ICE-closed subcategories and epibricks over one-point extensions

Xin Li, Hanpeng Gao*

School of Mathematical Sciences, Anhui University, Hefei 230601, *P.R. China*

Abstract Let B be the one-point extension algebra of A by an A-module M. We proved that every ICE-closed subcategory in $\operatorname{mod} A$ can be extended to be some ICE-closed subcategories in $\operatorname{mod} B$. In the same way, every epibrick in $\operatorname{mod} A$ can be extended to be some epibricks in $\operatorname{mod} B$. The number of ICE-closed subcategories in $\operatorname{mod} B$ and the number of ICE-closed subcategories in $\operatorname{mod} A$ are denoted respectively as m, n. We can conclude the following inequality:

$$m \geq 2n$$

This is the analogical in epibricks. As an application, we can get some wide τ-tilting modules of B by wide τ-tilting modules of A.

Keywords ICE-closed subcategory; epibrick; one-point extension

MR(2020) Subject Classification 16G20

1. Introduction

Several kinds of subcategories have been researched in the representation theory of algebras. For example, torsion class and torsion-free class are the key points of these subcategories. Torsion class is closed under quotients and extensions and can be classified by support τ-tilting modules in [7], which is an important breakthrough in classification of these subcategories. Similarly, Haruhisa Enomoto's paper given us a uniform way to classify torsion-free class by considering the information on monobricks.

Bricks and semibricks are considered in [6], [2]. Moreover, the semibrick has been studied from the point of view of τ-tilting theory in [1]. In 2021, Haruhisa Enomoto given the definition of monobrick in [4]: a set of bricks where every non-zero map between elements of bricks' isomorphism classes is an injection. The set of simple objects provides an effective approach to investigate torsion-free class and wide subcategories. Because monobricks are in bijiection with left Schur subcategories, which are same as subcategories closed under kernels, images and extensions. Without using τ-tilting theory, it infers several noted consequence on torsion class and wide subcategories via monobricks.

In 2022, the concept of ICE-closed subcategories of module categories have been introduced by Haruhisa Enomoto in [3]. The ICE-closed subcategory closed under images,cokernels and extensions correlates closely with torsion class and wide subcategory. It is worthy mentioning that representative instances of ICE-closed subcategory are torsion class and wide subcategory .

Supported by the National Natural Science Foundation of China (Grant No. 11871071, 12371015, 12301041) and the Science Fundation for Distinguished Young Scholars of Anhui Province (Grant No. 2108085J01).
* Corresponding author
E-mail address: lxin909@163.com(Xin LI); hpgao07@163.com(Hanpeng GAO)

Therefore, the ICE-closed subcategory can be seen as generation of these two classes. She proved that the number of ICE-closed subcategories does not dictated by the orientation of the quiver and given a clear formula for each Dynkin type.

In this paper, we construct ICE-closed subcategories and epibricks over the one-point extension B of an algebra A by an A-module M_A. The following is our main results of this article.

Theorem 1.1. *Let B be the one-point extension algebra of A by an A-module M_A and $\mathcal{T}_A$ be an ICE-closed subcategory in* $\operatorname{mod} A$.

(1) $\mathcal{T}_B := \{(N_A, 0, 0) | N_A \in \mathcal{T}_A\}$, *$\mathcal{T}_B$ is an ICE-closed subcategory in* $\operatorname{mod} B$.

(2) $\mathcal{T}_B := \{(N_A, k^n, f), (0, k^n, 0) | N_A \in \mathcal{T}_A, n \in \mathbb{N}, f : k^n \otimes_k M_A \to N_A\}$, *$\mathcal{T}_B$ is an ICE-closed subcategory in* $\operatorname{mod} B$.

Theorem 1.2. *Let B be the one-point extension algebra of A by an A-module M_A and $\mathcal{S}_A$ be an epibrick in* $\operatorname{mod} A$.

(1) $\mathcal{S}_B := \{(s, 0, 0) | s \in \mathcal{S}_A\}$, *$\mathcal{S}_B$ is an epibrick in* $\operatorname{mod} B$.

(2) $\mathcal{S'}_B := \{(s, 0, 0), (0, k, 0) | s \in \mathcal{S}_A\}$, *$\mathcal{S}_B$ is an epibrick in* $\operatorname{mod} B$.

Throughout this paper, all algebras will be basic, connected, finite dimensional K-algebras over an algebraically closed field K. Let A be an algebra, $\operatorname{mod} A$ will be the category of finitely generated right A-modules and τ the Auslander-Reiten translation of A. We also denote by $|M|$ the number of pairwise nonisomorphic indecomposable summands of M, $\operatorname{add} M$ the subcategory consisting of direct summands of finite direct sums of M for $M \in \operatorname{mod} A$. Given an algebra $A = KQ/I$, let P_i be the indecomposable projective module, S_i the simple module, e_i the primitive idempotent element of an algebra corresponding to the point i.

