Nonlinear Observer Design for the Discrete-time Systems: An Enhanced LMI Approach

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Abstract

This manuscript focuses on the $\mathcal{H}_\infty$ observer design for a class of nonlinear discrete systems under the presence of measurement noise or external disturbances. Two new Linear Matrix Inequality (LMI) conditions are developed in this method through the utilization of the reformulated Lipschitz property, a new variant of Young inequality and the well-known Linear Parameter Varying (LPV) approach. One of the key components of the proposed LMIs is the generalized matrix multipliers. The judicious use of these multipliers enables us to introduce more numbers of decision variables inside LMIs than the one illustrated in the literature. It aids in adding some extra degrees of freedom from a feasibility point of view, thus enhancing the LMI conditions. Thus, the established LMIs are less conservative than existing ones. Later on, the effectiveness of the developed LMIs and observer is highlighted through a numerical example and the application of state of charge (SoC) estimation in the Li-ion battery model.

Keywords: Nonlinear observer design, Lipschitz systems, $\mathcal{H}_\infty$ criterion and Linear Matrix Inequalities (LMIs).

1. Introduction

Over the past three decades, the topic of observer design for dynamical systems has received a significant amount of interest from researchers of control system engineering.

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This is because the state variables are essential in the system analysis and feedback control design. Although many of these variables can be measured through sensors, some remain inaccessible due to the unavailability of sensors. Thus, the observers play a pivotal role in modern-day applications that assist in capturing real-time information of systems, for example, the state-of-charge estimation of LI-ion battery model [1], autonomous vehicle tracking [2], and so on.

In the literature, numerous approaches have been proposed for linear observer design, and all have shown high reliability. However, in contrast to the linear observer, the development of nonlinear observers is an arduous problem. There is no systematic technique to construct these types of observers. Hence, an abundant amount of research has been carried out in this area. The several popular methods of nonlinear observers are as follows:

a) Transformation-based observers [3];

b) High-gain observers [4];

c) Sliding-mode observers [5];

d) Linear Matrix Inequality (LMI)-based observers [6].

All these methodologies are developed for continuous-time nonlinear systems, while the few methodologies related to discrete-time systems are showcased in [7, 8, 9] and [6]. Among these techniques, LMI-based techniques are extensively studied by researchers and some of them are provided in [10, 8, 11] and [9]. Recently, the authors of [12] had established a new matrix-multiplier-based LMI approach for continuous-time nonlinear observer design. The LMI condition presented in [12] is less conservative than the one shown in [2]. It inspires the authors to derive a new LMI condition for the development of nonlinear observers for discrete-time systems through the utilization of new matrix multipliers, reformulated Lipschitz property and Young inequalities. Further, the effectiveness of the developed LMI condition is shown through a numerical example. The performance of the observer is validated through the application of State-of-charge (SoC) estimation of Li-ion battery.

The rest of the paper is structured as follows: Some preliminaries related to the nonlinear observer design and the notations are illustrated in Section 2. Further, Section

3 encompasses the contextualization of the problem statement. Segment 4 contains the formulation of two new LMI conditions. Further, some comments and remarks related to the proposed methodology are outlined in Section 5. The effectiveness of the new LMI conditions and the observer performance is showcased in Section 6. Finally, Section 7 entails a few concluding remarks on the established approach.

2. Nomenclature and Prerequisites

This section encompasses the illustration of denotation used in this paper. Later on, we recaptured some mathematical tools related to nonlinear observer design.

2.1. Notations

Through this paper, the ensuing terminologies are utilized: The euclidean norm and the $\mathcal{L}_2$ norm of a vector e are depicted by $||e||$ and $||e||_{\mathcal{L}_2}$, respectively. The term e_0 denotes the initial values of $e(t)$ at $t = 0$. A vector of the canonical basis of $\mathbb{R}^s$ is illustrated as:

$$e_s(i) = (\underbrace{0, \ldots, 0, \overbrace{1}^{i^{\text{th}}}, 0, \ldots, 0}_{s \text{ components}})^\top \in \mathbb{R}^s, \; s \geq 1.$$

The identity matrix and the null matrix are represented by $\mathbb{I}$ and $\mathbb{O}$, respectively. The transpose of matrix A is symbolised as $A^\top$, while, $A \in \mathbf{S}^n$ infers that A is a symmetric matrix of dimension $n \times n$. The repeated blocks within a symmetric matrix are showcased by the symbol $(\star)$. $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ depict minimum and maximum eigenvalues of $A \in \mathbf{S}^n$, respectively. For the aforementioned matrix A, $A > 0$ ($A < 0$) indicates that it is a positive definite matrix (a negative definite matrix). Similarly, a positive semi-definite matrix (a negative semi-definite matrix) is showcased by $A \geq 0$ ($A \leq 0$). A block-diagonal matrix having elements $A_1, \ldots, A_n$ in the diagonal is described as $A =$ block-diag$(A_1, \ldots, A_n)$.

2.2. Preliminaries

This segment presents an overview of the mathematical tools and background results which will be needed in the development of the LMI conditions.

Lemma 1 (Reformulated Lipschitz property[13]). *Let us consider a global Lipschitz nonlinear function* $h : \mathbb{R}^n \to \mathbb{R}^n$. *Then, for all* $i, j \in \{1, \ldots, n\}$, *there exists functions* $h_{ij} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ *such that* $\forall\, \Psi,\, \Phi \in \mathbb{R}^n$,

$$h(\Psi) - h(\Phi) = \sum_{i=1}^{n} \sum_{j=1}^{n} h_{ij} \mathcal{H}_{ij} (\Psi - \Phi), \tag{1}$$

where $\mathcal{H}_{ij} = e_n(i) e_n^\top(j)$ *and* $h_{ij} \triangleq h_{ij}(\Psi^{\Phi_{j-1}}, \Psi^{\Phi_j})$. *Additionally, the functions* h_{ij} *hold*

$$h_{ij_{\min}} \leq h_{ij} \leq h_{ij_{\max}}, \tag{2}$$

where $h_{ij_{\min}}$ *and* $h_{ij_{\max}}$ *are constants.*

Lemma 2 (Young's inequalities). *For any two vectors* $X, Y \in \mathbb{R}^n$ *and a matrix* $Z > 0 \in \mathbf{S}^n$, *the ensuing inequalities are true:*

$$X^\top Y + Y^\top X \leq X^\top Z^{-1} X + Y^\top Z Y, \tag{3}$$

and

$$X^\top Y + Y^\top X \leq (X + ZY)^\top (2Z)^{-1} (X + ZY). \tag{4}$$

The inequality (3) *is known as the standard Young's inequality. However, the authors of [2] presented a variant of Young's inequality, represented by* (4).

