n}_0$ such that $\#\eta_{\mathcal{P}}(\mathcal{P}_{\mathbf{n}_0})x > 1$. If we fix some $w \in \mathcal{P}_{-\mathbf{n}_0}$ then

$$\#\eta_{\mathcal{P}}^w \eta_{\mathcal{P}}^{\mathcal{P}_{\mathbf{n}_0}} x = \#\eta_{\mathcal{P}}^{w\mathcal{P}_{\mathbf{n}_0}} x > 1.$$

For any $w' \in \mathcal{P}_{\mathbf{n}_0}$ we have $\Pi(ww') = \Pi(w) + \Pi(w') = 0$ so $w\mathcal{P}_{\mathbf{n}_0} \subset \mathcal{P}^c$, or $\#\eta_{\mathcal{P}}^{\mathcal{P}^c} x > 1$. Given any $w = w_1^s w_1^u ... w_N^s w_N^u \in \mathcal{P}^c$ we define a path $w_t = (tw_1^s)(tw_1^u)...(tw_N^s)(tw_N^u) \in \mathcal{P}^c$, $t \in [0,1]$, from 0 to w. It follows that the image $I = \eta_{\mathcal{P}}^{\mathcal{P}^c} x$ is an interval in $W^c(x)$, since it contains at least 2 distinct points and is connected. We claim that x is an interior point in this interval. If x is not an interior point, then $I = [x, y), [x, y]$ or $(y, x], [y, x]$ for some $y \in W^c(x)$. We will assume that one of the first two cases holds the other two cases are similar. Let $x \neq z \in I$ and let $\eta_{\mathcal{P}}^w x = z$. Then the interval $[x, z]$ lie within I, $[x, z] \subset I$. Since $w^{-1} \in \mathcal{P}^c$ and $\eta_{\mathcal{P}}^{w^{-1}}$ preserves orientation we have $I \supset \eta_{\mathcal{P}}^{w^{-1}}[x, z] = [\eta_{\mathcal{P}}^{w^{-1}} x, x]$ which would imply $\eta_{\mathcal{P}}^{w^{-1}} x = x$. After applying $\eta_{\mathcal{P}}^w$ on both sides of $\eta_{\mathcal{P}}^{w^{-1}} x = x$ we obtain $x = \eta_{\mathcal{P}}^w x = z$. This is a contradiction since we assumed $z \neq x$. The point x is interior in $\eta_{\mathcal{P}}^{\mathcal{P}^c} x$ and x was arbitrary, so the orbit of x under $\eta_{\mathcal{P}}^{\mathcal{P}^c}$ is open. This holds for every x and $W^c(x)$ is connected, so $\eta_{\mathcal{P}}^{\mathcal{P}^c} x = W^c(x)$. The second part of the lemma follows.

The first claim follows from the second part. Indeed the second part implies that for any $x \in G$ the $\mathcal{P}^c-$orbit of x is open in $\hat{W}^c(x)$. Connectedness of $\hat{W}^c(x)$ implies the first part of the lemma. $\blacksquare$

The following lemma is immediate from Lemma 4.2. Recall that $\rho : \mathbb{Z}^k \to \mathrm{GL}(d, \mathbb{Z})$ is defined by $\Phi \alpha^{\mathbf{n}} = \rho^{\mathbf{n}} \Phi$.

Lemma 4.7. *Let $\rho^{\mathbf{n}} : \mathcal{P} \to \mathcal{P}$ be the map*

$$\rho^{\mathbf{n}}(w_1^s w_1^u ... w_N^s w_N^u) = (\rho^{\mathbf{n}} w_1^s)(\rho^{\mathbf{n}} w_1^u)...(\rho^{\mathbf{n}} w_N^s)(\rho^{\mathbf{n}} w_N^u)$$

then $\alpha^{\mathbf{n}} \eta_{\mathcal{P}}^w = \eta_{\mathcal{P}}^{\rho^{\mathbf{n}} w} \alpha^{\mathbf{n}}$. Moreover, $\rho^{\mathbf{n}}$ preserve $\mathcal{P}^c$, $\mathcal{P}_{[\chi]}$, $\mathcal{Q}_{[\chi]}$, $\mathcal{P}_{[\chi]}^c$ and $\mathcal{Q}_{[\chi]}^c$.

5. An invariance principle for higher rank Anosov actions

In this section we prove an invariance principle for cocycles over higher rank Anosov actions. Before stating the invariance principle, we introduce cocycles. Let X_Γ be a nilmanifold (where, in this section, G is any simply connected nilpotent Lie group), and $\rho : \mathbb{Z}^k \to \mathrm{Aut}(X_\Gamma)$ a higher rank action. We will assume that ρ is the restriction of some map $Q : \mathbb{R}^k \to \mathrm{Aut}(G)$. Let $\Phi_\mathcal{X} : \mathcal{X} \to X_\Gamma$ be a Hölder fiber bundle over X_Γ with fibers $\mathcal{X}_x = \Phi_\mathcal{X}^{-1}(x)$ uniformly C^r for some $r > 1$ (we allow $r \in (1, 2)$). We will assume throughout this section that $\mathcal{X}$ is compact, and therefore have compact fibers.

Definition 5.1. *We say that $F : \mathbb{Z}^k \times \mathcal{X} \to \mathcal{X}$, written $F^\mathbf{n} : \mathcal{X} \to \mathcal{X}$, is a cocycle over ρ if F is a $\mathbb{Z}^k-$action covering ρ. That is, $F^{\mathbf{n}+\mathbf{m}} = F^\mathbf{n} F^\mathbf{m}$ and $\Phi_\mathcal{X} F^\mathbf{n} = \rho^\mathbf{n} \Phi_\mathcal{X}$. Moreover, F is a C^s-cocycle if $F^\mathbf{n} : \mathcal{X}_x \to \mathcal{X}_{\rho^\mathbf{n} x}$ is uniformly C^s.*

For each coarse exponent $[\chi]$ of ρ we have a translation action $T_{[\chi]} : G^{[\chi]} \times X_\Gamma \to X_\Gamma$, $T_{[\chi]}^g x = xg^{-1}$ for $g \in G^{[\chi]}$.

Definition 5.2. *Let F be a cocycle over ρ. The cocycle F admit $[\chi]-$translations if there is a Hölder action $\eta_{[\chi]} : G^{[\chi]} \times \mathcal{X} \to \mathcal{X}$ covering $T_{[\chi]}$ such that*

$$(5.1) \qquad F^\mathbf{n} \eta_{[\chi]}^g = \eta_{[\chi]}^{\rho^\mathbf{n} g} F^\mathbf{n},$$

for any $\mathbf{n} \in \mathbb{Z}^k$.

Our interest in cocycles over algebraic actions comes from the following lemma.

Lemma 5.1. *If α is as in Theorem A, with $\Phi \alpha^\mathbf{n} = \rho^\mathbf{n} \Phi$, then α is a C^2-cocycle over ρ that admit $[\chi]-$translations for every coarse $[\chi]$ where $\eta_{[\chi]} = \eta_\mathcal{P}|_{E_0^{[\chi]}}$.*

Proof. The lemma is immediate from Theorem 1.1 and Lemma 4.3. ∎

The main result of this section is a sufficient condition for the translation action $\eta_{[\chi]}$ to preserve a $F-$invariant measure.

Theorem 5.1. *Let F be a C^s-cocycle, $s > 1$, over ρ. Let ν be a $F-$invariant probability measure projecting onto μ_Γ, $(\Phi_\mathcal{X})_* \nu = \mu_\Gamma$. Let $\lambda_{F,\nu}^1, ..., \lambda_{F,\nu}^N : \mathbb{Z}^k \to \mathbb{R}$ be the $\nu-$Lyapunov exponents of F along the fibers of $\mathcal{X}$. If*

$$(5.2) \qquad \bigcap_{i=1}^N \ker \lambda_{F,\nu}^i \not\subset \ker[\chi]$$

then ν is $\eta_{[\chi]}-$invariant.

Remark 17. In Theorem 5.1 we assume that ρ is the restriction of some homomorphism $Q : \mathbb{R}^k \rightarrow \mathrm{Aut}(G)$. In practice this is not an issue, since if $\mathbb{Z}^k \rightarrow \mathrm{GL}(d, \mathbb{Z})$ is a homomorphism then there is a finite index subgroup of $\mathbb{Z}^k$ that satisfies this assumption.

To apply results from [3] it will be convenient to reformulate $\eta_{[\chi]}$−invariance of ν into essential holonomy invariance. Let $x \in X_\Gamma$, $g \in G^{[\chi]}$ and $y = T_{[\chi]}^g x$. Since $\eta_{[\chi]}^g$ cover $T_{[\chi]}^g$, we define

(5.3)
$$h_{x,y}^{[\chi]} : \mathcal{X}_x \rightarrow \mathcal{X}_y, \quad h_{x,y}^{[\chi]}(\xi) = \eta_{[\chi]}^g(\xi).$$

We say that $h_{x,y}^{[\chi]}$ is the $[\chi]$−holonomy between $\mathcal{X}_x$ and $\mathcal{X}_y$.

Definition 5.3. *Let ν be F−invariant such that $(\Phi_\mathcal{X})_* \nu = \mu_\Gamma$ and let $\{\nu_x\}_{x \in X_\Gamma}$ be the disintegration of ν over $\Phi_\mathcal{X}$. We say that ν, or $\{\nu_x\}_{x \in X_\Gamma}$, is essentially $[\chi]$−holonomy invariant if there is a μ_Γ−full measure set $Y \subset X_\Gamma$ such that $(h_{x,y}^{[\chi]})_* \nu_x = \nu_y$ for $x, y \in Y$.*

Lemma 5.2. *Let ν be F−invariant and projecting onto μ_Γ. If ν is essentially $[\chi]$−holonomy invariant then ν is $\eta_{[\chi]}$−invariant.*

Proof. Let $Y \subset X_\Gamma$ be a full measure subset such that $(h_{x,y}^{[\chi]})_* \nu_x = \nu_y$ for $x, y \in Y$. Let $g \in G^{[\chi]}$ and $\tilde{Y} = Y \cap \eta_{[\chi]}^{g^{-1}} Y$ so that $x, \eta_{[\chi]}^g x \in Y$ for $x \in \tilde{Y}$. If $\varphi \in C^0(\mathcal{X})$

$$\int_\mathcal{X} \varphi(\eta_{[\chi]}^g \xi) \mathrm{d}\nu(\xi) = \int_{\tilde{Y}} \left(\int_{\mathcal{X}_x} \varphi(\eta_{[\chi]}^g \xi) \mathrm{d}\nu_x(\xi) \right) \mathrm{d}\mu_\Gamma(x) =$$

$$\int_{\tilde{Y}} \left(\int_{\eta_{[\chi]}^g \mathcal{X}_x} \varphi(\xi) \mathrm{d}(h_{x, T_{[\chi]}^g x}^{[\chi]})_* \nu_x(\xi) \right) \mathrm{d}\mu_\Gamma(x) =$$

$$\int_{\tilde{Y}} \left(\int_{\mathcal{X}_{T_{[\chi]}^g x}} \varphi(\xi) \mathrm{d}\nu_{T_{[\chi]}^g x}(\xi) \right) \mathrm{d}\mu_\Gamma(x) =$$

$$\int_\mathcal{X} \varphi(x) \mathrm{d}\nu(x),$$

so ν is $\eta_{[\chi]}^g$−invariant. ∎

5.1. **The suspension construction.** Fix a higher rank action $\rho : \mathbb{Z}^k \rightarrow \mathrm{Aut}(X_\Gamma)$ and a cocycle $F^\mathbf{n} : \mathcal{X} \rightarrow \mathcal{X}$ over ρ as in Theorem 5.1. We recall the definition of the suspension of an action $\alpha : \mathbb{Z}^k \times M \rightarrow M$.

Definition 5.4. *Let $\tau : \mathbb{Z}^k \times (M \times \mathbb{R}^k) \rightarrow M \times \mathbb{R}^k$ be defined by $\tau^\mathbf{n}(x, \mathbf{s}) = (\alpha^\mathbf{n} x, \mathbf{s} - \mathbf{n})$. We define the suspension $\mathcal{S}$ of α as*

$$\mathcal{S} := (M \times \mathbb{R}^k) / \tau.$$

Given $(x, \mathbf{s}) \in M \times \mathbb{R}^k$ we denote by $[x, \mathbf{s}]$ the equivalence class of $(x, \mathbf{s})$ in $\mathcal{S}$. We also define a natural action on $\mathcal{S}$ by $\alpha_{\mathcal{S}}^{\mathbf{t}}(x, \mathbf{s}) = (x, \mathbf{s} + \mathbf{t})$. Since

$$\alpha_{\mathcal{S}}^{\mathbf{t}}\tau^{\mathbf{n}}(x, \mathbf{s}) = \alpha_{\mathcal{S}}^{\mathbf{t}}(\alpha^{\mathbf{n}}x, \mathbf{s} - \mathbf{n}) = (\alpha^{\mathbf{n}}x, \mathbf{t} + \mathbf{s} - \mathbf{n}) = \tau^{\mathbf{n}}(x, \mathbf{t} + \mathbf{s}) =$$
$$\tau^{\mathbf{n}}\alpha_{\mathcal{S}}^{\mathbf{t}}(x, \mathbf{s})$$

the action $\alpha_{\mathcal{S}}$ descends to an action on $\mathcal{S}$. Moreover the map $M \times \mathbb{R}^k \ni (x, \mathbf{s}) \mapsto \mathbf{s} + \mathbb{Z}^k \in \mathbb{T}^k$ descends to a map $\pi_{\mathcal{S}} : \mathcal{S} \to \mathbb{T}^k$ with fibers M. The map $\pi_{\mathcal{S}}$ semi-conjugates $\alpha_{\mathcal{S}}$ to the natural translation action on $\mathbb{T}^k$.

Given any α−invariant measure μ on M we define a measure $\mu_{\mathcal{S}}$ on $\mathcal{S}$ as follows. For each $x \in \mathbb{T}^k$ we choose some $\mathbf{s} \in \mathbb{R}^k$ such that $\mathbf{s} + \mathbb{Z}^k = x$, we then obtain a map $\iota_{\mathbf{s}} : M \to \pi_{\mathcal{S}}^{-1}(x) \subset \mathcal{S}$ defined by $\iota_{\mathbf{s}}(y) = [y, \mathbf{s}]$. Define a measure μ_x on $\pi_{\mathcal{S}}^{-1}(x)$ by

$$(5.4) \qquad\qquad\qquad\qquad (\iota_{\mathbf{s}})_*\mu = \mu_x.$$

Given any $\mathbf{n} \in \mathbb{Z}^k$ we have $\iota_{\mathbf{s}+\mathbf{n}}(y) = [y, \mathbf{s}+\mathbf{n}] = [\alpha^{\mathbf{n}}y, \mathbf{s}] = \iota_{\mathbf{s}}\alpha^{\mathbf{n}}y$. So α−invariance of μ implies

$$(\iota_{\mathbf{s}+\mathbf{n}})_*\mu = (\iota_{\mathbf{s}})_*\alpha_*^{\mathbf{n}}\mu = (\iota_{\mathbf{s}})_*\mu$$

showing that μ_x is well-defined. Define a suspended measure $\mu_{\mathcal{S}}$ by

$$(5.5) \qquad\qquad\qquad\qquad \mu_{\mathcal{S}} = \int_{\mathbb{T}^k} \mu_x \mathrm{dvol}_{\mathbb{T}^k}(x).$$

One checks that $\mu_{\mathcal{S}}$ is $\alpha_{\mathcal{S}}$−invariant.

