Functoriality in Finitary Vacuum Einstein Gravity and Free Yang-Mills Theories from an Abstract Differential Geometric Perspective*

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Abstract

We continue ongoing research work [59, 60, 61, 76, 77, 78, 79, 80, 81, 82, 83, 84] on applying the homological algebraic conceptual and technical machinery of Abstract Differential Geometry (ADG) towards formulating a finitary, causal and quantal version of vacuum Einstein Lorentzian gravity and free Yang-Mills theories, hitherto cumulatively referred to as *ADG-Gauge Theory* (ADG-GT). In particular, we unfold, express and highlight the inherently *functorial* character of ADG-GT both at the 'kinematical' and at the 'dynamical' level of the *aufbau* of the theory, although at the same time we observe that the traditional kinematics-*versus*-dynamics distinction becomes blurry in our ADG-theoretic approach as, in line with

*This paper is wholeheartedly dedicated to, and in loving memory of, my dearest teacher, mentor, friend, and companion in **The Quest**, *Professor Anastasios (Tasos) Mallios*. This is an invited paper contribution to a Special Contributory Volume/Issue titled *Physical Geometry: Unravelling the Weave of Quantum Geometry* in memory of Professor Anastasios Mallios, edited by Dr Elias Zafiris. The paper to be submitted will also be posted at the **General Relativity and Quantum Cosmology** website *www.arXiv.org/gr-qc* before Easter 2024. In turn, a longer version of the paper will constitute a chapter in a research monograph type of book that we have been working on, in collaboration with the late Professor Anastasios Mallios, since 2003 [62].

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[61, 62, 79, 81, 82, 67], we maintain that *there is no pre-existent geometrical/kinematical space in the quantum deep, but rather, that physical geometrical space derives from (or is an outcome of) field dynamics*. We moreover argue that the *gauge theory of the third kind* and the *third quantisation* schemes that ADG-GT has been seen to support [61, 81, 82], are also *functorial* in character. Furthermore, since our inherently algebraic ADG-theoretic scheme has been seen to be manifestly *background geometrical $\mathcal{C}^{\infty}$-smooth spacetime manifold independent* [59, 60, 61, 76, 77, 78, 79, 80, 81, 82], we entertain the idea, at the 'dynamical' level of functoriality, that there is both a *geometric morphism* and a *natural transformation* type of correspondences between the relevant Einstein and Yang-Mills field functor categories with the dynamical gauge connection and curvature sheaf morphisms implementing the homological algebraic dynamics within each, so as to further corroborate previous claims [78, 79, 80, 81, 82] that *from an ADG-theoretic perspective, ADG-gravity is an already finitistic, third quantised, $\mathcal{C}^{\infty}$-smooth geometrical background spacetime manifoldless, auto-dynamical and 'pure gauge' field theory of the third kind*. We also cast our formal canonical sheaf cohomological Third Quantisation heuristics originally formulated in [81] in a slightly different light so as to arrive at a new ADG-theoretic notion of *'Unitary' Quantal ADG-Gauge Field* which, in a tetrad of functorially and dynamically entwined structures $\mathbf{U} := (\mathcal{E}, \mathcal{D}, \mathcal{A}ut_{\mathbf{A}}\mathcal{E}, \mathcal{Q})$, it subsumes under a single coherent and inseparable 'unitary whole' all the four most important functorial structural traits of ADG-GT: *'local quantum particle states' represented by local sections of a vector sheaf $\mathcal{E}$, their 'dual-complementary' functorial ADG-gauge field dynamics generated by an algebraic $\mathbf{A}$-connection $\mathcal{D}$, the latter's local gauge invariance of the 3rd kind encoded in the principal structure sheaf $\mathcal{A}ut_{\mathbf{A}}\mathcal{E}$ of $\mathcal{E}$'s automorphisms, and the dual particle-field canonical-type of 3rd quantisation, represented by the functorial morphism $\mathcal{Q}$ between the sheaf categories involved*. At the end, we give a subjective (:from this author's viewpoint) account of certain key ideas, concepts and seminal mathematical results in the past that significantly motivated Professor Mallios to develop ADG, and how these ideas subsequently inspired this author to apply it to a finitistic and quantal theoretical scenario for Vacuum Einstein Gravity and Free Yang-Mills theories. Throughout the second half of the paper, we recall and analyse several pertinent quotes that Professor Mallios and this author used to repeatedly discuss and scrutinise in the course of the late nineties and early noughties, dur-

ing endless late night discussions over good food and wine at our favourite tavern, fittingly called *Algebra*, in Paleo Psychiko, Athens, Greece. Addendum 1 at the end recalls an important and telling early interaction that this author enjoyed with Professor Mallios at the end of last century. Addendum 2 at the end discusses *the importance of using poetic language*, plus imaginative and heuristic novel terminology, both of which emanate from the novel mathematical concepts, structures and techniques of ADG, in order to address, interpret and formulate new theoretical concepts and calculational techniques in the wildly speculative, glaringly non-intuitive and largely uncharted landscape of Quantum Gravity. An Appendix, defining, describing and explaining all the new ADG-theoretical concepts, concludes the paper.

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1 Technical Prolegomena: A Brief History of Finitary ADG-Gravity and Yang-Mills Theories with an Emphasis on the Functorial Character of our Concepts, Methods and Constructions

In this section we give a brief account of the main milestones reached along our way towards arriving at a purely algebraic, finitistic, causal and quantal theory of spacetime, gauge theories and gravity. As we outline the main results, we highlight and emphasise the homological algebraic (:category-theoretic), and especially *functorial*, nature of our basic concepts, structures and methods of their use in various constructions and associated (abstract differential geometric) calculations (:Differential Calculus), while *all this is accomplished purely algebraically, manifestly without any recourse to or dependence on a background $\mathcal{C}^\infty$-smooth geometrical base spacetime manifold.*

1.1 Finitary Substitutes of Continuous Spacetime Manifolds, their Incidence Algebras, and the Finitary Sheaves Thereof: 'Kinematical' Functoriality

1.1.1 Sorkin's Finitary Posets

Our journey begins with Sorkin's 'prophetic' *finitary substitutes of continuous spacetime manifolds* [89][1]. In that paper, with every *locally finite (:finitary) open cover* $\mathcal{U}_i = (U_i)$ of a (real) topological ($\mathcal{C}^0$) spacetime manifold M, Sorkin assigns a so-called *finitary partially ordered set* (finposet) $\mathcal{P}_i$:

$$\mathcal{U}_i \longrightarrow \mathcal{P}_i \tag{1}$$

The collection $\overleftarrow{\mathcal{P}} = (\mathcal{P}_i)_{i\in I}$ of such finposets is seen to constitute a so-called *inverse* or *projective* system, or *net*, of posets, which is seen to have a *projective limit space* effectively homeomorphic to the continuous $\mathcal{C}^0$-manifold M.[2]

[1] Refer to this paper for various mathematical concepts, structures and technical definitions thereof.

