Spatially regular charged black holes supporting charged massive scalar clouds

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Abstract

We prove that, as opposed to the familiar charged Reissner-Nordström black-hole spacetime, the spatially regular charged Ayón-Beato-García (ABG) black-hole spacetime can support charged scalar clouds, spatially regular stationary matter configurations which are made of linearized charged massive scalar fields. Interestingly, we reveal the fact that the composed black-hole-field system is amenable to an analytical treatment in the regime $Q/M \ll 1 \ll M\mu$ of weakly charged black holes and large-mass fields, in which case it is proved that the dimensionless physical parameter $\alpha \equiv \frac{qQ}{M\mu}$ must lie in the narrow interval $\alpha \in (\sqrt{\frac{3240}{6859}}, \frac{16}{23})$ [here $\{M, Q\}$ are the mass and electric charge of the central black hole and $\{\mu, q\}$ are the proper mass and charge coupling constant of the supported scalar field]. In particular, we explicitly prove that, for weakly charged black holes, the discrete resonance spectrum $\{\alpha(M, Q, \mu, q; n\}_{n=0}^{n=\infty}$ of the composed charged-ABG-black-hole-charged-massive-scalar-field cloudy configurations can be determined *analytically* in the eikonal large-mass regime.

I. INTRODUCTION

Bosonic field configurations that interact with spinning black holes can be amplified if their proper frequencies lie in the superradiant regime [1–4]

$$0 < \omega < m\Omega_{\rm H} \ , \tag{1}$$

where $\Omega_{\rm H}$ is the horizon angular velocity of the central black hole and the integer m (with $m > 0$) is the azimuthal harmonic index of the co-rotating bosonic field.

Interestingly, using analytical techniques in the linearized regime [5, 6] and numerical computations in the non-linear (self-gravitating) regime [7–9], it has been explicitly proved that the superradiant amplification phenomenon may allow spinning black-hole spacetimes to support spatially regular matter configurations which are made of stationary minimally-coupled massive bosonic fields which are characterized by the critical (marginal) frequency relation [10]

$$\omega = m\Omega_{\rm H} \ . \tag{2}$$

It should be emphasized, however, that not all massive bosonic fields can be supported by a central spinning black hole which is characterized by a given value $\Omega_{\rm H}$ of the horizon angular velocity. In particular, it was proved in [11] that the mathematically compact black-hole-field inequality

$$\mu < \sqrt{2}\cdot m\Omega_{\rm H} \tag{3}$$

provides a necessary condition for the existence of composed Kerr-black-hole-massive-scalar-field bound-state configurations, where μ is the proper mass of the supported scalar field.

As nicely pointed out in [12], a physically analogous phenomenon in which a charged bosonic field is superradiantly amplified by a charged black hole occurs if the proper frequency of the incident field lies in the superradiant regime

$$0 < \omega < q\Phi_{\rm H} \ , \tag{4}$$

where $\Phi_{\rm H}$ is the electric potential at the outer horizon of the central charged black hole and q is the charge coupling constant of the field [12].

Intriguingly, however, it has been explicitly proved in [13] that the canonical charged Reissner-Nordström (RN) black-hole spacetime cannot support spatially regular linearized matter configurations which are made of minimally coupled (static or stationary) charged massive scalar fields. In particular, it has been revealed in [13] that, as opposed to the case of spinning black holes [5–9], the mutual gravitational attraction between a central charged RN black hole and a charged massive scalar field cannot provide, in the superradiant regime (4), the confinement mechanism which is required in order to prevent the scalar field from radiating its energy to infinity [14–18].

It has recently been demonstrated numerically in the physically interesting paper [19] that charged scalar fields can be superradiantly amplified in the charged Ayón-Beato-García (ABG) [20] spacetime which describes a spatially regular black-hole solution of the coupled Einstein-non-linear-electrodynamics field equations. In particular, it has been intriguingly suggested in [19] that the charged ABG black-hole spacetime may be superradiantly unstable to linearized perturbations of charged massive scalar fields.

The main goal of the present paper is to explore, using *analytical* techniques, the physical and mathematical properties of the composed charged-ABG-black-hole-charged-massive-scalar-field system. Interestingly, we shall explicitly prove below that, as opposed to the charged RN black-hole spacetime, the spatially regular charged ABG black-hole spacetime can support stationary scalar clouds, spatially regular matter configurations which are made of linearized charged massive scalar fields. These composed black-hole-linearized-scalar-field stationary bound-state configurations, which are characterized by the critical (marginal) frequency relation $\omega = q\Phi_{\rm H}$ for the superradiant amplification phenomenon in the charged black-hole spacetime, mark the onset of the superradiant instabilities in the charged ABG black-hole spacetime to perturbations of charged massive scalar fields in the frequency regime (4).

