A NOTE ON RATIONAL FUNCTIONS WITH THREE BRANCHED POINTS ON THE RIEMANN SPHERE

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ABSTRACT. Studying the existence of rational functions with given branched data is a classical problem in the field of complex analysis and algebraic geometry. This problem dates back to Hurwitz and remains open to this day. In this paper, we utilize complex analysis to establish a property of rational functions with three branched points on the Riemann sphere. As applications, we present some new types of exceptional branched data. These results cover some previous results mentioned in [5, 18, 24]. We also establish the existence of a certain type of rational functions on the Riemann sphere.

Key words Branched cover, Hurwitz problem, Rational Function.

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1. Background and main results

Studying the existence of branched covers between two compact Riemann surfaces with given branched data is an important problem in the field of complex analysis and algebraic geometry. This problem is commonly referred to as the Hurwitz existence problem. In other words, given two compact Riemann surfaces M and N, along with a collection $\mathcal{D}$ of partitions of a positive integer d, the question is whether there exists a degree d branched cover $f : M \to N$ with $\mathcal{D}$ as the branched data.

Let M and N be a pair of compact Riemann surfaces. A smooth map $f : M \to N$ is a degree d branched cover if for each $x \in N$ there is a partition $\mu(x) = [\alpha_1, \ldots, \alpha_n]$ of d such that, over a neighborhood of x in N, f is equivalent to the map $\widetilde{f} : \{1, \ldots, n\} \times \mathbb{C} \to \mathbb{C}$ where $\widetilde{f}(i, z) = z^{\alpha_i}$ and x corresponds to 0 in $\mathbb{C}$ (here the square brackets are used to denote an unordered set with repetitions). The points $x \in N$ for which $\mu(x)$ is not the trivial partition $[1, 1, \ldots, 1]$ of d constitute the branch set B_f of f. The collection $\mathcal{D} = \{\mu(x)|x \in B_f\}$ (with repetitions allowed) is called the branched data of f. As is well known, the degree d and the branched data $\mathcal{D}$ of f should satisfy the Riemann-Hurwitz formula:

$$\nu(\mathcal{D}) = d \cdot \chi(N) - \chi(M), \tag{1.1}$$

where $\nu(\mathcal{D})$ denotes the total branching of f.

Explicitly, let $B_f = \{x_1, \ldots, x_n\} \subseteq N$ and $\mathcal{D} = \{[\alpha_1^1, \ldots, \alpha_1^{r_1}], \ldots, [\alpha_n^1, \ldots, \alpha_n^{r_n}]\}$ represent the branch set and the branched data of f, respectively. For each x_i, its pre-image under f consists of a finite number of points $y_i^1, \ldots, y_i^{r_i} \in M$, and near each y_i^j the map f equivalent to $\widetilde{f}(z) = z^{\alpha_i^j}$. The integer α_i^j is usually referred to as the local degree or multiplicity at the point y_i^j. Since x_i is a branching point, at least one of the α_i^j's should be greater than 1. It is evident that the set $\{y_i^j : i = 1, \ldots, n, j = 1, \ldots, r_i\} \subseteq M$ precisely corresponds to the set of ramification points. Thus, the total branching of f is $\nu(\mathcal{D}) = \sum_{i=1}^n \sum_{j=1}^{r_i} (\alpha_i^j - 1)$. Since

$$\sum_{j=1}^{r_i} \alpha_i^j = d, \ \forall i = 1, \ldots, n, \tag{1.2}$$

the Riemann-Hurwitz formula (1.1) can be expressed as

$$\sum_{i=1}^{n}\sum_{j=1}^{r_i}(\alpha_i^j-1)=\sum_{i=1}^{n}(d-r_i)=d\cdot\chi(N)-\chi(M). \tag{1.3}$$

In his classical work [7], Hurwitz reduced the problem of existence of a branched cover to a problem involving partitions realized by suitable permutations in symmetric groups. In [5], Edmonds, Kulkarni, and Stong proved that all data is realizable when $\chi(N)<0$. However, when $N=S^2$ the problem becomes much more complex. It is well known that there exist exceptional data $(d,\mathcal{D})$ satisfying (1.2) and (1.3) that cannot be realized by a branched cover. For example, $d=4,\mathcal{D}=\{[3,1],[2,2],[2,2]\}$. Characterizing all of such exceptional data remains an open problem to this day. In [23], Zheng determined all exceptional candidate branched covers with $n=3$ and $d\le 10$ by computer for the cases where $M=N=S^2$ and $M=T^2,N=S^2$.