[2] The index $i \in I$ in $\mathcal{P}_i$ is the so-called *refinement net index*, whereby $\mathcal{U}_i \prec \mathcal{U}_j$ (reads:

Three bullet points must be emphasised here in connection with (1) from [89]:

- First, Sorkin's original intuition that every geometrical point of a point-manifold is an ideal, operationally unrealistic and physically untenable plus problematic (:singular), dimensionless and structureless object, that better be 'smeared' and blown-up by 'enlarged' open sets (neighbourhoods) about it.[3] The open sets and their set-theoretic algebra are the carriers of the manifold's topology and its continuity, not its ideal points.[4]

- Sorkin's original idea strongly resonates with Grothendieck's pioneering idea to categorically abstract and generalise pointed topological spaces to *pointless* ones called *sites* by abstracting from the usual topological open covers like in $\overleftarrow{\mathcal{P}} = (\mathcal{U}_i) = (\mathcal{P}_i)_{i\in I}$ to *families (=sieves) of covering arrows in a category* defining a *Grothendieck topology* on the category [41, 42, 43].[5]

- Once we have done away with the pointed geometrical manifold continuum, we can work further with their poset substitutes $(\mathcal{P}_i)$ and build on them. That is what we do next.

First encounter with 'Kinematical' Functoriality. Before we go on to work with the finitary posets, we catch a first glimpse of *functoriality* of our constructions. Let $\mathcal{T}_i = \mathrm{span}\{U : \ U \in \mathcal{U}_i\}$ be *the topology 'spanned' or generated by arbitrary unions and finite intersections of the open sets in each locally finite open covering $\mathcal{U}_i$*. Then, a continuous map

$$f : \ \mathcal{T}_i \longrightarrow \mathcal{T}_j \tag{2}$$

the open cover $\mathcal{U}_j$ is finer than the open cover $\mathcal{U}_i$, and conversely, $\mathcal{U}_i$ is coarser than $\mathcal{U}_j$) if $\mathcal{U}_j$ has 'smaller' and more numerous open sets than $\mathcal{U}_i$. The aforementioned inverse limit now reads: *the $\mathcal{C}^0$-manifold is recovered at the inverse limit of infinite refinement of the open covers $\mathcal{U}_i$s in the net* (as $i \to \infty$).

[3]Let it be noted here that a pointed background geometrical spacetime continuum is a problematic, pathological structure that is arguably responsible both for the singularities plaguing General Relativity (GR) and for the pestilential non-renormalisable unphysical infinities marring Quantum (Gauge) Field Theories (QFT) of matter.

[4]In the ensuing discussion, by introducing *sheaf theory*, we will refine this statement even more.

[5]A *site* is by definition a category endowed with a Grothendieck topology.

induces (:maps functorially to) a *poset morphism* $\hat{f}: \mathcal{P}_i \to \mathcal{P}_j$,[6] such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{T}_i & \xrightarrow{f} & \mathcal{T}_j \\ \downarrow & & \downarrow \\ \mathcal{P}_i & \xrightarrow[\hat{f}]{} & \mathcal{P}j \end{array} \tag{3}$$

1.1.2 Differential Incidence (Rota) Algebras of Finitary Posets and their Simplicial Complexes

From [110, 111, 83, 84] we read that with every finitary poset $\mathcal{P}i$ one can straightforwardly associate a *finitary simplicial complex* $\mathcal{S}_i$, the so-called *Čech-Alexandrov nerve of the underlying finitary open covering*, á-la Čech Homology:

$$\mathcal{S}_i: \mathcal{P}_i \longrightarrow \mathcal{S}_i \tag{4}$$

Second encounter with 'Kinematical' Functoriality. The mapping $\mathcal{S}$ above is also *functorial* in the sense that a finitary poset morphism p_{ij} : $\mathcal{P}_i \to \mathcal{P}_j$ as before, induces (:maps functorially to) a *simplicial mapping*: $\hat{s}_{ij}: \mathcal{S}_i \to \mathcal{S}_j$, so that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{P}_i & \xrightarrow{p_{ij}} & \mathcal{P}_j \\ {\scriptstyle \mathcal{S}_i}\downarrow & & \downarrow{\scriptstyle \mathcal{S}_j} \\ \mathcal{S}_i & \xrightarrow[\hat{s}_{ij}]{} & \mathcal{S}_j \end{array} \tag{5}$$

Third encounter with 'Kinematical' Functoriality: Incidence (Rota) Algebras. More importantly for our considerations here, we read from [110, 111, 83, 84] that with every finitary poset $\mathcal{P}_i$ (or equivalently, with every finitary Čech simplicial complex $\mathcal{S}_i$) one can naturally associate a so-called *incidence (Rota) algebra* Ω_i over the complex numbers $\mathbb{C}$, as follows:

$$\mathcal{R}_i: \mathcal{P}_i \to \Omega_i \tag{6}$$

[6]By definition, *a poset morphism is a partial order preserving map.*

Like in the case of the simplicial mapping of finitary posets $\mathcal{S}$, the mapping $\mathcal{R}$ induces an *incidence algebra homomorphism* $\hat{r}$ in the following functorial commutative diagram sense:[7]

$$\begin{tikzcd} \mathcal{P}_i \arrow[r, "p_{ij}"] \arrow[d, "\mathcal{R}_i"'] & \mathcal{P}_j \arrow[d, "\mathcal{R}_j"] \\ \Omega_i \arrow[r, "\hat{r}_{ij}"'] & \Omega_j \end{tikzcd} \tag{7}$$

1.1.3 Fourth encounter with 'Kinematical' Functoriality: Gel'fand Duality

Without going into detailed technicalities here, we read from [110, 111, 83, 84] that one can go the other way around and extract from the finitary incidence algebras a topological space, endowed with a so-called *Rota topology*, by considering *irreducible representations of the incidence algebras*, the *kernels* of which correspond to *primitive ideals* in the algebras. In turn, the set of primitive ideals, the so-called *spectrum of the algebra* Spec(Ω), now regarded as a *generalised, 'blown up point set'*, is readily endowed with a Rota topology in such a way that *the incidence algebra homomorphisms in $\hat{r}_{ij}$ lift to continuous maps in the respective Rota topologies.*

This is another instance of *the functoriality of our constructions* and it corresponds to a finitary version of *Gel'fand Duality* according to which, very broadly speaking, from an algebraic structure A one can extract a 'geometrical space' Spec(A) carrying a 'natural continuity' (:a functorially imposed topology on it).[8]

[7] *Mutatis mutandis* for the finitary Čech simplicial complexes and the incidence Rota algebras thereof: their correspondence is manifestly functorial [110, 111, 83, 84].

[8] Furthermore, and again very broadly speaking, there is the *Gel'fand Representation theorem* that ensures that the $\mathbb{C}$-algebra of continuous complex valued functions on Spec(A) is naturally equivalent to the (complex) algebra A that one started with. This remark will prove crucial in the sequel when we recount the introduction of *finitary spacetime sheaves*.