2. Preliminaries

2.1 Basic definitions

In this section, we recall some basic definitions about ICE-closed subcategory in $\operatorname{mod} A$ and introduce the concept of epibrick in $\operatorname{mod} A$. First of all, We give several conditions for a subcategory of $\operatorname{mod} A$, including closed under images, cokernels, extension, quotients and so on.

Definition 2.1. ([3]) *Let A be an artin algebra and $\mathcal{T}$ a subcategory in* $\operatorname{mod} A$.

(1) $\mathcal{T}$ is closed under images (resp. kernels, cokernels) if for every map $f : M \to N$ with $M, N \in \mathcal{T}$, we have Im $f \in \mathcal{T}$ (resp. Ker $f \in \mathcal{T}$, Coker $f \in \mathcal{T}$).

(2) $\mathcal{T}$ is closed under extensions if for every short exact sequence in $\operatorname{mod} A$

$$0 \to N_1 \to N_2 \to N_3 \to 0$$

with $N_1, N_3 \in \mathcal{T}$, we have $N_2 \in \mathcal{T}$.

(3) $\mathcal{T}$ is closed under quotients if for every exact sequence in $\operatorname{mod} A$

$$N_1 \to N_2 \to 0$$

with $N_1 \in \mathcal{T}$, we have $N_2 \in \mathcal{T}$.

Then we can get the definitions of these subcategories.

Definition 2.2. ([3]) *Let A be an artin algebra and $\mathcal{T}$ a subcategory in* $\operatorname{mod} A$*.*

(1) $\mathcal{T}$ is a torsion class if $\mathcal{T}$ is closed under quotients and extensions.

(2) $\mathcal{T}$ is a wide subcategory if $\mathcal{T}$ is closed under kernels, cokernels, extensions.

(3) $\mathcal{T}$ is an ICE-closed subcategory if $\mathcal{T}$ is closed under images, cokernels, extensions.

Corollary 2.3. ([3]) *All torsion classes and wide subcategories are ICE-closed subcategories.*

Proof If $\mathcal{T}$ is a torsion class in $\operatorname{mod} A$, then $\mathcal{T}$ is closed under quotients and extensions. For every map $f : M \to N$ with $M, N \in \mathcal{T}$, We have

$$M \to Imf \to 0, N \to Cokerf \to 0$$

$Imf \in \mathcal{T}, Cokerf \in \mathcal{T}$. Since $M, N \in \mathcal{T}$ and $\mathcal{T}$ is closed under extensions. $\mathcal{T}$ is an ICE-closed subcategory.

In the same way, if $\mathcal{T}$ is a wide subcategory, then $\mathcal{T}$ is closed under kernels, cokernels and extensions. For every map $f : M \to N$ with $M, N \in \mathcal{T}$, We can get $Kerf \in \mathcal{T}$, $Cokerf \in \mathcal{T}$. Then for map $g : N \to Cokerf$ with $N, Cokerf \in \mathcal{T}$, we have $Kerg \in \mathcal{T}$ and $kerg = Imf$. That is $Imf \in \mathcal{T}$. $\mathcal{T}$ is an ICE-closed subcategory.

Next we give the definition of epibrick.

Definition 2.4. ([4]) *Let $\mathcal{S} \in$* $\operatorname{mod} A$*.*

(1) $\mathcal{S}$ is a brick if $\operatorname{End}_A(\mathcal{S})$ is a division ring. The set of isoclasses of bricks in $\operatorname{mod} A$ *is denoted by brick A.*

(2) A subset $\mathcal{S} \subseteq$ brick$A$ is called a semibrick if every morphism between elements of $\mathcal{S}$ is either zero or an isomorphism in A. The set of semibricks in $\operatorname{mod} A$ *is denoted by sbrick A.*

(3) A subset $\mathcal{S} \subseteq$ brick$A$ is called a monobrick if every morphism between elements of $\mathcal{S}$ is either zero or an injection in A. The set of monobricks in $\operatorname{mod} A$ *is denoted by mbrick A.*

(4) A subset $\mathcal{S} \subseteq$ brick$A$ is called an epibrick if every morphism between elements of $\mathcal{S}$ is either zero or a surjection in A. The set of epibricks in $\operatorname{mod} A$ *is denoted by ebrick A.*

It is easy to know that every semibrick is a monobrick or epibrick. By Schur's Lemma,every simple module is brick, and a set of isoclasses of simple modules is a semibrick.