Finding new types of exceptional data are of interest as it may provide insights towards establishing a universal criterion. However, the general pattern of realizable data remains unclear. Various approaches such as dessins d'enfant, Speiser graph and monodromy approach have been explored for studying branched cover. We refer the reader to [1, 2, 4, 5, 6, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 23] and the references cited in for more results. Particularly, we refer the reader to [4, 15, 20] for a review of available results and techniques. In [5], a conjecture proposing connections with number-theoretic facts has been put forward, which is supported by strong evidence in [14, 16].

Conjecture 1.1 (Prime degree conjecture)**.** *If $(d,\mathcal{D})$ is a set of branched data for $S^2\to S^2$ that satisfies (1.2) and (1.3) and the degree d is a prime, then the set of data is realizable.*

In [5], Edmonds, Kulkarni, and Stong reduced the Prime degree conjecture to the collections with exactly three partitions. Thus a collection with three partitions is important to study the existence of a branched cover. In this paper, we investigate exceptional data for the cases where $M=N=S^2\cong\overline{\mathbb{C}}$ in order to characterize all data that cannot be realized by a branched cover. We employ complex analysis techniques to derive the following property that provides insights into the structure of the exceptional data. Our main result is as follows and can be viewed as a generalization of the result in [18].

Theorem 1.1. *Suppose $f:\overline{\mathbb{C}}\to\overline{\mathbb{C}}$ is a rational function with three branched points and degree $d=rk$, where $r\ge 2,k\ge 2$. If the branched data of f is*

$$\{[\alpha_1,\dots,\alpha_A],[rx_1,\dots,rx_B],[ry_1,\dots,ry_C]\},$$

where $1\le x_1\le\dots\le x_B, 1\le y_1\le\dots\le y_C, \sum_{i=1}^{B}x_i=\sum_{j=1}^{C}y_j=k$ and $A=d-B-C+2$. If $GCD(x_1,\dots,x_B,y_1,\dots,y_C)=1$, then, up to two Möbius transformations on $\overline{\mathbb{C}}$, there exists a rational function $F:\overline{\mathbb{C}}\to\overline{\mathbb{C}}$ with degree k such that $f(z)=[F(z)]^r,\forall z\in\mathbb{C}$ and the branched data of F is one of the following:

(1) if $x_B=1,y_C=1$, $\{[\alpha_1^1,\dots,\alpha_1^{l_1}],\dots,[\alpha_s^1,\dots,\alpha_s^{l_s}]\}$,

(2) if $x_{S+1}\ge 2,x_S=1,y_C=1$, $\{[\alpha_1^1,\dots,\alpha_1^{l_1}],\dots,[\alpha_s^1,\dots,\alpha_s^{l_s}],[\underbrace{1,\dots,1}_{S},x_{S+1},\dots,x_B]\}$,

(3) if $x_B=1,y_{T+1}\ge 2,y_T=1$, $\{[\alpha_1^1,\dots,\alpha_1^{l_1}],\dots,[\alpha_s^1,\dots,\alpha_s^{l_s}],[\underbrace{1,\dots,1}_{T},y_{T+1},\dots,y_C]\}$,

(4) if $x_{S+1} \geq 2, x_S = 1, y_{T+1} \geq 2, y_T = 1$,

$$\{[\alpha_1^1, \dots, \alpha_1^{l_1}], \dots, [\alpha_s^1, \dots, \alpha_s^{l_s}], [\underbrace{1, \dots, 1}_{S}, x_{S+1}, \dots, x_B], [\underbrace{1, \dots, 1}_{T}, y_{T+1}, \dots, y_C]\},$$

for some $1 \leq s \leq r$. Moreover, $\alpha_1^1, \dots, \alpha_1^{l_1}, \dots, \alpha_s^1, \dots, \alpha_s^{l_s}$ belong to $\alpha_1, \dots, \alpha_A$ which means that up to a permutation $[\alpha_1^1, \dots, \alpha_1^{l_1}, \dots, \alpha_s^1, \dots, \alpha_s^{l_s}, 1, \dots, 1] = [\alpha_1, \dots, \alpha_A]$. Especially, we obtain $\alpha_i \leq k, \forall i$.

