

DERIVATION OF RENORMALIZED HARTREE-FOCK-BOGOLIUBOV AND QUANTUM BOLTZMANN EQUATIONS IN AN INTERACTING BOSE GAS

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ABSTRACT. Our previous work [37] presented a rigorous derivation of quantum Boltzmann equations near a Bose-Einstein condensate (BEC). Here, we extend it with a complete characterization of the leading order fluctuation dynamics. For this purpose, we correct the latter via an appropriate Bogoliubov rotation, in partial analogy to the approach by Grillakis-Machedon et al. [59], in addition to the Weyl transformation applied in [37]. Based on the analysis of the third order expansion of the BEC wave function, and the second order expansions of the pair-correlations, we show that through a renormalization strategy, various contributions to the effective Hamiltonian can be iteratively eliminated by an appropriate choice of the Weyl and Bogoliubov transformations. This leads to a separation of renormalized Hartree-Fock-Bogoliubov (HFB) equations and quantum Boltzmann equations. A multitude of terms that were included in the error term in [37] are now identified as contributions to the HFB renormalization terms. Thereby, the error bound in the work at hand is improved significantly. To the given order, it is now sharp, and matches the order or magnitude expected from scaling considerations. Consequently, we extend the time of validity to $t \sim (\log N)^2$ compared to $t \sim (\log N / \log \log N)^2$ before. We expect our approach to be extensible to smaller orders in $\frac{1}{N}$.

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1. INTRODUCTION

The Boltzmann (transport) equation describes the time-dependent behavior of the phase space probability distribution f of fluids and gases. It takes the form

$$(\partial_t + p \cdot \nabla_x) f = Q[f], \quad (1.1)$$

where, for a classical gas,

$$Q[f](p) = \int_{\mathbb{S}^2} d\omega \int_{\mathbb{R}^3} dp_* b(\omega, |p - p_*|) (f(p')f(p'_*) - f(p)f(p_*)) \Big|_{\substack{p' = p + [\omega \cdot (p_* - p)]\omega \\ p'_* = p_* - [\omega \cdot (p_* - p)]\omega}}. \quad (1.2)$$

In order to derive this equation, Boltzmann imposed the *Stosszahlansatz* or *molecular chaos assumption*, which requires that the joint distribution of N particles factorizes into the product of its one-particle marginals for particles that are *about to collide*. We will refer to this condition as *propagation of chaos*.

To this day, the rigorous derivation of the Boltzmann equation, and possible corrections, for all physically relevant regimes remains a widely open problem. However, in some special cases, and for sufficiently short times, there has been progress both from a heuristic and mathematically rigorous perspective. Hilbert [61] famously asked for a rigorous justification of Boltzmann's equations; the derivation of physical laws from a set

of mathematical axioms is referred to as *Hilbert's sixth problem*. A fundamental question that emerges in this context is how irreversibility of the (mesoscopic) Boltzmann equation, for which the entropy functional is non-decreasing, arises from the microscopic reversible many-body dynamics, see, e.g., [67].

In the case of classical gases, starting from the classical Liouville equation for an interacting N -particle gas, Boltzmann's idea to show the propagation of chaos in the derivation of Boltzmann's equation has been made rigorous, albeit for times that allow for at most one collision. The first rigorous, though incomplete, results go back to Cercignani [33] and Lanford [66], and were later completed [34, 35, 52, 89]. A crucial insight in the derivation is that the joint distribution function only needs to factorize for particles that are *about to collide*. The derivation of the classical Boltzmann equation remains an extraordinarily active area of research, see, e.g., [2, 17, 18, 43, 53, 86].

For a quantum gas or fluid, Nordheim [81] proposed a quantum analogue of (1.1), for which the collision operator Q is given by

$$\begin{aligned} Q_4[f](p) = & \int d\mathbf{p}_4 \delta(p_1 + p_2 - p_3 - p_4) \delta(E(p_1) + E(p_2) - E(p_3) - E(p_4)) \\ & |\mathcal{M}_{22}(\mathbf{p}_4)|^2 (\delta(p - p_1) + \delta(p - p_2) - \delta(p - p_3) - \delta(p - p_4)) \\ & ((1 \pm f(p_1))(1 \pm f(p_2))f(p_3)f(p_4) - f(p_1)f(p_2)(1 \pm f(p_3))(1 \pm f(p_4))), \end{aligned} \quad (1.3)$$

where '+' refers to the case of bosons, and '-' to fermions. Boldface letters with subscripts will denote multivectors, such as $\mathbf{p}_k = (p_1, p_2, \dots, p_k)$, where $p_j \in \mathbb{R}^3$, and $k \in \mathbb{N}$. $\mathcal{M}_{22}(\mathbf{p}_4)$ is the scattering cross section relevant for this process. In the case of bosons, it has been shown [48, 49, 74] that the solution to the quantum Boltzmann equation with collision operator (1.3) develops a δ -mass in finite time, corresponding to condensation in finite time, see also [1, 3, 4] for related works. If we then decompose $f = f^{(\text{ex})} + n_c \delta$ into its regular and singular part, we have that

$$Q_4[f] = n_c Q_3[f^{(\text{ex})}] + Q_4[f^{(\text{ex})}] - n_c \delta \int dq Q_3[f^{(\text{ex})}](q),$$

where

$$\begin{aligned} Q_3[f](p) = & \int d\mathbf{p}_3 \delta(p_1 + p_2 - p_3) \delta(E(p_1) + E(p_2) - E(p_3)) \\ & |\mathcal{M}_{22}(\mathbf{p}_3, 0)|^2 (\delta(p - p_1) + \delta(p - p_2) - \delta(p - p_3)) \\ & ((1 + f(p_1))(1 + f(p_2))f(p_3) - f(p_1)f(p_2)(1 + f(p_3))). \end{aligned} \quad (1.4)$$

This leads to the coupled system

$$\begin{cases} \partial_t f^{(\text{ex})} &= n_c Q_3[f^{(\text{ex})}] + Q_4[f^{(\text{ex})}] \\ \partial_t n_c &= -n_c \int dq Q_3[f^{(\text{ex})}](q). \end{cases} \quad (1.5)$$

If $n_c \gg 1$, $Q_3[f^{(\text{ex})}]$ determines the leading order dynamics. In this work, we are interested in studying the emergence of (1.4) for an interacting quantum Bose gas with initial condensate density $n_{C,0} = N$ and initial thermal excitation density $f_0^{(\text{ex})} \sim 1$.

In order to study the emergence of (1.3), we consider the Hamiltonian

$$H_{N,\varepsilon_1,\varepsilon_2} := \sum_{j=1}^N \frac{\varepsilon_1^2}{2} (-\Delta_{x_j}) + g_{N,\varepsilon_1,\varepsilon_2} \sum_{j < k}^N v\left(\frac{x_j - x_k}{\varepsilon_2}\right) \quad (1.6)$$

acting on the bosonic Hilbert space $L^2(\Lambda)^{\otimes_s N}$, where $\Lambda \subseteq \mathbb{R}^3$ is a 3-torus. Given the Schrödinger equation

$$i\varepsilon_1 \partial_t \Psi_{N,\varepsilon_1,\varepsilon_2,t} = H_{N,\varepsilon_1,\varepsilon_2} \Psi_{N,\varepsilon_1,\varepsilon_2,t}, \quad (1.7)$$

we are interested in the asymptotic behavior of the Wigner transforms

$$f_{N,\varepsilon_1,\varepsilon_2,t}^{(k)}(\mathbf{x}_k, \mathbf{p}_k) := \int_{\Lambda^N} d\mathbf{y}_N e^{i\mathbf{p}_k \cdot \mathbf{y}_k} \overline{\Psi}_{N,\varepsilon_1,\varepsilon_2,t}(\mathbf{x}_k + \frac{\varepsilon_1}{2}\mathbf{y}_k, \mathbf{y}_{N-k}) \Psi_{N,\varepsilon_1,\varepsilon_2,t}(\mathbf{x}_k - \frac{\varepsilon_1}{2}\mathbf{y}_k, \mathbf{y}_{N-k}),$$

where $\mathbf{y}_N = (\mathbf{y}_k, \mathbf{y}_{N-k})$. The propagation of chaos assumption in this case reads

$$f_{N,\varepsilon_1,\varepsilon_2,t}^{(k)} \approx (f_{N,\varepsilon_1,\varepsilon_2,t}^{(1)})^{\otimes k}.$$

The first mathematically rigorous results go back to Hugenholtz [63] and Ho-Landau [62], where it was shown that the terms proportional to g^2 in the Duhamel expansion of $f_{N,\varepsilon_1,\varepsilon_2,t}^{(1)}$ give rise to a Boltzmann collision term, see also works by Benedetto et al. [10, 11]. Under the assumption of propagation of restricted quasifreeness, a certain factorization property for quantum states, see Definition 2.1 below Erdös-Salmhofer-Yau showed that, for mesoscopic times $t \asymp g^{-2}$, the second order Duhamel expansion of $f_{N,\varepsilon_1,\varepsilon_2,t}^{(1)}$ yields the quantum Boltzmann equation (1.3). Lukkarinen and Spohn [75] later revisited this idea and stated conditions under which they derive a Boltzmann equation. However, they did not justify assumptions on the growth of certain moments for the evolution of $f_{N,t}^{(k)}$ that would correspond to a rigorous error control in the evolution.

X. Chen and Guo [39] showed in the case $\Omega = \mathbb{R}^3$ that, if one assumes a sufficiently regular potential v , and if one assumes the convergence and sufficient regularity of the marginals $f_{N,\varepsilon_1,\varepsilon_2,t}^{(k)}$, then, in the mesoscopic weak-coupling regime, i.e., $\varepsilon_1 = \varepsilon_2 = N^{-1/3}$, $g_{N,\varepsilon_1,\varepsilon_2} = \sqrt{\varepsilon_1}$, the limiting dynamics is given by a classical, quadratic Boltzmann equation (1.2). In a recent work [41], X. Chen and Holmer gave a derivation, conditional on a cycle regularity and a positivity condition for the Wigner transform. The regularity condition controls the growth of weighted Sobolev norms of the Wigner transforms $f_N^{(k)}(t)$. Both results suggest that a more detailed analysis of the fluctuation dynamics is required, in order to understand the emergence of (cubic or quartic) quantum Boltzmann dynamics.

In the case of fermions, Cárdenas and one of the authors [31] established the emergence of quantum Boltzmann fluctuations alongside bosonized self-interaction terms, beyond the Hartree-Fock approximation, while rigorously controlling the error. Their result is unconditional and holds for sufficiently short times. The bosonized self-interaction terms have been analyzed in more detail in [13, 15, 42] in the stationary case, and [12] for the dynamical case. Their results show that these can be characterized as *Random-Phase approximation*, as introduced by Bohm and Pines [19, 20, 84].

The *mean-field regime* is determined by $\varepsilon_1 = \varepsilon_2 = 1$ and $g = \frac{\lambda}{N}$, where N is the number of particles. For bosons, a question of special interest is the persistence of a Bose-Einstein condensate, i.e., the $N \gg 1$ asymptotic behavior of the wave function $\Psi_{N,t}$ satisfying (1.7) when, initially, $\Psi_{N,0} \approx \phi_0^{\otimes N}$. Going back to Ginibre-Velo [54] and Hepp [60], it has

been shown that the wave function remains factorized, i.e., $\Psi_{N,t} \approx \phi_t^{\otimes N}$, where ϕ satisfies a *nonlinear Hartree* (NLH) or *Schrödinger* (NLS) equation, dependent on the scaling regime. In particular, this establishes persistence of the condensate for microscopic times $t = O(1)$. These works have been extended, and convergence has been shown in different topologies [46, 47, 51, 59, 65, 71, 88] and for more singular scalings $\varepsilon_2 = N^{-\beta}$, $g = N^{3\beta-1}$, $\beta \in [0, 1]$ [27, 36, 38, 40, 44, 45, 57, 58, 83]. The case $\beta = 1$ is of particular physical interest, as it describes rare but strong interactions, and is referred to as the *Gross-Pitaevskii regime*. Another related problem is to study the asymptotic behavior of the ground state (energy) of (1.6), see, e.g., [9, 26, 32, 68, 69, 72, 77, 79, 80].

In order to study corrections to the leading order NLH/NLS BEC dynamics, it is necessary to include to study the dynamics of pair correlations, corresponding to thermal fluctuations. It has been shown [56, 57, 59, 76, 78, 82] that these are governed by the nonlinear *Hartree-Fock-Bogoliubov* (HFB) equations. Bach, Breteaux, Fröhlich, Sigal and one of the authors [7, 8] have shown that the *quasifree approximation* of the full dynamics is given by the HFB equations.

Another vibrant area of research is the (bosonic) ground state problem associated with (1.6), see [16, 28, 30, 70, 72, 73]. There, corrections to the leading order Gross-Pitaevskii energy are described by the *Lee-Huang-Yang formula*, see, e.g., [50, 55], and higher oder corrections have also been described [23, 30].

In our previous work [37], we studied the regime $g = \frac{\lambda_N}{N}$ for some $\lambda_N \ll 1$ and $N \gg 1$, on a 3-torus Λ , for mesoscopic times $t \propto \lambda_N^{-2}$. The Bose gas initially consisted of a translation invariant BEC of *density* N , and translation invariant thermal fluctuations close to equilibrium of density 1, described by a *quasifree* state, see Definition 2.1 below. Quasifreeness can be thought of as a quantum analogue of the classical molecular chaos.

Assuming $\hat{v}(0) = 0$, we showed that, subleading to the HFB fluctuation dynamics, for kinetic times $t \propto \lambda^{-2}$, the density of thermal fluctuations obeys a cubic Boltzmann equation with collision operator

$$Q_3[f](p) = \frac{1}{N |\Lambda|^3} \sum_{\substack{p_j \in \Lambda^* \\ j=1,2,3}} \delta_t(\Omega(p_1) + \Omega(p_2) - \Omega(p_3)) \delta(p_1 + p_2 - p_3) |\hat{v}(p_1) + \hat{v}(p_2)|^3 (1.8) \\ ((1 + f(p_1))(1 + f(p_2))f(p_3) - f(p_1)f(p_2)(1 + f(p_3))) ,$$

where Λ^* denotes the momentum (reciprocal) lattice. These collisions correspond to those, where a BEC particle is either absorbed or emitted in the collision. Here δ_t is a mollification of $\delta(p) \equiv \delta_{\Lambda^*}(p) \equiv |\Lambda| \delta_{p,0}$, $\Omega(p) = \sqrt{E(p)(E(p) + 2\lambda \hat{v}(p))}$ is the Bogoliubov dispersion, and we choose the BEC wavefunction to be constant to leading order. Given a time scale shorter than $O(|\Lambda|^{\frac{1}{3}})$, where we assumed Λ to be a square torus in three dimensions, we showed that the discrete Boltzmann operator in (1.8) can be approximated by a continuous Boltzmann operator, where summation over p is replaced by Riemann integrals and δ_t by a Dirac- δ ; for longer times, interference leads to additional resonance terms. Crucially to our work, we used the fact that the HFB dynamics preserves quasifreeness, and showed that the full dynamics approximately preserves restricted quasifreeness for times $t \propto \lambda^{-2} \propto (\frac{\log N}{\log \log N})^\alpha$, $\alpha > 0$. The result is unconditional and provides a rigorous error control for sufficiently short times.

In the present work, we revisit this problem from a different perspective. We show that corrections to the pair-absorption rate and to the BEC wave function can be absorbed into the HFB equations. This, in turn, leads to a renormalization of the HFB equations describing the coupled dynamics of the BEC wave function with the leading order thermal pair-excitations. After renormalizing, the corrections to the BEC and thermal fluctuation dynamics are given by pure Boltzmann collision terms. More precisely, let $(\phi_t, \gamma_t, \sigma_t)$ denote the renormalized HFB fields, and Ω_t be the corresponding renormalized Bogoliubov dispersion. Let $f_t^{(\text{tot})}$ denote the total density of the Bose gas, $\Phi_t^{(\text{tot})}$ denote the full BEC wavefunction, and $g_t^{(\text{tot})}$ the full pair-absorption rate. These fields can be expanded in the form

$$\begin{pmatrix} \Phi_t^{(\text{tot})} \\ f_t^{(\text{tot})} - |\Phi_t^{(\text{tot})}|^2 \delta \\ g_t^{(\text{tot})} - (\Phi_t^{(\text{tot})})^2 \delta \end{pmatrix} = \begin{pmatrix} \sqrt{N|\Lambda|} \phi_t \\ \gamma_t \\ \sigma_t \end{pmatrix} + \begin{pmatrix} a_t & 0 \\ 0 & A_t \end{pmatrix} \begin{pmatrix} \Phi_t \\ f_t - |\Phi_t|^2 \delta \\ g_t - \Phi_t^2 \delta \end{pmatrix}, \quad (1.9)$$

where $a_t \in \mathbb{C}$, $A_t \in \mathbb{C}^{2 \times 2}$ are linear operators dependent on the HFB fields $(\phi_t, \gamma_t, \sigma_t, \Omega_t)$, and γ_t and σ_t are related by $|\sigma_t|^2 = \gamma_t(1 + \gamma_t)$. (1.9) can be interpreted as a gradual centering process:

- (1) We collect all condensate terms to given order in the perturbation expansion and add them to the contribution coming from ϕ , and center the correlation functions f, g w.r.t. ϕ . Then Φ denotes the correction terms to given order.
- (2) Simultaneously, we subtract the leading order pair-correlations (γ, σ) from the centered pair correlations. Then f, g denote the pair correlation corrections to given order.

We show that, if $(\phi_t, \gamma_t, \sigma_t, \Omega_t)$ satisfy the renormalized HFB equations

$$\begin{cases} i\partial_t \phi_t &= \frac{\lambda}{N} [(\Gamma_t * (\hat{v} + \hat{v}(0)))(0)\phi_t + (\Sigma_t * \hat{v})(0)\bar{\phi}_t] - 2\lambda|\Lambda|\hat{v}(0)|\phi_t|^2\phi_t, \\ \partial_t \gamma_t &= \frac{2\lambda}{N} \text{Im}((\Sigma_t * \hat{v})\bar{\sigma}_t), \\ i\partial_t \sigma_t &= 2(E + \frac{\lambda}{N}\Gamma_t * (\hat{v} + \hat{v}(0)))\sigma_t + \frac{\lambda}{N}(\Sigma_t * \hat{v})(1 + 2\gamma_t), \\ \Omega_t &= E + \frac{\lambda}{N}(\Gamma_t * (\hat{v} + \hat{v}(0))) + \frac{\lambda \text{Re}((\Sigma_t * \hat{v})\sigma_t)}{1 + \gamma_t}, \end{cases}$$

where

$$\begin{pmatrix} \Gamma \\ \Sigma \end{pmatrix}(p) = N|\Lambda|\delta(p) \begin{pmatrix} |\phi|^2 \\ \phi^2 \end{pmatrix} + ((1 + f_0(p) + f_0(-p)) \begin{pmatrix} \gamma \\ \sigma \end{pmatrix}(p) + \begin{pmatrix} \frac{f_0(p) + f_0(-p)}{2} \\ 0 \end{pmatrix}),$$

and $E(p) = \frac{1}{2}|p|^2$, then (Φ, f, g) satisfy

$$\Phi_t - \Phi_0 = \frac{1}{N^{\frac{3}{2}}} \int_0^t ds (Q_3^{(\Phi)}[f](s) + Q_{3,3}^{(\Phi)}[f](s)) + O\left(\frac{e^{c|\Lambda|\lambda t}}{N^2}\right), \quad (1.10)$$

$$f_t - f_0 = \frac{1}{N} \int_0^t ds Q_3[f](s) + O\left(\frac{e^{c|\Lambda|\lambda t}}{N^{\frac{3}{2}}}\right),$$

$$g_t - g_0 = \frac{1}{N} \int_0^t ds Q_3^{(g)}[f](s) + O\left(\frac{e^{c|\Lambda|\lambda t}}{N^{\frac{3}{2}}}\right), \quad (1.11)$$

where each $Q_j^{(k)}$ denotes a collection of cubic Boltzmann collision terms, for which the collision kernels depend on the renormalized HFB fields $(\phi, \gamma, \sigma, \Omega)$. Crucially, the errors provided here are *sharp* in orders of N .

In particular, we show that, to leading orders, the BEC wave function $\Phi^{(\text{tot})}$ and the pair correlations $(f^{(\text{tot})}, g^{(\text{tot})})$ can each be decomposed into a part that satisfies renormalized HFB equations and a quantum Boltzmann correction part. This separation confirms the phenomenological paradigm that the HFB and QBE dynamics evolve according to different time scales; the HFB dynamics leads to fast oscillations, while the QBE determines the slow long-time dynamics. As in our previous work, our result is unconditional and rigorous, however, we extend the validity to times of order $t \propto \lambda^{-2} \sim (\log N)^2$, in contrast to $\lambda^{-1} \sim \log N / \log \log N$ before.

The term $N^{-3/2} \int_0^t ds Q_3^{(\Phi)}[f](s)$ in the evolution of the BEC wave function justifies the Boltzmann term in the evolution of the condensate density, see (1.5). However, the additional $N^{-3/2} \int_0^t ds Q_{3,3}^{(\Phi)}[f](s)$ has previously not been included, despite of being of the same order of magnitude as the cubic Boltzmann term, see also [85, 87, 90].

Boßmann et al. [21, 22, 24] have computed a full expansion of the correction dynamics in terms of the coupling constant. In comparison, in our current and past work [37], we have been able to characterize terms in the expansion as stemming either from a HFB contribution, or a quantum Boltzmann correction.

We are able to rigorously control the error for times $t \propto \lambda^{-2} \sim (\log N)^2$. A more detailed calculation shows that, at order $\frac{1}{N^2}$, terms emerge that are neither of HFB nor of quantum Boltzmann type.

2. STATEMENT OF RESULTS

Let $\Lambda = [-L/2, L/2]^3 / \sim$ denote a cubic torus of length L , and let $\Lambda^* = (\frac{2\pi}{L}\mathbb{Z})^3$ denote its reciprocal space. Let

$$\mathcal{F} := \mathbb{C} \oplus \bigoplus_{n \in \mathbb{N}} L^2(\Lambda)^{\otimes_{sn}}$$

denote the bosonic Fock space, endowed with the inner product

$$\langle \Phi, \Psi \rangle_{\mathcal{F}} = \sum_{n \in \mathbb{N}_0} \langle \Psi^{(n)}, \Phi^{(n)} \rangle_{L^2(\Lambda^n)}.$$

For $\psi \in L^2(\Lambda)$, let

$$(a(\psi)\Psi)^{(n-1)}(\mathbf{x}_{n-1}) = \sqrt{n} \int_{\Lambda} dx \overline{\psi}(x) \Psi^{(n)}(x, \mathbf{x}_{n-1})$$

denote the annihilation operator, and

$$\begin{aligned} (a^\dagger(\psi)\Psi)^{(n+1)} &:= \sqrt{n+1} P_{L^2(\Lambda)^{\otimes_{s(n+1)}}} \psi \otimes \Psi^{(n)} \\ &= \frac{\sqrt{n+1}}{(n+1)!} \sum_{\pi \in \mathcal{S}_{n+1}} \psi(x_{\pi(1)}) \Psi^{(n)}(x_{\pi(2)}, \dots, x_{\pi(n+1)}), \end{aligned}$$

the creation operator. In addition, we introduce the momentum-space annihilation/creation operators $a_p^\# := a^\#(e^{ip \cdot})$. They satisfy the CCR

$$[a_p, a_q] = [a_p^\dagger, a_q^\dagger] = 0, \quad [a_p, a_q^\dagger] = |\Lambda| \delta_{p,q} =: \delta_{\Lambda^*}(p - q).$$

In the following, we will omit the subscript Λ^* , when referring to a momentum- δ . For brevity, we will use the notation

$$\int_{\Lambda^*} dp \, h(p) := \frac{1}{|\Lambda|} \sum_{p \in \Lambda^*} h(p).$$

Again, we will omit the subscript Λ^* , unless it is ambiguous. Our convention for the Fourier transform is

$$\hat{h}(p) := \int_{\Lambda} dx \, e^{ip \cdot x} h(x).$$

We consider the mean-field Hamiltonian

$$\mathcal{H}_N := \int dp \, E(p) a_p^\dagger a_p + \frac{\lambda}{2N} \int d\mathbf{p}_4 \, \delta(p_1 + p_2 - p_3 - p_4) \hat{v}(p_1 - p_3) a_{p_1}^\dagger a_{p_2}^\dagger a_{p_3} a_{p_4},$$

where $E(p) := \frac{1}{2}|p|^2$ denotes the free dispersion, and $v \geq 0$ a pair potential such that $\hat{v} \geq 0$.

We are interested in determining the leading order dynamics for a Bose gas governed by the Hamiltonian \mathcal{H}_N . As an initial state, we will choose a Bose-Einstein condensate (BEC) of density N , surrounded by a thermal excitations that are described by a *quasifree* state with density $O(1)$.

2.1. Initial state.

Let

$$\mathcal{W}[f] := \exp(a^\dagger(f) - a(f))$$

denote the Weyl transform,

$$\mathcal{T}[k] := \exp\left(\frac{1}{2} \int dp \, (k(p) a_p^\dagger a_{-p}^\dagger - \bar{k}(p) a_p a_{-p})\right)$$

denote the Bogoliubov transformation in the translation invariant case. Without loss of generality, we assume that k is even. We have that

$$\mathcal{W}^\dagger[f] a_p \mathcal{W}[f] = a_p + \hat{f}(p), \quad (2.1)$$

$$\mathcal{T}^\dagger[k] a_p \mathcal{T}[k] = \cosh(|k(p)|) a_p + \sinh(|k(p)|) \frac{k(p)}{|k(p)|} a_{-p}^\dagger, \quad (2.2)$$

see, e.g., Lemma A.1.

Let \mathfrak{A} denotes the CCR algebra generated by the Weyl operators $\mathcal{W}[\psi]$, $\psi \in L^2(\Lambda)$, see [25, section 5.2.3].

Definition 2.1 (Quasifree state). *Let $\langle \cdot \rangle$ be a state and*

$$\langle A \rangle^{(\text{cen})} := \langle (\mathcal{W}[\langle a \rangle]) A \mathcal{W}^\dagger[\langle a \rangle] \rangle$$

denote its centering. We say $\langle \cdot \rangle$ is quasifree iff for all $n \in \mathbb{N}$

$$\begin{cases} \langle a^{\#1} a^{\#2} \dots a^{\#2n} \rangle^{(\text{cen})} &= \overbrace{a^{\#1} a^{\#2} \dots a^{\#2n}}^{\text{+ all pair contractions}}, \\ \langle a^{\#1} a^{\#2} \dots a^{\#2n-1} \rangle^{(\text{cen})} &= 0 \end{cases} \quad (2.3)$$

where $a^{\#1} a^{\#2} := \langle a^{\#1} a^{\#2} \rangle^{(\text{cen})}$. (2.3) is referred to as Wick's Theorem.

A state is restricted quasifree if (2.3) holds for $n \leq n_0$ for some $n_0 \in \mathbb{N}$.

Definition 2.2 (Number conserving state). A state $\langle \cdot \rangle$ is called number conserving iff $\langle [A, \mathcal{N}_b] \rangle = 0$ for every observable $A \in \mathfrak{A}$.

Definition 2.3 (Translation invariance). A state $\langle \cdot \rangle$ is called translation invariant iff

$$\left\langle \prod_{j=1}^n a_{p_j}^{(\sigma_j)} \right\rangle = \frac{\delta(\sum_{j=1}^n \sigma_j p_j)}{|\Lambda|} \left\langle \prod_{j=1}^n a_{p_j}^{(\sigma_j)} \right\rangle$$

for all p_1, \dots, p_n and $\sigma_j = \pm 1$.

Let $\langle \cdot \rangle_0$ be a number conserving, quasifree, and translation invariant. More precisely, let

$$\mathcal{K} := \int_{\Lambda^*} dp K(p) a_p^\dagger a_p$$

be such that $K(p) \geq \kappa_0$ for some $\kappa_0 > 0$, and let

$$\langle A \rangle_0 := \frac{\text{Tr}(e^{-\kappa} A)}{\text{Tr}(e^{-\kappa})}.$$

Observe that, by being number conserving, $\langle \cdot \rangle_0$ is already centered. $\langle \cdot \rangle_0$ determines the thermal excitations beyond the HFB fluctuations.

The full state describing the Bose gas is then given by

$$\langle A \rangle_0^{(\text{tot})} := \frac{1}{\text{Tr}(e^{-\kappa})} \text{Tr} \left(\mathcal{W}[\sqrt{N|\Lambda|}\phi_0] \mathcal{T}[k_0] e^{-\kappa} \mathcal{T}^\dagger[k_0] \mathcal{W}^\dagger[\sqrt{N|\Lambda|}\phi_0] A \right)$$

for all $A \in \mathfrak{A}$. Note that $\langle \cdot \rangle_0^{(\text{tot})}$ is quasifree. The initial value problem (IVP) associated with the Hamiltonian \mathcal{H}_N and the initial state $\langle \cdot \rangle_0^{(\text{tot})}$ is then given by the Liouville-von Neumann equation

$$i\partial_t \langle A \rangle_t^{(\text{tot})} = \langle [A, \mathcal{H}_N] \rangle_t^{(\text{tot})}$$

for all observables $A \in \mathfrak{A}$. Below, we impose assumptions on v ensuring that \mathcal{H}_N is self-adjoint and that it induces a unitary evolution $e^{-it\mathcal{H}_N}$. Notice that $\langle \cdot \rangle_t^{(\text{tot})}$ is not quasifree. However, we will show that for short enough times, $\langle \cdot \rangle_t^{(\text{tot})}$ is approximately quasifree. Consequently, the evolution of any expectation $\langle A \rangle_t^{(\text{tot})}$ is fully characterized by the first and second moments $\langle a_0 \rangle_t^{(\text{tot})}$, and $\langle a_p^\dagger a_p \rangle_t^{(\text{tot})}$, $\langle a_p a_{-p} \rangle_t^{(\text{tot})}$, respectively.

Of particular interest in the present work is the evolution of the density

$$f_t^{(\text{tot})}(p) := \frac{\langle a_p^\dagger a_p \rangle_t^{(\text{tot})}}{|\Lambda|}. \quad (2.4)$$

In order to study $f^{(\text{tot})}$, we will decompose the full dynamics into a BEC and a thermal fluctuation part. For the latter, we will further decompose the dynamics of the fluctuation particles into a quasifree part, which we will refer to as *Hartree-Fock-Bogoliubov fluctuations*, and a collision part, which we will identify as the *quantum Boltzmann fluctuations*.

Our approach includes the time-behavior of leading order pair correlations via Bogoliubov rotations as employed by Grillakis-Machedon et al. [56, 57, 59] to derive the leading order HFB dynamics. However, we extend the latter by a quasifree, number conserving,

centered (translation invariant) state. It is the presence of these additional excitation states that allow for a (cubic) Boltzmann equation to arise.

2.2. Fluctuation dynamics. Define the Bogoliubov propagation

$$\begin{cases} i\partial_t \mathcal{U}_{\text{Bog}}(t) &= \int dp \Omega_t(p) a_p^\dagger a_p \mathcal{U}_{\text{Bog}}(t), \\ \mathcal{U}_{\text{Bog}}(0) &= \mathbf{1}, \end{cases}$$

where Ω_t is an even function of p . We have that

$$\mathcal{U}_{\text{Bog}}^\dagger(t) a_p \mathcal{U}_{\text{Bog}}(t) = e^{-i \int_0^t ds \Omega_s(p)} a_p. \quad (2.5)$$

We introduce the *fluctuation dynamics* defined by

$$\mathcal{U}_N(t) := e^{iS_t} \mathcal{U}_{\text{Bog}}^\dagger(t) \mathcal{T}^\dagger[k_t] \mathcal{W}^\dagger[\sqrt{N|\Lambda|}\phi_t] e^{-it\mathcal{H}_N} \mathcal{W}[\sqrt{N|\Lambda|}\phi_0] \mathcal{T}[k_0], \quad (2.6)$$

where (ϕ_t, k_t) and S_t will be determined below. We will choose ϕ_t and k_t in such a way that the leading order and next-to-leading order contributions of the full dynamics $e^{-i\mathcal{H}_N t}$ are determined by (ϕ_t, k_t) .

We assume that $\phi_t(x) \equiv \phi_t$ is translation invariant. We show in Lemma A.2 that the fluctuation dynamics satisfies

$$\begin{cases} i\partial_t \mathcal{U}_N(t) &= \mathcal{H}_{\text{fluc}}(t) \mathcal{U}_N(t), \\ \mathcal{U}_N(0) &= \mathbf{1}, \end{cases} \quad (2.7)$$

where $\mathcal{H}_{\text{fluc}}(t)$ is given by

$$\mathcal{H}_{\text{fluc}}(t) = \mathcal{H}_{\text{BEC}}(t) + \mathcal{H}_{\text{HFB}}(t) + \mathcal{H}_{\text{cub}}(t) + \mathcal{H}_{\text{quart}}(t). \quad (2.8)$$

Each of the terms in $\mathcal{H}_{\text{fluc}}(t)$ is a normal-ordered polynomial in a and a^\dagger , and their explicit expressions are given in Lemma A.2. Here a monomial in a and a^\dagger is *normal-ordered* iff all creation operators a^\dagger are on the left of all annihilation operators a . We choose S_t in such a way that it absorbs all scalar terms. $\mathcal{H}_{\text{BEC}}(t)$ denotes the BEC Hamiltonian and it is linear in $a^\#$, \mathcal{H}_{HFB} is the HFB Hamiltonian, which is quadratic in $a^\#$, $\mathcal{H}_{\text{cub}}(t)$ is cubic in $a^\#$ and accounts for cubic scattering processes, where one of the particles is being absorbed into or emitted from the BEC, and $\mathcal{H}_{\text{quart}}(t)$ is quartic in $a^\#$ and describes pair interactions.

Let

$$u_t(p) := \cosh(|k_t(p)|), \quad (2.9)$$

$$v_t(p) := \sinh(|k_t(p)|) \frac{k_t(p)}{|k_t(p)|}.$$

In particular, we can rewrite (2.2) as

$$\mathcal{T}^\dagger[k_t] a_p \mathcal{T}[k_t] = u_t(p) a_p + v_t(p) a_{-p}^\dagger. \quad (2.10)$$

Note that we have $u_t(p)^2 - |v_t(p)|^2 = 1$. In addition, we introduce the scalar fields

$$\gamma_t(p) := |v_t(p)|^2, \quad (2.11)$$

$$\sigma_t(p) := u_t(p) v_t(p). \quad (2.12)$$

Observe that, from this definition, we have the relation

$$|\sigma_t(p)|^2 = u_t(p)^2 |v_t(p)|^2 = (1 + \gamma_t(p))\gamma_t(p). \quad (2.13)$$

Then the expressions for each of the normal-ordered terms in $\mathcal{H}_{\text{fluc}}(t)$ in (2.8) are given in Lemma A.2 in the Appendix.

We introduce the time-evolved *relative* state

$$\langle A \rangle_t := \langle \mathcal{U}_N^\dagger(t) A \mathcal{U}_N(t) \rangle_0.$$

It satisfies

$$i\partial_t \langle A \rangle_t = \langle [A, \mathcal{H}_{\text{fluc}}(t)] \rangle_t.$$

We introduce the relative moments

$$\Phi_t := \frac{\langle a_0 \rangle_t}{|\Lambda|}, \quad f_t(p) := \frac{\langle a_p^\dagger a_p \rangle_t}{|\Lambda|}, \quad g_t(p) := \frac{\langle a_p a_{-p} \rangle_t}{|\Lambda|}.$$

Our choice of $\langle \cdot \rangle_0$ implies $\Phi_0 = g_0 = 0$. Then we can rewrite the total density $f^{(\text{tot})}$, see (2.4) for its definition and section (3.1) for the derivation, as

$$\begin{aligned} f_t^{(\text{tot})}(p) &= \delta(p) \left[N|\Lambda| |\phi_t|^2 + \left(\sqrt{N|\Lambda|} e^{i \int_0^t ds \Omega_s(0)} (\phi_t u_t(0) + \bar{\phi}_t v_t(0)) \bar{\Phi}_t + \text{h.c.} \right) \right] \\ &\quad + \gamma_t(p) + (1 + \gamma_t(p)) f_t(p) + \gamma_t(p) f_t(-p) + \left(e^{2i \int_0^t ds \Omega_s(p)} \sigma_t(p) \bar{g}_t(p) + \text{h.c.} \right). \end{aligned}$$

Our goal is to show that if $(\phi_t, \gamma_t, \sigma_t, \Omega_t)$ satisfy the *renormalized* Hartree-Fock-Bogoliubov (HFB) equations, then the dynamics of (Φ_t, f_t, g_t) each are determined, to leading order, by a Boltzmann xn. More precisely, we will show that, to leading order, the dynamics of f is given by a cubic Boltzmann equation, while the dynamics of Φ and g , to leading order, are driven by f via a collision term, see (1.10)–(1.11).

2.3. Main result.

2.3.1. Renormalized HFB equations.

Let

$$f_0^{(+)}(p) := \frac{1}{2} (f_0(p) + f_0(-p)) \quad (2.14)$$

denote the even symmetrization of f_0 . We introduce the (second order) renormalized HFB fields

$$\begin{aligned} \Gamma^{(2)} &:= (1 + 2f_0^{(+)})\gamma + f_0^{(+)} + N|\Lambda||\phi|^2\delta, \\ \Sigma^{(2)} &:= (1 + 2f_0^{(+)})\sigma + N|\Lambda|\phi^2\delta. \end{aligned}$$

Then the renormalized HFB equations read

$$\begin{cases} i\partial_t \phi_t^{(2)} &= \frac{\lambda}{N} \left((\Gamma_t^{(2)} * (\hat{v} + \hat{v}(0))) (0) \phi_t^{(2)} + (\Sigma_t^{(2)} * \hat{v})(0) \bar{\phi}_t^{(2)} \right) \\ &\quad - 2\lambda |\Lambda| \hat{v}(0) |\phi_t^{(2)}|^2 \phi_t^{(2)}, \\ \partial_t \gamma_t^{(2)} &= \frac{2\lambda}{N} \text{Im} ((\Sigma_t^{(2)} * \hat{v}) \bar{\sigma}_t^{(2)}), \\ i\partial_t \sigma_t^{(2)} &= 2(E + \frac{\lambda}{N} \Gamma_t^{(2)} * (\hat{v} + \hat{v}(0))) \sigma_t^{(2)} + \frac{\lambda}{N} (\Sigma_t^{(2)} * \hat{v}) (1 + 2\gamma_t^{(2)}), \end{cases} \quad (2.15)$$

and the corresponding Bogoliubov dispersion is given by

$$\Omega_t^{(2)} := E + \frac{\lambda}{N} (\Gamma_t^{(2)} * (\hat{v} + \hat{v}(0))) + \frac{\lambda \operatorname{Re}(\bar{\Sigma}^{(2)} * \hat{v}) \sigma_t^{(2)}}{1 + \gamma_t^{(2)}}.$$

Remark 2.4. Observe that $\Omega_t^{(2)}$ is a modified Bogoliubov dispersion. Instead, the regular Bogoliubov dispersion, to leading order, is given by $\Omega_{\text{Bog}} = \sqrt{E(E + 2\lambda\hat{v})}$, where $E(p) = |p|^2/2$ is the free dispersion, see above and, e.g., [37, 82]. Observe that $\Omega_t^{(2)}$ is a modified Bogoliubov dispersion. Instead, the regular Bogoliubov dispersion, to leading order, is given by $\Omega_{\text{Bog}} = \sqrt{E(E + 2\lambda\hat{v})}$, where $E(p) = |p|^2/2$ is the free dispersion, see above and, e.g., [82].

Omitting superscripts, we note that the HFB equations (2.15) can be rewritten as

$$\begin{cases} i\partial_t \phi_t &= \frac{\lambda}{N} \left((\Gamma_t * (\hat{v} + \hat{v}(0)))(0) \phi_t + (\Sigma_t * \hat{v})(0) \bar{\phi}_t \right) - 2\lambda |\Lambda| \hat{v}(0) |\phi_t|^2 \phi_t, \\ \partial_t \Gamma_t &= \frac{2\lambda}{N} \operatorname{Im}((\Sigma_t * \hat{v}) \bar{\Sigma}_t), \\ i\partial_t \Sigma_t &= 2 \left(E + \frac{\lambda}{N} \Gamma_t * (\hat{v} + \hat{v}(0)) \right) \Sigma_t + \frac{\lambda}{N} (\Sigma_t * \hat{v}) (1 + 2\Gamma_t) \\ &\quad - 4N\lambda |\Lambda|^2 \hat{v}(0) |\phi_t|^2 \phi_t^2 \delta. \end{cases} \quad (2.16)$$

For any $1 \leq a \leq \infty$, we introduce the rescaled $L^a(\Lambda^*)$ -norms

$$\begin{aligned} \|f\|_{L^a(\Lambda^*)} &:= \|f\|_a := |\Lambda|^{-\frac{1}{a}} \|f\|_{\ell^a(\Lambda^*)}, \\ \|f\|_d &:= \|f\|_1 + \|f\|_\infty. \end{aligned}$$

For any weight $\tilde{w} : \Lambda^* \rightarrow \mathbb{R}^+$, define the weighted L^r space

$$L_{\tilde{w}}^r := \{f : \Lambda^* \rightarrow \mathbb{R} \mid \tilde{w}^{\frac{1}{r}} f \in L^r(\Lambda^*)\},$$

endowed with the norm

$$\|f\|_{L_{\tilde{w}}^r} := \|w^{\frac{1}{r}} f\|_r.$$

We abbreviate

$$\|\hat{v}\|_{w,d} := \|\hat{v}\|_{L^1_{\sqrt{1+E}}} + \|\hat{v}\|_\infty.$$

For all $j \in \mathbb{N}_0$, we introduce the function spaces

$$\mathcal{X}^j := \mathbb{C} \times (L^1_{(1+E)^j}(\Lambda^*) \cap L^\infty(\Lambda^*)) \times (L^2_{(1+E)^j}(\Lambda^*) \cap L^\infty(\Lambda^*)),$$

endowed with the norm

$$\|(\phi, \Gamma, \Sigma)\|_{\mathcal{X}^j} := |\phi| + \|\Gamma\|_{L^1_{(1+E)^j}} + \|\Gamma\|_\infty + \|\Sigma\|_{L^2_{(1+E)^j}} + \|\Sigma\|_\infty.$$

In addition, we define

$$\mathcal{X}^{-j} := (\mathcal{X}^j)',$$

in the sense of Banach space duals.

Definition 2.5 (Mild solution). We call (ϕ, Γ, Σ) a mild solution of (2.16) with initial datum $(\phi_0, \Gamma_0, \Sigma_0) \in \mathcal{X}^1$ iff there exists $T > 0$ such that $(\phi, \Gamma, \Sigma) \in C_t^0([0, T], \mathcal{X}^1) \cap C_t^1([0, T], \mathcal{X}^{-1})$ satisfies

$$\begin{aligned}\phi_t &= \phi_0 - i \int_0^t ds \left[\frac{\lambda}{N} \left((\Gamma_s * (\hat{v} + \hat{v}(0)))(0) \phi_s + (\Sigma_s * \hat{v})(0) \bar{\phi}_s \right) \right. \\ &\quad \left. - 2\lambda |\Lambda| \hat{v}(0) |\phi_s|^2 \phi_s \right], \\ \Gamma_t &= \Gamma_0 - \frac{2\lambda}{N} \int_0^t ds \operatorname{Im} \left((\Sigma_s * \hat{v}) \Sigma_s \right), \\ \Sigma_t &= e^{-2iEt} \Sigma_0 - i \int_0^t ds e^{-2iE(t-s)} \left[2 \left(\frac{\lambda}{N} \Gamma_s * (\hat{v} + \hat{v}(0)) \right) \Sigma_s \right. \\ &\quad \left. + \frac{\lambda}{N} (\Sigma_s * \hat{v}) (1 + 2\Gamma_s) - 4N\lambda |\Lambda|^2 \hat{v}(0) |\phi_s|^2 \phi_s^2 \delta \right]\end{aligned}$$

for all $t \in [0, T]$.

For the next result, we introduce the truncated fields

$$\Gamma^T := \Gamma - N|\Lambda| |\phi|^2 \delta, \quad \Sigma^T := \Sigma - N|\Lambda| \phi^2 \delta. \quad (2.17)$$

Proposition 2.6 (Global well-posedness). Assume that $\hat{v} \in L^1_{\sqrt{1+E}} \cap L^\infty(\Lambda^*)$, and that $v \geq 0$. Let $(\phi_0, \Gamma_0, \Sigma_0) \in \mathcal{X}^1$, such that $\Gamma_0^T \geq 0$ and $|\Sigma_0^T|^2 \leq (\Gamma_0^T + 1)\Gamma_0^T$. Let $(\phi, \Gamma, \Sigma) \in C_t^0([0, T_0], \mathcal{X}^1) \cap C_t^1([0, T_0], \mathcal{X}^{-1})$ be the associated unique maximal mild solution of (2.16) with existence time $T_0 > 0$. Then $T_0 = \infty$, and $\Gamma_t^T \geq 0$ and $|\Sigma_t^T|^2 \leq (\Gamma_t^T + 1)\Gamma_t^T$ for all $t \geq 0$.