Interestingly, in the present compact paper we shall explicitly prove that the composed charged-ABG-black-hole-charged-massive-scalar-field system is amenable to an analytical treatment in the dimensionless regime $Q/M \ll 1 \ll M\mu$ of weakly charged black holes and large-mass fields [21].

II. DESCRIPTION OF THE SYSTEM

We consider a physical system which is composed of a charged massive scalar field of proper mass μ and electric charge q which is linearly coupled to a central charged Ayón-Beato-García black hole [20] of mass M and electric charge Q [22, 23]. The ABG spacetime describes a spatially regular solution of the coupled Einstein-non-linear-electrodynamics field equations (see [19, 20] and references therein for details).

The black-hole spacetime is described, using the familiar Schwarzschild coordinates $\{t, r, \theta, \varphi\}$, by the curved line element [24]

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \ , \tag{5}$$

where the metric function in (5) is given by the radially-dependent expression [19, 20]

$$f^{\rm ABG}(r) = 1 - \frac{2Mr^2}{(r^2+Q^2)^{3/2}} + \frac{Q^2r^2}{(r^2+Q^2)^2} \ . \tag{6}$$

The horizon radii of the charged ABG spacetime are determined by the zeros of the metric function $f(r)$:

$$f(r) = 0 \quad \text{for} \quad r = r_{\rm H} \ . \tag{7}$$

The radially-dependent electrostatic potential of the ABG spacetime (5) is given by the non-trivial functional expression [19, 20]

$$\Phi^{\rm ABG}(r) = \frac{r^5}{2Q}\Big[\frac{3M}{r^5} + \frac{2Q^2}{(r^2+Q^2)^3} - \frac{3M}{(r^2+Q^2)^{5/2}}\Big] \ . \tag{8}$$

The dynamics of a linearized charged massive scalar field Ψ in the curved black-hole spacetime (5) is governed by the familiar Klein-Gordon wave equation [19, 25–27]

$$[(\nabla^\nu - iqA^\nu)(\nabla_\nu - iqA_\nu) - \mu^2]\Psi = 0 \ , \tag{9}$$

where $A_\nu = -\delta^0_\nu\Phi(r)$ is the electromagnetic potential [see Eq. (8)] of the charged black hole. Using the field decomposition

$$\Psi(t, r, \theta, \varphi) = \int\sum_{lm}\frac{1}{r}R_{lm}(r)Y_{lm}(\theta, \varphi)e^{im\varphi}e^{-i\omega t}d\omega \tag{10}$$

for the scalar wave function, where $Y_{lm}(\theta, \varphi)$ are the familiar spherical harmonic functions (with $l \geq |m|$ [28]), and using the tortoise radial coordinate $y(r)$, which is defined by the differential relation [29]

$$dy = \frac{dr}{f(r)} \ , \tag{11}$$

one finds from Eqs. (5), (6), and (9) the Schrödinger-like ordinary differential equation [30]

$$\frac{d^2R}{dy^2} - VR = 0 \tag{12}$$

for the scalar field. The effective potential $V[r(y)]$ of the composed charged-ABG-black-hole-charged-massive-scalar-field system is given by the radially-dependent functional expression [19]

$$V[r(y)] = f(r)\Big[\mu^2 + \frac{1}{r}\frac{df(r)}{dr} + \frac{l(l+1)}{r^2}\Big] - [\omega - q\Phi(r)]^2\ . \tag{13}$$

The differential equation (12) of the charged massive scalar field is supplemented by the large-r boundary condition [13]

$$R \sim e^{-\sqrt{\mu^2-\omega^2}y} \quad \text{for} \quad r \to \infty \ \ (y \to \infty) \tag{14}$$

which, in the bounded frequency regime

$$\omega^2 < \mu^2\ , \tag{15}$$

characterizes normalizable (spatially bounded) scalar eigenfunctions. In addition, the inner boundary condition [31]

$$R \sim e^{-i(\omega-q\Phi_{\text{H}})y} \quad \text{for} \quad r \to r_{\text{H}} \ \ (y \to -\infty) \tag{16}$$

for the scalar field, where [see Eqs. (8) and (10)]

$$\Phi_{\text{H}} \equiv \Phi(r = r_{\text{H}})\ , \tag{17}$$

describes purely ingoing waves at the outer horizon of the central black hole (as measured by a comoving observer).

