Holomorphic Laplacian on the Lie ball and the Penrose transform

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Abstract

We prove that any holomorphic function f on the Lie ball of even dimension satisfying $\Delta f = 0$ is obtained uniquely by the higher-dimensional Penrose transform of a Dolbeault cohomology for a twisted line bundle of a certain domain of the Grassmannian of isotropic subspaces. To overcome the difficulties arising from that the line bundle parameter is outside the *good range*, we use some techniques from algebraic representation theory.

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1 Introduction

We consider the holomorphic Laplacian $\Delta = \frac{\partial^2}{\partial z_1^2} + \cdots + \frac{\partial^2}{\partial z_n^2}$ on the Lie ball

$$D := \{z \in \mathbb{C}^n : |z\,{}^t z|^2 + 1 - 2\overline{z}\,{}^t z > 0, |z\,{}^t z| < 1\},$$

which is the bounded symmetric domain of type D IV in the É. Cartan classification.

The goal of this article is to prove that any holomorphic function f on D satisfying $\Delta f = 0$ can be obtained uniquely as the higher-dimensional Penrose transform of a Dolbeault cohomology of a non-compact complex manifold $X = SO_0(2, 2m)/U(1, m)$ when $n = 2m$.

To formulate our main results, let $m > 1$, $G := SO_0(2, 2m)$ be the identity component of the indefinite orthogonal group of signature $(2, 2m)$, $K = SO(2) \times SO(2m)$ a maximal compact subgroup, and θ the corresponding Cartan involution of G. Let T be a maximal torus of K, $\mathfrak{t}$ its Lie algebra, and $\mathfrak{t}^\vee$ the dual space. We take the standard basis $\{e_0, e_1, \ldots, e_m\}$ in $\sqrt{-1}\mathfrak{t}^\vee$ such that $\Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C}) = \{\pm e_i \pm e_j : 0 \le i < j \le m\}$. One has a decomposition of the complexified Lie algebra $\mathfrak{g}_\mathbb{C} = \mathfrak{so}(2m+2, \mathbb{C}) = \mathfrak{p}_- + \mathfrak{k}_\mathbb{C} + \mathfrak{p}_+$ as a $\mathfrak{k}_\mathbb{C}$-module with $\Delta(\mathfrak{p}_+, \mathfrak{t}_\mathbb{C}) := \{e_0 \pm e_j : 1 \le j \le m\}$. The bounded symmetric domain D may be identified with the Harish-Chandra realization of G/K in $\mathbb{C}^{2m} \simeq \mathfrak{p}_- \subset G_\mathbb{C}/K \exp(\mathfrak{p}_+)$.

We set $\mathbf{1}_{m+1} = e_0 + \cdots + e_m \in \sqrt{-1}\mathfrak{t}^\vee$, and define a θ-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l}_\mathbb{C} + \mathfrak{u}$ with $\mathfrak{l}_\mathbb{C} \supset \mathfrak{t}_\mathbb{C}$ such that the roots α for $\mathfrak{l}_\mathbb{C}$ and $\mathfrak{u}$ are given by $\langle \alpha, \mathbf{1}_{m+1} \rangle = 0$ and $\langle \alpha, \mathbf{1}_{m+1} \rangle > 0$, respectively. Let Q be the parabolic subgroup of the complexified Lie group $G_\mathbb{C} = SO(2m+2, \mathbb{C})$ with Lie algebra $\mathfrak{q}$, and $L := G \cap Q \simeq U(1, m)$. Then the homogeneous space $X := G/L = SO_0(2, 2m)/U(1, m)$ is identified with the set of indefinite Hermitian structures on $\mathbb{R}^{2m+2}$ of signature $(1, m)$, and becomes a complex manifold as an open set of $G_\mathbb{C}/Q \simeq SO(2m+2)/U(m+1)$, the Grassmannian of isotropic subspaces of $\mathbb{C}^{2m+2}$ equipped with non-degenerate quadratic form. For $\lambda \in \mathbb{Z}$, let $\mathbb{C}_\lambda$ denote the holomorphic character of $L_\mathbb{C} \simeq GL(m+1, \mathbb{C})$ given by $\det^\lambda$, and we form a $G_\mathbb{C}$-equivariant holomorphic line bundle $\mathcal{L}_\lambda := G_\mathbb{C} \times_Q \mathbb{C}_\lambda$ over $G_\mathbb{C}/Q$. We shall use the same letter $\mathcal{L}_\lambda$ to denote its restriction $\mathcal{L}_\lambda|_X \simeq G \times_L \mathbb{C}_\lambda$ to the open subset $X \simeq G/L$ of $G_\mathbb{C}/Q$. With this notation, the canonical bundle Ω_X of X is given by $\mathcal{L}_m$.

Let $H^j_{\bar\partial}(X, \mathcal{L}_\lambda)$ be the j-th Dolbeault cohomology group with coefficients in $\mathcal{L}_\lambda$, which carries a natural Fréchet topology by the closed range theorem of the $\bar\partial$ operator [26]. We set

$$\mathcal{S}ol(D, \Delta) = \{f \in \mathcal{O}(D) : \Delta f = 0\},$$

and equip it with the topology of uniform convergence on every compact sets. We prove:

Theorem 1.1 (see Theorem 2.1)**.** *Let $\mathcal{R}$ be the cohomological integral transform (Penrose transform) defined in* (2.2) *below. Then $\mathcal{R}$ gives a topological G-isomorphism:*

$$\mathcal{R}\colon H^{m(m-1)}_{\bar\partial}(X, \mathcal{L}_{m-1}) \to \mathcal{S}ol(D, \Delta).$$

In the case $m = 2$, via the double covering $SU(2,2) \to G = SO_0(2,4)$, the group G is of type A, X is biholomorphic to $SU(2,2)/U(1,2)$, and D is biholomorphic to the 4-dimensional bounded symmetric domain of type A III. In this case, the bijectivity of $\mathcal{R}$ in Theorem 1.1 was first proved in [4], and later generalized in [17] by a different approach.