2.3.2. *Boltzmann collision kernels.* In Lemma A.2, we compute the cubic collision kernels

$$\begin{aligned}\mathbf{w}_t^{(3,0)}(\mathbf{p}_3) &:= \\ &\sqrt{|\Lambda|} \left((u_t(p_1) u_t(p_2) v_t(p_3) \phi_t + v_t(p_1) v_t(p_2) u_t(p_3) \bar{\phi}_t) (\hat{v}(p_1) + \hat{v}(p_2)) \right. \\ &\quad + (v_t(p_1) u_t(p_2) u_t(p_3) \phi_t + u_t(p_1) v_t(p_2) v_t(p_3) \bar{\phi}_t) (\hat{v}(p_2) + \hat{v}(p_3)) \\ &\quad \left. + (u_t(p_1) v_t(p_2) u_t(p_3) \phi_t + v_t(p_1) u_t(p_2) v_t(p_3) \bar{\phi}_t) (\hat{v}(p_1) + \hat{v}(p_3)) \right),\end{aligned} \quad (2.18)$$

$$\begin{aligned}\mathbf{w}_t^{(2,1)}(\mathbf{p}_3) &:= \\ &\sqrt{|\Lambda|} \left((u_t(p_1) u_t(p_2) u_t(p_3) \phi_t + v_t(p_1) v_t(p_2) \bar{v}_t(p_3) \bar{\phi}_t) (\hat{v}(p_1) + \hat{v}(p_2)) \right. \\ &\quad + (v_t(p_1) u_t(p_2) \bar{v}_t(p_3) \phi_t + u_t(p_1) v_t(p_2) u_t(p_3) \bar{\phi}_t) (\hat{v}(p_2) + \hat{v}(p_3)) \\ &\quad \left. + (u_t(p_1) v_t(p_2) \bar{v}_t(p_3) \phi_t + v_t(p_1) u_t(p_2) u_t(p_3) \bar{\phi}_t) (\hat{v}(p_1) + \hat{v}(p_3)) \right)\end{aligned} \quad (2.19)$$

in the expression for $\mathcal{H}_{\text{cub}}(t)$. Analogously, we also obtain the quartic collision kernels

$$\begin{aligned}\mathbf{w}_t^{(4,0)}(\mathbf{p}_4) &:= \\ &(u_t(p_1) u_t(p_2) v_t(p_3) v_t(p_4) + v_t(p_1) v_t(p_2) u_t(p_3) u_t(p_4)) (\hat{v}(p_1 + p_3) + \hat{v}(p_2 + p_3)) \\ &+ (u_t(p_1) v_t(p_2) u_t(p_3) v_t(p_4) + v_t(p_1) u_t(p_2) v_t(p_3) u_t(p_4)) (\hat{v}(p_1 + p_2) + \hat{v}(p_2 + p_3))\end{aligned} \quad (2.20)$$

$$\begin{aligned}
& + (u_t(p_1)v_t(p_2)v_t(p_3)u_t(p_4) + v_t(p_1)u_t(p_2)u_t(p_3)v_t(p_4))(\hat{v}(p_1 + p_2) + \hat{v}(p_1 + p_3)), \\
\mathbf{w}_t^{(3,1)}(\mathbf{p}_4) & := \\
& (u_t(p_1)u_t(p_2)v_t(p_3)u_t(p_4) + v_t(p_1)v_t(p_2)u_t(p_3)\bar{v}_t(p_4))(\hat{v}(p_1 + p_3) + \hat{v}(p_2 + p_3)) \\
& + (u_t(p_1)v_t(p_2)u_t(p_3)u_t(p_4) + v_t(p_1)u_t(p_2)v_t(p_3)\bar{v}_t(p_4))(\hat{v}(p_1 + p_2) + \hat{v}(p_2 + p_3)) \\
& + (v_t(p_1)u_t(p_2)u_t(p_3)u_t(p_4) + u_t(p_1)v_t(p_2)v_t(p_3)\bar{v}_t(p_4))(\hat{v}(p_1 + p_2) + \hat{v}(p_1 + p_3)), \\
\mathbf{w}_t^{(2,2)}(\mathbf{p}_4) & := \\
& (u_t(p_1)u_t(p_2)u_t(p_3)u_t(p_4) + v_t(p_1)v_t(p_2)\bar{v}_t(p_3)\bar{v}_t(p_4))(\hat{v}(p_1 - p_3) + \hat{v}(p_2 - p_3)) \\
& + (u_t(p_1)v_t(p_2)\bar{v}_t(p_3)u_t(p_4) + v_t(p_1)u_t(p_2)u_t(p_3)\bar{v}_t(p_4))(\hat{v}(p_1 + p_2) + \hat{v}(p_2 - p_3)) \\
& + (v_t(p_1)u_t(p_2)\bar{v}_t(p_3)u_t(p_4) + u_t(p_1)v_t(p_2)u_t(p_3)\bar{v}_t(p_4))(\hat{v}(p_1 + p_2) + \hat{v}(p_1 - p_3)).
\end{aligned} \tag{2.21}$$

in the expression for $\mathcal{H}_{\text{quart}}(t)$. The Bogoliubov coefficients u and v are related to the HFB fields via

$$u_t(p) = \sqrt{1 + \gamma_t(p)}, \quad v_t(p) = \frac{\sigma_t(p)}{\sqrt{1 + \gamma_t(p)}}.$$

Moreover, we abbreviate

$$\tilde{h}(p) := h(p) + 1, \tag{2.22}$$

$$\bar{\mathbf{p}} := (p_1, p_2, -p_3), \tag{2.23}$$

$$\bar{\mathbf{p}}_3 := (p_3, p_2, p_1). \tag{2.23}$$

Then we introduce the cubic Boltzmann operators

$$\begin{aligned}
Q_3[h](t, p) & := \\
& 2\lambda^2 \operatorname{Re} \int_0^t ds \int d\mathbf{p}_3 \left(\frac{1}{2!} (\delta(p_1 - p) + \delta(p_2 - p) - \delta(p_3 - p)) \right. \\
& \mathbf{w}_{s_1}^{(2,1)}(\mathbf{p}_3) \bar{\mathbf{w}}_s^{(2,1)}(\mathbf{p}_3) e^{i \int_s^t d\tau (\Omega_\tau(p_1) + \Omega_\tau(p_2) - \Omega_\tau(p_3))} \delta(p_1 + p_2 - p_3) \\
& (\tilde{h}_s(p_1)\tilde{h}_s(p_2)h_s(p_3) - h_s(p_1)h_s(p_2)\tilde{h}_s(p_3)) \\
& + \frac{1}{3!} (\delta(p_1 - p) + \delta(p_2 - p) + \delta(p_3 - p)) \\
& \mathbf{w}_s^{(3,0)}(\mathbf{p}_3) \bar{\mathbf{w}}_s^{(3,0)}(\mathbf{p}_3) e^{i \int_s^t d\tau (\Omega_\tau(p_1) + \Omega_\tau(p_2) + \Omega_\tau(p_3))} \delta(p_1 + p_2 + p_3) \\
& \left. (\tilde{h}_s(p_1)\tilde{h}_s(p_2)\tilde{h}_s(p_3) - h_s(p_1)h_s(p_2)h_s(p_3)) \right)
\end{aligned}$$

and

$$\begin{aligned}
Q_3^{(g)}[h](t)[J] & := \tag{2.24} \\
& \lambda^2 \int dp J(p) \int_0^t ds \int d\mathbf{p}_3 \left[\delta(p_1 + p_2 - p_3) \left(\delta(p - p_3) e^{2i \int_0^t d\tau \Omega_\tau(p_3)} \right. \right. \\
& e^{i \int_s^t d\tau (\Omega_\tau(p_1) + \Omega_\tau(p_2) - \Omega_\tau(p_3))} \mathbf{w}_t^{(3,0)}(\bar{\mathbf{p}}_3) \bar{\mathbf{w}}_{s_2}^{(2,1)}(\mathbf{p}_3) - 2\delta(p - p_1) e^{-2i \int_0^t d\tau \Omega_\tau(p_1)} \\
& \left. \left. e^{-i \int_s^t d\tau (\Omega_\tau(p_1) + \Omega_\tau(p_2) - \Omega_\tau(p_3))} \bar{\mathbf{w}}_t^{(2,1)}(\bar{\mathbf{p}}_3) \mathbf{w}_s^{(2,1)}(\mathbf{p}_3) \right) \right]
\end{aligned}$$

$$\begin{aligned}
& \left(h_s(p_1)h_s(p_2)\tilde{h}_s(p_3) - \tilde{h}_s(p_1)\tilde{h}_s(p_2)h_s(p_3) \right) \\
& + \delta(p - p_3)e^{2i\int_0^t d\tau \Omega_\tau(p_3)}\delta(p_1 + p_2 + p_3)e^{-i\int_s^t d\tau (\Omega_\tau(p_1) + \Omega_\tau(p_2) + \Omega_\tau(p_3))} \\
& \quad \overline{\mathbf{w}}_t^{(2,1)}(\bar{\mathbf{p}}_3)\mathbf{w}_s^{(3,0)}(\mathbf{p}_3)\left(\tilde{h}_s(p_1)\tilde{h}_s(p_2)\tilde{h}_s(p_3) - h_s(p_1)h_s(p_2)h_s(p_3) \right) \Big]
\end{aligned}$$

and

$$\begin{aligned}
Q_3^{(\Phi)}[h](t) & := \\
& \lambda^2 \int_0^t ds e^{i\int_0^{s_1} d\tau \Omega_\tau(0)} \left[\frac{1}{2}\delta(p_1 + p_2 - p_3) \left(e^{i\int_s^t d\tau (\Omega_\tau(p_1) + \Omega_\tau(p_2) - \Omega_\tau(p_3))} \right. \right. \\
& \quad \mathbf{w}_t^{(3,1)}(0, \mathbf{p}_3)\overline{\mathbf{w}}_s^{(2,1)}(\mathbf{p}_3) - e^{-i\int_s^t d\tau (\Omega_\tau(p_1) + \Omega_\tau(p_2) - \Omega_\tau(p_3))} \mathbf{w}_t^{(2,2)}(0, \bar{\mathbf{p}}_3)\mathbf{w}_s^{(2,1)}(\bar{\mathbf{p}}_3) \Big) \\
& \quad \left. \left(h_s(p_1)h_s(p_2)\tilde{h}_s(p_3) - \tilde{h}_s(p_1)\tilde{h}_s(p_2)h_s(p_3) \right) \right. \\
& + \frac{1}{3!}\delta(p_1 + p_2 + p_3) \left(\mathbf{w}_t^{(4,0)}(0, \mathbf{p}_3)\overline{\mathbf{w}}_s^{(3,0)}(\mathbf{p}_3)e^{i\int_s^t d\tau (\Omega_\tau(p_1) + \Omega_\tau(p_2) + \Omega_\tau(p_3))} \right. \\
& \quad \left. \left. - \overline{\mathbf{w}}_{s_1}^{(3,1)}(\mathbf{p}_3, 0)\mathbf{w}_s^{(3,0)}(\mathbf{p}_3)e^{-i\int_s^t d\tau (\Omega_\tau(p_1) + \Omega_\tau(p_2) + \Omega_\tau(p_3))} \right) \right. \\
& \quad \left. \left(h_s(p_1)h_s(p_2)h_s(p_3) - \tilde{h}_s(p_1)\tilde{h}_s(p_2)\tilde{h}_s(p_3) \right) \right].
\end{aligned} \tag{2.25}$$

In the evolution of Φ , we obtain the additional collision term

$$\begin{aligned}
Q_{3,3}^{(\Phi)}[h](t) & := \lambda e^{i\int_0^t d\tau \Omega_\tau(0)} \int dp \left(\mathbf{w}_t^{(2,1)}(0, p, p)Q_3[h](t, p) \right. \\
& \quad \left. + \overline{\mathbf{w}}_t^{(2,1)}(p, p, 0)Q_3^{(g)}[h](t, p) + \mathbf{w}_t^{(3,0)}(p, p, 0)\overline{Q_3^{(g)}[h](t, p)} \right). \tag{2.26}
\end{aligned}$$

2.3.3. Main theorem.

Theorem 2.7. *Impose the same assumptions as in Proposition 2.6, and let $\Omega^{(2)}$ be the Bogoliubov dispersion defined in (3.13). In addition, let $|\Lambda| \geq 1$, $\lambda > 0$, and $t > 0$, and $N > 0$, and assume that $\|f_0\|_d, \|\hat{v}\|_{w,d}, \|\gamma_0\|_d < \infty$. Then there exist constants $C > 0$ dependent on $\|f_0\|_d, \|\gamma_0\|_d, \|\hat{v}\|_{w,d}, |\Lambda|$, and $K > 0$ dependent on $\|f_0\|_d, \|\gamma_0\|_d$ s.t. we have for all $J \in L^2 \cap L^\infty(\Lambda^*)$ that*

$$\left| \Phi_t - \frac{1}{N^{\frac{3}{2}}} \int_0^t ds (Q_3^{(\Phi)}[f](s) + Q_{3,3}^{(\Phi)}[f](s)) \right| \leq C e^{K\|\hat{v}\|_{w,d}|\Lambda|\lambda t} \frac{1}{N^2}, \tag{2.27}$$

$$\left| \int dp J(p)(f_t(p) - f_0(p) - \frac{1}{N} \int_0^t ds Q_3[f](s, p)) \right| \leq C e^{K\|\hat{v}\|_{w,d}|\Lambda|\lambda t} \frac{\|J\|_\infty}{N^{\frac{3}{2}}}, \tag{2.28}$$

$$\left| \int dp J(p)(g_t(p) - \frac{1}{N} \int_0^t ds Q_3^{(g)}[f](s, p)) \right| \leq C e^{K\|\hat{v}\|_{w,d}|\Lambda|\lambda t} \frac{\|J\|_2 + \|J\|_\infty}{N^{\frac{3}{2}}} \tag{2.29}$$

Remark 2.8. *The error bounds in Theorem 2.7 improve those obtained in [37] significantly. In the latter, the upper bounds were of order $O(\frac{\lambda}{N^{1/2}})$ for (2.27), and $O(\frac{\lambda}{N})$ for (2.28), (2.29), respectively. The t -dependence of the error remained the same in each case. In Theorem 2.7, the dependence of the error terms with respect to N is sharp.*

For the following statement, we introduce the mesoscopic fields

$$\Psi_T := \Phi_{T/\lambda^2}, \quad F_T := f_{T/\lambda^2}, \quad G_T := g_{T/\lambda^2},$$

as well as the mesoscopic Boltzmann operators

$$\begin{aligned}\mathcal{Q}_k^{(\Psi)}[F](S) &:= \lambda^{-2} Q_k^{(\Phi)}[f](S/\lambda^2), \quad \mathcal{Q}_3[F](S) := \lambda^{-2} Q_3[f](S/\lambda^2), \\ \mathcal{Q}_3^{(G)}[F](S) &:= \lambda^{-2} Q_3^{(g)}[f](S/\lambda^2).\end{aligned}$$

Corollary 2.9. *Under the same assumptions of Theorem 2.7, for any $\delta \in (0, \frac{1}{2})$, there exists a constant $C_\delta > 0$ dependent on $\|f_0\|_d$, $\|\gamma_0\|_d$, $\|\hat{v}\|_{w,d}$ and a constant K_δ dependent on $\|f_0\|_d$, $\|\gamma_0\|_d$, $\|\hat{v}\|_{w,d}$, $|\Lambda|$, such that for $t = \lambda^{-2}T$ and $\lambda = \frac{C_\delta}{\log N}$, we have that*

$$\begin{aligned}\left| \Psi_T - \frac{1}{N^{\frac{3}{2}}} \int_0^T dS (\mathcal{Q}_3^{(\Psi)}[F](S) + \mathcal{Q}_{3,3}^{(\Psi)}[F](S)) \right| &\leq \frac{K_\delta}{N^{\frac{3}{2}+\delta}}, \\ \left| \int dp J(p) (F_T(p) - F_0(p) - \frac{1}{N} \int_0^T dS \mathcal{Q}_3[F](S, p)) \right| &\leq \frac{K_\delta \|J\|_\infty}{N^{1+\delta}}, \\ \left| \int dp J(p) (G_T(p) - \frac{1}{N} \int_0^T dS \mathcal{Q}_3^{(G)}[F](S, p)) \right| &\leq \frac{K_\delta (\|J\|_2 + \|J\|_\infty)}{N^{1+\delta}}.\end{aligned}$$

Remark 2.10. Corollary 2.9 improves our previous time window, see [37], which was of order $t \sim (\frac{\log N}{\log \log N})^2$. Here, we obtain $t \sim (\log N)^2$.

Remark 2.11. In [37], we also studied the case $L = \lambda^{-2-} \gg \lambda^{-2} \sim t$. This is due to the reason, that for longer times/shorter system sizes, we observe superposition of waves, while for shorter times, we observe dispersion. In that case, the time window of validity was given by $t \sim \lambda^{-2}$, with $\lambda = O((\frac{\log \log N}{\log N})^{\frac{2}{7}-})$. For these times, we observed an elastic QBE, for which the dispersion relation is given by the Bogoliubov dispersion Ω . We do not provide such a result in the present work, due to the complicated phase structure in the quantum Boltzmann operators, involving the phases of the HFB fields (ϕ, σ) as well as the modified Bogoliubov dispersion $\Omega^{(2)}$, see Remark 2.4.

2.3.4. Sketch of the proof. In Section 3, we first derive the expansions for the total BEC wave function and total pair correlations. We then compute the perturbation expansion of f , g and Φ . In that expansion, we collect terms corresponding to the BEC evolution up to order $\frac{1}{\sqrt{N}}$. These can be eliminated by a suitable choice of the HFB field ϕ , due to the presence of the Weyl transform $\mathcal{W}[\sqrt{N|\Lambda|}\phi]$. The pair-absorption terms up to order $\frac{1}{N}$ contain some terms that can be characterized as HFB terms. These in turn can be eliminated by a suitable choice of (γ, σ) for the Bogoliubov rotation $\mathcal{T}[k_t]$. The remaining ‘free’ evolution terms can be eliminated by properly choosing the dispersion Ω in $\mathcal{U}_{\text{Bog}}(t)$. This procedure amounts to the renormalization of $(\phi, \gamma, \sigma, \Omega)$. In Section 4, we derive a priori bounds for the renormalized HFB fields (ϕ, γ, σ) . Those allow us to control the tail and other lower-order terms in the Duhamel expansions of (Φ, f, g) .

3. DERIVATION OF LEADING ORDER TERMS

3.1. Density expansion.

$$\mathcal{W}^\dagger[\sqrt{N|\Lambda|}\phi_t] a_p^\dagger a_p \mathcal{W}[\sqrt{N|\Lambda|}\phi_t] = a_p^\dagger a_p + \delta(p) \left(\sqrt{N|\Lambda|} (\phi_t a_0^\dagger + \bar{\phi}_t a_0) + N|\Lambda|^2 |\phi_t|^2 \right) \quad (3.1)$$

Analogously, (2.10), followed by (2.11), (2.12), yields

$$\begin{aligned}\mathcal{T}^\dagger[k_t]a_p^\dagger a_p \mathcal{T}[k_t] &= (u_t(p)a_p^\dagger + \bar{v}_t(p)a_{-p})(u_t(p)a_p + v_t(p)a_{-p}^\dagger) \\ &= u_t(p)^2 a_p^\dagger a_p + |v_t(p)|^2 a_{-p}^\dagger a_{-p} + (u_t(p)v_t(p)a_p^\dagger a_{-p}^\dagger + \text{h.c.}) + |\Lambda|v_t(p)\mathfrak{J}^{22} \\ &= (1 + \gamma_t(p))a_p^\dagger a_p + \gamma_t(p)a_{-p}^\dagger a_{-p} + (\sigma_t(p)a_p^\dagger a_{-p}^\dagger + \text{h.c.}) + |\Lambda|\gamma_t(p),\end{aligned}$$

and, similarly,

$$\begin{aligned}\mathcal{T}^\dagger[k_t](\phi_t a_0^\dagger + \bar{\phi}_t a_0) \mathcal{T}[k_t] &= \phi_t(u_t(0)a_0^\dagger + \bar{v}_t(0)a_0) + \bar{\phi}_t(u_t(0)a_0 + v_t(0)a_0^\dagger) \\ &= (\phi_t u_t(0) + \bar{\phi}_t v_t(0))a_0^\dagger + \text{h.c.}\end{aligned}\quad (3.3)$$

Next, (2.5) implies

$$\mathcal{U}_{Bog}^\dagger(t)a_0 \mathcal{U}_{Bog}(t) = e^{-i\int_0^t ds \Omega_s(0)}a_0, \quad (3.4)$$

$$\mathcal{U}_{Bog}^\dagger(t)a_p^\dagger a_{-p}^\dagger \mathcal{U}_{Bog}(t) = e^{2i\int_0^t ds \Omega_s(p)}a_p^\dagger a_{-p}^\dagger, \quad (3.5)$$

while $[a_p^\dagger a_p, \mathcal{U}_{Bog}(t)] = 0$.

Collecting (3.1), (3.2), (3.3), (3.4) and (3.5), and using the fact that f_t is even, we can rewrite the total density in terms of the relative densities

$$\begin{aligned}f_t^{(\text{tot})}(p) &= \delta(p) \left[N|\Lambda||\phi_t|^2 + \left(\sqrt{N|\Lambda|}e^{i\int_0^t ds \Omega_s(0)}(\phi_t u_t(0) + \bar{\phi}_t v_t(0))\bar{\Phi}_t + \text{h.c.} \right) \right] \\ &\quad + \gamma_t(p) + (1 + \gamma_t(p))f_t(p) + \gamma_t(p)f_t(-p) + \left(e^{2i\int_0^t ds \Omega_s(p)}\sigma_t(p)\bar{g}_t(p) + \text{h.c.} \right).\end{aligned}$$

Similarly, we have that

$$\mathcal{W}^\dagger[\sqrt{N|\Lambda|}\phi_t]a_p a_{-p} \mathcal{W}[\sqrt{N|\Lambda|}\phi_t] = a_p a_{-p} + \delta(p) \left(2\sqrt{N|\Lambda|}\phi_t a_0 + N|\Lambda|^2\phi_t^2 \right),$$

and that

$$\begin{aligned}\mathcal{T}^\dagger[k_t]a_p a_{-p} \mathcal{T}[k_t] &= (u_t(p)a_p + v_t(p)a_{-p}^\dagger)(u_t(p)a_{-p} + v_t(p)a_p^\dagger) \\ &= |\Lambda|\sigma_t(p) + \sigma_t(a_p^\dagger a_p + a_{-p}^\dagger a_{-p}) + (1 + \gamma_t(p))a_p a_{-p} \\ &\quad + \frac{\sigma_t(p)^2}{1 + \gamma_t(p)}a_p^\dagger a_{-p}^\dagger.\end{aligned}$$

Following analogous steps as above, we obtain

$$\begin{aligned}g_t^{(\text{tot})}(p) &= \delta(p) \left[N|\Lambda|\phi_t^2 + 2\sqrt{N|\Lambda|}\phi_t(u_t(0)e^{-i\int_0^t ds \Omega_s(0)}\Phi_t + v_t(0)e^{i\int_0^t ds \Omega_s(0)}\bar{\Phi}_t) \right] \\ &\quad + \sigma_t(p) + \sigma_t(f_t(p) + f_t(-p)) + (1 + \gamma_t(p))e^{2i\int_0^t ds \Omega_s(p)}g_t(p) \\ &\quad + \frac{\sigma_t(p)^2 e^{-2i\int_0^t ds \Omega_s(p)}\bar{g}_t(p)}{1 + \gamma_t(p)},\end{aligned}$$

and also

$$\Phi_t^{(\text{tot})} = \sqrt{N|\Lambda|}\phi_t + u_t(0)e^{-i\int_0^t d\tau \Omega_\tau(0)}\Phi_t + v_t(0)e^{i\int_0^t d\tau \Omega_\tau(0)}\bar{\Phi}.$$

3.2. Perturbation expansion. We are interested in the evolution of

$$f_t(p) = \frac{\langle \mathcal{U}_N^\dagger(t) a_p^\dagger a_p \mathcal{U}_N(t) \rangle_0}{|\Lambda|}.$$

Note that due to conjugation with $\mathcal{U}_N(t)$, the phase-factor e^{iS_t} in $\mathcal{H}_{\text{fluc}}(t)$ drops out. Using the Duhamel expansion, we obtain that

$$\begin{aligned} f_t(p) &= f_0(p) - i \int_0^t ds \frac{\langle [a_p^\dagger a_p, \mathcal{H}_{\text{fluc}}(s)] \rangle_0}{|\Lambda|} \\ &\quad - \int_{[0,t]^2} d\mathbf{s}_2 \mathbf{1}_{s_1 \geq s_2} \frac{\langle [[a_p^\dagger a_p, \mathcal{H}_{\text{fluc}}(s_1)], \mathcal{H}_{\text{fluc}}(s_2)] \rangle_0}{|\Lambda|} \\ &\quad + \int_{[0,t]^3} d\mathbf{s}_3 \frac{\langle [[[a_p^\dagger a_p, \mathcal{H}_{\text{fluc}}(s_1)], \mathcal{H}_{\text{fluc}}(s_2)], \mathcal{H}_{\text{fluc}}(s_3)] \rangle_{s_3}}{|\Lambda|}. \end{aligned}$$

Translation invariance and gauge invariance imply that

$$\langle [a_p^\dagger a_p, \mathcal{H}_{\text{fluc}}(s)] \rangle_0 = \overline{[a_p^\dagger a_p, \mathcal{H}_{\text{fluc}}(s)]} \propto f_0(p) \tilde{f}_0(p) - \tilde{f}_0(p) f_0(p) = 0. \quad (3.6)$$

Next, we have, due to gauge invariance, that

$$\begin{aligned} &\frac{\langle [[a_p^\dagger a_p, \mathcal{H}_{\text{fluc}}(s_1)], \mathcal{H}_{\text{fluc}}(s_2)] \rangle_0}{|\Lambda|} \\ &= \frac{\langle [[a_p^\dagger a_p, \mathcal{H}_{\text{BEC}}(s_1) + \mathcal{H}_{\text{cub}}(s_1)], \mathcal{H}_{\text{BEC}}(s_2) + \mathcal{H}_{\text{cub}}(s_2)] \rangle_0}{|\Lambda|} \\ &\quad + \frac{\langle [[a_p^\dagger a_p, \mathcal{H}_{\text{HFB}}(s_1) + \mathcal{H}_{\text{quart}}(s_1)], \mathcal{H}_{\text{HFB}}(s_2) + \mathcal{H}_{\text{quart}}(s_2)] \rangle_0}{|\Lambda|}. \end{aligned} \quad (3.7)$$

3.2.1. First order HFB renormalization. For general fields (ϕ, γ, σ) , the leading order contributions in (3.7) are generated by \mathcal{H}_{BEC} and \mathcal{H}_{HFB} . Thus, a possible choice is to set

$$\mathcal{H}_{\text{BEC}}(t) = 0, \quad (3.8)$$

$$\mathcal{H}_{\text{HFB}}(t) = 0. \quad (3.9)$$

Observe that it is sufficient for calculating the leading order expressions to assume

$$\mathcal{H}_{\text{BEC}}(t), \mathcal{H}_{\text{HFB}}(t) = O(\frac{1}{\sqrt{N}}).$$

Lemma A.2 implies that (3.8) is equivalent to

$$\begin{aligned} u_t(0) \Big(&- i\partial_t \phi_t + \lambda |\Lambda| |\phi_t|^2 \hat{v}(0) \phi_t + \frac{\lambda}{N} \int dp \hat{v}(p) \sigma_t(p) \bar{\phi}_t \\ &+ \frac{\lambda}{N} \int dp (\hat{v}(p) + \hat{v}(0)) \gamma_t(p) \phi_t \Big) + v_t(0) \Big(- i\overline{\partial_t \phi_t} + \lambda |\Lambda| |\phi_t|^2 \hat{v}(0) \bar{\phi}_t \\ &+ \frac{\lambda}{N} \int dp \hat{v}(p) \bar{\sigma}_t(p) \phi_t + \frac{\lambda}{N} \int dp (\hat{v}(p) + \hat{v}(0)) \gamma_t(p) \bar{\phi}_t \Big) = 0. \end{aligned}$$

Abbreviating

$$\Gamma_t^{(1)}(p) := \gamma_t(p) + N |\Lambda| |\phi_t|^2 \delta(p), \quad (3.10)$$

$$\Sigma_t^{(1)}(p) := \sigma_t(p) + N|\Lambda|\phi_t^2\delta(p), \quad (3.11)$$

this condition is satisfied if

$$\begin{aligned} i\partial_t\phi_t^{(1)} &= \frac{\lambda}{N} \left((\Gamma_t^{(1)} * (\hat{v} + \hat{v}(0)))(0)\phi_t^{(1)} + (\Sigma_t^{(1)} * \hat{v})(0)\bar{\phi}_t^{(1)} \right) \\ &\quad - 2\lambda|\Lambda|\hat{v}(0)|\phi_t^{(1)}|^2\phi_t^{(1)}. \end{aligned} \quad (3.12)$$

The superscript '⁽¹⁾' accounts for renormalization to first order, as detailed below. For now, they do not play a specific role.

Since $\mathcal{H}_{\text{HFB}}^{(\text{d})}(t)$ is a diagonal quadratic operator, we can absorb it into phase factors in $\mathcal{H}_{\text{BEC}}(t)$, $\mathcal{H}_{\text{cub}}(t)$, and $\mathcal{H}_{\text{quart}}(t)$. For that purpose, we set

$$\mathcal{H}_{\text{HFB}}^{(\text{d})}(t) = 0,$$

or equivalently,

$$\begin{aligned} \Omega_t^{(1)}(p) &= \\ &\left(E(p) + \frac{\lambda}{N} ((\gamma_t^{(1)} + N|\Lambda||\phi_t^{(1)}|^2\delta) * (\hat{v} + \hat{v}(0)))(p) \right) (1 + 2\gamma_t^{(1)}(p)) \\ &+ \frac{2\lambda}{N} \operatorname{Re} \left(((\bar{\sigma}_t^{(1)} + N|\Lambda|(\bar{\phi}_t^{(1)})^2\delta) * \hat{v})(p)\sigma_t^{(1)}(p) \right) - \frac{\operatorname{Re}(\bar{\sigma}_t^{(1)}(p)i\partial_t\sigma_t^{(1)}(p))}{1 + \gamma_t^{(1)}(p)}. \end{aligned} \quad (3.13)$$

see Lemma A.5.

Thus (3.9) is satisfied if

$$\mathcal{H}_{\text{HFB}}^{(\text{cor})}(t) = 0, \quad (3.14)$$

which has been elaborated on in [56–58]. Lemma A.2 implies that (3.14) is satisfied if

$$\begin{aligned} \frac{i\partial_t\sigma_t^{(1)}(p)}{2} - \frac{\sigma_t^{(1)}(p)i\partial_t\gamma_t^{(1)}(p)}{2(1 + \gamma_t^{(1)}(p))} &= \\ &\left(E(p) + \frac{\lambda}{N}\Gamma_t^{(1)} * (\hat{v} + \hat{v}(0))(p) \right) \sigma_t^{(1)}(p) \\ &+ \frac{\lambda}{2N} \left((\Sigma_t^{(1)} * \hat{v})(p)(1 + \gamma_t^{(1)}(p)) + (\bar{\Sigma}_t^{(1)} * \hat{v})(p)\frac{\sigma_t^{(1)}(p)^2}{1 + \gamma_t^{(1)}(p)} \right). \end{aligned} \quad (3.15)$$

We show in Lemma A.4 that (3.15) is equivalent to

$$\begin{aligned} i\partial_t\gamma_t^{(1)} &= \frac{\lambda}{N} \left[(\Sigma_t^{(1)} * \hat{v})\bar{\sigma}_t^{(1)} - (\bar{\Sigma}_t^{(1)} * \hat{v})\sigma_t^{(1)} \right], \\ i\partial_t\sigma_t^{(1)} &= 2(E + \frac{\lambda}{N}\Gamma_t^{(1)} * (\hat{v} + \hat{v}(0)))\sigma_t^{(1)} + \frac{\lambda}{N}(\Sigma_t^{(1)} * \hat{v})(1 + 2\gamma_t^{(1)}). \end{aligned} \quad (3.16)$$

Together with (3.12), we thus have shown that

$$\begin{aligned} i\partial_t \phi_t^{(1)} &= \frac{\lambda}{N} \left((\Gamma_t^{(1)} * (\hat{v} + \hat{v}(0)))(0) \phi_t^{(1)} + (\Sigma_t^{(1)} * \hat{v})(0) \overline{\phi}_t^{(1)} \right) \\ &\quad - 2\lambda |\Lambda| \hat{v}(0) |\phi_t^{(1)}|^2 \phi_t^{(1)}. \\ \partial_t \gamma_t^{(1)} &= \frac{2\lambda}{N} \operatorname{Im} ((\Sigma_t^{(1)} * \hat{v}) \overline{\sigma}_t^{(1)}), \\ i\partial_t \sigma_t^{(1)} &= 2 \left(E + \frac{\lambda}{N} \Gamma_t^{(1)} * (\hat{v} + \hat{v}(0)) \right) \sigma_t^{(1)} + \frac{\lambda}{N} (\Sigma_t^{(1)} * \hat{v}) (1 + 2\gamma_t^{(1)}). \end{aligned} \quad (3.17)$$

These are the well-known HFB equations in the translation invariant case, see, e.g., [8, 56]. Moreover, Lemma A.5 implies that, if $\sigma^{(1)}$ satisfies (3.17), then the Bogoliubov dispersion, see (3.13), satisfies

$$\Omega_t^{(1)} = E + \frac{\lambda}{N} \Gamma_t^{(1)} * (\hat{v} + \hat{v}(0)) + \frac{\lambda}{N} \frac{\operatorname{Re}((\Sigma_t^{(1)} * \hat{v}) \overline{\sigma}_t^{(1)})}{1 + \gamma_t^{(1)}}. \quad (3.18)$$

3.2.2. Second order HFB renormalization. In order to determine the cubic Boltzmann operator, we follow [37] and compute the second order Duhamel expansion.

Observe that for a self-adjoint operator A , operators B, C and any state ν we have that

$$\nu([[A, B + B^\dagger], C + C^\dagger]) = 2 \operatorname{Re}(\nu([[A, B], C]) + \nu([[A, B], C^\dagger])). \quad (3.19)$$

Remark 3.1 (Commutator rule). *We note that, due to the commutators, every right argument in a commutator needs to be connected to at least one argument to the left of it. We refer to this fact as the commutator rule.*

Remark 3.2. *Observe that we have that*

$$2 \operatorname{Re} \int_{[0,t]^2} ds_2 \mathbb{1}_{s_1 \geq s_2} A(s_1) \overline{A}(s_2) = \left| \int_0^t ds A(s) \right|^2.$$

Lemma A.2 implies that

$$\begin{aligned} \overline{\mathcal{H}}_{\text{cub}}(t) &= \frac{\lambda \sqrt{|\Lambda|}}{\sqrt{N}} e^{i \int_0^t d\tau \Omega_\tau(0)} a_0^\dagger \int dk \left[u_t(0) \left((1 + 2\gamma_t(k)) f_0(k) (\hat{v}(k) + \hat{v}(0)) \phi_t \right. \right. \\ &\quad \left. \left. + 2f_0(k) \sigma_t(k) \hat{v}(k) \overline{\phi}_t \right) + v_t(0) \left((1 + 2\gamma_t(k)) f_0(k) (\hat{v}(k) + \hat{v}(0)) \overline{\phi}_t \right. \right. \\ &\quad \left. \left. + 2f_0(k) \sigma_t(k) \hat{v}(k) \phi_t \right) \right] + \text{h.c.}. \end{aligned} \quad (3.20)$$

With that, we obtain

$$\begin{aligned} &- \int_{[0,t]^2} ds_2 \mathbb{1}_{s_1 \geq s_2} \frac{[[\overline{a_p^\dagger a_p}, \overline{\mathcal{H}}_{\text{cub}}(s_1)], \overline{\mathcal{H}}_{\text{cub}}(s_2)] + [[\overline{a_p^\dagger a_p}, \overline{\mathcal{H}}_{\text{cub}}(s_1)], \overline{\mathcal{H}}_{\text{cub}}(s_2)]}{|\Lambda|} \\ &= - 2 \operatorname{Re} \left(\int_{[0,t]^2} ds_2 \mathbb{1}_{s_1 \geq s_2} \frac{[[\overline{a_p^\dagger a_p}, \overline{\mathcal{H}}_{\text{cub}}(s_1)], \overline{\mathcal{H}}_{\text{cub}}(s_2)]}{|\Lambda|} \right) \end{aligned}$$

$$\begin{aligned}
&= -\delta(p) \int_{[0,t]^2} d\mathbf{s}_2 \mathbb{1}_{s_1 \geq s_2} \left(\frac{\langle [a_0^\dagger, \mathcal{H}_{\text{cub}}(s_1)] \rangle_0}{|\Lambda|} \frac{\langle [a_0, \mathcal{H}_{\text{cub}}(s_2)] \rangle_0}{|\Lambda|} + (s_1 \leftrightarrow s_2) \right) \\
&= \delta(p) \left| -i \int_0^t ds \frac{\langle [a_0, \mathcal{H}_{\text{cub}}(s)] \rangle_0}{|\Lambda|} \right|^2,
\end{aligned} \tag{3.21}$$

see Remark 3.2. As we show in [37], we have that this condensate contribution of size $\frac{\lambda^2 t^2}{N}$, and it dominates the cubic Boltzmann collision operator coming from

$$\begin{aligned}
\frac{1}{N} \int_0^t ds Q_3[f_0](s, p) &= - \int_{[0,t]^2} d\mathbf{s}_2 \mathbb{1}_{s_1 \geq s_2} \left(\frac{\langle [[a_p^\dagger a_p, \mathcal{H}_{\text{cub}}(s_1)], \mathcal{H}_{\text{cub}}(s_2)] \rangle_0}{|\Lambda|} \right. \\
&\quad \left. + \frac{\langle [[a_p^\dagger a_p, \mathcal{H}_{\text{cub}}(s_1)], \mathcal{H}_{\text{cub}}(s_2)] \rangle_0}{|\Lambda|} \right),
\end{aligned} \tag{3.22}$$

at least in the continuous approximation, when it is of size $\frac{\lambda^2 t}{N}$. Since (3.21) is proportional to $\delta(p)$, we can absorb it into the condensate contribution. With similar steps as in (3.21) and employing (3.6), we obtain

$$\begin{aligned}
&- \int_{[0,t]^2} d\mathbf{s}_2 \mathbb{1}_{s_1 \geq s_2} \left(\frac{\langle [[a_p^\dagger a_p, \mathcal{H}_{\text{BEC}}(s_1)], \mathcal{H}_{\text{BEC}}(s_2)] \rangle_0}{|\Lambda|} \right. \\
&\quad \left. + \frac{\langle [[a_p^\dagger a_p, \mathcal{H}_{\text{BEC}}(s_1)], \mathcal{H}_{\text{cub}}(s_2)] \rangle_0}{|\Lambda|} \right. \\
&\quad \left. + \frac{\langle [[a_p^\dagger a_p, \mathcal{H}_{\text{cub}}(s_1)], \mathcal{H}_{\text{BEC}}(s_2)] \rangle_0}{|\Lambda|} \right. \\
&\quad \left. + 2 \operatorname{Re} \frac{\langle [[a_p^\dagger a_p, \mathcal{H}_{\text{cub}}(s_1)], \mathcal{H}_{\text{cub}}(s_2)] \rangle_0}{|\Lambda|} \right) \\
&= \delta(p) \left| -i \int_0^t ds \frac{\langle [a_0, \mathcal{H}_{\text{BEC}}(s) + \mathcal{H}_{\text{cub}}(s)] \rangle_0}{|\Lambda|} \right|^2.
\end{aligned}$$

In order to eliminate this contribution, we choose

$$\langle [a_0, \mathcal{H}_{\text{BEC}}(t) + \mathcal{H}_{\text{cub}}(t)] \rangle_0 = 0. \tag{3.23}$$

Lemma A.2 implies that

$$\begin{aligned}
&\frac{\langle [a_0, \mathcal{H}_{\text{BEC}}(t)] \rangle_0}{|\Lambda|} = \\
&\sqrt{N|\Lambda|} \left[u_t(0) \left(-i\partial_t \phi_t + \lambda |\Lambda| |\phi_t|^2 \hat{v}(0) \phi_t + \frac{\lambda}{N} \int dp \hat{v}(p) \sigma_t(p) \bar{\phi}_t \right) \right. \\
&\quad \left. + \frac{\lambda}{N} \int dp (\hat{v}(p) + \hat{v}(0)) \gamma_t(p) \phi_t \right] + v_t(0) \left(-i\overline{\partial_t \phi_t} + \lambda |\Lambda| |\phi_t|^2 \hat{v}(0) \bar{\phi}_t \right. \\
&\quad \left. + \frac{\lambda}{N} \int dp \hat{v}(p) \bar{\sigma}_t(p) \phi_t + \frac{\lambda}{N} \int dp (\hat{v}(p) + \hat{v}(0)) \gamma_t(p) \bar{\phi}_t \right] e^{i \int_0^t ds \Omega_s(0)},
\end{aligned} \tag{3.24}$$

and (3.20) yields

$$\begin{aligned} \frac{\langle [a_0, \mathcal{H}_{\text{cub}}(t)] \rangle_0}{|\Lambda|} &= \frac{\langle [a_0, \overline{\mathcal{H}_{\text{cub}}}(t)] \rangle_0}{|\Lambda|} = \\ &\frac{\lambda\sqrt{|\Lambda|}}{\sqrt{N}} e^{i\int_0^t ds \Omega_s(0)} \left[u_t(0) \int dp \left((1 + 2\gamma_t(p))\phi_t(\hat{v}(p) + \hat{v}(0)) + \right. \right. \\ &2\sigma_t(p)\overline{\phi}_t\hat{v}(p) \Big) f_0(p) + v_t(0) \int dp \left((1 + 2\gamma_t(p))\overline{\phi}_t(\hat{v}(p) + \hat{v}(0)) + \right. \\ &\left. \left. 2\overline{\sigma}_t(p)\phi_t\hat{v}(p) \right) f_0(p) \right]. \end{aligned} \quad (3.25)$$

In order to satisfy (3.23), it suffices to equate the sum of the coefficients of $u_t(0)$ in (3.24) and (3.25) to zero. This condition is equivalent to

$$\begin{aligned} i\partial_t\phi_t &= \lambda|\Lambda||\phi|^2\hat{v}(0)\phi_t + \frac{\lambda}{N} \int dp \hat{v}(p)\sigma_t(p)\overline{\phi}_t + \frac{\lambda}{N} \int dp (\hat{v}(p) + \hat{v}(0))\gamma_t(p)\phi_t \\ &+ \frac{\lambda}{N} \int dp \left((1 + 2\gamma_t(p))\phi_t(\hat{v}(p) + \hat{v}(0)) + 2\sigma_t(p)\overline{\phi}_t\hat{v}(p) \right) f_0(p). \end{aligned} \quad (3.26)$$

Observe that all integrands except for f_0 are even. In anticipation of the evolution of γ and σ , recall from (2.14)

$$f_0^{(+)}(p) = \frac{1}{2}(f_0(p) + f_0(-p)),$$

the even symmetrization of f_0 . In analogy to (3.10), (3.11), we introduce the second order renormalized shifted expectations

$$\Gamma^{(2)} := (1 + 2f_0^{(+)})\gamma + f_0^{(+)} + N|\Lambda||\phi|^2\delta, \quad (3.27)$$

$$\Sigma^{(2)} := (1 + 2f_0^{(+)})\sigma + N|\Lambda|\phi^2\delta. \quad (3.28)$$

With these, and writing $\phi = \phi^{(2)}$, we can simplify (3.26) into

$$\begin{aligned} i\partial_t\phi_t^{(2)} &= \frac{\lambda}{N} \left((\Gamma_t^{(2)} * (\hat{v} + \hat{v}(0)))(0)\phi_t^{(2)} + (\Sigma_t^{(2)} * \hat{v})(0)\overline{\phi}_t^{(2)} \right) \\ &- 2\lambda|\Lambda|\hat{v}(0)|\phi_t^{(2)}|^2\phi_t^{(2)}. \end{aligned} \quad (3.29)$$

We recognize that (3.29) is a renormalization of (3.12), where we substituted $(\Gamma^{(1)}, \Sigma^{(1)})$ by the renormalized fields $(\Gamma^{(2)}, \Sigma^{(2)})$.

Next, observe that

$$[a_p^\dagger a_p, \mathcal{H}_{\text{HFB}}^{(\text{cor})}(t)] = A_t(p)a_p a_{-p} + \text{h.c.} \quad (3.30)$$

for some coefficient A_t . In particular, (3.19) yields

$$\begin{aligned} \langle [[a_p^\dagger a_p, \mathcal{H}_{\text{HFB}}^{(\text{cor})}(s_1)], \mathcal{H}_{\text{HFB}}^{(\text{cor})}(s_2) + \mathcal{H}_{\text{quart}}(s_2)] \rangle_0 \\ = 2\text{Re} (A_{s_1}(p)\langle [a_p a_{-p}, \mathcal{H}_{\text{HFB}}^{(\text{cor})}(s_2) + \mathcal{H}_{\text{quart}}(s_2)] \rangle_0). \end{aligned} \quad (3.31)$$

In addition, we have that

$$[a_p^\dagger a_p, \overline{\mathcal{H}_{\text{quart}}}(t)] = [a_p a_p^\dagger, \overline{\mathcal{H}_{\text{quart}}}(t)] \propto f_0(p)\tilde{f}_0(p) - \tilde{f}_0(p)f_0(p) = 0. \quad (3.32)$$

Lemma A.3 yields

$$\begin{aligned} \overline{\mathcal{H}_{\text{quart}}}(t) &= \frac{\lambda}{N} \int dp \left[\left((f_0^{(+)} \sigma_t) * \hat{v}(p)(1 + \gamma_t(p)) \right. \right. \\ &\quad \left. \left. + \frac{(f_0^{(+)} \bar{\sigma}_t) * \hat{v}(p)\sigma_t(p)}{1 + \gamma_t(p)} \right. \right. \\ &\quad \left. \left. + ((1 + 2\gamma_t)f_0^{(+)}) * (\hat{v} + \hat{v}(0))(p)\sigma_t(p) \right) e^{2i \int_0^t d\tau \Omega_\tau(p)} a_p^\dagger a_{-p}^\dagger + \text{h.c.} \right. \\ &\quad \left. + \left(((1 + 2\gamma_t)f_0^{(+)}) * (\hat{v} + \hat{v}(0))(p)(1 + 2\gamma_t(p)) \right. \right. \\ &\quad \left. \left. + 4 \operatorname{Re}((f_0^{(+)} \sigma_t) * \hat{v}(p)\bar{\sigma}_t(p)) \right) a_p^\dagger a_p \right]. \end{aligned} \quad (3.33)$$

Then the CCRs imply

$$[a_p^\dagger a_p, \overline{\mathcal{H}_{\text{quart}}}(t)] = B(t, p)a_p a_{-p} + \text{h.c.} \quad (3.34)$$

for some coefficient B . Using (3.19), (3.30), (3.32), and (3.34), we thus have

$$\begin{aligned} \langle [[a_p^\dagger a_p, \overline{\mathcal{H}_{\text{quart}}}(s_1)], \mathcal{H}_{\text{HFB}}^{(\text{cor})}(s_2) + \mathcal{H}_{\text{quart}}(s_2)] \rangle_0 \\ = 2 \operatorname{Re}(B(s_1, p) \langle [a_p a_{-p}, \mathcal{H}_{\text{HFB}}^{(\text{cor})}(s_2) + \mathcal{H}_{\text{quart}}(s_2)] \rangle_0). \end{aligned} \quad (3.35)$$

In order to eliminate the contributions coming from (3.31) and (3.35), we choose

$$\langle [a_p a_{-p}, \mathcal{H}_{\text{HFB}}^{(\text{cor})}(s_2) + \mathcal{H}_{\text{quart}}(s_2)] \rangle_0 = 0. \quad (3.36)$$

This condition relates to the *pair-absorption rate*. Observe that we have

$$[a_p a_{-p}, a_p^\dagger a_{-p}^\dagger] \propto 1 + 2f_0^{(+)}(p),$$

where $f_0^{(+)}(p) = f_0(p) + f_0(-p)$.