In the remainder of this section, we denote by $\mathcal{S}_0$ the suspension of ρ with action $\rho_{\mathcal{S}_0} : \mathbb{R}^k \times \mathcal{S}_0 \to \mathcal{S}_0$ and by $\mathcal{S}$ the suspension of F with action $F_{\mathcal{S}} : \mathbb{R}^k \times \mathcal{S} \to \mathcal{S}$. We also denote by $\mu_{\mathcal{S}_0}$ the suspension of μ_Γ and $\nu_{\mathcal{S}}$ the suspension of ν. Note that $\mu_{\mathcal{S}_0}$ is a volume on $\mathcal{S}_0$. Let $Q : \mathbb{R}^k \to \mathrm{Aut}(G)$ be a homomorphism such that

$$(5.6) \qquad\qquad\qquad\qquad Q|_{\mathbb{Z}^k} = \rho.$$

Suspend the actions $T_{[\chi]}^g$ and $\eta_{[\chi]}^g$ by

$$(5.7) \qquad \hat{T}_{[\chi]}^g\left([x, \mathbf{s}]\right) = \left[T_{[\chi]}^{Q^{-\mathbf{s}}g}x, \mathbf{s}\right], \quad \hat{\eta}_{[\chi]}^g\left([x, \mathbf{s}]\right) = \left[\eta_{[\chi]}^{Q^{-\mathbf{s}}g}x, \mathbf{s}\right].$$

For any $\mathbf{n} \in \mathbb{Z}^k$ we have

$$(5.8) \qquad \hat{T}_{[\chi]}^g([\rho^{\mathbf{n}}x, \mathbf{s} - \mathbf{n}]) = \left[T_{[\chi]}^{Q^{\mathbf{n}}Q^{-\mathbf{s}}g}\rho^{\mathbf{n}}x, \mathbf{s} - \mathbf{n}\right] = \left[\rho^{\mathbf{n}}T_{[\chi]}^{Q^{-\mathbf{s}}g}x, \mathbf{s} - \mathbf{n}\right]$$

so $\hat{T}_{[\chi]}$ is a well-defined action on $\mathcal{S}_0$ that acts in the fibers of $\pi_{\mathcal{S}_0} : \mathcal{S}_0 \to \mathbb{T}^k$. Similarly the action $\hat{\eta}_{[\chi]}$ is well-defined on $\mathcal{S}$. We define $\Phi_{\mathcal{S}} : \mathcal{S} \to \mathcal{S}_0$ by

$$(5.9) \qquad\qquad\qquad \Phi_{\mathcal{S}}([\xi, \mathbf{s}]) = [\Phi_{\mathcal{X}}(\xi), \mathbf{s}].$$

Since $\Phi_{\mathcal{X}}$ semi-conjugates ρ onto F, $\Phi_{\mathcal{S}}$ is well-defined. The following lemma is immediate from our definitions.

Lemma 5.3. *Let $\Phi_\mathcal{S}$, $F_\mathcal{S}$, $\rho_{\mathcal{S}_0}$, $\hat{T}_{[\chi]}$ and $\hat{\eta}_{[\chi]}$ be as above. The following holds*

(i) *the map $\Phi_\mathcal{S} : \mathcal{S} \to \mathcal{S}_0$ is a Hölder fiber bundle with uniformly C^r fibers. In fact, for any $\mathbf{s} \in \mathbb{T}^k$ the restriction $\Phi_\mathcal{S}|_{\pi_\mathcal{S}^{-1}(\mathbf{s})}$ coincides with $\Phi_\mathcal{X}$ using natural identifications of $\pi_\mathcal{S}^{-1}(\mathbf{s}) \cong \mathcal{X}$ and $\pi_{\mathcal{S}_0}^{-1}(\mathbf{s}) \cong X_\Gamma$,*

(ii) *the map $F_\mathcal{S}$ is a C^s cocycle over $\rho_{\mathcal{S}_0}$,*

(iii) *the Lyapunov exponents for $\rho_{\mathcal{S}_0}$ coincide with the Lyapunov exponents of ρ,*

(iv) *the $\nu_\mathcal{S}-$Lyapunov exponents along the fibers of $\Phi_\mathcal{S}$ for $F_\mathcal{S}$ coincide with the Lyapunov exponents of F,*

(v) *the map $\Phi_\mathcal{S}$ conjugates $\hat{\eta}_{[\chi]}$ to $\hat{T}_{[\chi]}$.*

Proof. Point (i) follows from the analogous properties of $\Phi_\mathcal{X}$ since $\Phi_\mathcal{S}$ is defined in the fibers of $\pi_\mathcal{S} : \mathcal{S} \to \mathbb{T}^k$. That $\Phi_\mathcal{S}$ conjugates $F_\mathcal{S}$ to $\rho_{\mathcal{S}_0}$ is immediate from its definition: $\Phi_\mathcal{S}[\xi, \mathbf{s} + \mathbf{t}] = [\Phi_\mathcal{X}(\xi), \mathbf{s} + \mathbf{t}]$. That $F_\mathcal{S}$ is C^s along the fibers of $\Phi_\mathcal{S}$ is immediate since F is C^s along the fibers of $\Phi_\mathcal{X}$ (note that the identifications $\iota_\mathbf{s}(\xi) = [\xi, \mathbf{s}]$, $\mathcal{X} \to \pi_\mathcal{S}^{-1}(\mathbf{s} + \mathbb{Z}^k)$, defines a smooth structure on the fibers of $\Phi_\mathcal{S}$ in which $F_\mathcal{S}$ is uniformly C^s). Points (iii) and (iv) holds for $\mathbb{Z}^k \subset \mathbb{R}^k$, and any functional is determined by its values on a lattice, proving (iii) and (iv). Point (v) is immediate from the definitions and the fact that $\Phi_\mathcal{X}$ conjugate $\eta_{[\chi]}$ to $T_{[\chi]}$. ∎

We define holonomies along the orbits of $\hat{T}_{[\chi]}$ as in Equation 5.3. That is, if $x \in \mathcal{S}_0$, $y = \hat{T}_{[\chi]}^g x$ then

$$(5.10) \qquad \hat{h}_{x,y}^{[\chi]}(\xi) = \hat{\eta}_{[\chi]}^g(\xi).$$

We say that $\nu_\mathcal{S}$ (or the disintegration $\{\nu_{\mathcal{S},x}\}_{x \in \mathcal{S}_0}$) is essentially $[\chi]-$holonomy invariant if there is a $\mu_{\mathcal{S}_0}$ full measure set $Y \subset \mathcal{S}_0$ such that

$$(5.11) \qquad (\hat{h}_{x,y}^{[\chi]})_* \nu_{\mathcal{S},x} = \nu_{\mathcal{S},y}, \quad x, y \in Y.$$

The key fact about the suspension, is that holonomy invariance of $\nu_\mathcal{S}$ implies holonomy invariance of ν. So, by Lemma 5.2, Theorem 5.1 follows by $[\chi]-$holonomy invariance of $\nu_\mathcal{S}$.

Lemma 5.4. *The measure ν is essentially $[\chi]-$holonomy invariant if and only if $\nu_\mathcal{S}$ is essentially $[\chi]-$holonomy invariant.*

Lemma 5.4 is immediate from the following lemma.

Lemma 5.5. *We have $(\Phi_\mathcal{S})_* \nu_\mathcal{S} = \mu_{\mathcal{S}_0}$. For $[x, \mathbf{s}] \in \mathcal{S}_0$ we have*

$$(5.12) \qquad \nu_{\mathcal{S},[x,\mathbf{s}]} = (\iota_\mathbf{s})_* \nu_x.$$

Proof. By construction we have $\nu_{\mathcal{S}} = (\iota_{\mathbf{s}})_* \nu \otimes d\mathbf{s}$, so

$$(\Phi_{\mathcal{S}})_* \nu_{\mathcal{S}} = \int_{\mathbb{T}^k} (\Phi_{\mathcal{S}})_* (\iota_{\mathbf{s}})_* \nu d\mathbf{s} = \int_{\mathbb{T}^k} (\Phi_{\mathcal{S}} \iota_{\mathbf{s}})_* \nu d\mathbf{s} = \int_{\mathbb{T}^k} (\iota_{\mathbf{s}} \Phi_{\mathcal{X}})_* \nu d\mathbf{s} =$$

$$\int_{\mathbb{T}^k} (\iota_{\mathbf{s}})_* \mu d\mathbf{s} = \mu_{\mathcal{S}_0}.$$

If we define $\nu_{\mathcal{S},[x,\mathbf{s}]} := (\iota_{\mathbf{s}})_* \nu_x$, then

(5.13) $$\nu_{\mathcal{S},[\rho^{\mathbf{n}} x, \mathbf{s}-\mathbf{n}]} = (\iota_{\mathbf{s}-\mathbf{n}})_* \nu_{\rho^{\mathbf{n}} x} = (\iota_{\mathbf{s}} \rho^{-\mathbf{n}})_* \nu_{\rho^{\mathbf{n}} x} = (\iota_{\mathbf{s}})_* \nu_x = \nu_{\mathcal{S},[x,\mathbf{s}]}$$

where we have used $\iota_{\mathbf{s}+\mathbf{n}}(x) = [x, \mathbf{s}+\mathbf{n}] = [\rho^{-\mathbf{n}} \rho^{\mathbf{n}} x, \mathbf{s}+\mathbf{n}] = [\rho^{\mathbf{n}} x, \mathbf{s}] = \iota_{\mathbf{s}}(\rho^{\mathbf{n}} x)$.
So $\nu_{\mathcal{S},[x,\mathbf{s}]}$ is well-defined. We calculate

$$\int_{\mathcal{S}_0} \nu_{\mathcal{S},[x,\mathbf{s}]} d\mu_{\mathcal{S}_0}([x,\mathbf{s}]) = \int_{\mathcal{S}_0} (\iota_{\mathbf{s}})_* \nu_x d\mu_{\mathcal{S}_0}([x,\mathbf{s}]) =$$

$$\int_{\mathbb{T}^k} \left[\int_{\iota_{\mathbf{s}} X_\Gamma} (\iota_{\mathbf{s}})_* \nu_x d(\iota_{\mathbf{s}})_* \mu_\Gamma([x,\mathbf{s}]) \right] d\mathbf{s} =$$

$$\int_{\mathbb{T}^k} \left[(\iota_{\mathbf{s}})_* \int_{X_\Gamma} \nu_x d\mu_\Gamma(x) \right] d\mathbf{s} =$$

$$\int_{\mathbb{T}^k} (\iota_{\mathbf{s}})_* \nu d\mathbf{s} = \nu_{\mathcal{S}}$$

which proves that $\nu_{\mathcal{S},[x,\mathbf{s}]}$ is a disintegration of $\nu_{\mathcal{S}}$ over $\Phi_{\mathcal{S}}$. ∎

Proof of Lemma 5.4. If ν is essentially $[\chi]$−holonomy invariant, then we find $Y \subset X_\Gamma$ such that $(h_{x,y}^{[\chi]})_* \nu_x = \nu_y$ for $x, y \in Y$. Letting $\tilde{Y} \subset \mathcal{S}_0$ be the image of $Y \times \mathbb{R}^k$ in $\mathcal{S}_0$, one direction in Lemma 5.4 follows from the formula in Lemma 5.5. For the converse direction, let $Y \subset \mathcal{S}_0$ be such that $(\hat{h}_{x,y})_* \nu_{\mathcal{S},x} = \nu_{\mathcal{S},y}$ for $x, y \in Y$. Since $\mu_{\mathcal{S}_0}(Y) = 1$ we have for $d\mathbf{s}$−almost every $\mathbf{s} \in \mathbb{T}^k$

$$\mu_\Gamma(\iota_{\mathbf{s}}^{-1}(Y \cap \pi_{\mathcal{S}_0}^{-1}(\mathbf{s}))) = 1.$$

For any $\mathbf{s} \in \mathbb{R}^k$, $g \in G^{[\chi]}$ and $x \in X_\Gamma$ we have

$$\iota_{\mathbf{s}} T_{[\chi]}^g(x) = \left[T_{[\chi]}^g(x), \mathbf{s} \right] = \left[T_{[\chi]}^{Q^{-\mathbf{s}} Q^{\mathbf{s}} g}(x), \mathbf{s} \right] = \hat{T}_{[\chi]}^{Q^{\mathbf{s}} g}[x, \mathbf{s}] = \hat{T}_{[\chi]}^{Q^{\mathbf{s}} g} \iota_{\mathbf{s}}(x).$$

Similarly, $\iota_{\mathbf{s}} \eta_{[\chi]}^g = \hat{\eta}_{[\chi]}^{Q^{\mathbf{s}} g} \iota_{\mathbf{s}}$. Since $\iota_{\mathbf{s}}$ maps fibers of $\Phi_{\mathcal{X}}$ to fibers of $\Phi_{\mathcal{S}}$, it follows that $\iota_{\mathbf{s}}^{-1} \hat{h}_{[x,\mathbf{s}],[y,\mathbf{s}]}^{[\chi]} \iota_{\mathbf{s}} = h_{x,y}^{[\chi]}$. With $\tilde{Y} = \iota_{\mathbf{s}}^{-1}(Y \cap \pi_{\mathcal{S}_0}^{-1}(\mathbf{s}))$ and Lemma 5.5

$$(h_{x,y}^{[\chi]})_* \nu_x = \left(\iota_{\mathbf{s}}^{-1} h_{[x,\mathbf{s}],[y,\mathbf{s}]}^{[\chi]} \iota_{\mathbf{s}} \right)_* \nu_x = \left(\iota_{\mathbf{s}}^{-1} h_{[x,\mathbf{s}],[y,\mathbf{s}]}^{[\chi]} \right)_* \nu_{\mathcal{S},[x,\mathbf{s}]} =$$

$$(\iota_{\mathbf{s}}^{-1})_* \nu_{\mathcal{S},[y,\mathbf{s}]} = \nu_y,$$

for $x, y \in \tilde{Y}$. Choosing $\mathbf{s}$ such that $\mu_\Gamma(\tilde{Y}) = 1$, it follows that ν is essentially $[\chi]$−holonomy invariant. ∎

5.2. **Proof of Theorem 5.1.** By Lemmas 5.2 and 5.4 it suffices to show that the disintegration of the suspension of ν is essentially $[\chi]-$holonomy invariant. We use the following general criteria for obtaining holonomy invariance, proved in [3, Proposition 4.2] (or [3, Corollary 4.3]).

Theorem 5.2. *Let $f : M \to M$ be a volume preserving diffeomorphism on a closed, smooth manifold with an invariant contracting smooth foliation W. Let $\mathcal{X} \to M$ be a Hölder fiber bundle with uniformly C^r fibers and $F : \mathcal{X} \to \mathcal{X}$ a map covering f such that F is uniformly C^s along the fibers of $\mathcal{X} \to M$. We assume that W admits holonomies in $\mathcal{X}$, that is for every $y \in W(x)$ there is a map $h_{x,y}^W : \mathcal{X}_x \to \mathcal{X}_y$ and satisfying $(sh1)$, $(sh2)$ and $(sh3)$ in [3, Section 2.4]. Let ν be a $F-$invariant measure on $\mathcal{X}$ projecting onto volume. If the $\nu-$exponent of F along the fibers of $\mathcal{X} \to M$ is 0 then the disintegration of ν is essentially $W-$holonomy invariant. That is, there is a full volume set $Y \subset M$ such that $(h_{x,y}^W)_* \nu_x = \nu_y$ for $x, y \in Y$.*

Proof. The theorem would follow immediately from [3, Proposition 4.2] if the foliation W coincided with the stable foliation of f (in the sense of [3, Section 4.1]). However, following the proof, it suffices that W is contracting. In fact, since W is a contracting foliation it is standard to produce a measurable partition subordinate to W, see for example [44], which simplifies the proof.   ∎

Proof of Theorem 5.1. Let $F_\mathcal{S}$ be the suspension of F and $\rho_{\mathcal{S}_0}$ the suspension of ρ. By assumption we find $\mathbf{t}_0 \subset \mathbb{R}^k \setminus 0$ such that $F_\mathcal{S}^{\mathbf{t}_0}$ has zero exponents along the fibers of $\Phi_\mathcal{S}$ and $[\chi](\mathbf{t}_0) < 0$. If $f = \rho_{\mathcal{S}_0}^{\mathbf{t}_0}$, $F = F_\mathcal{S}^{\mathbf{t}_0}$ and W is the orbit foliation of $\hat{T}_{[\chi]}$, then we apply Theorem 5.2 to conclude that $\nu_\mathcal{S}$ is essentially $[\chi]-$holonomy invariant. The theorem follows from Lemmas 5.2 and 5.4.   ∎

6. INVARIANT STRUCTURE IN THE CENTER DIRECTION

Let $\alpha : \mathbb{Z}^k \times X_\Gamma \to X_\Gamma$ be a smooth action satisfying the assumptions of Theorem A. In this section, we prove Theorem 1.3, that is: f, and α, have a unique measure of maximal entropy. Moreover, if μ is the measure of maximal entropy then $\Phi_* \mu = $ vol and the disintegration of μ is invariant under stable and unstable holonomy. Equivalently [3, 53] we show that the $\mu-$center exponent vanish, $\lambda_\mu^c = 0$. The proof of Theorem 1.3 is by contradiction, so we assume that $\lambda_\mu^c(f) \neq 0$. The proof splits into two cases. First, we have a generic case when the kernel of $\lambda_\mu^c : \mathbb{Z}^k \to \mathbb{R}$ does not coincide with the kernel of some exponent $\chi : \mathbb{Z}^k \to \mathbb{R}$ of ρ. Second, we have an exceptional case when $\ker \lambda_\mu^c = \ker \chi$ for some exponent χ of ρ. The first case is dealt with by using Theorem 5.1 and Lemma 5.1. The second, more technical, case is dealt with by studying the circle dynamics induced by the

holonomy maps on the center leaves. Suppose that λ_μ^c has the same kernel as $[\chi]$. We begin by showing that the $\mathcal{P}_{[\chi]}$−action commute with the $\mathcal{Q}_{[\chi]}$−action. This implies that either $\mathcal{P}_{[\chi]}^c$ or $\mathcal{Q}_{[\chi]}^c$ act transitively on center leaves (see Lemma 6.2). If $\mathcal{Q}_{[\chi]}^c$ act transitively on center leaves then Theorem 5.1 can be applied, as in the generic case. If $\mathcal{P}_{[\chi]}^c$ acts transitively on center leaves, then we show that $\mathcal{Q}_{[\chi]}$ acts minimally on X_Γ. We use the minimality of the $\mathcal{Q}_{[\chi]}$−action, and the fact that $\mathcal{Q}_{[\chi]}$ commute with $\mathcal{P}_{[\chi]}^c$ to produce a continuous $\mathbb{T}$−action preserving W^c that commutes with α. This shows that the exponent λ_μ^c must vanish, a contradiction.