Remark 1.1. *In **Theorem** 1.1, the sentence " up to two Möbius transformations" means that for two branched covers $f_1, f_2 : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ if there exist two Möbius transformations $\varphi, \psi : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ such that*

$$\varphi \circ f_1 = f_2 \circ \psi,$$

then we call f_1, f_2 equivalence. Obviously, if f_1, f_2 are equivalence, then they have same branched data.

As an application of **Theorem** 1.1, we present a type of exceptional data that can be viewed as a generalization of some results mentioned in [5, 18, 24]. This provides new insights into the nature of exceptional data and expands the range of possible configurations. Additionally, we offer new proof of some of the results presented in [18, 24], which provides further evidence and support for the validity of our findings. We also give some new exceptional data.

Corollary 1.1. *Suppose $d = rk$ is an integer where $r \geq 2, k \geq 2$ are two integers. Suppose*

$$[\alpha_1, \dots, \alpha_A], [rx_1, \dots, rx_B], [ry_1, \dots, ry_C]$$

are partitions of d with $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_A, \alpha_1 \geq 2, \sum_{i=1}^B x_i = \sum_{j=1}^C y_j = k$, $A = d - B - C + 2$ and $GCD(x_1, \dots, x_B, y_1, \dots, y_C) = 1$. If $\alpha_1 > k$, then $(d, \mathcal{D})$ is exceptional, where

$$\mathcal{D} = \{[\alpha_1, \dots, \alpha_A], [rx_1, \dots, rx_B], [ry_1, \dots, ry_C]\}.$$

Proof. By **Theorem** 1.1, if there exists a rational function that satisfied the condition, then $\alpha_1 \leq k$ which is in contradiction with $\alpha_1 > k$. $\square$

Proposition 1.1 ([24]). *If $d = 2k$ for $k \geq 2$, then the following set $(d, \mathcal{D})$ is exceptional:*

- $\mathcal{D} = \{[k_1, k_2], [\underbrace{2, \dots, 2}_{k}], [\underbrace{2, \dots, 2}_{k}]\}$, *where $k_1 + k_2 = 2k$ and $k_1 \neq k_2$.*

If in addition $k \geq 3$, then we also have the following exceptional $(d, \mathcal{D})$:

- $\mathcal{D} = \{[\underbrace{2, \dots, 2}_{k}], [\underbrace{2, \dots, 2}_{j_1}, 2k - 2j_1], [\underbrace{2, \dots, 2}_{j_2}, 2k - 2j_2]\}$, *where $j_1 + j_2 = k$, $j_1 \neq j_2$ and $j_1, j_2 \geq 1$.*

Proof. Since $k_1 + k_2 = 2k$ and $k_1 \neq k_2$, so $k_1 > k$ or $k_2 > k$. By **Corollary** 1.1, $(d, \mathcal{D})$ is exceptional, where

- $\mathcal{D} = \{[k_1, k_2], [\underbrace{2, \dots, 2}_{k}], [\underbrace{2, \dots, 2}_{k}]\}$, where $k_1 + k_2 = 2k$ and $k_1 \neq k_2$.

The proof of the second part is similar. $\square$

Proposition 1.2 ([24]). *When $d = 3k$ for k is odd and $k \geq 3$, the following set $(d, \mathcal{D})$ is exceptional:*

- $\mathcal{D} = \{[k - 2, \underbrace{2, \dots, 2}_{k+1}], [\underbrace{3, \dots, 3}_{k}], [\underbrace{3, \dots, 3}_{k}]\}$.

Proof. Since k is odd and 2 is even, by **Theorem** 1.1, $(d, \mathcal{D})$ is exceptional. $\square$

Proposition 1.3 ([24])**.** *When $d = rk$ where $r \geq 2, k \geq 2$, the following set $(d, \mathcal{D})$ is exceptional:*

- $\mathcal{D} = \{[2k-1, \underbrace{1, \ldots, 1}_{(r-2)k+1}], [\underbrace{r, \ldots, r}_{k}], [\underbrace{r, \ldots, r}_{k}]\}$.
- $\mathcal{D} = \{[j_1, j_2, \underbrace{1, \ldots, 1}_{(r-2)k}], [\underbrace{r, \ldots, r}_{k}], [\underbrace{r, \ldots, r}_{k}]\}$, *where $j_1 \neq j_2$ and $j_1 + j_2 = 2k$.*