We start by calculating

$$\begin{aligned} \frac{\langle [a_p a_{-p}, \mathcal{H}_{\text{HFB}}^{(\text{cor})}(t)] \rangle_0}{|\Lambda|} &= 2(1 + 2f_0^{(+)}(p)) e^{2i \int_0^t ds \Omega_s(p)} \\ &\quad \left[-\frac{i\partial_t \sigma_t(p)}{2} + \frac{\sigma_t(p)i\partial_t \gamma_t(p)}{2(1 + \gamma_t(p))} + \left(E(p) + \frac{\lambda}{N} \Gamma_t^{(1)} * (\hat{v} + \hat{v}(0))(p) \right) \sigma_t(p) \right. \\ &\quad \left. + \frac{\lambda}{2N} \left((\Sigma_t^{(1)} * \hat{v})(p)(1 + \gamma_t(p)) + (\bar{\Sigma}_t^{(1)} * \hat{v})(p) \frac{\sigma_t(p)^2}{1 + \gamma_t(p)} \right) \right], \end{aligned} \quad (3.37)$$

see Lemma A.2 for the expression for $\mathcal{H}_{\text{HFB}}^{(\text{cor})}$. We used the fact that all functions appearing here, except for f_0 , are even, and we replaced γ , σ by their respective shifts $\Gamma^{(1)}$, $\Sigma^{(1)}$, see (3.10), (3.11). Similarly, (3.33) yields

$$\begin{aligned} \frac{\langle [a_p a_{-p}, \mathcal{H}_{\text{quart}}(t)] \rangle_0}{|\Lambda|} &= \frac{\langle [a_p a_{-p}, \overline{\mathcal{H}_{\text{quart}}}(t)] \rangle_0}{|\Lambda|} = \\ &2 \frac{\lambda}{N} (1 + 2f_0^{(+)}(p)) e^{2i \int_0^t ds \Omega_s(p)} \left(((\sigma f_0^{(+)}) * \hat{v})(p)(1 + \gamma_t(p)) \right. \\ &\quad \left. + ((\bar{\sigma}_t f_0^{(+)}) * \hat{v})(p) \frac{\sigma_t(p)^2}{1 + \gamma_t(p)} + ((1 + 2\gamma_t)f_0^{(+)}) * (\hat{v} + \hat{v}(0))(p)\sigma_t(p) \right). \end{aligned} \quad (3.38)$$

Hence, substituting (3.37) and (3.38) into (3.36) yields

$$\begin{aligned} \frac{i\partial_t\sigma_t(p)}{2} - \frac{\sigma_t(p)i\partial_t\gamma_t(p)}{2(1+\gamma_t(p))} = \\ \left(E(p) + \frac{\lambda}{N}\Gamma_t^{(1)} * (\hat{v} + \hat{v}(0))(p)\right)\sigma_t(p) \\ + \frac{\lambda}{2N}\left((\Sigma_t^{(1)} * \hat{v})(p)(1+\gamma_t(p)) + (\bar{\Sigma}_t^{(1)} * \hat{v})(p)\frac{\sigma_t(p)^2}{1+\gamma_t(p)}\right) \\ + \frac{\lambda}{N}\left(((1+2\gamma_t)f_0^{(+)})*(\hat{v} + \hat{v}(0))(p)\sigma_t(p) + ((\sigma f_0^{(+)}) * \hat{v})(p)(1+\gamma_t(p))\right) \\ + ((\bar{\sigma}_tf_0^{(+)}) * \hat{v})(p)\frac{\sigma_t(p)^2}{1+\gamma_t(p)} . \end{aligned} \quad (3.39)$$

Using the renormalized shifted fields $\Gamma^{(2)}$ and $\Sigma^{(2)}$, see (3.27) and (3.28), we can rewrite (3.39) as

$$\begin{aligned} \frac{i\partial_t\sigma_t(p)}{2} - \frac{\sigma_t(p)i\partial_t\gamma_t(p)}{2(1+\gamma_t(p))} = \\ \left(E(p) + \frac{\lambda}{N}\Gamma_t^{(2)} * (\hat{v} + \hat{v}(0))(p)\right)\sigma_t(p) \\ + \frac{\lambda}{2N}\left((\Sigma_t^{(2)} * \hat{v})(p)(1+\gamma_t(p)) + (\bar{\Sigma}_t^{(2)} * \hat{v})(p)\frac{\sigma_t(p)^2}{1+\gamma_t(p)}\right) . \end{aligned} \quad (3.40)$$

We recognize that (3.40) is a renormalization of (3.15), where $(\Gamma^{(1)}, \Sigma^{(1)})$ is replaced by the renormalized fields $(\Gamma^{(2)}, \Sigma^{(2)})$. In particular, the proof of Lemma A.4 implies that (3.40) is equivalent to

$$\begin{aligned} i\partial_t\gamma_t &= \frac{\lambda}{N}[(\Sigma_t^{(2)} * \hat{v})\bar{\sigma}_t - (\bar{\Sigma}_t^{(2)} * \hat{v})\sigma_t], \\ i\partial_t\sigma_t &= 2\left(E + \frac{\lambda}{N}\Gamma_t^{(2)} * (\hat{v} + \hat{v}(0))\right)\sigma_t + \frac{\lambda}{N}(\Sigma_t^{(2)} * \hat{v})(1+2\gamma_t) . \end{aligned}$$

Recalling (3.27), (3.28) and (3.29), we thus have shown that

$$\begin{aligned} i\partial_t\phi_t^{(2)} &= \frac{\lambda}{N}\left((\Gamma_t^{(2)} * (\hat{v} + \hat{v}(0)))(0)\phi_t^{(2)} + (\Sigma_t^{(2)} * \hat{v})(0)\bar{\phi}_t^{(2)}\right) \\ &\quad - 2\lambda|\Lambda|\hat{v}(0)|\phi_t^{(2)}|^2\phi_t^{(2)}. \\ \partial_t\gamma_t^{(2)} &= \frac{2\lambda}{N}\text{Im}\left((\Sigma_t^{(2)} * \hat{v})\bar{\sigma}_t^{(2)}\right), \\ i\partial_t\sigma_t^{(2)} &= 2\left(E + \frac{\lambda}{N}\Gamma_t^{(2)} * (\hat{v} + \hat{v}(0))\right)\sigma_t^{(2)} + \frac{\lambda}{N}(\Sigma_t^{(2)} * \hat{v})(1+2\gamma_t^{(2)}) . \end{aligned} \quad (3.41)$$

(3.41) corresponds to the second order renormalization of (3.17) with the corresponding renormalized fields.

To complete the renormalization of the HFB fields, we need to also renormalize the Bogoliubov dispersion Ω . For that purpose, we recall the diagonal part of $\mathcal{H}_{\text{quart}}(t)$ from

(3.33), see also Lemma A.2,

$$\begin{aligned} \mathcal{H}_{\text{quart}}^{\square}(t)^{(d)} &:= \frac{\lambda}{N} \int dp \left(((1 + 2\gamma_t) f_0^{(+)} * (\hat{v} + \hat{v}(0))(p)(1 + 2\gamma_t(p)) \right. \\ &\quad \left. + 4 \operatorname{Re} ((f_0^{(+)} \sigma_t) * \hat{v}(p) \bar{\sigma}_t(p)) \right) a_p^\dagger a_p . \end{aligned}$$

Then we choose $\Omega^{(2)}$ such that

$$\mathcal{H}_{\text{HFB}}^{(d)}(t) + \mathcal{H}_{\text{quart}}^{\square}(t)^{(d)} = 0 . \quad (3.42)$$

Employing Lemma A.2, see also (3.13), we obtain

$$\begin{aligned} \Omega_t^{(2)}(p) &= \\ &\left(E(p) + \frac{\lambda}{N} ((\gamma_t^{(2)} + N|\Lambda| |\phi_t^{(2)}|^2 \delta) * (\hat{v} + \hat{v}(0))(p) \right) (1 + 2\gamma_t^{(2)}(p)) \\ &+ \frac{2\lambda}{N} \operatorname{Re} \left(((\bar{\sigma}_t^{(2)} + N|\Lambda| (\bar{\phi}_t^{(2)})^2 \delta) * \hat{v})(p) \sigma_t^{(2)}(p) \right) - \frac{\operatorname{Re} (\bar{\sigma}_t^{(2)}(p) i \partial_t \sigma_t^{(2)}(p))}{1 + \gamma_t^{(2)}(p)} \\ &+ \frac{\lambda}{N} ((1 + 2\gamma_t) f_0^{(+)} * (\hat{v} + \hat{v}(0))(p)(1 + 2\gamma_t(p)) \\ &+ \frac{4\lambda}{N} \operatorname{Re} \left((f_0^{(+)} \bar{\sigma}_t^{(2)}) * \hat{v}(p) \sigma_t^{(2)}(p) \right) . \end{aligned}$$

Recalling the total fields (3.27), (3.28), and employing (3.41), we thus conclude that

$$\Omega_t^{(2)} = E + \frac{\lambda}{N} (\Gamma_t^{(2)} * (\hat{v} + \hat{v}(0))) + \frac{\lambda}{N} \frac{\operatorname{Re} (\bar{\Sigma}^{(2)} * \hat{v}) \sigma_t^{(2)}}{1 + \gamma_t^{(2)}} . \quad (3.43)$$

(3.43) corresponds to (3.18) with renormalized fields.

We conclude this section by equivalently reformulating the collected renormalization conditions (3.23), (3.36), (3.42):

$$\begin{cases} \mathcal{H}_{\text{BEC}}(t) + \mathcal{H}_{\text{cub}}^{\square}(t) = 0 , \\ \mathcal{H}_{\text{HFB}}(t) + \mathcal{H}_{\text{quart}}(t) = 0 , \end{cases}$$

compared with the first-order renormalization conditions (3.8), (3.9).

3.3. Boltzmann collision terms. The discussion in section (3.2) yields

$$\begin{aligned} f_t(p) &= f_0(p) + \frac{1}{N} \int_0^t ds Q_3[f_0](s, p) \\ &+ \int_{[0,t]^3} ds_3 \frac{\langle [[[a_p^\dagger a_p, \mathcal{H}_{\text{fluc}}(s_1)], \mathcal{H}_{\text{fluc}}(s_2)], \mathcal{H}_{\text{fluc}}(s_3)] \rangle_{s_3}}{|\Lambda|} , \end{aligned}$$

where

$$\frac{1}{N} \int_0^t ds Q_3[f_0](s, p) = - \int_{[0,t]^2} d\mathbf{s}_2 \mathbb{1}_{s_1 \geq s_2} \left(\frac{\langle [[a_p^\dagger a_p, \mathcal{H}_{\text{cub}}(s_1)], \mathcal{H}_{\text{cub}}(s_2)] \rangle_0}{|\Lambda|} \right. \\ \left. + \frac{\langle [[a_p^\dagger a_p, \mathcal{H}_{\text{cub}}(s_1)], \mathcal{H}_{\text{cub}}(s_2)] \rangle_0}{|\Lambda|} \right),$$

see (3.22), denotes the cubic Boltzmann collision operator.

We show in Lemma C.1 that

$$\int_0^t ds Q_3[f_0](s, p) = \\ 2\lambda^2 \operatorname{Re} \int_{[0,t]^2} d\mathbf{s}_2 \mathbb{1}_{s_1 \geq s_2} \int d\mathbf{p}_3 \left(\frac{1}{2!} (\delta(p_1 - p) + \delta(p_2 - p) - \delta(p_3 - p)) \right. \\ \mathbf{w}_{s_1}^{(2,1)}(\mathbf{p}_3) \overline{\mathbf{w}}_{s_2}^{(2,1)}(\mathbf{p}_3) e^{i \int_{s_2}^{s_1} d\tau (\Omega_\tau(p_1) + \Omega_\tau(p_2) - \Omega_\tau(p_3))} \delta(p_1 + p_2 - p_3) \\ (\tilde{f}_0(p_1) \tilde{f}_0(p_2) f_0(p_3) - f_0(p_1) f_0(p_2) \tilde{f}_0(p_3)) \\ + \frac{1}{3!} (\delta(p_1 - p) + \delta(p_2 - p) + \delta(p_3 - p)) \\ \mathbf{w}_{s_1}^{(3,0)}(\mathbf{p}_3) \overline{\mathbf{w}}_{s_2}^{(3,0)}(\mathbf{p}_3) e^{i \int_{s_2}^{s_1} d\tau (\Omega_\tau(p_1) + \Omega_\tau(p_2) + \Omega_\tau(p_3))} \delta(p_1 + p_2 + p_3) \\ \left. (\tilde{f}_0(p_1) \tilde{f}_0(p_2) \tilde{f}_0(p_3) - f_0(p_1) f_0(p_2) f_0(p_3)) \right).$$

In Lemma C.1, we also give an expression for the quartic Boltzmann term $\frac{1}{N^2} \int_0^t ds Q_4[f_0](s, p)$.

3.4. Error terms.

We abbreviate

$$f[J] := \int dp J(p) a_p^\dagger a_p.$$

Then we define the error

$$\operatorname{Rem}[f](t)[J] := \\ - \int dp J(p) \int_0^t ds \left(\frac{1}{N} Q_3[f_0](s, p) + \frac{1}{N^2} Q_4[f_0](s, p) \right) \\ + \int_0^t ds \langle [f[J], \mathcal{H}_{\text{fluc}}(s)] \rangle_0 \\ + \int_{[0,t]^2} d\mathbf{s}_2 \langle [[f[J], \mathcal{H}_{\text{fluc}}(s_1)], \mathcal{H}_{\text{fluc}}(s_2)] \rangle_0 \quad (3.44)$$

$$- \frac{1}{N^2} \int_0^t ds Q_4[f_0](s, p) \quad (3.45)$$

$$+ \frac{1}{N} \int_0^t ds (Q_3[f](s, p) - Q_3[f_0](s, p)) \quad (3.46)$$

$$+ \int_{[0,t]^3} d\mathbf{s}_3 \langle [[[f[J], \mathcal{H}_{\text{fluc}}(s_1)], \mathcal{H}_{\text{fluc}}(s_2)], \mathcal{H}_{\text{fluc}}(s_3)] \rangle_{s_3}. \quad (3.47)$$

Using the fact that odd moments of $a^\#$ w.r.t. $\langle \cdot \rangle_0$ vanish and rearranging the terms in the first three lines of (3.44), yields

$$\begin{aligned}
& \int_0^t ds \langle [f[J], \mathcal{H}_{\text{BEC}}(s) + \mathcal{H}_{\text{cub}}(s)] \rangle_0 - \int dp J(p) \frac{1}{N} \int_0^t ds Q_3[f_0](s, p) \\
& + \int_{[0,t]^2} d\mathbf{s}_2 \langle [[f[J], \mathcal{H}_{\text{BEC}}(s_1) + \mathcal{H}_{\text{cub}}(s_1)], \mathcal{H}_{\text{BEC}}(s_2) + \mathcal{H}_{\text{cub}}(s_2)] \rangle_0 \\
& + \int_0^t ds \langle [f[J], \mathcal{H}_{\text{HFB}}^{(\text{cor})}(s) + \mathcal{H}_{\text{quart}}^{(\text{cor})}(s)] \rangle_0 \\
& + \int_{[0,t]^2} d\mathbf{s}_2 \langle [[f[J], \mathcal{H}_{\text{HFB}}^{(\text{cor})}(s_1) + \mathcal{H}_{\text{quart}}^{(\text{cor})}(s_1)], \mathcal{H}_{\text{HFB}}(s_2) + \mathcal{H}_{\text{quart}}(s_2)] \rangle_0 \\
& + \int_0^t ds \langle [f[J], \mathcal{H}_{\text{HFB}}^{(\text{d})}(s) + \mathcal{H}_{\text{quart}}^{(\text{d})}(s)] \rangle_0 \\
& + \int_{[0,t]^2} d\mathbf{s}_2 \langle [[f[J], \mathcal{H}_{\text{HFB}}^{(\text{d})}(s_1) + \mathcal{H}_{\text{quart}}^{(\text{d})}(s_1)], \mathcal{H}_{\text{HFB}}(s_2) + \mathcal{H}_{\text{quart}}(s_2)] \rangle_0 \\
& + \int_{[0,t]^2} d\mathbf{s}_2 \langle [[f[J], \mathcal{H}_{\text{quart}}(s_1) - \mathcal{H}_{\text{quart}}^{(\text{d})}(s_1)], \mathcal{H}_{\text{quart}}(s_2)] \rangle_0 \\
& - \int dp J(p) \frac{1}{N^2} \int_0^t ds Q_4[f_0](s, p).
\end{aligned} \tag{3.48}$$

Due to our choice of HFB fields $(\phi^{(2)}, \gamma^{(2)}, \sigma^{(2)})$ satisfying (3.41) and the Bogoliubov dispersion $\Omega^{(2)}$ satisfying (3.43), we have that (3.48) vanishes.

- (1) In order to estimate the terms coming from the tail (3.47), we compute \mathcal{H}_{BEC} , $\mathcal{H}_{\text{HFB}}^{(\text{cor})}$, and $\mathcal{H}_{\text{HFB}}^{(\text{d})}$ for the HFB fields $(\phi^{(2)}, \gamma^{(2)}, \sigma^{(2)}, \Omega^{(2)})$ in Lemma E.3. Then we use the following ideas:
 - (a) $\int dp a_p^\# \lesssim \mathcal{N}_b^{\frac{1}{2}}$ and that $a_p^\# \lesssim (|\Lambda| \mathcal{N}_b)^{\frac{1}{2}}$, see Lemmata D.2, D.3, and D.4.
 - (b) Proposition 5.1, followed by Lemma D.1 implies

$$\begin{aligned}
\langle (\mathcal{N}_b + |\Lambda|)^{\frac{\ell}{2}} \rangle_t & \lesssim_{\|f_0\|_d, \|\gamma_0\|_d, \ell} e^{K_\ell, \|f_0\|_d, \|\gamma_0\|_d \|\hat{v}\|_{w,d} |\Lambda| \lambda t} \\
\langle (\mathcal{N}_b + |\Lambda|)^{\frac{\ell}{2}} \left(1 + \frac{\mathcal{N}_b}{N|\Lambda|} \right) \rangle_0 & \\
\lesssim_{\|f_0\|_d, \|\gamma_0\|_d, |\Lambda|, \ell} e^{K_\ell, \|f_0\|_d, \|\gamma_0\|_d \|\hat{v}\|_{w,d} |\Lambda| \lambda t} |\Lambda|^{\frac{\ell}{2}}.
\end{aligned}$$

With these steps, we obtain

$$\begin{aligned}
& \left| \int_{[0,t]^3} d\mathbf{s}_3 \langle [[[f[J], \mathcal{H}_{\text{fluc}}(s_1)], \mathcal{H}_{\text{fluc}}(s_2)], \mathcal{H}_{\text{fluc}}(s_3)] \rangle_{s_3} \right| \\
& \lesssim_{\|f_0\|_d, \|\gamma_0\|_d, \|\hat{v}\|_{w,d}, |\Lambda|} e^{K_{\|f_0\|_d, \|\gamma_0\|_d} \|\hat{v}\|_{w,d} |\Lambda| \lambda t} \frac{1}{N^{\frac{3}{2}}} \|J\|_\infty
\end{aligned}$$

- (2) Due to the dependence of $\frac{1}{N^2} \int_0^t ds Q_4[f_0](s, p)$ on the HFB fields $(\phi_t, \gamma_t, \sigma_t)$, we will employ a priori estimates established in Corollary 4.6. With that, we can

control (3.45) by

$$\frac{1}{N^2} \left| \int dp J(p) \int_0^t ds Q_4[f_0](s, p) \right| \lesssim_{\|f_0\|_d, \|\hat{v}\|_{w,d}} \frac{\lambda^2 t^2}{N^2} e^{C_{\|f_0\|_d, \|\gamma_0\|_d} \lambda \|\hat{v}\|_{w,d} t} \|J\|_\infty.$$

- (3) In order to estimate (3.46), we write $Q_3[h] =: Q_3[h, h, h]$, in order to emphasize the dependence on three arguments, each evaluated at different momenta. Then we have that

$$\begin{aligned} Q_3[f, f, f] - Q_3[f_0, f_0, f_0] &= (Q_3[f, f, f] - Q_3[f_0, f, f]) \\ &\quad + (Q_3[f_0, f, f] - Q_3[f_0, f_0, f]) \\ &\quad + (Q_3[f_0, f_0, f] - Q_3[f_0, f_0, f_0]). \end{aligned}$$

Each of the differences contains a factor $f - f_0 \propto \frac{1}{N}$. Arguing as in [37, Chapter 5.3], we then obtain

$$\frac{1}{N} \left| \int_0^t ds (Q_3[f](s)[J] - Q_3[f_0](s)[J]) \right| \lesssim_{\|f_0\|_d, \|\gamma_0\|_d, \|\hat{v}\|_{w,d}, |\Lambda|} e^{K_{\|f_0\|_d, \|\gamma_0\|_d} \|\hat{v}\|_{w,d} |\Lambda| \lambda t} \frac{\|J\|_\infty}{N^{\frac{3}{2}}}.$$

With these estimates, we obtain

$$|\text{Rem}[f](t)[J]| \lesssim_{\|f_0\|_d, \|\gamma_0\|_d, \|\hat{v}\|_{w,d}, |\Lambda|} e^{K_{\|f_0\|_d, \|\gamma_0\|_d} \|\hat{v}\|_{w,d} |\Lambda| \lambda t} \frac{\|J\|_\infty}{N^{\frac{3}{2}}}.$$

3.5. Evolution of g . With analogous calculations as for f , one can show that the leading order term in the evolution of g is given by

$$-\frac{1}{|\Lambda|} \int dp J(p) \int_{[0,t]^2} d\mathbf{s}_2 \mathbb{1}_{s_1 \geq s_2} \langle [[a_p a_{-p}, \overline{\mathcal{H}_{\text{cub}}(s_1)}], \overline{\mathcal{H}_{\text{cub}}(s_2)}] \rangle_0.$$

We show in Lemma C.2 that

$$\begin{aligned} &-\frac{1}{|\Lambda|} \int dp J(p) \int_{[0,t]^2} d\mathbf{s}_2 \mathbb{1}_{s_1 \geq s_2} \langle [[a_p a_{-p}, \overline{\mathcal{H}_{\text{cub}}(s_1)}], \overline{\mathcal{H}_{\text{cub}}(s_2)}] \rangle_0 \\ &= \frac{1}{N} \int_0^t ds Q_3^{(g)}[f_0](s)[J], \end{aligned}$$

where $Q_3^{(g)}$ is given in (2.24).

With analogous calculations as in the case of f , and using the fact that $g_0 = 0$, we obtain that

$$\int dp J(p) g_t(p) = \frac{1}{N} \int_0^t ds Q_3^{(g)}[f](s)[J] + \text{Rem}[g](t)[J],$$

where

$$|\text{Rem}[g](t)[J]| \lesssim_{\|f_0\|_d, \|\gamma_0\|_d, \|\hat{v}\|_{w,d}, |\Lambda|} e^{K_{\|f_0\|_d, \|\gamma_0\|_d} \|\hat{v}\|_{w,d} |\Lambda| \lambda t} \frac{\|J\|_d}{N^{\frac{3}{2}}}.$$

3.6. Evolution of Φ . Recalling (2.25), we show in Lemma C.2 that

$$\frac{1}{N^{\frac{3}{2}}} \int_0^t ds Q_3^{(\Phi)}[f_0](s) = -\frac{1}{|\Lambda|} \int_{[0,t]^2} d\mathbf{s}_2 \mathbb{1}_{s_1 \geq s_2} \langle [[a_0, \overbrace{\mathcal{H}_{\text{quart}}(s_1)}], \overbrace{\mathcal{H}_{\text{cub}}(s_2)}] \rangle_0$$

We need to collect all terms involving a factor $N^{-\frac{3}{2}}$. Consequently, we need to consider the third-order Duhamel expansion of Φ .

(1) Due to $[a_0, \mathcal{H}_{\text{BEC}}(s_1)]$ being a scalar, we have that

$$\langle [[a_0, \mathcal{H}_{\text{BEC}}(s_1)], \mathcal{H}_{\text{fluc}}(s_2)], \mathcal{H}_{\text{fluc}}(s_3) \rangle = 0.$$

(2)

$$\begin{aligned} \langle [[a_0, \mathcal{H}_{\text{HFB}}(s_1)], \mathcal{H}_{\text{fluc}}(s_2)], \mathcal{H}_{\text{fluc}}(s_3) \rangle_0 &= \\ \frac{\lambda}{N} \langle [[a_{2,1}(s_1)a_0 + a_{2,2}a_0^\dagger], \mathcal{H}_{\text{fluc}}(s_2)], \mathcal{H}_{\text{fluc}}(s_3) \rangle_0 &\propto \frac{\lambda^3}{N^2}, \end{aligned}$$

for $\mathcal{H}_{\text{fluc}}(t) \sim \frac{\lambda}{\sqrt{N}}$, see Proposition 5.2.

(3) Similarly to the previous step, we have that

$$\langle [[a_0, \mathcal{H}_{\text{quart}}(s_1)], \mathcal{H}_{\text{fluc}}(s_2)], \mathcal{H}_{\text{fluc}}(s_3) \rangle_0 \propto \frac{\lambda^3}{N^2}.$$

(4) We are left with computing

$$\begin{aligned} \langle [[a_0, \mathcal{H}_{\text{cub}}(s_1)], \mathcal{H}_{\text{fluc}}(s_2)], \mathcal{H}_{\text{fluc}}(s_3) \rangle_0 &= \\ \frac{\lambda}{\sqrt{N}} \langle [[f[a_{3,1}(s_1)] + g[a_{3,2}(s_1)] + g^\dagger[a_{3,3}(s_1)]], \mathcal{H}_{\text{fluc}}(s_2)], \mathcal{H}_{\text{fluc}}(s_3) \rangle_0, \end{aligned}$$

where we abbreviated

$$g[J] := \int dp J(p) a_p a_{-p}, \quad g^\dagger[J] := \int dp J(p) a_p^\dagger a_{-p}^\dagger.$$

Employing Lemma A.2, we find that

$$\begin{aligned} a_{3,1}(t, p) &= e^{i \int_0^t d\tau \Omega_\tau(0)} \mathbf{w}_t^{(2,1)}(0, p, p), \\ a_{3,2}(t, p) &= \frac{1}{2} e^{i \int_0^t d\tau \Omega_\tau(0)} \overline{\mathbf{w}_t^{(2,1)}}(p, p, 0), \\ a_{3,3}(t, p) &= \frac{1}{2} e^{i \int_0^t d\tau \Omega_\tau(0)} \mathbf{w}_t^{(3,0)}(p, p, 0). \end{aligned}$$

We recognize the evolutions of f and g , resulting in

$$\begin{aligned} \int_{[0,t]^3} d\mathbf{s}_3 \mathbb{1}_{s_1 \geq s_2 \geq s_3} \langle [[a_0, \mathcal{H}_{\text{cub}}(s_1)], \mathcal{H}_{\text{fluc}}(s_2)], \mathcal{H}_{\text{fluc}}(s_3) \rangle_0 &= \\ \frac{\lambda}{N^{\frac{3}{2}}} \int_0^t ds e^{i \int_0^s d\tau \Omega_\tau(0)} \int dp &\left(\mathbf{w}_s^{(2,1)}(0, p, p) Q_3[f_0](s, p) \right. \\ &+ \left. \overline{\mathbf{w}_s^{(2,1)}}(p, p, 0) Q_3^{(g)}[f_0](s, p) + \mathbf{w}_s^{(3,0)}(p, p, 0) \overline{Q_3^{(g)}[f_0](s, p)} \right). \end{aligned}$$

With analogous steps as above and recalling (2.26), we then obtain that

$$\Phi_t = \frac{1}{N^{\frac{3}{2}}} \int_0^t ds (Q_3^{(\Phi)}[f](s) + Q_{3,3}^{(\Phi)}[f](s)) + \text{Rem}[\Phi](t),$$

where

$$|\text{Rem}[\Phi](t)| \lesssim_{\|f_0\|_d, \|\gamma_0\|_d, \|\hat{v}\|_{w,d}, |\Lambda|} e^{K\|f_0\|_d \cdot \|\gamma_0\|_d \|\hat{v}\|_{w,d} |\Lambda| \lambda t} \frac{1}{N^2}.$$

4. ESTIMATES ON THE HFB EVOLUTION

In order to proceed as in [37], we need to determine a priori bounds for u, v , and ϕ . Observe that it suffices to establish bounds for the HFB fields ϕ and γ , for $|\sigma|^2 = (1+\gamma)\gamma$.

We start by rewriting the renormalized HFB equations (3.17) and (3.41). In both cases, we get a closed system for the total fields $\phi^{(j)}$ and

$$\begin{pmatrix} \Sigma^{(j)} \\ \Gamma^{(j)} \end{pmatrix} = ((1 + \delta_{j,2} f_0^{(+)}) \begin{pmatrix} \sigma^{(j)} \\ \gamma^{(j)} \end{pmatrix} + \delta_{j,2} f_0^{(+)} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + N|\Lambda|\delta \begin{pmatrix} (\phi^{(j)})^2 \\ |\phi_t^{(j)}|^2 \end{pmatrix},$$

without having to explicitly refer to the reduced fields $(\phi^{(j)}, \sigma^{(j)}, \gamma^{(j)})$. Then the HFB equations (3.17) and (3.41) can be rewritten as

$$\begin{cases} i\partial_t \phi_t &= \frac{\lambda}{N} \left((\Gamma_t * (\hat{v} + \hat{v}(0)))(0) \phi_t + (\Sigma_t * \hat{v})(0) \bar{\phi}_t \right) - 2\lambda |\Lambda| \hat{v}(0) |\phi_t|^2 \phi_t, \\ \partial_t \Gamma_t &= \frac{2\lambda}{N} \text{Im}((\Sigma_t * \hat{v}) \bar{\Sigma}_t), \\ i\partial_t \Sigma_t &= 2(E + \frac{\lambda}{N} \Gamma_t * (\hat{v} + \hat{v}(0))) \Sigma_t + \frac{\lambda}{N} (\Sigma_t * \hat{v})(1 + 2\Gamma_t) \\ &\quad - 4N\lambda |\Lambda|^2 \hat{v}(0) |\phi_t|^2 \phi_t^2 \delta. \end{cases} \quad (4.1)$$

4.1. Well-posedness theory for the HFB system. Before establishing bounds on the HFB evolution, we need to ensure its well-posedness.

Lemma 4.1 (Local well-posedness). *Let $(\phi_0, \Gamma_0, \Sigma_0) \in \mathcal{X}^1$. Assume that $\hat{v} \in L^1_{\sqrt{1+E}} \cap L^\infty(\Lambda^*)$. Then there exists a maximal existence time $0 < T \leq \infty$ and a unique maximal mild solution (ϕ, Γ, Σ) of (4.1) on $[0, T)$. If $T < \infty$, then $\lim_{t \rightarrow T^-} \|(\phi_t, \Gamma_t, \Sigma_t)\|_{\mathcal{X}^1} = \infty$. The solution depends continuously on the initial datum.*

Proof. We start by abbreviating the nonlinearity

$$\begin{aligned} \mathcal{J}_1(\phi, \Gamma, \Sigma) := & -i \left[\frac{\lambda}{N} \left((\Gamma * (\hat{v} + \hat{v}(0)))(0) \phi + (\Sigma * \hat{v})(0) \bar{\phi} \right) \right. \\ & \left. - 2\lambda |\Lambda| \hat{v}(0) |\phi|^2 \phi \right] \end{aligned} \quad (4.2)$$

$$\begin{aligned} \mathcal{J}_2(\phi, \Gamma, \Sigma) := & -\frac{2\lambda}{N} \text{Im} \left((\bar{\Sigma} * \hat{v}) \Sigma \right), \\ \mathcal{J}_3(\phi, \Gamma, \Sigma) := & -i \left[2 \left(\frac{\lambda}{N} \Gamma * (\hat{v} + \hat{v}(0)) \right) \Sigma + \frac{\lambda}{N} (\Sigma * \hat{v})(1 + 2\Gamma) \right. \\ & \left. - 4N\lambda |\Lambda|^2 \hat{v}(0) |\phi|^2 \phi^2 \delta \right]. \end{aligned} \quad (4.3)$$

Let $\vec{\mathcal{J}} := (\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3)$, and $\text{diag}(a, b, c)$ denote the diagonal 3×3 matrix with entries a, b, c , and define

$$\begin{aligned} \mathcal{L}(\tilde{\phi}, \tilde{\Gamma}, \tilde{\Sigma}) := & \\ & (\phi_0, \Gamma_0, e^{-2iEt} \Sigma_0) + \int_0^t ds \vec{\mathcal{J}}(\tilde{\phi}_s, \tilde{\Gamma}_s, \tilde{\Sigma}_s) \text{diag}(1, 1, e^{-2iE(t-s)}). \end{aligned}$$

A standard calculation, explained in Lemma B.2, yields

$$\begin{aligned} & \|\vec{\mathcal{J}}(\phi_1, \Gamma_1, \Sigma_1) - \vec{\mathcal{J}}(\phi_2, \Gamma_2, \Sigma_2)\|_{\mathcal{X}^1} \\ & \leq C \left(\frac{\lambda}{N} (\|\hat{v}\|_{L^1_{\sqrt{1+E}}} + \|\hat{v}\|_\infty) (\|(\phi_1, \Gamma_1, \Sigma_1)\|_{\mathcal{X}^1} + \|(\phi_2, \Gamma_2, \Sigma_2)\|_{\mathcal{X}^1}) \right. \\ & \quad \|(\phi_1 - \phi_2, \Gamma_1 - \Gamma_2, \Sigma_1 - \Sigma_2)\|_{\mathcal{X}^1} + \lambda |\Lambda| \hat{v}(0) (|\phi_1|^2 + |\phi_2|^2) \\ & \quad \left. (1 + N |\Lambda|^{\frac{3}{2}} (|\phi_1| + |\phi_2|)) |\phi_1 - \phi_2| \right). \end{aligned}$$

With the previous estimates, we can apply the standard contraction mapping arguments to \mathcal{L} to show the existence of a unique solution $(\phi, \Gamma, \Sigma) \in C_t^0([0, T), \mathcal{X}^1) \cap C_t^1([0, T), \mathcal{X}^{-1})$ as long as $T > 0$ is small enough. In addition, in the usual way, one can show a blow-up alternative, i.e., if the maximal time $T > 0$ of existence is finite, we have that $\lim_{t \rightarrow T^-} \|(\phi_t, \Gamma_t, \Sigma_t)\|_{\mathcal{X}^1} = \infty$. With standard arguments, we also show the continuous dependence on initial data. \square

In order to obtain an expression for the HFB energy functional, we look at the phase factor

$$\begin{aligned} \partial_t S_t = & \\ & |\Lambda| \int dp \left[E(p) \gamma_t(p) + \frac{\lambda}{2N} \Gamma_t^{(1)} * (\hat{v} + \hat{v}(0))(p) \Gamma_t^{(1)}(p) \right. \\ & + \frac{\lambda}{2N} \bar{\Sigma}_t^{(1)} * \hat{v}(p) \Sigma_t^{(1)}(p) - N |\Lambda|^2 \lambda |\phi_t|^4 \hat{v}(p) \delta(p) \\ & \left. - \frac{N |\Lambda| \delta(p) \operatorname{Re}(\bar{\phi}_t i \partial_t \phi_t)}{2} - \frac{\operatorname{Re}(\bar{\sigma}_t(p) i \partial_t \sigma_t(p))}{2(1 + \gamma_t(p))} \right]. \end{aligned} \quad (4.4)$$

obtained in Lemma A.2. Observe that the terms not involving any time derivatives correspond to the sum over all complete contractions of \mathcal{H}_N with expectations

$$\begin{aligned} \langle a_p \rangle &= \phi \delta(p), \\ \langle a_p^\dagger a_p \rangle &= |\Lambda| \Gamma^{(1)}(p), \\ \langle a_p a_{-p} \rangle &= |\Lambda| \Sigma^{(1)}(p), \end{aligned}$$

see, e.g., [8]. Moreover, we can replace $E(p) \gamma_t(p)$ in (4.4) by $E(p) \Gamma_t^{(1)}(p)$. In particular, (4.4) gives rise to the *HFB energy (density) functional*

$$\begin{aligned} \mathcal{E}_{HFB}(\phi, \Gamma, \Sigma) := & \int dp \left(E(p) \Gamma(p) + \frac{\lambda}{2N} \Gamma * (\hat{v} + \hat{v}(0))(p) \Gamma(p) \right. \\ & \left. + \frac{\lambda}{2N} \bar{\Sigma} * \hat{v}(p) \Sigma(p) - N |\Lambda|^2 \lambda |\phi|^4 \hat{v}(p) \delta(p) \right). \end{aligned}$$

In addition, we define the (*total*) *HFB density*

$$\|\Gamma\|_1 := \int dp \Gamma(p).$$

Lemma 4.2 (Conservation laws). *Assume that $\hat{v} \in L^1_{\sqrt{1+E}} \cap L^\infty(\Lambda^*)$. Let $(\phi_0, \Gamma_0, \Sigma_0) \in \mathcal{X}^1$, and $(\phi, \Gamma, \Sigma) \in C_t^0([0, T), \mathcal{X}^1) \cap C_t^1([0, T), \mathcal{X}^{-1})$ denote the maximal mild solution of*

(4.1). Then the HFB density $\|\Gamma_t\|_1$ and HFB energy $\mathcal{E}_{HFB}(\phi_t, \Gamma_t, \Sigma_t)$ are differentiable in time and conserved.

Since the arguments needed for this result are standard, we omit the details at this point. For the interested reader, we elaborate on these details after the proof of Lemma B.2.

In order to close this section on the well-posedness of the HFB equations, recall that

$$\gamma_t = |v_t|^2 \geq 0, \quad |\sigma_t|^2 = u_t^2 |v_t|^2 = (1 + \gamma_t) \gamma_t.$$

Then we have that

$$\Gamma \geq \gamma \geq 0,$$

and

$$\begin{aligned} |\Sigma|^2 &= |\sigma + N|\Lambda||\phi|^2\delta|^2 \\ &\leq 2(|\sigma|^2 + (N|\Lambda||\phi|^2\delta)^2) \\ &= 2((\gamma + 1)\gamma + (N|\Lambda||\phi|^2\delta)^2) \\ &\leq 2(\Gamma + 1)\Gamma, \end{aligned}$$

where we applied Cauchy-Schwarz in the second line. As a consequence, Young's inequality implies

$$\begin{aligned} |\Sigma| &\leq \sqrt{2(\Gamma + 1)\Gamma} \\ &\leq \sqrt{2}(\Gamma + 1). \end{aligned} \tag{4.5}$$

For the next statement, we recall from (2.17) the truncated fields

$$\Gamma^T = \Gamma - N|\Lambda||\phi|^2\delta, \quad \Sigma^T = \Sigma - N|\Lambda||\phi|^2\delta.$$

Lemma 4.3 (Conditional global well-posedness). *Assume that $\hat{v} \in L^1_{\sqrt{1+E}} \cap L^\infty(\Lambda^*)$, and that $v \geq 0$. Let $(\phi_0, \Gamma_0, \Sigma_0) \in \mathcal{X}^1$. Let (ϕ, Γ, Σ) be the associated unique maximal mild solution of (4.1) with existence time $T_0 > 0$, and assume that the truncated expectations satisfy $\Gamma^T \geq 0$, $|\Sigma^T|^2 \leq (\Gamma^T + 1)\Gamma^T$. Then $T_0 = \infty$.*

Proof. Mass-conservation and $\Gamma^T \geq 0$ imply that

$$\begin{aligned} N|\Lambda||\phi_t|^2 &\leq \|\Gamma_t\|_1 \\ &= \|\Gamma_0\|_1, \end{aligned}$$

which is why

$$|\phi_t| \leq C_{\|\Gamma_0\|_1}.$$

Next,

$$\langle \Sigma_s * \hat{v}, \Sigma_s \rangle = \int dx |\check{\Sigma}_s(x)|^2 v(x),$$

see (B.12), and positivity of v yield

$$\mathcal{E}_{HFB}(\phi_t, \Sigma_t, \Gamma_t) \geq \int dp \left(E(p) \Gamma_t(p) + \frac{\lambda}{2N} \Gamma_t^T * (\hat{v} + \hat{v}(0))(p) \Gamma_t^T(p) \right)$$

$$\begin{aligned}
& + \frac{\lambda}{2N} \bar{\Sigma}_t * \hat{v}(p) \Sigma_t(p) \\
& \geq \int dp E(p) \Gamma_t(p).
\end{aligned} \tag{4.6}$$

Because of energy and mass conservation, we thus have that

$$\|\Gamma_t\|_{L^1_{1+E}} \leq C_{\|\Gamma_0\|_1, \mathcal{E}_{HFB}(\phi_0, \Gamma_0, \Sigma_0)}. \tag{4.7}$$

Using $\Gamma \geq 0$ and (4.5), we find that

$$\partial_t \Gamma_t(p) \leq \frac{4\lambda}{N} \|(\Gamma_t(p) + 1) * \hat{v}\|_\infty (\Gamma_t(p) + 1).$$

In particular, mass conservation implies

$$\partial_t \Gamma_t(p) \leq \frac{4\lambda}{N} (\|\Gamma_0\|_1 \|\hat{v}\|_\infty + \|\hat{v}\|_1) (\Gamma_t(p) + 1)$$

Gronwall's inequality then yields the point-wise bound

$$\Gamma_t(p) \leq e^{\frac{4\lambda}{N} (\|\Gamma_0\|_1 \|\hat{v}\|_\infty + \|\hat{v}\|_1)t} (\Gamma_t(p) + 1),$$

which, in turn, gives the uniform bound

$$\|\Gamma_t\|_\infty \leq e^{\frac{4\lambda}{N} (\|\Gamma_0\|_1 \|\hat{v}\|_\infty + \|\hat{v}\|_1)t} (\|\Gamma_0\|_\infty + 1). \tag{4.8}$$

Finally, employing (2.13), we find that

$$\|\Sigma_t\|_{L^2_{1+E}}^2 \leq 2 \int dp (\Gamma_t(p) + 1) \Gamma_t(p) (1 + E(p)).$$

Collecting (4.7) and (4.8), we thus obtain

$$\|\Sigma_t\|_{L^2_{1+E}}^2 \leq C_{\|\Gamma_0\|_1, \mathcal{E}_{HFB}(\phi_0, \Gamma_0, \Sigma_0), \|\Gamma_0\|_\infty} e^{\frac{4\lambda}{N} (\|\Gamma_0\|_1 \|\hat{v}\|_\infty + \|\hat{v}\|_1)t}.$$

(4.5) and (4.8) imply

$$\|\Sigma_t\|_\infty \leq 2e^{\frac{4\lambda}{N} (\|\Gamma_0\|_1 \|\hat{v}\|_\infty + \|\hat{v}\|_1)t} (\|\Gamma_0\|_\infty + 1).$$

In particular, we have that $\|(\phi_t, \Gamma_t, \Sigma_t)\|_{\mathcal{X}^1} < \infty$ for all $t \geq 0$, which, by the blow-up alternative in Lemma 4.1, implies $T_0 = \infty$. \square

4.2. Symplectic description of HFB equations. In this section, we follow closely the ideas in [8]. Let

$$\mathcal{R}_t := \begin{pmatrix} \Gamma_t^T & \Sigma_t^T \\ \bar{\Sigma}_t^T & 1 + \Gamma_t^T \end{pmatrix},$$

where we recall the definitions (2.17) of Γ^T and Σ^T . Observe that we have that

$$\mathcal{R} \geq 0 \iff \Gamma^T \geq 0 \wedge |\Sigma^T|^2 \leq (\Gamma^T + 1)\Gamma^T. \tag{4.9}$$

The goal of this section is to prove that $\mathcal{R}_0 \geq 0$ implies $\mathcal{R}_t \geq 0$ along the flow induced by the HFB equations (4.1). More precisely, let

$$\begin{aligned}
h_\Gamma &:= E + \frac{\lambda}{N} \Gamma * (\hat{v} + \hat{v}(0)), \\
h_\Sigma &:= \frac{\lambda}{N} \Sigma * \hat{v},
\end{aligned}$$

$$\begin{aligned}\mathcal{H}^{(\Gamma, \Sigma)} &:= \begin{pmatrix} h_\Gamma & h_\Sigma \\ h_\Sigma & h_\Gamma \end{pmatrix}, \\ \mathcal{S} &:= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.\end{aligned}$$

A straightforward calculation yields

$$i\partial_t \mathcal{R}_t = \mathcal{S} \mathcal{H}^{(\Gamma_t, \Sigma_t)} \mathcal{R}_t - \mathcal{R}_t \mathcal{H}^{(\Gamma_t, \Sigma_t)} \mathcal{S}.$$

Let

$$\begin{cases} i\partial_t \mathcal{V}_t^\dagger &= \mathcal{H}^{(\Gamma_t, \Sigma_t)} \mathcal{S} \mathcal{V}_t^\dagger, \\ \mathcal{V}_0^\dagger &= \mathbf{1}. \end{cases} \quad (4.10)$$

We introduce the function space

$$\mathcal{Y}^j := \left\{ \begin{pmatrix} a & b \\ \bar{b} & a \end{pmatrix} \mid a \in L_{(1+E)^j}^1(\Lambda^*), b \in L_{(1+E)^j}^2(\Lambda^*) \right\}.$$

Lemma 4.4 (Well-posedness of (4.10)). *Assume that $\hat{v} \in L_{\sqrt{1+E}}^1 \cap L^\infty(\Lambda^*)$. Let $(\phi, \Gamma, \Sigma) \in C_t^0([0, T), \mathcal{X}^1)$. Then there is a unique solution $\mathcal{V}^\dagger \in C_t^1([0, T), \mathcal{Y}^1)$ of (4.10).*

Proof. We have that $E\mathcal{S}$ is the generator of a continuous semi-group on \mathcal{Y}^1 . In addition, $(\phi, \Gamma, \Sigma) \in C_t^0([0, T), \mathcal{X}^1)$ and the estimates in the proof of Lemma B.2 imply the continuity of $t \mapsto \mathcal{H}^{(\Gamma_t, \Sigma_t)} - E\mathcal{S} \in \mathcal{Y}^1$. Finally, we apply standard functional results, e.g., as found in [64], in order to show the well-posedness and regularity of \mathcal{V}_t^\dagger . \square

Now let $(\phi, \Gamma, \Sigma) \in C_t^0([0, T), \mathcal{X}^1) \cap C_t^1([0, T), \mathcal{X}^{-1})$ be the solution of (4.1). Using Lemma 4.4, we have that

$$\begin{aligned}i\partial_t (\mathcal{V}_t \mathcal{R}_t \mathcal{V}_t^\dagger) &= -\mathcal{V}_t \mathcal{S} \mathcal{H}^{(\Gamma_t, \Sigma_t)} \mathcal{R}_t \mathcal{V}_t^\dagger + \mathcal{V}_t \mathcal{R}_t \mathcal{H}^{(\Gamma_t, \Sigma_t)} \mathcal{S} \mathcal{V}_t^\dagger \\ &\quad + \mathcal{V}_t \mathcal{S} \mathcal{H}^{(\Gamma_t, \Sigma_t)} \mathcal{R}_t \mathcal{V}_t^\dagger - \mathcal{V}_t \mathcal{R}_t \mathcal{H}^{(\Gamma_t, \Sigma_t)} \mathcal{S} \mathcal{V}_t^\dagger \\ &= 0.\end{aligned}$$

In particular, provided $\mathcal{R}_0 \geq 0$, we have that

$$\mathcal{R}_t = \mathcal{V}_t^\dagger \mathcal{R}_0 \mathcal{V}_t \geq 0.$$

Together with (4.9), we have thus proved Proposition 2.6.