Lemma 3 ([12]). *Let us consider*

$$\mathbb{X}^\top = \begin{bmatrix} a_1 \mathbb{I}_n & a_2 \mathbb{I}_n & \ldots & a_n \mathbb{I}_n \end{bmatrix}, \tag{5}$$

$$\mathbb{Y}^\top = \begin{bmatrix} b_1 \mathbb{I}_n & b_2 \mathbb{I}_n & \ldots & b_n \mathbb{I}_n \end{bmatrix}, \tag{6}$$

along with

$$Z = \begin{bmatrix} Z_1 & Z_{a_1^2} & \ldots & Z_{a_1^n} \\ \star & Z_2 & \ldots & Z_{a_2^n} \\ \star & \star & \ddots & \vdots \\ \star & \star & \ldots & Z_n \end{bmatrix}, \tag{7}$$

where $0 \leq a_i \leq b_i$ $\forall i \in \{1, \ldots, n\}$ *and* $Z_i > 0 \in \mathbf{S}^n$, $Z_{a_i^j} \geq 0 \in \mathbf{S}^n$ $\forall\, i \in \{1, \ldots, n\}$ *so that* $Z > 0$. *Then, the subsequent inequality is fulfilled:*

$$\mathbb{X}^\top Z \mathbb{X} - \mathbb{Y}^\top Z \mathbb{Y} \leq 0. \tag{8}$$

For the proof of Lemma 3, one can refer [12].

Lemma 4. *For any given matrices A, B and C of appropriate dimension, the subsequent equality is true:*

$$\begin{bmatrix} A^\top CA & A^\top CB \\ B^\top CA & B^\top PB \end{bmatrix} = \begin{bmatrix} A & B \end{bmatrix}^\top C \begin{bmatrix} A & B \end{bmatrix} \tag{9}$$

The proof of this lemma is very straightforward. Through matrix multiplication, one can easily derive (9).

3. Problem statement

Let us consider that the following sets of equations represent a class of disturbance-affected nonlinear system dynamics with nonlinear output:

$$\begin{aligned} x_{k+1} &= Ax_k + Gf(x_k) + B_1 u_k + E\omega_k, \\ y_k &= Cx_k + Fg(x_k) + B_2 u_k + D\omega_k, \end{aligned} \tag{10}$$

where $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^p$ denote a state vector of the system, and its output, respectively. The input provided to the system is illustrated by $u \in \mathbb{R}^s$. However, $\omega \in \mathbb{R}^q$ depicts $\mathcal{L}_2$ bounded noise/disturbance vectors affecting the system dynamics and measurements. The matrices $A \in \mathbb{R}^{n\times n}$, $G \in \mathbb{R}^{n\times m}$, $B_1 \in \mathbb{R}^{n\times s}$, $B_2 \in \mathbb{R}^{p\times s}$, $C \in R^{p\times n}$, $E \in \mathbb{R}^{n\times q}$, $D \in \mathbb{R}^{p\times q}$ and $F \in \mathbb{R}^{p\times r}$ are known and constant. The functions $f(\cdot) : \mathbb{R}^n \to \mathbb{R}^m$ and $g(\cdot) : \mathbb{R}^n \to \mathbb{R}^r$ are the nonlinearities present in the dynamics and outputs of the system, respectively. Both functions are presumed to be globally Lipschitz. Further, we have expressed $f(\cdot)$ and $g(\cdot)$ in the ensuing manner:

$$f(x_k) = \begin{bmatrix} f_1(F_1 x_k) \\ \vdots \\ f_i(\underbrace{F_i x_k}_{\theta_i}) \\ \vdots \\ f_m(\theta_m) \end{bmatrix}, \tag{11a}$$

$$g(x_k) = \begin{bmatrix} g_1(G_1 x_k) \\ \vdots \\ f_i(\underbrace{G_i x_k}_{\nu_i}) \\ \vdots \\ g_r(\nu_r) \end{bmatrix}, \tag{11b}$$

where $F_i \in \mathbb{R}^{\bar{n} \times n}\ \forall\ i \in \{1, \ldots, m\}$ and $G_i \in \mathbb{R}^{\bar{p} \times n}\ \forall\ i \in \{1, \ldots, r\}$.

Remark 1. *If system dynamics and output of* (10) *are influenced by two different noises,* ω_1 *and* ω_2*, through* E_1 *and* D_1 *respectively, then the system can be rewritten in the form of* (10) *by considering matrices* $E = \begin{bmatrix} E_1 & \mathbb{O} \end{bmatrix}$, $D = \begin{bmatrix} \mathbb{O} & D_1 \end{bmatrix}$, *and the noise vector* $\omega = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}$.

For the state estimation purposes of the system (10), we have deployed the subsequent Luenberger observer form:

$$\hat{x}_{k+1} = A\hat{x}_k + Gf(\hat{x}_k) + Bu_k + L\big(y_k - C\hat{x}_k - Fg(\hat{x}_k)\big), \tag{12}$$

where $\hat{x}_k$ and $L \in \mathbb{R}^{n \times p}$ are the estimated states and the observer gain matrix, respectively. The estimation error of the proposed observer (12) is defined as

$$e_k = x_k - \hat{x}_k.$$

Thus, from (10) and (12), the estimation error dynamic is computed and expressed as:

$$e_{k+1} = (A - LC)e_k + G\big(f(x_k) - f(\hat{x}_k)\big) - LF\big(g(x_k) - g(\hat{x}_k)\big) + (E - LD)\omega_k. \tag{13}$$

Since $f(\cdot)$ and $g(\cdot)$ are globally Lipschitz, the implementation of Lemma 1 on the terms $(f(x_k) - f(\hat{x}_k))$ and $(g(x_k) - g(\hat{x}_k))$ yields:

1) There exist functions $f_{ij} : \mathbb{R}^{\bar{n}} \times \mathbb{R}^{\bar{n}} \to \mathbb{R}$, $g_{ij} : \mathbb{R}^{\bar{p}} \times \mathbb{R}^{\bar{p}} \to \mathbb{R}$ such that

$$f(x_k) - f(\hat{x}_k) = \sum_{i,j=1}^{m,\bar{n}} f_{ij}\mathcal{H}_{ij}F_i e_k, \tag{14a}$$

$$g(x_k) - g(\hat{x}_k) = \sum_{i,j=1}^{r,\bar{p}} g_{ij}\mathcal{G}_{ij}G_i e_k, \tag{14b}$$

where $f_{ij} \triangleq f_{ij}(\theta_i^{\hat{\theta}_{i,j-1}}, \theta_i^{\hat{\theta}_{i,j}})$ and $g_{ij} \triangleq g_{ij}(\nu_i^{\hat{\nu}_{i,j-1}}, \nu_i^{\hat{\nu}_{i,j}})$.

2) The functions f_{ij} and g_{ij} satisfy

$$f_{a_{ij}} \leq f_{ij} \leq f_{b_{ij}}, \tag{15}$$

$$g_{a_{ij}} \leq g_{ij} \leq g_{b_{ij}}, \tag{16}$$

where $f_{a_{ij}}$, $f_{b_{ij}}$, $g_{a_{ij}}$, and $g_{b_{ij}}$ are known constants.

Without loss of generality, let us presume that $f_{a_{ij}} = 0$ and $g_{a_{ij}} = 0$. Thus, the inequalities (15) and (16) are reformulated as:

$$0 \leq f_{ij} \leq f_{b_{ij}}, \tag{17}$$

$$0 \leq g_{ij} \leq g_{b_{ij}}. \tag{18}$$

One can refer [13] for additional information about this.