Proof. Since $2k - 1 > k$ and $j_1 > k$ or $j_2 > k$, so by **Corollary** 1.1, $(d, \mathcal{D})$ is exceptional. $\square$

Proposition 1.4. *When $d = 3k$ for k is odd and $k \geq 3$, the following set $(d, \mathcal{D})$ is exceptional:*

- $\mathcal{D} = \{[j_1, j_2, \underbrace{2, \ldots, 2}_{k}], [\underbrace{3, \ldots, 3}_{k}], [\underbrace{3, \ldots, 3}_{k}]\}$, *where $j_1 + j_2 = k$.*

Proof. Since k is odd and $j_1 + j_2 = k$, j_1, j_2 have different parity. Since 2 is even, $(d, \mathcal{D})$ is exceptional. $\square$

Proposition 1.5. *The following set $(d = 3k, \mathcal{D})$ $(k = 2 + 3l, l \geq 1)$ is exceptional:*

- $\mathcal{D} = \{[\underbrace{3, \ldots, 3}_{k-1}, \underbrace{1, 1, 1}_{3}], [\underbrace{3, \ldots, 3}_{k}], [\underbrace{3, \ldots, 3}_{k}]\}$.

Proof. Since $k = 2 + 3l \geq 5$ and the number of 1s in $[\underbrace{3, \ldots, 3}_{k-1}, \underbrace{1, 1, 1}_{3}]$ equals 3, which is $2 + 1$, $(d, \mathcal{D})$ is exceptional.

$\square$

From the constructed exceptional data mentioned above, we obtain an additional result regarding non-prime degrees, which has been proven in [5, 24] using a different methodology.

Corollary 1.2 ([5],[24])**.** *For every d that is not a prime, there exists at least one set of data $(d, \mathcal{D})$ that is exceptional.*

2. Proof of **Theorem** 1.1

Firstly, we prove the following lemma.

Lemma 2.1. *Consider a holomorphic function $f : \Delta \to \mathbb{C}$ defined on the disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$, where $f(0) \neq 0$. Suppose the local degree of f at $z = 0$ is $n \geq 1$. If $f(z) = [F(z)]^r$, where $r \geq 1$ is an integer and $F : \Delta \to \mathbb{C}$ is a holomorphic function, then the local degree of F at $z = 0$ is also n.*

Proof. Suppose the Taylor expansion of f near $z = 0$ is

$$f(z) = a_0 + a_n z^n + a_{n+1} z^{n+1} + \ldots,$$

and the Taylor expansion of F near $z = 0$ is

$$F(z) = b_0 + b_{n_1} z^{n_1} + b_{n_1+1} z^{n_1+1} + \ldots,$$

where $a_0 a_n b_0 b_{n_1} \neq 0$, then using $f(z) = [F(z)]^r$, we obtain

$$n_1 = n.$$

$\square$

Now the proof of **Theorem** 1.1 is as follows.

Up to two Möbius transformations on $\overline{\mathbb{C}}$, we can suppose the expression of $f:\overline{\mathbb{C}}\to\overline{\mathbb{C}}$ is

$$f(z)=\frac{(z-z_1)^{rx_1}(z-z_2)^{rx_2}\cdots(z-z_B)^{rx_B}}{(z-w_1)^{ry_1}(z-w_2)^{ry_2}\cdots(z-w_C)^{ry_C}},$$

where $z_1,\ldots,z_B,w_1,\ldots,w_C\in\mathbb{C}$ are distinct complex numbers.

Set $F(z)=\frac{(z-z_1)^{x_1}(z-z_2)^{x_2}\cdots(z-z_B)^{x_B}}{(z-w_1)^{y_1}(z-w_2)^{y_2}\cdots(z-w_C)^{y_C}}$, then F is a rational function with degree k on $\overline{\mathbb{C}}$. It is obvious that $f(z)=[F(z)]^r,\forall z\in\mathbb{C}$, and $f'(z)=r[F(z)]^{r-1}F'(z),\forall z\in\mathbb{C}$.