Theorem 1.1 in the case $m \ge 3$ consists of the following assertions:

(a) the range of $\mathcal{R}$ satisfies the differential equation;

(b) (surjectivity) the Penrose transform $\mathcal{R}$ constructs all the solutions;
(c) (injectivity) the kernel of $\mathcal{R}$ is zero;
(d) (non-vanishing) the $m(m-1)$-th cohomology does not vanish;
(e) (cohomological purity) the j-th cohomology vanishes if $j \neq m(m-1)$;
(f) (topology) $\mathcal{R}$ is not only a bijection but also a topological isomorphism.

There have been various approaches to (a) in certain settings, *e.g.,* Eastwood–Penrose–Wells [4], Mantini [15], Marastoni–Tanisaki [16] for sufficiently positive parameter λ. We note that the representations on the cohomologies with coefficients in the "good range" in the sense of Vogan [24] are well-understood by the Beilinson–Bernstein theory [1] or by the algebraic representation theory (*e.g.,* [24, 25]). However, Theorem 1.1 needs to treat the parameter λ **outside the good range** for which the general theory does not apply. It should be noted that there are **counterexamples** to what we may expect from (c)–(d):

(c)′ the Penrose transform may have an infinite-dimensional kernel;
(d)′ the Dolbeault cohomologies of all degrees may vanish.

See, *e.g.,* [18, 19] for (c)′; Kobayashi [6] and Trapa [23] for (d)′ for some classical groups. In fact, (d) is a special case of a long-standing problem in algebraic representation theory about when Zuckerman's derived functor module $A_{\mathfrak{q}}(\lambda) \neq 0$ for singular λ outside the good range.

For $m \geq 3$, there are two minimal representations of the group $G = SO_0(2, 2m)$ in the sense that their annihilator is the Joseph ideal [5] of the enveloping algebra $U(\mathfrak{g}_{\mathbb{C}})$, and they are dual to each other in our setting, see *e.g.,* [22]. Theorem 1.1 gives their complex-geometric realization.

Corollary 1.2. *For $m \geq 3$, the two minimal representations of $G = SO_0(2, 2m)$ are realized in the cohomologies $H^{m(m-1)}_{\overline{\partial}}(X, \mathcal{L}_{m-1})$ where $\mathfrak{q}$ is taken to be $\mathfrak{l}_{\mathbb{C}} + \mathfrak{u}$ or $\mathfrak{l}_{\mathbb{C}} + \mathfrak{u}^-$.*

Remark 1.3. Kobayashi–Ørsted [11] proposed yet another complex-geometric realization of the minimal representation of $O(p, q)$ by using Dolbeault cohomologies on a non-compact complex manifold when $p+q$ is even and $p, q \geq 2$. Our complex manifold X is different from the one in [11] with $p = 2$.

Last but not least, the representation of G on the Fréchet space of Dolbeault cohomologies in Theorem 1.1 contains the unique Hilbert space as its dense subspace on which G acts as an irreducible unitary representation, to be denoted by π, by [24] and by Proposition 4.4. In his paper [2], van Dijk classified "generalized Gelfand pairs" (G, H) under the assumption that G/H is a semisimple symmetric space of rank one.

In particular, $(G,H)=(SO_0(2,2n),SO_0(2,2n-1))$ is a generalized Gelfand pair, and thus $\dim \mathrm{Hom}_G(\pi,\mathcal{D}'(G/H))\le 1$.

2 Penrose transform

The morphism in Theorem 1.1 is the higher-dimensional Penrose transform, of which we review quickly from [18] the definition adapted to our specific situation.

Let K be a maximal compact subgroup of a linear reductive Lie group G, θ a complexified Cartan involution, $\mathfrak{q}=\mathfrak{l}_{\mathbb{C}}+\mathfrak{u}$ a θ-stable parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}}$, and $X=G/L$ the open G-orbit in the flag variety $G_{\mathbb{C}}/Q$ through the origin $o=eQ$, see *e.g.,* [10]. We consider a compact submanifold $C:=K/L\cap K\simeq K_{\mathbb{C}}/Q\cap K_{\mathbb{C}}$ in G/L, and write $\iota\colon C\hookrightarrow X$ for the natural embedding. Let S denote the complex dimension of C, and T be a maximal torus of $L\cap K$, hence that of K, too. We take a positive system $\Delta^+(\mathfrak{k}_{\mathbb{C}},\mathfrak{t}_{\mathbb{C}})$ containing the weights $\Delta(\mathfrak{u}\cap\mathfrak{k}_{\mathbb{C}},\mathfrak{t}_{\mathbb{C}})$. For a dominant character μ of T, we let V_μ denote the irreducible K-module with highest weight μ, and form a G-equivariant vector bundle $\mathcal{V}_\mu:=G\times_K V_\mu$ over the Riemannian symmetric space G/K. We write ℓ_g for the action of $g\in G$ on the line bundle $\mathcal{L}_\lambda$ over G/L. Then the natural map

$$\mathcal{E}^{0,S}(G/L,\mathcal{L}_\lambda)\times G\to\mathcal{E}^{0,S}(K/L\cap K,\iota^*\mathcal{L}_\lambda),\quad (\alpha,g)\mapsto \iota^*\ell_g^*\alpha$$

induces the one for Dolbeault cohomologies:

$$H^S_{\bar\partial}(G/L,\mathcal{L}_\lambda)\times G\to H^S_{\bar\partial}(K/L\cap K,\iota^*\mathcal{L}_\lambda),\quad ([\alpha],g)\mapsto[\iota^*\ell_g^*\alpha].$$