By incorporating (14) into (13), the error dynamic (13) is reformulated as:

$$e_{k+1} = \underbrace{\left((A - LC) + \sum_{i,j=1}^{m,\bar{n}} f_{ij} G\mathcal{H}_{ij} H_i - \sum_{i,j=1}^{r,\bar{p}} g_{ij} LF\mathcal{G}_{ij} G_i\right)}_{\mathbb{A}} e_k + \underbrace{(E - LD)}_{\mathbb{E}} \omega_k. \tag{19}$$

Remark 2. *In various practical applications, it is possible to have $f_{a_{ij}}, g_{a_{ij}} \neq 0$. In such cases, the system* (19) *is rewritten as*

$$\begin{aligned} e_{k+1} = &\left(\underbrace{A - LC + \sum_{i,j=1}^{m,\bar{n}} f_{a_{ij}} G\mathcal{H}_{ij} H_i - \sum_{i,j=1}^{r,\bar{p}} g_{a_{ij}} LF\mathcal{G}_{ij} G_i}_{\tilde{A}}\right) e_k \\ &+ \left(\sum_{i,j=1}^{m,\bar{n}} \underbrace{(f_{ij} - f_{a_{ij}})}_{\tilde{f}_{ij}} G\mathcal{H}_{ij} H_i \tilde{x} - \sum_{i,j=1}^{r,\bar{p}} \underbrace{(g_{ij} - g_{a_{ij}})}_{\tilde{g}_{ij}} LF\mathcal{G}_{ij} G_i\right) e_k + \mathbb{E}\omega_k. \end{aligned}$$

It yields:

$$e_{k+1} = \underbrace{\left(\tilde{A} + \sum_{i,j=1}^{m,\bar{n}} \tilde{f}_{ij} G\mathcal{H}_{ij} H_i \tilde{x} - \sum_{i,j=1}^{r,\bar{p}} \tilde{g}_{ij} LF\mathcal{G}_{ij} G_i\right)}_{\tilde{\mathbb{A}}} e_k + \mathbb{E}\omega_k. \tag{20}$$

For the error dynamic (20)*, the functions $\tilde{f}_{ij}$ and $\tilde{g}_{ij}$ hold* (17) *and* (18)*, respectively. It is easy to notice that both forms* (19) *and* (20) *are analogous.*

The objective of the proposed methodology is to estimate the gain matrix L so that

1. The estimation error dynamic (19) is asymptotically stable in the absence of the disturbances/noise, i.e., at $\omega = 0$.

2. When $\omega \neq 0$, the closed-loop system (19) fulfills the ensuing $\mathcal{H}_\infty$ criterion:

$$||e||_{\mathcal{L}_2} \leq \sqrt{\mu||\omega||^2_{\mathcal{L}_2} + \nu||e_0||^2}, \tag{21}$$

   where the positive scalar $\sqrt{\mu}$ is known as the noise attenuation level.

The aforementioned problem statement has garnered a significant amount of interest from researchers in the domain of control system engineering, resulting in the establishment of numerous LMI-based methods, for example, [8], [14], [15], and so on. The LMI conditions provided by each of these methods are based on several mathematical tools, such as the Schur lemma and the Young inequality. Though all of these techniques yield less conservative LMI conditions, there is a scope for further enhancements. In the sequel, through the exploration of Lemma 2, Lemma 3 and newly defined matrix multipliers, two new LMI criteria are derived.

4. Main result

This section of the manuscript is devoted to the formulation of the $\mathcal{H}_\infty$ criterion-based LMI conditions which ensures the asymptotic stability of the error dynamic (19).

For the stability analysis of the system (19), the following quadratic Lyapunov function is used:

$$V(e_k) = e_k^\top P e_k, \text{ where } P > 0 \in \mathbf{S}^n \tag{22}$$

Let us consider

$$\Delta V_k = V(e_{k+1}) - V(e_k).$$

Through the utilization of (19) and (22), one can obtain:

$$\begin{aligned}\Delta V_k = {} & e_k^\top\Big(-P + \mathbb{A}^\top P\mathbb{A}\Big)e_k + e_k^\top\Big(\mathbb{A}^\top P\mathbb{E}\Big)\omega_k + \omega_k^\top\Big(\mathbb{E}^\top P\mathbb{A}\Big)e_k \\ & + \omega_k^\top\mathbb{E}^\top P\mathbb{E}\omega_k.\end{aligned} \tag{23}$$

According to [14], the $\mathcal{H}_\infty$ criterion (21) is satisfied if the ensuing inequality is true:

$$\mathcal{W}_k \triangleq \Delta V_k + ||e_k||^2 - \mu||\omega_k||^2 \leq 0. \tag{24}$$

From (23), the inequality (24) is modified as:

$$\begin{aligned}\mathcal{W}_k &= e_k^\top\Big(\mathbb{I}-P+\mathbb{A}^\top P\mathbb{A}\Big)e_k+e_k^\top\Big(\mathbb{A}^\top P\mathbb{E}\Big)\omega_k+\omega_k^\top\Big(\mathbb{E}^\top P\mathbb{A}\Big)e_k \\ &\quad+\omega_k^\top\Big(\mathbb{E}^\top P\mathbb{E}-\mu\mathbb{I}\Big)\omega_k.\end{aligned} \tag{25}$$

Further, $\mathcal{W}_k \leq 0$ if

$$\begin{bmatrix}\mathbb{I}-P+\mathbb{A}^\top P\mathbb{A} & \mathbb{A}^\top P\mathbb{E}\\ \mathbb{E}^\top P\mathbb{A} & \mathbb{E}^\top P\mathbb{E}-\mu\mathbb{I}\end{bmatrix}\leq 0. \tag{26}$$

Through the deployment of Lemma 4, the inequality (26) is equivalent to

$$\begin{bmatrix}\mathbb{I}-P & \mathbb{O}\\ \mathbb{O} & -\mu\mathbb{I}\end{bmatrix}+\begin{bmatrix}\mathbb{A}^\top\\ \mathbb{E}^\top\end{bmatrix}P\begin{bmatrix}\mathbb{A} & \mathbb{E}\end{bmatrix}\leq 0. \tag{27}$$

The use of Schur Lemma on (27) resulted in

$$\Sigma_1+\mathbf{NL}\leq 0, \tag{28}$$

where

$$\Sigma_1=\begin{bmatrix}\begin{bmatrix}\mathbb{I}-P & \mathbb{O}\\ \mathbb{O} & -\mu\mathbb{I}\end{bmatrix} & \begin{bmatrix}(A-LC)^\top P\\ (E-LD)^\top P\end{bmatrix}\\ \star & -P\end{bmatrix}, \tag{29}$$

$$\mathbf{NL}=\begin{bmatrix}\mathbb{O} & \mathbb{O} & \Big(\sum_{i,j=1}^{m,\bar{n}} f_{ij}G\mathcal{H}_{ij}H_i-\sum_{i,j=1}^{r,\bar{p}} g_{ij}LF\mathcal{G}_{ij}G_i\Big)^\top P\\ \star & \mathbb{O} & \mathbb{O}\\ \star & \star & \mathbb{O}\end{bmatrix}. \tag{30}$$