Since the branched data of f is

$$\{[\alpha_1,\ldots,\alpha_A],[rx_1,\ldots,rx_B],[ry_1,\ldots,ry_C]\},$$

where $1\le x_1\le\ldots\le x_B, 1\le y_1\le\ldots\le y_C$, then by **Lemma** 2.1 the branched data of F is

$$\{[\alpha_1^1,\ldots,\alpha_1^{l_1}],\ldots,[\alpha_s^1,\ldots,\alpha_s^{l_s}]\},\ if\ x_B=1,y_C=1$$

or

$$\{[\alpha_1^1,\ldots,\alpha_1^{l_1}],\ldots,[\alpha_s^1,\ldots,\alpha_s^{l_s}],[\underbrace{1,\ldots,1}_{S},x_{S+1},\ldots,x_B]\},\ if\ x_{S+1}\ge 2,x_S=1,y_C=1$$

or

$$\{[\alpha_1^1,\ldots,\alpha_1^{l_1}],\ldots,[\alpha_s^1,\ldots,\alpha_s^{l_s}],[\underbrace{1,\ldots,1}_{T},y_{T+1},\ldots,y_C]\},\ if\ x_B=1,y_{T+1}\ge 2,y_T=1$$

or

$$\{[\alpha_1^1,\ldots,\alpha_1^{l_1}],\ldots,[\alpha_s^1,\ldots,\alpha_s^{l_s}],[\underbrace{1,\ldots,1}_{S},x_{S+1},\ldots,x_B],[\underbrace{1,\ldots,1}_{T},y_{T+1},\ldots,y_C]\},$$
$$if\ x_{S+1}\ge 2,x_S=1,y_{T+1}\ge 2,y_T=1,$$

where $1\le s\le r$. Obviously, $\alpha_1^1,\ldots,\alpha_1^{l_1},\ldots,\alpha_s^1,\ldots,\alpha_s^{l_s}$ belong to $\alpha_1,\ldots,\alpha_A$ which means that, up to a permutation, $[\alpha_1^1,\ldots,\alpha_1^{l_1},\ldots,\alpha_s^1,\ldots,\alpha_s^{l_s},1\ldots,1]=[\alpha_1,\ldots,\alpha_A]$. In particular, we obtain $\alpha_i\le k,\forall i$.

We note that by **Lemma** 2.1, the proof of the following theorem is easy.

Theorem 2.1. *Suppose $F:\overline{\mathbb{C}}\to\overline{\mathbb{C}}$ is a rational function with degree $d\ge 3$ and three branched points such that its branded data is*

$$\{[\alpha_1,\ldots,\alpha_A],[\beta_1,\ldots,\beta_B],[\gamma_1,\ldots,\gamma_C]\},$$

then for any integer $k\ge 2$, up to two Möbius transformations $f=F^k$ is a rational function with degree kd and three branched points such that its branched data is

$$\{[\alpha_1,\ldots,\alpha_A,\underbrace{1,\ldots,1}_{(k-1)d}],[k\beta_1,\ldots,k\beta_B],[k\gamma_1,\ldots,k\gamma_C]\}.$$

3. Some existence results

We conclude the paper by presenting some existence results for a certain type of rational functions on $\overline{\mathbb{C}}$ with three branched points. First, we give a new proof of the following theorem which was proved in [5] and which is a special case in [4](Theorem 3.5).

Theorem 3.1 ([5]). *There exists a rational function $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ with branched data*

$$\{[\alpha_1, \alpha_2], [\underbrace{2, \dots, 2}_{k}], [\underbrace{2, \dots, 2}_{k}]\},$$

where $k \geq 2, \alpha_1 + \alpha_2 = 2k$, if and only if $\alpha_1 = \alpha_2 = k$.

Proof. If $r = 2, \alpha_1 = \alpha_2 = k$, set $F(z) = \frac{z^k - 1}{z^k + 1}$, then by direct calculation the branched data of $f(z) = [F(z)]^2$ is

$$\{[k, k], [\underbrace{2, \dots, 2}_{k}], [\underbrace{2, \dots, 2}_{k}]\}.$$

Suppose there exists a rational function $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ with branched data

$$\{[\alpha_1, \alpha_2], [\underbrace{2, \dots, 2}_{k}], [\underbrace{2, \dots, 2}_{k}]\},$$

where $k \geq 2, \alpha_1 + \alpha_2 = 2k$. By **Theorem** 1.1, we obtain $\alpha_1 \leq k, \alpha_2 \leq k$. Thus $\alpha_1 + \alpha_2 \leq 2k$. Since $\alpha_1 + \alpha_2 = 2k$, we obtain $\alpha_1 = \alpha_2 = k$. □

Secondly, we can derive the following theorem.