Denote by $\mathcal{M}_{\mathrm{vol}}^f(X_\Gamma)$ the f−invariant measures projecting to volume

$$(6.1) \qquad\qquad \Phi_*\mu = \mathrm{vol}, \quad \mu \in \mathcal{M}^f(X_\Gamma).$$

Equivalently the measures $\mathcal{M}_{\mathrm{vol}}^f(X_\Gamma)$ are precisely the measures of maximal entropy for f [53]. From [53] the set $\mathcal{M}_{\mathrm{vol}}^f(X_\Gamma)$ is finite so we may assume, after possibly dropping to a finite index subgroup, that $\mathcal{M}_{\mathrm{vol}}^f(X_\Gamma)$ consist of α−invariant measures. We assume, for contradiction, that $\lambda_\mu^c(f) \neq 0$ for some $\mu \in \mathcal{M}_{\mathrm{vol}}^f(X_\Gamma)$, which implies $\lambda_\nu^c(f) \neq 0$ for all $\nu \in \mathcal{M}_{\mathrm{vol}}^f(X_\Gamma)$ [53].

Lemma 6.1. *For any two $\mu, \nu \in \mathcal{M}_{\mathrm{vol}}^f(X_\Gamma)$ we have $\ker \lambda_\mu^c = \ker \lambda_\nu^c$ where $\lambda_\mu^c, \lambda_\nu^c : \mathbb{Z}^k \to \mathbb{R}$. Moreover, there is a $\mathrm{vol}-full$ measure set $Y \subset \mathbb{T}^d$ such that for any $y \in Y$ we have $x_1, ..., x_N \in \Phi^{-1}(x)$ with $x_1 < x_2 < ... < x_N$ in the orientation of $\Phi^{-1}(y)$ such that (x_{j-1}, x_{j+1}) is the stable or unstable manifold for some $\nu' \in \mathcal{M}_{\mathrm{vol}}(X_\Gamma)$ in $\Phi^{-1}(y)$.*

Proof. We sketch the construction of measures in [53]. Let $\mu \in \mathcal{M}_{\mathrm{vol}}^f(X_\Gamma)$ and $(\mu_x^c)_{x \in X_\Gamma}$ the disintegration of μ with respect to the center foliation. The measures μ_x^c are atomic μ−almost everywhere, since $\lambda_\mu^c(f) \neq 0$ and W^c has 1−dimensional leaves. Denote by $\mathcal{F}_\mu$ the f−invariant foliation in W^c, the stable foliation if $\lambda_\mu^c(f) < 0$ and the unstable foliation if $\lambda_\mu^c(f) > 0$. Let $Y \subset \mathbb{T}^d$ be such that μ_x^c exists and is atomic for each $x \in \Phi^{-1}(y)$. For $y \in Y$ let

$$(6.2) \qquad\qquad \mu_x^c = (\delta_{p_1(x)} + ... + \delta_{p_k(x)})/k.$$

We define $q_j(x)$ as the positively oriented end point of $\mathcal{F}_\mu(p_j(x))$. Define a new measure ν by $\nu_y = \delta_{q_1(x)} + ... + \delta_{q_k(x)}$ and $\nu = \nu_y \otimes \mathrm{dvol}(y)$. Then $\Phi_*\nu = \mathrm{vol}$ and ν is ergodic, so $\nu \in \mathcal{M}_{\mathrm{vol}}^f(X_\Gamma)$. For any $\mathbf{n} \in \mathbb{Z}^k$ such that $\lambda_\mu^c(\mathbf{n}) < 0$, $\mathcal{F}_\mu(p_j(x))$ is a stable manifold for $\alpha^{\mathbf{n}}$. If $\lambda_\nu^c(\mathbf{n}) < 0$, then any point in $(p_j(x), q_j(x))$ lie in the stable manifold for both $p_j(x)$ and $q_j(x)$, which is a contradiction. It follows that $\lambda_\mu^c(\mathbf{n}) < 0$ implies $\lambda_\nu^c(\mathbf{n}) \geq 0$, or $\lambda_\mu^c = -c\lambda_\nu^c$ for some $c > 0$ (note that $c \neq 0$ since there are no measure $\nu \in \mathcal{M}_{\mathrm{vol}}^f(X_\Gamma)$ with zero center exponent [53]). The first part of the lemma follows for measures constructed as above. Since $\#\mathcal{M}_{\mathrm{vol}}(X_\Gamma) < \infty$ [53] the construction of new invariant measures outlined above can only produce new measures finitely many times. This proves the last part of the lemma, since

if the invariant manifolds did not cover the center leaves, then we could proceed the construction. This also proves the first part of the lemma since the measures constructed above have invariant manifolds covering the entire center leaves for vol−almost every $\Phi^{-1}(y)$. ∎

6.1. Generic case of Theorem 1.3.

Let $\mu \in \mathcal{M}_{\mathrm{vol}}^{f}(X_\Gamma)$. If $\ker \lambda_\mu^c$ does not coincide with $\ker[\chi]$ for any coarse exponent of ρ, then $\eta_\mathcal{P}|_{E_0^{[\chi]}}$ preserve μ for all coarse $[\chi]$ by Theorem 5.1 and Lemma 5.1. So, μ is $\eta_\mathcal{P}$−invariant. Accessibility of f implies that $\eta_\mathcal{P}$ acts transitively, which implies that the disintegration of μ, μ_x^c, can not be atomic. This is a contradiction since we assumed that $\lambda_\mu^c(f) \neq 0$.

6.2. Exceptional case of Theorem 1.3.

Now we deal with the exceptional case of Theorem 1.3. We fix $\mu \in \mathcal{M}_{\mathrm{vol}}^{f}(X_\Gamma)$ and assume that $\lambda_\mu^c = r\chi$ for some Lyapunov exponent χ of $\rho^{\mathbf{n}}$. Denote by $[\chi]$ the corresponding coarse Lyapunov exponent. We will need two preliminary result on circle maps.

Lemma 6.2. *If $G, H \subset \mathrm{Homeo}_+(\mathbb{T})$ are two path-connected groups such that $GH.x = HG.x = \mathbb{T}$ for all $x \in \mathbb{T}$ then either $G.x = \mathbb{T}$ or $H.x = \mathbb{T}$.*

Proof. We have $G.x = (a,b)$, $G.x = (a,b]$, $G.x = [a,b)$ or $G.x = [a,b]$ since G is path-connected. We write $I = G.x$. If $I \neq \mathbb{T}$ then we claim that a and b are fixed by G. Indeed, if $g \in G$ then $g(I) = I$ and since g is an orientation preserving homeomorphism, g fix the endpoints of I. So, $G.a = a$ and $G.b = b$. If $H.x \subset I$ then $\mathbb{T} = GH.x = G.x = I \neq \mathbb{T}$, so $H.x \not\subset I$. Since H is path-connected $J = H.x$ is an interval that contain x and $J \not\subset I$, so $a \in H.x$ or $b \in H.x$. Assume that $a \in H.x$. Since a is fixed by G we have $\mathbb{T} = HG.a = H.a \subset H.x$ so $H.x = \mathbb{T}$. ∎

Lemma 6.3. *Let $K \subset \mathbb{T}$ be compact such that*

    (i) there is a subgroup $\mathcal{G} \subset \mathrm{Homeo}_+(\mathbb{T})$ acting transitively, with subgroup $\mathcal{G}_K = \mathrm{Stab}_K(\mathcal{G}) = \{g \in \mathcal{G}_K : gK = K\}$,
    (ii) if $x \in K$, $g \in \mathcal{G}$ satisfy $gx \in K$ then $g \in \mathcal{G}_K$,
    (iii) if $g \in \mathcal{G}_K$ satisfy $gx = x$ for $x \in K$ then $g|_K = \mathrm{id}_K$,
    (iv) there is a compact subset $\mathcal{G}_0 \subset \mathcal{G}$ such that $\mathcal{G}_0 x = \mathbb{T}$ for every $x \in \mathbb{T}$.

Then $K = \mathbb{T}$ or K is finite.

Proof. The group $\mathcal{G}_K$ act transitively and freely on K by (i), (ii) and (iii). Combining this with (iv) it follows that $\mathcal{G}_K$ is a compact group, so $\mathcal{G}_K$ preserves a measure ν on $\mathbb{T}$. From Lemma 3.7 it follows that the rotation number $\omega : \mathcal{G}_K \to \mathbb{T}$ is a homomorphism. If $\omega(g) = 0$ for $g \in \mathcal{G}_K$, then g fix a point in K by Lemma 3.6, so by (iii) we have $g|_K = \mathrm{id}_K$. It follows that, if we view $\mathcal{G}_K \subset \mathrm{Homeo}(K)$,

$\omega : \mathcal{G}_K \to \mathbb{T}$ is injective. The image $T = \omega(\mathcal{G}_K)$ is compact, so either $T = \mathbb{T}$ or $\#T < \infty$. In the second case, since $\omega|_{\mathcal{G}_K}$ is injective, we have $\#K = \#\mathcal{G}_K < \infty$. In the first case, $\omega : \mathcal{G}_K \to \mathbb{T}$ is injective and surjective, so a homeomorphism. It follows that K is homeomorphic to a circle, so $K \hookrightarrow \mathbb{T}$ is a local homeomorphism by invariance of domain. In particular, K is both open and closed. Since $\mathbb{T}$ is connected it follows that $K = \mathbb{T}$. ∎

Lemma 6.4. *The action of $\mathcal{P}_{[\chi]}$ commute with the action of $\mathcal{Q}_{[\chi]}$.*

Proof. By induction it suffices to consider $v \in E_0^{\pm[\chi]}$ and $w \in E_0^{[\eta]}$ for $[\eta]$ independent of $[\chi]$. Assume $v \in E_0^{[\chi]}$, the other case is identical. Let $g = \eta_\mathcal{P}^v \eta_\mathcal{P}^w \eta_\mathcal{P}^{-v} \eta_\mathcal{P}^{-w}$. Fix

$$\mathcal{C}_{v,w}^- = \left\{ \mathbf{n} \in \mathbb{Z}^2 : \lim_{\ell \to \infty} \frac{1}{\ell} \log \left\| \rho^{\ell\mathbf{n}} v \right\|, \frac{1}{\ell} \log \left\| \rho^{\ell\mathbf{n}} w \right\| \to 0 \right\}$$

which is a non-empty cone since $[\chi]$ and $[\eta]$ are independent. Let $Y \subset \mathbb{T}^d$ be the full measure set from Lemma 6.1. Given $y \in Y$ let $x \in \Phi^{-1}(y)$. We have

$$\alpha^{\ell\mathbf{n}} g x = \alpha^{\ell\mathbf{n}} \eta_\mathcal{P}^v \eta_\mathcal{P}^w \eta_\mathcal{P}^{-v} \eta_\mathcal{P}^{-w} x =$$

$$\eta_\mathcal{P}^{\rho^{\ell\mathbf{n}} v} \eta_\mathcal{P}^{\rho^{\ell\mathbf{n}} w} \eta_\mathcal{P}^{-\rho^{\ell\mathbf{n}} v} \eta_\mathcal{P}^{-\rho^{\ell\mathbf{n}} w} \alpha^{\ell\mathbf{n}} x.$$

The action $\eta_\mathcal{P}$ is Hölder by Lemma 4.2, so from or choice of $\mathcal{C}_{v,w}^-$ there is uniform $\kappa > 0$ such that for any $\mathbf{n} \in \mathcal{C}_{v,w}^-$

$$\limsup_{\ell \to \infty} \frac{1}{\ell} \log d_c \left(\alpha^{\ell\mathbf{n}} x, \alpha^{\ell\mathbf{n}} g x \right) \leq \kappa \min_{[\tilde{\eta}]=[\eta],[\tilde{\chi}]=[\chi]} (\tilde{\chi}(\mathbf{n}) + \tilde{\eta}(\mathbf{n})) \leq$$

$$\kappa \min_{[\tilde{\eta}]=[\eta]} \tilde{\eta}(\mathbf{n})$$

where the last inequality is immediate since $[\chi](\mathbf{n}) < 0$. In particular, gx lie in the stable manifold of x for any $\mathbf{n} \in \mathcal{C}_{v,w}^-$. Assume for contradiction that $gx \neq x$. Since $gx, x \in W^c(x)$ lie in the same stable manifold, if $x \neq gx$ then Lemma 6.1 implies

$$\limsup_{\ell \to \infty} \frac{1}{\ell} \log d_c \left(\alpha^{\ell\mathbf{n}} x, \alpha^{\ell\mathbf{n}} g x \right) = \lambda_\nu^c(\mathbf{n})$$

for some $\nu \in \mathcal{M}_{\text{vol}}^f(X_\Gamma)$. So, $\lambda_\nu^c(\mathbf{n}) \leq \kappa \min_{[\tilde{\eta}]=[\eta]} \tilde{\eta}(\mathbf{n})$ for any $\mathbf{n} \in \mathcal{C}_{v,w}^-$. Since $[\chi]$ is independent of $[\eta]$ we find $\mathbf{n}_j \in \mathcal{C}_{v,w}^-$ such that $\tilde{\chi}(\mathbf{n}_j) \to 0$ for any $[\tilde{\chi}] = [\chi]$. By independence we may also choose $\mathbf{n}_j$ such that

$$\lim_{j \to \infty} \min_{[\tilde{\eta}]=[\eta]} \tilde{\eta}(\mathbf{n}_j) \to -\infty.$$

Since λ_ν^c has the same kernel as $[\chi]$ we also have $\lambda_\nu^c(\mathbf{n}_j) \to 0$. This is a contradiction since

$$0 = \lim_{j \to \infty} \lambda_\nu^c(\mathbf{n}_j) \leq \lim_{j \to \infty} \kappa \min_{[\tilde{\chi}]=[\chi]} \tilde{\eta}(\mathbf{n}_j) = -\infty.$$

We conclude that $gx = x$. Since $gx = x$ holds on a dense set, and g is continuous, we have $g = \mathrm{id}_{X_\Gamma}$. ∎

By Lemmas 6.2 and 6.4 either $\mathcal{P}^c_{[\chi]}$ or $\mathcal{Q}^c_{[\chi]}$ must act transitively on every $W^c(x)$. We will prove that neither $\mathcal{P}^c_{[\chi]}$ or $\mathcal{Q}^c_{[\chi]}$ can act transitively on $W^c(x)$, which is a contradiction.

Lemma 6.5. *The group $\mathcal{Q}^c_{[\chi]}$ can not act transitively on any $W^c(x)$.*

Proof. Let $Y \subset X_\Gamma$ be the subset where $\mathcal{Q}^c_{[\chi]}$ act transitively on $W^c(y)$, $y \in Y$. Since $\mathcal{Q}^c_{[\chi]}$ is normal in $\mathcal{Q}_{[\chi]}$ (it is the kernel of a homomorphism) it is immediate that Y is $\mathcal{Q}_{[\chi]}$−invariant. But, by Lemma 6.4 the set Y is also $\mathcal{P}_{[\chi]}$−invariant. Using Lemma 6.4 again, it follows that Y is $\mathcal{P}$-invariant. Since f is accessible it follows that, either $Y = \emptyset$ or $Y = X_\Gamma$.

Assume for contradiction that $Y \neq \emptyset$, so $Y = X_\Gamma$. The remainder of the proof is similar to the proof of the generic case of Theorem 1.3. Indeed, by Theorem 5.1 and Lemma 5.1 the action $\mathcal{Q}^c_{[\chi]}$ preserve μ. Since $\mathcal{Q}^c_{[\chi]}$ act transitively on every $W^c(x)$, then the disintegration of μ, μ^c_x, is not atomic. This is a contradiction, so we conclude that $Y = \emptyset$. ∎

Before proceeding, we define the space

$$(6.3) \qquad\qquad V = \Pi \mathcal{Q}_{[\chi]} = \bigoplus_{\eta \neq \pm[\chi]} E_0^{[\eta]}$$

and the associated translation action on $\mathbb{T}^d$ by $R_v(x) = x + v$. By Lemma 2.2, R_v is a minimal action.