Let $M \in \operatorname{mod} A$. The *one-point extension* of A by M_A is given by the following matrix algebra

$$B := \begin{pmatrix} A & 0 \\ M_A & k \end{pmatrix}$$

with the ordinary matrix and the multiplication induced by the module structure of M_A. All B-modules can be seen as (N_A, k^n, f), where $N_A \in \operatorname{mod} A$, $n \in \mathbb{N}$ and $f : k^n \otimes_k M_A \to N_A$. The morphisms from (N_A, k^{n_1}, f_1) to (N'_A, k^{n_2}, f_2) are pairs of (f, g),where $f \in \operatorname{Hom}_A(N_A, N'_A)$ and $g \in \operatorname{Hom}(k^{n_1}, k^{n_2})$, such that the following diagram commutes,

$$\begin{tikzcd} k^{n_1} \otimes_k M_A \arrow[r, "f_1"] \arrow[d, "g\otimes M_A"'] & N_A \arrow[d, "f"] \\ k^{n_2} \otimes_k M_A \arrow[r, "f_2"] & N'_A \end{tikzcd}$$

A sequence

$$0 \to (N_A, k^{n_1}, f_1) \stackrel{(h_1,g_1)}{\longrightarrow} (N'_A, k^{n_2}, f_2) \stackrel{(h_2,g_2)}{\longrightarrow} (N''_A, k^{n_3}, f_3) \to 0$$

in $\operatorname{mod} B$ is exact if and only if

$$0 \to N_A \xrightarrow{h_1} N'_A \xrightarrow{h_2} N''_A \to 0$$

is exact in $\operatorname{mod} A$ and

$$0 \to k^{n_1} \xrightarrow{g_1} k^{n_2} \xrightarrow{g_2} k^{n_3} \to 0$$

is exact in $\operatorname{mod} k$.

3. Main Result

In this section, we will give ICE-closed subcategories (resp. epibricks) of $\operatorname{mod} B$ via an ICE-closed subcategory (resp. epibrick) of $\operatorname{mod} A$ in two different ways, where B is one-point extension algebra of A by an A-module M_A.

Theorem 3.1. *Let B be the one-point extension algebra of A by an A-module M_A and $\mathcal{T}_A$ be an ICE-closed subcategory in* $\operatorname{mod} A$.

(1) $\mathcal{T}_B := \{(N_A, 0, 0) | N_A \in \mathcal{T}_A\}$, $\mathcal{T}_B$ *is an ICE-closed subcategory in* $\operatorname{mod} B$.

(2) $\mathcal{T}_B := \{(N_A, k^n, f), (0, k^n, 0) | N_A \in \mathcal{T}_A, n \in \mathbb{N}, f : k^n \otimes_k M_A \to N_A\}$, $\mathcal{T}_B$ *is an ICE-closed subcategory in* $\operatorname{mod} B$.

Proof (1) Firstly, we check $\mathcal{T}_B$ is closed under extensions. Given an arbitrary short exact sequence in $\operatorname{mod} B$: $0 \to (N_1, 0, 0) \longrightarrow (N, k^n, f) \longrightarrow (N_2, 0, 0) \to 0$, $(N_1, 0, 0)$, $(N_2, 0, 0) \in \mathcal{T}_B$, we have $0 \to N_1 \longrightarrow N \longrightarrow N_2 \to 0$ is exact in $\operatorname{mod} A$ and $0 \to 0 \longrightarrow k^n \longrightarrow 0 \to 0$ is exact in $\operatorname{mod} k$. Then $N \in \mathcal{T}_A$ and $n = 0$. Since $N_1,\ N_2 \in \mathcal{T}_A$ and $\mathcal{T}_A$ is closed under extension . Therefore, $(N, k^n, f) = (N, 0, 0) \in \mathcal{T}_B$.

Secondly, we check $\mathcal{T}_B$ is closed under images. Given a map $F : (N_1, 0, 0) \to (N_2, 0, 0)$, $(N_1, 0, 0),\ (N_2, 0, 0) \in \mathcal{T}_B$. $F = (f, g)$, where $f : N_1 \to N_2,\ g = 0$. Obviously, $Img = 0$. $Imf \in \mathcal{T}_A$. Because $N_1,\ N_2 \in \mathcal{T}_A$ and $\mathcal{T}_A$ is closed under images. $ImF = (Imf, Img, h) = (Imf, 0, 0) \in \mathcal{T}_B$.

Finally, we check $\mathcal{T}_B$ is closed under cokernels. Given a map $F : (N_1, 0, 0) \to (N_2, 0, 0)$, $(N_1, 0, 0), (N_2, 0, 0) \in \mathcal{T}_B$. $F = (f, g)$, where $f : N_1 \to N_2,\ g = 0$. It is easy to know that $Cokerg = 0$ and $Cokerf \in \mathcal{T}_A$. Because $N_1,\ N_2 \in \mathcal{T}_A$ and $\mathcal{T}_A$ is closed under cokernels. $CokerF = (Cokerf, Cokerg, h) = (Cokerf, 0, 0) \in \mathcal{T}_B$.