By the Borel–Weil–Bott theorem, the target space is finite-dimensional, and is K-isomorphic to the irreducible representation V_{μ_λ} as far as $\mu_\lambda:=\mathbb{C}_\lambda\otimes\Lambda^S(\mathfrak{k}_{\mathbb{C}}/(\mathfrak{q}\cap\mathfrak{k}_{\mathbb{C}}))$ is $\Delta^+(\mathfrak{k}_{\mathbb{C}},\mathfrak{t}_{\mathbb{C}})$-dominant. In turn, the above map yields a continuous G-homomorphism:

$$\mathcal{R}\colon H^S_{\bar\partial}(G/L,\mathcal{L}_\lambda)\to C^\infty(G/K,\mathcal{V}_{\mu_\lambda}),\quad [\alpha]\mapsto(g\mapsto[\iota^*\ell_g^*\alpha]),$$

which is referred to as a (higher-dimensional) Penrose transform, or to the Penrose transform in short ([18, Thm. 2.6]).

In our setting, the compact submanifold $C\simeq SO(2m)/U(m)$, $S=m(m-1)$, and $\mathfrak{u}=(\mathfrak{u}\cap\mathfrak{k}_{\mathbb{C}})\oplus(\mathfrak{u}\cap\mathfrak{p}_{\mathbb{C}})$ with

$$(2.1)\quad \Delta(\mathfrak{u}\cap\mathfrak{k}_{\mathbb{C}})=\{e_i+e_j:1\le i<j\le m\},\quad \Delta(\mathfrak{u}\cap\mathfrak{p}_{\mathbb{C}})=\{e_0+e_j:1\le j\le m\}.$$

Then halves the sums of the roots in $\Delta(\mathfrak{u}, \mathfrak{t}_{\mathbb{C}})$ and $\Delta(\mathfrak{u}\cap\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ are given respectively by

$$\rho(\mathfrak{u}) = \frac{m}{2}\mathbf{1}_{m+1}, \quad \rho(\mathfrak{u}\cap\mathfrak{k}_{\mathbb{C}}) = 0 \oplus \frac{m-1}{2}\mathbf{1}_m$$

in the standard coordinates $\mathfrak{t}_{\mathbb{C}}^* \simeq \mathbb{C}^{m+1}$. Since $\mu_\lambda = (m-1, 0, \ldots, 0)$ for $\lambda = (m-1)\mathbf{1}_{m+1}$, one has a G-intertwining operator

$$\mathcal{R}\colon H^{m(m-1)}_{\overline{\partial}}(G/L, \mathcal{L}_{m-1}) \to C^\infty(G/K, \mathcal{V}_{m-1}). \tag{2.2}$$

Here, by an abuse of notation we write $\mathcal{V}_{m-1}$ for the line bundle $\mathcal{V}_{(m-1,0,\ldots,0)}$ over G/K, which is isomorphic to $(\Omega_{G/K})^{\frac{m-1}{2m}}$ where $\Omega_{G/K}$ is the canonical bundle of G/K. Trivializing the line bundle $\mathcal{V}_{m-1}$ via the Harish-Chandra realization $G/K \xrightarrow{\sim} D \subset \mathfrak{p}_-$ of the Hermitian symmetric space, one may identify $\mathcal{F}(G/K, \mathcal{V}_{m-1})$ with $\mathcal{F}(D)$ for $\mathcal{F} = C^\infty$ or $\mathcal{O}$.

Now our theorem is formulated as follows.

Theorem 2.1. *The higher-dimensional Penrose transform $\mathcal{R}$ in* (2.2) *is injective, and its image coincides with $\mathcal{S}ol(D, \Delta)$ via the identification $\mathcal{O}(D) \simeq \mathcal{O}(G/K, \mathcal{V}_{m-1})$ where $G = SO_0(2, 2m)$. Moreover, $\mathcal{R}$ gives a G-equivariant topological isomorphism:*

$$\mathcal{R}\colon H^{m(m-1)}_{\overline{\partial}}(G/L, \mathcal{L}_{m-1}) \xrightarrow{\sim} \mathcal{S}ol(D, \Delta).$$

Remark 2.2. For $G = SO_0(2, 2m+1)$ or its covering group, the higher-dimensional Penrose transform can be defined in a similar geometric setting, however, we do not expect an analogous theorem holds. For instance, if $m = 1$, then via the double covering $Sp(2, \mathbb{R}) \to SO_0(2, 3)$, one sees that the range of the Penrose transform does not satisfy the differential equation $\Delta f = 0$ as was proved in [18] in the $Sp(n, \mathbb{R})$ case.

3 Differential operators on G/K

In this section we analyze the space $\mathcal{S}ol(D, \Delta)$ and see that it is realized naturally as a G-submodule of $\mathcal{O}(G/K, \mathcal{V}_{m-1})$ and compute its K-type formula.