1.1.4 Finitary Incidence (Rota) Algebras as Finitary Differential Algebras/Modules

We read again directly from [83, 84, 60] that *the finitary incidence algebras* Ω_i are $\mathbb{Z}_+$*-graded discrete differential algebras/modules of finite rank*,[9] as follows:

$$\Omega = \bigoplus_{n\in\mathbb{Z}_+} \Omega^n = \Omega^0 \oplus \Omega^1 \oplus \ldots = \mathbf{A} \oplus \mathcal{R} \tag{8}$$

The Ω^ns above are seen to be the reticular analogues of the usual linear spaces of n-grade (Grassmann exterior) differential forms [60] on a $\mathcal{C}^\infty$-smooth manifold. The grade 0 commutative linear subalgebra $\mathbf{A} = \Omega^0$ is the discrete analogue of the algebra of (smooth) functions (0-forms) on the continuum, while $\mathcal{R} = \bigoplus_{n\geq 1} \Omega^n$ serves as the $\mathbf{A}$-module of discrete differential forms on it.

Furthermore, we witness in [60, 61] that there is a discrete analogue of the (flat) Cartan-Kähler (exterior) differential d operator:

$$d : \ \Omega^n \longrightarrow \Omega^{n+1} \tag{9}$$

that is a *linear map* and it obeys the *Leibniz rule*.

1.1.5 The Differential Caveat: Finitary Spacetime Sheaves of Incidence Algebras and Preliminary Vibes of ADG

As soon as this author realised that *the incidence algebras encode not only topological, but also differential geometric, information in their structure*, the next tenable position would be to somehow make them *(dynamically) variable*, thus he envisaged to employ in the longer run the full ADG-theoretic panoply towards formulating a finitary and quantal version of Gravity and Gauge Theory.

A first step to that end would be to '*sheafify*' them; that is, to consider *sheaves* thereof.[10] Thus *finitary spacetime sheaves* [77] $\mathcal{E}_i$ of incidence algebras[11] over Sorkin's finitary poset discretisations were born.

[9] See also [11, 12, 13] for an early study of such differential spaces.

[10] Along very similar lines of thought, the reader should refer to the Introduction of [67] to read how the notion of a *sheaf* comes hand in hand with the notion of *variable structure*.

[11] Now the locally finite posets being interpreted as *causal and quantal* versions of *Sorkin's causal sets* [5, 90, 91, 88], as expounded in [76].

Fifth encounter with 'Kinematical' Functoriality: Sheafification. That is, this author realised in [77] that the mapping $\mathcal{R}_i : \ \mathcal{P}_i \rightarrow \Omega_i$ in 6 above is actually a *contravariant functor*[12] which, when subjected to suitable compatibility (glueing) conditions, it can be promoted to a *local homeomorphism* (between the corresponding covering topologies $\mathcal{T}_i$ generated by the finitary open covers $\mathcal{U}_i$), the very definition of a sheaf [6].

All in all, when the resulting finitary spacetime sheaves $\mathcal{E}_i$ have Ω_is in their stalks, they were recognised as being the reticular analogues of Mallios's *vector sheaves*.[13] Hence, the whole enterprise of applying Mallios's Abstract Differential Geometry (ADG) to to a finitistic, causal and quantal version of Lorentzian vacuum Einstein Gravity and free Yang-Mills (gauge) theories of matter commenced.

In this line of thought, in [59, 60, 61] we defined *finitary differential triads*, as the following triplets:

$$\mathcal{T}_i := (\mathbf{A}_i, d, \Omega_i) \tag{10}$$

which are the 'discrete' analogues of the ADG-theoretic differential triads $(\mathbf{A}, \partial, \mathcal{E} \simeq \Omega)$.[14]

1.1.6 Sixth encounter with 'Kinematical' Functoriality: The Category of Differential Triads

In [72, 73], Papatriantafillou observed that the ADG-theoretic differential triads form a very homologically rich category: *the category of differential triads*, whose objects are differential triads and whose arrows are differential structure preserving *sheaf morphisms*.

What is very interesting for us here, as observed in [59, 60, 61, 79, 82], is that *there is a contravariant functor between the category of finitary posets and the category of finitary differential triads of incidence algebras*. As a

[12] Effectively, the definition of a *presheaf* [6].

[13] By definition, Mallios's vector sheaves are locally free (differential) $\mathbf{A}$-modules of finite rank n [44, 45, 50]. That is, locally for every open set $U \in \mathcal{U}i$ in an open covering (:set of local gauges $\mathcal{U}_i$) of the base topological space X of a $\mathbb{C}$-*algebraized space* $(X, \mathbf{A})$, one has by definition the following $\mathbf{A}|_U$-isomorphisms: $\mathcal{E}|_U = \mathbf{A}^n|_U = (\mathbf{A}|_U)^n$ and, concomitantly, the following equalities section-wise: $\mathcal{E}(U) = \mathbf{A}^n(U) = \mathbf{A}(U)^n$ (with $\mathbf{A}^n$ the n-fold Whitney sum of $\mathbf{A}$ with itself).

[14] Refer to [44, 45, 50, 59, 60, 61] for more detailed definitions, further interpretational discussion and relevant results.

result, as the inverse system of Sorkin's finitary posets was seen to possess an inverse (or projective) limit space, the corresponding, categorically dual, inductive system of finitary incidence algebras was seen to form a direct (or inductive) limit space, consistent and in agreement with Papatriantafillou's results in [73].[15]

1.2 Enter ADG: 'Dynamical' Functoriality

(**Note:** Henceforth in this paper, all our finitary considerations, constructions and results presented thus far carry on *mutatis mutandis* to the general ADG-theoretic constructions. Thus, the finitary case is obtained from the general ADG-theory [44, 45, 50] simply by adjoining a finitarity index-subscript 'i' to all the ADG-symbols and constructions.[16])

With the identification of the aforementioned finitary sheaves of quantum causal sets as reticular versions of *vector sheaves* in ADG, we swiftly moved on to mathematically model *dynamical variations* thereof.

1.2.1 Briefly revisiting A-connections in ADG

To that end, we readily appreciated that the Cartan-Kähler differential operator in (9) is a special example of an ADG-theoretic *connection* on the sheaves of 'differential forms' that it functorially acts as a *sheaf morphism*, albeit a *flat connection* [44, 45, 50, 59, 60, 61].

In order to dynamically vary the quantum causal sets that dwell as (germs of) local sections in the stalks of the aforementioned sheaves $\mathcal{E}_i$, we need to 'gauge'[17] the flat d to a more general (:curved) connection $\mathcal{D}$,

$$d \longrightarrow \mathcal{D} \qquad (11)$$

[15] Subsequently, this observation was crucial in our idea of promoting our category of finitary differential triads into a *topos-like structure* [43, 79, 82], having (at least finite) categorical limits (direct/inductive) and colimits (inverse/projective).

[16] For instance, a general open cover $\mathcal{U}$ of the background topological space X in ADG (:there coined *open coordinate gauge of* X), becomes $\mathcal{U}_i$ in our finitary domain of the theory. Similarly, as we shall see next, the finitary version of the Cartan-Kähler differential operator d or the general ADG $\mathbf{A}$-connection operator $\mathcal{D}$, become d_i and $\mathcal{D}_i$ respectively in our finitary realm *without any loss of generality whatsoever* [59, 60, 61, 78, 79, 80, 81, 82].