Interestingly, in the next section we shall explicitly prove that the set of equations (12), (13), (14), and (16) determine the discrete resonance spectrum of the composed charged-ABG-black-hole-charged-massive-scalar-field cloudy configurations.

III. THE DISCRETE RESONANCE SPECTRUM OF THE COMPOSED ABG-BLACK-HOLE-LINEARIZED-CHARGED-MASSIVE-SCALAR-FIELD CLOUDY CONFIGURATIONS

In the present section we shall reveal the fact that the physical and mathematical properties of the composed charged-ABG-black-hole-linearized-charged-massive-scalar-field system can be studied *analytically* in the dimensionless regime

$$\frac{Q}{M} \ll 1 \ll M\mu \ll Mq\ . \tag{18}$$

The strong inequalities (18) characterize weakly charged ABG black holes and massive scalar fields whose Compton wavelengths are much smaller than the characteristic lengthscale M which is set by the radius of the central black hole.

In particular, we shall explicitly prove that the discrete resonance spectrum of the dimensionless charge-mass parameter

$$\alpha \equiv \frac{qQ}{M\mu}\ , \tag{19}$$

which characterizes the composed black-hole-scalar-field bound-state configurations, can be determined analytically in the regime (18).

The stationary (marginally-stable) bound-state scalar configurations in the ABG black-hole spacetime (5) are characterized by the critical frequency

$$\omega = q\Phi_{\text{H}} \tag{20}$$

for the superradiant amplification phenomenon of bosonic fields in the charged black-hole spacetime. In particular, taking cognizance of Eq. (16) one deduces that, for scalar fields with the critical frequency (20), there is no net flux of energy through the outer horizon of the central supporting black hole.

Taking cognizance of the strong inequalities (18), the metric function (6) can be approximated by

$$f(r) = 1 - \frac{2M}{r} + O(Q^2/r^2)\ , \tag{21}$$

which yields the expression [see Eq. (7)]

$$r_{\text{H}} = 2M\cdot[1 + O(Q^2/M^2)] \tag{22}$$

for the radius of the black-hole outer horizon.

In addition, from Eqs. (8) and (18) one finds the relation

$$\Phi(r) = \frac{Q}{r}\cdot\Big[1 + \frac{15M}{4r} + O(Q^2/r^2)\Big] \tag{23}$$

for the black-hole electric potential, which yields the simple horizon relation [see Eq. (22)]

$$\Phi_{\text{H}} = \frac{23}{16}\cdot\frac{Q}{M}\cdot[1 + O(Q^2/M^2)]\ . \tag{24}$$

Substituting Eqs. (19), (20), (21), (23), and (24) into Eq. (13), one finds that the composed black-hole-field radial potential can be approximated by

$$V(r) = \mu^2\cdot\Big(1 - \frac{2M}{r}\Big)\Big[1 - \alpha^2\cdot\Big(1 - \frac{2M}{r}\Big)\Big(\frac{23r + 30M}{16r}\Big)^2\Big]\ . \tag{25}$$

A. Upper and lower bounds on the allowed values of the dimensionless physical parameter α

In the present subsection we shall derive two necessary conditions for the existence of composed charged-ABG-black-hole-charge-field bound-state configurations in the dimensionless regime (18).

To this end, we first point out that Eqs. (15), (20), and (24) imply that, in the regime (18), spatially bounded scalar configurations are characterized by the dimensionless inequality

$$\Big(\frac{23}{16}\cdot\frac{qQ}{M}\Big)^2<\mu^2\ , \tag{26}$$

which yields the upper bound [see Eq. (19)]

$$\alpha<\frac{16}{23}\ . \tag{27}$$

In addition, we point out that the radial potential (25) is characterized by the asymptotic functional behaviors [see Eq. (22)] [32]

$$V(r\to r^+_{\text{H}})\to 0^+ \tag{28}$$

and

$$V(r\to\infty)\to\mu^2\cdot\Big[1-\Big(\frac{23\alpha}{16}\Big)^2\Big]>0\ . \tag{29}$$

The existence of a binding potential well outside the black-hole horizon provides a necessary condition for the existence of stationary bound-state configurations of the charged massive scalar fields in the curved spacetime (5). In particular, taking cognizance of the asymptotic properties (28) and (29) of the composed black-hole-field radial potential, one deduces that the requirement

$$\min_r\{V(r)\}<0 \tag{30}$$

provides a necessary condition for the existence of composed ABG-black-hole-scalar-field bound-state configurations.