We begin with some useful results about parabolic Verma modules. Let $\mathfrak{g}_{\mathbb{C}} = \mathfrak{so}(2m+2, \mathbb{C})$ with $m > 1$, $\mathbb{C}_\lambda$ be a character of $\mathfrak{k}_{\mathbb{C}} + \mathfrak{p}_+$ that takes the form $(\lambda, 0, \ldots, 0)$ on $\mathfrak{t}_{\mathbb{C}} \simeq \mathbb{C}^{m+1}$ for $\lambda \in \mathbb{C}$. By [14, Lem. 10.1], for $\lambda, \nu \in \mathbb{C}$, one has

$$\mathrm{Hom}_{\mathfrak{g}_{\mathbb{C}}}(U(\mathfrak{g}_{\mathbb{C}}) \otimes_{U(\mathfrak{k}_{\mathbb{C}}+\mathfrak{p}_+)} \mathbb{C}_{-\nu}, U(\mathfrak{g}_{\mathbb{C}}) \otimes_{U(\mathfrak{k}_{\mathbb{C}}+\mathfrak{p}_+)} \mathbb{C}_{-\lambda}) \neq \{0\}$$

if and only if

$$(\lambda, \nu) = (m-\ell, m+\ell) \qquad \text{for some } \ell \in \mathbb{N}. \tag{3.1}$$

In turn, it follows from the duality theorem, see *e.g.,* [12, Thm. 2.12] applied to $G_{\mathbb{C}} = G'_{\mathbb{C}} = SO(2m+2,\mathbb{C})$, that there is a holomorphic $\mathfrak{g}_{\mathbb{C}}$-intertwining differential operator between the $\mathfrak{g}_{\mathbb{C}}$-equivariant sheaves $\mathcal{O}(G_{\mathbb{C}}/K_{\mathbb{C}}\exp(\mathfrak{p}_+),\mathcal{V}_{m-\ell})$ and $\mathcal{O}(G_{\mathbb{C}}/K_{\mathbb{C}}\exp(\mathfrak{p}_+),\mathcal{V}_{m+\ell})$. Such an operator is unique up to scalar multiplication, and is given as the ℓ-th power of the holomorphic Laplacian on the Bruhat open cell $\mathfrak{p}_-$.

$$\Delta^\ell\colon \mathcal{O}(G_{\mathbb{C}}/K_{\mathbb{C}}\exp(\mathfrak{p}_+),\mathcal{V}_{m-\ell}) \to \mathcal{O}(G_{\mathbb{C}}/K_{\mathbb{C}}\exp(\mathfrak{p}_+),\mathcal{V}_{m+\ell}), \tag{3.2}$$

see [13]. See also Remark 3.3 for analogous operators in another real form.

In the case $\ell = 1$, one has a G-intertwining operator

$$\Delta\colon \mathcal{O}(D,\mathcal{V}_{m-1}) \to \mathcal{O}(D,\mathcal{V}_{m+1}). \tag{3.3}$$

Here is the K-structure of the kernel:

Proposition 3.1. *The kernel of the holomorphic Laplacian Δ is a G-submodule of $\mathcal{O}(D,\mathcal{V}_{m-1})$ with the following K-structure:*

$$\mathcal{S}ol(D,\Delta)_{K\text{-finite}} \simeq \bigoplus_{\ell=0}^{\infty} \mathbb{C}_{\ell+m-1} \boxtimes F^{SO(2m)}(\ell,0,\dots,0).$$

Remark 3.2. As we shall see in Theorem 2.1 and Proposition 4.4, the G-module $\mathcal{S}ol(D,\Delta)$ is irreducible.

Proof of Proposition 3.1. Let $\mathrm{Pol}^\ell(\mathfrak{p}_-)$ denote the space of homogeneous polynomials in $\mathfrak{p}_-$ of degree ℓ. We set

$$\mathcal{H}(\mathfrak{p}_-) := \{f \in \mathrm{Pol}(\mathfrak{p}_-) : \Delta f = 0\}, \quad \mathcal{H}^\ell(\mathfrak{p}_-) := \mathcal{H}(\mathfrak{p}_-) \cap \mathrm{Pol}^\ell(\mathfrak{p}_-).$$

Then $SO(2m)$ acts irreducibly on $\mathcal{H}^\ell(\mathfrak{p}_-)$ for every $\ell \in \mathbb{N}$ when $m > 1$, and its highest weight is given by $(\ell,0,\dots,0)$. Since the first factor $SO(2)$ of K acts on $\mathrm{Pol}^\ell(\mathfrak{p}_-) \simeq S^\ell(\mathfrak{p}_+)$ as the character $\mathbb{C}_\ell$, the irreducible decomposition of the K-module $\mathcal{H}(\mathfrak{p}_-)$ is given as $\bigoplus_{\ell=0}^\infty \mathcal{H}^\ell(\mathfrak{p}_-) \simeq \bigoplus_{\ell=0}^\infty \mathbb{C}_\ell \boxtimes F^{SO(2m)}(\ell,0,\dots,0)$. Now the proposition follows from the observation that the K-module structure on the underlying $(\mathfrak{g}_{\mathbb{C}},K)$-module $\mathcal{O}(G/K,\mathcal{V}_{m-1})_{K\text{-finite}}$ is given as the multiplicity-free direct sum $\mathrm{Pol}(\mathfrak{p}_-)\otimes(\mathbb{C}_{m-1}\boxtimes\mathbf{1}) \simeq \bigoplus_{\ell=0}^{\infty} S^\ell(\mathfrak{p}_+)\otimes(\mathbb{C}_{m-1}\boxtimes\mathbf{1})$. □