(2) Firstly, we check $\mathcal{T}_B$ is closed under extensions. Given an arbitrary short exact sequence in $\operatorname{mod} B$: $0 \to (N_1, k^{n_1}, f_1) \longrightarrow (N_2, k^{n_2}, f_2) \longrightarrow (N_3, k^{n_3}, f_3) \to 0$, (N_1, k^{n_1}, f_1), $(N_3, k^{n_3}, f_3) \in \mathcal{T}_B$, we have $0 \to N_1 \longrightarrow N_2 \longrightarrow N_3 \to 0$ is exact in $\operatorname{mod} A$ and $0 \to k^{n_1} \longrightarrow k^{n_2} \longrightarrow k^{n_3} \to 0$ is exact in $\operatorname{mod} k$. Then $N_2 \in \mathcal{T}_A$ since $N_1,\ N_3 \in \mathcal{T}_A$ and $\mathcal{T}_A$ is closed under extension. And $n_2 = n_1 + n_3 \in \mathbb{N}$. Therefore, $(N_2, k^{n_2}, f_2) \in \mathcal{T}_A$. In the same way, we can proof that $\{(0, k^n, 0)\}$ is closed under extensions.

Secondly, we check $\mathcal{T}_B$ is closed under images. Given a map $F : (N_1, k^{n_1}, f_1) \to (N_2, k^{n_2}, f_2)$, $(N_1, k^{n_1}, f_1),\ (N_2, k^{n_2}, f_2) \in \mathcal{T}_B$. $F = (f, g)$, where $f : N_1 \to N_2,\ g = k^{n_1} \to k^{n_2}$. $N_1,\ N_2 \in \mathcal{T}_A$ and $\mathcal{T}_A$ is closed under images. So $Imf \in \mathcal{T}_A$. Img is subspace of k^{n_2}. Then Img is n dimensional vector space, $n \in \mathbb{N}$. $ImF = (Imf, Img, h) \in \mathcal{T}_B$, $h : Img \otimes_k M_A \to Imf$. Similarly, we can proof that $\{(0, k^n, 0)\}$ is closed under images($f = 0$).

Finally, we check $\mathcal{T}_B$ is closed under cokernels. Given a map $F : (N_1, k^{n_1}, f_1) \to (N_2, k^{n_2}, f_2)$, $(N_1, k^{n_1}, f_1),\ (N_2, k^{n_2}, f_2) \in \mathcal{T}_B$. $F = (f, g)$, where $f : N_1 \to N_2,\ g = k^{n_1} \to k^{n_2}$. $N_1,\ N_2 \in \mathcal{T}_A$ and $\mathcal{T}_A$ is closed under cokernels. So $Cokerf \in \mathcal{T}_A$. Obviously $Cokerg$ is n dimensional vector space, $n \in \mathbb{N}$. $CokerF = (Cokerf, Cokerg, h) \in \mathcal{T}_B$, $h : Cokerg \otimes_k M_A \to Cokerf$. Similarly, we can proof that $\{(0, k^n, 0)\}$ is closed under cokernels($f = 0$).

Example 3.2. (1) $B := KQ_B$, $Q_B : 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$.Let $A := KQ_A$, $Q_A : 2 \xrightarrow{\beta} 3$, $M_A = \langle \alpha \rangle \cong P_2$. Then $B := \begin{pmatrix} A & 0 \\ M_A & k \end{pmatrix}$. The irreducible representations of A are $P_2 : k \to k$, $S_2 : k \to 0$, $S_3 : 0 \to k$. The ICE-closed subcategories in $\operatorname{mod} A$ are $\operatorname{add}\left\{\begin{smallmatrix}2\\3\end{smallmatrix}, 2, 3\right\}$, $\operatorname{add}\left\{\begin{smallmatrix}2\\3\end{smallmatrix}, 2\right\}$, $\operatorname{add}\{2\}$, $\operatorname{add}\{3\}$, $\operatorname{add}\{0\}$. Then we can get ICE-closed subcategories in $\operatorname{mod} B$: $\operatorname{add}\left\{\begin{smallmatrix}2\\3\end{smallmatrix}, 2, 3\right\}$, $\operatorname{add}\left\{\begin{smallmatrix}2\\3\end{smallmatrix}, 2\right\}$, $\operatorname{add}\{2\}$, $\operatorname{add}\{3\}$, $\operatorname{add}\{0\}$, $\operatorname{add}\left\{\begin{smallmatrix}1\\2\\3\end{smallmatrix}, 1, \begin{smallmatrix}2\\3\end{smallmatrix}, \begin{smallmatrix}1\\2\end{smallmatrix}, 2, 3\right\}$, $\operatorname{add}\left\{\begin{smallmatrix}1\\2\\3\end{smallmatrix}, 1, \begin{smallmatrix}2\\3\end{smallmatrix}, \begin{smallmatrix}1\\2\end{smallmatrix}, 2\right\}$, $\operatorname{add}\left\{\begin{smallmatrix}1\\2\end{smallmatrix}, 1, 2\right\}$, $\operatorname{add}\{1, 3\}$, $\operatorname{add}\{1\}$ by Theorem 3.1.