[17] That is, we need to localise and relativise differential changes relative to arbitrary sets of (covering) open gauges ($\mathcal{U}_i$) [44, 50, 59, 60, 61].

which is also defined functorially as an $\mathbf{A}$*-linear, Leibnizian sheaf morphism*, acting on the relevant module sheaves as follows:

$$\mathcal{D}: \ \mathcal{E} \longrightarrow \mathcal{E} \otimes_{\mathbf{A}} \mathbf{\Omega} \cong \mathbf{\Omega} \otimes_{\mathbf{A}} \mathcal{E} \equiv \mathbf{\Omega}(\mathcal{E}) \tag{12}$$

With the introduction of $\mathcal{D}$ upon localising or 'gauging' the flat differential d relative to a set of local open gauges $\mathcal{U}_i$, the latter acquires locally an additional term—the so-called *gauge vector field potentials'* term $\mathcal{A}$,[18] as follows:

$$d \longrightarrow \mathcal{D}|_{U \in \mathcal{U}_i} = d + \mathcal{A}|_U \tag{13}$$

As alluded to in the last footnote, from an ADG-theoretic point of view, the connection $\mathcal{D}$ is viewed as a 'unitary', autonomous dynamical entity [61, 81, 82], regardless of its local gauge split as in (13). Which brings us to arguably the most important ADG-theoretic definition.

1.2.2 First encounter with 'Dynamical' Functoriality: ADG-theoretic Fields

In ADG, a *dynamical field* is defined as the following pair:

$$\mathcal{F} := (\mathcal{E}, \mathcal{D}) \tag{14}$$

That is to say:

> *A dynamical field is a pair consisting of a vector sheaf* $\mathcal{E}$*, localised on an in principle arbitrary* $\mathbb{C}$*-algebraized space* $(X, \mathbf{A})$*, and a connection* $\mathcal{D}$ *acting functorially on its (local) sections as a sheaf morphism.*

Four things to highlight here in connection with the fundamental definition above:

[18] Traditionally, in the Classical Differential Geometry (CDG) $\mathcal{C}^\infty$-smooth manifolds M used by Physics [27], the term *connection* is normally reserved for the so-called *gauge vector field potentials* $\mathcal{A}^i_\mu$ (with μ an external spacetime index, and i an internal gauge symmetry index). On the other hand, from an ADG-theoretic perspective, the denomination *connection* is a 'holistic', 'unitary' one, pertaining to $\mathcal{D}$ as a whole, and not referring to its 'contingent' local split by a choice of gauge $\mathcal{U}_i$ as in (13) above. Read on.

- An ADG-field consists of both the source and the agent of dynamical variability—the connection $\mathcal{D}$, and the recipient of the agent's dynamical action—the vector sheaf $\mathcal{E}$, as *an autonomous and indivisible/inseparable unit*.

- From a geometric (pre)quantization perspective, the local sections of $\mathcal{E}$ correspond to quantum particle states [46, 51, 50, 57, 60, 61, 81, 82]. If $\mathcal{E}$ is a *line sheaf* $\mathcal{L}$ (:a vector sheaf of rank $n = 1$), its local sections represent the quantum particle states of a boson like the 'photon', hence the ADG-field $\mathcal{F}_{Max} = (\mathcal{L}, \mathcal{D})$ is coined the *Maxwell field*. More general *Yang-Mills* ADG-fields are represented as connections $\mathcal{D}$ on vector sheaves $\mathcal{E}$ of rank $n > 1$. ADG-theoretically, we represent them by the pair: $\mathcal{F}_{YM} = (\mathcal{E}, \mathcal{D})$. Finally, the gravitational connections constitute ADG-theoretic *Einstein fields*: $\mathcal{F}_{Einst} = (\mathcal{E}, \mathcal{D})$.

- It is important to stress here that, in a very technical and rigorous sense, *the vector sheaves $\mathcal{E}$ correspond to the associated or representation sheaves of the principal group sheaf $\mathcal{A}ut(\mathcal{E})$ of the reversible endomorphisms (:the automorphisms) of $\mathcal{E}$* [44, 99, 100, 101, 102, 50].[19] In turn, *$\mathcal{A}ut(\mathcal{E})$ is the local relativity and gauge invariance structure group sheaf of the functorial dynamics effectuated by the connection sheaf morphism $\mathcal{D}$ acting dynamically on (the local sections of) $\mathcal{E}$.*

- The Maxwell $\mathcal{F}_{Max} = (\mathcal{L}, \mathcal{D})$, Yang-Mills $\mathcal{F}_{YM} = (\mathcal{E}, \mathcal{D})$ and Einstein $\mathcal{F}_{Einst} = (\mathcal{E}, \mathcal{D})$ ADG-fields can be organised, as objects, into respective *categories* with categorical sheaf morphisms as arrows between them, coined: the *Maxwell Category* $\mathcal{T}_{Max}$, the *Yang-Mills Category* $\mathcal{T}_{YM}$ and the *Einstein Category* $\mathcal{T}_{Einst}$ categories [50, 55, 57]. In contradistinction to the 'flat', ungauged and 'static-kinematical' categories of differential triads that we alluded to earlier [72, 73, 50], these three categories are 'dynamical' in character, in the sense that object-fields in them obey and satisfy certain dynamical laws of motion[20] and the **A**-

[19] We recall from [44, 61, 50] that for any vector sheaf $\mathcal{E}$, $\mathcal{E}nd\mathcal{E} \equiv \mathcal{H}om_{\mathbf{A}}(\mathcal{E}, \mathcal{E}) \cong \mathcal{E} \otimes_{\mathbf{A}} \mathcal{E}^* = \mathcal{E}^* \otimes_{\mathbf{A}} \mathcal{E}$, so that $\mathcal{A}ut(\mathcal{E}) \simeq \mathcal{E}nd(\mathcal{E})^*$. It follows that, for a choice of local open gauges $U \in \mathcal{U}_i$, $\mathcal{A}ut(\mathcal{E})_U \equiv \mathcal{A}ut(\mathcal{E})(U) \simeq \mathcal{E}nd(\mathcal{E})^*(U) = M_n(\mathbf{A}_U) \equiv M_n(\mathbf{A}(U))$, the *non-commutative gauge structure group sheaf* of $(n \times n)$-matrices having for entries local sections in the structure algebra sheaf $\mathbf{A}$: $\Gamma(U, \mathbf{A}) = \mathbf{A}_U \equiv \mathbf{A}(U)$.

[20] The Maxwell, Yang-Mills and Einstein dynamical (differential) equations, which in turn derive from Lagrangian variation of corresponding action functionals. Read on.

functoriality thereof corresponds to the **A**-invariance and the $\mathcal{A}ut_{\mathbf{A}}\mathcal{E}$ local gauge/generalised coordinates' invariance of the respective dynamical laws of motion [55, 57].[21]

We close this subsection with an important observation regarding the three *ADG*-field categories defined above:

> All three aforementioned ADG-field categories, the *Maxwell* $\mathcal{T}_{Max}$, the *Yang-Mills* $\mathcal{T}_{YM}$, and the *Einstein* category $\mathcal{T}_{Einst}$, are by definition *functor categories* [43].[22]

1.2.3 Second encounter with 'Dynamical' Functoriality: A-invariance is a generalised, functorial form of gauge invariance of the ADG-field dynamics

The discussion above brings us to the all-important issue of gauge covariance, local gauge invariance and their intimate relation to the basic ADG-theoretic notion of $\mathcal{A}$-invariance.[23] In this subsection, we will focus only on [**?**] Einstein and Yang-Mills ADG-fields on higher rank vector sheaves, leaving the *abelian* case of Maxwell fields on line sheaves to their exhaustive treatment in the monograph references [44, 50].[24]

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[21] We shall return to discuss further the categorical implications and the deeper physical interpretation of the generalised ADG-theoretic conception of local gauge invariance as **A**-invariance and the $\otimes_{\mathbf{A}}$-functoriality of the ADG-theoretic gauge dynamics in the sequel.