Remark 3.3. Let $P_{\mathbb{R}}$ a minimal parabolic subgroup of $G_{\mathbb{R}} = SO_0(2m+1,1)$. Then $G = SO_0(2,2m)$ and $G_{\mathbb{R}} = SO_0(2m+1,1)$ have the common complexifications $G_{\mathbb{C}} = SO(2m+2,\mathbb{C})$. We may regard $(G_{\mathbb{R}},P_{\mathbb{R}})$ as a real from of $(G_{\mathbb{C}},Q)$. Let $I(\lambda) =$

$\mathrm{Ind}_{P_{\mathbb{R}}}^{G_{\mathbb{R}}}(\mathbb{C}_\lambda)$ be the unnormalized spherical principal series representation induced from a character $\mathbb{C}_\lambda$ for $\lambda \in \mathbb{C}$. Our parametrization is taken to be the same with the one in the monograph [14] so that the trivial representation $\mathbf{1}$ of $G_{\mathbb{R}}$ occurs as the unique subrepresentation $I(0)$ and also as the unique quotient of $I(n)$, see [14, (2.11)]. Then the Knapp–Stein intertwining operator $I(\lambda) \to I(n-\lambda)$ has a pole at $\lambda \in \{\frac{n}{2}, \frac{n}{2}-1, \frac{n}{2}-2, \dots\}$ and its residue is a scalar multiple of the ℓ-th power of the (Riemannian) Laplacian Δ on the open Bruhat cell $\mathbb{R}^n$ [14, (4.29) and Remark 10.3] if we set $\ell := \frac{n}{2} - \lambda \in \mathbb{N}$. In particular, for $n = 2m$, the residue operator

$$\Delta\colon I(m-1) \to I(m+1)$$

is a $G_{\mathbb{R}}$-intertwining operator. This operator may be regarded as a "real form" of the holomorphic differential operator (3.3).

Remark 3.4. With the notation of the classification [3] of irreducible unitarizable lowest weight modules for $\mathfrak{g} = \mathfrak{so}(2,n)$, one has $A(\lambda_0) = \frac{n}{2}$ and $B(\lambda_0) = n-1$ if $\lambda_0 = (1-n)e_0$. Accordingly, the lowest weight $(\mathfrak{g}_{\mathbb{C}}, K)$-module $L(-(\lambda_0 + ze_0)) = L((n-1-z)e_0)$ is unitarizable if and only if $z \leq 0$ or $z \in \{\frac{n}{2}, n-1\}$. Its $\mathfrak{Z}(\mathfrak{g}_{\mathbb{C}})$-infinitesimal character is given by $(n-1-z)e_0 - \rho_G$. The underlying $(\mathfrak{g}_{\mathbb{C}}, K)$-module of $\mathcal{S}ol(D, \Delta)$ is isomorphic to $L((\frac{n}{2}-1)e_0)$ corresponding to the first reduction point $z = A(\lambda_0)$.

4 Generalized Blattner formula

In this section we examine the K-type formula of the Dolbeault cohomology group.

We recall $K = SO(2) \times SO(2m)$. Irreducible K-modules are parametrized by $\mu_0 \in \mathbb{Z}$ and $\mu = (\mu_1, \dots, \mu_m) \in \mathbb{Z}^m$ satisfying $\mu_1 \geq \cdots \geq \mu_{m-1} \geq |\mu_m|$. We write $\mathbb{C}_{\mu_0} \boxtimes F^{SO(2m)}(\mu)$ for the irreducible K-module with highest weight $(\mu_0; \mu_1, \dots, \mu_m)$.

Proposition 4.1. *As a K-module, $H^{m(m-1)}_{\bar{\partial}}(G/L, \mathcal{L}_{-1})_{K\text{-finite}}$ is multiplicity-free, and its K-type formula is given by*

$$\bigoplus_{\ell=0}^{\infty} \mathbb{C}_{\ell+m-1} \boxtimes F^{SO(2m)}(\ell, 0, \dots, 0).$$

Remark 4.2. In connection to the theory of visible actions on complex manifolds ([8]), Kobayashi raised a cohomological multiplicity-free conjecture in the general setting for branching problems of Zuckerman's derived functor modules [9, Conj. 4.2]. We observe that both (G, K) and (G, L) are reductive symmetric pairs, hence Proposition 4.1 gives an evidence of his conjecture.

The proof of Proposition 4.1 is based on a combinatorial computation of a generalized Blattner formula as in [6, Chap. 4]. Since the statement of this section is purely algebraic, we need only the underlying $(\mathfrak{g}_{\mathbb{C}}, K)$-modules of the cohomologies, namely, Zuckerman's derived functor modules. As an algebraic analogue of the Dolbeault cohomology with coefficients in a G-equivariant holomorphic vector bundle over the complex manifold G/L, Zuckerman introduced a derived functor $\mathcal{R}_{\mathfrak{q}}^j \equiv (\mathcal{R}_{\mathfrak{q}}^{\mathfrak{g}_{\mathbb{C}}})^j$ ($j \in \mathbb{N}$) as a cohomological parabolic induction. We follow [6, 7, 25] for the normalization so that $\mathcal{R}_{\mathfrak{q}}^j$ is a covariant functor from the category of metaplectic $(\mathfrak{l}_{\mathbb{C}}, (L \cap K)^{\sim})$-modules to the category of $(\mathfrak{g}_{\mathbb{C}}, K)$-modules and that if $\nu \in \mathfrak{h}_{\mathbb{C}}^*/W(\mathfrak{l}_{\mathbb{C}})$ is the $\mathfrak{Z}(\mathfrak{l}_{\mathbb{C}})$-infinitesimal character of an $(\mathfrak{l}_{\mathbb{C}}, (L \cap K)^{\sim})$-module V then the $\mathfrak{Z}(\mathfrak{g}_{\mathbb{C}})$-infinitesimal character of $\mathcal{R}_{\mathfrak{q}}^S(V)$ equals $\nu \in \mathfrak{h}_{\mathbb{C}}^*/W(\mathfrak{g}_{\mathbb{C}})$.