4.3. Bounds on the HFB fields. After establishing global well-posedness of (4.1), as well as energy and mass conservation, we are ready to establish bounds on Γ^T and ϕ . As described at the beginning of section, these, in turn, will yield a bound on γ , and thus on u and v . For that, we establish the next result. Recall from (2.17) that $\Gamma^T = \Gamma - N|\Lambda||\phi|^2\delta$, $\Sigma^T = \Sigma - N|\Lambda|\phi^2\delta$.

Lemma 4.5 (Mass transfer bound). *Assume that $\hat{v} \in L_{\sqrt{1+E}}^1 \cap L^\infty(\Lambda^*)$, and that $v \geq 0$. Let $(\phi_0, \Gamma_0, \Sigma_0) \in \mathcal{X}^1$ with $\Gamma_0^T \geq 0$, $|\Sigma_0^T|^2 \leq \sqrt{(\Gamma_0^T + 1)\Gamma_0^T}$, and $(\phi, \Gamma, \Sigma) \in C_t^0([0, T), \mathcal{X}^1) \cap C_t^1([0, T), \mathcal{X}^{-1})$ denote the global solution of (4.1). Then we have that*

$$|\phi_t|^2 = |\phi_0|^2 + \frac{\|\Gamma_0^T\|_1 - \|\Gamma_t^T\|_1}{N|\Lambda|},$$

$$\begin{aligned}\Gamma_t^T &\leq e^{2\lambda\|\hat{v}\|_{w,d}(\frac{\|\Gamma_0^T\|_1}{N}+2)t}(\Gamma_0^T + 1), \\ \|\Gamma_t^T\|_1 &\leq e^{2\lambda\|\hat{v}\|_{w,d}(\frac{\|\Gamma_0^T\|_1}{N}+1)t}(\|\Gamma_0^T\|_1 + 1), \\ \int dp E(p)\Gamma_t(p) &\leq \mathcal{E}_{HFB}(\phi_0, \Gamma_0, \Sigma_0),\end{aligned}$$

and $|\Sigma_t^T| \leq \sqrt{(\Gamma_t^T + 1)\Gamma_t^T}$.

Proof. Observe that we have $\|\Gamma\|_1 = \int dp \Gamma(p)$ for $\Gamma \geq 0$. Mass conservation implies

$$|\phi_t|^2 = \frac{\|\Gamma_0\|_1 - \|\Gamma_t^T\|_1}{N|\Lambda|} = |\phi_0|^2 + \frac{\|\Gamma_0^T\|_1 - \|\Gamma_t^T\|_1}{N|\Lambda|}. \quad (4.11)$$

Notice that Lemma 2.6 yields

$$|\Sigma_t^T| \leq \sqrt{(\Gamma_t^T + 1)\Gamma_t^T} \leq \Gamma_t^T + 1, \quad (4.12)$$

due to Young's inequality. (4.1) implies

$$\partial_t \Gamma_t^T = \frac{2\lambda}{N} \operatorname{Im}((\Sigma_t * \hat{v}) \Sigma_t^T). \quad (4.13)$$

Using (4.12), Young's inequality yields

$$\begin{aligned}\partial_t \Gamma_t^T &\leq \frac{2\lambda}{N} (\|\hat{v}\|_1 + \|\hat{v}\|_\infty) (\|\Gamma_t\|_1 + 1) (\Gamma_t^T + 1) \\ &= \frac{2\lambda}{N} \|\hat{v}\|_{w,d} (\|\Gamma_0\|_1 + 1) (\Gamma_t^T + 1),\end{aligned}$$

where we employed mass conservation. We thus obtain, via Gronwall's inequality and using the fact $\|\Gamma_0\|_1 = \|\Gamma_0^T\|_1 + N$, that

$$\Gamma_t^T \leq e^{2\lambda\|\hat{v}\|_{w,d}(\frac{\|\Gamma_0^T\|_1}{N}+2)t}(\Gamma_0^T + 1).$$

Similarly, (4.13), together with (4.12) and (4.11), yields

$$\begin{aligned}\partial_t \|\Gamma_t^T\|_1 &= 2\lambda|\Lambda| \operatorname{Im} \left(\int dp (\Sigma_t * \hat{v})(p) \bar{\Sigma}_t^T(p) \right) \\ &= 2\lambda|\Lambda| \operatorname{Im} ((\bar{\Sigma}_t^T * \hat{v})(0) \phi_t^2) \\ &\leq 2\lambda \|\hat{v}\|_{w,d} \frac{\|\Gamma_0\|_1 - \|\Gamma_t^T\|_1}{N} (\|\Gamma_t^T\|_1 + 1),\end{aligned}$$

where, analogously to (B.12), we used that $\int dp (\Sigma_t^T * \hat{v})(p) \bar{\Sigma}_t^T(p) \in \mathbb{R}$. Then Gronwall's inequality implies

$$\|\Gamma_t^T\|_1 \leq e^{2\lambda\|\hat{v}\|_{w,d}(\frac{\|\Gamma_0^T\|_1}{N}+1)t}(\|\Gamma_0^T\|_1 + 1).$$

(4.6) together with energy conservation implies

$$\int dp E(p)\Gamma_t(p) \leq \mathcal{E}_{HFB}(\phi_t, \Gamma_t, \Sigma_t) = \mathcal{E}_{HFB}(\phi_0, \Gamma_0, \Sigma_0).$$

This concludes the proof. \square

As a direct consequence, we obtain the following result.

Corollary 4.6 (Bounds on Bogoliubov coefficients). *Assume that $\hat{v} \in L^1_{\sqrt{1+E}} \cap L^\infty(\Lambda^*)$. Let $(\phi_0, \Gamma_0, \Sigma_0) \in \mathcal{X}^1$ with $\Gamma_0^T \geq 0$, $|\Sigma_0^T|^2 \leq \sqrt{(\Gamma_0^T + 1)\Gamma_0^T}$, and $(\phi, \Gamma, \Sigma) \in C_t^0([0, T), \mathcal{X}^1) \cap C_t^1([0, T), \mathcal{X}^{-1})$ denote the global solution of (4.1). Let $\gamma_t, \sigma_t : \Lambda^* \rightarrow \mathbb{C}$ with $0 \leq \gamma_t \leq \Gamma_t^T$ and $|\sigma_t|^2 = (\gamma_t + 1)\gamma_t$ be given, and define*

$$\begin{aligned} u_t(p) &:= \sqrt{1 + \gamma_t(p)}, \\ v_t(p) &:= \frac{\sigma_t(p)}{\sqrt{1 + \gamma_t(p)}}, \end{aligned}$$

i.e., $|v_t|^2 = \gamma_t$. Then we have that

$$\begin{aligned} 1 + \|\gamma_t\|_\infty &= 1 + \|v_t\|_\infty^2 = \|u_t\|_\infty^2 \leq 1 + e^{2\lambda\|\hat{v}\|_{w,d}(\frac{\|\Gamma_0^T\|_1}{N}+2)t}(\|\Gamma_0^T\|_\infty + 1), \\ \|v_t\|_2^2 &= \|\gamma_t\|_1 \leq e^{2\lambda\|\hat{v}\|_{w,d}(\frac{\|\Gamma_0^T\|_1}{N}+1)t}\|\Gamma_0^T\|_1, \\ \|v_t\|_{L_E^2}^2 &= \|\gamma_t\|_{L_E^1} \leq \mathcal{E}_{HFB}(\phi_0, \Gamma_0, \Sigma_0). \end{aligned}$$

5. TAIL ESTIMATES

In this section, we bound the tail in the perturbation expansion. Recall from (3.27) and (3.28) that

$$\begin{aligned} \Gamma^{(2)} &= (1 + 2f_0^{(+)})\gamma^{(2)} + f_0^{(+)} + N|\Lambda||\phi^{(2)}|^2\delta, \\ \Sigma^{(2)} &= (1 + 2f_0^{(+)})\sigma^{(2)} + N|\Lambda|(\phi^{(2)})^2\delta. \end{aligned}$$

Proposition 5.1 (Bound on Fluctuation dynamics). *Assume that $\hat{v} \in L^1_{\sqrt{1+E}} \cap L^\infty(\Lambda^*)$, and that $|\Lambda| \geq 1$. Let $(\phi^{(2)}, \Gamma^{(2)}, \Sigma^{(2)}) \in C_t^0([0, T), \mathcal{X}^1) \cap C_t^1([0, T), \mathcal{X}^{-1})$ denote the global solution of (4.1) with initial datum*

$$(\phi_0^{(2)}, \Gamma_0^{(2)}, \Sigma_0^{(2)}) = (\phi_0, (1 + 2f_0^{(+)})\gamma_0 + f_0^{(+)} + N|\Lambda||\phi_0|^2\delta, (1 + 2f_0^{(+)})\sigma_0 + N|\Lambda|\phi_0^2\delta) \in \mathcal{X}^1.$$

Then for any $\ell \in \mathbb{N}$, there exist constants $C_\ell, K_\ell > 0$ such that for all $t > 0$, we have that

$$\begin{aligned} &\left\| (\mathcal{N}_b + |\Lambda|)^{\frac{\ell}{2}} \mathcal{U}_N(t)(\mathcal{N}_b + |\Lambda|)^{-\frac{\ell}{2}} \left(1 + \frac{\mathcal{N}_b}{N|\Lambda|}\right)^{-\frac{1}{2}} \right\| \\ &\leq C_\ell \left(1 + \frac{(1 + \|f_0\|_d)^2(1 + \|\gamma_0\|_d)^2}{N}\right)^{\frac{\ell}{2}} (1 + \|f_0\|_d)^{\frac{\ell}{2}} (1 + \|\gamma_0\|_d)^\ell \\ &\quad e^{K_\ell \|\hat{v}\|_{w,d} \lambda |\Lambda| (1 + \frac{(1 + \|f_0\|_d)(1 + \|\gamma_0\|_d)}{N}) t}. \end{aligned}$$

Proof. Lemma E.1 implies

$$\begin{aligned} &\left\| (\mathcal{N}_b + |\Lambda|)^{\frac{\ell}{2}} \mathcal{T}[k_0](\mathcal{N}_b + |\Lambda|)^{-\frac{\ell}{2}} \right\| \lesssim_\ell (1 + \|\gamma_0\|_1 + \|\gamma_0\|_\infty)^{\frac{\ell}{2}}, \\ &\left\| (\mathcal{N}_b + |\Lambda|)^{\frac{\ell}{2}} \mathcal{T}[k_t](\mathcal{N}_b + |\Lambda|)^{-\frac{\ell}{2}} \right\| \lesssim_\ell (1 + \|\gamma_t\|_1 + \|\gamma_t\|_\infty)^{\frac{\ell}{2}}. \end{aligned} \tag{5.1}$$

By definition (2.17), we have that $(\Gamma^{(2)})^T = (1 + 2f_0^{(+)})\gamma^{(2)} + f_0^{(+)}$. Corollary 4.6 then implies

$$\begin{aligned} & \|(\mathcal{N}_b + |\Lambda|)^{\frac{\ell}{2}} \mathcal{T}[k_t](\mathcal{N}_b + |\Lambda|)^{-\frac{\ell}{2}}\| \\ & \lesssim_{\ell} e^{\ell\lambda\|\hat{v}\|_{w,d}(\frac{\|\Gamma_0^{(2)}\|_1}{N}+1)t}(1 + \|(\Gamma_0^{(2)})^T\|_1 + \|(\Gamma_0^{(2)})^T\|_{\infty})^{\frac{\ell}{2}} \\ & \lesssim_{\ell} e^{\ell\lambda\|\hat{v}\|_{w,d}(\frac{(1+2\|f_0\|_{\infty})\|\gamma_0\|_1+\|f_0\|_1}{N}+1)t}(1 + \|f_0\|_d)^{\frac{\ell}{2}}(1 + \|\gamma_0\|_d)^{\frac{\ell}{2}}. \end{aligned} \quad (5.2)$$

Lemma E.2 implies that for $\widehat{\mathcal{U}}_N(t) := \mathcal{W}^\dagger[\sqrt{N|\Lambda|}\phi_t]e^{-it\mathcal{H}_N}\mathcal{W}[\sqrt{N|\Lambda|}\phi_0]$, we have that

$$\begin{aligned} & \|(\mathcal{N}_b + |\Lambda|)^{\frac{\ell}{2}} \widehat{\mathcal{U}}_N(t)(\mathcal{N}_b + |\Lambda|)^{-\frac{\ell}{2}}(1 + \frac{\mathcal{N}_b}{N|\Lambda|})^{-\frac{1}{2}}\| \\ & \leq C_{\ell} \left(1 + \frac{(1 + \|f_0\|_d)^2(1 + \|\gamma_0\|_d)^2}{N}\right)^{\frac{\ell}{2}} e^{K_{\ell}\|\hat{v}\|_{w,d}\lambda|\Lambda|(1 + \frac{(1 + \|f_0\|_d)(1 + \|\gamma_0\|_d)}{N})t}. \end{aligned} \quad (5.3)$$

Collecting (5.1), (5.2), and (5.3), we obtain

$$\begin{aligned} & \|(\mathcal{N}_b + |\Lambda|)^{\frac{\ell}{2}} \mathcal{U}_N(t)(\mathcal{N}_b + |\Lambda|)^{-\frac{\ell}{2}}(1 + \frac{\mathcal{N}_b}{N|\Lambda|})^{-\frac{1}{2}}\| \\ & \leq C_{\ell} \left(1 + \frac{(1 + \|f_0\|_d)^2(1 + \|\gamma_0\|_d)^2}{N}\right)^{\frac{\ell}{2}} (1 + \|f_0\|_d)^{\frac{\ell}{2}}(1 + \|\gamma_0\|_d)^{\ell} \\ & \quad e^{K_{\ell}\|\hat{v}\|_{w,d}\lambda|\Lambda|(1 + \frac{(1 + \|f_0\|_d)(1 + \|\gamma_0\|_d)}{N})t}, \end{aligned}$$

which concludes the proof. \square

Proposition 5.2. *Under the same assumptions as in Proposition 5.1, we have that*

$$\begin{aligned} & \frac{\|\mathcal{H}_{\text{BEC}}(t)P_n\|}{\sqrt{n+|\Lambda|}}, \sqrt{N} \frac{\|\mathcal{H}_{\text{HFB}}^{(\text{cor})}(t)P_n\|}{n+|\Lambda|}, \sqrt{N} \frac{\|\mathcal{H}_{\text{HFB}}^{(\text{d})}(t)P_n\|}{n}, \frac{\|\mathcal{H}_{\text{cub}}(t)P_n\|}{(n+|\Lambda|)^{\frac{3}{2}}}, \sqrt{N} \frac{\|\mathcal{H}_{\text{quart}}(t)P_n\|}{(n+|\Lambda|)^2} \\ & \leq C \frac{\lambda}{\sqrt{N}} e^{C\lambda\|\hat{v}\|_{w,d}(1 + \frac{(1 + \|f_0\|_d)(1 + \|\gamma_0\|_d)}{N})t} \|\hat{v}\|_{w,d} \|f_0\|_d (1 + \|f_0\|_d)^2 (1 + \|\gamma_0\|_d)^2. \end{aligned}$$

Proof. Recalling from (2.13) that $|\sigma|^2 = (1 + \gamma)\gamma$, Lemmata E.3 and D.4 imply

$$\begin{aligned} \|\mathcal{H}_{\text{BEC}}(t)P_n\| & \leq C \frac{\lambda|\Lambda|}{\sqrt{N}} (|u_t(0)| + |v_t(0)|) |\phi_t| \\ & \quad (|f_0^{(+)}\sigma_t * \hat{v}(0)| + |((1 + 2\gamma_t)f_0^{(+)}) * (\hat{v} + \hat{v}(0))(0)|) \sqrt{n+1} \\ & \leq C \frac{\lambda|\Lambda|}{\sqrt{N}} \|\hat{v}\|_{w,d} (|u_t(0)| + |v_t(0)|) |\phi_t| \|f_0\|_d (1 + \|\gamma_t\|_d) \sqrt{n+1}. \end{aligned}$$

Now we employ Lemma 4.5 and Corollary 4.6, and obtain

$$\begin{aligned} \frac{\|\mathcal{H}_{\text{BEC}}(t)P_n\|}{\sqrt{n+1}} & \leq C \frac{\lambda}{\sqrt{N}} e^{C\lambda\|\hat{v}\|_{w,d}(1 + \frac{\|\Gamma_0^T\|_1}{N})t} \|\hat{v}\|_{w,d} \|f_0\|_d \left(1 + \frac{\|\Gamma_0^T\|_1}{N}\right)^{\frac{1}{2}} (1 + \|\Gamma_0^T\|_d)^{\frac{3}{2}} \\ & \leq C \frac{\lambda}{\sqrt{N}} e^{C\lambda\|\hat{v}\|_{w,d}(1 + \frac{(1 + \|f_0\|_d)(1 + \|\gamma_0\|_d)}{N})t} \|\hat{v}\|_{w,d} \|f_0\|_d (1 + \|f_0\|_d)^2 (1 + \|\gamma_0\|_d)^2. \end{aligned}$$

The bounds on $\|\mathcal{H}_{\text{HFB}}^{(\text{cor})}(t)P_n\|$ and $\|\mathcal{H}_{\text{HFB}}^{(\text{d})}(t)P_n\|$, see Lemma E.3, and on $\|\mathcal{H}_{\text{cub}}(t)P_n\|$ and $\|\mathcal{H}_{\text{quart}}(t)P_n\|$, see Lemma A.2, can be obtained in an analogous fashion, using Lemmata D.3 and D.4, respectively. This concludes the proof. \square

Corollary 5.3. *Under the same assumptions as in Proposition 5.1, and for $|\Lambda| \leq N$, we have that*

$$\begin{aligned} & \left| \int_{[0,t]^3} d\mathbf{s}_3 \langle [[[f[J], \mathcal{H}_{\text{fluc}}(s_1)], \mathcal{H}_{\text{fluc}}(s_2)], \mathcal{H}_{\text{fluc}}(s_3)] \rangle_{s_3} \right| \\ & \leq C_{\|f_0\|_d} \|J\|_2 \frac{1}{N^{\frac{3}{2}}} e^{C\lambda|\Lambda|\|\hat{v}\|_{w,d}(1+\frac{(1+\|f_0\|_d)(1+\|\gamma_0\|_d)}{N})t} (1 + \|\gamma_0\|_d)^6 |\Lambda|^{\frac{11}{2}}. \end{aligned}$$

Proof. Proposition 5.2 and Lemma D.2 imply

$$\begin{aligned} & \left| \int_{[0,t]^3} d\mathbf{s}_3 \langle [[[f[J], \mathcal{H}_{\text{fluc}}(s_1)], \mathcal{H}_{\text{fluc}}(s_2)], \mathcal{H}_{\text{fluc}}(s_3)] \rangle_{s_3} \right| \\ & \leq C \|J\|_2 \frac{\lambda^3 t^3}{N^{\frac{3}{2}}} e^{C\lambda\|\hat{v}\|_{w,d}(1+\frac{(1+\|f_0\|_d)(1+\|\gamma_0\|_d)}{N})t} \|\hat{v}\|_{w,d}^3 \|f_0\|_d^3 \\ & \quad (1 + \|f_0\|_d)^6 (1 + \|\gamma_0\|_d)^6 \sup_{s \in [0,t]} \langle (\mathcal{N}_b + |\Lambda|)^{1+\frac{9}{2}} \left(1 + \frac{(\mathcal{N}_b + |\Lambda|)^{\frac{1}{2}}}{\sqrt{N}}\right)^3 \rangle_s. \end{aligned}$$

Proposition 5.1 then yields

$$\begin{aligned} & \left| \int_{[0,t]^3} d\mathbf{s}_3 \langle [[[f[J], \mathcal{H}_{\text{fluc}}(s_1)], \mathcal{H}_{\text{fluc}}(s_2)], \mathcal{H}_{\text{fluc}}(s_3)] \rangle_{s_3} \right| \\ & \leq C \|J\|_2 \frac{\lambda^3 t^3}{N^{\frac{3}{2}}} e^{C\lambda|\Lambda|\|\hat{v}\|_{w,d}(1+\frac{(1+\|f_0\|_d)(1+\|\gamma_0\|_d)}{N})t} \|\hat{v}\|_{w,d}^3 \|f_0\|_d^3 \\ & \quad (1 + \|f_0\|_d)^6 (1 + \|\gamma_0\|_d)^6 \langle (\mathcal{N}_b + |\Lambda|)^{\frac{11}{2}} \left(1 + \frac{(\mathcal{N}_b + |\Lambda|)^{\frac{1}{2}}}{\sqrt{N}}\right)^3 \left(1 + \frac{\mathcal{N}_b}{N|\Lambda|}\right) \rangle_0. \end{aligned}$$

Finally, we employ Lemma D.1 and obtain

$$\begin{aligned} & \left| \int_{[0,t]^3} d\mathbf{s}_3 \langle [[[f[J], \mathcal{H}_{\text{fluc}}(s_1)], \mathcal{H}_{\text{fluc}}(s_2)], \mathcal{H}_{\text{fluc}}(s_3)] \rangle_{s_3} \right| \\ & \leq C_{\|f_0\|_d} \|J\|_2 \frac{\lambda^3 t^3}{N^{\frac{3}{2}}} e^{C\lambda|\Lambda|\|\hat{v}\|_{w,d}(1+\frac{(1+\|f_0\|_d)(1+\|\gamma_0\|_d)}{N})t} \|\hat{v}\|_{w,d}^3 \\ & \quad (1 + \|\gamma_0\|_d)^6 |\Lambda|^{\frac{11}{2}} \left(1 + \frac{|\Lambda|^{\frac{1}{2}}}{N^{\frac{1}{2}}}\right)^3 \\ & \leq C_{\|f_0\|_d} \|J\|_2 \frac{1}{N^{\frac{3}{2}}} e^{C\lambda|\Lambda|\|\hat{v}\|_{w,d}(1+\frac{(1+\|f_0\|_d)(1+\|\gamma_0\|_d)}{N})t} (1 + \|\gamma_0\|_d)^6 |\Lambda|^{\frac{11}{2}}. \end{aligned}$$

□

APPENDIX A. CALCULATION OF THE FLUCTUATION DYNAMICS

Lemma A.1. *We have that*

$$\mathcal{T}^\dagger[k] a_p \mathcal{T}[k] = \cosh(|k(p)|) a_p + \sinh(|k(p)|) \frac{k(p)}{|k(p)|} a_{-p}^\dagger.$$

Proof. Observe that

$$\partial_\tau \left(\begin{array}{l} \mathcal{T}^\dagger[\tau k] a_p \mathcal{T}[\tau k] \\ \mathcal{T}^\dagger[\tau k] a_{-p}^\dagger \mathcal{T}[\tau k] \end{array} \right) = \left(\begin{array}{l} \frac{1}{2} \int dq k(q) \mathcal{T}^\dagger[\tau k][a_p, a_q^\dagger a_{-q}^\dagger] \mathcal{T}[\tau k] \\ -\frac{1}{2} \int dq k(q) \mathcal{T}^\dagger[\tau k][a_{-p}^\dagger, a_q a_{-q}] \mathcal{T}[\tau k] \end{array} \right)$$

$$= \begin{pmatrix} 0 & k(p) \\ \bar{k}(p) & 0 \end{pmatrix} \begin{pmatrix} \mathcal{T}^\dagger[\tau k] a_p \mathcal{T}[\tau k] \\ \mathcal{T}^\dagger[\tau k] a_{-p}^\dagger \mathcal{T}[\tau k] \end{pmatrix},$$

where we used the fact that k is even. Employing the fact that $\mathcal{T}[0] = \mathbf{1}$ and $\begin{pmatrix} 0 & k(p) \\ \bar{k}(p) & 0 \end{pmatrix}^2 = |k(p)|^2 \mathbf{1}$, we find that

$$\begin{aligned} \begin{pmatrix} \mathcal{T}^\dagger[k] a_p \mathcal{T}[k] \\ \mathcal{T}^\dagger[k] a_{-p}^\dagger \mathcal{T}[k] \end{pmatrix} &= \exp \left(\begin{pmatrix} 0 & k(p) \\ \bar{k}(p) & 0 \end{pmatrix} \right) \begin{pmatrix} a_p \\ a_{-p}^\dagger \end{pmatrix} \\ &= \left[\cosh(|k(p)|) \mathbf{1} + \frac{\sinh(|k(p)|)}{|k(p)|} \begin{pmatrix} 0 & k(p) \\ \bar{k}(p) & 0 \end{pmatrix} \right] \begin{pmatrix} a_p \\ a_{-p}^\dagger \end{pmatrix}. \end{aligned}$$

This finishes the proof. \square

Lemma A.2. *Let $\mathcal{H}_{\text{fluc}}(t)$ be defined by (2.6) and (2.7). Recall from (2.18)–(2.21) the definitions of $\mathbf{w}^{(j,j)}$. By choosing*

$$\begin{aligned} S_t &= |\Lambda| \int_0^t ds \int dp \left[E(p) \gamma_s(p) + \frac{\lambda}{2N} \Gamma_s^{(1)} * (\hat{v} + \hat{v}(0))(p) \Gamma_s^{(1)}(p) \right. \\ &\quad + \frac{\lambda}{2N} \bar{\Sigma}_s^{(1)} * \hat{v}(p) \Sigma_s^{(1)}(p) - N|\Lambda|^2 \lambda |\phi_s|^4 \hat{v}(0) \\ &\quad \left. - \frac{N|\Lambda| \delta(p) \operatorname{Re}(\bar{\phi}_s i \partial_s \phi_s)}{2} - \frac{\operatorname{Re}(\bar{\sigma}_s(p) i \partial_s \sigma_s(p))}{2(1 + \gamma_s(p))} \right], \end{aligned}$$

we then have that

$$\mathcal{H}_{\text{fluc}}(t) = \mathcal{H}_{\text{BEC}}(t) + \mathcal{H}_{\text{HFB}}(t) + \mathcal{H}_{\text{cub}}(t) + \mathcal{H}_{\text{quart}}(t),$$

where

$$\begin{aligned} \mathcal{H}_{\text{BEC}}(t) &= \sqrt{N|\Lambda|} \left[u_t(0) \left(-i \partial_t \phi_t + \lambda |\Lambda| \hat{v}(0) |\phi_t|^2 \phi_t + \frac{\lambda}{N} (\sigma_t * \hat{v})(0) \bar{\phi}_t \right. \right. \\ &\quad + \frac{\lambda}{N} (\gamma_t * (\hat{v} + \hat{v}(0)))(0) \phi_t \Big) + v_t(0) \left(-i \partial_t \phi_t + \lambda |\Lambda| \hat{v}(0) |\phi_t|^2 \bar{\phi}_t \right. \\ &\quad \left. \left. + \frac{\lambda}{N} (\bar{\sigma}_t * \hat{v})(0) \phi_t + \frac{\lambda}{N} (\gamma_t * (\hat{v} + \hat{v}(0)))(0) \bar{\phi}_t \right) \right] e^{i \int_0^t ds \Omega_s(0)} a_0^\dagger + \text{h.c.} \end{aligned}$$

is the BEC-Hamiltonian,

$$\mathcal{H}_{\text{HFB}}(t) = \mathcal{H}_{\text{HFB}}^{(\text{d})}(t) + \mathcal{H}_{\text{HFB}}^{(\text{cor})}(t)$$

is the HFB-Hamiltonian,

$$\begin{aligned} \mathcal{H}_{\text{HFB}}^{(\text{d})}(t) &= \int dp \left[-\Omega_t(p) - \frac{\operatorname{Re}(\bar{\sigma}_t(p) i \partial_t \sigma_t(p))}{1 + \gamma_t(p)} \right. \\ &\quad + \left(E(p) + \frac{\lambda}{N} ((\gamma_t + N|\Lambda| |\phi_t|^2 \delta) * (\hat{v} + \hat{v}(0)))(p) \right) (1 + 2\gamma_t(p)) \\ &\quad \left. + \frac{2\lambda}{N} \operatorname{Re} \left(((\bar{\sigma}_t + N|\Lambda| \bar{\phi}_t^2 \delta) * \hat{v})(p) \sigma_t(p) \right) \right] a_p^\dagger a_p \end{aligned}$$

refers to its diagonal part,

$$\begin{aligned} \mathcal{H}_{\text{HFB}}^{(\text{cor})}(t) = & \int dp \left[-\frac{i\partial_t\sigma_t(p)}{2} + \frac{\sigma_t(p)i\partial_t\gamma_t(p)}{2(1+\gamma_t(p))} \right. \\ & + \left(E(p) + \frac{\lambda}{N}((\gamma_t + N|\Lambda||\phi_t|^2\delta) * (\hat{v} + \hat{v}(0)))(p) \right) \sigma_t(p) \\ & + \frac{\lambda}{2N} \left(((\sigma_t + N|\Lambda|\phi_t^2\delta) * \hat{v})(p)(1 + \gamma_t(p)) \right. \\ & \left. \left. + ((\bar{\sigma}_t + N|\Lambda|\bar{\phi}_t^2\delta) * \hat{v})(p) \frac{\sigma_t(p)^2}{1 + \gamma_t(p)} \right) \right] e^{2i\int_0^t d\tau \Omega_\tau(p)} a_p^\dagger a_{-p}^\dagger + \text{h.c.} \end{aligned}$$

to its off-diagonal part,

$$\begin{aligned} \mathcal{H}_{\text{cub}}(t) = & \frac{\lambda}{\sqrt{N}} \int d\mathbf{p}_3 \left(\delta(p_1 + p_2 + p_3) e^{i\int_0^t d\tau (\Omega_\tau(p_1) + \Omega_\tau(p_2) + \Omega_\tau(p_3))} \right. \\ & \frac{1}{3!} \mathbf{w}_t^{(3,0)}(\mathbf{p}_3) a_{p_1}^\dagger a_{p_2}^\dagger a_{p_3}^\dagger \\ & + \delta(p_1 + p_2 - p_3) e^{i\int_0^t d\tau (\Omega_\tau(p_1) + \Omega_\tau(p_2) - \Omega_\tau(p_3))} \\ & \frac{1}{2!} \mathbf{w}_t^{(2,1)}(\mathbf{p}_3) a_{p_1}^\dagger a_{p_2}^\dagger a_{p_3} \\ & \left. + \text{h.c.} \right) \end{aligned}$$

describes the cubic processes, and

$$\begin{aligned} \mathcal{H}_{\text{quart}}(t) = & \frac{\lambda}{N} \int d\mathbf{p}_4 \left(\delta(p_1 + p_2 + p_3 + p_4) e^{i\int_0^t d\tau (\Omega_\tau(p_1) + \Omega_\tau(p_2) + \Omega_\tau(p_3) + \Omega_\tau(p_4))} \right. \\ & \frac{1}{4!} \mathbf{w}_t^{(4,0)}(\mathbf{p}_4) a_{p_1}^\dagger a_{p_2}^\dagger a_{p_3}^\dagger a_{p_4}^\dagger \\ & + \delta(p_1 + p_2 + p_3 - p_4) e^{i\int_0^t d\tau (\Omega_\tau(p_1) + \Omega_\tau(p_2) + \Omega_\tau(p_3) - \Omega_\tau(p_4))} \\ & \frac{1}{3!} \mathbf{w}_t^{(3,1)}(\mathbf{p}_4) a_{p_1}^\dagger a_{p_2}^\dagger a_{p_3}^\dagger a_{p_4} \\ & + \text{h.c.} \\ & \left. + \delta(p_1 + p_2 - p_3 - p_4) e^{i\int_0^t d\tau (\Omega_\tau(p_1) + \Omega_\tau(p_2) - \Omega_\tau(p_3) - \Omega_\tau(p_4))} \right. \\ & \left. \frac{1}{(2!)^2} \mathbf{w}_t^{(2,2)}(\mathbf{p}_4) a_{p_1}^\dagger a_{p_2}^\dagger a_{p_3} a_{p_4} \right) \end{aligned}$$

describes quartic interactions.

Proof. By definition, we have that

$$\begin{aligned} \mathcal{H}_{\text{fluc}}(t) = & -\partial_t S_t - \int dp \Omega_t(p) a_p^\dagger a_p + \mathcal{U}_{\text{Bog}}^\dagger(t) [i\partial_t \mathcal{T}^\dagger[k_t]] \mathcal{T}[k_t] \mathcal{U}_{\text{Bog}}(t) \\ & + \mathcal{U}_{\text{Bog}}^\dagger(t) \mathcal{T}^\dagger[k_t] [i\partial_t \mathcal{W}^\dagger[\sqrt{N|\Lambda|}\phi_t]] \mathcal{W}[\sqrt{N|\Lambda|}\phi_t] \mathcal{T}[k_t] \mathcal{U}_{\text{Bog}}(t) \\ & + \mathcal{U}_{\text{Bog}}^\dagger(t) \mathcal{T}^\dagger[k_t] \mathcal{W}^\dagger[\sqrt{N|\Lambda|}\phi_t] \mathcal{H}_N \mathcal{W}[\sqrt{N|\Lambda|}\phi_t] \mathcal{T}[k_t] \mathcal{U}_{\text{Bog}}(t). \quad (\text{A.1}) \end{aligned}$$

In ascending order, we define $\partial_t S_t$ till $\mathcal{H}_{\text{quart}}(t)$ to be the normal-ordered zeroth till fourth order polynomials in $\mathcal{H}_{\text{fluc}}(t)$.

In order to calculate $[i\partial_t \mathcal{T}^\dagger(t)]\mathcal{T}(t)$, we use the fact that

$$\begin{aligned} [\partial_t e^{-A(t)}]e^{A(t)} &= \lim_{h \rightarrow 0} \frac{1}{h} \int_0^1 d\tau \partial_\tau \left(e^{-A(t+h)\tau} e^{A(t)\tau} \right) \\ &= \int_0^1 d\tau e^{-A(t)\tau} [-\partial_t A(t)] e^{A(t)\tau}, \end{aligned} \quad (\text{A.2})$$

see, e.g., the proof of Proposition 3.2 in [14]. Accordingly, we have that

$$\begin{aligned} [\partial_t \mathcal{T}^\dagger[k_t]]\mathcal{T}[k_t] &= \frac{1}{2} \int_0^1 d\tau \mathcal{T}^\dagger[\tau k_t] \int dp (\partial_t \bar{k}_t(p) a_p a_{-p} - \partial_t k_t(p) a_p^\dagger a_{-p}^\dagger) \mathcal{T}[\tau k_t] \\ &= \frac{1}{2} \int dp \int_0^1 d\tau \left((\cosh(\tau|k_t(p)|)^2 \partial_t \bar{k}_t(p) \right. \\ &\quad - \sinh(\tau|k_t(p)|)^2 \frac{\bar{k}_t(p)^2}{|k_t(p)|^2} \partial_t k_t(p)) a_p a_{-p} - h.c. \\ &\quad + \sinh(\tau|k_t(p)|) \cosh(\tau|k_t(p)|) \frac{k_t(p) \partial_t \bar{k}_t(p) - \partial_t k_t(p) \bar{k}_t(p)}{|k_t(p)|} \\ &\quad \left. (a_p^\dagger a_p + a_p a_p^\dagger) \right). \end{aligned}$$

Using the trigonometric identities for sinh and cosh, we can simplify the expression to be

$$\begin{aligned} [\partial_t \mathcal{T}^\dagger[k_t]]\mathcal{T}[k_t] &= \frac{1}{4} \int dp \left[\left(\frac{\sinh(2|k_t(p)|) + 1}{2|k_t(p)|} \partial_t \bar{k}_t(p) \right. \right. \\ &\quad - \frac{\sinh(2|k_t(p)|) - 1}{2|k_t(p)|} \frac{\bar{k}_t(p)^2}{|k_t(p)|^2} \partial_t k_t(p) \Big) a_p a_{-p} - h.c. \\ &\quad + 4 \frac{\cosh(2|k_t(p)|) - 1}{2|k_t(p)|} \frac{i \operatorname{Im}(k_t(p) \partial_t \bar{k}_t(p))}{|k_t(p)|} (a_p^\dagger a_p + \frac{|\Lambda|}{2}) \Big] \\ &= \frac{1}{2} \int dp \left[\left(u_t(p) \bar{v}_t(p) \frac{i \operatorname{Im}(k_t(p) \partial_t \bar{k}_t(p))}{|k_t(p)|^2} \right. \right. \\ &\quad + \frac{\bar{k}_t(p) \operatorname{Re}(k_t(p) \partial_t \bar{k}_t(p))}{|k_t(p)|^2} \Big) a_p a_{-p} - h.c. \\ &\quad \left. \left. + 2|v_t(p)|^2 \frac{i \operatorname{Im}(k_t(p) \partial_t \bar{k}_t(p))}{|k_t(p)|^2} (a_p^\dagger a_p + \frac{|\Lambda|}{2}) \right]. \right. \end{aligned} \quad (\text{A.3})$$

A straightforward calculation yields

$$\begin{aligned} \partial_t u_t(p) &= v_t(p) \frac{\bar{k}_t(p) \operatorname{Re}(k_t(p) \partial_t \bar{k}_t(p))}{|k_t(p)|^2}, \\ \partial_t \bar{v}_t(p) &= u_t(p) \frac{\bar{k}_t(p) \operatorname{Re}(k_t(p) \partial_t \bar{k}_t(p))}{|k_t(p)|^2} + \bar{v}_t(p) \frac{i \operatorname{Im}(k_t(p) \partial_t \bar{k}_t(p))}{|k_t(p)|^2}. \end{aligned}$$

In particular, we have that

$$\begin{aligned}
& \frac{\overline{k_t}(p) \operatorname{Re}(k_t(p) \partial_t \overline{k_t}(p))}{|k_t(p)|^2} + u_t(p) \overline{v_t}(p) \frac{i \operatorname{Im}(k_t(p) \partial_t \overline{k_t}(p))}{|k_t(p)|^2} \\
&= u_t(p) \partial_t \overline{v_t}(p) - \overline{v_t}(p) \partial_t u_t(p) \\
&= u_t(p)^2 \partial_t \left(\frac{\overline{v_t}(p)}{u_t(p)} \right). \tag{A.4}
\end{aligned}$$

Because $u_t(p)^2 = 1 + |v_t(p)|^2$, we can invoke that

$$u_t(p) \partial_t u_t(p) = \frac{1}{2} \partial_t u_t(p)^2 = \frac{1}{2} \partial_t |v_t(p)|^2 = \operatorname{Re}(v_t(p) \partial_t \overline{v_t}(p)),$$

in order to obtain that

$$\begin{aligned}
& |v_t(p)|^2 \frac{i \operatorname{Im}(k_t(p) \partial_t \overline{k_t}(p))}{|k_t(p)|^2} \\
&= v_t(p) \partial_t \overline{v_t}(p) - u_t(p) \partial_t u_t(p) \\
&= i \operatorname{Im}(v_t(p) \partial_t \overline{v_t}(p)) \\
&= v_t(p)^2 \partial_t \left(\frac{\overline{v_t}(p)}{v_t(p)} \right). \tag{A.5}
\end{aligned}$$

Using the notation $\gamma_t(p) = |v_t(p)|^2$, $\sigma_t(p) = u_t(p)v_t(p)$, and employing (A.4), (A.5), we can hence simplify (A.3) as

$$\begin{aligned}
[i \partial_t \mathcal{T}^\dagger[k_t]] \mathcal{T}[k_t] &= \frac{i}{2} \int dp \left((1 + \gamma_t(p)) \partial_t \left(\frac{\overline{\sigma}_t(p)}{1 + \gamma_t(p)} \right) a_p a_{-p} - h.c. \right. \\
&\quad \left. + 2 \frac{\sigma_t(p)^2}{1 + \gamma_t(p)} \partial_t \left(\frac{\overline{\sigma}_t(p)}{\sigma_t(p)} \right) (a_p^\dagger a_p + \frac{|\Lambda|}{2}) \right) \\
&= - \int dp \left[\left(\frac{i \partial_t \overline{\sigma}_t(p)}{2} + \frac{\overline{\sigma}_t(p) i \partial_t \gamma_t(p)}{2(1 + \gamma_t(p))} \right) a_p a_{-p} + h.c. \right. \\
&\quad \left. - \frac{\operatorname{Im}(\sigma_t(p) \partial_t \overline{\sigma}_t(p))}{1 + \gamma_t(p)} \left(a_p^\dagger a_p + \frac{|\Lambda|}{2} \right) \right] \tag{A.6}
\end{aligned}$$

Observing that $\operatorname{Im}(z) = \operatorname{Re}(i\bar{z})$, and using (2.5), (A.6) implies

$$\begin{aligned}
& \mathcal{U}_{Bog}^\dagger(t) [i \partial_t \mathcal{T}^\dagger[k_t]] \mathcal{T}[k_t] \mathcal{U}_{Bog}(t) \\
&= - \int dp \left[\left(\frac{i \partial_t \overline{\sigma}_t(p)}{2} + \frac{\overline{\sigma}_t(p) i \partial_t \gamma_t(p)}{2(1 + \gamma_t(p))} \right) e^{-2i \int_0^t ds \Omega_s(p)} a_p a_{-p} + h.c. \right. \\
&\quad \left. + \frac{\operatorname{Re}(\overline{\sigma}_t(p) i \partial_t \sigma_t(p))}{1 + \gamma_t(p)} \left(a_p^\dagger a_p + \frac{|\Lambda|}{2} \right) \right] \tag{A.7}
\end{aligned}$$

Next, using (A.2) again, we have that

$$\begin{aligned}
& [i\partial_t \mathcal{W}^\dagger[\sqrt{N|\Lambda|}\phi_t]]\mathcal{W}[\sqrt{N|\Lambda|}\phi_t] \\
&= -\sqrt{N|\Lambda|} \int_0^1 d\tau \mathcal{W}^\dagger[\tau\sqrt{N|\Lambda|}\phi_t](i\partial_t\phi_t a_0^\dagger + i\overline{\partial_t\phi_t}a_0)\mathcal{W}[\tau\sqrt{N|\Lambda|}\phi_t] \\
&= -\sqrt{N|\Lambda|}(i\partial_t\phi_t a_0^\dagger + i\overline{\partial_t\phi_t}a_0) - \frac{N|\Lambda|^2}{2} \operatorname{Re}(\overline{\phi}_t i\partial_t\phi_t),
\end{aligned} \tag{A.8}$$

where we also employed that $\delta(0) = |\Lambda|$.