[22] That is to say, *the objects in those categories are sheaf morphisms, while the arrows between them are themselves functors*. It follows, that if there are functorial correspondences between them and other functor categories, these correspondences will be some kind of *natural transformations*, and especially, some kind of *geometric morphisms* [43]. As we will see in the sequel, of special interest to us will be a geometric morphism asssociated with the $\otimes_{\mathbf{A}}$-Hom *adjunction*, which effectuates a kind of natural transformation between ADG-field and ADG-curvature space categories. Read on.

[23] As it has also been observed in past publications [59, 60, 61], in our work we use the symbol **A** for the *structure sheaf of algebras of generalised ADG-theoretic coordinate functions*, as opposed to $\mathcal{A}$ used throughout Mallios's work [44, 45, 50, 55, 57], as we have reserved the symbol $\mathcal{A}$ for local *Einstein gravitational and Yang-Mills local gauge potentials* as in (13).

[24] The epithets *abelian* and *non-abelian* above pertain, as in the usual theory [27], to the structure gauge groups being *commutative* and *non-commutative*, respectively. Indeed, the principal group sheaves associated with the line sheaves $\mathcal{L}$ of the ADG-theoretic Maxwell fields $\mathcal{F}_{Max}$ above carry abelian unitary ($:\equiv U_1(\mathbf{A})$) groups in their stalks, while as we saw couple of footnotes earlier, the structure group sheaf $\mathcal{A}ut_{\mathbf{A}}\mathcal{E}$ of dynamical automorphisms

The A-Functoriality of the Curvature of an A-Connection: R is an $\otimes_{\mathbf{A}}$-tensor. To that end, we first recall from [44, 45, 50, 59, 60, 61] the general *functorial* ADG-theoretic definition of the curvature R of an $\mathbf{A}$-connection $\mathcal{D}$ as the following $\mathbf{A}$*)-morphism of* $\mathbf{A}$*-modules*:[25]
We first define the *1st prolongation of* $\mathcal{D}$ to be the following $\mathbf{C}$-linear vector sheaf morphism:

$$\mathcal{D}^1 : \ \mathbf{\Omega}^1(\mathcal{E}) \longrightarrow \mathbf{\Omega}^2(\mathcal{E}) \tag{15}$$

satisfying section-wise relative to $\mathcal{D}$:

$$\mathcal{D}^1(s \otimes t) := s \otimes \mathrm{d}t - t \wedge \mathcal{D}s, \ (s \in \mathcal{E}(U), t \in \mathbf{\Omega}^1(U), U \text{ open in } X) \tag{16}$$

We are now in a position to define the curvature R of an $\mathbf{A}$-connection $\mathcal{D}$ by the following triangular commutative diagram:

$$\begin{tikzcd} \mathcal{E} \arrow[r, "\mathcal{D}"] \arrow[d, "R(\mathcal{D})=\mathcal{D}^1\circ\mathcal{D}"'] & \mathbf{\Omega}^1(\mathcal{E}) \equiv \mathcal{E} \otimes_{\mathbf{A}} \mathbf{\Omega}^1 \arrow[dl, "\mathcal{D}^1"] \\ \mathbf{\Omega}^2(\mathcal{E}) \equiv \mathcal{E} \otimes_{\mathbf{A}} \mathbf{\Omega}^2 & \end{tikzcd} \tag{17}$$

from which we read directly that:

$$R \equiv R(\mathcal{D}) := \mathcal{D}^1 \circ \mathcal{D} \tag{18}$$

Therefore, any time we have the $\mathbf{C}$-linear morphism $\mathcal{D}$ and its prolongation $\mathcal{D}^1$ at our disposal, we can define the curvature $R(\mathcal{D})$ of the connection $\mathcal{D}$.[26]

As a matter of fact, it is rather straightforward to see that, for $\mathcal{E}$ a vector sheaf, $R(\mathcal{D})$ *is functorially defined as an* $\mathbf{A}$*-morphism of* $\mathbf{A}$*-modules*, in the following sense:

---

of the vector sheaf $\mathcal{E}$ of, say, the Einstein field $\mathcal{F}_{Einst}$ is locally homomorphic to $M_n(\mathbf{A}(U))$, *which is manifestly noncommutative*.

[25] With a vector sheaf $\mathcal{E}$, as explained before, regarded as a*sheaf of differential* $\mathbf{A}$*-modules, with structure sheaf* $\mathbf{A}$*, that is locally isomorphic to* $\mathbf{A}^n(U)$.

[26] In connection with (18), one can justify our earlier remark that the standard Cartan-Kähler (exterior) differential operator $d \equiv d^0$ is a *flat* type of connection, since: $R(d) = d \circ d \equiv d^2 = 0$, which is secured by the well known *nilpotency* of the usual Cartan-Kähler (exterior) differential operator d [27, 44, 50, 61]). In a (co)homological-algebraic sense, *the curvature of an algebraic connection measures the 'obstruction' to or the 'deviation' from the nilpotency of the connection (:differential)* [59, 60, 61].

$$\begin{array}{c} R \in \mathrm{Hom}_{\mathbf{A}}(\mathcal{E}, \mathbf{\Omega}^2(\mathcal{E})) = \mathcal{H}om_{\mathbf{A}}(\mathcal{E}, \mathbf{\Omega}^2(\mathcal{E}))(X) \\ \mathbf{\Omega}^2(\mathcal{E}nd\mathcal{E})(X) = Z^0(\mathcal{U}, \mathbf{\Omega}^2(\mathcal{E}nd\mathcal{E})) \end{array} \tag{19}$$

where, as usual, $\mathcal{U}_i = \{U_\alpha\}_{\alpha \in I}$ is an open cover of the base topological space X and $Z^0(\mathcal{U}, \mathbf{\Omega}^2(\mathcal{E}nd\mathcal{E}))$ the $\mathbf{A}(U)$-module of 0-*cocycles* of $\mathbf{\Omega}^2(\mathcal{E}nd\mathcal{E})$ relative to the $\mathcal{U}_i$-covering of X.[27]

1.2.4 The Non-Tensorial and the Tensorial Character of $\mathcal{D}$ and R, respectively, under Local 'Gauge-Coordinate' Transformations

(**Note:** In the following presentation and discussion, we are not going to specify what ADG-connections and their curvatures we are talking about. The reader can assume that the connections are either Einstein-Lorentzian or Yang-Mills, in the sense that the arguments below apply *mutatis mutandis* to both.)