The dimensionless function [see Eq. (25)]

$$\mathcal{F}(r)\equiv 1-\alpha^2\cdot\Big(1-\frac{2M}{r}\Big)\Big(\frac{23r+30M}{16r}\Big)^2 \tag{31}$$

is characterized by the radial minimum (for $r\geq r_{\text{H}}$)

$$\min_r\{\mathcal{F}(r)\}=1-\frac{6859}{3240}\cdot\alpha^2\quad\text{ for }\quad r_{\text{min}}=\frac{90}{7}M\ . \tag{32}$$

Thus, from Eqs. (25), (30), and (32) one finds that the dimensionless lower bound

$$\alpha > \sqrt{\frac{3240}{6859}} \tag{33}$$

provides a necessary condition for the existence of composed ABG-black-hole-scalar-field bound-state configurations.

Taking cognizance of the analytically derived necessary conditions (27) and (33) one deduces that, in the regime (18), the dimensionless physical parameter α which characterizes the composed charged-ABG-black-hole-charged-massive-scalar-field cloudy configurations must lie in the narrow [33] interval

$$\sqrt{\frac{3240}{6859}} < \alpha < \frac{16}{23}\ . \tag{34}$$

B. The resonance spectrum of the composed charged-ABG-black-hole-charged-massive-scalar-field cloudy configurations

In the present subsection we shall analyze the resonance spectrum $\{\alpha(M,Q,\mu,q;n\}_{n=0}^{n=\infty}$ of the dimensionless charge-mass parameter which characterizes the composed ABG-black-hole-charged-massive-scalar-field system. Interestingly, we shall explicitly prove that the discrete resonance spectrum can be determined *analytically* in the near-critical regime [see Eq. (33)]

$$\alpha \gtrsim \sqrt{\frac{3240}{6859}}\ . \tag{35}$$

In particular, the Schrödinger-like radial differential equation (12) of the supported charged massive scalar fields in the charged ABG black-hole spacetime (5) is characterized by the well known second-order WKB quantization condition [34–36]

$$\int_{y_{t_-}}^{y_{t_+}} dy\sqrt{-V(y;M,Q,\mu,q)} = (n+\frac{1}{2})\cdot\pi \quad ; \quad n=0,1,2,...\ , \tag{36}$$

where the integration limits $\{y_{t_-},y_{t_+}\}$, which are determined by the radial relations

$$V(y_{t_-}) = V(y_{t_+}) = 0\ , \tag{37}$$

are the classical turning points of the binding potential (25). The integer $n\in\{0,1,2,...\}$ in the WKB integral relation (36) is the discrete resonance parameter of the composed black-hole-field system. Taking cognizance of the differential relation (11), one can express the

WKB resonance condition (36) in the form

$$\int_{r_{t_-}}^{r_{t_+}} dr\sqrt{-\frac{V(r;M,Q,\mu,q)}{[f(r)]^2}} = (n+\frac{1}{2})\cdot\pi \quad ; \quad n=0,1,2,...\ . \tag{38}$$

Using the dimensionless physical variables $\{\epsilon, x\}$, which are defined by the relations [see Eqs. (32) and (35)]

$$\alpha \equiv \sqrt{\frac{3240}{6859}}\cdot(1+\epsilon) \quad ; \quad 0\leq\epsilon\ll 1 \tag{39}$$

and

$$r \equiv r_{\text{min}}\cdot(1+x) \quad ; \quad |x|\ll 1\ , \tag{40}$$

one can write the effective radial potential (25) of the composed charged-ABG-black-hole-charged-massive-scalar-field configurations in the dimensionless form [37]

$$V(x) = \mu^2\cdot\frac{38}{45}\cdot\Big(-2\epsilon+\frac{147}{5776}\cdot x^2\Big)\cdot[1+O(x,\epsilon)]\ . \tag{41}$$