For enhancement of comprehensibility of the method, let us introduce $R^\top = PL$. Now, one can express the term Σ_1 in the ensuing form:

$$\Sigma=\begin{bmatrix}\begin{bmatrix}\mathbb{I}-P & \mathbb{O}\\ \mathbb{O} & -\mu\mathbb{I}\end{bmatrix} & \begin{bmatrix}A^\top P-C^\top R\\ E^\top P-D^\top R\end{bmatrix}\\ \star & -P\end{bmatrix}. \tag{31}$$

Similarly, the term **NL** is rewritten as:

$$
\begin{aligned}
\mathbf{NL} = \sum_{i,j=1}^{m,\bar{n}} & \left(\underbrace{\begin{bmatrix} \mathbb{O} \\ \mathbb{O} \\ PG\mathcal{H}_{ij} \end{bmatrix}}_{\mathbb{U}_{ij}^\top} \underbrace{f_{ij} \overbrace{\begin{bmatrix} H_i & \mathbb{O} & \mathbb{O} \end{bmatrix}}^{\mathbb{H}_i}}_{\mathbb{V}_{ij}} + \mathbb{V}_{ij}^\top \mathbb{U}_{ij} \right) \\
+ \sum_{i,j=1}^{r,\bar{p}} & \left(\underbrace{\begin{bmatrix} \mathbb{O} \\ \mathbb{O} \\ -R^\top F\mathcal{G}_{ij} \end{bmatrix}}_{\mathbb{M}_{ij}^\top} \underbrace{g_{ij} \overbrace{\begin{bmatrix} G_i & \mathbb{O} & \mathbb{O} \end{bmatrix}}^{\mathbb{G}_i}}_{\mathbb{N}_{ij}} + \mathbb{N}_{ij}^\top \mathbb{M}_{ij} \right).
\end{aligned}
\tag{32}
$$

Recently, the topic of LMI-based observers has been extensively investigated to handle Lipschitz nonlinearities. One can go through [8, 6, 14] and so on. The authors of [6, 16] have used the global form of nonlinearities (i.e., $\tilde{f}(x, \hat{x}) = f(x) - f(\hat{x})$). However, the detailed form of nonlinearities (that is, (12)) was deployed in papers[14]. The use of nonlinearities in their detailed form enables the inclusion of additional decision variables in the LMI approach. In this paper, we propose two new LMI conditions inspired by the method outlined in [17].

In order to avoid cumbersome equations, the term **NL** is reformulated as:

$$\mathbf{NL} = \mathbb{U}^\top(\mathbb{H}\Phi) + \Phi^\top\mathbb{H}^\top\mathbb{U} + \mathbb{M}^\top(\mathbb{G}\Psi) + \Psi^\top\mathbb{G}^\top\mathbb{M}, \tag{35}$$

where

$$\mathbb{U} = \begin{bmatrix} \mathbb{U}_{11}^\top & \dots & \mathbb{U}_{1\bar{n}}^\top & \dots & \mathbb{U}_{m1}^\top & \dots & \mathbb{U}_{m\bar{n}}^\top \end{bmatrix}^\top, \tag{36}$$

$$\mathbb{H} = \text{block-diag}(\mathbb{H}_1, \dots, \mathbb{H}_1, \dots, \mathbb{H}_m, \dots, \mathbb{H}_m), \tag{37}$$

$$\Phi = \begin{bmatrix} f_{11}\mathbb{I} & \dots & f_{1\bar{n}}\mathbb{I} & \dots & f_{m1}\mathbb{I} & \dots & f_{m\bar{n}}\mathbb{I} \end{bmatrix}^\top, \tag{38}$$

$$\mathbb{M} = \begin{bmatrix} \mathbb{M}_{11}^\top & \dots & \mathbb{M}_{1\bar{p}}^\top & \dots & \mathbb{M}_{r1}^\top & \dots & \mathbb{M}_{r\bar{p}}^\top \end{bmatrix}^\top, \tag{39}$$

$$\mathbb{G} = \text{block-diag}(\mathbb{G}_1, \dots, \mathbb{G}_1, \dots, \mathbb{G}_r, \dots, \mathbb{G}_r), \tag{40}$$

$$\Psi = \begin{bmatrix} g_{11}\mathbb{I} & \dots & g_{1\bar{p}}\mathbb{I} & \dots & g_{r1}\mathbb{I} & \dots & g_{r\bar{p}}\mathbb{I} \end{bmatrix}^\top. \tag{41}$$

From (28), (31) and (35), $\mathcal{W} \leq 0$ if

$$\Sigma + \underbrace{\mathbb{U}^\top(\mathbb{H}\Phi) + \Phi^\top\mathbb{H}^\top\mathbb{U}}_{\mathbf{NL}_1} + \underbrace{\mathbb{M}^\top(\mathbb{G}\Psi) + \Psi^\top\mathbb{G}^\top\mathbb{M}}_{\mathbf{NL}_2} \leq 0. \tag{42}$$

$$
\mathbb{Z} = \begin{bmatrix}
Z_{11} & Z_{a^1_{12}} & \dots & Z_{a^1_{1\bar{n}}} & Z_{b^{11}_{21}} & Z_{b^{11}_{22}} & \dots & Z_{b^{11}_{2\bar{n}}} & \dots & Z_{b^{11}_{m1}} & Z_{b^{11}_{m2}} & \dots & Z_{b^{11}_{m\bar{n}}} \\
Z_{a^1_{12}} & Z_{12} & \dots & Z_{a^2_{1\bar{n}}} & Z_{b^{12}_{21}} & Z_{b^{12}_{22}} & \dots & Z_{b^{12}_{2\bar{n}}} & \dots & Z_{b^{12}_{m1}} & Z_{b^{12}_{m2}} & \dots & Z_{b^{12}_{m\bar{n}}} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \dots & \vdots & \vdots & \ddots & \vdots \\
Z_{a^1_{1\bar{n}}} & Z_{a^2_{1\bar{n}}} & \dots & Z_{1\bar{n}} & Z_{b^{1\bar{n}}_{21}} & Z_{b^{1\bar{n}}_{22}} & \dots & Z_{b^{1\bar{n}}_{2\bar{n}}} & \dots & Z_{b^{1\bar{n}}_{m1}} & Z_{b^{1\bar{n}}_{m2}} & \dots & Z_{b^{1\bar{n}}_{m\bar{n}}} \\
Z_{b^{11}_{21}} & Z_{b^{12}_{21}} & \dots & Z_{b^{1\bar{n}}_{21}} & Z_{21} & Z_{a^1_{22}} & \dots & Z_{a^1_{2\bar{n}}} & \dots & Z_{b^{21}_{m1}} & Z_{b^{21}_{m2}} & \dots & Z_{b^{21}_{m\bar{n}}} \\
Z_{b^{11}_{22}} & Z_{b^{12}_{22}} & \dots & Z_{b^{1\bar{n}}_{22}} & Z_{a^1_{22}} & Z_{22} & \dots & Z_{a^2_{2\bar{n}}} & \dots & Z_{b^{22}_{m1}} & Z_{b^{22}_{m2}} & \dots & Z_{b^{22}_{m\bar{n}}} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \dots & \vdots & \vdots & \ddots & \vdots \\
Z_{b^{11}_{2\bar{n}}} & Z_{b^{12}_{2\bar{n}}} & \dots & Z_{b^{1\bar{n}}_{2\bar{n}}} & Z_{a^1_{2\bar{n}}} & Z_{a^2_{2\bar{n}}} & \dots & Z_{2\bar{n}} & \dots & Z_{b^{2\bar{n}}_{m1}} & Z_{b^{2\bar{n}}_{m2}} & \dots & Z_{b^{2\bar{n}}_{m\bar{n}}} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
Z_{b^{11}_{m1}} & Z_{b^{12}_{m1}} & \dots & Z_{b^{1\bar{n}}_{m1}} & Z_{b^{21}_{m1}} & Z_{b^{22}_{m1}} & \dots & Z_{b^{2\bar{n}}_{m1}} & \dots & Z_{m1} & Z_{a^1_{m2}} & \dots & Z_{a^1_{m\bar{n}}} \\
Z_{b^{11}_{m2}} & Z_{b^{12}_{m2}} & \dots & Z_{b^{1\bar{n}}_{m2}} & Z_{b^{21}_{m2}} & Z_{b^{22}_{m2}} & \dots & Z_{b^{2\bar{n}}_{m2}} & \dots & Z_{a^1_{m2}} & Z_{m2} & \dots & Z_{a^2_{m\bar{n}}} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \dots & \vdots & \vdots & \ddots & \vdots \\
Z_{b^{11}_{m\bar{n}}} & Z_{b^{12}_{m\bar{n}}} & \dots & Z_{b^{1\bar{n}}_{m\bar{n}}} & Z_{b^{21}_{m\bar{n}}} & Z_{b^{22}_{m\bar{n}}} & \dots & Z_{b^{2\bar{n}}_{m\bar{n}}} & \dots & Z_{a^1_{m\bar{n}}} & Z_{a^2_{m\bar{n}}} & \dots & Z_{m\bar{n}}
\end{bmatrix}, \tag{33}
$$