In this subsection, we recall from [44, 61, 50] a very subtle and important for our arguments in the sequel ADG-theoretic result, which may be distilled down to the following two statements:

- *The ADG-theoretic connection $\mathcal{D}$ is only a $\mathbf{C}$-linear sheaf morphism (hence not an $\otimes_{\mathbf{A}}$-tensor);*[28] while;
- *The ADG-theoretic curvature R is a full $\mathbf{A}$-structure sheaf morphism (hence a pure $\otimes_{\mathbf{A}}$-tensor).*[29]

Two equivalent statements to the ones above, which the theoretical physicist/mathematician who is familiar with the usual differential geometry of gauge theory, which employs smooth fiber bundles over a $\mathcal{C}^\infty$-smooth differential spacetime manifold M can straightforwardly understand [27], are the following:

---

[27]One may wish to recall again that, for a vector sheaf $\mathcal{E}$ like the one involved in (19) above: $\mathcal{E}nd\mathcal{E} \equiv \mathcal{H}om_{\mathbf{A}}(\mathcal{E}, \mathcal{E}) \cong \mathcal{E} \otimes_{\mathbf{A}} \mathcal{E}^* = \mathcal{E}^* \otimes_{\mathbf{A}} \mathcal{E}$.

[28]Where $\mathbf{C}$ is just *the constant sheaf of* $\mathbb{C}$-*numbers* [44, 61, 50].

[29]With $\otimes_{\mathbf{A}}$ the *homological algebraic (:categorical) tensor product functor*. In the sequel, we will return to explain and discuss in more detail the paramount importance of $\otimes_{\mathbf{A}}$ and its adjoint functor Hom [41, 42, 43] for the *dynamical functoriality* of our (finitary) ADG-perspective on gravity and gauge theories, and its cogent physical interpretation in the quantum deep.

- *$\mathcal{D}$ does transforms inhomogeneously (:non-tensorially or affinely) under local gauge-coordinate transformations*; while;
- *R does indeed transform homogeneously (:tensorially) under local gauge-coordinate transformations.*

To explicate in detail what the above statements mean, let us recall briefly from [44, 61, 50] how the ADG-connection $\mathcal{D}$ and the ADG-curvature R behave (:transform) respectively under local general coordinate-gauge changes.

1.2.5 Local Gauge Transformation of $\mathcal{D}$

Let $\mathcal{E}$ be a differential $\mathbf{A}$-module (:an ADG-theoretic vector sheaf) of rank n. Let $e^U \equiv \{U;\ e_{i=1\cdots n}\}$ and $f^V \equiv \{V;\ f_{i=1\cdots n}\}$ be local gauges[30] of $\mathcal{E}$ over the open set gauges U and V of X[31] which, in turn, we assume have non-empty intersection ($U \cap V \neq \emptyset$). Let us also denote by $g \equiv (g_{ij})$ the following *change of local gauge matrix*:

$$f_j = \sum_{i=1}^{n} g_{ij} e_i \tag{20}$$

which, plainly, is a local (*i.e.*, relative to $U \cap V$) section of the 'natural' structure group sheaf $\mathcal{GL}(n, \mathbf{A})$ of $\mathcal{E}$[32]—that is to say, $g_{ij} \in \mathrm{GL}(n, \mathbf{A}(U \cap V)) = \mathcal{GL}(n, \mathbf{A})(U \cap V)$.

Without going into the details of the derivation, which can be found in [44, 45, 50], we note that under such a local gauge transformation g, the gauge potential part $\mathcal{A}$ of $\mathcal{D}$ in (13) transforms as follows:

$$\mathcal{A}' = g^{-1}\mathcal{A}g + g^{-1}\partial g \tag{21}$$

a way we are familiar with from the usual differential geometry of the smooth fiber bundles of gauge theories [27]. For completeness, it must be noted here that, in (21), $\mathcal{A} \equiv (\mathcal{A}_{ij}) \in M_n(\mathbf{\Omega}^1(U)) = M_n(\mathbf{\Omega}^1)(U)$ and $\mathcal{A}' \equiv (\mathcal{A}'_{ij}) \in M_n(\mathbf{\Omega}^1(V)) = M_n(\mathbf{\Omega}^1)(V)$.

---

[30] A general gauge-coordinate *n-frame* (or *n-bein*).

[31] $U, V \in \mathcal{U}_i$, with $\mathcal{U}_i$ an open covering of the underlying topological space X of the $\mathbf{C}$-algebraized space $(X, \mathbf{A})$, as assumed throughout this paper.

[32] As noted earlier, one may recognise $\mathcal{GL}(n, \mathbf{A})$ above as the local version of the automorphism principal group sheaf $\mathcal{A}ut_{\mathbf{A}}\mathcal{E}$ of $\mathcal{E}$. The adjective 'local' here pertains to the fact mentioned earlier that ADG assumes that $\mathcal{E}$ is locally isomorphic to $\mathbf{A}^n$.

The transformation of $\mathcal{A}$ under local gauge changes is called *inhomogeneous*, *non-tensorial* or *affine* in the usual gauge-theoretic parlance [27] precisely because of the (additional to the homogeneous) term $g^{-1}\partial g$.

1.2.6 Local Gauge Transformation of R

On the other hand, we read directly from [44, 61, 50] that under similar local gauge-coordinate changes, the curvature $R(\mathcal{D})$ of the ADG-connection $\mathcal{D}$ transforms *purely homogeneously* or *tensorially*, as follows:
To that end, let again $g \equiv g_{ij} \in \mathcal{GL}(n, \mathbf{A})(U \cap V)$ be the change-of-gauge matrix we considered in (20) in connection with the transformation law of gauge potentials $\mathcal{A}_{ij}$. Again, without going into the technical details of the derivation, we bring forth from [44, 61, 50] the following *local transformation law of gauge field strengths*:

$$\begin{aligned} \text{for a local frame change}: \ e^U \stackrel{g}{\longrightarrow} e^V (U, V \in \mathcal{U}i \text{ covering } X), \\ \text{the curvature transforms as}: \ R \stackrel{g}{\longrightarrow} R^{'} = g^{-1} R g \end{aligned} \tag{22}$$

the form of which we are familiar with from the usual differential geometric (*i.e.*, smooth fiber bundle-theoretic) treatment of gauge theories [27]. For completeness, we remind ourselves here that, in (21) above, $R^{U\cap V} \equiv (R^{U\cap V}_{ij}) \in M_n(\mathbf{\Omega}^2(U \cap V))$—an $(n \times n)$-matrix of sections of local 2-forms in $\mathbf{\Omega}^2$.

The transformation of R under local gauge-coordinate changes is called *homogeneous*, *tensorial* or *covariant* in the usual smooth fiber bundle gauge-theoretic parlance [27].