Finally, we calculate

$$\begin{aligned}
& \mathcal{W}^\dagger[\sqrt{N|\Lambda|}\phi_t]\mathcal{H}_N\mathcal{W}[\sqrt{N|\Lambda|}\phi_t] \\
&= \frac{N|\Lambda|^3\lambda|\phi_t|^4\hat{v}(0)}{2} \\
&\quad + \sqrt{N|\Lambda|^3}\lambda|\phi_t|^2\hat{v}(0)(\phi_t a_0^\dagger + \overline{\phi}_t a_0) \\
&\quad + \int dp (E(p) + \lambda|\Lambda|(\hat{v}(p) + \hat{v}(0))|\phi_t|^2)a_p^\dagger a_p \\
&\quad + \frac{\lambda|\Lambda|}{2} \int dp \hat{v}(p)(\phi_t^2 a_p^\dagger a_{-p}^\dagger + \overline{\phi}_t^2 a_p a_{-p}) \\
&\quad + \frac{\lambda\sqrt{|\Lambda|}}{\sqrt{N}} \int d\mathbf{p}_2 \hat{v}(p_2)(a_{p_1}^\dagger a_{p_2}^\dagger a_{p_{12}}\phi_t + a_{p_{12}}^\dagger a_{p_2} a_{p_1} \overline{\phi}_t) \\
&\quad + \frac{\lambda}{2N} \int d\mathbf{p}_4 \delta(p_1 + p_2 - p_3 - p_4)\hat{v}(p_1 - p_3)a_{p_1}^\dagger a_{p_2}^\dagger a_{p_3} a_{p_4}.
\end{aligned} \tag{A.9}$$

In order to calculate

$$\mathcal{T}^\dagger[k_t]\mathcal{W}^\dagger[\sqrt{N|\Lambda|}\phi_t]\mathcal{H}_N\mathcal{W}[\sqrt{N|\Lambda|}\phi_t]\mathcal{T}[k_t],$$

we calculate the transformations for the scalar terms, the terms linear in a , a^\dagger , etc. in $\mathcal{W}^\dagger[\sqrt{N|\Lambda|}\phi_t]\mathcal{H}_N\mathcal{W}[\sqrt{N|\Lambda|}\phi_t]$. We have that

$$\begin{aligned}
& \sqrt{N|\Lambda|^3}\lambda|\phi_t|^2\hat{v}(0)\mathcal{T}^\dagger[k_t](\phi_t a_0^\dagger + \overline{\phi}_t a_0)\mathcal{T}[k_t] \\
&= \sqrt{N|\Lambda|^3}\lambda|\phi_t|^2\hat{v}(0)\left((\phi_t u_t(0) + \overline{\phi}_t v_t(0))a_0^\dagger + (\overline{\phi}_t u_t(0) + \phi_t \overline{v}_t(0))a_0\right).
\end{aligned} \tag{A.10}$$

For the quadratic terms, we have that

$$\begin{aligned}
& \int dp (E(p) + \lambda|\Lambda|(\hat{v}(p) + \hat{v}(0))|\phi_t|^2)\mathcal{T}^\dagger[k_t]a_p^\dagger a_p\mathcal{T}[k_t] \\
&\quad + \frac{\lambda|\Lambda|}{2} \int dp \hat{v}(p)\mathcal{T}^\dagger[k_t](\phi_t^2 a_p^\dagger a_{-p}^\dagger + \overline{\phi}_t^2 a_p a_{-p})\mathcal{T}[k_t] \\
&= \int dp \left[(E(p) + \lambda|\Lambda|(\hat{v}(p) + \hat{v}(0))|\phi_t|^2)(u_t(p)^2 a_p^\dagger a_p + |v_t(p)|^2 a_p a_p^\dagger)\right. \\
&\quad + \frac{\lambda|\Lambda|}{2}\hat{v}(p)(u_t(p)\overline{v}_t(p)\phi_t^2 + u_t(p)v_t(p)\overline{\phi}_t^2)(a_p^\dagger a_p + a_p a_p^\dagger) \\
&\quad \left.+ (E(p) + \lambda|\Lambda|(\hat{v}(p) + \hat{v}(0))|\phi_t|^2)u_t(p)(v_t(p)a_p^\dagger a_{-p}^\dagger + \overline{v}_t(p)a_p a_{-p})\right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{\lambda|\Lambda|}{2} \hat{v}(p) \left((u_t(p)^2 \phi_t^2 + v_t(p)^2 \bar{\phi}_t^2) a_p^\dagger a_{-p}^\dagger \right. \\
& \quad \left. + (\bar{v}_t(p)^2 \phi_t^2 + u_t(p)^2 \bar{\phi}_t^2) a_p a_{-p} \right) \Big] \\
& = \int dp \left[\left((E(p) + \lambda|\Lambda|(\hat{v}(p) + \hat{v}(0))|\phi_t|^2) \gamma_t(p) \right. \right. \\
& \quad + \frac{\lambda|\Lambda|}{2} \hat{v}(p) (\bar{\sigma}_t(p) \phi_t^2 + \sigma_t(p) \bar{\phi}_t^2) \Big) |\Lambda| \\
& \quad + \left((E(p) + \lambda|\Lambda|(\hat{v}(p) + \hat{v}(0))|\phi_t|^2) (1 + 2\gamma_t(p)) \right. \\
& \quad + \lambda|\Lambda| \hat{v}(p) (\bar{\sigma}_t(p) \phi_t^2 + \sigma_t(p) \bar{\phi}_t^2) \Big) a_p^\dagger a_p \\
& \quad + \left. \left. \left((E(p) + \lambda|\Lambda|(\hat{v}(p) + \hat{v}(0))|\phi_t|^2) \sigma_t(p) \right. \right. \right. \\
& \quad + \left. \left. \left. + \frac{\lambda|\Lambda|}{2} \hat{v}(p) ((1 + \gamma_t(p)) \phi_t^2 + \frac{\sigma_t(p)^2}{1 + \gamma_t(p)} \bar{\phi}_t^2) \right) a_p^\dagger a_{-p}^\dagger + \text{h.c.} \right], \tag{A.11}
\end{aligned}$$

where the hermitian conjugate 'h.c.' only refers to the terms proportional to $a_p^\dagger a_{-p}^\dagger$. Bogoliubov-rotating the cubic terms yields

$$\begin{aligned}
& \frac{\lambda\sqrt{|\Lambda|}}{\sqrt{N}} \int d\mathbf{p}_2 \hat{v}(p_2) \mathcal{T}^\dagger[k_t] (a_{p_1}^\dagger a_{p_2}^\dagger a_{p_{12}} \phi_t + a_{p_{12}}^\dagger a_{p_2} a_{p_1} \bar{\phi}_t) \mathcal{T}[k_t] \\
& = \frac{\lambda\sqrt{|\Lambda|}}{\sqrt{N}} \int d\mathbf{p}_2 \frac{\hat{v}(p_1) + \hat{v}(p_2)}{2} \left((u_t(p_1) a_{p_1}^\dagger + \bar{v}_t(p_1) a_{-p_1}) \right. \\
& \quad \left. (u_t(p_2) a_{p_2}^\dagger + \bar{v}_t(p_2) a_{-p_2}) (u_t(p_{12}) a_{p_{12}} + v_t(p_{12}) a_{-p_{12}}^\dagger) \phi_t + \text{h.c.} \right). \tag{A.12}
\end{aligned}$$

After normal-ordering, the expressions linear in a, a^\dagger in (A.12) are given by

$$\begin{aligned}
& \frac{\lambda\sqrt{|\Lambda|}}{\sqrt{N}} \int dp \left[\left(\hat{v}(p) (u_t(p) \bar{v}_t(p) v_t(0) \phi_t + u_t(p) v_t(p) u_t(0) \bar{\phi}_t) \right. \right. \\
& \quad \left. \left. + |v_t(p)|^2 (\hat{v}(p) + \hat{v}(0)) (u_t(0) \phi_t + v_t(0) \bar{\phi}_t) \right) a_0^\dagger + \text{h.c.} \right] \\
& = \frac{\lambda\sqrt{|\Lambda|}}{\sqrt{N}} \int dp \left[u_t(0) \left(\hat{v}(p) \sigma_t(p) \bar{\phi}_t + (\hat{v}(p) + \hat{v}(0)) \gamma_t(p) \phi_t \right) \right. \\
& \quad \left. + v_t(0) \left(\hat{v}(p) \bar{\sigma}_t(p) \phi_t + (\hat{v}(p) + \hat{v}(0)) \gamma_t(p) \bar{\phi}_t \right) \right] a_0^\dagger + \text{h.c.}. \tag{A.13}
\end{aligned}$$

Similarly, the cubic terms obtained from normal-ordering (A.12) are given by

$$\begin{aligned}
& \frac{\lambda\sqrt{|\Lambda|}}{2\sqrt{N}} \int d\mathbf{p}_3 (\hat{v}(p_1) + \hat{v}(p_2)) \delta(p_1 + p_2 - p_3) \\
& \quad \left((u_t(p_1) u_t(p_2) u_t(p_3) \phi_t + v_t(p_1) v_t(p_2) \bar{v}_t(p_3) \bar{\phi}_t) a_{p_1}^\dagger a_{p_2}^\dagger a_{p_3} \right. \\
& \quad + (u_t(p_1) u_t(p_2) v_t(p_3) \phi_t + v_t(p_1) v_t(p_2) u_t(p_3) \bar{\phi}_t) a_{p_1}^\dagger a_{p_2}^\dagger a_{-p_3}^\dagger \\
& \quad \left. + 2(\bar{v}_t(p_1) u_t(p_2) v_t(p_3) \phi_t + u_t(p_1) v_t(p_2) u_t(p_3) \bar{\phi}_t) a_{-p_3}^\dagger a_{p_2}^\dagger a_{-p_1} \right)
\end{aligned}$$

$$+ h.c. \Big) . \quad (\text{A.14})$$

Here, we already exploited the symmetry w.r.t. $p_1 \leftrightarrow p_2$. Relabeling momenta yields

$$\begin{aligned} & \frac{\lambda\sqrt{|\Lambda|}}{2\sqrt{N}} \int d\mathbf{p}_3 \left[\left((u_t(p_1)u_t(p_2)u_t(p_3)\phi_t + v_t(p_1)v_t(p_2)\bar{v}_t(p_3)\bar{\phi}_t)(\hat{v}(p_1) + \hat{v}(p_2)) \right. \right. \\ & + 2(v_t(p_1)u_t(p_2)\bar{v}_t(p_3)\phi_t + u_t(p_1)v_t(p_2)u_t(p_3)\bar{\phi}_t)(\hat{v}(p_2) + \hat{v}(p_3)) \Big) \\ & \delta(p_1 + p_2 - p_3)a_{p_1}^\dagger a_{p_2}^\dagger a_{p_3} \\ & + (u_t(p_1)u_t(p_2)v_t(p_3)\phi_t + v_t(p_1)v_t(p_2)u_t(p_3)\bar{\phi}_t)(\hat{v}(p_1) + \hat{v}(p_2)) \\ & \delta(p_1 + p_2 + p_3)a_{p_1}^\dagger a_{p_2}^\dagger a_{p_3}^\dagger \\ & \left. \left. + h.c. \right) . \right] \end{aligned}$$

To conclude the calculation of these cubic terms, we symmetrize the coefficient of $a_{p_1}^\dagger a_{p_2}^\dagger a_{p_3}$ w.r.t. $p_1 \leftrightarrow p_2$, and the coefficient of $a_{p_1}^\dagger a_{p_2}^\dagger a_{p_3}^\dagger$ w.r.t. permutations of (p_1, p_2, p_3) to obtain

$$\begin{aligned} & \frac{\lambda\sqrt{|\Lambda|}}{\sqrt{N}} \int d\mathbf{p}_3 \left[\frac{1}{3!} \delta(p_1 + p_2 + p_3)a_{p_1}^\dagger a_{p_2}^\dagger a_{p_3}^\dagger \right. \\ & \left((u_t(p_1)u_t(p_2)v_t(p_3)\phi_t + v_t(p_1)v_t(p_2)u_t(p_3)\bar{\phi}_t)(\hat{v}(p_1) + \hat{v}(p_2)) \right. \\ & + (v_t(p_1)u_t(p_2)u_t(p_3)\phi_t + u_t(p_1)v_t(p_2)v_t(p_3)\bar{\phi}_t)(\hat{v}(p_2) + \hat{v}(p_3)) \\ & + (u_t(p_1)v_t(p_2)u_t(p_3)\phi_t + v_t(p_1)u_t(p_2)v_t(p_3)\bar{\phi}_t)(\hat{v}(p_1) + \hat{v}(p_3)) \Big) \\ & + \frac{1}{2!} \delta(p_1 + p_2 - p_3)a_{p_1}^\dagger a_{p_2}^\dagger a_{p_3} \\ & \left((u_t(p_1)u_t(p_2)u_t(p_3)\phi_t + v_t(p_1)v_t(p_2)\bar{v}_t(p_3)\bar{\phi}_t)(\hat{v}(p_1) + \hat{v}(p_2)) \right. \\ & + (v_t(p_1)u_t(p_2)\bar{v}_t(p_3)\phi_t + u_t(p_1)v_t(p_2)u_t(p_3)\bar{\phi}_t)(\hat{v}(p_2) + \hat{v}(p_3)) \\ & + (u_t(p_1)v_t(p_2)\bar{v}_t(p_3)\phi_t + v_t(p_1)u_t(p_2)u_t(p_3)\bar{\phi}_t)(\hat{v}(p_1) + \hat{v}(p_3)) \Big) \\ & \left. + h.c. \right) . \quad (\text{A.15}) \end{aligned}$$

We are left with calculating

$$\begin{aligned} & \frac{\lambda}{2N} \int d\mathbf{p}_4 \delta(p_1 + p_2 - p_3 - p_4)\hat{v}(p_1 - p_3)\mathcal{T}^\dagger[k_t]a_{p_1}^\dagger a_{p_2}^\dagger a_{p_3} a_{p_4} \mathcal{T}[k_t] \\ & = \frac{\lambda}{2N} \int d\mathbf{p}_4 \delta(p_1 + p_2 - p_3 - p_4)\hat{v}(p_1 - p_3)(u_t(p_1)a_{p_1}^\dagger + \bar{v}_t(p_1)a_{-p_1}) \\ & (u_t(p_2)a_{p_2}^\dagger + \bar{v}_t(p_2)a_{-p_2})(u_t(p_3)a_{p_3} + v_t(p_3)a_{-p_3}^\dagger) \\ & (u_t(p_4)a_{p_4} + v_t(p_4)a_{-p_4}^\dagger) . \quad (\text{A.16}) \end{aligned}$$

After normal-ordering, the scalar terms come from pairs of annihilation operators a left to a pair of creation operators a^\dagger in (A.16), and they are given by

$$\begin{aligned} & \frac{\lambda}{2N} \int d\mathbf{p}_4 \delta(p_1 + p_2 - p_3 - p_4) \hat{v}(p_1 - p_3) \\ & \quad \left(\delta(p_1 + p_2) \delta(p_3 + p_4) \bar{v}_t(p_1) u_t(p_2) u_t(p_3) v_t(p_4) \right. \\ & \quad + \left. (\delta(p_1 - p_4) \delta(p_2 - p_3) + \delta(p_1 - p_3) \delta(p_2 - p_4)) \right. \\ & \quad \left. \bar{v}_t(p_1) \bar{v}_t(p_2) v_t(p_3) v_t(p_4) \right) \\ & = \frac{\lambda |\Lambda|}{2N} \int dp dq \left(\bar{\sigma}_t(p) \hat{v}(p - q) \sigma_t(q) + \gamma_t(p) (\hat{v}(p - q) + \hat{v}(0)) \gamma_t(q) \right). \end{aligned} \quad (\text{A.17})$$

Next, we determine all terms proportional to aa in the normal-ordering of (A.16), that come from having a creation operator a left of an annihilation operator a^\dagger , are given by

$$\begin{aligned} & \frac{\lambda}{2N} \int d\mathbf{p}_4 \delta(p_1 + p_2 - p_3 - p_4) \hat{v}(p_1 - p_3) \\ & \quad \left(\bar{v}_t(p_1) u_t(p_2) u_t(p_3) u_t(p_4) \delta(p_1 + p_2) a_{p_3} a_{p_4} + \bar{v}_t(p_1) \bar{v}_t(p_2) u_t(p_3) v_t(p_4) \right. \\ & \quad \left(\delta(p_3 + p_4) a_{p_1} a_{p_2} + \delta(p_2 - p_4) a_{p_1} a_{p_3} + \delta(p_1 - p_4) a_{p_2} a_{p_3} \right) \\ & \quad + \bar{v}_t(p_1) \bar{v}_t(p_2) v_t(p_3) u_t(p_4) \left(\delta(p_1 - p_3) a_{p_2} a_{p_4} + \delta(p_2 - p_3) a_{p_1} a_{p_4} \right) \Big) \\ & = \frac{\lambda}{2N} \int dp dq \left(\bar{\sigma}_t(q) \hat{v}(q - p) (1 + \gamma_t(p)) + \sigma_t(q) \hat{v}(q - p) \frac{\bar{\sigma}_t(p)^2}{1 + \gamma_t(p)} \right. \\ & \quad \left. + 2\gamma_t(q) (\hat{v}(q - p) + \hat{v}(0)) \bar{\sigma}_t(p) \right) a_p a_{-p}. \end{aligned} \quad (\text{A.18})$$

In a similar fashion, we determine all terms involving $a^\dagger a$ in (A.16) after normal-ordering to be

$$\begin{aligned} & \frac{\lambda}{2N} \int d\mathbf{p}_4 \delta(p_1 + p_2 - p_3 - p_4) \hat{v}(p_1 - p_3) \\ & \quad \left(u_t(p_1) \bar{v}_t(p_2) v_t(p_3) u_t(p_4) \delta(p_2 - p_3) a_{p_1}^\dagger a_{p_4} + u_t(p_1) \bar{v}_t(p_2) u_t(p_3) v_t(p_4) \right. \\ & \quad \left(\delta(p_3 + p_4) a_{p_1}^\dagger a_{-p_2} + \delta(p_2 - p_4) a_{p_1}^\dagger a_{p_3} \right) + \bar{v}_t(p_1) u_t(p_2) v_t(p_3) u_t(p_4) \\ & \quad \left(\delta(p_1 + p_2) a_{-p_3}^\dagger a_{p_4} + \delta(p_1 - p_3) a_{p_2}^\dagger a_{p_4} \right) + \bar{v}_t(p_1) u_t(p_2) u_t(p_3) v_t(p_4) \\ & \quad \left(\delta(p_1 + p_2) a_{-p_4}^\dagger a_{p_3} + \delta(p_1 - p_4) a_{p_2}^\dagger a_{p_3} + \delta(p_3 + p_4) a_{-p_2}^\dagger a_{p_1} \right) \\ & \quad + \bar{v}_t(p_1) \bar{v}_t(p_2) v_t(p_3) v_t(p_4) \left(\delta(p_1 - p_4) a_{-p_3}^\dagger a_{-p_2} + \delta(p_1 - p_3) a_{-p_4}^\dagger a_{-p_2} \right. \\ & \quad \left. + \delta(p_2 - p_3) a_{-p_4}^\dagger a_{-p_1} + \delta(p_2 - p_4) a_{-p_3}^\dagger a_{-p_2} \right) \\ & = \frac{\lambda}{N} \int dp dq \left(\gamma_t(q) (\hat{v}(q - p) + \hat{v}(0)) (1 + 2\gamma_t(p)) \right. \\ & \quad \left. + \sigma_t(q) \hat{v}(q - p) \bar{\sigma}_t(p) + \bar{\sigma}_t(q) \hat{v}(q - p) \sigma_t(p) \right) a_p^\dagger a_p. \end{aligned} \quad (\text{A.19})$$

Here recall that we need to contract each a with each a^\dagger to the right of it. Finally, the quartic, normal-ordered terms coming from (A.16) are given by

$$\begin{aligned} & \frac{\lambda}{4N} \int d\mathbf{p}_4 \delta(p_1 + p_2 - p_3 - p_4) (\hat{v}(p_1 - p_3) + \hat{v}(p_1 - p_4)) \\ & \left(u_t(p_1)u_t(p_2)v_t(p_3)v_t(p_4)a_{p_1}^\dagger a_{p_2}^\dagger a_{-p_3}^\dagger a_{-p_4}^\dagger \right. \\ & + 2u_t(p_1)u_t(p_2)v_t(p_3)u_t(p_4)a_{p_1}^\dagger a_{p_2}^\dagger a_{-p_3}^\dagger a_{p_4} \\ & + 2u_t(p_1)\bar{v}_t(p_2)v_t(p_3)v_t(p_4)a_{p_1}^\dagger a_{-p_3}^\dagger a_{-p_4}^\dagger a_{-p_2} \\ & + h.c. \\ & + (u_t(p_1)u_t(p_2)u_t(p_3)u_t(p_4) + v_t(p_1)v_t(p_2)\bar{v}_t(p_3)\bar{v}_t(p_4))a_{p_1}^\dagger a_{p_2}^\dagger a_{p_3} a_{p_4} \\ & \left. + 4u_t(p_1)\bar{v}_t(p_2)v_t(p_3)u_t(p_4)a_{p_1}^\dagger a_{-p_3}^\dagger a_{-p_2} a_{p_4} \right), \end{aligned}$$

where the hermitian conjugate refers to the preceding terms. Analogously to (A.14), we already symmetrized the expression w.r.t. $p_1 \leftrightarrow p_2$ and $p_3 \leftrightarrow p_4$. Relabeling momenta again, we have

$$\begin{aligned} & \frac{\lambda}{4N} \int d\mathbf{p}_4 \left[\delta(p_1 + p_2 + p_3 + p_4)a_{p_1}^\dagger a_{p_2}^\dagger a_{p_3}^\dagger a_{p_4}^\dagger \right. \\ & u_t(p_1)u_t(p_2)v_t(p_3)v_t(p_4)(\hat{v}(p_1 + p_3) + \hat{v}(p_2 + p_3)) \\ & + 2\delta(p_1 + p_2 + p_3 - p_4)a_{p_1}^\dagger a_{p_2}^\dagger a_{p_3}^\dagger a_{p_4} \\ & \left(u_t(p_1)u_t(p_2)v_t(p_3)u_t(p_4)(\hat{v}(p_1 + p_3) + \hat{v}(p_2 + p_3)) \right. \\ & + u_t(p_1)v_t(p_2)v_t(p_3)\bar{v}_t(p_4)(\hat{v}(p_1 + p_2) + \hat{v}(p_1 + p_3)) \\ & + h.c. \\ & + \delta(p_1 + p_2 - p_3 - p_4)a_{p_1}^\dagger a_{p_2}^\dagger a_{p_3} a_{p_4} \\ & \left((u_t(p_1)u_t(p_2)u_t(p_3)u_t(p_4) + v_t(p_1)v_t(p_2)\bar{v}_t(p_3)\bar{v}_t(p_4))(\hat{v}(p_1 - p_3) + \hat{v}(p_2 - p_3)) \right. \\ & \left. + 4u_t(p_1)v_t(p_2)\bar{v}_t(p_3)u_t(p_4)(\hat{v}(p_1 + p_2) + \hat{v}(p_2 - p_3)) \right). \end{aligned} \tag{A.20}$$

Let

$$a^{(\sigma)}(\mathbf{q}_\ell) := \prod_{j=1}^{\ell} a^{(\sigma)}(q_j),$$

see also (D.1). Symmetrizing the coefficients of $a^\dagger(\mathbf{p}_n)a(\mathbf{k}_m)$ in (A.20) w.r.t. \mathbf{p}_n and \mathbf{k}_m , we find that

$$\begin{aligned} & \frac{\lambda}{N} \int d\mathbf{p}_4 \left[\frac{1}{4!} \delta(p_1 + p_2 + p_3 + p_4)a_{p_1}^\dagger a_{p_2}^\dagger a_{p_3}^\dagger a_{p_4}^\dagger \right. \\ & \left((u_t(p_1)u_t(p_2)v_t(p_3)v_t(p_4) + v_t(p_1)v_t(p_2)u_t(p_3)u_t(p_4))(\hat{v}(p_1 + p_3) + \hat{v}(p_2 + p_3)) \right. \\ & + (u_t(p_1)v_t(p_2)u_t(p_3)v_t(p_4) + v_t(p_1)u_t(p_2)v_t(p_3)u_t(p_4))(\hat{v}(p_1 + p_2) + \hat{v}(p_2 + p_3)) \\ & \left. + (u_t(p_1)v_t(p_2)v_t(p_3)u_t(p_4) + v_t(p_1)u_t(p_2)u_t(p_3)v_t(p_4))(\hat{v}(p_1 + p_2) + \hat{v}(p_1 + p_3)) \right) \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{3!} \delta(p_1 + p_2 + p_3 - p_4) a_{p_1}^\dagger a_{p_2}^\dagger a_{p_3}^\dagger a_{p_4} \\
& \left((u_t(p_1)u_t(p_2)v_t(p_3)u_t(p_4) + v_t(p_1)v_t(p_2)u_t(p_3)\bar{v}_t(p_4)) (\hat{v}(p_1 + p_3) + \hat{v}(p_2 + p_3)) \right. \\
& + (u_t(p_1)v_t(p_2)u_t(p_3)u_t(p_4) + v_t(p_1)u_t(p_2)v_t(p_3)\bar{v}_t(p_4)) (\hat{v}(p_1 + p_2) + \hat{v}(p_2 + p_3)) \\
& + (v_t(p_1)u_t(p_2)u_t(p_3)u_t(p_4) + u_t(p_1)v_t(p_2)v_t(p_3)\bar{v}_t(p_4)) (\hat{v}(p_1 + p_2) + \hat{v}(p_1 + p_3)) \\
& + h.c. \\
& + \frac{1}{(2!)^2} \delta(p_1 + p_2 - p_3 - p_4) a_{p_1}^\dagger a_{p_2}^\dagger a_{p_3} a_{p_4} \\
& \left. \left((u_t(p_1)u_t(p_2)u_t(p_3)u_t(p_4) + v_t(p_1)v_t(p_2)\bar{v}_t(p_3)\bar{v}_t(p_4)) (\hat{v}(p_1 - p_3) + \hat{v}(p_2 - p_3)) \right. \right. \\
& + (u_t(p_1)v_t(p_2)\bar{v}_t(p_3)u_t(p_4) + v_t(p_1)u_t(p_2)u_t(p_3)\bar{v}_t(p_4)) (\hat{v}(p_1 + p_2) + \hat{v}(p_2 - p_3)) \\
& \left. \left. + (v_t(p_1)u_t(p_2)\bar{v}_t(p_3)u_t(p_4) + u_t(p_1)v_t(p_2)u_t(p_3)\bar{v}_t(p_4)) (\hat{v}(p_1 + p_2) + \hat{v}(p_1 - p_3)) \right) \right). \tag{A.21}
\end{aligned}$$

With these calculations, we are ready to identify each of the terms in $\mathcal{H}_{\text{fluc}}$, in ascending order of involved number of annihilation and creation operators. Collecting (A.1), (A.7), (A.8), (A.9), (A.11), and (A.17), we obtain that we need to choose

$$\begin{aligned}
\partial_t S_t = & \\
| \Lambda | \left[\frac{N | \Lambda | \lambda | \phi_t |^4 \hat{v}(0)}{2} - \frac{N | \Lambda |}{2} \operatorname{Re}(\bar{\phi}_t i \partial_t \phi_t) - \frac{1}{2} \int dp \frac{\operatorname{Re}(\bar{\sigma}_t(p) i \partial_t \sigma_t(p))}{1 + \gamma_t(p)} \right. & \\
+ \int dp \left(\frac{\lambda}{2N} (\bar{\sigma}_t * \hat{v}(p) \sigma_t(p) + \gamma_t * (\hat{v} + \hat{v}(0))(p) \gamma_t(p)) \right. & \\
(E(p) + \lambda | \Lambda | (\hat{v}(p) + \hat{v}(0)) | \phi_t |^2) \gamma_t(p) + \frac{\lambda | \Lambda |}{2} \hat{v}(p) (\bar{\sigma}_t(p) \phi_t^2 + \sigma_t(p) \bar{\phi}_t^2) \Big) &
\end{aligned}$$

in order to eliminate the scalar contributions in $\mathcal{H}_{\text{fluc}}(t)$.

Using the shifted expectations $\Gamma^{(1)}$ and $\Sigma^{(1)}$, see (3.10), (3.11), we can rewrite $\partial_t S_t$ as

$$\begin{aligned}
\partial_t S_t = & \\
| \Lambda | \int dp \left[E(p) \gamma_t(p) + \frac{\lambda}{2N} \Gamma_t^{(1)} * (\hat{v} + \hat{v}(0))(p) \Gamma_t^{(1)}(p) \right. & \\
+ \frac{\lambda}{2N} \bar{\Sigma}_t^{(1)} * \hat{v}(p) \Sigma_t^{(1)}(p) - N | \Lambda | \lambda | \phi_t |^4 \hat{v}(0) & \\
\left. - \frac{N | \Lambda | \delta(p) \operatorname{Re}(\bar{\phi}_t i \partial_t \phi_t)}{2} - \frac{\operatorname{Re}(\bar{\sigma}_t(p) i \partial_t \sigma_t(p))}{2(1 + \gamma_t(p))} \right]. &
\end{aligned}$$

Similarly, collecting (2.5), (A.8), (A.10), (A.13), we have, after conjugating with $\mathcal{U}_{\text{Bog}}(t)$, that

$$\begin{aligned}
\mathcal{H}_{\text{BEC}}(t) = & \\
\sqrt{N | \Lambda |} \left[- (i \partial_t \phi_t u_t(0) + i \partial_t \bar{\phi}_t v_t(0)) \right. & \\
+ \lambda | \Lambda | | \phi_t |^2 \hat{v}(0) (\phi_t u_t(0) + \bar{\phi}_t v_t(0)) &
\end{aligned}$$

$$\begin{aligned}
& + \frac{\lambda}{N} u_t(0) \left(\int dp \hat{v}(p) \sigma_t(p) \bar{\phi}_t + \int dp (\hat{v}(p) + \hat{v}(0)) \gamma_t(p) \phi_t \right) \\
& + v_t(0) \left(\int dp \hat{v}(p) \bar{\sigma}_t(p) \phi_t + \int dp (\hat{v}(p) + \hat{v}(0)) \gamma_t(p) \bar{\phi}_t \right) e^{i \int_0^t ds \Omega_\tau(s)} a_0^\dagger \\
& + h.c. \\
= & \sqrt{N|\Lambda|} \left[u_t(0) \left(-i \partial_t \phi_t + \lambda |\Lambda| \hat{v}(0) |\phi_t|^2 \phi_t + \frac{\lambda}{N} (\sigma_t * \hat{v})(0) \bar{\phi}_t \right. \right. \\
& + \frac{\lambda}{N} (\gamma_t * (\hat{v} + \hat{v}(0))(0) \phi_t) + v_t(0) \left(-i \partial_t \bar{\phi}_t + \lambda |\Lambda| \hat{v}(0) |\phi_t|^2 \bar{\phi}_t \right. \\
& \left. \left. + \frac{\lambda}{N} (\bar{\sigma}_t * \hat{v})(0) \phi_t + \frac{\lambda}{N} (\gamma_t * (\hat{v} + \hat{v}(0))(0) \bar{\phi}_t) \right) e^{i \int_0^t d\tau \Omega_\tau(\tau)} a_0^\dagger + h.c. \right]
\end{aligned}$$

In order to give an expression for $\mathcal{H}_{\text{HFB}}(t)$, we decompose it into

$$\mathcal{H}_{\text{HFB}}(t) = \mathcal{H}_{\text{HFB}}^{(\text{d})}(t) + \mathcal{H}_{\text{HFB}}^{(\text{cor})}(t),$$

where $\mathcal{H}_{\text{HFB}}^{(\text{d})}(t)$ refers to the diagonal part, involving terms proportional to $a^\dagger a$, and $\mathcal{H}_{\text{HFB}}^{(\text{cor})}(t)$ to the off-diagonal part, involving terms proportional to aa and $a^\dagger a^\dagger$. Then (A.1), (A.7), (A.11), (A.19), after conjugation with $\mathcal{U}_{\text{Bog}}(t)$ using (2.5), imply

$$\begin{aligned}
\mathcal{H}_{\text{HFB}}^{(\text{d})}(t) = & \\
& \int dp \left[-\Omega_t(p) - \frac{\text{Re}(\bar{\sigma}_t(p) i \partial_t \sigma_t(p))}{1 + \gamma_t(p)} \right. \\
& + \left(E(p) + \frac{\lambda}{N} ((\gamma_t + N|\Lambda| |\phi_t|^2 \delta) * (\hat{v} + \hat{v}(0)))(p) \right) (1 + 2\gamma_t(p)) \\
& \left. + \frac{2\lambda}{N} \text{Re} \left(((\bar{\sigma}_t + N|\Lambda| |\bar{\phi}_t|^2 \delta) * \hat{v})(p) \sigma_t(p) \right) \right] a_p^\dagger a_p.
\end{aligned}$$

Similarly, we obtain, using, in addition (A.18), that

$$\begin{aligned}
\mathcal{H}_{\text{HFB}}^{(\text{cor})}(t) = & \\
& \int dp \left[-\frac{i \partial_t \sigma_t(p)}{2} + \frac{\sigma_t(p) i \partial_t \gamma_t(p)}{2(1 + \gamma_t(p))} \right. \\
& + \left(E(p) + \frac{\lambda}{N} ((\gamma_t + N|\Lambda| |\phi_t|^2 \delta) * (\hat{v} + \hat{v}(0)))(p) \right) \sigma_t(p) \\
& + \frac{\lambda}{2N} \left(((\sigma_t + N|\Lambda| |\phi_t|^2 \delta) * \hat{v})(p) (1 + \gamma_t(p)) \right. \\
& \left. + ((\bar{\sigma}_t + N|\Lambda| |\bar{\phi}_t|^2 \delta) * \hat{v})(p) \frac{\sigma_t(p)^2}{1 + \gamma_t(p)} \right] e^{2i \int_0^t d\tau \Omega_\tau(p)} a_p^\dagger a_{-p}^\dagger \\
& + h.c.
\end{aligned}$$

Conjugating (A.15) with $\mathcal{U}_{\text{Bog}}(t)$ and using the notation (2.18)–(2.19), (2.5) implies

$$\begin{aligned}
\mathcal{H}_{\text{cub}}(t) = & \\
& \frac{\lambda}{\sqrt{N}} \int d\mathbf{p}_3 \left[\delta(p_1 + p_2 + p_3) e^{i \int_0^t d\tau (\Omega_\tau(p_1) + \Omega_\tau(p_2) + \Omega_\tau(p_3))} \right.
\end{aligned}$$

$$\begin{aligned}
& \frac{1}{3!} \mathbf{w}_t^{(3,0)}(\mathbf{p}_3) a_{p_1}^\dagger a_{p_2}^\dagger a_{p_3}^\dagger \\
& + \delta(p_1 + p_2 - p_3) e^{i \int_0^t d\tau (\Omega_\tau(p_1) + \Omega_\tau(p_2) - \Omega_\tau(p_3))} \\
& \frac{1}{2!} \mathbf{w}_t^{(2,1)}(\mathbf{p}_3) a_{p_1}^\dagger a_{p_2}^\dagger a_{p_3} \\
& + h.c. \Big).
\end{aligned}$$

Similarly, conjugating (A.21) with $\mathcal{U}_{\text{Bog}}(t)$ using (2.5) and the notation (2.20)–(2.21), we obtain that

$$\begin{aligned}
\mathcal{H}_{\text{quart}}(t) = & \\
& \frac{\lambda}{N} \int d\mathbf{p}_4 \left[\delta(p_1 + p_2 + p_3 + p_4) e^{i \int_0^t d\tau (\Omega_\tau(p_1) + \Omega_\tau(p_2) + \Omega_\tau(p_3) + \Omega_\tau(p_4))} \right. \\
& \frac{1}{4!} \mathbf{w}_t^{(4,0)}(\mathbf{p}_4) a_{p_1}^\dagger a_{p_2}^\dagger a_{p_3}^\dagger a_{p_4}^\dagger \\
& + \delta(p_1 + p_2 + p_3 - p_4) e^{i \int_0^t d\tau (\Omega_\tau(p_1) + \Omega_\tau(p_2) + \Omega_\tau(p_3) - \Omega_\tau(p_4))} \\
& \frac{1}{3!} \mathbf{w}_t^{(3,1)}(\mathbf{p}_4) a_{p_1}^\dagger a_{p_2}^\dagger a_{p_3}^\dagger a_{p_4} \\
& + h.c. \\
& + \delta(p_1 + p_2 - p_3 - p_4) e^{i \int_0^t d\tau (\Omega_\tau(p_1) + \Omega_\tau(p_2) - \Omega_\tau(p_3) - \Omega_\tau(p_4))} \\
& \left. \frac{1}{(2!)^2} \mathbf{w}_t^{(2,2)}(\mathbf{p}_4) a_{p_1}^\dagger a_{p_2}^\dagger a_{p_3} a_{p_4} \right].
\end{aligned}$$

This finishes the proof. \square

Lemma A.3 (Contracted vertices). *Let \mathcal{H}_{cub} and $\mathcal{H}_{\text{quart}}$ be defined as in Lemma A.2. Then we have*

$$\begin{aligned}
\overline{\mathcal{H}}_{\text{cub}}(t) = & \frac{\lambda \sqrt{|\Lambda|}}{\sqrt{N}} e^{i \int_0^t d\tau \Omega_\tau(0)} a_0^\dagger \int dk \left[u_t(0) \left((1 + 2\gamma_t(k)) f_0(k) (\hat{v}(k) + \hat{v}(0)) \phi_t \right. \right. \\
& + 2f_0(k) \sigma_t(k) \hat{v}(k) \bar{\phi}_t \Big) + v_t(0) \left((1 + 2\gamma_t(k)) f_0(k) (\hat{v}(k) + \hat{v}(0)) \bar{\phi}_t \right. \\
& \left. \left. + 2f_0(k) \bar{\sigma}_t(k) \hat{v}(k) \phi_t \right) \right] + h.c.,
\end{aligned}$$

and

$$\begin{aligned}
\overline{\mathcal{H}}_{\text{quart}}(t) = & \frac{\lambda}{N} \int dp \left[\left((f_0^{(+)} \sigma_t) * \hat{v}(p) (1 + \gamma_t(p)) + \frac{(f_0^{(+)} \bar{\sigma}_t) * \hat{v}(p) \sigma_t(p)}{1 + \gamma_t(p)} \right. \right. \\
& + ((1 + 2\gamma_t) f_0^{(+)}) * (\hat{v} + \hat{v}(0))(p) \sigma_t(p) \Big) e^{2i \int_0^t d\tau \Omega_\tau(p)} a_p^\dagger a_{-p}^\dagger + h.c. \\
& + \left. \left. \left(((1 + 2\gamma_t) f_0^{(+)}) * (\hat{v} + \hat{v}(0))(p) (1 + 2\gamma_t(p)) \right. \right. \right. \\
& \left. \left. \left. + 4 \operatorname{Re} ((f_0^{(+)} \sigma_t) * \hat{v}(p) \bar{\sigma}_t(p)) \right) a_p^\dagger a_p \right].
\end{aligned}$$

Proof. Using symmetry of $\mathbf{w}_t^{(2,1)}(p_1, p_2, p_3)$ in $(p_1 \leftrightarrow p_2)$, see Lemma A.2, we obtain that

$$\mathcal{H}_{\text{cub}}^{\square}(t) = \frac{\lambda}{\sqrt{N}} \int dk \mathbf{w}_t^{(2,1)}(k, 0, k) f_0(k) e^{i \int_0^t d\tau \Omega_\tau(0)} a_0^\dagger + \text{h.c.},$$

where

$$\begin{aligned} \mathbf{w}_t^{(2,1)}(\mathbf{p}_3) &= \\ &\sqrt{|\Lambda|} \left((u_t(p_1)u_t(p_2)u_t(p_3)\phi_t + v_t(p_1)v_t(p_2)\bar{v}_t(p_3)\bar{\phi}_t)(\hat{v}(p_1) + \hat{v}(p_2)) \right. \\ &+ (v_t(p_1)u_t(p_2)\bar{v}_t(p_3)\phi_t + u_t(p_1)v_t(p_2)u_t(p_3)\bar{\phi}_t)(\hat{v}(p_2) + \hat{v}(p_3)) \\ &\left. + (u_t(p_1)v_t(p_2)\bar{v}_t(p_3)\phi_t + v_t(p_1)u_t(p_2)u_t(p_3)\bar{\phi}_t)(\hat{v}(p_1) + \hat{v}(p_3)) \right), \end{aligned}$$

Using the facts that $u_t = \sqrt{1 + \gamma_t}$ and that $v_t = \frac{\sigma_t}{\sqrt{1 + \gamma_t}}$, we thus find that

$$\begin{aligned} \mathcal{H}_{\text{cub}}^{\square}(t) &= \frac{\lambda\sqrt{|\Lambda|}}{\sqrt{N}} \int dk f_0(k) e^{i \int_0^t d\tau \Omega_\tau(0)} a_0^\dagger \\ &\left((u_t(0)(1 + \gamma_t(k))\phi_t + v_t(0)\gamma_t(k)\bar{\phi}_t)(\hat{v}(k) + \hat{v}(0)) \right. \\ &+ (u_t(0)\gamma_t(k)\phi_t + v_t(0)(1 + \gamma_t(k))\bar{\phi}_t)(\hat{v}(k) + \hat{v}(0)) \\ &\left. + (v_t(0)\bar{v}_t(k)\phi_t + u_t(0)\sigma_t(k)\bar{\phi}_t)2\hat{v}(k) \right) + \text{h.c.} \\ &= \frac{\lambda\sqrt{|\Lambda|}}{\sqrt{N}} \int dk f_0(k) e^{i \int_0^t d\tau \Omega_\tau(0)} a_0^\dagger \\ &\left((\hat{v}(k) + \hat{v}(0))(1 + 2\gamma_t(k))(u_t(0)\phi_t + v_t(0)\bar{\phi}_t) \right. \\ &\left. + 2\hat{v}(k)(v_t(0)\bar{v}_t(k)\phi_t + u_t(0)\sigma_t(k)\bar{\phi}_t) \right) + \text{h.c.}, \end{aligned}$$

where we also used the evenness of σ_t and γ_t . Sorting the terms by the coefficients $u_t(0)$ and $v_t(0)$, we obtain

$$\begin{aligned} \mathcal{H}_{\text{cub}}^{\square}(t) &= \frac{\lambda\sqrt{|\Lambda|}}{\sqrt{N}} e^{i \int_0^t d\tau \Omega_\tau(0)} a_0^\dagger \int dk \left[u_t(0) \left((1 + 2\gamma_t(k))f_0(k)(\hat{v}(k) + \hat{v}(0))\phi_t \right. \right. \\ &+ 2f_0(k)\sigma_t(k)\hat{v}(k)\bar{\phi}_t \left. \right) + v_t(0) \left((1 + 2\gamma_t(k))f_0(k)(\hat{v}(k) + \hat{v}(0))\bar{\phi}_t \right. \\ &\left. \left. + 2f_0(k)\bar{v}_t(k)\hat{v}(k)\phi_t \right) \right] + \text{h.c..} \end{aligned}$$

Similarly, the symmetries of $\mathbf{w}_t^{(3,1)}(\mathbf{p}_4)$ in $(p_1 \leftrightarrow p_2 \leftrightarrow p_3)$, and of $\mathbf{w}_t^{(2,2)}(\mathbf{p}_4)$ in $(p_1 \leftrightarrow p_2)$ and $(p_3 \leftrightarrow p_4)$, see Lemma A.2, imply

$$\begin{aligned} \mathcal{H}_{\text{quart}}^{\square}(t) &= \frac{\lambda}{N} \int dp dk f_0(k) \left(\frac{\mathbf{w}_t^{(3,1)}(p, -p, k, k)}{2} e^{2i \int_0^t d\tau \Omega_\tau(p)} a_p^\dagger a_{-p}^\dagger + \text{h.c.} \right. \\ &\left. + \mathbf{w}_t^{(2,2)}(p, k, k, p) a_p^\dagger a_p \right), \end{aligned}$$

where

$$\mathbf{w}_t^{(3,1)}(\mathbf{p}_4) :=$$

$$\begin{aligned}
& (u_t(p_1)u_t(p_2)v_t(p_3)u_t(p_4) + v_t(p_1)v_t(p_2)u_t(p_3)\bar{v}_t(p_4))(\hat{v}(p_1+p_3) + \hat{v}(p_2+p_3)) \\
& + (u_t(p_1)v_t(p_2)u_t(p_3)u_t(p_4) + v_t(p_1)u_t(p_2)v_t(p_3)\bar{v}_t(p_4))(\hat{v}(p_1+p_2) + \hat{v}(p_2+p_3)) \\
& + (v_t(p_1)u_t(p_2)u_t(p_3)u_t(p_4) + u_t(p_1)v_t(p_2)v_t(p_3)\bar{v}_t(p_4))(\hat{v}(p_1+p_2) + \hat{v}(p_1+p_3)), \\
\mathbf{w}_t^{(2,2)}(\mathbf{p}_4) := & \\
& (u_t(p_1)u_t(p_2)u_t(p_3)u_t(p_4) + v_t(p_1)v_t(p_2)\bar{v}_t(p_3)\bar{v}_t(p_4))(\hat{v}(p_1-p_3) + \hat{v}(p_2-p_3)) \\
& + (u_t(p_1)v_t(p_2)\bar{v}_t(p_3)u_t(p_4) + v_t(p_1)u_t(p_2)u_t(p_3)\bar{v}_t(p_4))(\hat{v}(p_1+p_2) + \hat{v}(p_2-p_3)) \\
& + (v_t(p_1)u_t(p_2)\bar{v}_t(p_3)u_t(p_4) + u_t(p_1)v_t(p_2)u_t(p_3)\bar{v}_t(p_4))(\hat{v}(p_1+p_2) + \hat{v}(p_1-p_3)).
\end{aligned}$$

Again using the relations between u_t , v_t and γ_t , σ_t , we obtain

$$\begin{aligned}
\mathcal{H}_{\text{quart}}^{\square}(t) = & \\
& \frac{\lambda}{N} \int dp dk f_0(k) \\
& \left[\left(((1 + \gamma_t(p))\sigma_t(k) + \frac{\sigma_t(p)\bar{\sigma}_t(k)}{1 + \gamma_t(p)}) 2\hat{v}(p - k) + 2(\sigma_t(p)(1 + \gamma_t(k)) \right. \right. \\
& \quad \left. \left. + \sigma_t(p)\gamma_t(k)) (\hat{v}(p - k) + \hat{v}(0)) \right) \frac{1}{2} e^{2i \int_0^t d\tau \Omega_\tau(p)} a_p^\dagger a_{-p}^\dagger + \text{h.c.} \right. \\
& \quad \left. + \left(((1 + \gamma_t(p))(1 + \gamma_t(k)) + \gamma_t(p)\gamma_t(k)) (\hat{v}(p - k) + \hat{v}(0)) \right. \right. \\
& \quad \left. \left. + ((1 + \gamma_t(p))\gamma_t(k) + \gamma_t(p)(1 + \gamma_t(k))) (\hat{v}(p - k) + \hat{v}(0)) \right. \right. \\
& \quad \left. \left. + (\sigma_t(p)\bar{\sigma}_t(k) + \bar{\sigma}_t(p)\sigma_t(k)) 2\hat{v}(p - k) \right) a_p^\dagger a_p \right] \\
= & \frac{\lambda}{N} \int dp dk f_0(k) \left[\left(((1 + \gamma_t(p))\sigma_t(k) + \frac{\sigma_t(p)\bar{\sigma}_t(k)}{1 + \gamma_t(p)}) \hat{v}(p - k) \right. \right. \\
& \quad \left. \left. + (\sigma_t(p)(1 + \gamma_t(k)) + \bar{\sigma}_t(p)\gamma_t(k)) \hat{v}(p - k) \right) e^{2i \int_0^t d\tau \Omega_\tau(p)} a_p^\dagger a_{-p}^\dagger + \text{h.c.} \right. \\
& \quad \left. + \left((1 + 2\gamma_t(p))(1 + 2\gamma_t(k)) (\hat{v}(p - k) + \hat{v}(0)) \right. \right. \\
& \quad \left. \left. + 4 \operatorname{Re}(\sigma_t(p)\bar{\sigma}_t(k)) \hat{v}(p - k) \right) a_p^\dagger a_p \right].
\end{aligned}$$