(2) $B := KQ_B$, $Q_B : 1 \overset{\alpha}{\to} 2 \overset{\beta}{\to} 3$ with relation $\alpha\beta = 0$. Let $A := KQ_A$, $Q_A : 2 \overset{\beta}{\to} 3$, $M_A = \langle \alpha \rangle \cong S_2$. Then $B := \begin{pmatrix} A & 0 \\ M_A & k \end{pmatrix}$. The irreducible representations of A are the same as (1). The ICE-closed subcategories in $\operatorname{mod} A$ are also identical to (1). However, the ICE-closed subcategories in $\operatorname{mod} B$ are $\operatorname{add}\left\{\begin{smallmatrix}2\\3\end{smallmatrix}, 2, 3\right\}$, $\operatorname{add}\left\{\begin{smallmatrix}2\\3\end{smallmatrix}, 2\right\}$, $\operatorname{add}\{2\}$, $\operatorname{add}\{3\}$, $\operatorname{add}\{0\}$, $\operatorname{add}\left\{\begin{smallmatrix}2\\3\end{smallmatrix}, 1, \begin{smallmatrix}1\\2\end{smallmatrix}, 2, 3\right\}$, $\operatorname{add}\left\{\begin{smallmatrix}2\\3\end{smallmatrix}, 1, \begin{smallmatrix}1\\2\end{smallmatrix}, 2\right\}$, $\operatorname{add}\left\{\begin{smallmatrix}1\\2\end{smallmatrix}, 1, 2\right\}$, $\operatorname{add}\{1, 3\}$, $\operatorname{add}\{1\}$ by Theorem 3.1.

Remark 3.3. Applying Theorem 3.1, we can give a part of ICE-closed subcategories in $\operatorname{mod} B$. But more computation is required to give all the ICE-closed subcategories in $\operatorname{mod} B$.

Corollary 3.4. *The number of ICE-closed subcategories in* $\operatorname{mod} B$ *and the number of ICE-closed subcategories in* $\operatorname{mod} A$ *are denoted respectively as* m, n. *Then we have :*

$$m \geq 2n.$$

Theorem 3.5. *Let B be the one-point extension algebra of A by an A-module M_A and $\mathcal{S}_A$ be an epibrick in* $\operatorname{mod} A$.

(1) $\mathcal{S}_B := \{(s, 0, 0) | s \in \mathcal{S}_A\}$, $\mathcal{S}_B$ *is an epibrick in* $\operatorname{mod} B$.

(2) $\mathcal{S}'_B := \{(s, 0, 0), (0, k, 0) | s \in \mathcal{S}_A\}$, $\mathcal{S}'_B$ *is an epibrick in* $\operatorname{mod} B$.

Proof (1)For an arbitrary morphism $F : w_1 \to w_2$, where $w_1 = (s_1, 0, 0)$, $w_2 = (s_2, 0, 0) \in \mathcal{S}_B$, it is easy to know $F = (f, 0)$ with $f : s_1 \to s_2$. So $F \cong f$. f is either zero or a surjection. Since s_1, $s_2 \in \mathcal{S}_A$ and $\mathcal{S}_A$ is an epibrick in $\operatorname{mod} A$. Therefore F is either zero or a surjection. $\mathcal{S}_B$ is an epibrick in $\operatorname{mod} B$.

(2)For an arbitrary morphism $F : w_1 \to w_2$, where $w_1 = (s, 0, 0)$, $w_2 = (0, k, 0) \in \mathcal{S}_B$, $F = (f, g)$ with $f : s \to 0$, $g : 0 \to k$. So $F = 0$. According to (1), $\mathcal{S}_B$ is an epibrick in $\operatorname{mod} B$ and $(0, k, 0)$ is also an epibrick in $\operatorname{mod} B$. Therefore, $\mathcal{S}'_B$ is an epibrick in $\operatorname{mod} B$.

Remark 3.6. Let B be the one-point extension algebra of A by an A-module M_A and $\mathcal{S}_A$ be a monobrick in $\operatorname{mod} A$.