where $Z_{ij} > 0 \in \mathbf{S}^{\bar{n}}$, $Z_{a^k_{ij}} \geq 0 \in \mathbf{S}^{\bar{n}}$ $\forall i, k \in \{1, \dots, m\}, \& j \in \{1, \dots, \bar{n}\}$; $Z_{b^{kj}_{ij}} \geq 0 \in \mathbf{S}^{\bar{n}}, \forall i \in \{2, \dots, m\}, k \in \{1, \dots, m-1\}, \& j \in \{1, \dots, \bar{n}\}$ such that $\mathbb{Z} > 0$.

Through the use of Lemma 2 and Lemma 3, two new LMI conditions are developed in the sequels, which ensures the asymptotic stability of the system (19).

Theorem 1. *Let us introduce two matrices, $\mathbb{Z}$ and $\mathbb{S}$, illustrated by* (33) *and* (34)*, respectively. If there exist matrices $P > 0 \in \mathbf{S}^n$, $R \in \mathbb{R}^{p \times n}$ and a positive scalar μ such that the following optimization problem is solvable:*

$$
\textit{minimize } \mu \textit{ subject to}
$$

$$
\begin{bmatrix}
\Sigma & \mathbb{U}^\top & (\mathbb{Z}\mathbb{H}\Phi_m)^\top & \mathbb{M}^\top & (\mathbb{S}\mathbb{G}\Psi_m)^\top \\
\star & -\mathbb{Z} & \mathbb{O} & \mathbb{O} & \mathbb{O} \\
\star & \star & -\mathbb{Z} & \mathbb{O} & \mathbb{O} \\
\star & \star & \star & -\mathbb{S} & \mathbb{O} \\
\star & \star & \star & \star & -\mathbb{S}
\end{bmatrix} \leq 0, \tag{43}
$$

$$\mathbb{S} = \begin{bmatrix} S_{11} & S_{a^1_{12}} & \dots & S_{a^1_{1\bar{p}}} & S_{b^{11}_{21}} & S_{b^{11}_{22}} & \dots & S_{b^{11}_{2\bar{p}}} & \dots & S_{b^{11}_{r1}} & S_{b^{11}_{r2}} & \dots & S_{b^{11}_{r\bar{p}}} \\ S_{a^1_{12}} & S_{12} & \dots & S_{a^2_{1\bar{p}}} & S_{b^{12}_{21}} & S_{b^{12}_{22}} & \dots & S_{b^{12}_{2\bar{p}}} & \dots & S_{b^{12}_{r1}} & S_{b^{12}_{r2}} & \dots & S_{b^{12}_{r\bar{p}}} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \dots & \vdots & \vdots & \ddots & \vdots \\ S_{a^1_{1\bar{p}}} & S_{a^2_{1\bar{p}}} & \dots & S_{1\bar{p}} & S_{b^{1\bar{p}}_{21}} & S_{b^{1\bar{p}}_{22}} & \dots & S_{b^{1\bar{p}}_{2\bar{p}}} & \dots & S_{b^{1\bar{p}}_{r1}} & S_{b^{1\bar{p}}_{r2}} & \dots & S_{b^{1\bar{p}}_{r\bar{p}}} \\ S_{b^{11}_{21}} & S_{b^{12}_{21}} & \dots & S_{b^{1\bar{p}}_{21}} & S_{21} & S_{a^1_{22}} & \dots & S_{a^1_{2\bar{p}}} & \dots & S_{b^{21}_{r1}} & S_{b^{21}_{r2}} & \dots & S_{b^{21}_{r\bar{p}}} \\ S_{b^{11}_{22}} & S_{b^{12}_{22}} & \dots & S_{b^{1\bar{p}}_{22}} & S_{a^1_{22}} & S_{22} & \dots & S_{a^2_{2\bar{p}}} & \dots & S_{b^{22}_{r1}} & S_{b^{22}_{r2}} & \dots & S_{b^{22}_{r\bar{p}}} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \dots & \vdots & \vdots & \ddots & \vdots \\ S_{b^{11}_{2\bar{p}}} & S_{b^{12}_{2\bar{p}}} & \dots & S_{b^{1\bar{p}}_{2\bar{p}}} & S_{a^1_{2\bar{p}}} & S_{a^2_{2\bar{p}}} & \dots & S_{2\bar{p}} & \dots & S_{b^{2\bar{p}}_{r1}} & S_{b^{2\bar{p}}_{r2}} & \dots & S_{b^{2\bar{p}}_{r\bar{p}}} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ S_{b^{11}_{r1}} & S_{b^{12}_{r1}} & \dots & S_{b^{1\bar{p}}_{r1}} & S_{b^{21}_{r1}} & S_{b^{22}_{r1}} & \dots & S_{b^{2\bar{p}}_{r1}} & \dots & S_{r1} & S_{a^1_{r2}} & \dots & S_{a^1_{r\bar{p}}} \\ S_{b^{11}_{r2}} & S_{b^{12}_{r2}} & \dots & S_{b^{1\bar{p}}_{r2}} & S_{b^{21}_{r2}} & S_{b^{22}_{r2}} & \dots & S_{b^{2\bar{p}}_{r2}} & \dots & S_{a^1_{r2}} & S_{r2} & \dots & S_{a^2_{r\bar{p}}} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \dots & \vdots & \vdots & \ddots & \vdots \\ S_{b^{11}_{r\bar{p}}} & S_{b^{12}_{r\bar{p}}} & \dots & S_{b^{1\bar{p}}_{r\bar{p}}} & S_{b^{21}_{r\bar{p}}} & S_{b^{22}_{r\bar{p}}} & \dots & S_{b^{2\bar{p}}_{r\bar{p}}} & \dots & S_{a^1_{r\bar{p}}} & S_{a^2_{r\bar{p}}} & \dots & S_{r\bar{p}} \end{bmatrix}, \tag{34}$$