Substituting Eqs. (40) and (41) into Eq. (38) and defining the dimensionless variable

$$z = \sqrt{\frac{147}{11552\epsilon}}\cdot x\ , \tag{42}$$

one obtains the remarkably compact WKB resonance condition

$$\epsilon\cdot M\mu\cdot\frac{360\sqrt{570}}{49}\int_{-1}^{1}dz\sqrt{1-z^2} = (n+\frac{1}{2})\cdot\pi \quad ; \quad n=0,1,2,... \tag{43}$$

for the composed charged-black-hole-charged-massive-scalar-field cloudy configurations. Performing the integration in (43), one finds the discrete resonance spectrum [38]

$$\epsilon_n = \frac{49}{180\sqrt{570}M\mu}\cdot(n+\frac{1}{2}) \quad ; \quad n=0,1,2,...\ . \tag{44}$$

As a consistency check we point out that one deduces from Eq. (44) the strong inequality $\epsilon\ll 1$ in the large-mass $M\mu\gg 1$ regime [see Eqs. (18) and (39)].

Substituting the analytically derived relation (44) into Eq. (39), one obtains the discrete large-mass resonance spectrum

$$\alpha_n = \sqrt{\frac{3240}{6859}}+\frac{49\sqrt{3}}{10830M\mu}\cdot(n+\frac{1}{2}) \quad ; \quad n=0,1,2,... \tag{45}$$

of the composed charged-ABG-black-hole-charged-massive-scalar-field cloudy configurations.

IV. THE EFFECTIVE RADIAL WIDTHS OF THE SUPPORTED CHARGED MASSIVE SCALAR CLOUDS

In the present section we shall determine the effective radial widths of the stationary charged massive scalar clouds that are supported in the charged ABG black-hole spacetime (5). In particular, we shall reveal the physically interesting fact that, in the dimensionless large-mass $M\mu \gg 1$ regime [see Eq. (18)], the supported scalar configurations can be made arbitrarily thin.

The effective widths of the supported charged scalar clouds in the charged ABG black-hole spacetime are determined by the classically allowed radial region [see Eq. (37)]

$$\Delta r(M,Q,\mu,q) \equiv r_{t_+} - r_{t_-} \tag{46}$$

of the composed black-hole-field binding potential (25). In particular, from Eq. (41) one finds the simple functional relation

$$\Delta x(M,Q,\mu,q) \equiv x_{t_+} - x_{t_-} = \sqrt{\frac{46208}{147}} \cdot \sqrt{\epsilon}\ , \tag{47}$$

which, taking cognizance of Eqs. (32) and (40), yields the expression

$$\frac{\Delta r}{M} = \frac{4560\sqrt{6}}{49} \cdot \sqrt{\epsilon} \tag{48}$$

for the effective dimensionless widths of the charged massive scalar field configurations that are supported in the charged ABG black-hole spacetime (5).

Substituting the resonance relation (44) into Eq. (48), one obtains the dimensionless expression

$$\frac{\Delta r}{M} = \frac{8\sqrt{19}\sqrt[4]{570}}{7} \cdot \sqrt{n+\frac{1}{2}} \cdot \frac{1}{\sqrt{M\mu}} \tag{49}$$

for the effective widths of the scalar clouds. From the analytically derived functional expression (49) one finds that the stationary charged scalar clouds, which are supported in the charged ABG black-hole spacetime (5), can be made arbitrarily thin in the dimensionless large-mass $M\mu \gg 1$ regime [39].

V. SUMMARY AND DISCUSSION

It has recently been demonstrated in the physically important paper [19] that charged scalar fields can be superradiantly amplified in the charged ABG black-hole spacetime [20] which describes a spatially regular solution of the coupled Einstein-non-linear-electrodynamics field equations.

Motivated by the interesting numerical results presented in [19], we have studied, using analytical techniques, the physical and mathematical properties of charged massive scalar field configurations (stationary scalar clouds) that are supported by charged ABG black holes. In particular, we have explicitly proved that the composed charged-ABG-black-hole-charged-massive-scalar-field system can be studied *analytically* in the dimensionless regime $Q/M \ll 1 \ll M\mu$ of weakly charged black holes and large-mass fields.