Lemma 6.6. *The group $\mathcal{P}^c_{[\chi]}$ can not act transitively on any $W^c(x_0)$.*

As in the proof of Lemma 6.5 we may assume that $\mathcal{P}^c_{[\chi]}$ act transitively on every $W^c(x)$, we will do this in the remainder. We split the proof of Lemma 6.6 into parts. We begin by proving that $\mathcal{Q}_{[\chi]}$ act minimally.

Given $x \in X_\Gamma$ we define

$$(6.4) \qquad\qquad K_x := \overline{\{\eta^w_{\mathcal{P}} x \, : \, w \in \mathcal{Q}_{[\chi]}\}}.$$

First, we show that K_x is a minimal set for the $\mathcal{Q}_{[\chi]}$−action for every $x \in X_\Gamma$.

Lemma 6.7. *For any $y \in K_x$ we have*

$$(6.5) \qquad\qquad K_x = \overline{\{\eta^w_{\mathcal{P}} y \, : \, w \in \mathcal{Q}_{[\chi]}\}},$$

that is K_x is a minimal set for the $\mathcal{Q}_{[\chi]}$−action. In particular, the set $\{K_x\}_x$ form a partition of X_Γ.

Proof. Define $Y \subset X_\Gamma$ as those $x \in X_\Gamma$ such that K_x is minimal for the $\mathcal{Q}_{[\chi]}$-action. A standard application of Zorn's lemma shows that there is some compact set $M \subset X_\Gamma$ such that M is minimal for the $\mathcal{Q}_{[\chi]}$-action. Since $M \subset Y$ it follows that Y is non-empty. Given any $w \in \mathcal{Q}_{[\chi]}$ we have $\eta_\mathcal{P}^w K_x \subset K_x$, so Y is $\mathcal{Q}_{[\chi]}$-invariant. By Lemma 6.4 the action of $\mathcal{P}_{[\chi]}$ commute with the action of $\mathcal{Q}_{[\chi]}$. So, given $w \in \mathcal{P}_{[\chi]}$ the map $\eta_\mathcal{P}^w : K_x \to \eta_\mathcal{P}^w K_x$ conjugates the $\mathcal{Q}_{[\chi]}$-action on K_x to the $\mathcal{Q}_{[\chi]}$-action on $\eta_\mathcal{P}^w K_x = K_{\eta_\mathcal{P}^w x}$. It follows that $x \in Y$ if and only if $\eta_\mathcal{P}^w x \in Y$. Since Y is $\mathcal{P}_{[\chi]}$-invariant and $\mathcal{Q}_{[\chi]}$-invariant, Lemma 6.4 implies that Y is su-saturated. Accessibility and the fact that $Y \neq \emptyset$ implies that $Y = X_\Gamma$. ∎

Lemma 6.8. *The action of $\mathcal{Q}_{[\chi]}$ is minimal.*

Proof. Since K_x is $\mathcal{Q}_{[\chi]}$-invariant, Lemma 4.3 implies

$$(6.6) \qquad \Phi(K_x) = \Phi(\eta_\mathcal{P}^w K_x) = \Phi(K_x) + \Pi(w), \quad w \in \mathcal{Q}_{[\chi]}$$

so $\Phi(K_x)$ is compact and invariant by the translation action R_v from Equation 6.3. The translation action by V is minimal, so $\Phi(K_x) = \mathbb{T}^d$. It follows that $K_x \cap W^c(y) \neq \emptyset$ for every $y \in X_\Gamma$.

If we let $\mathcal{G} = \eta_\mathcal{P}^{\mathcal{P}_{[\chi]}^c}$ and $K = K_x \cap W^c(y)$ then the assumptions in Lemma 6.3 are satisfied (after identifying $W^c(y) \cong \mathbb{T}$). Indeed, $\mathcal{G}$ act transitively by Lemma 4.6. Property (ii) and (iii) follows from Lemmas 6.4 and 6.7. To show property (iv), we follow [2, Section 8.3]. For $N \in \mathbb{N}$, write

$$(6.7) \qquad \mathcal{P}_{[\chi],N}^c = \{w = v_1 v_2 ... v_N \in \mathcal{P}_{[\chi]}^c : v_j \in E_0^{\pm[\chi]}, \ \|v_j\| \leq N, \ n \leq N\}.$$

That is $\mathcal{P}_{[\chi],N}^c$ consist of those words in $\mathcal{P}_{[\chi]}^c$ with at most N letters, and each letter has length of at most N. The following is immediate from the definition

$$(6.8) \qquad \mathcal{P}_{[\chi],N+1}^c \supset \mathcal{P}_{[\chi],N}^c, \quad \mathcal{P}_{[\chi]}^c = \bigcup_{N \geq 1} \mathcal{P}_{[\chi],N}^c.$$

Let $\mathcal{K}_N = \{\eta_\mathcal{P}^w : w \in \mathcal{P}_{[\chi],N}^c\}$. Let $K_N = \mathcal{K}_N x_0 = \{k(x_0) : k \in \mathcal{K}_N\}$ for fixed $x_0 \in W^c(y)$. Then $K_N \subset K_{N+1}$ is an ascending sequence of compact sets. If $K_{N_0} = W^c(y)$ and $x \in W^c(x)$ then $x_0 \in \mathcal{K}_{N_0} x$ so $\mathcal{K}_{2N_0} x \supset \mathcal{K}_{N_0} x_0 = W^c(y)$. That is, property (iv) of Lemma 6.3 holds if $K_{N_0} = W^c(y)$ for some N_0. Since

$$(6.9) \qquad \bigcup_{N \geq 1} K_N = \{\eta_\mathcal{P}^w x_0 : w \in \mathcal{P}_{[\chi]}^c\} = W^c(y),$$

Baire's category theorem implies that there is N_1 such that $\text{int}(K_N) \neq \emptyset$ for $N \geq N_1$ (where $\text{int}(K_N)$ is the interior in $W^c(y)$). Define $U_N = \text{int}(K_N)$ and $C_N = U_N^c$ (where the complement is in $W^c(y)$). If $N \geq N_1$ then $x_0 \in \mathcal{K}_N U_N \subset U_{2N}$, so $K_N \subset \mathcal{K}_N U_{2N} \subset U_{3N}$. From Equation 6.9 we obtain

$$(6.10) \qquad W^c(y) = \bigcup_{N \geq 1} K_N = \bigcup_{N \geq N_1} K_N \subset \bigcup_{N \geq N_1} U_{3N} = \bigcup_{N \geq 1} U_N$$

or

(6.11) $$\emptyset = \bigcap_{N \geq 1} C_N.$$

Since $C_{N+1} \subset C_N$ is a descending sequence of compact sets it follows that there is N_0 such that $C_{N_0} = \emptyset$. Equivalently $U_{N_0} = K_{N_0} = W^c(y)$, proving that property (iv) of Lemma 6.3 holds. Lemma 6.3 implies that $\#K_x \cap W^c(y) < \infty$ or $K_x \cap W^c(y) = W^c(y)$ for every $x, y \in X_\Gamma$.

We claim that $\#K_x \cap W^c(y)$ is independent of y. Indeed, as in the proof of Lemma 6.7, for any $w \in \mathcal{P}$ we have $\eta_\mathcal{P}^w K_x = K_{\eta_\mathcal{P}^w x}$. So given any $z \in K_x \cap W^c(y)$ we find, by accessibility, some $w \in \mathcal{P}$ that satisfy $\eta_\mathcal{P}^w x = z$ and in extension $\eta_\mathcal{P}^w K_x = K_z = K_x$ (since $z \in K_x$, and $\{K_x\}_{x \in X_\Gamma}$ is a partition, Lemma 6.7). We also have $\eta_\mathcal{P}^w W^c(x) = W^c(y)$ (since $z \in W^c(y)$) so $\eta_\mathcal{P}^w(K_x \cap W^c(x)) = K_x \cap W^c(y)$. It follows that

(6.12) $$\#K_x \cap W^c(y) = \#K_x \cap W^c(x)$$

for all $y \in X_\Gamma$. If $\#K_x \cap W^c(x) = q < \infty$, then the fibers of Φ intersect K_x precisely q times. Given $z_0 \in \mathbb{T}^d$ we define

$$\varepsilon_0 = \inf_{y \neq y' \; y, y' \in \Phi^{-1}(z_0) \cap K_x} d(y, y') > 0.$$

Let $\{y_1, ..., y_q\} = K_x \cap \Phi^{-1}(z_0)$ and $\delta > 0$ small. For $z \in B_\delta(z_0)$, we define $y_j(z)$ as the element in $\Phi^{-1}(z) \cap K_x$ that minimize $d(y_j(z), y_j(z_0))$ (if this does not define a unique point, choose an arbitrary minimizer). If $z_n \to z_0$ then any convergent subsequence of $y_j(z_n)$ converges to an element of $K_x \cap \Phi^{-1}(z_0)$. From our choice of $y_j(z)$ it is clear that any convergent subsequence converge to $y_j(z_0)$. So, $y_j(z_n) \to y_j(z_0)$ and y_j is continuous at z_0. Choose $\delta > 0$ small enough such that $d(y_j(z), y_j(z_0)) < \varepsilon_0/100$ for all $z \in B_\delta(z_0)$ and $j = 1, ..., q$. For $z, z' \in B_\delta(z_0)$ and $i \neq j$ the reverse triangle inequality implies

$$d(y_i(z), y_j(z')) \geq d(y_i(z_0), y_j(z')) - d(y_i(z_0), y_i(z)) \geq$$
$$d(y_i(z_0), y_j(z_0)) - d(y_j(z_0), y_j(z')) - d(y_i(z_0), y_i(z)) >$$
$$\varepsilon_0 - \frac{\varepsilon_0}{50} > \frac{49\varepsilon_0}{50} > 0.$$

where the last inequality use the definition of ε_0. In particular, $y_i(z) \neq y_j(z)$ for every $z \in B_\delta(z_0)$, so $K_x \cap \Phi^{-1}(z) = \{y_1(z), ..., y_q(z)\}$. Moreover, we have

(6.13) $$d(y_i(z), y_i(z')) \leq d(y_i(z), y_i(z_0)) + d(y_i(z_0), y_i(z')) < \frac{\varepsilon_0}{50}.$$

Since $\varepsilon_0/50 < 49\varepsilon_0/50$, $y_i(z')$ is the element $y \in K_x \cap \Phi^{-1}(z')$ that minimize $d(y_i(z), y)$. The same argument that showed continuity at z_0, now show that y_j is continuous at any $z \in B_\delta(z_0)$. So the functions $y_1, ..., y_q : B_\delta(z_0) \to K_x$ are continuous. Note that Φ restricted to $K_x \cap \Phi^{-1}(B_\delta(z_0)) \cap B_{\varepsilon_0}(y_j(z_0))$ has inverse

given by y_j. So, $\Phi : K_x \to \mathbb{T}^d$ is a finite covering map. In particular, K_x is homeomorphic to $\mathbb{T}^d$. It follows that the map

$$(6.14) \qquad (\Phi|_{K_x})_* : \pi_1\mathbb{T}^d \cong \mathbb{Z}^d \to \Gamma \cong \pi_1 X_\Gamma$$

is injective with image of finite index. The map

$$(6.15) \qquad \mathbb{Z}^d \cong \pi_1 K_x \xrightarrow{i_{K_x}} \pi_1 X_\Gamma = \Gamma$$

is injective since its injective after composition with $\Phi_* = \pi_*$. The group $\Gamma' = \mathrm{Im}(i_{K_x})_* \times \ker \Phi_* = \mathrm{Im}(i_{K_x})_* \times [\Gamma, \Gamma]$ has finite index in Γ and is abelian. Since Γ is not virtually abelian, this is a contradiction. We conclude that $K_x \cap W^c(y) = W^c(y)$ for every y which implies that $K_x = X_\Gamma$ for every x. ∎

Lemma 6.9. *The map* $r : \mathcal{P}^c_{[\chi]} \to \mathbb{T}$, $r : w \mapsto \omega(\eta^w_{\mathcal{P}}|_{W^c(x)})$ *is a well-defined surjective homomorphism. Moreover,* $r(w) = 0$ *if and only if* $\eta^w_{\mathcal{P}} = \mathrm{id}_{X_\Gamma}$.

Proof. We fix $w \in \mathcal{P}^c_{[\chi]}$, $g = \eta^w_{\mathcal{P}}$. Let $\varphi : \mathbb{T}^d \to \mathbb{T}$ be defined by

$$(6.16) \qquad \varphi(y) := \omega(g|_{\Phi^{-1}(y)}).$$

The function φ is continuous since rotation numbers vary continuously in the C^0−topology [40, Proposition 11.1.6]. Given any $w \in \mathcal{Q}_{[\chi]}$ we have $g\eta^w_{\mathcal{P}} = \eta^w_{\mathcal{P}} g$, by Lemma 6.4, and $\Phi\eta^w_{\mathcal{P}} = R_{\Pi(w)}\Phi$ by Lemma 4.3. Let V be the space defined in Equation 6.3. Given any $v \in V$ let $w \in \mathcal{Q}_{[\chi]}$ be such that $\Pi(w) = v$. We have

$$\varphi(R_v(x)) = \omega(g|_{\Phi^{-1}(R_v(x))}) = \omega(g|_{\eta^w_{\mathcal{P}}\Phi^{-1}(x)}) =$$
$$\omega(\eta^w_{\mathcal{P}} g \eta^{w^{-1}}_{\mathcal{P}}|_{\eta^w_{\mathcal{P}}\Phi^{-1}(x)}) = \omega(g|_{\Phi^{-1}(x)}) = \varphi(x)$$

where the second to last equality uses that the rotation number is conjugacy invariant [40, Proposition 11.1.3]. Since φ is invariant of the translation action of V and the translation action of V is minimal, φ is constant. That is, $g : W^c(x) \to W^c(x)$ has a rotation number independent of $x \in X_\Gamma$. This shows that $r : \mathcal{P}^c_{[\chi]} \to \mathbb{T}$ is well-defined. If $r(w) = 0$ then $\eta^w_{\mathcal{P}}$ fix some x, so Lemmas 6.4 and 6.8 implies that $\eta^w_{\mathcal{P}} = \mathrm{id}_{X_\Gamma}$.

Since $H = \eta_{\mathcal{P}}|_{\mathcal{P}^c_{[\chi]}} \cong \mathcal{P}^c_{[\chi]}/\{w \in \mathcal{P}^c_{[\chi]} : \eta^w_{\mathcal{P}} = \mathrm{id}_{X_\Gamma}\}$ act transitively and freely on each $W^c(x)$, it follows that H is homeomorphic to $W^c(x) \cong \mathbb{T}$. So, G is a compact topological group. Since H is compact the H−action preserve a measure. That r is a homomorphism follows from Lemma 3.7. The image $r(H) \subset \mathbb{T}$ is a compact subgroup, so it is either finite or $\mathbb{T}$. If $r(H)$ is finite then every orbit is finite since $r(w) = 0$ implies that $\eta^w_{\mathcal{P}} = \mathrm{id}_{X_\Gamma}$. But H act transitively on every $W^c(x)$, so the H−orbits can not be finite. We conclude that $r(H) = \mathbb{T}$. ∎

We can now prove Lemma 6.6.

Proof of Lemma 6.6. Suppose that $\mathcal{P}^c_{[\chi]}$ act transitively on some $W^c(x_0)$. By Lemma 6.9 we can define an action $\eta_c : \mathbb{T} \times X_\Gamma \to X_\Gamma$ by $\eta^s_c x = \eta^w_{\mathcal{P}} x$ with w chosen such that $r(w) = s$. The action η_c is free, and transitive on the center leaves. Moreover, for $\mathbf{n} \in \mathbb{Z}^k$ and $w \in \mathcal{P}^c_{[\chi]}$ we have

$$(6.17) \qquad\qquad \eta^{\rho^{\mathbf{n}}w}_{\mathcal{P}} = \alpha^{\mathbf{n}} \eta^w_{\mathcal{P}} \alpha^{-\mathbf{n}}$$

so by conjugacy invariance of the rotation number, $r(\rho^{\mathbf{n}}w) = r(w) = s$. It follows

$$(6.18) \qquad\qquad \eta^s_c = \eta^{\rho^{\mathbf{n}}w}_{\mathcal{P}} = \alpha^{\mathbf{n}} \eta^w_{\mathcal{P}} \alpha^{-\mathbf{n}} = \alpha^{\mathbf{n}} \eta^s_c \alpha^{-\mathbf{n}}$$

which proves that α commute with η_c. This implies that α have vanishing Lyapunov exponent along W^c, which is a contradiction. We conclude that $\mathcal{P}^c_{[\chi]}$ can not act transitively on $W^c(x_0)$. ∎

We finish the proof of Theorem 1.3 in the exceptional case.