Theorem 3.2. *There exists a rational function $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ with branched data*

$$\{[\alpha_1, \alpha_2, \alpha_3], [\underbrace{2, \dots, 2}_{k}], [\underbrace{2, \dots, 2}_{k-2}, 4]\},$$

where $k \geq 3, \alpha_1 + \alpha_2 + \alpha_3 = 2k, \alpha_1 \geq \alpha_2 \geq \alpha_3$, if and only if $\alpha_1 = k$.

Proof. The necessary is from **Theorem** 1.1. Now we give the proof of the sufficiency.

By a result of Boccara [3] or Thom [22], there exists a rational function $F : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ with branched data

$$\{[k], [\alpha_2, \alpha_3], [\underbrace{1, \dots, 1}_{k-2}, 2]\}.$$

Up to a Möbius transformation, we may suppose the branched points of F are $-1, 1, 0$. Then $f(z) = [F(z)]^2$ satisfies the condition of **Theorem** 3.2. □

Similarly, one can prove the following theorem.

Theorem 3.3. *There exists a rational function $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ with branched data*

$$\{[\alpha_1, \alpha_2, \dots, \alpha_{x+1}], [\underbrace{2, \dots, 2}_{k}], [\underbrace{2, \dots, 2}_{k-x}, 2x]\},$$

where $x \geq 1, k \geq 3, \alpha_1 + \ldots + \alpha_{x+1} = 2k$, if and only if $\alpha_1, \alpha_2, \dots, \alpha_{x+1}$ can be divided into two partitions of k.

Using the result of Song-Xu in [21], one can prove the following theorem.

Theorem 3.4. *There exists a rational function $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ with branched data*

$$\{[\alpha_1, \alpha_2, \dots, \alpha_{x+y}], [\underbrace{2, \dots, 2}_{k-y}, 2y], [\underbrace{2, \dots, 2}_{k-x}, 2x]\},$$

where $x \geq 1, y \geq 1, k \geq 3, \alpha_1 + \ldots + \alpha_{x+y} = 2k$, if and only if $\alpha_1, \alpha_2, \dots, \alpha_{x+y}$ can be divided into two partitions of k.

Finally, we provide two examples to explain our results.

Example 3.1. *There exists a rational function* $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ *with branched data*

$$\{[\alpha_1, \alpha_2, \alpha_3, 2, 2], [3, 3, 3], [3, 3, 3]\},$$

where $\alpha_1 + \alpha_2 + \alpha_3 = 5, \alpha_1 \geq \alpha_2 \geq \alpha_3$*, if and only if* $\alpha_1 = 3$.

Proof. "Only if" . Obviously $\alpha_1 \leq 3$. If $\alpha_1 = 2$, then $\alpha_2 = 2, \alpha_3 = 1$. It is a contradiction. So $\alpha_1 = 3$.

"If". If $\alpha_1 = 3$, then $\alpha_2 = \alpha_3 = 1$. Since there exists a rational function F with branched data

$$\{[3], [2, 1], [2, 1]\}.$$

Without loss of generality, suppose the branched points of F are $1, e^{\frac{2\pi}{3}i}, e^{\frac{4\pi}{3}i}$, then $f(z) = F^3(z), \forall z \in \mathbb{C}$ is a rational function with branched data

$$\{[3, 2, 2, 1, 1], [3, 3, 3], [3, 3, 3]\}.$$

□

Similar as the example above, one can prove the following example.

Example 3.2. *There exists a rational function* $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ *with branched data*

$$\{[\alpha_1, \alpha_2, \alpha_3, 2, 2, 2, 2], [3, 3, 3, 3, 3], [3, 3, 3, 3, 3]\},$$

where $\alpha_1 + \alpha_2 + \alpha_3 = 7, \alpha_1 \geq \alpha_2 \geq \alpha_3$*, if and only if* $\alpha_1 = 5$ *or* $\alpha_1 = \alpha_2 = 3$.

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Declarations

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