where $S_{ij} > 0 \in \mathbf{S}^{\bar{p}}$, $S_{a^k_{ij}} \geq 0 \in \mathbf{S}^{\bar{p}}\ \forall i,k \in \{1,\dots,r\}, \& j \in \{1,\dots,\bar{p}\}$; $S_{b^{kj}_{ij}} \geq 0 \in \mathbf{S}^{\bar{p}}$, $\forall i \in \{2,\dots,r\}, k \in \{1,\dots,r-1\}, \& j \in \{1,\dots,\bar{p}\}$ so that $\mathbb{S} > 0$.

---

where Σ, $\mathbb{U}$, and $\mathbb{M}$ are described by (31), (36) and (39), respectively. Additionally,

$$\Phi_m = \begin{bmatrix} f_{b_{11}}\mathbb{I} & \dots & f_{b_{1\bar{n}}}\mathbb{I} & \dots & f_{b_{m1}}\mathbb{I} & \dots & f_{b_{m\bar{n}}}\mathbb{I} \end{bmatrix}^\top, \tag{44}$$

$$\Psi_m = \begin{bmatrix} g_{b_{11}}\mathbb{I} & \dots & g_{b_{1\bar{p}}}\mathbb{I} & \dots & g_{b_{r1}}\mathbb{I} & \dots & g_{b_{r\bar{p}}}\mathbb{I} \end{bmatrix}^\top. \tag{45}$$

Then, the error dynamic (19) is $\mathcal{H}_\infty$ asymptotically stable. The gain matrix L is computed by utilising $L = P^{-1}R^\top$.

Proof. The deployment of the inequality (3) on the terms $\mathbf{NL}_1$ and $\mathbf{NL}_2$ yield:

$$\mathbf{NL}_1 \leq \mathbb{U}^\top(\mathbb{Z})^{-1}\mathbb{U} + \Phi^\top\mathbb{H}^\top(\mathbb{Z})\mathbb{H}\Phi, \tag{46}$$

and

$$\mathbf{NL}_2 \leq \mathbb{M}^\top(\mathbb{S})^{-1}\mathbb{M} + \Psi^\top\mathbb{G}^\top(\mathbb{S})\mathbb{G}\Psi, \tag{47}$$

where $\mathbb{Z} > 0$ and $\mathbb{S} > 0$ are defined in (33) and (34), respectively.

Since $f_{ij} \leq f_{b_{ij}}$, we obtain the following inequality by implementing Lemma 3 on the matrices Φ and Φ_m:

$$\Phi^\top \mathbb{H}^\top(\mathbb{Z})\mathbb{H}\Phi \leq \Phi_m^\top \mathbb{H}^\top(\mathbb{Z})\mathbb{H}\Phi_m, \tag{48}$$

where Φ_m is defined in (44).

Similarly,

$$\Psi^\top \mathbb{G}^\top(\mathbb{S})\mathbb{G}\Psi \leq \Psi_m^\top \mathbb{G}^\top(\mathbb{S})\mathbb{G}\Psi_m, \tag{49}$$

where Ψ_m is described in (45).

From (48) and (49), we get

$$\mathbf{NL}_1 \leq \mathbb{U}^\top(\mathbb{Z})^{-1}\mathbb{U} + \Phi_m^\top \mathbb{H}^\top(\mathbb{Z})\mathbb{H}\Phi_m, \tag{50}$$

$$\mathbf{NL}_2 \leq \mathbb{M}^\top(\mathbb{S})^{-1}\mathbb{M} + \Psi_m^\top \mathbb{G}^\top(\mathbb{S})\mathbb{G}\Psi_m. \tag{51}$$

Thus, the inequality (42) is satisfied if

$$\Sigma + \mathbb{U}^\top(\mathbb{Z})^{-1}\mathbb{U} + \Phi_m^\top \mathbb{H}^\top(\mathbb{Z})\mathbb{H}\Phi_m + \mathbb{M}^\top(\mathbb{S})^{-1}\mathbb{M} + \Psi_m^\top \mathbb{G}^\top(\mathbb{S})\mathbb{G}\Psi_m \leq 0. \tag{52}$$

The LMI (43) is deduced by deploying Schur's Lemma on (52). If the LMI (43) is feasible, then the condition specified in (27) is fulfilled. Thus, the estimation error dynamic (19) satisfies $\mathcal{H}_\infty$ criterion (21), ensuring the asymptotic stability of (19). $\square$

In the next theorem, an LPV-based LMI condition is showcased.

Theorem 2. *The system* (19) *is $\mathcal{H}_\infty$ asymptotically stable if there exist matrices $\mathbb{Z}$ and $\mathbb{S}$ under the form* (33) *and* (34)*, respectively, along with $P > 0 \in \mathbf{S}^n$, $R \in \mathbb{R}^{p\times n}$ and a positive scalar μ, such that the following optimization problem is solvable:*

$$\textit{minimize } \mu \textit{ subject to}$$

$$\begin{bmatrix} \Sigma & (\mathbb{U}+\mathbb{Z}\mathbb{H}\Phi)^\top & (\mathbb{M}+\mathbb{S}\mathbb{G}\Psi)^\top \\ \star & -2\mathbb{Z} & \mathbb{O} \\ \star & \star & -2\mathbb{S} \end{bmatrix} < 0,\ \forall \Phi \in \mathcal{F}_{H_m}, \forall \Psi \in \mathcal{G}_{H_m}, \tag{53}$$

where

$$\mathcal{F}_{H_m} = \Big\{\{\mathcal{F}_{11}, \ldots, \mathcal{F}_{1\bar{n}}, \ldots, \mathcal{F}_{m1}, \ldots, \mathcal{F}_{m\bar{n}}\} : \mathcal{F}_{ij} \in [0, f_{b_{ij}}]\Big\}, \tag{54}$$

$$\mathcal{G}_{H_m} = \Big\{\{\mathcal{F}_{11}, \ldots, \mathcal{F}_{1\bar{p}}, \ldots, \mathcal{F}_{r1}, \ldots, \mathcal{F}_{r\bar{p}}\} : \mathcal{F}_{ij} \in [0, g_{b_{ij}}]\Big\}. \tag{55}$$

The remaining terms remain the same as the one outlined in Theorem 1. The gain matrix L is calculated by using $L = P^{-1}R^{\top}$.