Proof of Theorem 1.3 in exceptional case. By Lemma 6.4 the action of $\mathcal{P}^c_{[\chi]}$ commute with the action of $\mathcal{Q}^c_{[\chi]}$. Since f is accessible $\eta^{\mathcal{P}^c_{[\chi]}}_{\mathcal{P}} \eta^{\mathcal{Q}^c_{[\chi]}}_{\mathcal{P}} x = W^c(x)$ for all $x \in X_\Gamma$. By Lemma 6.2 either $\mathcal{P}^c_{[\chi]}$ or $\mathcal{Q}^c_{[\chi]}$ act transitively on $W^c(x)$. Lemma 6.6 shows that $\mathcal{P}^c_{[\chi]}$ does not act transitively on $W^c(x)$ so $\mathcal{Q}^c_{[\chi]}$ must act transitively on $W^c(x)$. This is a contradiction by Lemma 6.5. ∎

7. Compatible algebraic structure

Let $\alpha : \mathbb{Z}^k \times X_\Gamma \to X_\Gamma$ and $f : X_\Gamma \to X_\Gamma$ be as in the previous section. From Theorem 1.3 there is a unique $\alpha-$invariant measure μ that projects to volume, and this measure satisfies $\lambda^c_\mu = 0$. We use this measure to construct an algebraic structure on X_Γ that is compatible with $f : X_\Gamma \to X_\Gamma$. This algebraic structure is then used to produce a topological conjugacy between α and an affine action α_0, Theorem 7.1.

We begin by using the measure μ to construct a circle action $\eta_c : \mathbb{T} \times X_\Gamma \to X_\Gamma$ that commutes with α and $\eta_{\mathcal{P}}$, preserves center leaves and acts transitively and freely on each center leaf. Moreover, if $w \in \mathcal{P}^c$ then $\eta^w_{\mathcal{P}} = \eta^t_c$ for some $t \in \mathbb{T}$. This shows that the action of $\mathcal{P}$ on X_Γ factor through a nilpotent group with base $E^u_0 \oplus E^s_0$ and center isomorphic to $\mathbb{T}$. Lifting this action to G we obtain a transitive free action of a $2-$step nilpotent group, N, on G.

Using that $\lambda^c_\mu = 0$, the following two lemma are proved as in Lemmas 6.4 and 6.9.

Lemma 7.1. *Given two independent course Lyapunov path groups $\mathcal{P}_{[\chi]}$ and $\mathcal{P}_{[\eta]}$ with $w_1 \in \mathcal{P}_{[\chi]}$ and $w_2 \in \mathcal{P}_{[\eta]}$ then*

$$(7.1) \qquad \eta_{\mathcal{P}}^{w_1} \eta_{\mathcal{P}}^{w_2} = \eta_{\mathcal{P}}^{w_2} \eta_{\mathcal{P}}^{w_1}.$$

Lemma 7.2. *For any $w \in \mathcal{P}^c$ the rotation number of*

$$(7.2) \qquad \eta_{\mathcal{P}}^{w} : W^c(x) \to W^c(x)$$

is independent of x. The map $r(w)$ mapping $w \in \mathcal{P}^c$ to the rotation number of $\eta_{\mathcal{P}}^{w}$ is a homomorphism with kernel $\{w \in \mathcal{P}^c : \eta_{\mathcal{P}}^{w} = \mathrm{id}_{X_\Gamma}\}$.

Using Lemmas 7.1 and 7.1 we produce an $N-$action, with N nilpotent, on G.

Lemma 7.3. *The action of $\mathcal{P}$ on G factor through a nilpotent Lie group N that acts transitively and freely on G, the action of N descends to an action on X_Γ.*

Proof. We denote by N the image of $\mathcal{P}$ in $\mathrm{Homeo}(G)$, and $\tilde{N}$ the image of $\mathcal{P}$ in $\mathrm{Homeo}(X_\Gamma)$. We begin by showing that the $\mathcal{P}^c-$action on X_Γ factor through an abelian group. As in the proof of Lemma 6.6, we define $\eta_c^s : X_\Gamma \to X_\Gamma$ by $\eta_c^s = \eta_{\mathcal{P}}^{w}$ for any $w \in \mathcal{P}^c$ that satisfies $\omega(\eta_{\mathcal{P}}^{w}|_{W^c(x)}) = s$. By Lemma 7.2 η_c is well-defined, continuous and acts transitively and freely on each $W^c(x)$.

As in the proof of Lemma 6.6, it follows that α commute with η_c. Let r be from Lemma 7.2. Given $w \in \mathcal{P}$ and $w_c \in \mathcal{P}^c$ we have $r(w w_c w^{-1}) = r(w)$ since $\eta_{\mathcal{P}}^{w} \eta_{\mathcal{P}}^{w_c} (\eta_{\mathcal{P}}^{w})^{-1}$ has the same rotation number as $\eta_{\mathcal{P}}^{w}$. It follows that

$$(7.3) \qquad \eta_c^{r(w_c)} = \eta_{\mathcal{P}}^{w} \eta_{\mathcal{P}}^{w_c} (\eta_{\mathcal{P}}^{w})^{-1} = \eta_{\mathcal{P}}^{w} \eta_c^{r(w_c)} (\eta_{\mathcal{P}}^{w})^{-1}$$

which shows that η_c commute with $\eta_{\mathcal{P}}$. We let $\tilde{r} : \mathcal{P}^c \to \mathbb{R}$ be defined by $\tilde{r}(w) = \tilde{\omega}(\eta_{\mathcal{P}}^{w})$ where $\tilde{\omega}(\eta_{\mathcal{P}}^{w}) \in \mathbb{R}$ is the rotation number of $\eta_{\mathcal{P}}^{w}$ with respect to the natural lift to G from Definition 4.3. Lift η_c to G, and note that $\eta_{\mathcal{P}}^{w} = \eta_c^{\tilde{r}(w)}$. It is immediate from the analogous properties of $\eta_c : \mathbb{T} \times X_\Gamma \to X_\Gamma$ that $\eta_c : \mathbb{T} \times G \to G$ commute with α and $\eta_{\mathcal{P}}$.

Denote by $N \subset \mathrm{Homeo}(G)$ the image of $\mathcal{P}$ under $\eta_{\mathcal{P}}$. We denote by $N^c \subset N$ the image of $\mathcal{P}^c$. Since η_c commute with $\eta_{\mathcal{P}}$ and since $\eta_{\mathcal{P}}^{w} = \eta_{\mathcal{P}}^{\tilde{r}(w)}$ for $w \in \mathcal{P}^c$, it holds that N^c is central in N. Moreover, Lemma 4.1 implies

$$(7.4) \qquad \mathcal{P}^c = [\mathcal{P}, \mathcal{P}]$$

so $N^c = [N, N]$. It follows that N is 2−step nilpotent. By Lemma 4.1 we can write any $w \in \mathcal{P}$ as $w = w_c v_s v_u$ with $v_\sigma \in E_0^\sigma$, $\sigma = s, u$, and $w_c \in \mathcal{P}^c$. It follows that $\eta_{\mathcal{P}}^{w} = \eta_c^{\tilde{r}(w_c)} \eta_s^{v_s} \eta_u^{v_u}$, so $T : E_0^s \times E_0^u \times \mathbb{R} \to N$, $T(v_s, v_u, t) = \eta_c^{t} \eta_s^{v_s} \eta_u^{v_u}$ is surjective. If $\eta_c^{t} \eta_s^{v_s} \eta_u^{v_u} = \eta_c^{t'} \eta_s^{v_s'} \eta_u^{v_u'}$ then we apply Φ and obtain $v_s = v_s'$, $v_u = v_u'$, Lemma 4.3. If $v_s = v_s'$, $v_u = v_u'$ then we can simplify and obtain $\eta_c^{t} = \eta_c^{t'}$. Since η_c is a free action this implies $t = t'$. So $T : E_0^s \times E_0^u \times \mathbb{R} \to N$ is bijective. It is also

clear that T is continuous. Let (v_s, v_u, t) and (v_s', v_u', t') be such that $T(v_s, v_u, t)$ is close to $T(v_s', v_u', t')$. After applying Φ, it is immediate that v_s is close to v_s' and v_u is close to v_u'. Writing $h = T(v_s, v_u, t)T(v_s', v_u', t')^{-1}$ we obtain

$$ h = \eta_s^{v_s}\eta_u^{v_u - v_u'}\eta_s^{v_s'}\eta_c^{t-t'}, \quad \eta_c^{t-t'} = \eta_s^{-v_s'}\eta_u^{-v_u+v_u'}\eta_s^{-v_s}h $$

so $\eta_c^{t-t'}$ is close to id_G. Since η_c act freely, it follows that $t - t'$ is close to 0. It follows that T has a continuous inverse, so T is a homeomorphism. Since N is a topological group, homeomorphic to $E_0^s \times E_0^u \times \mathbb{R}$, N has a unique structure as a Lie group [46, 29]. ∎

Theorem 7.1. *The diffeomorphism $f : X \to X$ is bi-Hölder conjugate to some affine map $f_0 : X \to X$ where $f_0(x) = L(x)z_0$ with $z_0 \in G^c$.*

Proof. Let N be the group from Lemma 7.3 and F a lift of f. For $w \in \mathcal{P}$ we have $F\eta_{\mathcal{P}}^w F^{-1} = \eta_{\mathcal{P}}^{L_{su}w}$ by Lemma 4.7. It follows that $\tilde{L}(n) = FnF^{-1} \in N$ for every $n \in N$. So, $\tilde{L} : N \to N$ is an autormophism such that $Fn = \tilde{L}(n)F$ for $n \in N$.

Define $\Lambda \subset N$ by $\lambda \in \Lambda$ if $\lambda(e) \in \Gamma$ for the identity element $e \in G$. Since any two $\lambda, \lambda' \in \Lambda$ are lifted from X_Γ we have some $\gamma \in \Gamma$ such that $\lambda'(\lambda(e)) = \lambda'(\gamma) = \gamma\lambda'(e) \in \Gamma$. Similarly if $\lambda \in \Lambda$ then $\lambda^{-1}(\lambda(e)) = \lambda(e)\lambda^{-1}(e)$ so $\lambda^{-1}(e) = (\lambda(e))^{-1} \in \Gamma$. It follows that Λ is a subgroup of N. Moreover, Γ is closed in G so, since the action of N is continuous, $\Lambda \subset N$ is closed.

The map $q : N \ni h \mapsto h(e) \in G$ is a homeomorphism. Indeed, q is bijective so it is a homeomorphism by invariance of domain. Let $h \in N$ and $\lambda \in \Lambda$. Then

$$ q(h\lambda) = h(\lambda(e)) = \lambda(e)h(e) = \lambda(e)q(h) = q(\lambda)q(h) $$

so q descends to a map $N/\Lambda \to \Gamma \backslash G = X_\Gamma$. If $q(h) = \gamma q(h')$ for some $h, h' \in N$ and $\gamma \in \Gamma$ then we find some $\lambda \in \Lambda$ such that $\gamma = \lambda(e)$ (since the action of N on G is transitive). It follows that

$$ q(h) = \lambda(e)q(h') = \lambda(e)h'(e) = h'(\lambda(e)) = q(h'\lambda) $$

or $h(e) = h'(\lambda(e))$. The action of N is free, so $h = h'\lambda$. That is, we have a diagram

$$
\begin{array}{ccc}
N & \xrightarrow{\ q\ } & G \\
\downarrow & & \downarrow \\
N/\Lambda & \xrightarrow{\ q\ } & X_\Gamma
\end{array}
$$

where both horizontal maps are homeomorphisms. The group N is a nilpotent Lie group by Lemma 7.3, so N/Λ is a nilmanifold. It follows that N is isomorphic to G as a Lie group and Λ is isomorphic to Γ under this map $N \to G$ [48].

We claim that the induced map $\tilde{f} : N/\Lambda \to N/\Lambda$, $\tilde{f} = q^{-1} \circ f \circ q$ is affine. Equivalently, $\tilde{F} : N \to N$, $\tilde{F} = q^{-1} \circ F \circ q$ is affine. From the relation $Fn = (\tilde{L}n)F$, $n \in N$, we obtain

$$q\left(\tilde{F}(n)\right) = F(q(n)) = F(n(e)) = \tilde{L}(n)(fe) = \tilde{L}(n)\left(n_0(e)\right) = q\left(\tilde{L}(n) \cdot n_0\right)$$

where $n_0 \in N$ is chosen such that $n_0(e) = Fe$. It follows that $\tilde{F}$, and therefore also $\tilde{f}$, is affine.

Denote by $H : X_\Gamma \to X_\Gamma$ the conjugacy such that $H(fx) = f_0(h(x)) = L(H(x))z_0$. After conjugating with a translation, we may assume that $z_0 \in G^c = [G, G]$. To finish the proof, we show that $H : X_\Gamma \to X_\Gamma$ is bi-Hölder. By uniqueness of Φ we have $\Phi(x) = \pi(H(x)) = H(x)G^c$. Since $\Phi : X_\Gamma \to \mathbb{T}^d$ is Hölder and the inverse of Φ restricted to stable and unstable leaves is Hölder, Lemma 3.3, H is bi-Hölder along W^s and W^u. So, it suffices to show that H is Hölder along $W^c(x)$ and H^{-1} is Hölder along $W_0^c(x)$. Write

(7.5) $$H(x) = xe^{-h(x)} = xe^{-(h_s(x)+h_c(x)+h_u(x))}$$

with $h : X_\Gamma \to \mathfrak{g}$ and $h_\sigma : X_\Gamma \to E_0^\sigma$, $\sigma = s, c, u$. Since $\pi(H(x)) = \Phi(x)$ and Φ is Hölder, both h_s and h_u are Hölder. The functional equation for H implies

$$H(fx) = (fx)e^{-h(fx)} = L(x)e^{-v(x)}e^{-h(fx)} = L(x)e^{-v(x)-h(fx)+\frac{[v(x),h(fx)]}{2}} =$$
$$L(H(x))z_0 = L(x)e^{-Lh(x)+Z_0}$$

or

(7.6) $$h(fx) - Lh(x) = \frac{[v(x), h(fx)]}{2} - v(x) + Z_0.$$

We obtain an equation for $h_c(x)$

$$h_c(fx) - h_c(x) = \frac{[v_s(x), h_u(fx)] + [v_u(x), h_s(fx)]}{2} - v_c(x) + Z_0 = w(x)$$

where $w : X_\Gamma \to E_0^c$ is Hölder. It follows that $h_c(x)$ is Hölder [67], so H is Hölder. Next we show that H^{-1} is Hölder along $W_0^c(x)$. Fix $X \in E_0^s$, $Y \in E_0^u$ such that $[X, Y] = Z \neq 0$ with $Z \in E_0^c$. Given any $t \geq 0$ the Baker–Campbell–Hausdorff formula implies $e^{tX}e^{tY}e^{-tX}e^{-tY} = e^{t^2[X,Y]}$. We have

$$H^{-1}\left(xe^{t^2 Z}\right) - H^{-1}(x) = H^{-1}\left(xe^{tX}e^{tY}e^{-tX}e^{-tY}\right) - H^{-1}(x) =$$
$$H^{-1}\left(xe^{tX}e^{tY}e^{-tX}e^{-tY}\right) - H^{-1}\left(xe^{tX}e^{tY}e^{-tX}\right) +$$
$$H^{-1}\left(xe^{tX}e^{tY}e^{-tX}\right) - H^{-1}\left(xe^{tX}e^{tY}\right) +$$
$$H^{-1}\left(xe^{tX}e^{tY}\right) - H^{-1}\left(xe^{tX}\right) +$$
$$H^{-1}\left(xe^{tX}\right) - H^{-1}(x).$$

Using that H^{-1} is Hölder along W_0^s and W_0^u we have, for t small, some exponent θ and some constant C

$$\mathrm{d}\left(H^{-1}\left(xe^{t^2 Z}\right), H^{-1}(x)\right) \le Ct^\theta.$$

Since $\mathrm{d}(xe^{t^2 Z}, x) \ge ct^2$ for small t

$$\mathrm{d}\left(H^{-1}\left(xe^{t^2 Z}\right), H^{-1}(x)\right) \le C\mathrm{d}\left(x, xe^{t^2 Z}\right)^{\theta/2}.$$

For $y \in W_0^c(x)$ close, it follows $\mathrm{d}(H^{-1}(x), H^{-1}(y)) \le C\mathrm{d}(x,y)^{\theta/2}$. ∎

8. Rigidity: Smoothness of the bi-Hölder conjugacy

In this section, we prove Theorem 1.2. Let α, α_0 and $f = \alpha^{\mathbf{n}_0}$, $f_0 = \alpha_0^{\mathbf{n}_0}$ be as in Theorem 1.2. We prove Theorem 1.2 under the assumption that f is accessible, this is done for two reasons. First, the proof simplifies because we can apply [67] to obtain regularity of the conjugacy along the center direction. In particular, there is no loss of generality in assuming that the center of f_0 coincides with the joint center of α_0, see Remark 19. Second, if f is accessible then α naturally preserves a volume form μ and $H_*\mu = \mu_\Gamma$, see Lemma 8.2.