As a last important observation before we move on to explicate the $\mathbf{A}$-invariant and its associated $\otimes_{\mathbf{A}}$-functorial character of the dynamical equations of motion from Einstein gravity and free Yang-Mills gauge theories, we note:

1.2.7 ADG-curvature spaces and ADG-curvature field categories

As we increase by a notch the level of abstraction and generality, from [44, 61, 50] we note the definition of *ADG-curvature spaces* as the following quintuples:

$$(\mathbf{A}, \mathcal{D}, \mathbf{\Omega}^1, \mathcal{D}^2, \mathbf{\Omega}^2) \tag{23}$$

consisting of $\mathbf{A}$-modules and $\mathbf{C}$-linear morphisms between them, which, in turn, by the very definition of the ADG-curvature field in (17), reduce to the following duet representing *the ADG-curvature fields*:

$$\mathcal{R} := (\mathbf{A}, R(\mathcal{D})) \tag{24}$$

1.2.8 Third encounter with 'Dynamical' Functoriality: three ADG-curvature field functor categories

In much the same way that we defined earlier the three functor categories of ADG-connection fields: the *Maxwell* $\mathcal{T}_{Max}$, the *Yang-Mills* $\mathcal{T}_{YM}$, and the *Einstein* category $\mathcal{T}_{Einst}$ earlier, we can similarly define here:

> *Three ADG-curvature field functor categories*: $\mathcal{C}_{Max}$, $\mathcal{C}_{YM}$ and $\mathcal{C}_{Einst}$, whose objects are ADG-curvature fields as in (24), and whose arrows are *natural transformation* type of correspondences between their $\otimes_{\mathbf{A}}$-functorial objects.[33]

1.2.9 Lagrangean Action Derivation of Vacuum Einstein Gravity and Free Yang-Mills Theories

Now that we have recalled the essential characteristics and local gauge transformation behaviour of the affine ADG-connections and their curvature $\otimes_{\mathbf{A}}$-tensors, we note that the ADG-theoretic versions of *the dynamical free Yang-Mills and vacuum Einstein equations* both derive from (the variation of) respective Yang-Mills ($\mathcal{YM}$) and Einstein-Hilbert ($\mathcal{EH}$) *Lagrangean action functionals*, as follows:[34]

[33]We are going to return to this important definition shortly, when we explicate the $\mathbf{A}$-invariant and $\otimes_{\mathbf{A}}$-functorial Einstein and Yang-Mills ADG-theoretic local gauge dynamical laws of motion. Of special interest and semantic importance will be the pair of adjoint functors Hom-$\otimes_{\mathbf{A}}$, which will be seen to be a *geometric morphism/natural transformation type of correspondence* between the corresponding ADG-categories of connection fields $\mathcal{D}$ and their curvatures $R(\mathcal{D})$: for gravity, for instance, the mapping: $\mathcal{T}_{Einst} \xrightarrow{\mathrm{Hom}-\otimes_{\mathbf{A}}} \mathcal{C}_{Einst}$ will be seen to be such a *geometric morphism* of huge physical significance for the cogent physical semantics of our ADG-theoretic perspective on gravity and gauge theories as Mallios had originally envisaged.

[34]For the equations above, see [44, 50] for technical definitions and details.

$$\begin{aligned}\mathcal{EH}_{\mathcal{E}}(\mathcal{D}) = \int_X tr(R_{Ric}(\mathcal{D})) &\xrightarrow{\delta\mathcal{A}} R_{Einst}(\mathcal{E}) = 0 \\ \mathcal{YM}_{\mathcal{E}}(\mathcal{D}) = \frac{1}{2}\int_X tr(R_{YM} \wedge \star R_{YM}) &\xrightarrow{\delta\mathcal{A}} \Delta^2_{\mathcal{E}nd\mathcal{E}}(R_{YM}) = 0\end{aligned} \tag{25}$$

in which we read from [44, 61, 50] that R_{Einst} is the *Ricci Scalar*[35] of the ADG-theoretic Einstein-Lorentzian metric connection field $\mathcal{D}$, while R_{YM} is the *Yang-Mills gauge field strength* of the homonymous ADG-theoretic Yang-Mills connection field $\mathcal{D}$.

1.3 Miscellaneous Remarks on 'Dynamical Functoriality' and A-Invariance-cum-Covariance of the ADG-Field Autodynamics: A Unified, Pure Gauge, Smooth Base Spacetime Manifoldless and Finitistic ADG-Theoretic Quantum Field 'Solipsism' of the 3rd Kind

In this subsection we make eight conceptual and technical remarks on the ADG-theoretic perspective on the Einstein and Yang-Mills field dynamics in (25) above. We itemise our remarks as follows:

1.3.1 ADG-Kinematics: The Affine Space of A-Connections

From the vacuum Einstein and the free Yang-Mills dynamical equations à-la ADG in (25) above, it follows that the sole dynamical variable in our ADG-theoretic perspective on gravity and gauge theories is the local **A**-connection $\mathcal{D}$. That is, as equation (25) depicts above, the dynamical equations derive from the variation ($\delta\mathcal{A}$) of the Einstein-Hilbert and Yang-Mills Lagrangian action functionals with respect to the local gauge potential part

[35]For expository completeness, we briefly recall from [61] that given a (real) Lorentzian vector sheaf $(\mathcal{E}, \rho)$ of rank n equipped with a non-flat Lorentzian ρ-metric **A**-connection $\mathcal{D}$, one can define the following *Ricci curvature operator* R_{Ric} relative to a local gauge $U \in \mathcal{U}_i$ of $\mathcal{E}$: $R_{Ric}(\,.\,, s)t \in (\mathcal{E}nd\mathcal{E})(U) = M_n(\mathbf{A}(U))$, for local sections s and t of $\mathcal{E}$ in $\mathcal{E}(U) = \mathbf{A}^n(U) = \mathbf{A}(U)^n$. Thus, the Ricci curvature here R_{Ric} is an $\mathcal{E}nd\mathcal{E}$-valued operator, a *curvature endomorphism of* $\mathcal{E}$. Moreover, *since* R_{Ric} *is matrix-valued, one can take its trace*, thus define the following *Ricci scalar curvature operator* $R_{Einst} := tr(R_{Ric}(\,.\,, s)t)$, which, plainly, is $\mathbf{A}(U)$-valued.

$(\mathcal{A})$ of the gravitational and Yang-Mills $\mathbf{A}$-connections $\mathcal{D}$ on their respective vector sheaves $\mathcal{E}$.

Thus, as emphasised in [61], the sole dynamical variable in our ADG-theoresis of vacuum Einstein gravity and free Yang-Mills theories is the ADG-theoretic Einstein connection field pairs $\mathcal{F}_{Einst} = (\mathcal{E}, \mathcal{D})$ and $\mathcal{F}_{YM} = (\mathcal{E}, \mathcal{D})$ defined by (14) earlier, within their respective categories $\mathcal{T}_{Einst}$ and $\mathcal{T}_{YM}$.