Proof. The subsequent inequalities are achieved through the employment of the new variant of Young's inequality (4) on the term $\mathbb{NL}_1$ and $\mathbb{NL}_2$:

$$\mathbf{NL}_1 \leq (\mathbb{U} + \mathbb{ZH}\Phi)^{\top}(2\mathbb{Z})^{-1}(\mathbb{U} + \mathbb{ZH}\Phi), \tag{56}$$

and

$$\mathbf{NL}_2 \leq (\mathbb{M} + \mathbb{SG}\Psi)^{\top}(2\mathbb{S})^{-1}(\mathbb{M} + \mathbb{SG}\Psi), \tag{57}$$

where $\mathbb{Z} > 0$ and $\mathbb{S} > 0$ are defined in (33) and (34), respectively.
From (56) and (57), the inequality (42) is true if

$$\Sigma + (\mathbb{U} + \mathbb{ZH}\Phi)^{\top}(2\mathbb{Z})^{-1}(\mathbb{U} + \mathbb{ZH}\Phi) + (\mathbb{M} + \mathbb{SG}\Psi)^{\top}(2\mathbb{S})^{-1}(\mathbb{M} + \mathbb{SG}\Psi) \leq 0. \tag{58}$$

The inequalities (17) and (18) infer that each element f_{ij} and g_{ij} inside $\mathbb{V}$ and $\mathbb{N}$, respectively, are bounded and belongs to its respective convex sets, whose vertices are described in (54) and (55), respectively. Thus, the condition specified in (58) is fulfilled if

$$\begin{aligned} \Sigma + &\left[(\mathbb{U} + \mathbb{ZH}\Phi)^{\top}(2\mathbb{Z})^{-1}(\mathbb{U} + \mathbb{ZH}\Phi)\right]_{\Phi\in\mathcal{F}_{H_m}} \\ + &\left[(\mathbb{M} + \mathbb{SG}\Psi)^{\top}(2\mathbb{S})^{-1}(\mathbb{M} + \mathbb{SG}\Psi)\right]_{\Psi\in\mathcal{G}_{H_m}} \leq 0. \end{aligned} \tag{59}$$

The Schur's compliment of (59) resulted in the LMI (53). From convexity principle proposed in [18], the error dynamic (19) holds $\mathcal{H}_\infty$ criterion (21) if the LMI (53) is solved for all $\Phi \in \mathcal{F}_{H_m}$ and $\Psi \in \mathcal{G}_{H_m}$. □

5. Comments related to the proposed techniques

In this section, we have outlined some remarks related to the established methodology.

5.1. Case of the nonlinear systems with linear outputs

This segment focuses on the observer design for the nonlinear systems having linear outputs in the presence of disturbances/noise. The system (10) with linear output is reformulated as:

$$\begin{aligned} x_{k+1} &= Ax_k + Gf(x_k) + Bu_k + E\omega_k, \\ y_k &= Cx_k + D\omega_k, \end{aligned} \tag{60}$$

where all variables and parameters remain consistent with those specified in (10). The nonlinear function $f(\cdot)$ is presumed to be globally Lipschitz and holds the detailed form (11a). Analogous to the previous Section 3, the states of the system (60) are estimated by deploying the ensuing observer:

$$\hat{x}_{k+1} = A\hat{x}_k + Gf(\hat{x}_k) + Bu_k + L\big(y_k - C\hat{x}_k\big), \tag{61}$$

where all the parameters and variables are the same as the one illustrated in (12). If one follows the steps (13)- (19) showcased in Section 3, it is easy to obtain the subsequent error dynamics of the observer (61):

$$e_{k+1} = (A - LC)e_k + \sum_{i,j=1}^{m,\bar{n}} f_{ij} G\mathcal{H}_{ij} F_i e_k + (E - LD)\omega_k. \tag{62}$$

The following corollaries present two new LMI conditions which guarantee the $\mathcal{H}_\infty$ stability of the closed-loop system (62).

Corollary 1. *If there exist matrices $P > 0 \in \mathbf{S}^n$, $R \in \mathbf{S}^{p\times n}$, along with $\mathbb{Z}$ in the form of* (33) *and a positive scalar μ, such that, the ensuing optimization problem is solvable:*

$$\begin{gathered} \textit{minimize } \mu \textit{ subject to} \\ \begin{bmatrix} \Sigma & \mathbb{U}^\top & (\mathbb{Z}\mathbb{H}\Phi_m)^\top \\ \star & -\mathbb{Z} & \mathbb{O} \\ \star & \star & -\mathbb{Z} \end{bmatrix} \leq 0, \end{gathered} \tag{63}$$

where all variables and parameters are the same as the one described in the LMI (43)*. Then, the estimation error dynamic* (62) *satisfied $\mathcal{H}_\infty$ criterion* (21)*.*

Corollary 2. *Let us introduce the matrices $P > 0 \in \mathbf{S}^n$, $R \in \mathbf{S}^{p\times n}$, the matrix $\mathbb{Z}$ defined by* (33)*, a positive scalar μ and the following optimization problem:*

$$\begin{gathered} \textit{minimize } \mu \textit{ subject to} \\ \begin{bmatrix} \Sigma & (\mathbb{U} + \mathbb{Z}\mathbb{H}\Phi)^\top \\ \star & -2\mathbb{Z} \end{bmatrix} < 0, \ \forall \Phi \in \mathcal{F}_{H_m}, \end{gathered} \tag{64}$$

where all the terms and variables remain consistent with the one specified in the LMI (53). *If the aforementioned optimization problem is solvable, then the estimation error dynamic* (62) *is* $\mathcal{H}_\infty$ *asymptotically stable.*

For the proof of both corollaries, one can follow the proof of Theorem 1 and Theorem 2.

5.2. Case of absence of exogenous input

At $\omega = 0$, the inequality (24) is reformulated as:

$$\Delta V_k + ||e_k||^2 \leq 0.$$

It yields the exponential stability condition

$$\Delta V_k \leq -\sigma V(e_k),$$

along with $\sigma = \frac{1}{\lambda_{\max}(P)} > 0$. Hence, the proposed LMIs ensure the exponential stability of the error dynamic (19) when $\omega = 0$.

6. Illustrative examples

This section is dedicated to the analysis of the established LMI-based observer methodology. The first part of this segment emphasises the superiority of the proposed LMI approach through a numerical example. Later on, the performance of the observer is demonstrated by applying it to SoC estimation in Li-ion batteries.

6.1. Numerical example 1

Let us consider a nonlinear system represented in the form (60) with the ensuing parameters: $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$, $G = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $B = E = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 \\ -0.1 \end{bmatrix}$. In addition to this, $f(x) = \begin{bmatrix} \sin(\theta_1 x_1) \\ \cos(\theta_2 x_2) \end{bmatrix}$ along with $H_1 = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ and $H_2 = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$. Hence, we get $m = 2$ and $\bar{n} = 3$.