(1) $\mathcal{S}_B := \{(s, 0, 0) | s \in \mathcal{S}_A\}$, $\mathcal{S}_B$ is an monobrick in $\operatorname{mod} B$

(2) $\mathcal{S}'_B := \{(s, 0, 0), (0, k, 0) | s \in \mathcal{S}_A\}$, $\mathcal{S}'_B$ is an monobrick in $\operatorname{mod} B$

Example 3.7. $B := KQ_B$, Q_B :

$$\begin{tikzcd} & & 3 \arrow[dl, "\beta"'] \\ 1 \arrow[r, "\alpha"] & 2 \arrow[dr, "\gamma"] & \\ & & 4 \end{tikzcd}$$

Let $A := KQ_A$, $Q_A : 4 \xrightarrow{\gamma} 2 \xrightarrow{\beta} 3$, $M_A = \langle \alpha \rangle = k\{\alpha, \alpha\beta\} \cong P_2 : 0 \to k \to k$.Then $B := \begin{pmatrix} A & 0 \\ M_A & k \end{pmatrix}$. The irreducible representations of A : 4, 2, 3, $\begin{smallmatrix}4\\2\end{smallmatrix}$, $\begin{smallmatrix}2\\3\end{smallmatrix}$, $\begin{smallmatrix}4\\2\\3\end{smallmatrix}$. The epibricks in $\operatorname{mod} A$ are:

$\{4\}$, $\{4,2\}$, $\{4,3\}$, $\left\{4, \begin{smallmatrix}4\\2\end{smallmatrix}\right\}$, $\left\{4, \begin{smallmatrix}2\\3\end{smallmatrix}\right\}$, $\left\{4, \begin{smallmatrix}4\\2\\3\end{smallmatrix}\right\}$, $\{4,2,3\}$, $\left\{4,2, \begin{smallmatrix}2\\3\end{smallmatrix}\right\}$, $\left\{4,2, \begin{smallmatrix}4\\2\\3\end{smallmatrix}\right\}$, $\left\{4,3, \begin{smallmatrix}4\\2\end{smallmatrix}\right\}$, $\left\{4, \begin{smallmatrix}4\\2\end{smallmatrix}, \begin{smallmatrix}4\\2\\3\end{smallmatrix}\right\}$,
$\{2\}$, $\{2,3\}$, $\left\{2, \begin{smallmatrix}2\\3\end{smallmatrix}\right\}$, $\left\{2, \begin{smallmatrix}4\\2\\3\end{smallmatrix}\right\}$, $\left\{2, \begin{smallmatrix}4\\2\\3\end{smallmatrix}, \begin{smallmatrix}4\\2\end{smallmatrix}\right\}$, $\{3\}$, $\left\{3, \begin{smallmatrix}4\\2\end{smallmatrix}\right\}$, $\left\{\begin{smallmatrix}4\\2\end{smallmatrix}, \begin{smallmatrix}4\\2\\3\end{smallmatrix}\right\}$, $\left\{\begin{smallmatrix}2\\3\end{smallmatrix}\right\}$, $\left\{\begin{smallmatrix}4\\2\\3\end{smallmatrix}\right\}$, $\{0\}$.

Then we can get some epibricks in $modB$ by Theorem 3.5:

$\{4\}$, $\{4,2\}$, $\{4,3\}$, $\left\{4, \begin{smallmatrix}4\\2\end{smallmatrix}\right\}$, $\left\{4, \begin{smallmatrix}2\\3\end{smallmatrix}\right\}$, $\left\{4, \begin{smallmatrix}4\\2\\3\end{smallmatrix}\right\}$, $\{4,2,3\}$, $\left\{4,2, \begin{smallmatrix}2\\3\end{smallmatrix}\right\}$, $\left\{4,2, \begin{smallmatrix}4\\2\\3\end{smallmatrix}\right\}$, $\left\{4,3, \begin{smallmatrix}4\\2\end{smallmatrix}\right\}$, $\left\{4, \begin{smallmatrix}4\\2\end{smallmatrix}, \begin{smallmatrix}4\\2\\3\end{smallmatrix}\right\}$,
$\{2\}$, $\{2,3\}$, $\left\{2, \begin{smallmatrix}2\\3\end{smallmatrix}\right\}$, $\left\{2, \begin{smallmatrix}4\\2\\3\end{smallmatrix}\right\}$, $\left\{2, \begin{smallmatrix}4\\2\\3\end{smallmatrix}, \begin{smallmatrix}4\\2\end{smallmatrix}\right\}$, $\{3\}$, $\left\{3, \begin{smallmatrix}4\\2\end{smallmatrix}\right\}$, $\left\{\begin{smallmatrix}4\\2\end{smallmatrix}, \begin{smallmatrix}4\\2\\3\end{smallmatrix}\right\}$, $\left\{\begin{smallmatrix}2\\3\end{smallmatrix}\right\}$, $\left\{\begin{smallmatrix}4\\2\\3\end{smallmatrix}\right\}$, $\{0\}$, $\{4,1\}$, $\{4,2,1\}$,
$\{4,3,1\}$, $\left\{4, \begin{smallmatrix}4\\2\end{smallmatrix}, 1\right\}$, $\left\{4, \begin{smallmatrix}2\\3\end{smallmatrix}, 1\right\}$, $\left\{4, \begin{smallmatrix}4\\2\\3\end{smallmatrix}, 1\right\}$, $\{4,2,3,1\}$, $\left\{4,2, \begin{smallmatrix}2\\3\end{smallmatrix}, 1\right\}$, $\left\{4,2, \begin{smallmatrix}4\\2\\3\end{smallmatrix}, 1\right\}$, $\left\{4,3, \begin{smallmatrix}4\\2\end{smallmatrix}, 1\right\}$, $\left\{4, \begin{smallmatrix}4\\2\end{smallmatrix}, \begin{smallmatrix}4\\2\\3\end{smallmatrix}, 1\right\}$,
$\{2,1\}$, $\{2,3,1\}$, $\left\{2, \begin{smallmatrix}2\\3\end{smallmatrix}, 1\right\}$, $\left\{2, \begin{smallmatrix}4\\2\\3\end{smallmatrix}, 1\right\}$, $\left\{2, \begin{smallmatrix}4\\2\\3\end{smallmatrix}, \begin{smallmatrix}4\\2\end{smallmatrix}, 1\right\}$, $\{3,1\}$, $\left\{3, \begin{smallmatrix}4\\2\end{smallmatrix}, 1\right\}$, $\left\{\begin{smallmatrix}4\\2\end{smallmatrix}, \begin{smallmatrix}4\\2\\3\end{smallmatrix}, 1\right\}$, $\left\{\begin{smallmatrix}2\\3\end{smallmatrix}, 1\right\}$, $\left\{\begin{smallmatrix}4\\2\\3\end{smallmatrix}, 1\right\}$,
$\{1\}$.