Abbreviating $f_0^{(+)} = \frac{1}{2}(f_0(p) + f_0(-p))$, see (2.14) and using evenness of σ , γ , \hat{v} , we can simplify the expression as

$$\begin{aligned}
\mathcal{H}_{\text{quart}}^{\square}(t) = & \frac{\lambda}{N} \int dp \left[\left((f_0^{(+)}\sigma_t) * \hat{v}(p)(1 + \gamma_t(p)) + \frac{(f_0^{(+)}\bar{\sigma}_t) * \hat{v}(p)\sigma_t(p)}{1 + \gamma_t(p)} \right. \right. \\
& \quad \left. \left. + ((1 + 2\gamma_t)f_0^{(+)}) * (\hat{v} + \hat{v}(0))(p)\sigma_t(p) \right) e^{2i \int_0^t d\tau \Omega_\tau(p)} a_p^\dagger a_{-p}^\dagger + \text{h.c.} \right. \\
& \quad \left. + \left(((1 + 2\gamma_t)f_0^{(+)}) * (\hat{v} + \hat{v}(0))(p)(1 + 2\gamma_t(p)) \right. \right. \\
& \quad \left. \left. + 4 \operatorname{Re}((f_0^{(+)}\sigma_t) * \hat{v}(p)\bar{\sigma}_t(p)) \right) a_p^\dagger a_p \right].
\end{aligned}$$

This concludes the proof. \square

Lemma A.4 (HFB equations). *Let*

$$\begin{aligned}\Gamma_t^{(1)}(p) &:= \gamma_t(p) + N|\Lambda||\phi_t|^2\delta(p), \\ \Sigma_t^{(1)}(p) &:= \sigma_t(p) + N|\Lambda|\phi_t^2\delta(p).\end{aligned}$$

Then

$$\begin{aligned}\frac{i\partial_t\sigma_t(p)}{2} - \frac{\sigma_t(p)i\partial_t\gamma_t(p)}{2(1+\gamma_t(p))} &= \\ \left(E(p) + \frac{\lambda}{N}\Gamma_t^{(1)} * (\hat{v} + \hat{v}(0))(p)\right)\sigma_t(p) \\ + \frac{\lambda}{2N}\left((\Sigma_t^{(1)} * \hat{v})(p)(1+\gamma_t(p)) + (\bar{\Sigma}_t^{(1)} * \hat{v})(p)\frac{\sigma_t(p)^2}{1+\gamma_t(p)}\right)\end{aligned}\quad (\text{A.22})$$

is equivalent to

$$\begin{aligned}i\partial_t\gamma_t &= \frac{\lambda}{N}\left[(\Sigma_t^{(1)} * \hat{v})\bar{\sigma}_t - (\bar{\Sigma}_t^{(1)} * \hat{v})\sigma_t\right], \\ i\partial_t\sigma_t &= 2\left(E + \frac{\lambda}{N}\Gamma_t^{(1)} * (\hat{v} + \hat{v}(0))\right)\sigma_t + \frac{\lambda}{N}(\Sigma_t^{(1)} * \hat{v})(1+2\gamma_t).\end{aligned}\quad (\text{A.23})$$

Proof. Multiplying (A.22) by $\bar{\sigma}_t(p)$, and taking imaginary parts, the l.h.s. of (A.22) reads

$$\begin{aligned}\frac{\text{Re}(\bar{\sigma}_t(p)\partial_t\sigma_t(p))}{2} - \frac{|\sigma_t(p)|^2\partial_t\gamma_t(p)}{2(1+\gamma_t(p))} \\ = \frac{1}{4}\partial_t(|\sigma_t(p)|^2 - \gamma_t(p)^2) \\ = \frac{1}{4}\partial_t\gamma_t(p),\end{aligned}$$

where we employed (2.13). Thus, we have that

$$\frac{1}{4}\partial_t\gamma_t = \frac{\lambda}{2N}\text{Im}\left((\Sigma_t^{(1)} * \hat{v})(1+\gamma_t)\bar{\sigma}_t + (\bar{\Sigma}_t^{(1)} * \hat{v})\frac{|\sigma_t|^2\sigma_t}{1+\gamma_t}\right),$$

which, using (2.13), is equivalent to

$$\begin{aligned}\partial_t\gamma_t(p) &= \frac{2\lambda}{N}\text{Im}\left((\Sigma_t^{(1)} * \hat{v})(1+\gamma_t)\bar{\sigma}_t + (\bar{\Sigma}_t^{(1)} * \hat{v})\gamma_t\sigma_t\right) \\ &= \frac{2\lambda}{N}\text{Im}\left((\Sigma_t^{(1)} * \hat{v})\bar{\sigma}_t\right).\end{aligned}$$

Substituting this identity into (A.22), we find that

$$\begin{aligned}\frac{i\partial_t\sigma_t(p)}{2} - \frac{\lambda}{N}\frac{\sigma_t(p)}{2(1+\gamma_t(p))}\left((\Sigma_t^{(1)} * \hat{v})\bar{\sigma}_t - (\bar{\Sigma}_t^{(1)} * \hat{v})\sigma_t\right) &= \\ \left(E(p) + \frac{\lambda}{N}\Gamma_t^{(1)} * (\hat{v} + \hat{v}(0))(p)\right)\sigma_t(p) \\ + \frac{\lambda}{2N}\left((\Sigma_t^{(1)} * \hat{v})(p)(1+\gamma_t(p)) + (\bar{\Sigma}_t^{(1)} * \hat{v})(p)\frac{\sigma_t(p)^2}{1+\gamma_t(p)}\right).\end{aligned}$$

Isolating $i\partial_t\sigma_t$ on the l.h.s. and using (2.13), we obtain that

$$i\partial_t\sigma_t = 2\left(E + \frac{\lambda}{N}\Gamma_t^{(1)} * (\hat{v} + \hat{v}(0))\right)\sigma_t + \frac{\lambda}{N}(\Sigma_t^{(1)} * \hat{v})(1 + 2\gamma_t).$$

This concludes the proof. \square

Lemma A.5 (Bogoliubov dispersion). *We have that $\mathcal{H}_{\text{HFB}}^{(\text{d})}(t) = 0$ if*

$$\begin{aligned} \Omega_t(p) &= \\ &\left(E(p) + \frac{\lambda}{N}((\gamma_t + N|\Lambda||\phi_t|^2\delta) * (\hat{v} + \hat{v}(0)))(p)\right)(1 + 2\gamma_t(p)) \\ &+ \frac{2\lambda}{N}\text{Re}\left(((\bar{\sigma}_t + N|\Lambda|\bar{\phi}_t^2\delta) * \hat{v})(p)\sigma_t(p)\right) - \frac{\text{Re}(\bar{\sigma}_t(p)i\partial_t\sigma_t(p))}{1 + \gamma_t(p)}. \end{aligned}$$

If σ_t satisfies (A.23), we have that

$$\Omega_t^{(1)} = E + \frac{\lambda}{N}\Gamma_t^{(1)} * (\hat{v} + \hat{v}(0)) + \frac{\lambda}{N}\frac{\text{Re}((\Sigma_t^{(1)} * \hat{v})\bar{\sigma}_t^{(1)})}{1 + \gamma_t^{(1)}}.$$

Proof. Lemma A.2 implies that $\mathcal{H}_{\text{HFB}}^{(\text{d})}(t) = 0$ if

$$\begin{aligned} \Omega_t(p) &= \\ &\left(E(p) + \frac{\lambda}{N}((\gamma_t + N|\Lambda||\phi_t|^2\delta) * (\hat{v} + \hat{v}(0)))(p)\right)(1 + 2\gamma_t(p)) \\ &+ \frac{2\lambda}{N}\text{Re}\left(((\bar{\sigma}_t + N|\Lambda|\bar{\phi}_t^2\delta) * \hat{v})(p)\sigma_t(p)\right) - \frac{\text{Re}(\bar{\sigma}_t(p)i\partial_t\sigma_t(p))}{1 + \gamma_t(p)}. \end{aligned}$$

Substituting (3.16) in this expression and recalling definitions (3.10) of $\Gamma^{(1)}$ and (3.11) of $\Sigma^{(1)}$, we obtain

$$\begin{aligned} \Omega_t^{(1)} &= -2\left(E + \frac{\lambda}{N}\Gamma_t^{(1)} * (\hat{v} + \hat{v}(0))\right)\frac{|\sigma_t^{(1)}|^2}{1 + \gamma_t^{(1)}} + \left[E + \frac{\lambda}{N}\Gamma_t^{(1)} * (\hat{v} + \hat{v}(0))\right](1 + 2\gamma_t^{(1)}) \\ &- \frac{\lambda}{N}\text{Re}\left((\Sigma_t^{(1)} * \hat{v})\bar{\sigma}_t^{(1)}\right)\frac{1 + 2\gamma_t}{1 + \gamma_t^{(1)}} + \frac{2\lambda}{N}\text{Re}\left((\Sigma_t^{(1)} * \hat{v})\bar{\sigma}_t^{(1)}\right). \end{aligned}$$

Recalling that $|\sigma_t|^2 = \gamma_t(1 + \gamma_t)$, see (2.13), this results in

$$\Omega_t^{(1)} = E + \frac{\lambda}{N}\Gamma_t^{(1)} * (\hat{v} + \hat{v}(0)) + \frac{\lambda}{N}\frac{\text{Re}((\Sigma_t^{(1)} * \hat{v})\bar{\sigma}_t^{(1)})}{1 + \gamma_t^{(1)}}.$$

This finishes the proof. \square

APPENDIX B. HFB SYSTEM ANALYSIS

Lemma B.1 (Generalized HFB equations). *Let (ϕ, γ, σ) satisfy either of the renormalized equations*

$$\begin{aligned} i\partial_t\phi_t &= \frac{\lambda}{N}\left(\left(\Gamma_t^{(j)} * (\hat{v} + \hat{v}(0))\right)(0)\phi_t + (\Sigma_t^{(j)} * \hat{v})(0)\bar{\phi}_t\right) \\ &- 2\lambda|\Lambda|\hat{v}(0)|\phi_t|^2\phi_t, \end{aligned} \tag{B.1}$$

$$i\partial_t \gamma_t = \frac{\lambda}{N} [(\Sigma_t^{(j)} * \hat{v}) \bar{\sigma}_t - (\overline{\Sigma_t^{(j)}} * \hat{v}) \sigma_t], \quad (\text{B.2})$$

$$i\partial_t \sigma_t = 2(E + \frac{\lambda}{N} \Gamma_t^{(j)} * (\hat{v} + \hat{v}(0))) \sigma_t + \frac{\lambda}{N} (\Sigma_t^{(j)} * \hat{v})(1 + 2\gamma_t) \quad (\text{B.3})$$

for $j = 0, 1$, where

$$\begin{aligned} \Gamma^{(j)} &= (1 + 2\delta_{1,j} f_0^{(+)}) \gamma + \delta_{1,j} f_0^{(+)} + N|\Lambda| |\phi|^2 \delta, \\ \Sigma^{(j)} &= (1 + 2\delta_{1,j} f_0^{(+)}) \sigma + N|\Lambda| \phi^2 \delta, \end{aligned}$$

see definitions (3.10), (3.11), (3.27) and (3.28). Then $(\phi, \Gamma^{(j)}, \Sigma^{(j)})$ satisfies

$$\begin{aligned} i\partial_t \phi_t &= \frac{\lambda}{N} \left((\Gamma_t^{(j)} * (\hat{v} + \hat{v}(0)))(0) \phi_t + (\Sigma_t^{(j)} * \hat{v})(0) \bar{\phi}_t \right) \\ &\quad - 2\lambda |\Lambda| \hat{v}(0) |\phi_t|^2 \phi_t, \\ i\partial_t \Gamma_t^{(j)} &= -\frac{2\lambda}{N} \operatorname{Im} \left((\overline{\Sigma_t^{(j)}} * \hat{v}) \Sigma_t^{(j)} \right), \\ i\partial_t \Sigma_t^{(j)} &= 2(E + \frac{\lambda}{N} \Gamma_t^{(j)} * (\hat{v} + \hat{v}(0))) \Sigma_t^{(j)} + \frac{\lambda}{N} (\Sigma_t^{(j)} * \hat{v})(1 + 2\Gamma_t^{(j)}) \\ &\quad - 4N\lambda |\Lambda|^2 \hat{v}(0) |\phi_t|^2 \phi_t^2 \end{aligned}$$

for $j = 0, 1$.

Proof. We start with

$$\begin{aligned} \partial_t |\phi_t|^2 &= 2 \operatorname{Re}(\bar{\phi}_t \partial_t \phi_t) \\ &= 2 \operatorname{Im}(\bar{\phi}_t i \partial_t \phi_t) \\ &= \frac{2\lambda}{N} \operatorname{Im} \left((\Sigma_t^{(j)} * \hat{v})(0) \bar{\phi}_t^2 \right). \end{aligned}$$

As a consequence, (B.2) implies that

$$\begin{aligned} \partial_t \Gamma_t^{(j)} &= (1 + 2\delta_{1,j} f_0^{(+)}) \partial_t \gamma_t + N|\Lambda| \partial_t |\phi_t|^2 \delta \\ &= \frac{2\lambda}{N} \operatorname{Im} \left((\Sigma_t^{(j)} * \hat{v}) ((1 + 2\delta_{1,j} f_0^{(+)}) \bar{\sigma}_t + N|\Lambda| \bar{\phi}_t^2 \delta) \right) \\ &= -\frac{2\lambda}{N} \operatorname{Im} \left((\overline{\Sigma_t^{(j)}} * \hat{v}) \Sigma_t^{(j)} \right). \end{aligned}$$

Similarly, we obtain, using (B.1) and (B.3), that

$$\begin{aligned} i\partial_t \Sigma_t^{(j)} &= (1 + 2\delta_{1,j} f_0^{(+)}) i\partial_t \sigma_t + 2N|\Lambda| |\phi_t| i\partial_t \phi_t \delta \\ &= (1 + 2\delta_{1,j} f_0^{(+)}) \left(2(E + \frac{\lambda}{N} \Gamma_t^{(j)} * (\hat{v} + \hat{v}(0))) \sigma_t \right. \\ &\quad \left. + \frac{\lambda}{N} (\Sigma_t^{(j)} * \hat{v})(1 + 2\gamma_t) \right) + 2\lambda |\Lambda| |\phi_t| \delta \left((\Gamma_t^{(j)} * (\hat{v} + \hat{v}(0))) \phi_t \right. \\ &\quad \left. + (\Sigma_t^{(j)} * \hat{v}) \bar{\phi}_t \right) - 4N\lambda |\Lambda|^2 \hat{v}(0) |\phi_t|^2 \phi_t^2 \\ &= 2(E + \frac{\lambda}{N} \Gamma_t^{(j)} * (\hat{v} + \hat{v}(0))) \Sigma_t^{(j)} + \frac{\lambda}{N} (\Sigma_t^{(j)} * \hat{v})(1 + 2\Gamma_t^{(j)}) \end{aligned}$$

$$- 4N\lambda|\Lambda|^2\hat{v}(0)|\phi|^2\phi^2\delta.$$

This concludes the proof. \square

Lemma B.2. *Assume that $\hat{v} \in L^1_{\sqrt{1+E}} \cap L^\infty(\Lambda^*)$. Recall the definition of the nonlinearity $\vec{\mathcal{J}} = (\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3)$ in the HFB equations, where*

$$\begin{aligned}\mathcal{J}_1(\phi, \Gamma, \Sigma) &= -i\left[\frac{\lambda}{N}\left((\Gamma * (\hat{v} + \hat{v}(0)))(0)\phi + (\Sigma * \hat{v})(0)\bar{\phi}\right)\right. \\ &\quad \left.- 2\lambda|\Lambda|\hat{v}(0)|\phi|^2\phi\right] \\ \mathcal{J}_2(\phi, \Gamma, \Sigma) &= -\frac{2\lambda}{N}\text{Im}\left((\bar{\Sigma} * \hat{v})\Sigma\right), \\ \mathcal{J}_3(\phi, \Gamma, \Sigma) &= -i\left[2\left(\frac{\lambda}{N}\Gamma * (\hat{v} + \hat{v}(0))\right)\Sigma + \frac{\lambda}{N}(\Sigma * \hat{v})(1 + 2\Gamma)\right. \\ &\quad \left.- 4N\lambda|\Lambda|^2\hat{v}(0)|\phi|^2\phi^2\delta\right],\end{aligned}$$

see (4.2)–(4.3). Then there is a constant $C > 0$ such that we have that

$$\begin{aligned}\|\vec{\mathcal{J}}(\phi_1, \Gamma_1, \Sigma_1) - \vec{\mathcal{J}}(\phi_2, \Gamma_2, \Sigma_2)\|_{\mathcal{X}^1} \\ \leq C\left(\frac{\lambda}{N}(\|\hat{v}\|_{L^1_{\sqrt{1+E}}} + \|\hat{v}\|_\infty)(\|(\phi_1, \Gamma_1, \Sigma_1)\|_{\mathcal{X}^1} + \|(\phi_2, \Gamma_2, \Sigma_2)\|_{\mathcal{X}^1})\right. \\ \left.\|(\phi_1 - \phi_2, \Gamma_1 - \Gamma_2, \Sigma_1 - \Sigma_2)\|_{\mathcal{X}^1} + \lambda|\Lambda|\hat{v}(0)(|\phi_1|^2 + |\phi_2|^2)\right. \\ \left.(1 + N|\Lambda|^{\frac{3}{2}}(|\phi_1| + |\phi_2|))|\phi_1 - \phi_2|\right)\end{aligned}$$

Proof. \mathcal{J}_1 satisfies

$$\begin{aligned}|\mathcal{J}_1(\phi_1, \Gamma_1, \Sigma_1) - \mathcal{J}_1(\phi_2, \Gamma_2, \Sigma_2)| \\ \leq \frac{\lambda}{N}\left(|\Gamma_1 - \Gamma_2| * (|\hat{v}| + |\hat{v}(0)|)(0)|\phi_1| + |\Gamma_2| * (|\hat{v}| + |\hat{v}(0)|)(0)|\phi_1 - \phi_2|\right. \\ \left.+ (|\Sigma_1 - \Sigma_2| * |\hat{v}|)(0)|\phi_1| + (|\Sigma_2| * |\hat{v}|)(0)|\phi_1 - \phi_2|\right) \\ + 2\lambda|\Lambda||\hat{v}(0)|||\phi_1|^2\phi_1 - |\phi_2|^2\phi_2|.\end{aligned}$$

Using the fact that

$$|\phi_1|^2 - |\phi_2|^2 = \text{Re}[(\bar{\phi}_1 - \bar{\phi}_2)(\phi_1 + \phi_2)], \quad (\text{B.4})$$

we thus obtain that

$$\begin{aligned}|\mathcal{J}_1(\phi_1, \Gamma_1, \Sigma_1) - \mathcal{J}_1(\phi_2, \Gamma_2, \Sigma_2)| \\ \leq \frac{\lambda}{N}\left(2\|\hat{v}\|_\infty(|\phi_1|\|\Gamma_1 - \Gamma_2\|_1 + \|\Gamma_2\|_1|\phi_1 - \phi_2|)\right. \\ \left.+ \|\hat{v}\|_\infty(\|\Sigma_1 - \Sigma_2\|_1|\phi_1| + \|\Sigma_2\|_1|\phi_1 - \phi_2|)\right) \\ + 4\lambda|\Lambda|\hat{v}(0)(|\phi_1|^2 + |\phi_2|^2)|\phi_1 - \phi_2|.\end{aligned}$$

Next, we find that

$$\|\mathcal{J}_2(\phi_1, \Gamma_1, \Sigma_1) - \mathcal{J}_2(\phi_2, \Gamma_2, \Sigma_2)\|_{L^1_{1+E}}$$

$$\begin{aligned}
&= \frac{2\lambda}{N} \int dp |(\bar{\Sigma}_1 - \bar{\Sigma}_2) * \hat{v}(p) \Sigma_1(p) + \bar{\Sigma}_2 * \hat{v}(p) (\Sigma_1 - \Sigma_2)(p)| (1 + E(p)) \\
&\leq \frac{2\lambda}{N} \|(\Sigma_1 - \Sigma_2) * \hat{v}\|_{L^2_{1+E}} (\|\Sigma_1\|_{L^2_{1+E}} + \|\Sigma_2\|_{L^2_{1+E}})
\end{aligned} \tag{B.5}$$

by Cauchy-Schwarz, where we also used that $\int dp f * g(p)h(p) = \int dp h * g(p)f(p)$. In addition, observe that Young's inequality implies

$$\|(\Sigma_1 - \Sigma_2) * \hat{v}\|_2 \leq \|\hat{v}\|_1 \|\Sigma_1 - \Sigma_2\|_2. \tag{B.6}$$

Using $E(p) = \frac{1}{2}|p|^2$ and

$$\sqrt{E(p)} \leq \sqrt{E(p-q)} + \sqrt{E(q)},$$

Young's inequality yields

$$\begin{aligned}
\|((\Sigma_1 - \Sigma_2) * \hat{v})\sqrt{E}\|_2 &\leq \|(\sqrt{E}(\Sigma_1 - \Sigma_2)) * \hat{v}\|_2 + \|(\Sigma_1 - \Sigma_2) * (\sqrt{E}\hat{v})\|_2 \\
&\leq \|\sqrt{E}(\Sigma_1 - \Sigma_2)\|_2 \|\hat{v}\|_1 + \|\Sigma_1 - \Sigma_2\|_2 \|\sqrt{E}\hat{v}\|_1.
\end{aligned} \tag{B.7}$$

Collecting (B.5), (B.6), and (B.7), we conclude that

$$\begin{aligned}
&\|\mathcal{J}_2(\phi_1, \Gamma_1, \Sigma_1) - \mathcal{J}_2(\phi_2, \Gamma_2, \Sigma_2)\|_{L^1_{1+E}} \\
&\leq \frac{2\lambda}{N} (\|\Sigma_1\|_{L^2_{1+E}} + \|\Sigma_2\|_{L^2_{1+E}}) \|\hat{v}\|_{L^1_{\sqrt{1+E}}} \\
&\quad (\|\Sigma_1 - \Sigma_2\|_{L^2_{1+E}} + \|\Sigma_1 - \Sigma_2\|_\infty).
\end{aligned}$$

With similar steps, we arrive at

$$\begin{aligned}
&\|\mathcal{J}_2(\phi_1, \Gamma_1, \Sigma_1) - \mathcal{J}_2(\phi_2, \Gamma_2, \Sigma_2)\|_\infty \\
&\leq \frac{2\lambda}{N} \|(\Sigma_1 - \Sigma_2) * \hat{v}\|_\infty (\|\Sigma_1\|_\infty + \|\Sigma_2\|_\infty) \\
&\leq \frac{2\lambda}{N} \|\hat{v}\|_1 (\|\Sigma_1\|_\infty + \|\Sigma_2\|_\infty) \|\Sigma_1 - \Sigma_2\|_\infty.
\end{aligned}$$

Finally, \mathcal{J}_3 satisfies

$$\begin{aligned}
&\|\mathcal{J}_3(\phi_1, \Gamma_1, \Sigma_1) - \mathcal{J}_3(\phi_2, \Gamma_2, \Sigma_2)\|_{L^2_{1+E}} \\
&\leq \frac{2\lambda}{N} \left(\|(\Gamma_1 - \Gamma_2) * (\hat{v} + \hat{v}(0))\|_{L^2_{1+E}} + \|\Gamma_2 * (\hat{v} + \hat{v}(0))\|_{L^2_{1+E}} \right) \|\Sigma_1 - \Sigma_2\|_{L^2_{1+E}} \\
&\quad + \|(\Sigma_1 - \Sigma_2) * \hat{v}\|_{L^2_{1+E}} + \|\Sigma_2 * \hat{v}\|_{L^2_{1+E}} + \frac{1}{2} \|(\Sigma_1 - \Sigma_2) * \hat{v}\|_{L^2_{1+E}} \\
&\quad + 4N\lambda |\Lambda|^{\frac{5}{2}} \hat{v}(0) |\phi_1|^2 \phi_1^2 - |\phi_2|^2 \phi_2^2|.
\end{aligned} \tag{B.8}$$

Again, Cauchy-Schwarz implies

$$\begin{aligned}
&\|(\Gamma_1 - \Gamma_2) * (\hat{v} + \hat{v}(0))\|_{L^2_{1+E}} \|\Sigma_1\|_{L^2_{1+E}} \\
&\leq \|(\Gamma_1 - \Gamma_2) * \hat{v}\|_{L^2_{1+E}} \|\Sigma_1\|_{L^2_{1+E}} + \|\hat{v}(0)\|_{L^2_{1+E}} \|\Gamma_1 - \Gamma_2\|_1 \|\Sigma_1\|_{L^2_{1+E}} \\
&\leq \left(\|(\Gamma_1 - \Gamma_2) * \hat{v}\|_{L^2_{1+E}} + \|\hat{v}(0)\|_{L^2_{1+E}} \right) \|\Sigma_1\|_{L^2_{1+E}}.
\end{aligned}$$

(B.6) and (B.7) hence yield

$$\|(\Gamma_1 - \Gamma_2) * (\hat{v} + \hat{v}(0))\|_{L^2_{1+E}} \|\Sigma_1\|_{L^2_{1+E}}$$

$$\leq \left(\|\hat{v}\|_{L^1_{\sqrt{1+E}}} \|\Gamma_1 - \Gamma_2\|_{L^2_{1+E}} + \hat{v}(0) \|\Gamma_1 - \Gamma_2\|_1 \right) \|\Sigma_1\|_{L^2_{1+E}}. \quad (\text{B.9})$$

Moreover, by (B.4), we have that

$$\begin{aligned} |\phi_1|^2 \phi_1^2 - |\phi_2|^2 \phi_2^2 &\leq |\phi_1|^2 |\phi_1^2 - \phi_2^2| + |\phi_2^2| |\phi_1|^2 - |\phi_2|^2| \\ &\leq (|\phi_1|^2 + |\phi_2|^2) (|\phi_1| + |\phi_2|) |\phi_1 - \phi_2|. \end{aligned} \quad (\text{B.10})$$

Then (B.8), (B.9), (B.10) and analogous estimates to those for \mathcal{J}_2 imply that

$$\begin{aligned} \|\mathcal{J}_3(\phi_1, \Gamma_1, \Sigma_1) - \mathcal{J}_3(\phi_2, \Gamma_2, \Sigma_2)\|_{L^2_{1+E}} &\leq \frac{2\lambda}{N} (\|\hat{v}\|_{L^1_{\sqrt{1+E}}} + \|\hat{v}\|_\infty) ((\|\Sigma_1\|_{L^2_{1+E}} + \|\Sigma_2\|_{L^2_{1+E}}) \|\Gamma_1 - \Gamma_2\|_{L^2_{1+E}} \\ &+ (\|\Gamma_1\|_{L^2_{1+E}} + \|\Gamma_2\|_{L^2_{1+E}} + \frac{1}{2}) \|\Sigma_1 - \Sigma_2\|_{L^2_{1+E}}) \\ &+ 4N\lambda|\Lambda|^{\frac{5}{2}}\hat{v}(0)(|\phi_1|^2 + |\phi_2|^2)(|\phi_1| + |\phi_2|) |\phi_1 - \phi_2|. \end{aligned}$$

Observe that

$$\begin{aligned} \|\Gamma\|_{L^2_{1+E}} &\leq \|\Gamma\|_{L^1_{1+E}}^{\frac{1}{2}} \|\Gamma\|_\infty^{\frac{1}{2}} \\ &\leq \frac{1}{2} (\|\Gamma\|_{L^1_{1+E}} + \|\Gamma\|_\infty), \end{aligned}$$

which is why

$$\begin{aligned} \|\mathcal{J}_3(\phi_1, \Gamma_1, \Sigma_1) - \mathcal{J}_3(\phi_2, \Gamma_2, \Sigma_2)\|_{L^2_{1+E}} &\leq \frac{\lambda}{N} (\|\hat{v}\|_{L^1_{\sqrt{1+E}}} + \|\hat{v}\|_\infty) ((\|\Sigma_1\|_{L^2_{1+E}} + \|\Sigma_2\|_{L^2_{1+E}}) (\|\Gamma_1 - \Gamma_2\|_{L^1_{1+E}} + \|\Gamma_1 - \Gamma_2\|_\infty) \\ &+ (\|\Gamma_1\|_{L^1_{1+E}} + \|\Gamma_1\|_\infty + \|\Gamma_2\|_{L^1_{1+E}} + \|\Gamma_2\|_\infty + 1) \|\Sigma_1 - \Sigma_2\|_{L^2_{1+E}}) \\ &+ 4N\lambda|\Lambda|^{\frac{5}{2}}\hat{v}(0)(|\phi_1|^2 + |\phi_2|^2)(|\phi_1| + |\phi_2|) |\phi_1 - \phi_2|. \end{aligned}$$

Similarly, we obtain that

$$\begin{aligned} \|\mathcal{J}_3(\phi_1, \Gamma_1, \Sigma_1) - \mathcal{J}_3(\phi_2, \Gamma_2, \Sigma_2)\|_\infty &\leq \frac{4\lambda}{N} \|\hat{v}\|_\infty ((\|\Sigma_1\|_\infty + \|\Sigma_2\|_\infty) \|\Gamma_1 - \Gamma_2\|_1 \\ &+ (\|\Gamma_1\|_1 + \|\Gamma_2\|_1 + 1) \|\Sigma_1 - \Sigma_2\|_\infty) \\ &+ 4N\lambda|\Lambda|^{\frac{5}{2}}\hat{v}(0)(|\phi_1|^2 + |\phi_2|^2)(|\phi_1| + |\phi_2|) |\phi_1 - \phi_2|. \end{aligned}$$

□

Proof of Lemma 4.2. As a consequence of (ϕ, Γ, Σ) being a mild solution, we have that

$$\int dp \Gamma_t(p) = \int dp \Gamma_0(p) - \frac{2\lambda}{N} \int_0^t ds \operatorname{Im} \langle \Sigma_s * \hat{v}, \Sigma_s \rangle. \quad (\text{B.11})$$

Let \check{f} denote the inverse Fourier transform of f . Observe that due to Plancherel,

$$\langle \Sigma_s * \hat{v}, \Sigma_s \rangle = \int dx |\check{\Sigma}_s(x)|^2 v(x) \quad (\text{B.12})$$

is a real number, as $\|v\|_\infty \leq \|\hat{v}\|_1 < \infty$, v is real-valued, and $\Sigma_s \in \ell^2(\Lambda^*)$. As a consequence, (B.11) implies

$$\int dp \Gamma_t(p) = \int dp \Gamma_0(p),$$

as desired.

In order to show energy-conservation, we split the energy into its individual terms. From equation (3.41), we have that

$$\int dp E(p)\Gamma_t(p) = \int dp E(p)\Gamma_0(p) - \frac{2\lambda}{N} \int_0^t ds \langle \Sigma_s * \hat{v}, E\Sigma_s \rangle, \quad (\text{B.13})$$

where we notice that $\Sigma_s * \hat{v} \in L^2_{1+E} \cap L^\infty(\Lambda^*)$. In particular, all terms in (B.13) are finite. Next, we have that

$$\begin{aligned} & \frac{\lambda}{2N} \langle \Gamma_t * (\hat{v} + \hat{v}(0)), \Gamma_t \rangle - \frac{\lambda}{2N} \langle \Gamma_0 * (\hat{v} + \hat{v}(0)), \Gamma_0 \rangle \\ &= \frac{\lambda}{N} \int_0^t ds \langle \Gamma_s * (\hat{v} + \hat{v}(0)), \partial_s \Gamma_s \rangle \\ &= -\frac{2\lambda^2}{N^2} \int_0^t ds \langle \Gamma_s * (\hat{v} + \hat{v}(0)), \text{Im}((\bar{\Sigma}_s * \hat{v})\Sigma_s) \rangle \\ &= -\frac{2\lambda^2}{N^2} \int_0^t ds \text{Im} \langle \Sigma_s * \hat{v}, (\Gamma_s * \hat{v})\Sigma_s \rangle, \end{aligned} \quad (\text{B.14})$$

where the expressions in each line are finite. Using the fact that $\Sigma_s * \hat{v} \in L^2_{1+E} \cap L^\infty(\Lambda^*)$, equation (3.41) for Σ

$$\begin{aligned} & \frac{\lambda}{2N} \langle \Sigma_t * \hat{v}, \Sigma_t \rangle - \frac{\lambda}{2N} \langle \Sigma_0 * \hat{v}, \Sigma_0 \rangle \\ &= \frac{\lambda}{N} \int_0^t ds \text{Im} \langle \Sigma_s * \hat{v}, i\partial_s \Sigma_s \rangle \\ &= \frac{\lambda}{N} \text{Im} \int_0^t ds \int dp (\bar{\Sigma}_s * \hat{v})(p) \left(2(E(p) + \frac{\lambda}{N} \Gamma_s * (\hat{v}(p) + \hat{v}(0))) \Sigma_s(p) \right) \\ & \quad + \frac{\lambda}{N} (\Sigma_s * \hat{v}(p)) (1 + 2\Gamma_s(p)) - 4N\lambda |\Lambda|^2 \hat{v}(0) |\phi_s|^2 \phi_s^2 \delta(p) \\ &= \frac{2\lambda}{N} \text{Im} \int_0^t ds \left(\langle \Sigma_s * \hat{v}, (E + \frac{\lambda}{N} \Gamma_s * \hat{v})\Sigma_s \rangle \right. \\ & \quad \left. - 2N\lambda |\Lambda|^2 \hat{v}(0) |\phi_s|^2 (\bar{\Sigma}_s * \hat{v})(0) \phi_s^2 \right). \end{aligned} \quad (\text{B.15})$$

In addition, equation (3.41) for ϕ implies

$$\begin{aligned} & -N|\Lambda|^2 \lambda \hat{v}(0) (|\phi_t|^4 - |\phi_0|^4) \\ &= -2N|\Lambda|^2 \lambda \hat{v}(0) \text{Im} \int_0^t ds |\phi_s|^2 \bar{\phi}_s i\partial_s \phi_s \\ &= -2|\Lambda|^2 \lambda^2 \hat{v}(0) \text{Im} \int_0^t ds |\phi_s|^2 (\Sigma_s * \hat{v})(0) \bar{\phi}_s^2 \end{aligned}$$

$$= 2|\Lambda|^2 \lambda^2 \hat{v}(0) \operatorname{Im} \int_0^t ds |\phi_s|^2 (\bar{\Sigma}_s * \hat{v})(0) \phi_s^2. \quad (\text{B.16})$$

Collecting (B.13), (B.14), (B.15), (B.16), we obtain that $\mathcal{E}_{HFB}(\phi_t, \Gamma_t, \Sigma_t)$ is differentiable w.r.t. t, and that

$$\partial_t \mathcal{E}_{HFB}(\phi_t, \Gamma_t, \Sigma_t) = 0.$$

This concludes the proof. \square

APPENDIX C. BOLTZMANN EVOLUTION

Lemma C.1 (Calculation of collision operators). *Using the notation in Lemma A.2, the cubic Boltzmann operator is given by*

$$\begin{aligned} \frac{1}{N} \int_0^t ds Q_3[f_0](s, p) &= \\ \frac{2\lambda^2}{N} \operatorname{Re} \int_{[0,t]^2} d\mathbf{s}_2 \mathbb{1}_{s_1 \geq s_2} \int d\mathbf{p}_3 &\left(\frac{1}{2!} (\delta(p_1 - p) + \delta(p_2 - p) - \delta(p_3 - p)) \right. \\ \mathbf{w}_{s_1}^{(2,1)}(\mathbf{p}_3) \bar{\mathbf{w}}_{s_2}^{(2,1)}(\mathbf{p}_3) e^{i \int_{s_2}^{s_1} d\tau (\Omega_\tau(p_1) + \Omega_\tau(p_2) - \Omega_\tau(p_3))} \delta(p_1 + p_2 - p_3) \\ (\tilde{f}_0(p_1) \tilde{f}_0(p_2) f_0(p_3) - f_0(p_1) f_0(p_2) \tilde{f}_0(p_3)) \\ + \frac{1}{3!} (\delta(p_1 - p) + \delta(p_2 - p) + \delta(p_3 - p)) \\ \mathbf{w}_{s_1}^{(3,0)}(\mathbf{p}_3) \bar{\mathbf{w}}_{s_2}^{(3,0)}(\mathbf{p}_3) e^{i \int_{s_2}^{s_1} d\tau (\Omega_\tau(p_1) + \Omega_\tau(p_2) + \Omega_\tau(p_3))} \delta(p_1 + p_2 + p_3) \\ \left. (\tilde{f}_0(p_1) \tilde{f}_0(p_2) \tilde{f}_0(p_3) - f_0(p_1) f_0(p_2) f_0(p_3)) \right), \end{aligned}$$

and the quartic Boltzmann operator is given by

$$\begin{aligned} \frac{1}{N^2} \int_0^t ds Q_4[f_0](s, p) &= \\ \frac{2\lambda^2}{N^2} \operatorname{Re} \int_{[0,t]^2} d\mathbf{s}_2 \mathbb{1}_{s_1 \geq s_2} \int d\mathbf{p}_4 &\left(\frac{1}{(2!)^2} \mathbf{w}_{s_1}^{(2,2)}(\mathbf{p}_4) \bar{\mathbf{w}}_{s_2}^{(2,2)}(\mathbf{p}_4) \right. \\ (\delta(p_1 - p) + \delta(p_2 - p) - \delta(p_3 - p) - \delta(p_4 - p)) \\ \delta(p_1 + p_2 - p_3 - p_4) e^{i \int_{s_2}^{s_1} d\tau (\Omega_\tau(p_1) + \Omega_\tau(p_2) - \Omega_\tau(p_3) - \Omega_\tau(p_4))} \\ (\tilde{f}_0(p_1) \tilde{f}_0(p_2) f_0(p_3) f_0(p_4) - f_0(p_1) f_0(p_2) \tilde{f}_0(p_3) \tilde{f}_0(p_4)) \\ + \frac{1}{3!} \mathbf{w}_{s_1}^{(3,1)}(\mathbf{p}_4) \bar{\mathbf{w}}_{s_2}^{(3,1)}(\mathbf{p}_4) \\ (\delta(p_1 - p) + \delta(p_2 - p) + \delta(p_3 - p) - \delta(p_4 - p)) \\ \delta(p_1 + p_2 + p_3 - p_4) e^{i \int_{s_2}^{s_1} d\tau (\Omega_\tau(p_1) + \Omega_\tau(p_2) + \Omega_\tau(p_3) - \Omega_\tau(p_4))} \\ (\tilde{f}_0(p_1) \tilde{f}_0(p_2) \tilde{f}_0(p_3) f_0(p_4) - f_0(p_1) f_0(p_2) f_0(p_3) \tilde{f}_0(p_4)) \\ + \frac{1}{4!} \mathbf{w}_{s_1}^{(4,0)}(\mathbf{p}_4) \bar{\mathbf{w}}_{s_2}^{(4,0)}(\mathbf{p}_4) \end{aligned}$$

$$\begin{aligned}
& (\delta(p_1 - p) + \delta(p_2 - p) + \delta(p_3 - p) + \delta(p_4 - p)) \\
& \delta(p_1 + p_2 + p_3 + p_4) e^{i \int_{s_2}^{s_1} d\tau \left(\Omega_\tau(p_1) + \Omega_\tau(p_2) + \Omega_\tau(p_3) + \Omega_\tau(p_4) \right)} \\
& \left(\tilde{f}_0(p_1) \tilde{f}_0(p_2) \tilde{f}_0(p_3) \tilde{f}_0(p_4) - f_0(p_1) f_0(p_2) f_0(p_3) f_0(p_4) \right).
\end{aligned}$$

Proof. Recall that Lemma A.2 implies

$$\begin{aligned}
\mathcal{H}_{\text{cub}}(t) := & \\
& \frac{\lambda}{\sqrt{N}} \int d\mathbf{p}_3 \left(\delta(p_1 + p_2 + p_3) e^{i \int_0^t d\tau \left(\Omega_\tau(p_1) + \Omega_\tau(p_2) + \Omega_\tau(p_3) \right)} \right. \\
& \frac{1}{3!} \mathbf{w}_t^{(3,0)}(\mathbf{p}_3) a_{p_1}^\dagger a_{p_2}^\dagger a_{p_3}^\dagger \\
& + \delta(p_1 + p_2 - p_3) e^{i \int_0^t d\tau \left(\Omega_\tau(p_1) + \Omega_\tau(p_2) - \Omega_\tau(p_3) \right)} \\
& \frac{1}{2!} \mathbf{w}_t^{(2,1)}(\mathbf{p}_3) a_{p_1}^\dagger a_{p_2}^\dagger a_{p_3} \\
& \left. + h.c. \right).
\end{aligned}$$

Using (3.19), we hence have that

$$\begin{aligned}
& \frac{1}{N} \int_0^t ds Q_3[f_0](s, p) = \\
& - \int_{[0,t]^2} ds_2 \mathbb{1}_{s_1 \geq s_2} \left(\frac{\langle [[a_p^\dagger a_p, \mathcal{H}_{\text{cub}}(s_1)], \mathcal{H}_{\text{cub}}(s_2)] \rangle_0}{|\Lambda|} \right. \\
& \left. + \frac{\langle [[a_p^\dagger a_p, \mathcal{H}_{\text{cub}}(s_1)], \mathcal{H}_{\text{cub}}(s_2)] \rangle_0}{|\Lambda|} \right) = \tag{C.1}
\end{aligned}$$

$$\begin{aligned}
& - \frac{2\lambda^2}{N|\Lambda|} \operatorname{Re} \int_{[0,t]^2} ds_2 \mathbb{1}_{s_1 \geq s_2} \int d\mathbf{p}_3 d\mathbf{k}_3 \left[\langle [[a_p^\dagger a_p, a_{p_1}^\dagger a_{p_2}^\dagger a_{p_3}], a_{k_3}^\dagger a_{k_2} a_{k_1}] \rangle_0 \right. \\
& \frac{1}{(2!)^2} \mathbf{w}_{s_1}^{(2,1)}(\mathbf{p}_3) \bar{\mathbf{w}}_{s_2}^{(2,1)}(\mathbf{k}_3) \delta(p_1 + p_2 - p_3) \delta(k_1 + k_2 - k_3) \\
& e^{i \int_0^{s_1} d\tau \left(\Omega_\tau(p_1) + \Omega_\tau(p_2) - \Omega_\tau(p_3) \right)} - i \int_0^{s_2} d\tau \left(\Omega_\tau(k_1) + \Omega_\tau(k_2) - \Omega_\tau(k_3) \right) \\
& + \langle [[a_p^\dagger a_p, a_{p_1}^\dagger a_{p_2}^\dagger a_{p_3}], a_{k_1} a_{k_2} a_{k_3}] \rangle_0 \\
& \frac{1}{(3!)^2} \mathbf{w}_{s_1}^{(3,0)}(\mathbf{p}_3) \bar{\mathbf{w}}_{s_2}^{(3,0)}(\mathbf{k}_3) \delta(p_1 + p_2 + p_3) \delta(k_1 + k_2 + k_3) \\
& e^{i \int_0^{s_1} d\tau \left(\Omega_\tau(p_1) + \Omega_\tau(p_2) + \Omega_\tau(p_3) \right)} - i \int_0^{s_2} d\tau \left(\Omega_\tau(k_1) + \Omega_\tau(k_2) + \Omega_\tau(k_3) \right) \\
& \left. + \text{all contractions of the form (C.1)–(C.2)} \right]. \tag{C.3}
\end{aligned}$$

We recognize the collision kernels $\mathbf{w}^{(2,1)}$ and $\mathbf{w}^{(3,0)}$. Now observe that

$$\begin{aligned}\langle [[\overline{a_p^\dagger a_p}, \overline{A_1 a_{p_j}^\dagger A_2}], \overline{B_1 a_{k_\ell} B_2}] \rangle_0 &= \delta(p - p_j) (\tilde{f}_0(p) - f_0(p)) \langle [[\overline{A_1 a_{p_j}^\dagger A_2}], \overline{B_1 a_{k_\ell} B_2}] \rangle_0 \\ &= \delta(p - p_j) \langle [[\overline{A_1 a_{p_j}^\dagger A_2}], \overline{B_1 a_{k_\ell} B_2}] \rangle_0, \\ \langle [[\overline{a_p^\dagger a_p}, \overline{A_1 a_{p_j} A_2}], \overline{B_1 a_{k_\ell}^\dagger B_2}] \rangle_0 &= -\delta(p - p_j) \langle [[\overline{A_1 a_{p_j}^\dagger A_2}], \overline{B_1 a_{k_\ell} B_2}] \rangle_0.\end{aligned}$$

In particular, we have that

$$\begin{aligned}\langle [[\overline{a_p^\dagger a_p}, \overline{a_{p_1}^\dagger a_{p_2}^\dagger a_{p_3}}], \overline{a_{k_1} a_{k_2} a_{k_3}}] \rangle_0 &+ \text{all contractions of the form (C.1)–(C.2)} \\ &= (\delta(p_1 - p) + \delta(p_2 - p) - \delta(p_3 - p)) \\ &\quad \left(\sum_{\pi \in S_2} \delta(p_1 - k_{\pi(1)}) \delta(p_2 - k_{\pi(2)}) \right) \delta(p_3 - k_3) \\ &\quad (f_0(p_1) f_0(p_2) \tilde{f}_0(p_3) - \tilde{f}_0(p_1) \tilde{f}_0(p_2) f_0(p_3)),\end{aligned}$$

and similarly

$$\begin{aligned}\langle [[\overline{a_p^\dagger a_p}, \overline{a_{p_1}^\dagger a_{p_2}^\dagger a_{p_3}}], \overline{a_{k_3}^\dagger a_{k_2} a_{k_1}}] \rangle_0 &+ \text{all contractions of the form (C.1)–(C.2)} \\ &= (\delta(p_1 - p) + \delta(p_2 - p) + \delta(p_3 - p)) \\ &\quad \sum_{\pi \in S_3} \delta(p_1 - k_{\pi(1)}) \delta(p_2 - k_{\pi(2)}) \delta(p_3 - k_{\pi(3)}) \\ &\quad (f_0(p_1) f_0(p_2) f_0(p_3) - \tilde{f}_0(p_1) \tilde{f}_0(p_2) \tilde{f}_0(p_3)).\end{aligned}$$