Retain the setting as in Section 2. By an abuse of notation, we write $\mathbb{C}_{\lambda-\rho(\mathfrak{u})}$ for the metaplectic $(\mathfrak{l}_{\mathbb{C}}, (L \cap K)^{\sim})$-character with its differential $\lambda \mathbf{1}_{m+1} - \rho(\mathfrak{u})$ when $\lambda \in \mathbb{Z}$. Since $\lambda \mathbf{1}_{m+1} - \rho(\mathfrak{u}) = (\lambda - \frac{m}{2})\mathbf{1}_{m+1}$, one has the following $(\mathfrak{g}_{\mathbb{C}}, K)$-isomorphisms [26]:

$$(4.1) \qquad H_{\bar{\partial}}^j(G/L, \mathcal{L}_\lambda)_{K\text{-finite}} \simeq \mathcal{R}_{\mathfrak{q}}^j(\mathbb{C}_{\lambda-\rho(\mathfrak{u})}) = \mathcal{R}_{\mathfrak{q}}^j(\mathbb{C}_{(\lambda-\frac{m}{2})\mathbf{1}_{m+1}}) \qquad \text{for all } j,$$

which has $\mathfrak{Z}(\mathfrak{g}_{\mathbb{C}})$-infinitesimal character $(\lambda - \frac{m}{2})\mathbf{1}_{m+1} + \rho_{\mathfrak{l}} = \lambda \mathbf{1}_{m+1} + (0, -1, \ldots, -m)$. Then the generalized Blattner formula for Zuckerman's derived functor modules asserts the following identity

$$(4.2) \quad \sum_i (-1)^i \dim \operatorname{Hom}_K(\pi, \mathcal{R}_{\mathfrak{q}}^{S-i}(\mathbb{C}_{\lambda-\rho(\mathfrak{u})})) \\ = \sum_j (-1)^j \dim \operatorname{Hom}_{L\cap K}(H^j(\mathfrak{u} \cap \mathfrak{k}_{\mathbb{C}}, \pi), S(\mathfrak{u} \cap \mathfrak{p}_{\mathbb{C}}) \otimes \mathbb{C}_{\mu_\lambda}),$$

for any $\pi \in \widehat{K}$, where $S = \dim_{\mathbb{C}}(\mathfrak{u} \cap \mathfrak{k}_{\mathbb{C}}) = m(m-1)$ and $S(\mathfrak{u} \cap \mathfrak{p}_{\mathbb{C}})$ denotes the space of symmetric tensors.

We are particularly interested in the case $\lambda = m - 1$ with $m \geq 2$. Then $\mu_\lambda = \mathbb{C}_\lambda \otimes \Lambda^S(\mathfrak{k}_{\mathbb{C}}/(\mathfrak{q} \cap \mathfrak{k}_{\mathbb{C}}))$ amounts to $(m-1) \oplus 0\mathbf{1}_m$. On the other hand, the parameter of the metaplectic $(\mathfrak{l}_{\mathbb{C}}, (L \cap K)^{\sim})$-character $\mathbb{C}_{\lambda-\rho(\mathfrak{u})} = \mathbb{C}_{(\frac{m}{2}-1)\mathbf{1}_{m+1}}$, see (4.1), lies in the weakly fair range with respect to $\mathfrak{q}$, namely, $\langle \lambda \mathbf{1}_{m+1} - \rho(\mathfrak{u}), \alpha \rangle \geq 0$ for any $\alpha \in \Delta(\mathfrak{u})$. The general theory [26] guarantees neither the irreducibility nor the non-vanishing of $\mathcal{R}_{\mathfrak{q}}^S(\mathbb{C}_{\lambda-\rho(\mathfrak{u})})$ in the weakly fair range, but implies $\mathcal{R}_{\mathfrak{q}}^{S-i}(\mathbb{C}_{\lambda-\rho(\mathfrak{u})}) = 0$ for $i \neq 0$ and the unitarizability of $\mathcal{R}_{\mathfrak{q}}^S(\mathbb{C}_{\lambda-\rho(\mathfrak{u})})$ unless it vanishes. In particular, the left-hand side of (4.2) is equal to $\dim \operatorname{Hom}_K(\pi, \mathcal{R}_{\mathfrak{q}}^S(\mathbb{C}_{\lambda-\rho(\mathfrak{u})}))$.