Remark 3.8. Applying Theorem 3.5, we can give a part of epibricks in $\operatorname{mod} B$. But more computation is required to give all the epibricks in $\operatorname{mod} B$.

Corollary 3.9. *The number of epibricks in* $\operatorname{mod} B$ *and the number of epibricks in* $\operatorname{mod} A$ *are denoted respectively as* m, n. *Then we have :*

$$m \geq 2n.$$

Applications. Let Λ be an algebra and $M \in \operatorname{mod} \Lambda$. M is τ-tilting if $\operatorname{Hom}_\Lambda(M, \tau M) = 0$ and $|M| = |\Lambda|$. M is support τ-tilting if it is a τ-tilting $\Lambda/\Lambda e \Lambda$-module for some idempotent e of Λ. Enomoto shown that every every functorially finite wide subcategory $\mathcal{W}$ is equivalent to a module category (i.e, there is an algebra Γ such that $\mathcal{W}$ is equivalent to $\operatorname{mod} \Gamma$), and then he introduced the definition of wide τ-tilting modules as follows.

Definition 3.10. ([3])

(1) *Given a functorially finite wide subcategory* $\mathcal{W}$ *of* $\operatorname{mod} \Lambda$ *and* $M \in \mathcal{W}$, *fix a equivalent* $F : \mathcal{W} \simeq \operatorname{mod} \Gamma$. *We say* M *is* $\tau_{\mathcal{W}}$*-tilting if* $F(M)$ *is a* τ*-tilting* Γ*-module.*

(2) A Λ-module M is called wide τ-tilting *if there is a functorially finite wide subcategory $\mathcal{W}$ of $\operatorname{mod}\Lambda$ such that M is $\tau_{\mathcal{W}}$-tilting. The set of all wide τ-tilting Λ-modules will be denoted by wτ-tilt Λ.*

Suppose that A, B are Nakayama algebras and B is the one-point extension of A by an A-module M_A. In [5], the authors get the following bijections:

$$\text{w}\tau\text{−tilt } \Lambda \underset{P(-)}{\overset{cok(-)}{\rightleftarrows}} \text{ice } \Lambda \underset{Filt(-)}{\overset{Sim(-)}{\rightleftarrows}} \text{ebrick } \Lambda$$

where Λ is either A or B, $\operatorname{cok}(M)$ denote the subcategory of mod A consisting of cokernels of morphisms in add M, $\operatorname{Filt}(\mathcal{S})$ denote the minimal Extension-closed subcategory which contains $\mathcal{S}$ for $\mathcal{S} \in$ ebrick A, $\operatorname{Sim}(B)$ denote the set of all simple object of ice B, $\operatorname{P}(\mathcal{C})$ denote the maximal Ext-projective object of $\mathcal{C}$. Then we have two different ways to construct wide τ-tilting B-modules from wide τ-tilting A-modules as follows:

$$\begin{tikzcd}
\text{w}\tau\text{-tilt } A \arrow[d, "cok(-)"'] & & \text{w}\tau\text{-tilt } B \\
\text{ice } A \arrow[rr, "\text{Theorem 3.1}"] & & \text{ice } B \arrow[u, "P(-)"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
\text{w}\tau\text{-tilt } A \arrow[d, "Sim(cok(-))"'] & & \text{w}\tau\text{-tilt } B \\
\text{ebrick } A \arrow[rr, "\text{Theorem 3.5}"] & & \text{ebrick } B \arrow[u, "P(Filt(-))"']
\end{tikzcd}$$

Example 3.11. $B := KQ_B$, $Q_B : 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$. Let $A := KQ_A$, $Q_A : 2 \xrightarrow{\beta} 3$, $M_A = \langle\alpha\rangle \cong P_2$. Then $B := \begin{pmatrix} A & 0 \\ M_A & k \end{pmatrix}$.