Exploiting the symmetry of $\mathbf{w}_s^{(2,1)}(\mathbf{k}_3)$ w.r.t. permutations of (k_1, k_2) and the symmetry of $\mathbf{w}_s^{(3,0)}(\mathbf{k}_3)$ w.r.t. permutations of (k_1, k_2, k_3) , (C.3) yields

$$\begin{aligned}\frac{1}{N} \int_0^t ds Q_3[f_0](s, p) &= \\ \frac{2\lambda^2}{N} \operatorname{Re} \int_{[0,t]^2} ds_1 \mathbb{1}_{s_1 \geq s_2} \int d\mathbf{p}_3 &\left(\frac{1}{2!} (\delta(p_1 - p) + \delta(p_2 - p) - \delta(p_3 - p)) \right. \\ \mathbf{w}_{s_1}^{(2,1)}(\mathbf{p}_3) \overline{\mathbf{w}}_{s_2}^{(2,1)}(\mathbf{p}_3) e^{i \int_{s_2}^{s_1} d\tau (\Omega_\tau(p_1) + \Omega_\tau(p_2) - \Omega_\tau(p_3))} &\delta(p_1 + p_2 - p_3) \\ (\tilde{f}_0(p_1) \tilde{f}_0(p_2) f_0(p_3) - f_0(p_1) f_0(p_2) \tilde{f}_0(p_3)) &\\ + \frac{1}{3!} (\delta(p_1 - p) + \delta(p_2 - p) + \delta(p_3 - p)) &\\ \mathbf{w}_{s_1}^{(3,0)}(\mathbf{p}_3) \overline{\mathbf{w}}_{s_2}^{(3,0)}(\mathbf{p}_3) e^{i \int_{s_2}^{s_1} d\tau (\Omega_\tau(p_1) + \Omega_\tau(p_2) + \Omega_\tau(p_3))} &\delta(p_1 + p_2 + p_3) \\ (\tilde{f}_0(p_1) \tilde{f}_0(p_2) \tilde{f}_0(p_3) - f_0(p_1) f_0(p_2) f_0(p_3)) \Big). &\end{aligned}$$

With analogous steps, one obtains

$$\begin{aligned}
& \frac{1}{N^2} \int_0^t ds Q_4[f_0](s, p) = \\
& \frac{2\lambda^2}{N^2} \operatorname{Re} \int_{[0,t]^2} d\mathbf{s}_2 \mathbb{1}_{s_1 \geq s_2} \int d\mathbf{p}_4 \left(\frac{1}{(2!)^2} \mathbf{w}_{s_1}^{(2,2)}(\mathbf{p}_4) \overline{\mathbf{w}}_{s_2}^{(2,2)}(\mathbf{p}_4) \right. \\
& (\delta(p_1 - p) + \delta(p_2 - p) - \delta(p_3 - p) - \delta(p_4 - p)) \\
& \delta(p_1 + p_2 - p_3 - p_4) e^{i \int_{s_2}^{s_1} d\tau (\Omega_\tau(p_1) + \Omega_\tau(p_2) - \Omega_\tau(p_3) - \Omega_\tau(p_4))} \\
& (\tilde{f}_0(p_1) \tilde{f}_0(p_2) f_0(p_3) f_0(p_4) - f_0(p_1) f_0(p_2) \tilde{f}_0(p_3) \tilde{f}_0(p_4)) \\
& + \frac{1}{3!} \mathbf{w}_{s_1}^{(3,1)}(\mathbf{p}_4) \overline{\mathbf{w}}_{s_2}^{(3,1)}(\mathbf{p}_4) \\
& (\delta(p_1 - p) + \delta(p_2 - p) + \delta(p_3 - p) - \delta(p_4 - p)) \\
& \delta(p_1 + p_2 + p_3 - p_4) e^{i \int_{s_2}^{s_1} d\tau (\Omega_\tau(p_1) + \Omega_\tau(p_2) + \Omega_\tau(p_3) - \Omega_\tau(p_4))} \\
& (\tilde{f}_0(p_1) \tilde{f}_0(p_2) \tilde{f}_0(p_3) f_0(p_4) - f_0(p_1) f_0(p_2) f_0(p_3) \tilde{f}_0(p_4)) \\
& + \frac{1}{4!} \mathbf{w}_{s_1}^{(4,0)}(\mathbf{p}_4) \overline{\mathbf{w}}_{s_2}^{(4,0)}(\mathbf{p}_4) \\
& (\delta(p_1 - p) + \delta(p_2 - p) + \delta(p_3 - p) + \delta(p_4 - p)) \\
& \delta(p_1 + p_2 + p_3 + p_4) e^{i \int_{s_2}^{s_1} d\tau (\Omega_\tau(p_1) + \Omega_\tau(p_2) + \Omega_\tau(p_3) + \Omega_\tau(p_4))} \\
& \left. (\tilde{f}_0(p_1) \tilde{f}_0(p_2) \tilde{f}_0(p_3) \tilde{f}_0(p_4) - f_0(p_1) f_0(p_2) f_0(p_3) f_0(p_4)) \right).
\end{aligned}$$

This concludes the proof. \square

Lemma C.2 (Calculation of collision operators for (Φ, g)). *Recall from (2.22) and (2.23) that $\tilde{\mathbf{p}}_3 = (p_3, p_2, p_1)$ and $\overline{\mathbf{p}} = (p_1, p_2, -p_3)$. Using the notation in Lemma A.2, the cubic Boltzmann operator for Φ is given by*

$$\begin{aligned}
& \int_0^t ds Q_3^{(\Phi)}[f_0](s) = \\
& \lambda^2 \int_{[0,t]^2} d\mathbf{s}_2 \mathbb{1}_{s_1 \geq s_2} e^{i \int_0^{s_1} d\tau \Omega_\tau(0)} \left[\frac{1}{2} \delta(p_1 + p_2 - p_3) \left(e^{i \int_{s_2}^{s_1} d\tau (\Omega_\tau(p_1) + \Omega_\tau(p_2) - \Omega_\tau(p_3))} \right. \right. \\
& \mathbf{w}_{s_1}^{(3,1)}(0, \mathbf{p}_3) \overline{\mathbf{w}}_{s_2}^{(2,1)}(\mathbf{p}_3) - e^{-i \int_{s_2}^{s_1} d\tau (\Omega_\tau(p_1) + \Omega_\tau(p_2) - \Omega_\tau(p_3))} \mathbf{w}_{s_1}^{(2,2)}(0, \tilde{\mathbf{p}}_3) \mathbf{w}_{s_2}^{(2,1)}(\tilde{\mathbf{p}}_3) \Big) \\
& (f_0(p_1) f_0(p_2) \tilde{f}_0(p_3) - \tilde{f}_0(p_1) \tilde{f}_0(p_2) f_0(p_3)) \\
& + \frac{1}{3!} \delta(p_1 + p_2 + p_3) \left(\mathbf{w}_{s_1}^{(4,0)}(0, \mathbf{p}_3) \overline{\mathbf{w}}_{s_2}^{(3,0)}(\mathbf{p}_3) e^{i \int_{s_2}^{s_1} d\tau (\Omega_\tau(p_1) + \Omega_\tau(p_2) + \Omega_\tau(p_3))} \right. \\
& \left. \left. - \overline{\mathbf{w}}_{s_1}^{(3,1)}(\mathbf{p}_3, 0) \mathbf{w}_{s_2}^{(3,0)}(\mathbf{p}_3) e^{-i \int_{s_2}^{s_1} d\tau (\Omega_\tau(p_1) + \Omega_\tau(p_2) + \Omega_\tau(p_3))} \right) \right. \\
& (f_0(p_1) f_0(p_2) f_0(p_3) - \tilde{f}_0(p_1) \tilde{f}_0(p_2) \tilde{f}_0(p_3)) \Big],
\end{aligned}$$

and for g , it is given by

$$\begin{aligned} \int_0^t ds Q_3^{(g)}[f_0](s)[J] = & \lambda^2 \int dp J(p) \int_{[0,t]^2} d\mathbf{s}_2 \mathbb{1}_{s_1 \geq s_2} \int d\mathbf{p}_3 \left[\delta(p_1 + p_2 - p_3) \left(\delta(p - p_3) e^{2i \int_0^{s_1} d\tau \Omega_\tau(p_3)} \right. \right. \\ & e^{i \int_{s_2}^{s_1} d\tau (\Omega_\tau(p_1) + \Omega_\tau(p_2) - \Omega_\tau(p_3))} \mathbf{w}_{s_1}^{(3,0)}(\bar{\mathbf{p}}_3) \overline{\mathbf{w}}_{s_2}^{(2,1)}(\mathbf{p}_3) - 2\delta(p - p_1) e^{-2i \int_0^{s_1} d\tau \Omega_\tau(p_1)} \\ & e^{-i \int_{s_2}^{s_1} d\tau (\Omega_\tau(p_1) + \Omega_\tau(p_2) - \Omega_\tau(p_3))} \overline{\mathbf{w}}_{s_1}^{(2,1)}(\bar{\mathbf{p}}_3) \mathbf{w}_{s_2}^{(2,1)}(\mathbf{p}_3) \left. \left. \right) \right. \\ & (f_0(p_1) f_0(p_2) \tilde{f}_0(p_3) - \tilde{f}_0(p_1) \tilde{f}_0(p_2) f_0(p_3)) \\ & + \delta(p - p_3) e^{2i \int_0^{s_1} d\tau \Omega_\tau(p_3)} \delta(p_1 + p_2 + p_3) e^{-i \int_{s_2}^{s_1} d\tau (\Omega_\tau(p_1) + \Omega_\tau(p_2) + \Omega_\tau(p_3))} \\ & \left. \left. \mathbf{w}_{s_1}^{(2,1)}(\bar{\mathbf{p}}_3) \mathbf{w}_{s_2}^{(3,0)}(\mathbf{p}_3) (\tilde{f}_0(p_1) \tilde{f}_0(p_2) \tilde{f}_0(p_3) - f_0(p_1) f_0(p_2) f_0(p_3)) \right] \right]. \end{aligned}$$

Proof. With analogous calculations as in Lemma C.1, we obtain

$$\begin{aligned} \frac{1}{N^{\frac{3}{2}}} \int_0^t ds Q_3^{(\Phi)}[f_0](s) = & -\frac{1}{|\Lambda|} \int_{[0,t]^2} d\mathbf{s}_2 \mathbb{1}_{s_1 \geq s_2} \langle [[a_0, \overbrace{\mathcal{H}_{\text{quart}}(s_1)}], \mathcal{H}_{\text{cub}}(s_2)] \rangle_0 \\ = & \frac{\lambda^2}{N^{\frac{3}{2}}} \int_{[0,t]^2} d\mathbf{s}_2 \mathbb{1}_{s_1 \geq s_2} e^{i \int_0^{s_1} d\tau \Omega_\tau(0)} \left[\frac{1}{2} \delta(p_1 + p_2 - p_3) e^{i \int_{s_2}^{s_1} d\tau (\Omega_\tau(p_1) + \Omega_\tau(p_2) - \Omega_\tau(p_3))} \right. \\ & \mathbf{w}_{s_1}^{(3,1)}(0, \mathbf{p}_3) \overline{\mathbf{w}}_{s_2}^{(2,1)}(\mathbf{p}_3) (f_0(p_1) f_0(p_2) \tilde{f}_0(p_3) - \tilde{f}_0(p_1) \tilde{f}_0(p_2) f_0(p_3)) \\ & + \frac{1}{3!} \delta(p_1 + p_2 + p_3) \left(\mathbf{w}_{s_1}^{(4,0)}(0, \mathbf{p}_3) \overline{\mathbf{w}}_{s_2}^{(3,0)}(\mathbf{p}_3) e^{i \int_{s_2}^{s_1} d\tau (\Omega_\tau(p_1) + \Omega_\tau(p_2) + \Omega_\tau(p_3))} \right. \\ & \left. \left. - \overline{\mathbf{w}}_{s_1}^{(3,1)}(\mathbf{p}_3, 0) \mathbf{w}_{s_2}^{(3,0)}(\mathbf{p}_3) e^{-i \int_{s_2}^{s_1} d\tau (\Omega_\tau(p_1) + \Omega_\tau(p_2) + \Omega_\tau(p_3))} \right) \right. \\ & (f_0(p_1) f_0(p_2) f_0(p_3) - \tilde{f}_0(p_1) \tilde{f}_0(p_2) \tilde{f}_0(p_3)) \left. \right] \\ & + \frac{1}{2} \mathbf{w}_{s_1}^{(2,2)}(0, \mathbf{p}_3) \mathbf{w}_{s_2}^{(2,1)}(\mathbf{p}_3) \delta(p_1 - p_2 - p_3) e^{i \int_{s_2}^{s_1} d\tau (\Omega_\tau(p_1) - \Omega_\tau(p_2) - \Omega_\tau(p_3))} \\ & (f_0(p_1) \tilde{f}_0(p_2) \tilde{f}_0(p_3) - \tilde{f}_0(p_1) f_0(p_2) f_0(p_3)). \end{aligned}$$

In order to conclude the computation of $Q_3^{(\Phi)}[f_0](t)$, we substitute $p_1 \leftrightarrow p_3$ in the last term.

Similarly, we compute

$$\begin{aligned} \int_0^t ds Q_3^{(g)}[f_0](s)[J] = & -\frac{N}{|\Lambda|} \int dp J(p) \int_{[0,t]^2} d\mathbf{s}_2 \mathbb{1}_{s_1 \geq s_2} \langle [[a_p a_{-p}, \overbrace{\mathcal{H}_{\text{cub}}(s_1)}], \mathcal{H}_{\text{cub}}(s_2)] \rangle_0 \\ = & \lambda^2 \int dp J(p) \int_{[0,t]^2} d\mathbf{s}_2 \mathbb{1}_{s_1 \geq s_2} \int d\mathbf{p}_3 \left[\delta(p - p_3) e^{2i \int_0^{s_1} d\tau \Omega_\tau(p_3)} \right. \\ & \left(\delta(p_1 + p_2 - p_3) e^{i \int_{s_2}^{s_1} d\tau (\Omega_\tau(p_1) + \Omega_\tau(p_2) - \Omega_\tau(p_3))} \mathbf{w}_{s_1}^{(3,0)}(\bar{\mathbf{p}}_3) \overline{\mathbf{w}}_{s_2}^{(2,1)}(\mathbf{p}_3) \right. \\ & (f_0(p_1) f_0(p_2) \tilde{f}_0(p_3) - \tilde{f}_0(p_1) \tilde{f}_0(p_2) f_0(p_3)) \left. \right. \end{aligned}$$

$$\begin{aligned}
& + \delta(p_1 + p_2 + p_3) e^{-i \int_{s_2}^{s_1} d\tau (\Omega_\tau(p_1) + \Omega_\tau(p_2) + \Omega_\tau(p_3))} \bar{\mathbf{w}}_{s_1}^{(2,1)}(\bar{\mathbf{p}}_3) \mathbf{w}_{s_2}^{(3,0)}(\mathbf{p}_3) \\
& \left(\tilde{f}_0(p_1) \tilde{f}_0(p_2) \tilde{f}_0(p_3) - f_0(p_1) f_0(p_2) f_0(p_3) \right) \\
& + 2\delta(p - p_1) e^{-2i \int_0^{s_1} d\tau \Omega_\tau(p_1)} \delta(p_1 + p_2 - p_3) e^{-i \int_{s_2}^{s_1} d\tau (\Omega_\tau(p_1) + \Omega_\tau(p_2) - \Omega_\tau(p_3))} \\
& \bar{\mathbf{w}}_{s_1}^{(2,1)}(\bar{\mathbf{p}}_3) \mathbf{w}_{s_2}^{(2,1)}(\mathbf{p}_3) \left(\tilde{f}_0(p_1) \tilde{f}_0(p_2) f_0(p_3) - f_0(p_1) f_0(p_2) \tilde{f}_0(p_3) \right).
\end{aligned}$$

Rearranging the terms yields the result. \square

APPENDIX D. TRACE ESTIMATES

Lemma D.1 (Number operator moments). *We have for all $\ell \in \mathbb{N}_0$ that*

$$\langle (\mathcal{N}_b + |\Lambda|)^{\frac{\ell}{2}} \rangle_0 \leq C_{\ell, \|f_0\|_d} |\Lambda|^{\frac{\ell}{2}}.$$

Lemma D.2 (Operator product bound). *Let $A_j \in \mathcal{P}[a, a^\dagger]$ be monomials in a, a^\dagger , $\gamma_j > 0$, and $k_j \in \mathbb{N}$ be such that*

$$\|P_m A_j P_{m-\text{sign}(A_j)}\| \leq \gamma_j (m + |\Lambda|)^{k_j/2}$$

for all $j \in \{1, \dots, \ell\}$ and all $m \in \mathbb{N}_0$. Then we have that

$$|\nu\left(\prod_{j=1}^{\ell} A_j\right)| \leq \left(\prod_{j=1}^{\ell} \gamma_j\right) \nu\left((\mathcal{N} + \sum_{m=1}^{\ell} |\text{sign}(A_m)| + |\Lambda|)^{\sum_{j=1}^{\ell} k_j/2}\right)$$

for any state ν .

Lemma D.3. *Given a test function $J \in L^2 \cap L^\infty(\Lambda^*)$, let*

$$\begin{aligned}
f[J] &:= \int_{\Lambda^*} dp J(p) a_p^\dagger a_p, \\
g[J] &:= \int_{\Lambda^*} dp J(p) a_{-p} a_p.
\end{aligned}$$

Then we have

$$\begin{aligned}
\|P_m f[J] P_n\| &\leq \delta_{m,n} \|J\|_\infty m, \\
\|P_m g[J] P_n\| &\leq \delta_{n,m+2} (\|J\|_2 + \|J\|_\infty) (m + 1 + |\Lambda|).
\end{aligned}$$

For the following standard result, we need to introduce some notation. For a proof of the statement, we refer, e.g., to [5, 6]. Denote

$$a_p^{(1)} := a_p^\dagger, \quad a_p^{(-1)} := a_p.$$

Given a finite ordered subset $J = \{j_1 < j_2 < \dots < j_r\} \subset \mathbb{N}$ and $\sigma_{j_k} \in \{\pm 1\}$, we define the ordered product

$$\prod_{j \in J} a_{p_j}^{(\sigma_j)} := a_{p_{j_1}}^{(\sigma_{j_1})} \dots a_{p_{j_r}}^{(\sigma_{j_r})}.$$

In addition, we abbreviate

$$\mathbf{p}_J := (p_{j_k})_{k=1}^r,$$

as well as

$$a^{(\sigma)}(\mathbf{p}_J) := \prod_{j \in J} a_{p_j}^{(\sigma)}. \quad (\text{D.1})$$

Note that we have that

$$[\mathcal{N}_b, \prod_{j \in J} a_{p_j}^{(\sigma_j)}] = \sum_{j \in J} \sigma_j \prod_{j \in J} a_{p_j}^{(\sigma_j)}. \quad (\text{D.2})$$

Furthermore, we define the sets

$$J_{\pm} := \{j \in J \mid \sigma_j = \pm 1\}$$

and the Wick-ordered product

$$:\prod_{j \in J} a_{p_j}^{(\sigma_j)}: := a^{\dagger}(\mathbf{p}_{J_+}) a(\mathbf{p}_{J_-})$$

with all creation operators to the left, and all annihilation operators to the right.

Finally, in order to keep track of the correct scaling, it is useful to work with the rescaled $\ell^2(\Lambda^*)$ -norm

$$\|H\|_{L^2(\Lambda^*)} = \frac{1}{\sqrt{|\Lambda|}} \|H\|_{\ell^2(\Lambda^*)}.$$

More generally, we also define

$$\begin{aligned} \|H\|_{L_{\mathbf{p}_m}^{\infty} L_{\mathbf{k}_n}^2((\Lambda^*)^{m+n})} &:= \sup_{\mathbf{p}_m \in (\Lambda^*)^m} \left(\int_{(\Lambda^*)^n} d\mathbf{k}_n |H(\mathbf{p}_m, \mathbf{k}_n)|^2 \right)^{\frac{1}{2}}, \\ \|H\|_{L_{\mathbf{k}_n}^2 L_{\mathbf{p}_m}^{\infty}((\Lambda^*)^{m+n})} &:= \left(\int_{(\Lambda^*)^n} d\mathbf{k}_n \sup_{\mathbf{p}_m \in (\Lambda^*)^m} |H(\mathbf{p}_m, \mathbf{k}_n)|^2 \right)^{\frac{1}{2}}, \end{aligned}$$

where in the case $n = 0$, this norm reduces to $\|H\|_{L_{\mathbf{p}_m}^{\infty}((\Lambda^*)^m)}$, and in the case $m = 0$, to $\|H\|_{L_{\mathbf{k}_n}^2((\Lambda^*)^n)}$.

Lemma D.4 (Wick-ordered operator bound). *Let $M \in \mathbb{N}_0$, $n \in \mathbb{N}$, $J := \{1, \dots, n\}$, $\sigma_j \in \{\pm 1\}$ for all $j \in J$. Let $H : (\Lambda^*)^n \rightarrow \mathbb{C}$, and $g_j : \Lambda^* \rightarrow \mathbb{C}$ be given functions. Then the following holds true*

(1) *If $J_{\pm} \neq \emptyset$, we have that*

$$\begin{aligned} &\left\| \int_{(\Lambda^*)^n} d\mathbf{p}_n H(\mathbf{p}_n) \delta\left(\sum_{j=1}^n p_j \sigma_j\right) \prod_{j=1}^n g_j(p_j) : \prod_{j=1}^n a_{p_j}^{(\sigma_j)} : P_M \right\| \\ &\leq \|H|^{\frac{1}{2}} \prod_{j \in J_-} g_j(p_j) \delta\left(\sum_{j=1}^n \sigma_j p_j\right)^{\frac{1}{2}} \|_{L_{\mathbf{p}_{J_+}}^{\infty} L_{\mathbf{p}_{J_-}}^2} \|H|^{\frac{1}{2}} \prod_{j \in J_+} g_j(p_j) \delta\left(\sum_{j=1}^n \sigma_j p_j\right)^{\frac{1}{2}} \|_{L_{\mathbf{p}_{J_-}}^{\infty} L_{\mathbf{p}_{J_+}}^2} \\ &\quad (M + \sum_{j=1}^n \sigma_j)^{\frac{1}{2}} |J_+| (M)^{\frac{1}{2}} |J_-| \mathbb{1}_{M \geq |J_-|}, \end{aligned}$$

where

$$(x)_m := \prod_{k=0}^{m-1} (x - k)$$

denotes the falling factorial.

(2) If $J_+ = \emptyset$ and $n \geq 2$, we find that

$$\begin{aligned} & \left\| \int_{(\Lambda^*)^n} d\mathbf{p}_n H(\mathbf{p}_n) \delta\left(\sum_{j=1}^n p_j\right) a(\mathbf{p}_n) P_M \right\| \\ & \leq \left(\|H\|_{L_{\mathbf{p}_{n-2} L_{p_{n-1}, p_n}^\infty}^2}^2 (M-n+1) + |\Lambda| \|\delta\left(\sum_{j=1}^n p_j\right)^{\frac{1}{2}} H\|_{L_{\mathbf{p}_n}^2}^2 \right)^{\frac{1}{2}} (M)_{n-1}^{\frac{1}{2}}. \end{aligned}$$

Similarly, in the case $J_- = \emptyset$, we have that

$$\begin{aligned} & \left\| \int_{(\Lambda^*)^n} d\mathbf{p}_n H(\mathbf{p}_n) \delta\left(\sum_{j=1}^n p_j\right) a^\dagger(\mathbf{p}_n) P_M \right\| \\ & \leq \left(\|H\|_{L_{\mathbf{p}_{n-2} L_{p_{n-1}, p_n}^\infty}^2}^2 (M+1) + |\Lambda| \|\delta\left(\sum_{j=1}^n p_j\right)^{\frac{1}{2}} H\|_{L_{\mathbf{p}_n}^2}^2 \right)^{\frac{1}{2}} (M+n)_{n-1}^{\frac{1}{2}}. \end{aligned}$$

(3) If $n = 1$, we obtain

$$\|a_0 P_M\| = \sqrt{M|\Lambda|}.$$

APPENDIX E. BOUNDS ON FLUCTUATION DYNAMICS

Lemma E.1 (Bogoliubov dynamics). *Assume that $|\Lambda| \geq 1$. For any $t \geq 0$, let u_t, v_t be defined as in (2.9), such that $\gamma_t := |v_t|^2 \in L^1 \cap L^\infty(\Lambda^*)$. Then, for any $\ell \in \mathbb{N}$, there exist constants $C_\ell > 0$ such that for any $t > 0$ we have that*

$$\begin{aligned} \|(\mathcal{N}_b + |\Lambda|)^{\frac{\ell}{2}} \mathcal{T}[k_0](\mathcal{N}_b + |\Lambda|)^{-\frac{\ell}{2}}\| & \leq C_\ell (1 + \|\gamma_0\|_1 + \|\gamma_0\|_\infty)^{\frac{\ell}{2}}, \\ \|(\mathcal{N}_b + |\Lambda|)^{\frac{\ell}{2}} \mathcal{T}^\dagger[k_t](\mathcal{N}_b + |\Lambda|)^{-\frac{\ell}{2}}\| & \leq C_\ell (1 + \|\gamma_t\|_1 + \|\gamma_t\|_\infty)^{\frac{\ell}{2}}. \end{aligned}$$

Proof. Recall the facts that $u_t(p)^2 = 1 + |v_t(p)|^2 = 1 + \gamma_t$, see (2.11), and that γ_t is even. Employing (2.10) and Lemma D.3, we obtain

$$\begin{aligned} \mathcal{T}^\dagger[k_0] \mathcal{N}_b \mathcal{T}[k_0] & = \int_{\Lambda^*} dp (u_0(p) a_p^\dagger + \bar{v}_0(p) a_{-p})(u_0(p) a_p + v_0(p) a_{-p}^\dagger) \\ & = \mathcal{N}_b + \int_{\Lambda^*} dp (2\gamma_0(p) a_p^\dagger a_p + (\sigma_0(p) a_p^\dagger a_{-p}^\dagger + \text{h.c.}) + |\Lambda| \gamma_0(p)). \end{aligned}$$

Lemma D.3 implies

$$\begin{aligned} \|P_m f[1 + 2\gamma_0] P_n\| & \leq \delta_{m,n} (1 + 2\|\gamma_0\|_\infty) m, \\ \|P_m g[\bar{\sigma}_0] P_n\| & \leq \delta_{n,m+2} (\|\sigma_0\|_2 + \|\sigma_0\|_\infty) (m+1 + |\Lambda|). \end{aligned} \tag{E.1}$$

Due to $|\sigma|^2 = \gamma(\gamma + 1)$, see (2.13), we have

$$\begin{aligned} \|\sigma\|_2 & \leq \|\gamma\|_1^{\frac{1}{2}} (1 + \|\gamma\|_\infty)^{\frac{1}{2}} \leq \frac{1}{2} (1 + \|\gamma\|_1 + \|\gamma\|_\infty), \\ \|\sigma\|_\infty & \leq \|\gamma\|_\infty^{\frac{1}{2}} (1 + \|\gamma\|_\infty)^{\frac{1}{2}} \leq \frac{1}{2} (1 + 2\|\gamma\|_\infty). \end{aligned}$$

As a consequence, (E.1) implies

$$\|P_m g[\bar{\sigma}_0] P_n\| \leq \frac{\delta_{n,m+2}}{2} (2 + \|\gamma_0\|_1 + 3\|\gamma_0\|_\infty) (m+1 + |\Lambda|).$$

Using $|\Lambda| \geq 1$, Lemma D.2 then yields

$$\begin{aligned} \langle \psi, \mathcal{T}^\dagger[k_0](\mathcal{N}_b + |\Lambda|)^\ell \mathcal{T}[k_0]\psi \rangle &= \langle \psi, (\mathcal{T}^\dagger[k_0]\mathcal{N}_b\mathcal{T}[k_0] + |\Lambda|)^\ell \psi \rangle \\ &\lesssim_\ell (1 + \|\gamma_0\|_1 + \|\gamma_0\|_\infty)^\ell \langle \psi, (\mathcal{N}_b + |\Lambda| + \ell)^\ell \psi \rangle \\ &\lesssim_\ell (1 + \|\gamma_0\|_1 + \|\gamma_0\|_\infty)^\ell \langle \psi, (\mathcal{N}_b + |\Lambda|)^\ell \psi \rangle. \end{aligned}$$

With analogous steps, we obtain

$$\langle \psi, \mathcal{T}[k_t](\mathcal{N}_b + |\Lambda|)^\ell \mathcal{T}^\dagger[k_t]\psi \rangle \lesssim_\ell (1 + \|\gamma_t\|_1 + \|\gamma_t\|_\infty)^\ell \langle \psi, (\mathcal{N}_b + |\Lambda|)^\ell \psi \rangle.$$

This concludes the proof. \square

Lemma E.2 (BEC fluctuation dynamics). *Assume $|\Lambda| \geq 1$. Let $\phi \in L^\infty(0, t)$ for all $t > 0$. Define $\hat{\mathcal{U}}_N(t) := \mathcal{W}^\dagger[\sqrt{N|\Lambda|}\phi_t]e^{-it\mathcal{H}_N}\mathcal{W}[\sqrt{N|\Lambda|}\phi_0]$. Then there are $C_\ell, K_\ell > 0$ such that*

$$\begin{aligned} &\left\| (\mathcal{N}_b + |\Lambda|)^{\frac{\ell}{2}} \hat{\mathcal{U}}_N(t)(\mathcal{N}_b + |\Lambda|)^{-\frac{\ell}{2}} \left(1 + \frac{\mathcal{N}_b}{N|\Lambda|}\right)^{-\frac{1}{2}} \right\| \\ &\leq C_\ell \left(1 + \frac{(\|\Gamma_0^T\|_1 + \|\Gamma_0^T\|_\infty + 1)^2}{N}\right)^\ell e^{K_\ell \|\hat{v}\|_{w,d} \lambda |\Lambda| (1 + \frac{\|\Gamma_0^T\|_1}{N})t} \end{aligned}$$

for all $\ell \in \mathbb{N}$.

Proof. We need to reprove this statement as, compared to [37], here, $\hat{v}(0) \neq 0$ and ϕ is not stationary. Nonetheless, we follow the steps of the proof in [29]. We show the statement by induction on ℓ . Let $\psi \in \mathcal{F}$ be arbitrary with $\|\psi\| = 1$.

(1) Induction basis: Define

$$\begin{aligned} \mathcal{H}_{\text{BEC}}^{(\phi_t)} &:= \sqrt{N|\Lambda|} a_0^\dagger (-i\partial_t \phi_t + \lambda |\Lambda| \hat{v}(0) |\phi_t|^2 \phi_t) + \text{h.c.} \\ \mathcal{H}_{\text{HFB}}^{(\text{cor}, \phi_t)} &:= \frac{\lambda |\Lambda|}{2} \int dp \hat{v}(p) (\phi_t^2 a_p^\dagger a_{-p}^\dagger + \bar{\phi}_t^2 a_p a_{-p}) \\ \mathcal{H}_{\text{cub}}^{(\phi_t)} &:= \frac{\lambda \sqrt{|\Lambda|}}{\sqrt{N}} \int d\mathbf{p}_2 \hat{v}(p_2) (a_{p_1}^\dagger a_{p_2}^\dagger a_{p_1 p_2} \phi_t + a_{p_1 p_2}^\dagger a_{p_2} a_{p_1} \bar{\phi}_t). \end{aligned}$$

Substituting (4.1) and recalling definitions (2.17), we find that

$$\mathcal{H}_{\text{BEC}}^{(\phi_t)} = -\frac{\lambda \sqrt{|\Lambda|}}{\sqrt{N}} a_0^\dagger \left((\Gamma_t^T * (\hat{v} + \hat{v}(0)))(0) \phi_t + (\Sigma_t^T * \hat{v})(0) \phi_t \right) + \text{h.c.}$$

Then (A.8) and (A.9) imply that

$$\begin{aligned} &i\partial_t \langle \hat{\mathcal{U}}_N(t)\psi, (\mathcal{N}_b + |\Lambda|) \hat{\mathcal{U}}_N(t)\psi \rangle \\ &= \langle \hat{\mathcal{U}}_N(t)\psi, [\mathcal{N}_b, \mathcal{H}_{\text{BEC}}^{(\phi_t)} + \mathcal{H}_{\text{HFB}}^{(\text{cor}, \phi_t)} + \mathcal{H}_{\text{cub}}^{(\phi_t)}] \hat{\mathcal{U}}_N(t)\psi \rangle. \end{aligned} \tag{E.2}$$

Recall from (D.2) that for any monomial $A \in \mathcal{P}[a, a^\dagger]$ we have

$$[\mathcal{N}_b, A] = \text{sign}(A)A.$$

Employing Lemmata D.3 and D.4 together with Young's inequality, (E.2) yields

$$|\partial_t \langle \hat{\mathcal{U}}_N(t)\psi, (\mathcal{N}_b + |\Lambda|) \hat{\mathcal{U}}_N(t)\psi \rangle|$$

$$\begin{aligned}
&\leq C(\|\hat{v}\|_1 + \|\hat{v}\|_\infty)\lambda \left(\frac{|\Lambda|}{\sqrt{N}} (\|\Gamma_t^T\|_1 + \|\Sigma_t^T\|_2 + \|\Sigma_t^T\|_\infty) |\phi_t| \right. \\
&\quad \langle \hat{\mathcal{U}}_N(t)\psi, (\mathcal{N}_b + |\Lambda|)^{\frac{1}{2}} \hat{\mathcal{U}}_N(t)\psi \rangle \\
&\quad + |\phi_t|^2 |\Lambda| \langle \hat{\mathcal{U}}_N(t)\psi, (\mathcal{N}_b + |\Lambda|) \hat{\mathcal{U}}_N(t)\psi \rangle \\
&\quad \left. + \frac{|\phi_t| \sqrt{|\Lambda|}}{\sqrt{N}} \langle \hat{\mathcal{U}}_N(t)\psi, (\mathcal{N}_b + |\Lambda|)^{\frac{3}{2}} \hat{\mathcal{U}}_N(t)\psi \rangle \right) \\
&\leq C(\|\hat{v}\|_1 + \|\hat{v}\|_\infty)\lambda |\Lambda| \left(\frac{(\|\Gamma_t^T\|_1 + \|\Gamma_t^T\|_\infty)^2}{N} \right. \\
&\quad + |\phi_t|^2 \langle \hat{\mathcal{U}}_N(t)\psi, (\mathcal{N}_b + |\Lambda|) \hat{\mathcal{U}}_N(t)\psi \rangle \\
&\quad \left. + \frac{1}{N|\Lambda|} \langle \hat{\mathcal{U}}_N(t)\psi, (\mathcal{N}_b + |\Lambda|)^2 \hat{\mathcal{U}}_N(t)\psi \rangle \right), \tag{E.3}
\end{aligned}$$

where in the last step, we used Cauchy-Schwarz together with Lemma 4.5. Our goal now is to bound $\frac{1}{N} \langle \hat{\mathcal{U}}_N(t)\psi, (\mathcal{N}_b + |\Lambda|)^2 \hat{\mathcal{U}}_N(t)\psi \rangle$ in terms of $\langle \hat{\mathcal{U}}_N(t)\psi, (\mathcal{N}_b + |\Lambda|) \hat{\mathcal{U}}_N(t)\psi \rangle$ and time-dependent ψ -independent terms, in order to close the estimate. Using

$$\mathcal{W}[\sqrt{N|\Lambda|}\phi_t]a_p \mathcal{W}^\dagger[\sqrt{N|\Lambda|}\phi_t] = a_p - \sqrt{N|\Lambda|}\phi_t \delta(p),$$

we derive that

$$\begin{aligned}
[\mathcal{N}_b, \mathcal{W}^\dagger[\sqrt{N|\Lambda|}\phi_t]] &= -\sqrt{N|\Lambda|} \mathcal{W}^\dagger[\sqrt{N|\Lambda|}\phi_t] (\bar{\phi}_t a_0 + \phi_t a_0^\dagger) \\
&\quad + N|\Lambda| \mathcal{W}^\dagger[\sqrt{N|\Lambda|}\phi_t] \\
&= -(\sqrt{N|\Lambda|}(\bar{\phi}_t a_0 + \phi_t a_0^\dagger) + N|\Lambda|) \mathcal{W}^\dagger[\sqrt{N|\Lambda|}\phi_t], \\
[\mathcal{N}_b, \mathcal{W}[\sqrt{N|\Lambda|}\phi_0]] &= \mathcal{W}[\sqrt{N|\Lambda|}\phi_0] (\sqrt{N}(a_0 + a_0^\dagger) + N|\Lambda|).
\end{aligned}$$

From these identities and using $[\mathcal{N}_b, \mathcal{H}_N] = 0$ and $\phi_0 = |\Lambda|^{-\frac{1}{2}}$, we obtain that

$$\begin{aligned}
[\mathcal{N}_b, \hat{\mathcal{U}}_N(t)] &= [\mathcal{N}_b, \mathcal{W}^\dagger[\sqrt{N|\Lambda|}\phi_t]] e^{-it\mathcal{H}_N} \mathcal{W}[\sqrt{N|\Lambda|}\phi_0] \\
&\quad + \mathcal{W}^\dagger[\sqrt{N|\Lambda|}\phi_0] e^{-it\mathcal{H}_N} [\mathcal{N}_b, \mathcal{W}[\sqrt{N|\Lambda|}\phi_0]] \\
&= -\sqrt{N|\Lambda|}(\bar{\phi}_t a_0 + \phi_t a_0^\dagger) \hat{\mathcal{U}}_N(t) + \sqrt{N} \hat{\mathcal{U}}_N(t)(a_0 + a_0^\dagger). \tag{E.4}
\end{aligned}$$

As a consequence, we have that

$$\begin{aligned}
&\langle \hat{\mathcal{U}}_N(t)\psi, \mathcal{N}_b^2 \hat{\mathcal{U}}_N(t)\psi \rangle \\
&= \langle \mathcal{N}_b \hat{\mathcal{U}}_N(t)\psi, \hat{\mathcal{U}}_N(t)\mathcal{N}_b \psi \rangle + \sqrt{N} \langle \mathcal{N}_b \hat{\mathcal{U}}_N(t)\psi, \hat{\mathcal{U}}_N(t)(a_0 + a_0^\dagger) \psi \rangle \\
&\quad - \sqrt{N|\Lambda|} \langle \mathcal{N}_b \hat{\mathcal{U}}_N(t)\psi, (\bar{\phi}_t a_0 + \phi_t a_0^\dagger) \hat{\mathcal{U}}_N(t)\psi \rangle.
\end{aligned}$$

Using Cauchy-Schwarz and employing the fact that $\hat{\mathcal{U}}_N(t) : \mathcal{F} \rightarrow \mathcal{F}$ is a unitary transformation, we thus obtain

$$\begin{aligned} & \langle \hat{\mathcal{U}}_N(t)\psi, \mathcal{N}_b^2 \hat{\mathcal{U}}_N(t)\psi \rangle \\ & \leq \|\mathcal{N}_b \hat{\mathcal{U}}_N(t)\psi\| \left(\|\hat{\mathcal{U}}_N(t)\mathcal{N}_b\psi\| + \sqrt{N}(\sqrt{|\Lambda|}\|(\bar{\phi}_t a_0 + \phi_t a_0^\dagger)\hat{\mathcal{U}}_N(t)\psi\| \right. \\ & \quad \left. + \|\hat{\mathcal{U}}_N(t)(a_0 + a_0^\dagger)\psi\|) \right) \\ & \leq \|\mathcal{N}_b \hat{\mathcal{U}}_N(t)\psi\| \left(\|\mathcal{N}_b\psi\| + \sqrt{N}(\sqrt{|\Lambda|}|\phi_t|(\|a_0 \hat{\mathcal{U}}_N(t)\psi\| \right. \\ & \quad \left. + \|a_0^\dagger \hat{\mathcal{U}}_N(t)\psi\|) + \|a_0\psi\| + \|a_0^\dagger\psi\|) \right). \end{aligned} \quad (\text{E.5})$$

Lemma D.4 implies

$$\begin{aligned} \|a_0\psi\|^2 &= \sum_{M=0}^{\infty} \|P_M a_0\psi\|^2 = \sum_{M=1}^{\infty} \|a_0 P_{M-1}\psi\|^2 \\ &\leq \sum_{M=1}^{\infty} (M-1)|\Lambda| \|P_{M-1}\psi\|^2 = |\Lambda| \|\sqrt{\mathcal{N}_b}\psi\|^2. \end{aligned} \quad (\text{E.6})$$

Similarly, we have that

$$\|a_0^\dagger\psi\| \leq \sqrt{|\Lambda|} \|\sqrt{\mathcal{N}_b+1}\psi\| \quad (\text{E.7})$$

Employing (E.6) and (E.7), (E.5) implies

$$\begin{aligned} & \langle \hat{\mathcal{U}}_N(t)\psi, \mathcal{N}_b^2 \hat{\mathcal{U}}_N(t)\psi \rangle \\ & \leq \|\mathcal{N}_b \hat{\mathcal{U}}_N(t)\psi\| \left(\|\mathcal{N}_b\psi\| + 2\sqrt{N|\Lambda|}(\sqrt{|\Lambda|}|\phi_t| \|\sqrt{\mathcal{N}_b+1}\hat{\mathcal{U}}_N(t)\psi\| \right. \\ & \quad \left. + \|\sqrt{\mathcal{N}_b+1}\psi\|) \right). \end{aligned}$$

Young's inequality then implies

$$\begin{aligned} & \langle \hat{\mathcal{U}}_N(t)\psi, \mathcal{N}_b^2 \hat{\mathcal{U}}_N(t)\psi \rangle \\ & \leq \frac{1}{2} \langle \hat{\mathcal{U}}_N(t)\psi, \mathcal{N}_b^2 \hat{\mathcal{U}}_N(t)\psi \rangle + C \left(\langle \psi, (\mathcal{N}_b^2 + N|\Lambda|(\mathcal{N}_b + 1))\psi \rangle \right. \\ & \quad \left. + N|\Lambda|^2|\phi_t|^2 \langle \hat{\mathcal{U}}_N(t)\psi, (\mathcal{N}_b + 1)\hat{\mathcal{U}}_N(t)\psi \rangle \right) \end{aligned}$$

Rearranging terms, we thus obtain that

$$\begin{aligned} & \frac{1}{N|\Lambda|} \langle \hat{\mathcal{U}}_N(t)\psi, \mathcal{N}_b^2 \hat{\mathcal{U}}_N(t)\psi \rangle \\ & \leq C \left(|\Lambda||\phi_t|^2 \langle \hat{\mathcal{U}}_N(t)\psi, (\mathcal{N}_b + 1)\hat{\mathcal{U}}_N(t)\psi \rangle + \langle \psi, (\mathcal{N}_b + 1 + \frac{\mathcal{N}_b^2}{N|\Lambda|})\psi \rangle \right). \end{aligned} \quad (\text{E.8})$$

Substituting this into (E.3) and using $|\Lambda| \geq 1$, we obtain that

$$\begin{aligned} & |\partial_t \langle \hat{\mathcal{U}}_N(t)\psi, (\mathcal{N}_b + |\Lambda|)\hat{\mathcal{U}}_N(t)\psi \rangle| \\ & \leq C \|\hat{v}\|_{w,d} |\Lambda| \left(\frac{(\|\Gamma_t^T\|_1 + \|\Gamma_t^T\|_\infty)^2}{N} \right) \end{aligned}$$

$$+ |\Lambda| |\phi_t|^2 \langle \hat{\mathcal{U}}_N(t) \psi, (\mathcal{N}_b + |\Lambda|) \hat{\mathcal{U}}_N(t) \psi \rangle + \langle \psi, (\mathcal{N}_b + |\Lambda| + \frac{\mathcal{N}_b^2}{N|\Lambda|}) \psi \rangle \Big).$$

Lemma 4.5 then implies

$$\begin{aligned} & |\partial_t \langle \hat{\mathcal{U}}_N(t) \psi, (\mathcal{N}_b + |\Lambda|) \hat{\mathcal{U}}_N(t) \psi \rangle| \\ & \leq C \|\hat{v}\|_{w,d} \lambda |\Lambda| \left(e^{C\|\hat{v}\|_{w,d}(1+\frac{\|\Gamma_0^T\|_1}{N})\lambda t} \frac{(\|\Gamma_0^T\|_1 + \|\Gamma_0^T\|_\infty + 1)^2}{N} \right. \\ & \quad \left. + \left(1 + \frac{\|\Gamma_0^T\|_1}{N} \right) \langle \hat{\mathcal{U}}_N(t) \psi, (\mathcal{N}_b + |\Lambda|) \hat{\mathcal{U}}_N(t) \psi \rangle + \langle \psi, (\mathcal{N}_b + |\Lambda| + \frac{\mathcal{N}_b^2}{N|\Lambda|}) \psi \rangle \right) \end{aligned}$$

Grönwall's inequality then implies

$$\begin{aligned} & \langle \hat{\mathcal{U}}_N(t) \psi, (\mathcal{N}_b + |\Lambda|) \hat{\mathcal{U}}_N(t) \psi \rangle \\ & \leq C_1 \left(1 + \frac{(\|\Gamma_0^T\|_1 + \|\Gamma_0^T\|_\infty + 1)^2}{N} \right) e^{K_1 \|\hat{v}\|_{w,d}(1+\frac{\|\Gamma_0^T\|_1}{N})\lambda|\Lambda|t} \\ & \quad \langle \psi, (\mathcal{N}_b + |\Lambda| + \frac{\mathcal{N}_b^2}{N|\Lambda|}) \psi \rangle. \end{aligned}$$