Further, the developed LMI conditions (63) and (64) are solved using MATLAB toolbox to compute the observer parameter L and the optimal noise attenuation index $\sqrt{\mu}$ for different values of θ_1 and θ_2. The obtained minimised value $\sqrt{\mu}$ is outlined in Table 1. The proposed LMI provides a better noise attenuation level than the one obtained from [14, LMI (50)] and [19, LMI (45)] for all considered values of θ_1 and θ_2. It implies that the noise mitigation achieved by the developed LMI-based observer is more efficient than the existing methods. Hence, it emphasizes that the established LMI condition gives better results than the LMIs of [14] and [19]. Therefore, the effectiveness of the derived LMI condition is highlighted through this comparison. In the sequel, the

Table 1: Portraying the optimal values of $\sqrt{\mu}$ in several cases

LMI approaches	$\theta_1 = 0.1$ $\theta_2 = 0.5$	$\theta_1 = 0.2$ $\theta_2 = 0.1$	$\theta_1 = 0.2$ $\theta_2 = 0.4$	$\theta_1 = 0.4$ $\theta_2 = 0.1$	$\theta_1 = 0.5$ $\theta_2 = 0.5$
LMI (63)	3.4167×10^{-6}	1.9600×10^{-6}	1.1503×10^{-6}	1.7356	inf
LMI (64)	1.2145×10^{-6}	1.0941×10^{-6}	3.2263×10^{-8}	1.9462×10^{-6}	2.1295×10^{-6}
[14, LMI (50)]	0.0393	0.0994	0.1438	0.4885	2.4415
[19, LMI (45)]	2.1801	2.1149	2.3706	2.4415	4.4248

validation of observer performance is showcased.

6.2. Application: SoC estimation of Li-ion batteries

In this segment, the authors have demonstrated the effectiveness of the established observer methodology through the deployment of the developed observer for the State-of-Charge (SoC) estimation of the Li-ion battery model. Let us consider the subsequent

2^{nd} order equivalent circuit model proposed in [20]:

$$\begin{aligned}
\dot{V}_1 &= \frac{1}{R_1C_1}V_1 + \frac{1}{C_1}I, \\
\dot{V}_2 &= \frac{1}{R_2C_2}V_2 + \frac{1}{C_2}I, \\
\dot{s} &= -\frac{1}{C_n}s, \\
V_t &= OCV(s) - V_1 - V_2 + R_0 I
\end{aligned} \tag{65}$$

where V_1 and V_2 represent voltages across polarisation resistances R_1 and R_2, respectively, whereas s indicates the SoC of the battery. I infers the current flowing through the load. The terms C_1 and C_2 denote polarisation capacitors, however, C_n depict the capacity of the battery. The terminal voltage (V_t) is considered as the output of the model. The term $OCV(s)$ depicts the open circuit voltage (OCV) of the battery, and it is illustrated as:

$$OCV(s) = 0.9206 \cdot s^3 - 1.3781 \cdot s^2 + 1.3905 \cdot s + 3.2416. \tag{66}$$

The details of the remaining parameters of the model are as follows:

- Battery capacity: $C_n = 5$ A $\cdot$ hour; Battery resistance: $R_0 = 0.0314\ \Omega$; Sampling time: $T_s = 0.02$ hour;
- Polarisation resistances: $R_1 = 0.0181\ \Omega,\ R_2 = 0.0281\ \Omega$;
- Polarisation capacitors: $C_1 = 1712$ F, $C_2 = 55257$ F;

Through the utilization of the Euler forward method, the system (65) is transformed into the discrete-time model represented (10), whose parameters are showcased as: $x(k) = \begin{bmatrix} V_1(k) \\ V_2(k) \\ s(k) \end{bmatrix}$, $A = \begin{bmatrix} 1 - \frac{T_s}{R_1C_1} & 0 & 0 \\ 0 & 1 - \frac{T_s}{R_2C_2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $B_1 = \begin{bmatrix} \frac{T_s}{C_1} \\ \frac{T_s}{C_2} \\ -\frac{T_s}{C_n} \end{bmatrix}$, $B_2 = -R_0,\ F = 1,\ g(x(k)) = OCV(x_3(k))$, $E = B_1$ and $D = 1$. Let us assume that the system dynamics and outputs are corrupted with the Gaussian noise $\omega \rightsquigarrow (0, 0.1)$.

Since $0 \leq x_3(k) \leq 1$, it is easy to infer that the partial derivative of $g(x)$ satisfies (18).

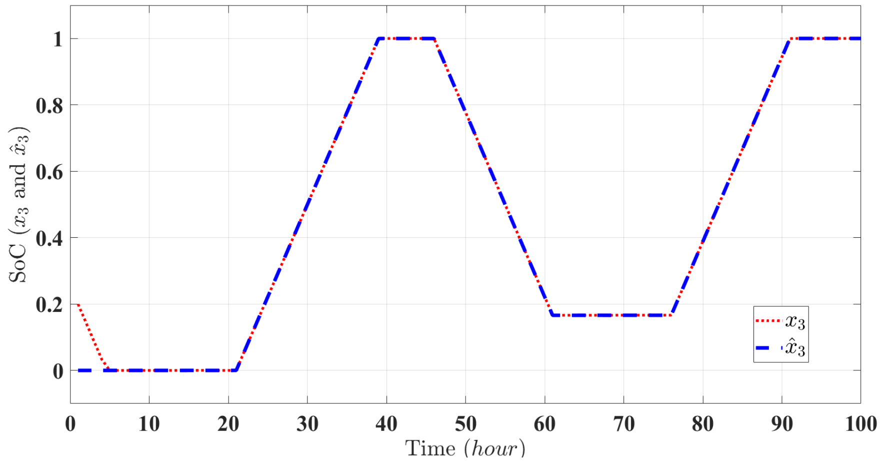

Figure 1: Graph of estimated and actual SoC

The developed LMI (53) is solved by using YALMIP toolbox [21], and we obtain:

$$\sqrt{\mu} = 4.0907 \times 10^{-6} \text{ and } L = \begin{bmatrix} -0.0005 \\ -0.0000 \\ 0.1769 \end{bmatrix}.$$

By using the aforementioned matrix L, the proposed observer (12) is implemented in MATLAB environment for the state estimation purpose. The plot of estimated SoC and actual SoC is shown in Figure 1. It shows that the developed observer performs the efficient estimation of SoC along with optimal noise attenuation. Further, for the same task, the extended Kalman filter (EKF) methodology of [22] is implemented in MATLAB. The RMSE values of the estimation error obtained from EKF and the established observer are summarised in Table 2. It emphasises that the accuracy of the SoC obtained from the proposed approach is relatively better than the one achieved from EKF. Thus, the effectiveness of the new LMI-based observer over the well-known EKF technique is highlighted.

7. Conclusion

In this paper, the problem of nonlinear observer design for discrete-time systems is addressed. It is tackled by formulating two new LMI conditions which provide the observer

Table 2: Comparison of RMSE values of the estimation error

Methodology	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$ (SoC)
Proposed observer (12) with LMI (53)	7.71×10^{-4}	1.97×10^{-7}	0.0014
EKF approach [22]	7.78×10^{-4}	7.92×10^{-4}	0.0028

gain and ensure the $\mathcal{H}_\infty$ stability of the estimation error of the proposed observer. The established matrix-multiplier-based LMIs are derived by incorporating the reformulated Lipschitz property, a new variant of Young inequality. The obtained LMIs encompass some additional decision variables as compared to the existing ones due to the deliberate use of matrix multipliers and a new variant of Young inequality. It resulted in an improvement in LMI feasibility. Thus, the introduction of generalized matrix multipliers inside LMIs plays a vital role in their enhancement. Further, the performance of the observer and the efficacy of the LMI are demonstrated by using numerical examples. From a future perspective, the authors plan to implement the proposed strategy for the stabilization of the same class of systems used in this article.

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