1. (1) wτ-tilt $A \subseteq$ wτ-tilt B by Theorem 3.1(1).

   (2) We list wτ-tilt A, ice A, ice B and wτ-tilt B in table 1 by Theorem 3.1(2).

wτ-tilt A	ice A	ice B	wτ-tilt B
0	add$\{0\}$	add$\{1\}$	1
2	add$\{2\}$	add$\left\{\begin{smallmatrix}1\\2\end{smallmatrix}, 1, 2\right\}$	$2 \oplus \begin{smallmatrix}1\\2\end{smallmatrix}$
3	add$\{3\}$	add$\{1,\ 3\}$	$1 \oplus 3$
$\begin{smallmatrix}2\\3\end{smallmatrix}$	add$\left\{\begin{smallmatrix}2\\3\end{smallmatrix}\right\}$	add$\left\{\begin{smallmatrix}1\\2\\3\end{smallmatrix}, 1, \begin{smallmatrix}2\\3\end{smallmatrix}\right\}$	$\begin{smallmatrix}2\\3\end{smallmatrix} \oplus \begin{smallmatrix}1\\2\\3\end{smallmatrix}$
$\begin{smallmatrix}2\\3\end{smallmatrix} \oplus 2$	add$\left\{\begin{smallmatrix}2\\3\end{smallmatrix}, 2\right\}$	add$\left\{\begin{smallmatrix}1\\2\\3\end{smallmatrix}, 1, \begin{smallmatrix}2\\3\end{smallmatrix}, \begin{smallmatrix}1\\2\end{smallmatrix}, 2\right\}$	$\begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus 2$
$\begin{smallmatrix}2\\3\end{smallmatrix} \oplus 3$	add$\left\{\begin{smallmatrix}2\\3\end{smallmatrix}, 3, 2\right\}$	mod B	$\begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus 3$

Table 1 wτ-tilt A ice A ice B wτ-tilt B

2. (1) wτ-tilt $A \subseteq$ wτ-tilt B by Theorem 3.5(1).

   (2) We list wτ-tilt A, ebrick A, ebrick B and wτ-tilt B in table 2 by Theorem 3.5(2).

wτ-tilt A	ebrick A	ebrick B	wτ-tilt B
0	$\{0\}$	$\{1\}$	1
2	$\{2\}$	$\{1,2\}$	$2 \oplus \begin{smallmatrix}1\\2\end{smallmatrix}$
3	$\{3\}$	$\{1,\ 3\}$	$1 \oplus 3$
$\begin{smallmatrix}2\\3\end{smallmatrix}$	$\left\{\begin{smallmatrix}2\\3\end{smallmatrix}\right\}$	$\left\{1, \begin{smallmatrix}2\\3\end{smallmatrix}\right\}$	$\begin{smallmatrix}2\\3\end{smallmatrix} \oplus \begin{smallmatrix}1\\2\\3\end{smallmatrix}$
$\begin{smallmatrix}2\\3\end{smallmatrix} \oplus 2$	$\left\{\begin{smallmatrix}2\\3\end{smallmatrix}, 2\right\}$	$\left\{1, \begin{smallmatrix}2\\3\end{smallmatrix}, 2\right\}$	$\begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus 2$
$\begin{smallmatrix}2\\3\end{smallmatrix} \oplus 3$	$\{2,3\}$	$\{1,2,3\}$	$\begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus 3$

Table 2 wτ-tilt A ebrick A ebrick B wτ-tilt B

Acknowledgements. The authors would like to thank Yanhong Bao for helpful discussions. The second author was supported by NSFC (Grant No.12301041).

References

[1] S. ASAI. *Semibricks*. Int.Math.Res.Not.IMRN., 2020, **16**: 4993–5054.

[2] C. M. RINGEL. *Representations of K-species and bimodules*. J.Algebra., 1976, **41**(2): 269–302.

[3] H. ENOMOTO. *Rigid modules and ICE-closed subcategories in quiver representations*. Journal of Algebra., 2022, **594**: 364–388.

[4] H. ENOMOTO. *Monobrick, a uniform approach to torsion-free classes and wide subcategories*. Advances in Mathematics., 2021, **393**: 108–113.

[5] Hanpeng GAO, Dajun LIU. *A note on wide τ-tilting modules and epibricks*. prepublication.

[6] P. GABRIEL. *Des catégories abéliennes*. Bull.Soc.Math.France., 1962, **90**: 323–448.

[7] T. ADACHI, O. IYAMA, I. REITEN. *τ-tilting theory*. Compos.Math., 2014, **150**(3): 415–452.