(2) Induction Step: Assume that

$$\begin{aligned} & \langle \hat{\mathcal{U}}_N(t) \psi, (\mathcal{N}_b + |\Lambda|)^j \hat{\mathcal{U}}_N(t) \psi \rangle \\ & \leq C_j \left(1 + \frac{(\|\Gamma_0^T\|_1 + \|\Gamma_0^T\|_\infty + 1)^2}{N} \right)^j e^{K_j \|\hat{v}\|_{w,d}\lambda|\Lambda|(1+\frac{\|\Gamma_0^T\|_1}{N})t} \\ & \quad \langle \psi, (\mathcal{N}_b + |\Lambda|)^j \left(1 + \frac{\mathcal{N}_b^2}{N|\Lambda|} \right) \psi \rangle \end{aligned} \tag{E.9}$$

for all $1 \leq j \leq \ell$ and some constants C_j, K_j , and any $\psi \in \mathcal{F}$. We compute

$$\begin{aligned} & i\partial_t \langle \hat{\mathcal{U}}_N(t) \psi, (\mathcal{N}_b + |\Lambda|)^{\ell+1} \hat{\mathcal{U}}_N(t) \psi \rangle \\ & = \langle \hat{\mathcal{U}}_N(t) \psi, [(\mathcal{N}_b + |\Lambda|)^{\ell+1}, \mathcal{H}_{\text{BEC}}^{(\phi_t)} + \mathcal{H}_{\text{HFB}}^{(\text{cor}, \phi_t)} + \mathcal{H}_{\text{cub}}^{(\phi_t)}] \hat{\mathcal{U}}_N(t) \psi \rangle \\ & = \sum_{j=1}^{\ell+1} \langle \hat{\mathcal{U}}_N(t) \psi, (\mathcal{N}_b + |\Lambda|)^{j-1} [\mathcal{N}_b, \mathcal{H}_{\text{BEC}}^{(\phi_t)} + \mathcal{H}_{\text{HFB}}^{(\text{cor}, \phi_t)} + \mathcal{H}_{\text{cub}}^{(\phi_t)}] \\ & \quad (\mathcal{N}_b + |\Lambda|)^{\ell+1-j} \hat{\mathcal{U}}_N(t) \psi \rangle. \end{aligned} \tag{E.10}$$

Let

$$A_{\text{cub}}[\hat{v}] := \int d\mathbf{p}_3 \hat{v}(p_2) \delta(p_1 + p_2 - p_3) a_{p_1}^\dagger a_{p_2}^\dagger a_{p_3}.$$

Using (D.2) and recalling Lemma D.3, (E.10) yields

$$\begin{aligned}
& i\partial_t \langle \hat{\mathcal{U}}_N(t)\psi, (\mathcal{N}_b + |\Lambda|)^{\ell+1} \hat{\mathcal{U}}_N(t)\psi \rangle \\
&= 2\lambda \sum_{j=1}^{\ell+1} \operatorname{Im} \left(-\frac{\sqrt{|\Lambda|}}{\sqrt{N}} ((\Gamma_t^T * (\hat{v} + \hat{v}(0)))(0)\phi_t + (\Sigma_t^T * \hat{v})(0)\bar{\phi}_t) \right. \\
&\quad \left. \langle \hat{\mathcal{U}}_N(t)\psi, (\mathcal{N}_b + |\Lambda|)^{j-1} a_0^\dagger (\mathcal{N}_b + |\Lambda|)^{\ell+1-j} \hat{\mathcal{U}}_N(t)\psi \rangle \right. \\
&\quad \left. + |\Lambda| \bar{\phi}_t^2 \langle \hat{\mathcal{U}}_N(t)\psi, (\mathcal{N}_b + |\Lambda|)^{j-1} g[\hat{v}] (\mathcal{N}_b + |\Lambda|)^{\ell+1-j} \hat{\mathcal{U}}_N(t)\psi \rangle \right. \\
&\quad \left. - \frac{\sqrt{|\Lambda|}\phi_t}{\sqrt{N}} \langle \hat{\mathcal{U}}_N(t)\psi, (\mathcal{N}_b + |\Lambda|)^{j-1} A_{\text{cub}}[\hat{v}] (\mathcal{N}_b + |\Lambda|)^{\ell+1-j} \hat{\mathcal{U}}_N(t)\psi \rangle \right) \\
&= 2\lambda \sum_{j=1}^{\ell+1} \operatorname{Im} \sum_{m,n=0}^{\infty} (m + |\Lambda|)^{j-1} (n + |\Lambda|)^{\ell+1-j} \\
&\quad \left\langle \hat{\mathcal{U}}_N(t)\psi, P_m \left(-\frac{\sqrt{|\Lambda|}}{\sqrt{N}} ((\Gamma_t^T * (\hat{v} + \hat{v}(0)))(0)\phi_t \right. \right. \\
&\quad \left. \left. + (\Sigma_t^T * \hat{v})(0)\bar{\phi}_t) a_0^\dagger + |\Lambda| \bar{\phi}_t^2 g[\hat{v}] - \frac{\sqrt{|\Lambda|}\phi_t}{\sqrt{N}} A_{\text{cub}}[\hat{v}] \right) P_n \hat{\mathcal{U}}_N(t)\psi \right\rangle.
\end{aligned}$$

Observe that we have

$$\begin{aligned}
P_m g[\hat{v}] P_n &= P_m g[\hat{v}] P_{m+2} \delta_{n,m+2}, \\
P_m A_{\text{cub}}[\hat{v}] P_n &= P_m A_{\text{cub}}[\hat{v}] P_{m-1} \delta_{n,m-1}.
\end{aligned}$$

Lemmata D.3 and D.4, and (E.11) then imply

$$\begin{aligned}
& |\partial_t \langle \hat{\mathcal{U}}_N(t)\psi, (\mathcal{N}_b + |\Lambda|)^{\ell+1} \hat{\mathcal{U}}_N(t)\psi \rangle| \\
&\leq C\lambda \sum_{j=1}^{\ell+1} \sum_{m,n=0}^{\infty} (m + |\Lambda|)^{j-1} (n + |\Lambda|)^{\ell+1-j} \|P_m \hat{\mathcal{U}}_N(t)\psi\| \|P_n \hat{\mathcal{U}}_N(t)\psi\| \\
&\quad \left(\frac{\sqrt{|\Lambda|}}{\sqrt{N}} \|\hat{v}\|_{w,d} (\|\Gamma_t^T\|_1 + \|\Gamma_t^T\|_\infty) |\phi_t| \|a_0^\dagger P_{m-1}\| \delta_{n,m-1} \right. \\
&\quad \left. + |\Lambda| |\phi_t|^2 \|g[\hat{v}] P_{m+2}\| \delta_{n,m+2} + \frac{\sqrt{|\Lambda|} |\phi_t| \|A_{\text{cub}}[\hat{v}] P_{m-1}\|}{\sqrt{N}} \delta_{n,m-1} \right) \\
&\leq C \|\hat{v}\|_{w,d} \lambda \sum_{j=1}^{\ell+1} \sum_{m,n=0}^{\infty} (m + |\Lambda|)^{j-1} (n + |\Lambda|)^{\ell+1-j} \\
&\quad \left((\|P_m \hat{\mathcal{U}}_N(t)\psi\|^2 + \|P_n \hat{\mathcal{U}}_N(t)\psi\|^2) \right. \\
&\quad \left(\frac{|\Lambda|}{\sqrt{N}} (\|\Gamma_t^T\|_1 + \|\Gamma_t^T\|_\infty) |\phi_t| m^{\frac{1}{2}} \delta_{n,m-1} \right. \\
&\quad \left. + |\Lambda| |\phi_t|^2 (m + 2 + |\Lambda|) \delta_{n,m+2} + \frac{\sqrt{|\Lambda|} |\phi_t| m^{\frac{3}{2}}}{\sqrt{N}} \delta_{n,m-1} \right)
\end{aligned}$$

$$\begin{aligned} &\leq C\|\hat{v}\|_{w,d}\lambda|\Lambda|\sum_{j=1}^{\ell+1}\sum_{m=0}^{\infty}(m+|\Lambda|)^{j-1}(m+2+|\Lambda|)^{\ell+1-j}\|P_m\hat{\mathcal{U}}_N(t)\psi\|^2 \\ &\quad \left(\frac{(\|\Gamma_t^T\|_1+\|\Gamma_t^T\|_\infty)^2}{N}+|\phi_t|^2(m+2+|\Lambda|)+\frac{(m+1)^2}{N|\Lambda|}\right), \end{aligned}$$

where, in the last step, we applied Cauchy-Schwarz. We can rewrite the last inequality as

$$\begin{aligned} &|\partial_t\langle\hat{\mathcal{U}}_N(t)\psi, (\mathcal{N}_b+|\Lambda|)^{\ell+1}\hat{\mathcal{U}}_N(t)\psi\rangle| \\ &\leq K_\ell\|\hat{v}\|_{w,d}\lambda|\Lambda|\left(\frac{(\|\Gamma_t^T\|_1+\|\Gamma_t^T\|_\infty)^2}{N}\langle\hat{\mathcal{U}}_N(t)\psi, (\mathcal{N}_b+|\Lambda|)^\ell\hat{\mathcal{U}}_N(t)\psi\rangle\right. \\ &\quad +|\phi_t|^2\langle\hat{\mathcal{U}}_N(t)\psi, (\mathcal{N}_b+|\Lambda|)^{\ell+1}\hat{\mathcal{U}}_N(t)\psi\rangle \\ &\quad \left.+\frac{1}{N|\Lambda|}\langle\hat{\mathcal{U}}_N(t)\psi, (\mathcal{N}_b+|\Lambda|)^{\ell+2}\hat{\mathcal{U}}_N(t)\psi\rangle\right). \end{aligned}$$

The induction hypothesis (E.9) and Lemma 4.5 then imply

$$\begin{aligned} &|\partial_t\langle\hat{\mathcal{U}}_N(t)\psi, (\mathcal{N}_b+|\Lambda|)^{\ell+1}\hat{\mathcal{U}}_N(t)\psi\rangle| \\ &\leq C_\ell\|\hat{v}\|_{w,d}\lambda|\Lambda|\left[\left(1+\frac{(\|\Gamma_0^T\|_1+\|\Gamma_0^T\|_\infty+1)^2}{N}\right)^\ell e^{K_\ell\|\hat{v}\|_{w,d}\lambda|\Lambda|(1+\frac{\|\Gamma_0^T\|_1}{N})t}\right. \\ &\quad \langle\psi, (\mathcal{N}_b+|\Lambda|)^\ell(1+\frac{\mathcal{N}_b}{N|\Lambda|})\psi\rangle \\ &\quad +\left(1+\frac{\|\Gamma_0^T\|_1}{N}\right)\langle\hat{\mathcal{U}}_N(t)\psi, (\mathcal{N}_b+|\Lambda|)^{\ell+1}\hat{\mathcal{U}}_N(t)\psi\rangle \\ &\quad \left.+\frac{1}{N}\langle\hat{\mathcal{U}}_N(t)\psi, (\mathcal{N}_b+|\Lambda|)^{\ell+2}\hat{\mathcal{U}}_N(t)\psi\rangle\right]. \end{aligned} \tag{E.11}$$

We claim that

$$\begin{aligned} &\frac{1}{N|\Lambda|}\langle\hat{\mathcal{U}}_N(t)\psi, (\mathcal{N}_b+|\Lambda|)^{j+1}\hat{\mathcal{U}}_N(t)\psi\rangle \\ &\leq C_j\left(\left(1+\frac{(\|\Gamma_0^T\|_1+\|\Gamma_0^T\|_\infty+1)^2}{N}\right)^je^{K_j\|\hat{v}\|_{w,d}\lambda|\Lambda|(1+\frac{\|\Gamma_0^T\|_1}{N})t}\right. \\ &\quad \langle\psi, (\mathcal{N}_b+|\Lambda|)^j(1+\frac{\mathcal{N}_b}{N|\Lambda|})\psi\rangle \\ &\quad \left.+\left(1+\frac{\|\Gamma_0^T\|_1}{N}\right)\langle\hat{\mathcal{U}}_N(t)\psi, (\mathcal{N}_b+|\Lambda|)^j\hat{\mathcal{U}}_N(t)\psi\rangle\right) \end{aligned} \tag{E.12}$$

for all $1 \leq j \leq \ell+1$ and all $\psi \in \mathcal{F}$. (E.12) for $j = \ell+1$ together with (E.11) then implies

$$\begin{aligned} &|\partial_t\langle\hat{\mathcal{U}}_N(t)\psi, (\mathcal{N}_b+|\Lambda|)^{\ell+1}\hat{\mathcal{U}}_N(t)\psi\rangle| \\ &\leq C_\ell\|\hat{v}\|_{w,d}\left(1+\frac{\|\Gamma_0^T\|_1}{N}\right)\lambda|\Lambda| \\ &\quad \left(\langle\hat{\mathcal{U}}_N(t)\psi, (\mathcal{N}_b+|\Lambda|)^{\ell+1}\hat{\mathcal{U}}_N(t)\psi\rangle+e^{K_{\ell+1}\|\hat{v}\|_{w,d}\lambda|\Lambda|(1+\frac{\|\Gamma_0^T\|_1}{N})t}\right. \end{aligned}$$

$$\left(1 + \frac{(\|\Gamma_0^T\|_1 + \|\Gamma_0^T\|_\infty + 1)^2}{N}\right)^{\ell+1} \langle \psi, (\mathcal{N}_b + |\Lambda|)^{\ell+1} \left(1 + \frac{\mathcal{N}_b}{N|\Lambda|}\right) \psi \rangle.$$

Grönwall's Lemma then implies

$$\begin{aligned} & \langle \hat{\mathcal{U}}_N(t)\psi, (\mathcal{N}_b + |\Lambda|)^{\ell+1} \hat{\mathcal{U}}_N(t)\psi \rangle \\ & \leq C_{\ell+1} e^{K_{\ell+1} \|\hat{v}\|_{w,d} \lambda |\Lambda| (1 + \frac{\|\Gamma_0^T\|_1}{N}) t} \left(1 + \frac{(\|\Gamma_0^T\|_1 + \|\Gamma_0^T\|_\infty + 1)^2}{N}\right)^{\ell+1} \\ & \quad \langle \psi, (\mathcal{N}_b + |\Lambda|)^{\ell+1} \left(1 + \frac{\mathcal{N}_b}{N|\Lambda|}\right) \psi \rangle. \end{aligned}$$

Thus, proving (E.12) for $j = \ell + 1$ concludes the proof. We have proved (E.12) for $j = 1$ in Step 1, (E.8). We have that (E.12) also holds for $j = 0$: In fact, for

$$\begin{aligned} \langle \Psi, (\bar{\phi}_t a_0 + \phi_t a_0^\dagger) \Psi \rangle & \leq |\phi_t| \|\Psi\| (\|a_0 \Psi\| + \|a_0^\dagger \Psi\|) \\ & \leq 2|\phi_t| \sqrt{|\Lambda|} \|\Psi\| \sqrt{\mathcal{N}_b + 1} \|\Psi\| \\ & \leq \langle \Psi, (\mathcal{N}_b + 1 + |\Lambda| |\phi_t|^2) \Psi \rangle, \end{aligned}$$

due to (E.6) and (E.7) followed by Young's inequality, we find that

$$\begin{aligned} & \mathcal{W}[\sqrt{N|\Lambda|} \phi_t] \mathcal{N}_b \mathcal{W}^\dagger[\sqrt{N|\Lambda|} \phi_t] \\ & = \mathcal{N}_b - \sqrt{N|\Lambda|} (\bar{\phi}_t a_0 + \phi_t a_0^\dagger) + N|\Lambda|^2 |\phi_t|^2 \\ & \leq 2(\mathcal{N}_b + |\Lambda| + N|\Lambda|^2 |\phi_t|^2), \end{aligned}$$

which then commutes with $e^{-it\mathcal{H}_N}$.

Suppose (E.12) holds up to some $1 \leq j \leq \ell$. Applying (E.4), we have that

$$\begin{aligned} & \frac{1}{N|\Lambda|} \langle \hat{\mathcal{U}}_N(t)\psi, (\mathcal{N}_b + |\Lambda|)^{j+2} \hat{\mathcal{U}}_N(t)\psi \rangle \\ & = \frac{1}{N|\Lambda|} \langle (\mathcal{N}_b + |\Lambda|)^{j+1} \hat{\mathcal{U}}_N(t)\psi, \hat{\mathcal{U}}_N(t)(\mathcal{N}_b + |\Lambda|)\psi \rangle \\ & \quad - \frac{1}{\sqrt{N|\Lambda|}} \langle (\mathcal{N}_b + |\Lambda|)^{j+1} \hat{\mathcal{U}}_N(t)\psi, (\bar{\phi}_t a_0 + \phi_t a_0^\dagger) \hat{\mathcal{U}}_N(t)\psi \rangle \\ & \quad + \frac{1}{\sqrt{N|\Lambda|}} \langle (\mathcal{N}_b + |\Lambda|)^{j+1} \hat{\mathcal{U}}_N(t)\psi, \hat{\mathcal{U}}_N(t)(a_0 + a_0^\dagger)\psi \rangle. \end{aligned} \tag{E.13}$$

Whenever $\phi_t \neq 0$, we can bound the second term by

$$\begin{aligned} & \frac{1}{\sqrt{N|\Lambda|}} \langle (\mathcal{N}_b + |\Lambda|)^{j+1} \hat{\mathcal{U}}_N(t)\psi, (\bar{\phi}_t a_0 + \phi_t a_0^\dagger) \hat{\mathcal{U}}_N(t)\psi \rangle \\ & \leq \alpha |\Lambda| |\phi_t|^2 \langle \hat{\mathcal{U}}_N(t)\psi, (\mathcal{N}_b + |\Lambda|)^{j+1} \hat{\mathcal{U}}_N(t)\psi \rangle \\ & \quad + \frac{\langle \hat{\mathcal{U}}_N(t)\psi, (\bar{\phi}_t a_0 + \phi_t a_0^\dagger)(\mathcal{N}_b + |\Lambda|)^{j+1} (\bar{\phi}_t a_0 + \phi_t a_0^\dagger) \hat{\mathcal{U}}_N(t)\psi \rangle}{\alpha N|\Lambda|^2 |\phi_t|^2}. \end{aligned} \tag{E.14}$$

Employing (E.6) and (E.7), we find that, for any $\tilde{\psi} \in \mathcal{F}$ and $k \in \mathbb{N}_0$,

$$\begin{aligned} & \langle \tilde{\psi}, (\bar{\phi}_t a_0 + \phi_t a_0^\dagger)(\mathcal{N}_b + |\Lambda|)^k (\bar{\phi}_t a_0 + \phi_t a_0^\dagger) \tilde{\psi} \rangle \\ & \leq |\phi_t|^2 (\|(\mathcal{N}_b + |\Lambda|)^{\frac{k}{2}} a_0 \tilde{\psi}\| + \|(\mathcal{N}_b + |\Lambda|)^{\frac{k}{2}} a_0^\dagger \tilde{\psi}\|)^2 \\ & = |\phi_t|^2 (\|a_0(\mathcal{N}_b + |\Lambda| - 1)^{\frac{k}{2}} \tilde{\psi}\| + \|a_0^\dagger(\mathcal{N}_b + |\Lambda| + 1)^{\frac{k}{2}} \tilde{\psi}\|)^2 \\ & \leq C_k |\Lambda| |\phi_t|^2 \langle \tilde{\psi}, (\mathcal{N}_b + |\Lambda|)^{k+1} \tilde{\psi} \rangle, \end{aligned}$$

i.e.,

$$(\bar{\phi}_t a_0 + \phi_t a_0^\dagger)(\mathcal{N}_b + |\Lambda|)^k (\bar{\phi}_t a_0 + \phi_t a_0^\dagger) \leq C_k |\Lambda| |\phi_t|^2 (\mathcal{N}_b + |\Lambda|)^{k+1}. \quad (\text{E.15})$$

Analogously, we have that

$$(a_0 + a_0^\dagger)(\mathcal{N}_b + |\Lambda|)^k (a_0 + a_0^\dagger) \leq C_k |\Lambda| (\mathcal{N}_b + |\Lambda|)^{k+1}. \quad (\text{E.16})$$

Employing (E.15) and choosing $\alpha > 0$ sufficiently large, (E.14) implies

$$\begin{aligned} & \frac{1}{\sqrt{N|\Lambda|}} \langle (\mathcal{N}_b + |\Lambda|)^{j+1} \hat{\mathcal{U}}_N(t) \psi, (\bar{\phi}_t a_0 + \phi_t a_0^\dagger) \hat{\mathcal{U}}_N(t) \psi \rangle \\ & \leq C_j |\Lambda| |\phi_t|^2 \langle \hat{\mathcal{U}}_N(t) \psi, (\mathcal{N}_b + |\Lambda|)^{j+1} \hat{\mathcal{U}}_N(t) \psi \rangle \\ & \quad + \frac{1}{4N|\Lambda|} \langle \hat{\mathcal{U}}_N(t) \psi, (\mathcal{N}_b + |\Lambda|)^{j+2} \hat{\mathcal{U}}_N(t) \psi \rangle. \end{aligned} \quad (\text{E.17})$$

We bound the third term in (E.13) by

$$\begin{aligned} & \frac{1}{\sqrt{N|\Lambda|}} | \langle (\mathcal{N}_b + |\Lambda|)^{j+1} \hat{\mathcal{U}}_N(t) \psi, \hat{\mathcal{U}}_N(t) (a_0 + a_0^\dagger) \psi \rangle | \\ & \leq \frac{1}{|\Lambda|} \langle \hat{\mathcal{U}}_N(t) (a_0 + a_0^\dagger) \psi, (\mathcal{N}_b + |\Lambda|)^j \hat{\mathcal{U}}_N(t) (a_0 + a_0^\dagger) \psi \rangle \\ & \quad + \frac{1}{4N|\Lambda|} \langle \hat{\mathcal{U}}_N(t) \psi, (\mathcal{N}_b + |\Lambda|)^{j+2} \hat{\mathcal{U}}_N(t) \psi \rangle. \end{aligned} \quad (\text{E.18})$$

In particular, (E.13), (E.17), and (E.18) imply

$$\begin{aligned} & \frac{1}{N|\Lambda|} \langle \hat{\mathcal{U}}_N(t) \psi, (\mathcal{N}_b + |\Lambda|)^{j+2} \hat{\mathcal{U}}_N(t) \psi \rangle \\ & \leq \frac{2}{N|\Lambda|} | \langle (\mathcal{N}_b + |\Lambda|)^{j+1} \hat{\mathcal{U}}_N(t) \psi, \hat{\mathcal{U}}_N(t) (\mathcal{N}_b + |\Lambda|) \psi \rangle | \\ & \quad + C_j |\Lambda| |\phi_t|^2 \langle \hat{\mathcal{U}}_N(t) \psi, (\mathcal{N}_b + |\Lambda|)^{j+1} \hat{\mathcal{U}}_N(t) \psi \rangle \\ & \quad + \frac{1}{|\Lambda|} \langle \hat{\mathcal{U}}_N(t) (a_0 + a_0^\dagger) \psi, (\mathcal{N}_b + |\Lambda|)^j \hat{\mathcal{U}}_N(t) (a_0 + a_0^\dagger) \psi \rangle. \end{aligned} \quad (\text{E.19})$$

For the first term in (E.19), we apply (E.4) to the left and obtain

$$\begin{aligned}
& \frac{1}{N|\Lambda|} \langle (\mathcal{N}_b + |\Lambda|)^{j+1} \hat{\mathcal{U}}_N(t) \psi, \hat{\mathcal{U}}_N(t) (\mathcal{N}_b + |\Lambda|) \psi \rangle \\
&= \frac{1}{N|\Lambda|} \langle \hat{\mathcal{U}}_N(t) (\mathcal{N}_b + |\Lambda|) \psi, (\mathcal{N}_b + |\Lambda|)^j \hat{\mathcal{U}}_N(t) (\mathcal{N}_b + |\Lambda|) \psi \rangle \\
&\quad - \frac{1}{\sqrt{N|\Lambda|}} \langle (\mathcal{N}_b + |\Lambda|)^j (\bar{\phi}_t a_0 + \phi_t a_0^\dagger) \hat{\mathcal{U}}_N(t) \psi, \hat{\mathcal{U}}_N(t) (\mathcal{N}_b + |\Lambda|) \psi \rangle \\
&\quad + \frac{1}{\sqrt{N|\Lambda|}} \langle (\mathcal{N}_b + |\Lambda|)^j \hat{\mathcal{U}}_N(t) (a_0 + a_0^\dagger) \psi, \hat{\mathcal{U}}_N(t) (\mathcal{N}_b + |\Lambda|) \psi \rangle.
\end{aligned} \tag{E.20}$$

With analogous steps to above, we bound the second term in (E.20) by

$$\begin{aligned}
& \frac{1}{\sqrt{N|\Lambda|}} \left| \langle (\mathcal{N}_b + |\Lambda|)^j (\bar{\phi}_t a_0 + \phi_t a_0^\dagger) \hat{\mathcal{U}}_N(t) \psi, \hat{\mathcal{U}}_N(t) (\mathcal{N}_b + |\Lambda|) \psi \rangle \right| \\
&\leq C_j |\Lambda| |\phi_t|^2 \langle \hat{\mathcal{U}}_N(t) \psi, (\mathcal{N}_b + |\Lambda|)^{j+1} \hat{\mathcal{U}}_N(t) \psi \rangle \\
&\quad + \frac{1}{N|\Lambda|} \langle \hat{\mathcal{U}}_N(t) (\mathcal{N}_b + |\Lambda|) \psi, (\mathcal{N}_b + |\Lambda|)^j \hat{\mathcal{U}}_N(t) (\mathcal{N}_b + |\Lambda|) \psi \rangle,
\end{aligned}$$

and the third term by

$$\begin{aligned}
& \frac{1}{\sqrt{N|\Lambda|}} \left| \langle (\mathcal{N}_b + |\Lambda|)^j \hat{\mathcal{U}}_N(t) (a_0 + a_0^\dagger) \psi, \hat{\mathcal{U}}_N(t) (\mathcal{N}_b + |\Lambda|) \psi \rangle \right| \\
&\leq \frac{1}{|\Lambda|} \langle \hat{\mathcal{U}}_N(t) (a_0 + a_0^\dagger) \psi, (\mathcal{N}_b + |\Lambda|)^j \hat{\mathcal{U}}_N(t) (a_0 + a_0^\dagger) \psi \rangle \\
&\quad + \frac{1}{N|\Lambda|} \langle \hat{\mathcal{U}}_N(t) (\mathcal{N}_b + |\Lambda|) \psi, (\mathcal{N}_b + |\Lambda|)^j \hat{\mathcal{U}}_N(t) (\mathcal{N}_b + |\Lambda|) \psi \rangle.
\end{aligned}$$

In particular, we can bound (E.20) by

$$\begin{aligned}
& \frac{1}{N|\Lambda|} \left| \langle (\mathcal{N}_b + |\Lambda|)^{j+1} \hat{\mathcal{U}}_N(t) \psi, \hat{\mathcal{U}}_N(t) (\mathcal{N}_b + |\Lambda|) \psi \rangle \right| \\
&\leq \frac{3}{N|\Lambda|} \langle \hat{\mathcal{U}}_N(t) (\mathcal{N}_b + |\Lambda|) \psi, (\mathcal{N}_b + |\Lambda|)^j \hat{\mathcal{U}}_N(t) (\mathcal{N}_b + |\Lambda|) \psi \rangle \\
&\quad + C_j |\Lambda| |\phi_t|^2 \langle \hat{\mathcal{U}}_N(t) \psi, (\mathcal{N}_b + |\Lambda|)^{j+1} \hat{\mathcal{U}}_N(t) \psi \rangle \\
&\quad + \frac{1}{|\Lambda|} \langle \hat{\mathcal{U}}_N(t) (a_0 + a_0^\dagger) \psi, (\mathcal{N}_b + |\Lambda|)^j \hat{\mathcal{U}}_N(t) (a_0 + a_0^\dagger) \psi \rangle.
\end{aligned}$$

As a consequence, (E.19) implies

$$\begin{aligned}
& \frac{1}{N|\Lambda|} \langle \hat{\mathcal{U}}_N(t) \psi, (\mathcal{N}_b + |\Lambda|)^{j+2} \hat{\mathcal{U}}_N(t) \psi \rangle \\
&\leq \frac{6}{N|\Lambda|} \langle \hat{\mathcal{U}}_N(t) (\mathcal{N}_b + |\Lambda|) \psi, (\mathcal{N}_b + |\Lambda|)^j \hat{\mathcal{U}}_N(t) (\mathcal{N}_b + |\Lambda|) \psi \rangle \\
&\quad + C_j |\Lambda| |\phi_t|^2 \langle \hat{\mathcal{U}}_N(t) \psi, (\mathcal{N}_b + |\Lambda|)^{j+1} \hat{\mathcal{U}}_N(t) \psi \rangle \\
&\quad + \frac{2}{|\Lambda|} \langle \hat{\mathcal{U}}_N(t) (a_0 + a_0^\dagger) \psi, (\mathcal{N}_b + |\Lambda|)^j \hat{\mathcal{U}}_N(t) (a_0 + a_0^\dagger) \psi \rangle.
\end{aligned} \tag{E.21}$$

For the first term in (E.20), we use the induction hypothesis (E.12), and obtain

$$\begin{aligned}
& \frac{1}{N|\Lambda|} \langle \hat{\mathcal{U}}_N(t)(\mathcal{N}_b + |\Lambda|)\psi, (\mathcal{N}_b + |\Lambda|)^j \hat{\mathcal{U}}_N(t)(\mathcal{N}_b + |\Lambda|)\psi \rangle \\
& \leq C_{j-1} \left(\left(1 + \frac{(\|\Gamma_0^T\|_1 + \|\Gamma_0^T\|_\infty + 1)^2}{N} \right)^{j-1} e^{K_{j-1}\|\hat{v}\|_{w,d}\lambda|\Lambda|(1+\frac{\|\Gamma_0^T\|_1}{N})t} \right. \\
& \quad \langle \psi, (\mathcal{N}_b + |\Lambda|)^{j+1} \left(1 + \frac{\mathcal{N}_b}{N|\Lambda|} \right) \psi \rangle \\
& \quad \left. + \left(1 + \frac{\|\Gamma_0^T\|_1}{N} \right) \langle \hat{\mathcal{U}}_N(t)(\mathcal{N}_b + |\Lambda|)\psi, (\mathcal{N}_b + |\Lambda|)^{j-1} \hat{\mathcal{U}}_N(t)(\mathcal{N}_b + |\Lambda|)\psi \rangle \right) \\
& \leq C_j \left(1 + \frac{(\|\Gamma_0^T\|_1 + \|\Gamma_0^T\|_\infty + 1)^2}{N} \right)^j e^{K_j\|\hat{v}\|_{w,d}\lambda|\Lambda|(1+\frac{\|\Gamma_0^T\|_1}{N})t} \\
& \quad \langle \psi, (\mathcal{N}_b + |\Lambda|)^{j+1} \left(1 + \frac{\mathcal{N}_b}{N|\Lambda|} \right) \psi \rangle
\end{aligned} \tag{E.22}$$

where in the last step, we employed the induction hypothesis (E.9).

For the third term in (E.20), we use the induction hypothesis (E.9) to obtain

$$\begin{aligned}
& \frac{2}{|\Lambda|} \langle \hat{\mathcal{U}}_N(t)(a_0 + a_0^\dagger)\psi, (\mathcal{N}_b + |\Lambda|)^j \hat{\mathcal{U}}_N(t)(a_0 + a_0^\dagger)\psi \rangle \\
& \leq \frac{C_j}{|\Lambda|} \left(1 + \frac{(\|\Gamma_0^T\|_1 + \|\Gamma_0^T\|_\infty + 1)^2}{N} \right)^j e^{K_j\|\hat{v}\|_{w,d}\lambda|\Lambda|(1+\frac{\|\Gamma_0^T\|_1}{N})t} \\
& \quad \langle (a_0 + a_0^\dagger)\psi, (\mathcal{N}_b + |\Lambda|)^j \left(1 + \frac{\mathcal{N}_b}{N|\Lambda|} \right) (a_0 + a_0^\dagger)\psi \rangle \\
& \leq C_j \left(1 + \frac{(\|\Gamma_0^T\|_1 + \|\Gamma_0^T\|_\infty + 1)^2}{N} \right)^j e^{K_j\|\hat{v}\|_{w,d}\lambda|\Lambda|(1+\frac{\|\Gamma_0^T\|_1}{N})t} \\
& \quad \langle \psi, (\mathcal{N}_b + |\Lambda|)^{j+1} \left(1 + \frac{\mathcal{N}_b}{N|\Lambda|} \right) \psi \rangle,
\end{aligned} \tag{E.23}$$

where, in the last step, we employed (E.16).

Substituting (E.22) and (E.23) into (E.21), we arrive at

$$\begin{aligned}
& \frac{1}{N|\Lambda|} \langle \hat{\mathcal{U}}_N(t)\psi, (\mathcal{N}_b + |\Lambda|)^{j+2} \hat{\mathcal{U}}_N(t)\psi \rangle \\
& \leq C_j \left(1 + \frac{(\|\Gamma_0^T\|_1 + \|\Gamma_0^T\|_\infty + 1)^2}{N} \right)^{j+1} e^{K_j\|\hat{v}\|_{w,d}\lambda|\Lambda|(1+\frac{\|\Gamma_0^T\|_1}{N})t} \\
& \quad \langle \psi, (\mathcal{N}_b + |\Lambda|)^{j+1} \left(1 + \frac{\mathcal{N}_b}{N|\Lambda|} \right) \psi \rangle \\
& \quad + C_j |\Lambda| |\phi_t|^2 \langle \hat{\mathcal{U}}_N(t)\psi, (\mathcal{N}_b + |\Lambda|)^{j+1} \hat{\mathcal{U}}_N(t)\psi \rangle.
\end{aligned}$$

This concludes the proof. □

Lemma E.3 (Expressions for \mathcal{H}_{BEC} and \mathcal{H}_{HFB}). *Assume that $(\phi^{(2)}, \gamma^{(2)}, \sigma^{(2)})$ satisfy (3.41) and that $\Omega^{(2)}$ satisfies (3.43). Then we have*

$$\mathcal{H}_{\text{BEC}}(t) =$$

$$\begin{aligned}
& - \frac{\lambda\sqrt{|\Lambda|}}{\sqrt{N}} \left[u_t(0) \left(2(f_0^{(+)} \sigma_t * \hat{v})(0) \bar{\phi}_t + ((1+2\gamma_t)f_0^{(+)}) * (\hat{v} + \hat{v}(0))(0) \phi_t \right) \right. \\
& + v_t(0) \left(2(f_0^{(+)} \sigma_t * \hat{v})(0) \phi_t + ((1+2\gamma_t)f_0^{(+)}) * (\hat{v} + \hat{v}(0))(0) \bar{\phi}_t \right) e^{i \int_0^t ds \Omega_s(0)} a_0^\dagger \\
& \left. + \text{h.c.}, \right] \\
\mathcal{H}_{\text{HFB}}^{(\text{d})}(t) & = - \frac{\lambda}{N} \int dp \left[2 \operatorname{Re} \left(((f_0^{(+)} \bar{\sigma}_t^{(2)}) * \hat{v})(p) \sigma_t^{(2)}(p) \right) \right. \\
& \left. + ((f_0^{(+)} (1+2\gamma^{(2)})) * (\hat{v} + \hat{v}(0)))(p) (1+2\gamma_t^{(2)}(p)) \right] a_p^\dagger a_p, \\
\mathcal{H}_{\text{HFB}}^{(\text{cor})}(t) & = - \frac{\lambda}{N} \int dp \left[\left((f_0^{(+)} (1+2\gamma_t^{(2)})) * (\hat{v} + \hat{v}(0)) \right)(p) \sigma_t^{(2)}(p) \right. \\
& \left. + ((f_0^{(+)} \sigma_t^{(2)}) * \hat{v})(p) (1+\gamma_t^{(2)}(p)) \right. \\
& \left. + ((f_0^{(+)} \bar{\sigma}_t^{(2)}) * \hat{v})(p) \frac{\sigma_t^{(2)}(p)^2}{1+\gamma_t^{(2)}(p)} \right] e^{2i \int_0^t d\tau \Omega_\tau^{(2)}(p)} a_p^\dagger a_{-p}^\dagger + \text{h.c.}
\end{aligned}$$

Proof. We start by computing $\mathcal{H}_{\text{BEC}}(t)$. By Lemma A.2, we have that

$$\begin{aligned} \mathcal{H}_{\text{BEC}}(t) = & \sqrt{N|\Lambda|} \left[u_t(0) \left(-i\partial_t \phi_t + \lambda |\Lambda| \hat{v}(0) |\phi_t|^2 \phi_t + \frac{\lambda}{N} (\sigma_t * \hat{v})(0) \bar{\phi}_t \right. \right. \\ & + \frac{\lambda}{N} (\gamma_t * (\hat{v} + \hat{v}(0)))(0) \phi_t \Big) + v_t(0) \left(-i\overline{\partial_t \phi_t} + \lambda |\Lambda| \hat{v}(0) |\phi_t|^2 \bar{\phi}_t \right. \\ & \left. \left. + \frac{\lambda}{N} (\bar{\sigma}_t * \hat{v})(0) \phi_t + \frac{\lambda}{N} (\gamma_t * (\hat{v} + \hat{v}(0)))(0) \bar{\phi}_t \right) \right] e^{i \int_0^t ds \Omega_s(0)} a_0^\dagger + \text{h.c.} \end{aligned} \quad (\text{E.24})$$

Recalling (3.41), $\phi^{(2)}$ satisfies

$$\begin{aligned} i\partial_t \phi_t^{(2)} &= \frac{\lambda}{N} \left((\Gamma_t^{(2)} * (\hat{v} + \hat{v}(0)))(0) \phi_t^{(2)} + (\Sigma_t^{(2)} * \hat{v})(0) \bar{\phi}_t^{(2)} \right) \\ &\quad - 2\lambda |\Lambda| \hat{v}(0) |\phi_t^{(2)}|^2 \phi_t^{(2)}. \end{aligned}$$

With that, (E.24) implies

$$\begin{aligned} \mathcal{H}_{\text{BEC}}(t) = & -\frac{\lambda\sqrt{|\Lambda|}}{\sqrt{N}} \left[u_t(0) \left(2(f_0^{(+)} \sigma_t * \hat{v})(0) \bar{\phi}_t + ((1+2\gamma_t) f_0^{(+)}) * (\hat{v} + \hat{v}(0))(0) \phi_t \right) \right. \\ & + v_t(0) \left(2(f_0^{(+)} \sigma_t * \hat{v})(0) \phi_t + ((1+2\gamma_t) f_0^{(+)}) * (\hat{v} + \hat{v}(0))(0) \bar{\phi}_t \right) e^{i \int_0^t ds \Omega_s(0)} a_0^\dagger \\ & + \text{h.c.} \end{aligned}$$

Similarly, Lemma A.2 implies

$$\mathcal{H}_{\text{HFB}}^{(\text{d})}(t) = \int dp \left[-\Omega_t(p) - \frac{\text{Re}(\bar{\sigma}_t(p)i\partial_t\sigma_t(p))}{1 + \gamma_t(p)} \right] \quad (\text{E.25})$$

$$\begin{aligned}
& + \left(E(p) + \frac{\lambda}{N} ((\gamma_t + N|\Lambda||\phi_t|^2\delta) * (\hat{v} + \hat{v}(0))) (p) \right) (1 + 2\gamma_t(p)) \\
& + \frac{2\lambda}{N} \operatorname{Re} \left(((\bar{\sigma}_t + N|\Lambda|\bar{\phi}_t^2\delta) * \hat{v})(p) \sigma_t(p) \right) \Big] a_p^\dagger a_p
\end{aligned}$$

By (3.41), we have that

$$i\partial_t \sigma_t^{(2)} = 2(E + \frac{\lambda}{N} \Gamma_t^{(2)} * (\hat{v} + \hat{v}(0))) \sigma_t^{(2)} + \frac{\lambda}{N} (\Sigma_t^{(2)} * \hat{v}) (1 + 2\gamma_t^{(2)}),$$

while (3.43) implies

$$\Omega_t^{(2)} = E + \frac{\lambda}{N} (\Gamma_t^{(2)} * (\hat{v} + \hat{v}(0))) + \frac{\lambda}{N} \frac{\operatorname{Re}(\bar{\Sigma}^{(2)} * \hat{v}) \sigma_t^{(2)}}{1 + \gamma_t^{(2)}}.$$

Using the fact that $|\sigma|^2 = \gamma(\gamma + 1)$, (E.25) thus implies

$$\begin{aligned}
\mathcal{H}_{\text{HFB}}^{(\text{d})}(t) &= \\
& \int dp \left[-\frac{\lambda}{N} \frac{\operatorname{Re}(\bar{\Sigma}^{(2)} * \hat{v})(p) \sigma_t^{(2)}(p)}{1 + \gamma_t^{(2)}(p)} + \frac{2\lambda}{N} \operatorname{Re} \left(((\bar{\sigma}_t^{(2)} + N|\Lambda|(\bar{\phi}_t^{(2)})^2\delta) * \hat{v})(p) \sigma_t^{(2)}(p) \right) \right. \\
& - \frac{\lambda}{N} \operatorname{Re} \left((\bar{\Sigma}_t^{(2)} * \hat{v})(p) \sigma_t^{(2)}(p) \right) \frac{1 + 2\gamma_t^{(2)}(p)}{1 + \gamma_t^{(2)}(p)} - \frac{\lambda}{N} (\Gamma_t^{(2)} * (\hat{v} + \hat{v}(0)))(p) (1 + 2\gamma_t^{(2)}(p)) \\
& \left. + \frac{\lambda}{N} ((\gamma_t^{(2)} + N|\Lambda||\phi_t^{(2)}|^2\delta) * (\hat{v} + \hat{v}(0)))(p) (1 + 2\gamma_t^{(2)}(p)) \right] a_p^\dagger a_p.
\end{aligned}$$

Simplifying the terms, recalling definitions (3.27) and (3.28), we obtain

$$\begin{aligned}
\mathcal{H}_{\text{HFB}}^{(\text{d})}(t) &= -\frac{\lambda}{N} \int dp \left[2 \operatorname{Re} \left(((f_0^{(+)} \bar{\sigma}_t^{(2)}) * \hat{v})(p) \sigma_t^{(2)}(p) \right) \right. \\
& \left. + ((f_0^{(+)} (1 + 2\gamma^{(2)})) * (\hat{v} + \hat{v}(0)))(p) (1 + 2\gamma_t^{(2)}(p)) \right] a_p^\dagger a_p.
\end{aligned}$$

Finally, Lemma A.2 yields

$$\begin{aligned}
\mathcal{H}_{\text{HFB}}^{(\text{cor})}(t) &= \int dp \left[-\frac{i\partial_t \sigma_t(p)}{2} + \frac{\sigma_t(p) i\partial_t \gamma_t(p)}{2(1 + \gamma_t(p))} \right. \\
& + \left(E(p) + \frac{\lambda}{N} ((\gamma_t + N|\Lambda||\phi_t|^2\delta) * (\hat{v} + \hat{v}(0)))(p) \right) \sigma_t(p) \\
& + \frac{\lambda}{2N} \left(((\sigma_t + N|\Lambda|\phi_t^2\delta) * \hat{v})(p) (1 + \gamma_t(p)) \right. \\
& \left. + ((\bar{\sigma}_t + N|\Lambda|\bar{\phi}_t^2\delta) * \hat{v})(p) \frac{\sigma_t(p)^2}{1 + \gamma_t(p)} \right) e^{2i \int_0^t d\tau \Omega_\tau(p)} a_p^\dagger a_{-p}^\dagger + \text{h.c.}
\end{aligned} \tag{E.26}$$

By (3.41), we have that

$$\begin{aligned}
\partial_t \gamma_t^{(2)} &= \frac{2\lambda}{N} \operatorname{Im} ((\Sigma_t^{(2)} * \hat{v}) \bar{\sigma}_t^{(2)}) = \frac{\lambda}{iN} ((\Sigma_t^{(2)} * \hat{v}) \bar{\sigma}_t^{(2)} - (\bar{\Sigma}_t^{(2)} * \hat{v}) \sigma_t^{(2)}), \\
i\partial_t \sigma_t^{(2)} &= 2(E + \frac{\lambda}{N} \Gamma_t^{(2)} * (\hat{v} + \hat{v}(0))) \sigma_t^{(2)} + \frac{\lambda}{N} (\Sigma_t^{(2)} * \hat{v}) (1 + 2\gamma_t^{(2)}).
\end{aligned}$$

Substituting these into (E.26), yields

$$\begin{aligned}
\mathcal{H}_{\text{HFB}}^{(\text{cor})}(t) = & \\
& \int dp \left[- \left(E(p) + \frac{\lambda}{N} \Gamma_t^{(2)} * (\hat{v} + \hat{v}(0))(p) \right) \sigma_t^{(2)}(p) \right. \\
& - \frac{\lambda}{2N} (\Sigma_t^{(2)} * \hat{v})(p) (1 + 2\gamma_t^{(2)}(p)) \\
& + \frac{\lambda}{2N} \left((\Sigma_t^{(2)} * \hat{v})(p) \gamma_t^{(2)}(p) - (\bar{\Sigma}_t^{(2)} * \hat{v})(p) \frac{\sigma_t^{(2)}(p)^2}{1 + \gamma_t^{(2)}(p)} \right) \\
& + \left(E(p) + \frac{\lambda}{N} ((\gamma_t^{(2)} + N|\Lambda| |\phi_t^{(2)}|^2 \delta) * (\hat{v} + \hat{v}(0)))(p) \right) \sigma_t^{(2)}(p) \\
& + \frac{\lambda}{2N} \left(((\sigma_t^{(2)} + N|\Lambda| (\phi_t^{(2)})^2 \delta) * \hat{v})(p) (1 + \gamma_t^{(2)}(p)) \right. \\
& \left. \left. + ((\bar{\sigma}_t^{(2)} + N|\Lambda| (\bar{\phi}_t^{(2)})^2 \delta) * \hat{v})(p) \frac{\sigma_t^{(2)}(p)^2}{1 + \gamma_t^{(2)}(p)} \right) \right] e^{2i \int_0^t d\tau \Omega_\tau^{(2)}(p)} a_p^\dagger a_{-p}^\dagger + \text{h.c.}
\end{aligned}$$

We simplify this expression as

$$\begin{aligned}
\mathcal{H}_{\text{HFB}}^{(\text{cor})}(t) = & - \frac{\lambda}{N} \int dp \left[\left((f_0^{(+)}(1 + 2\gamma_t^{(2)})) * (\hat{v} + \hat{v}(0)) \right)(p) \sigma_t^{(2)}(p) \right. \\
& + ((f_0^{(+)} \sigma_t^{(2)}) * \hat{v})(p) (1 + \gamma_t^{(2)}(p)) \\
& \left. + ((f_0^{(+)} \bar{\sigma}_t^{(2)}) * \hat{v})(p) \frac{\sigma_t^{(2)}(p)^2}{1 + \gamma_t^{(2)}(p)} \right] e^{2i \int_0^t d\tau \Omega_\tau^{(2)}(p)} a_p^\dagger a_{-p}^\dagger + \text{h.c.}
\end{aligned}$$

□

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