If α is assumed to preserve a volume form μ such that $H_*\mu = \mu_\Gamma$ then a result similar to Theorem 1.2 still holds, without accessibility. We give a brief sketch of the proof. Let $H : G \to G$ be a lift of the conjugacy. By the argument below, H is uniformly C^∞ along W^σ and $D_x H : E^\sigma(x) \to E_0^\sigma$, $\sigma = s, u$, is invertible at each x. Define

$$(8.1) \qquad \hat{H} : G \xrightarrow{H} G \to G/G^{cs}$$

then the fibers of $\hat{H}$ coincides with $\hat{W}^{cs}$. In particular, $\hat{H}$ is uniformly C^∞ along $\hat{W}^{cs}$. Moreover, since H is uniformly C^∞ along W^u, $\hat{H}$ is uniformly C^∞ along W^u. By Journé's lemma [36] $\hat{H}$ is smooth. The map $D_x\hat{H}|_{E^u(x)}$ is invertible, so $\hat{H}$ is a submersion. It follows that the leaves of $\hat{H}$ form a C^∞−foliation, so W^{cs} is a C^∞−foliation. Similarly, W^{cu} is a C^∞−foliation. Once we know that W^c is a C^∞−foliation the argument in [52, 26] to obtain regularity of H along W^s and W^u can also be used along W^c (note that, a priori, the assumptions of [52, Theorem A.1] are not satisfied for W^c).

8.1. Dynamical coherence and regularity of center leaves.
We begin by proving that f is dynamically coherent with W^{cs}, W^{cu} and W^c all uniquely integrable. We use this without mention in the remainder.

Lemma 8.1. *The map f is dynamically coherent with W^{cs}, W^{cu} and W^c all uniquely integrable with uniformly C^∞ leaves.*

Proof. Since f is conjugated to $f_0 = \alpha_0^{\mathbf{n}_0}$, and f_0 is uniformly subexponetial along its center E_0^c, f is also uniformly subexponential along its center E^c. Indeed, for any f−invariant measure ν the stable Pesin manifold $W_\nu^s(x)$ maps into the stable manifold, $W_0^s(H(x))$, of f_0 under H. Since H is invertible $\dim W_\nu^s(x) \leq \dim W_0^s(H(x))$. It follows that no ν−exponents are negative along E^c. By exchanging f for f^{-1}, no ν−exponent is positive along E^c. By [55] f is uniformly subexponential along E^c. We fix lifts F, F_0 and H to G, such that $H(Fx) = F_0(H(x))$. If $\gamma : I \to G$ is a C^1−curve tangent to E^{cs} then, for any $\varepsilon > 0$, the length $|F^n \circ \gamma|$ satisfies

$$(8.2) \qquad\qquad |F^n \circ \gamma| \leq C_\varepsilon e^{n\varepsilon}.$$

The conjugacy $H : G \to G$ can be written $H(x) = xh(x)^{-1}$ with $h : G \to G$ being Γ−invariant. We estimate

$$
\begin{aligned}
\mathrm{d}(H(x), H(y)) =& \mathrm{d}(xh(x)^{-1}, yh(y)^{-1}) \leq \\
\leq & \mathrm{d}(x, xh(x)^{-1}) + \mathrm{d}(y, yh(y)^{-1}) + \mathrm{d}(x,y) = \\
& \mathrm{d}(e, h(x)^{-1}) + \mathrm{d}(e, h(y)^{-1}) + \mathrm{d}(x,y) \leq \mathrm{d}(x,y) + C
\end{aligned}
$$

for some uniform C. With $x = H^{-1}(z)$ and $y = H^{-1}(w)$ we obtain

$$\mathrm{d}(H^{-1}(z), H^{-1}(w)) \geq \mathrm{d}(z, w) - C.$$

Now, $\mathrm{d}(F^n(\gamma(0)), F^n(\gamma(1))) \leq C_\varepsilon e^{n\varepsilon}$, so

$$
\begin{aligned}
C_\varepsilon e^{n\varepsilon} \geq & \mathrm{d}(F^n(\gamma(0)), F^n(\gamma(1))) = \mathrm{d}(H^{-1}HF^n(\gamma(0)), H^{-1}HF^n(\gamma(1))) \geq \\
& \mathrm{d}(HF^n(\gamma(0)), HF^n(\gamma(1))) - C = \mathrm{d}(F_0^n H(\gamma(0)), F_0^n(H(\gamma(1)))) - C.
\end{aligned}
$$

With ε sufficiently small it follows that $H(\gamma(1)) \in W_0^{cs}(H(\gamma(1)))$. We conclude that E^{cs} is uniquely integrable with leaves given by

$$(8.3) \qquad\qquad W^{cs}(x) = H^{-1}(W_0^{cs}(H(x))).$$

Similarly, E^{cu} is uniquely integrable with $W^{cu}(x) = H^{-1}(W_0^{cu}(H(x)))$. Since f is uniformly subexponential along E^c each foliation W^{cs}, W^{cu} and $W^c = W^{cs} \cap W^{cu}$ have uniformly C^∞ leaves, see [20, Theorem 7] (or [35]). $\blacksquare$

8.2. **Volume preservation and smoothness along the center direction.**
To apply arguments using exponential mixing we need to show that α as in Theorem 1.2 preserve a smooth volume form. Since we assume that the action α is accessible, we show that H is C^∞ along W^c without using the exponential mixing argument from [26], and instead relying on results from [67].

Lemma 8.2. *Let α be as in Theorem 1.2, then α preserve a smooth volume form μ. Moreover, the conjugacy $H : X_\Gamma \to X_\Gamma$ is volume preserving in the sense that $H_*\mu = \mu_\Gamma$ where μ_Γ is the Haar measure on X_Γ.*

Proof. Existence of an invariant Hölder continuous volume form, μ, follows from [30, Theorem 1.5]. We assume accessibility, so smoothness of μ follows from [67, Theorem A, case (IV)]. As in [25, Proposition 2.4], $H_*\mu$ is a measure of maximal entropy for the α_0−action. From [61] it follows that $H_*\mu = \mu_\Gamma$.   ■

Remark 18. By Moser's trick, there is no loss of generality if we assume that $\mu = \mu_\Gamma$.

We show that H is uniformly C^∞ along W^c following [67].

Lemma 8.3. *The restriction $H : W^c(x) \to W^c_0(H(x))$ is uniformly C^∞.*

Proof. Let $M = W^c(x) \times W^c_0(H(x))$ and let $N \subset M$ be the graph

$$(8.4) \qquad\qquad N = \{(y, H(y)) : y \in W^c(x)\}.$$

Given any two $z, w \in W^c(x)$ we fix a su−path γ from z to w, such a path always exists by accessibility. Denote by $h^\gamma_{z,w} : W^c(x) \to W^c(x)$ the composition of holonomy maps along γ (note that $z, w \in W^c(x)$ so $W^c(z) = W^c(w) = W^c(x)$). Since f is uniformly subexponential along W^c, f is ∞−bunching so $h^\gamma_{z,w}$ is C^∞ [47]. Since H map W^σ, $\sigma = s, u$, onto W^σ_0 we have $Hh^\gamma_{z,w} = h^{H\gamma,0}_{H(z),H(w)}$ where $h^{H\gamma,0}_{H(z),H(w)} : W^c_0(H(x)) \to W^c_0(H(x))$ is the composition of holonomy maps along $H\gamma$. We define

$$(8.5) \qquad \hat{h}_{z,w} : M \to M, \quad \hat{h}_{z,w}(p, q) = \left(h^\gamma_{z,w}(p), h^{H\gamma,0}_{H(z),H(w)}(q) \right).$$

Since $Hh^\gamma_{z,w} = h^{H\gamma,0}_{H(z),H(w)}H$ we have $\hat{h}_{z,w}(N) = N$. Moreover, $\hat{h}_{z,w}(z, H(z)) = (w, H(w))$. Since z and w were arbitrary, and $\hat{h}_{z,w}$ is smooth, it follows that N is C^∞−homogeneous. By [67, Corollary 1.3] N is a C^∞ submanifold. The graph of $H : W^c(x) \to W^c(H(x))$ is C^∞, so the map $H : W^c(x) \to W^c_0(H(x))$ is also C^∞. Finally, H is uniformly C^∞ along W^c since it intertwines the holonomies of f with the holonomies of f_0.   ■

Remark 19. Using Lemma 8.3 we may assume, without loss of generality, that the center E^c_0 of f_0 coincides with the joint center of α.

8.3. **Smoothness of coarse components along the stable foliation.** Let $\mathfrak{g} = E^s_0 \oplus E^c_0 \oplus E^u_0$ be the splitting of $\mathfrak{g}$ with respect to f_0. Denote by G^σ, $\sigma = s, c, u, cs, cu$, the subgroup associated to E^σ_0. We have a C^∞−diffeomorphism $G^s \times G^c \times G^u \to G$ defined by $(g^s, g^c, g^u) \mapsto g_s g_c g_u$. Write $H(x) = xh(x)^{-1}$ with $h : X_\Gamma \to G$ satisfying $h(\gamma x) = h(x)$ for all $\gamma \in \Gamma$. We decompose $h(x)$ with respect to the map $G^s \times G^c \times G^u \to G$

$$(8.6) \qquad\qquad h(x) = h_s(x)h_c(x)h_u(x).$$

It is immediate that each h_σ is a Hölder map. Given a coarse exponent $[\chi]$ we denote by $G^{[\chi]}$ the subgroup associated to $E_0^{[\chi]}$.

Let $[\chi]$ be a coarse exponent along E_0^s. We decompose $h_s(x)$ further

$$(8.7) \qquad\qquad h_s(x) = h_{ss}(x) h_{[\chi]}(x)$$

where $h_{[\chi]}(x)$ is the component of $h_s(x)$ along $G^{[\chi]}$ and $h_{ss}(x)$ is the component of $h_s(x)$ along the complementary group G^{ss}, see [52, Lemma 3.1]. The following is proved in [52, Section 3].

Lemma 8.4. *Let α be as in Theorem 1.2 with $\alpha^{\mathbf{n}_0} = f$ partially hyperbolic and $\ker[\chi]$, $[\chi](\mathbf{n}_0) < 0$, a wall of the chamber that contains $\mathbf{n}_0$. The map $h_{[\chi]}(x)$ in the Equation 8.7 is C^∞ along W^s, with all derivatives along W^s uniformly Hölder.*

Proof. The proof follows as in [52, Section 3] once we note that we do not need f to be Anosov. Indeed, once restricted to the foliation W^s, the argument only requires the action α to be exponentially mixing with respect to volume, which is proved in [30]. ∎

8.4. **New partially hyperbolic elements: passing the chamber wall.** Let $\mathbf{n}_0$ be in the Weyl chamber $\mathcal{C}$ and let $\ker[\chi]$ be a chamber wall for $\mathcal{C}$. Now we start the work of passing the Weyl chamber wall $\ker[\chi]$ by constructing a partially hyperbolic element in the chamber $\mathcal{C}'$ adjacent to $\mathcal{C}$ through $\ker[\chi]$. We initially follow [52], but change the argument from Section 4 in [52], by not relying on smooth ergodic theory. If $x \in X_\Gamma$ then the map $H : W^s(x) \to W_0^s(H(x)) = H(x)G^s$ is a homeomorphism. We define

$$(8.8) \qquad H_{s,x} : W^s(x) \to G^s, \quad H(y) = H(x)\left(H_{s,x}(y)\right)^{-1}.$$

If $\alpha_0^{\mathbf{n}}(x) = \rho^{\mathbf{n}}(x)\eta_{\mathbf{n}}^{-1}$ with $\rho^{\mathbf{n}} \in \mathrm{Aut}(X_\Gamma)$ and $\eta_{\mathbf{n}} \in G$ then $H_{s,x}$ satisfy

$$H_{s,\alpha^{\mathbf{n}x}}(\alpha^{\mathbf{n}}y) = \eta_{\mathbf{n}}\rho^{\mathbf{n}}\left(H_{s,x}(y)\right)\eta_{\mathbf{n}}^{-1}.$$

We obtain a formula for $H_{s,x}$ in terms of h

$$yh(y)^{-1} = H(y) = H(x)H_{s,x}(y)^{-1} = xh(x)^{-1}H_{s,x}(y)^{-1}$$

or, using the decomposition $h(x) = h_s(x)h_{cu}(x)$,

$$yh_{cu}(y)^{-1}h_s(y)^{-1} = xh_{cu}(x)^{-1}h_s(x)^{-1}H_{s,x}(y)^{-1}.$$

For $y \in W^s(x)$ we write $y = xg_x(y)^{-1}$ where $(x,y) \mapsto g_x(y)$ is chosen continuously and such that $g_x(x) = e$. Then $g_x : W^s(x) \to G^s$ is C^∞. With this notation

$$xg_x(y)^{-1}h(y)^{-1} = yh(y)^{-1} = H(y) = H(x)H_{s,x}(y)^{-1} = xh(x)^{-1}H_{s,x}(y)^{-1}$$

or

$$H_{s,x}(y) = h(y)g_x(y)h(x)^{-1} =$$
$$h_s(y)h_{cu}(y)\left[(h(x)g_x(y)^{-1})_s(h(x)g_x(y)^{-1})_{cu}\right]^{-1} =$$
$$h_s(y)\left[h_{cu}(y)\left(h(x)g_x(y)^{-1}\right)_{cu}^{-1}\right]\left(h(x)g_x(y)^{-1}\right)_s^{-1}.$$

If we multiply both sides of the equality, on the left by $h_s(y)^{-1}$ and on the right by $(h(x)g_x(y))_s$, then the right-hand side of the equality lie in G^{cu}, but the left-hand side lies in G^s. It follows that both sides of the equality are identity, so for $y \in W^s(x)$ we have

(8.9)
$$h_{cu}(y) = \left(h(x)g_x(y)^{-1}\right)_{cu},$$

(8.10)
$$H_{s,x}(y) = h_s(y)\left(h(x)g_x(y)^{-1}\right)_s^{-1}.$$

Using the fact that the map $a \mapsto a_\sigma$, $\sigma = s, cu$, and the map $g_x : W^s(x) \to G$ are both smooth the following is immediate, see also [52, Corollary 3.14].

Lemma 8.5. *The map h_{cu} is uniformly C^∞ along W^s.*

Define $H_{s,x}(y) = (H_{s,x}(y))_{ss}H_{s,x}^{[\chi]}(y)$ with $H_{s,x}^{[\chi]}(y) \in G^{[\chi]}$. From our definitions it is immediate

$$\{y \in W^s(x) : H_{s,x}^{[\chi]}(y) = e\} = H^{-1}(W_0^{ss}(H(x))), \quad W_0^{ss}(y) = yG^{ss}.$$

So, if we prove that $H_{s,x}^{[\chi]}$ is a (local) C^∞ submersion for every x, then the fibers of $H_{s,x}^{[\chi]}$ defines a smooth foliation (within W^s). This shows that the, a priori only Hölder, foliation $W^{ss}(x) = H^{-1}(W_0^{ss}(H(x)))$ is a Hölder foliation with uniformly smooth leaves.

Lemma 8.6. *The map $H_{s,x}^{[\chi]} : W^s(x) \to G^{[\chi]}$ is uniformly C^∞.*

Proof. Since G^{ss} is normal in G^s, [52, Lemma 3.1], we have

$$H_{s,x}(y) = h_s(y)\left(h(x)g_x(y)^{-1}\right)_s^{-1} =$$
$$h_{ss}(y)h_{[\chi]}(y)\left[\left(h(x)g_x(y)^{-1}\right)_s^{-1}\right]_{ss}\left[\left(h(x)g_x(y)^{-1}\right)_s^{-1}\right]_{[\chi]} =$$
$$a_{ss} \cdot h_{[\chi]}(y)\left[\left(h(x)g_x(y)^{-1}\right)_s^{-1}\right]_{[\chi]}$$

for some $a_{ss} \in G^{ss}$. That is, we obtain the formula

(8.11)
$$H_{s,x}^{[\chi]}(y) = h_{[\chi]}(y)\left[\left(h(x)g_x(y)^{-1}\right)_s^{-1}\right]_{[\chi]}$$

which is uniformly C^∞ along W^s since $h_{[\chi]}$ and g_x are, Lemma 8.4 . ∎

Lemma 8.7. *The map $D_xH_{s,x}^{[\chi]} : E^s(x) \to E_0^{[\chi]}$ is surjective at every $x \in X_\Gamma$. In particular, the foliation W^{ss} has uniformly C^∞ leaves.*

Denote by $K = \{D_x H_{s,x}^{[\chi]} : \text{is not surjective}\}$, then K is compact and $\alpha-$invariant. We wish to show that $K = \emptyset$. Since $H : W^s(x) \to W_0^s(H(x))$ is surjective, it follows by Sard's theorem that $K \neq X_\Gamma$. We will show that if K is non-empty then K contains a $W^{ss}-$leaf, which is a contradiction since every $W^{ss}-$leaf is dense (Lemma A.1).