Let us compute the alternating sum in the right-hand side of (4.2). We recall $(K, L \cap K) = (SO(2) \times SO(2m), \mathbb{T} \times U(m))$. The first factor $SO(2) \simeq \mathbb{T}$ sits in the center of K, hence it does not affect the computation of the Weyl group $W(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ below. We recall $\mathfrak{u} \cap \mathfrak{k}_{\mathbb{C}}$ from (2.1), and set

$$\begin{aligned}
\Delta^+(w) :=& \Delta^+(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) \cap w \cdot \Delta^-(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}), \\
\ell(w) :=& \# \Delta^+(w), \\
W_K^{\mathfrak{l}\cap\mathfrak{k}} :=& \{w \in W(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) : \Delta^+(w) \subset \Delta(\mathfrak{u} \cap \mathfrak{k}_{\mathbb{C}})\}, \\
=& \{w \in W(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) : w\mu \text{ is dominant for } \Delta^+(\mathfrak{l}_{\mathbb{C}} \cap \mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) \\
& \qquad \text{whenever } \mu \text{ is dominant for } \Delta^+(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})\}.
\end{aligned} \tag{4.3}$$

Let $\mu_0 \in \mathbb{Z}$ and $\mu \in \mathbb{Z}^m$ satisfy $\mu_1 \geq \cdots \geq \mu_{m-1} \geq |\mu_m|$. For $\pi = \mathbb{C}_{\mu_0} \boxtimes F^{SO(2m)}(\mu)$, the j-th cohomology group $H^j(\mathfrak{u} \cap \mathfrak{k}_{\mathbb{C}}, \pi)$ is isomorphic to

$$\mathbb{C}_{\mu_0} \boxtimes \bigoplus_{\substack{w \in W_K^{\mathfrak{l}\cap\mathfrak{k}} \\ \ell(w)=j}} F^{L\cap K}(w(\mu + \rho_c) - \rho_c) \tag{4.4}$$

as $(L \cap K)$-modules by Kostant's Borel–Weil–Bott theorem.

On the other hand, since $\mathfrak{u} \cap \mathfrak{p}_{\mathbb{C}}$ is isomorphic to $\mathbb{C}_1 \boxtimes \mathbb{C}^m$ as an $L \cap K \simeq \mathbb{T} \times U(m)$-module, one has

$$S(\mathfrak{u} \cap \mathfrak{p}_{\mathbb{C}}) \simeq \bigoplus_{\ell=0}^{\infty} \mathbb{C}_\ell \boxtimes S^\ell(\mathbb{C}^m) = \bigoplus_{\ell=0}^{\infty} \mathbb{C}_\ell \boxtimes F^{U(m)}(\ell, 0, \ldots, 0). \tag{4.5}$$

To compute the alternating sum in the right-hand side of (4.2), we compare irreducible $(L \cap K)$-modules occurring in (4.4) and (4.5). Then the following combinatorial result plays a key role.

Lemma 4.3. *For $\mu_0 \in \mathbb{Z}$, $\mu \in \mathbb{Z}^m$ satisfying $\mu_1 \geq \cdots \geq \mu_{m-1} \geq |\mu_m|$ and $w \in W_K^{\mathfrak{l}\cap\mathfrak{k}}$, the following two conditions are equivalent:*
(i) $\mu_0 = \ell + m - 1$, $w(\mu + \rho_c) - \rho_c = (\ell, 0, \ldots, 0)$,
(ii) $w = e$, $\mu_0 = \ell + m - 1$, *and* $\mu = (\ell, 0, \ldots, 0)$.

Proof. Let us verify (i) $\Rightarrow$ (ii). By the definition (4.3), any element $w \in W_K^{\mathfrak{l}\cap\mathfrak{k}}$ is of the form

$$w\mu = (\mu_1, \ldots, \widehat{\mu_{j_1}}, \ldots, \widehat{\mu_{j_{2r}}}, \ldots, \mu_m, -\mu_{j_{2r}}, \ldots, -\mu_{j_1})$$

for some $1 \leq j_1 < j_2 < \cdots < j_{2r} \leq m$ with $0 \leq r \leq \frac{m}{2}$. If $r > 0$, then the last component of $w(\mu + \rho_c) - \rho_c$ amounts to $-(\mu_{j_1} + m - j_1)$ which is negative, whence $w(\mu + \rho_c) - \rho_c \neq (\ell, 0, \ldots, 0)$. Thus the implication (i) $\Rightarrow$ (ii) follows.

The converse is implication (ii) $\Rightarrow$ (i) is obvious. $\square$

Proof of Proposition 4.1. Suppose π is a K-type of the form $\mathbb{C}_{\mu_0} \boxtimes F^{SO(2m)}(\mu_1, \ldots, \mu_m)$ with $\mu_1 \geq \cdots \geq \mu_{m-1} \geq |\mu_m|$. Then it follows from (4.4) and Lemma 4.3 that the right-hand side of (4.2) equals

$$\begin{cases} 1 & \text{if } \mu_0 + 1 - m \in \mathbb{N} \text{ and } \mu = (\mu_0 + 1 - m, 0, \ldots, 0), \\ 0 & \text{otherwise.} \end{cases}$$

Thus the proposition is shown by (4.1). $\square$

As we have mentioned, the general theory [25] of Zuckerman's derived functor does not guarantee the irreducibility of $\mathcal{R}^S_{\mathfrak{q}}(\mathbb{C}_{\lambda-\rho(\mathfrak{u})})$ because $\lambda - \rho(\mathfrak{u})$ is not in the good range, namely, $\langle \lambda \mathbf{1}_{m+1} - \rho(\mathfrak{u}) + \rho_{\mathfrak{l}}, \alpha \rangle$ is not necessarily positive for all $\alpha \in \Delta(\mathfrak{u})$ as $\lambda \mathbf{1}_{m+1} - \rho(\mathfrak{u}) + \rho_{\mathfrak{l}} = (m-1, m-2, \ldots, 0, -1)$. Nevertheless, in our specific setting, one has the following irreducibility result:

Proposition 4.4. *The G-module $H^{m(m-1)}_{\overline{\partial}}(G/L, \mathcal{L}_{m-1})$ is non-zero and irreducible.*

Proof. Our proof utilizes the multiplicity-free K-type formula in Proposition 4.1. Let $W_\ell := \mathbb{C}_{\ell+m-1} \boxtimes F^{SO(2m)}(\ell, 0, \ldots, 0)$. Suppose V is an irreducible $(\mathfrak{g}_{\mathbb{C}}, K)$-submodule in $H^{m(m-1)}_{\overline{\partial}}(G/L, \mathcal{L}_{m-1})_{K\text{-finite}}$. Since the K-type formula in Proposition 4.1 is multiplicity-free, the K-type of V is of the form $\oplus_{\ell \in J} W_\ell$ for some subset $J \subset \mathbb{N}$. We shall show $J = \mathbb{N}$. Assume it were not the case. Since the underlying $(\mathfrak{g}_{\mathbb{C}}, K)$-module of the G-module $H^{m(m-1)}_{\overline{\partial}}(G/L, \mathcal{L}_{m-1})$ is unitarizable, it is completely reducible. Therefore, by replacing J with $\mathbb{N} \setminus J$ if necessary, we may find $N \in J$ such that $N + 1 \notin J$. Then $V \cap \oplus_{\ell \leq N} W_\ell$ would be a $(\mathfrak{g}_{\mathbb{C}}, K)$-submodule of V because the $SO(2m)$-type in $\mathfrak{p}_{\mathbb{C}} W_\ell := \mathbb{C}\text{-span}\{Xv : X \in \mathfrak{p}_{\mathbb{C}},\, v \in W_\ell\}$ must be either $W_{\ell+1}$ or $W_{\ell-1}$ but $W_{N+1} \not\subset V$ by the choice of N.

On the other hand, the G-module $H^{m(m-1)}_{\overline{\partial}}(G/L, \mathcal{L}_{m-1})$ cannot have a non-trivial finite-dimensional submodule except for the trivial one-dimensional representation because it is unitarizable. But the trivial one-dimensional representation cannot be a submodule because the $\mathfrak{Z}(\mathfrak{g}_{\mathbb{C}})$-infinitesimal character of the cohomology is $(m-1, m-2, \ldots, 0, -1)$ in the Harish-Chandra parametrization. Hence the proposition is proved. $\square$

5 Proof of Theorem 2.1 and Corollary 1.2

This section completes the proof of our main results. We have seen in Section 2 that the Penrose transform is a G-intertwining operator:

$$\mathcal{R} \colon H^{m(m-1)}_{\overline{\partial}}(G/L, \mathcal{L}_{m-1}) \to C^\infty(G/K, \mathcal{V}_{m-1}).$$

Since $\Delta(\mathfrak{u}\cap\mathfrak{p}_{\mathbb{C}}) \subset \Delta(\mathfrak{p}_+)$, X has a K-equivariant holomorphic fiber bundle structure over C, one has from [20, 21] that $\mathcal{R}$ is non-zero on $W_0 = \mathbb{C}_{m-1} \boxtimes \mathbb{C}$, and Image $\mathcal{R} \subset \mathcal{O}(G/K, \mathcal{V}_{m-1})$.

By the irreducibility of the G-module $H_{\overline{\partial}}^{m(m-1)}(G/L, \mathcal{L}_{m-1})$ given by Proposition 4.4, one sees that $\mathcal{R}$ is injective. Since $\mathcal{O}(G/K, \mathcal{V}_{m-1})$ is K-multiplicity-free, one obtains the following proposition from Propositions 3.1 and 4.1.

Proposition 5.1. *The Penrose transform $\mathcal{R}$ in* (2.2) *induces a $(\mathfrak{g}_{\mathbb{C}}, K)$-isomorphism between the underlying $(\mathfrak{g}_{\mathbb{C}}, K)$-modules of $H_{\overline{\partial}}^{m(m-1)}(G/L, \mathcal{L}_{m-1})$ and $\mathcal{S}ol(D, \Delta)$.*

Now Theorem 2.1 follows from the general argument on the maximal globalization as in [18] because both the G-module $H_{\overline{\partial}}^{m(m-1)}(G/L, \mathcal{L}_{m-1})$ and $\mathcal{S}ol(D, \Delta)$ are the maximal globalizations of their underlying $(\mathfrak{g}_{\mathbb{C}}, K)$-modules.

Since the G-module $H_{\overline{\partial}}^{m(m-1)}(G/L, \mathcal{L}_{m-1})$ is irreducible (Proposition 4.4), and its underlying $(\mathfrak{g}_{\mathbb{C}}, K)$-module is a lowest weight module [20, 21] with the K-type formula as in Proposition 4.1, it is identified with one of the two minimal $(\mathfrak{g}_{\mathbb{C}}, K)$-module, see [22]. The same argument applies if we replace $\mathfrak{q} = \mathfrak{l}_{\mathbb{C}} + \mathfrak{u}$ by $\mathfrak{l}_{\mathbb{C}} + \mathfrak{u}^-$. Thus Corollary 1.2 is also shown.

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