The main analytical results derived in this paper and their physical implications are as follows:

(1) We have proved that, in the regime (18), the dimensionless physical parameter $\alpha \equiv qQ/M\mu$, which characterizes the composed charged-black-hole-charged-field bound-state configurations, must lie in the narrow interval [see Eqs. (19) and (34)]

$$\frac{qQ}{M\mu} \in \Big(\sqrt{\frac{3240}{6859}}, \frac{16}{23}\Big) . \tag{50}$$

(2) Using a WKB analysis in the eikonal large-mass regime (18), we have derived the remarkably compact analytical resonance formula [see Eqs. (19) and (45)]

$$\Big(\frac{qQ}{M\mu}\Big)_n = \sqrt{\frac{3240}{6859}} + \frac{49\sqrt{3}}{10830M\mu} \cdot (n + \frac{1}{2}) \quad ; \quad n = 0, 1, 2, ... \tag{51}$$

for the dimensionless charge-mass parameter $qQ/M\mu$ which characterizes the composed charged-ABG-black-hole-charged-massive-scalar-field bound-state configurations.

(3) We have proved that the stationary charged massive scalar clouds are characterized by the effective dimensionless widths [40]

$$\frac{\Delta r}{M} = \frac{4\sqrt{38}\sqrt[4]{570}}{7} \cdot \frac{1}{\sqrt{M\mu}} . \tag{52}$$

Interestingly, the analytically derived functional expression (52) reveals the fact that the charged scalar configurations, which are supported in the charged ABG black-hole spacetime (5), are extremely thin (with $\Delta r/M \ll 1$) in the dimensionless large-mass $M\mu \gg 1$ regime.

(4) It is worth emphasizing that the analytically derived critical existence-line [41]

$$\frac{qQ}{M\mu} = \sqrt{\frac{3240}{6859}} + \frac{49\sqrt{3}}{21660M\mu} , \tag{53}$$

which characterizes the stationary charged-ABG-black-hole-charged-massive-scalar-field bound-state configurations, marks in the dimensionless large-mass regime (18) the sharp boundary between bald ABG black-hole spacetimes and charged black holes that are superradiantly unstable to perturbations of charged massive scalar fields.

In particular, the expected onset of superradiant instabilities in the composed ABG-black-hole-scalar-field system above the critical existence-line (53) hints that this physically interesting system may be characterized by the existence of charged hairy black-hole configurations that support non-linear (self-gravitating) charged massive scalar fields.

Finally, we would like to stress again that, in the present analysis, the minimally coupled charged massive scalar fields were treated at the linearized level. As we explicitly shown, the main advantage of this approach lies in the fact that the composed charged-ABG-black-hole-charged-massive-linearized-scalar-field system is amenable to an *analytical* treatment. We believe that it would be highly interesting (and physically important) to use non-linear *numerical* techniques in order to prove the existence of genuine spatially regular hairy (scalarized) charged ABG black-hole spacetimes.

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[21] Here M and Q are respectively the mass and electric charge of the ABG black hole and μ is the proper mass of the scalar field.

[22] We shall assume, without loss of generality, the relations $q > 0$ and $Q > 0$ for the electric charges of the scalar field and the central black hole.

[23] Note that the physical parameters q and μ of the charged massive scalar field stand respectively for $q/\hbar$ and $\mu/\hbar$ and therefore have the dimensions of $(\text{length})^{-1}$.

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[28] Here the integers l and m are respectively the spherical harmonic index and the azimuthal harmonic index which characterize the scalar field mode.

[29] Note that the differential relation (11) maps the radial coordinate $r \in [r_{\rm H},\infty]$ to the radial coordinate $y \in [-\infty,\infty]$.

[30] For brevity, we shall henceforth omit the angular harmonic indices $\{l,m\}$ of the charged massive scalar field.

[31] We shall henceforth use the notation $r_{\rm H}$ for the radius of the black-hole outer horizon.

[32] It is worth noting that the exact radial potential (13) is also characterized by the horizon behavior (28) [see Eqs. (7), (17), and (20)].

[33] Interestingly, one finds that the upper bound 16/23 on the value of the dimensionless physical parameter α is larger than the lower bound $\sqrt{3240/6859}$ by only $\sim 1\%$ [see Eq. (34)].

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[38] Here we have used the integral relation $\int_{-1}^{1} dz\sqrt{1-z^2} = \pi/2$.

[39] It is worth noting that the charged massive scalar field is exponentially suppressed outside the classically allowed narrow radial interval (49).

[40] Here we have used the dimensionless functional relation (49) with the value $n=0$ for the fundamental resonant mode of the composed charged-black-hole-charged-scalar-field system.

[41] Here we have used the analytically derived resonance spectrum (51) with the value $n=0$ for the fundamental resonant mode of the composed charged-black-hole-charged-scalar-field system.