Proof of Lemma 8.7. In the remainder, we fix some background metric $\langle \cdot, \cdot \rangle$, and will calculate determinants with respect to the top form induced by $\langle \cdot, \cdot \rangle$. Assume for contradiction that $K \neq \emptyset$. Fix $\mathbf{n}$ such that $\alpha^{\mathbf{n}}$ contract W^{ss} and $W^{[\chi]}$, with $\mathbf{n}$ close to the kernel $\ker[\chi]$ (we specify later how close). Fix $y \in W^{ss}(x)$ and some subspace $V_0 \subset E^s(y)$ of dimension $\dim(E_0^{[\chi]})$. We also define $V_n = D_y \alpha^{n\mathbf{n}} V_0$. The relation $H_{s,\alpha^{\mathbf{m}}x}^{[\chi]}(\alpha^{\mathbf{m}} y) = \eta_{\mathbf{m}} \rho^{\mathbf{m}} \left(H_{s,x}^{[\chi]}(y) \right) \eta_{\mathbf{m}}^{-1}$ implies

$$(8.12) \qquad \det(D_y H_{s,y}^{[\chi]}|_{V_0}) = \det(\rho^{-n\mathbf{n}}|_{E_0^{[\chi]}}) \det(D_{\alpha^{n\mathbf{n}}y} H_{s,\alpha^{n\mathbf{n}}y}^{[\chi]}|_{V_n}) \det(D_y \alpha^{n\mathbf{n}}|_{V_0}).$$

If χ is a representative of $[\chi]$ then the estimate $\det(D\rho^{-n\mathbf{n}}) \leq Ce^{-rn\chi(\mathbf{n})}$, for some uniform $r > 0$ depending on the dimension of $E_0^{[\chi]}$, is immediate. If $\ell = \dim(E_0^{[\chi]})$ and $\mathrm{Gr}_\ell(E^s)$ is the $\ell-$grassmannian bundle of E^s then $(V, x) \mapsto \det(D_x H_{s,x}^{[\chi]}|_V)$, $(V, x) \in \mathrm{Gr}_\ell(E^s)$, is uniformly smooth along W^s. For $x \in K$ we have $\det(D_x H_{s,x}^{[\chi]}|_V) = 0$ since $D_x H_{s,x}^{[\chi]}$ is not surjective. It follows that there is some constant C such that

$$(8.13) \qquad\qquad\qquad |\det(D_y H_{s,y}^{[\chi]}|_V)| \leq C \mathrm{d}_s(y, K)$$

for $y \in W^s(K)$. Since $y \in W^{ss}(x)$, we have

$$(8.14) \qquad \det(D_{\alpha^{n\mathbf{n}}y} H_{\alpha^{n\mathbf{n}}y}^{[\chi]}|_{V_n}) \leq C\mathrm{d}(\alpha^{n\mathbf{n}}y, K) \leq C\mathrm{d}(\alpha^{n\mathbf{n}}y, \alpha^{n\mathbf{n}}x) \leq C'e^{-\lambda n}$$

where $\lambda > 0$ can be chosen independently of $\mathbf{n}$ if we let $\mathbf{n}$ be close to the kernel of $[\chi]$. Equations 8.12 and 8.14 implies

$$|\det(D_y H_{s,y}^{[\chi]}|_{V_0})| \leq Ce^{-n(\lambda + r\chi(\mathbf{n}))}|\det(D_y \alpha^{n\mathbf{n}}|_{V_0})|.$$

Since $\alpha_0^{\mathbf{n}}$ contract the foliations W^{ss} and $W^{[\chi]}$, $\alpha^{\mathbf{n}}$ contracts W^s, we have a uniform bound $|\det(D_y \alpha^{n\mathbf{n}}|_{V_0})| \leq C$. We obtain an estimate

$$|\det(D_y H_{s,y}^{[\chi]}|_{V_0})| \leq Ce^{-n(\lambda + r\chi(\mathbf{n}))}.$$

With $\mathbf{n}$ be sufficiently close to $\ker[\chi]$ we have $\lambda + r\chi(\mathbf{n}) > 0$ and finally if $n \to \infty$ then

$$\det(D_y H_{s,y}^{[\chi]}|_{V_0}) = 0.$$

The point y was arbitrary, so $W^{ss}(x) \subset K$ for $x \in K$. The foliation W^{ss} is minimal, Lemma A.1, and K is closed, so $K = X_\Gamma$ which contradicts Sard's theorem. ∎

To show that $E^{[\chi]}$ exists as a continuous bundle we will apply a linear graph transform argument. Denote by $E^{ss}(x)$ the $D\alpha^{\mathbf{n}}$−invariant, continuous subbundle tangent to $W^{ss}(x)$. Let $F(x)$ be any continuous bundle that is complementary to E^{ss} within E^s. We can, for example, fix a metric and let F be the complement of E^{ss} within E^s. Fix $\mathbf{n} \in \mathbb{Z}^k$ such that $[\chi](\mathbf{n}) < 0$ and $\alpha^{\mathbf{n}}$ expands W^{ss} (that is, we chose $\mathbf{n}$ such that $-\mathbf{n}$ have passed the chamber wall $\ker[\chi]$ from the chamber that contains $\mathbf{n}_0$). Write $g = \alpha^{\mathbf{n}}$. With respect to the splitting $E^s = F \oplus E^{ss}$ we write

$$D_x g(u, v) = (A(x)u, B(x)v + C(x)u)$$

where $A(x)u$ is $D_x g(u)$ projected onto $F(gx)$ and $C(x)$ is $D_x g(u)$ projected onto $E^{ss}(gx)$.

Lemma 8.8. *We have* $\|A\|_{C^0} < 1$.

Proof. Note that $D_x H_{s,x}^{[\chi]} D_x g = D_e \alpha_0^{\mathbf{n}} D_x H_{s,x}^{[\chi]}$. There is $\mu < 1$, depending on our choice of $\mathbf{n} \in \mathbb{Z}^k$, such that for any $u \in F(x)$ we have

$$(8.15) \quad \left\| D_{gx} H_{s,gx}^{[\chi]} (A(x)u + B(x)u) \right\| = \left\| D_e \alpha_0^{\mathbf{n}} D_x H_{s,x}^{[\chi]}(u) \right\| \le \mu \left\| D_x H_{s,x}^{[\chi]}(u) \right\|.$$

Since $B(x)u \in E^{ss}(gx)$ and $\ker D_x H_{s,x}^{[\chi]} = E^{ss}(x)$ (the fibers of $H_{s,x}^{[\chi]}$ are the foliation $W^{ss}(x)$) it follows that $D_{gx} H_{s,gx}^{[\chi]} B(x) = 0$. That is, Equation 8.15 simplifies

$$(8.16) \quad \left\| D_{gx} H_{s,gx}^{[\chi]} A(x)u \right\| \le \mu \left\| D_x H_{s,x}^{[\chi]}(u) \right\|.$$

Since $D_x H_{s,x}^{[\chi]} : F(x) \to E_0^{[\chi]}$ is an isomorphism (of vector bundles after identifying $E_0^{[\chi]}$ with the trivial bundle $X_\Gamma \times E_0^{[\chi]} \to X_\Gamma$) the lemma follows (after possibly altering the norm used or exchanging $\mathbf{n}$ for $N\mathbf{n}$ with N sufficiently large). $\blacksquare$

Lemma 8.9. *The map* $T : \Gamma^0(\mathrm{Hom}(F, E^{ss})) \to \Gamma^0(\mathrm{Hom}(F, E^{ss}))$ *defined by*

$$(TP)(x) = B(x)^{-1} (P(gx)A(x) - C(x))$$

has a unique fixed point.

Proof. Since $\|A\|_{C^0}, \|B^{-1}\|_{C^0} < 1$ the lemma follows from Banach's fixed point theorem. $\blacksquare$

Lemma 8.10. *There exists an* $\alpha−$*invariant continuous subbundle* $E^{[\chi]} \subset E^s$ *such that* $E^s = E^{[\chi]} \oplus E^{ss}$.

Proof. Let $P \in \Gamma^0(\mathrm{Hom}(F, E^{ss}))$ be the unique $T−$fixed point from Lemma 8.9. Define $E^{[\chi]}(x) := \mathrm{Graph}(P(x)) = \{(u, P(x)u) : u \in F(x)\}$. It is immediate that

$E^{[\chi]} \oplus E^{ss} = E^s$. Given $u \in F(x)$

$$
\begin{aligned}
D_x g(u, P(x)u) &= (A(x)u, B(x)P(x)u + C(x)u) = \\
&\quad (A(x)u, P(gx)A(x)u - C(x)u + C(x)u) = \\
&\quad (A(x)u, P(gx)A(x)u) \in \mathrm{Graph}(P(gx))
\end{aligned}
$$

so $D_x g E^{[\chi]}(x) \subset E^{[\chi]}(gx)$. That $D_x g E^{[\chi]}(x) = E^{[\chi]}(gx)$ follows since $D_x g$ is invertible. That $E^{[\chi]}(x)$ is $\alpha^{\mathbf{m}}-$invariant for all $\mathbf{m} \in \mathbb{Z}^k$ follows by applying the graph transform of $\alpha^{\mathbf{m}}$ on the element $P \in \Gamma^0(\mathrm{Hom}(F, E^{ss}))$. This defines a $T-$fixed point, since g commute with $\alpha^{\mathbf{m}}$, and the $T-$fixed point is unique, Lemma 8.9, so the $\alpha^{\mathbf{m}}-$graph transform of P is P. Equivalently, $E^{[\chi]}$ is $D_x \alpha^{\mathbf{m}}-$invariant.   ∎

Since $D_x H^{[\chi]}_{s,x} : E^{[\chi]}(x) \to E^{[\chi]}_0$ conjugates $D_x \alpha^{\mathbf{n}}$ to $\mathrm{Ad}(\eta_{\mathbf{n}})D\rho^{\mathbf{n}}$ the following lemma follows immediately by induction.

Lemma 8.11. *Every element $\mathbf{n} \in \mathbb{Z}^k \setminus 0$ defines a partially hyperbolic $\alpha^{\mathbf{n}} : X_\Gamma \to X_\Gamma$ where the center of $\alpha^{\mathbf{n}}$ has the same dimension as the center of $\alpha^{\mathbf{n}}_0$.*

Remark 20. Lemma 8.11 passes the Weyl chamber containing $\mathbf{n}_0$, and produce new partially hyperbolic element in adjacent chambers.

8.5. **Finishing the proof of Theorem 1.2.** We finish the proof of Theorem 1.2 by showing that H is a C^∞ diffeomorphism. We begin by proving that h_σ, $\sigma = s, u$, are C^∞.

Lemma 8.12. *The maps $h_\sigma : X_\Gamma \to G^\sigma$, $\sigma = s, u$, are C^∞.*

Proof. It suffices to consider the case $\sigma = s$. By Lemma 8.5 and Journé's lemma [36] it suffices to show that h_s is uniformly smooth along W^s. Number all coarse exponents along W^s by $[\chi_1], ..., [\chi_N]$. The function h_s can be decomposed with respect to the map $G^{[\chi_1]} \times ... \times G^{[\chi_N]} \to G^s$, see [52, Lemma 3.2], as

(8.17) $$h_s(x) = h_{[\chi_1]}(x)...h_{[\chi_N]}(x).$$

Using Lemmas 8.11 and 8.4 it is immediate that $h_{[\chi_N]}(x)$ is C^∞. Moreover, we use Lemmas 8.11 and 8.4 to show that $h^{(1)}_{[\chi_j]}(x) = h_{[\chi_j]}(x)/G^{(1)}$ is smooth for all $j = 1, ..., N$. If we define $h^{(i)}_{[\chi_j]}(x) = h_{[\chi_j]}(x)/G^{(i)}$ and assume that $h^{(i-1)}_{[\chi_j]}(x)$ is smooth, then we can change the order of products in 8.17 modulo a polynomial in $h^{(i-1)}_{[\chi_j]}$. Once more applying Lemmas 8.11 and 8.4, $h^{(i)}_{[\chi_j]}$ is C^∞ for each $j = 1, ..., N$. For i large enough, using that G is nilpotent, $h_{[\chi_j]}$ is C^∞ along W^s.   ∎

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. If we show that H is C^∞ then H is automatically a diffeomorphism since the Jacobian can never vanish, this would contradict volume preservation of α.

By Lemma A.1 the map H is uniformly C^∞ along W^c. From equation 8.9 we have $H_{s,x}(y) = h_s(y) \left(h(x)g_x(y)^{-1}\right)_s^{-1}$ and since $g_x(y)$ and $h_s(y)$ are both uniformly C^∞ along W^s, by Lemma 8.12, it follows that $H_{s,x}$ is uniformly C^∞ along W^s. That is H is uniformly C^∞ along W^s. Similarly, H is uniformly C^∞ along W^u. Using Journé's lemma along W^c and W^s, H is uniformly C^∞ along W^{cs}. Using Journé's lemma once more along W^u and W^{cs} it follows that H is C^∞. ∎

9. Proof of main theorems

We complete the proofs of Theorems A and B.

Proof of Theorem A. By Theorem 7.1 the action by α is bi-Hölder conjugate to some affine action α_0. Produce the conjugacy H for the special element $f = \alpha^{\mathbf{n}_0}$, but it then also conjugates the full action into an affine action. This is easily seen to hold directly by the construction of H but also follows from an argument as in [1]. Applying Theorem 1.2 proves Theorem A. ∎

Proof of Theorem B. Let $f_0 \in \mathrm{Aff}(X_\Gamma)$ and $f \in \mathrm{Diff}^\infty(X_\Gamma)$ be C^1-close to f_0. Write $L_{su} : \mathbb{T}^d \to \mathbb{T}^d$ for the induced hyperbolic automorphism. Note that f satisfy assumptions (i) and (ii) of Theorem A since it is close to f_0. In particular, we obtain Φ from Theorem 1.1. We denote by $Z^\infty(f)$ the $C^\infty-$centralizer of f and let $Z_c^\infty(f) \subset Z^\infty(f)$ be the center fixing, normal subgroup of $Z^\infty(f)$ from Equation 1.4. Define the quotient

$$(9.1) \qquad\qquad Z_{su}^\infty(f) = Z^\infty(f)/Z_c^\infty(f).$$

If $g \in Z_c^\infty(f)$ then $\Phi(gx) = \Phi(x)$ so if we consider the induced map on $H_1 X_\Gamma$ for g then $\Phi_* g_* = \Phi_*$ or since $\Phi_* : H_1 X_\Gamma \to H_1 \mathbb{T}^d$ is an isomorphism, it follows that $g_* = \mathrm{id}$. Conversely, if $g_* = \mathrm{id}$ then, since Φ semiconjugates g to g_* Theorem 1.1, it holds that $\Phi(gx) = \Phi(x)$ so $g \in Z_c^\infty(f)$. It follows that each non-trivial $g \in Z_{su}^\infty(f)$ represent an element of $Z^\infty(f)$ that project onto a non-trivial automorphism on $\mathbb{T}^d$. In particular, if $\mathrm{rank}(Z_{su}^\infty(f)) > 1$ then the image of $Z^\infty(f)$ in $Z_{\mathrm{Aut}}(L_{su})$ contains a subgroup isomorphic to $\mathbb{Z}^2$. Irreducibility of L_{su} implies that this $\mathbb{Z}^2-$subgroup in $Z^\infty(f)$ is higher rank. The action of $Z^\infty(f)$ is $C^\infty-$conjugate to some affine action by Theorem A. If $\mathrm{rank}(Z_{su}^\infty(f)) = 1$ and $\#Z_c^\infty(f) = \infty$ then [21, Corollary 18] implies that case (ii) of Theorem B holds. Finally, if $\mathrm{rank}(Z_{su}^\infty(f)) = 1$ and $\#Z_c^\infty(f) < \infty$ then $Z^\infty(f)$ is virtually $\mathbb{Z}$ so case (i) of Theorem B holds. ∎

Appendix A. Some algebraic lemmas

In this appendix, we show some basic properties of higher rank, algebraic actions on nilmanifolds. The first two lemmas, 2.1 and 2.2, are stated in Section 2.2.

Proof of Lemma 2.1. If the conclusion does not hold then there is a decomposition $\mathfrak{g} = E_0^s \oplus E_0^c \oplus E_0^u$ so that every $\alpha_0^{\mathbf{n}}$ is subexponential along E_0^c and either contract or expand E_0^s and E_0^u. Consider the projected action on the base. If this is rank-1, then the whole action is rank-1 (and has a rank-1 factor). Recall that for any $n \in \mathbb{N}$ there is some $\varepsilon_n > 0$ such that if $L \in \mathrm{GL}(n, \mathbb{Z})$ satisfies that the eigenvalues of L with modulus larger than one have a product bounded by $1 + \varepsilon_n$, then L have no eigenvalues of modulus larger than 1.

Let $W \subset \mathbb{R}^k$ be the kernel of the unique pair of negatively proportional exponents of α_0. For any $\mathbf{n} \in \mathbb{Z}^k$ sufficiently close to W, all eigenvalues of $\alpha_0^{\mathbf{n}} \in \mathrm{GL}(n, \mathbb{Z})$ will be close to the unit circle, which implies that all eigenvalues of $\alpha_0^{\mathbf{n}}$ lie on the unit circle. It follows that any $\mathbf{n} \in \mathbb{Z}^k$ sufficiently close to W lies in W. In particular

$$\text{(A.1)} \qquad \qquad \mathrm{rank}(\mathbb{Z}^k \cap W) = \dim(W) = k - 1.$$

Elements in W have all eigenvalues on the unit circle, dropping to a finite index subgroup of $\mathbb{Z}^k$, we may assume that all eigenvalues are 1. After taking a quotient to remove Jordan blocks, the action of W is trivial. Since $\dim(W) = k - 1$, the action of $\mathbb{Z}^k$ is rank-1. ∎

Proof of Lemma 2.2. The translation action of V is minimal if and only if the induced translation action on the base is minimal, so we assume without loss of generality that X_Γ is a torus. Let W be the rational span of V, then V acts minimally if and only if $W = \mathbb{R}^d$. If $W \neq \mathbb{R}^d$ then $\mathbb{T}^\ell \cong \mathbb{T}^d/W$ with $\ell \geq 1$, and α_0 descend to $\mathbb{T}^\ell$. We have quotiened out all coarse exponents except for one negatively proportional pair, so the factor $\mathbb{T}^\ell$ has only one pair of negatively proportional exponents. By Lemma 2.1 $\mathbb{T}^\ell$ is a rank-1 factor, which is a contradiction. We conclude that $W = \mathbb{R}^d$, so V act minimally. ∎

We will need a lemma, like Lemma 2.2, but only considering coarse directions that lie in the same stable direction for some element of the action $\alpha_0^{\mathbf{n}_0}$. The following lemma is a consequence of Lemma 2.1.

Lemma A.1. *Let $\alpha_0 : \mathbb{Z}^k \to \mathrm{Aff}(X_\Gamma)$ be higher rank. We say that $\mathbf{n} \in \mathbb{Z}^k$ is regular if the center of $\alpha_0^{\mathbf{n}}$ coincide with the joint center of α_0. Let $\mathbf{n}_0$ be regular and E_0^s be the stable space associated to $\alpha_0^{\mathbf{n}_0}$. Let $[\chi]$ be a coarse exponent such that $\ker[\chi]$ is a wall for the Weyl chamber that contains $\mathbf{n}_0$ and $E_0^{[\chi]} \subset E_0^s$. Either*

$E_0^s = E_0^{[\chi]}$ *or the complementary subspace* E_0^{ss}, $E_0^{ss} \oplus E_0^{[\chi]} = E_0^s$, *defines a minimal foliation in* X_Γ.

Proof. After projecting to the base, we assume without loss of generality that X_Γ is a torus $\mathbb{T}^d$. If $E_0^{[\chi]} = E_0^s$ for $[\chi](\mathbf{n}_0) < 0$, then there is nothing to prove, so suppose that there is at least one coarse exponent $[\eta]$ satisfying $[\eta] \neq [\chi]$ and $[\eta](\mathbf{n}_0) < 0$. Write E_0^{ss} for the complementary subspace defined by

$$\text{(A.2)} \qquad E_0^{ss} = \bigoplus_{\substack{[\eta](\mathbf{n}_0)<0 \\ [\eta]\neq[\chi]}} E_0^{[\eta]}.$$

Let W be the rational closure of E_0^{ss}, then W is α_0-invariant and rational. Each $\alpha_0^{\mathbf{n}}|_W$ preserves the lattice $W \cap \mathbb{Z}^d$ so $\det(\alpha_0^{\mathbf{n}}|_W) = \pm 1$. Given a coarse exponent $[\eta]$, define $\eta' = r(\eta')\eta$ for $[\eta'] = [\eta]$ and $d_{[\eta]}^W \in [0,1]$ by

$$\text{(A.3)} \qquad d_{[\eta]}^W = \frac{\sum_{\eta'\in[\eta]} r(\eta') \dim(E_0^{\eta'} \cap W)}{\sum_{\eta'\in[\eta]} r(\eta') \dim(E_0^{\eta'})}.$$

From this definition it is immediate that

$$\text{(A.4)} \qquad \left| \det(\alpha_0^{\mathbf{n}}|_{E_0^{[\eta]}\cap W}) \right| = \left| \det(\alpha_0^{\mathbf{n}}|_{E_0^{[\eta]}}) \right|^{d_{[\eta]}^W}, \quad \mathbf{n} \in \mathbb{Z}^k.$$

Our choice of W implies that $d_{[\eta]}^W = 1$ if $[\eta](\mathbf{n}_0) < 0$ and $[\eta] \neq [\chi]$. Rewrite

$$1 = |\det(\alpha_0^{\mathbf{n}}|_W)| = \prod_{[\eta]} \left| \det(\alpha_0^{\mathbf{n}}|_{E_0^{[\eta]}}) \right|^{d_{[\eta]}^W} =$$

$$\left(\prod_{\substack{[\eta](\mathbf{n}_0)<0 \\ [\eta]\neq[\chi]}} |\det(\alpha_0^{\mathbf{n}}|_{E_0^{[\eta]}})| \right) \cdot \left| \det(\alpha_0^{\mathbf{n}}|_{E_0^{[\chi]}}) \right|^{d_{[\chi]}^W} \cdot \left(\prod_{[\eta](\mathbf{n}_0)>0} |\det(\alpha_0^{\mathbf{n}}|_{E_0^{[\eta]}})|^{d_{[\eta]}^W} \right).$$

Using $|\det(\alpha_0^{\mathbf{n}})| = 1$

$$\prod_{\substack{[\eta](\mathbf{n}_0)<0 \\ [\eta]\neq[\chi]}} |\det(\alpha_0^{\mathbf{n}}|_{E_0^{[\eta]}})| = \frac{1}{\left| \det(\alpha_0^{\mathbf{n}}|_{E_0^{[\chi]}}) \right|} \cdot \prod_{[\eta](\mathbf{n}_0)>0} \frac{1}{\left| \det(\alpha_0^{\mathbf{n}}|_{E_0^{[\eta]}}) \right|}$$

and combining estimates

$$\text{(A.5)} \qquad 1 = \left| \det(\alpha_0^{\mathbf{n}}|_{E_0^{[\chi]}}) \right|^{d_{[\chi]}^W - 1} \cdot \prod_{[\eta](\mathbf{n}_0)>0} |\det(\alpha_0^{\mathbf{n}}|_{E_0^{[\eta]}})|^{d_{[\eta]}^W - 1}.$$

If $\mathbf{n}_j \in \mathbb{Z}^k$ is a sequence in the same Weyl chamber as $\mathbf{n}_0$ such that $\mathbf{n}_j \to \ker[\chi]$, then

$$\left| \det(\alpha_0^{\mathbf{n}_j}|_{E_0^{[\chi]}}) \right|^{d_{[\chi]}^W - 1} \to 1.$$

Each $[\eta] \neq \pm[\chi]$ satisfies $|\det(\alpha_0^{\mathbf{n}_j}|_{E_0^{[\eta]}})| \geq \mu > 1$ for some uniform μ. Letting $j \to \infty$, $d_{[\eta]}^W = 1$ for $[\eta](\mathbf{n}_0) > 0$, $[\eta] \neq -[\chi]$. It follows that $E_0^{[\eta]} \subset W$ for $[\eta] \neq \pm[\chi]$. If $\mathbb{T}^d/W$ is non-trivial, then the projected action on $\mathbb{T}^d/W$ does not have two independent coarse Lyapunov exponents. By Lemma 2.1, $\mathbb{T}^d/W$ is a rank-1 factor of α_0. This contradicts the assumption that α_0 is higher rank so $W = \mathbb{R}^d$ which proves the lemma. ∎

Lemma A.2. *Let $A \subset \mathrm{GL}(n, \mathbb{Z})$ be a free abelian subgroup with Lyapunov exponents $\mathrm{Lyap}(A)$. Let N be the maximal number of linearly independent Lyapunov exponents (that is, $N = \dim(\mathrm{span}(\mathrm{Lyap}(A)))$). If the intersection of the kernels of all $\chi \in \mathrm{Lyap}(A)$ is trivial in A then $\mathrm{rank}(A) \leq N$.*

Proof. Let $p(t) \in \mathbb{Z}[t]$ be of degree d, monic and with constant term ± 1. Let $\lambda_1, ..., \lambda_d$ be the roots of $p(t)$ (possibly with multiplicity). There is a constant $\mu_d > 1$ such that either $p(t)$ has only roots on the unit circle or

$$(A.6) \qquad M(p(t)) = \prod_{j=1}^{d} \max(1, |\lambda_j|) \geq \mu_d$$

see for example [22]. We number $\{\chi_1, ..., \chi_n\} = \mathrm{Lyap}(A)$. Let d_j be the dimension of the Lyapunov space associated to χ_j. Given $a \in A$ let $p_a(t)$ be the corresponding characteristic polynomial. We obtain

$$(A.7) \qquad \log M(p_a(t)) = \sum_{\chi_j(a)>0} d_j \chi_j(a).$$

Suppose that $\mathrm{rank}(A) > N$ for contradiction. We let $\chi_1, ..., \chi_N$ be chosen such that every χ_j lie in $\mathrm{span}(\chi_1, ..., \chi_N)$. If $a_n \in A$ is such that $\chi_1(a_n), ..., \chi_N(a_n) \to 0$ then $\chi_j(a_n) \to 0$ for all $j = 1, ..., n$. The intersection

$$(A.8) \qquad V := \bigcap_{j=1}^{N} \ker \chi_j \subset A \otimes \mathbb{R}$$

is non-trivial since $\mathrm{rank}(A) > N$. The set A is a lattice in $A \otimes \mathbb{R}$, so we find a sequence $a_n \in A$ such that $\mathrm{d}(a_n, V) \to 0$ as $n \to \infty$ but $a_n \not\to e$ (where e is the identity in $A \subset \mathrm{GL}(n, \mathbb{Z})$). It follows that

$$(A.9) \qquad \log M(p_{a_n}(t)) = \sum_{\chi_j(a)>0} \chi_j(a_n)$$

tends to 0 as $n \to \infty$. With n big enough $\log M(p_{a_n}(t)) < \log \mu_d$ which implies that a_n has only roots on the unit circle. It follows that a_n lie in the kernel of all χ_j, so $a_n = e$ by assumption. This implies that $a_n \to e$ which is a contradiction. ∎

Lemma A.3. *Let $A \in \mathrm{Sp}(d, \mathbb{Z})$ be hyperbolic with irreducible characteristic polynomial. If $r_1(A)$ denotes the number of real eigenvalues of A and $r_2(A)$ the number of pairs of complex eigenvalues of A then $\mathrm{rank}(Z_{\mathrm{Sp}(d,\mathbb{Z})}(A)) = r_1(A)/2 + r_2(A)/2$.*

Proof. Fix $A \in \mathrm{Sp}(d, \mathbb{Z})$ with irreducible characteristic polynomial. Let

$$(A.10) \qquad \qquad \Lambda_2(\mathbb{R}^d) = \mathbb{R}^d \wedge \mathbb{R}^d$$

be the vector space of 2−vectors from $\mathbb{R}^d$. Consider the induced map $A \wedge A :$ $\Lambda_2(\mathbb{R}^d) \to \Lambda_2(\mathbb{R}^d)$. Note that $\Lambda_2(\mathbb{Z}^d)$ is a $A \wedge A$−invariant lattice in $\Lambda_2(\mathbb{R}^d)$. Denote by $W \le \Lambda_2(\mathbb{R}^d)$ the subspace defined by $A \wedge A|_W = \mathrm{id}_W$, that is W is the eigenspace of 1 for $A \wedge A$. We write $\Gamma := W \cap \Lambda_2(\mathbb{Z}^d)$, since W is a rational subspace of $\Lambda_2(\mathbb{R}^d)$ the subgroup $\Gamma \le W$ is a lattice.

Given $B \in Z_{\mathrm{GL}(d,\mathbb{Z})}(A)$ the wedge $B \wedge B$ preserve W and stabilize Γ in W. So, after identifying $W \cong \mathbb{R}^{d/2}$ and $\Gamma \cong \mathbb{Z}^{d/2}$ we obtain a map $\Psi : Z_{\mathrm{GL}(d,\mathbb{Z})}(A) \to$ $\mathrm{GL}(d/2, \mathbb{Z})$ defined by $\Psi(B) := (B \wedge B)|_W$. It is immediate that Ψ is a homomorphism. Fix eigenvectors $e_1, ..., e_{d/2}, \tilde{e}_1, ..., \tilde{e}_{d/2} \in \mathbb{C}^d$ such that $Ae_j = \lambda_j e_j$ and $A\tilde{e}_j = \tilde{e}_j/\lambda_j$. We can identify (the complexification of) W by $W = \mathrm{span}(e_1 \wedge \tilde{e}_1, ..., e_{d/2} \wedge \tilde{e}_{d/2})$, so for $B \in Z_{\mathrm{GL}(d,\mathbb{Z})}(A)$ we have

$$(A.11) \qquad \qquad B \wedge B(e_j \wedge \tilde{e}_j) = \mu_j(B)\tilde{\mu}_j(B)e_j \wedge \tilde{e}_j$$

where $Be_j = \mu_j(B)e_j$ and $B\tilde{e}_j = \tilde{\mu}_j(B)\tilde{e}_j$. If $\Psi(B) = e$, then $\mu_j(B)\tilde{\mu}_j(B) = 1$ which implies that B preserve the symplectic form that A preserve (note that $e_1, ..., e_{d/2}, \tilde{e}_1, ..., \tilde{e}_{d/2}$ can be chosen such that the symplectic form can be written $e^1 \wedge \tilde{e}^1 + ... + e^{d/2} \wedge \tilde{e}^{d/2}$). Conversely, if $B \in Z_{\mathrm{Sp}(d,\mathbb{Z})}(A)$ then $\Psi(B) = e$. Equation A.11 also implies that the Lyapunov exponents of $\mathrm{Im}(\Psi)$ are given by

$$(A.12) \qquad \qquad \chi_j^\Psi(\Psi(B)) = \log|\mu_j(B)| + \log|\tilde{\mu}_j(B)|.$$

It follows that $\mathrm{Im}(\Psi)$ has $r_1(A)/2 + r_2(A)/2$ Lyapunov exponents. Indeed, if $\mu_j(B), \tilde{\mu}_j(B)$ takes values in $\mathbb{R}$ then $\log|\mu_j(B)| + \log|\tilde{\mu}_j(B)|$ defines one Lyapunov exponent. If $\mu_j(B), \tilde{\mu}_j(B)$ takes values in $\mathbb{C} \setminus \mathbb{R}$ then $\overline{\mu_j(B)} = \mu_{j'}(B)$ and $\overline{\tilde{\mu}_j(B)} = \tilde{\mu}_{j'}(B)$ are eigenvalues of $Z_{\mathrm{GL}(d,\mathbb{Z})}(A)$, and

$$(A.13) \qquad \qquad \chi_j^\Psi(B) = \chi_{j'}^\Psi(B).$$

By Lemma A.2, $\mathrm{rank}(\mathrm{Im}(\Psi)) \le r_1(A)/2 + r_2(A)/2 - 1$ since $|\det(\Psi(B))| = 1$ implies that, at least, one Lyapunov exponent of $\mathrm{Im}(\Psi)$ can be written as a combination of the other Lyapunov exponents. By [41], the rank of $Z_{\mathrm{GL}(d,\mathbb{Z})}(A)$ is $r_1(A) + r_2(A) - 1$, so

$$\mathrm{rank}(\ker(\Psi)) = \mathrm{rank}(Z_{\mathrm{GL}(d,\mathbb{Z})}(A)) - \mathrm{rank}(\mathrm{Im}(\Psi)) \ge$$

$$r_1(A) + r_2(A) - 1 - \left[\frac{r_1(A) + r_2(A)}{2} - 1\right] =$$

$$\frac{r_1(A) + r_2(A)}{2}$$

or since $\ker(\Psi) = Z_{\mathrm{Sp}(d,\mathbb{Z})}(A)$, $\mathrm{rank}(Z_{\mathrm{Sp}(d,\mathbb{Z})}(A)) \ge (r_1(A) + r_2(A))/2$. The converse inequality is clear from Lemma A.2 since if χ is a Lyapunov exponent of $Z_{\mathrm{Sp}(d,\mathbb{Z})}(A)$ then so is $-\chi$. ∎

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