As also highlighted in [61], it follows that *the generalised 'kinematical' space of the theory is the affine space $\mathsf{A}_{\mathbf{A}}(\mathcal{E})$ of $\mathbf{A}$-connections $\mathcal{A}$ on $\mathcal{E}$*. Moreover, since the Lagrangians involved in (25) are invariant under the group sheaf $\mathcal{G}(\mathcal{E}) = \mathcal{A}ut_{\mathbf{A}}\mathcal{E}$ of local automorphisms (:local gauge transformations) of $\mathcal{E}$,

> *the relevant kinematical space is the moduli space $\mathsf{A}/\mathcal{G} = \mathsf{A}_{\mathbf{A}}(\mathcal{E})/\mathcal{A}ut_{\mathbf{A}}\mathcal{E}$ of gauge equivalent $\mathbf{A}$-connections $\mathcal{A}$ on $\mathcal{E}$.*[36]

Hence the integration sign in the dynamical action functionals in (25), which supposedly extends over the base topological space X, in effect extends over the moduli space $\mathsf{A}/\mathcal{G} = \mathsf{A}_{\mathbf{A}}(\mathcal{E})/\mathcal{A}ut_{\mathbf{A}}\mathcal{E}$ of gauge equivalent connections.[37]

1.3.2 A-Invariance and 'Dynamical' Functoriality

We noted earlier a fundamental difference in ADG between an $\mathbf{A}$-connection $\mathcal{D}$ and its curvature $R(\mathcal{D})$, namely that,

> *The curvature $R(\mathcal{D})$ is an $\otimes_{\mathbf{A}}$-tensor, while the connection $\mathcal{D}$ itself is not.*

In other words,

> *The curvature $R(\mathcal{D})$ respects our (algebras) of generalised measurements in $\mathbf{A}$, while the connection $\mathcal{D}$ itself does not—it 'eludes' them.*

[36] Recall from [61] that $\mathcal{G}(\mathcal{E})|_U = \mathcal{A}ut_{\mathbf{A}}\mathcal{E}|_U = (\mathrm{Aut}_{\mathbf{A}}\mathcal{E})(U) := \Gamma(U, \mathcal{A}ut_{\mathbf{A}}\mathcal{E}) \equiv M^n_U(\mathbf{A})$. Thus, $\mathsf{A}/\mathcal{G} = \mathsf{A}_{\mathbf{A}}(\mathcal{E})/\mathcal{A}ut_{\mathbf{A}}\mathcal{E}$ is the so-called *$\mathcal{G}$-orbit space* as the structure gauge group sheaf $\mathcal{A}ut_{\mathbf{A}}\mathcal{E}(\mathbf{A})$ cuts through the affine space $\mathsf{A}_{\mathbf{A}}(\mathcal{E})$, carving out 'paths' or 'orbits' of gauge equivalent connections in the process, which, in turn, leave the corresponding curvature Lagrangians in (25) invariant under (local) gauge transformations.

[37] From [61, 55, 57, 50] we read that *a suitably defined ADG-theoretic Radon-type of $\mathbf{A}$-linear continuous integration measure $d\mu$ on a suitably topologised $\mathsf{A}/\mathcal{G}$* ($d\mu : \mathsf{A}_{\mathbf{A}}(\mathcal{E})/\mathcal{A}ut_{\mathbf{A}}\mathcal{E} \longrightarrow \mathbf{A}$) *is expected to render rigorous the dynamical action integrals in* (25).

In still equivalent parlance,

> *The curvature $R(\mathcal{D})$ is a 'geometrical object', while the connection $\mathcal{D}$ itself is an 'algebraic object'.*[38]

Yet,

> *the dynamics on $\mathcal{E}$, generated by the connection field $\mathcal{D}$ acting on $\mathcal{E}$ and expressed as a differential equation on it as in (25), is derived from an action principle involving the curvature of the connection.*

As such,

> *the dynamical equations of motion on $\mathcal{E}$, which are derived from an action principle involving the curvature of the connection, is gauge invariant, hence our free gauge choices of generalised coordinate measurements in* $\mathbf{A}$ *respect the dynamics, and the physical laws are independent of our measurements in* $\mathbf{A}$.

Thus, *in toto*,

> *The physical laws are* $\mathbf{A}$*-invariant.*

Which brings us to a fundamental observation, in connection with *Utiyama's Theorem*, that we read directly from [57].

1.3.3 The Algebra-Geometry Duality: The $\mathrm{Hom}_{\mathbf{A}} - \otimes_{\mathbf{A}}$ Functorial Adjunction between the ADG-Field and Curvature Categories

Below, we quote Mallios *verbatim* from [57]:

[38] We will make this statement mathematically much more precise and rigorous in the sequel when we discuss the fundamental $\otimes_{\mathbf{A}} -$Hom-adjunction '*geometric morphism equivalence*' between the ADG Einstein and/or Yang-Mills connection field and curvature field functor categories $\mathcal{T}_{Einst/YM}$ and $\mathcal{C}_{Einst/YM}$, respectively.

> "...*Utiyama's theorem*, relates/characterizes the '**A**-*invariance*' of what we may call **A**-*connection Lagrangian through that one of* the corresponding *curvature Lagrangian*. So *the* aforementioned *two notions* (of '*Lagrangians*') are, in effect (physically) *equivalent*, through/due to the '**A**-*invariance*'..."[39]

which leads us to Mallios's telling remarks of what he calls *The Fundamental (Physical) Adjunction*:[40]

> "...**The Fundamental (Physical) Adjunction:** Thus, the basic Homological (:categorical) $Hom - \otimes$ *adjunction*, corresponds, *within the context of ADG*, to the *fundamental physical adjunction*, effectuated by the following '*adjoint pair of functors*':
>
> $$\mathbf{A}{-}connection\ (:field,\ `potential') \leftrightarrows\ curvature\ (:`field\ strength')$$
>
> The above can actually be perceived, as describing the whole *function of a physical law*, hence, in fact, *of the Nature herself*..."[41]

And Mallios concludes Section 2 of the paper [57] with the following intuitively telling paragraph:

> "...On the other hand, the *connecting function* of a given *adjunction*, is in effect a *natural transformation of functors*. Consequently, the latter should still preserve '**A**-*invariance*' of the adjunction, with respect to any '**A**-*invariant function', referring to any one* of the two associated functors through the adjunction: One gets at it, just, based on the very definitions[42] and on the *'functorial nature' of ADG*..."[43]

[39]Throughout this quotation we have been faithful to the *emphasis* placed by Mallios on certain key words in the original paper [57].

[40]Again quoting Mallios *verbatim* from [57].

[41]Again,throughout the quotation above we have been faithful to the *emphasis* placed by Mallios on certain key words in the original paper [57].

[42]Given before in the paper [57].

[43]Once again, throughout the quotation above, we have been faithful to the *emphasis* placed by the author on certain key words in the original text [57].

Now, in view of our presentation and arguments in the present paper, we are in a position to distill and further mathematically formalise and explicate Mallios's remarks above on *functoriality, adjunction* and *natural transformation of functors.*

1.3.4 Fourth Encounter with 'Dynamical' Functoriality: Mallios's Fundamental (Physical) Adjunction Explicated and Interpreted

The *categorical adjunction between the connection* $\mathcal{D}$ *and its curvature* $R(\mathcal{D})$

$$\mathbf{A} - connection \rightleftarrows \mathbf{A} - curvature \tag{26}$$

that Mallios emphasises in the excerpt from [57] in connection with Utiyama's Theorem displayed above, can now be cast in a mathematically rigorous and precise *functorial form*, as *a functorial correspondence between the respective categories of Einstein (or Yang-Mills) connection fields* $\mathcal{T}_{Einst}$ (or $\mathcal{T}_{YM}$) *and their corresponding categories of Einstein (or Yang-Mills) curvature field strengths* $\mathcal{C}_{Einst}$ (or $\mathcal{C}_{YM}$), as follows:

$$\begin{gathered} \mathcal{T}_{Einst} \underset{\mathrm{Hom}_{\mathbf{A}}}{\overset{\otimes_{\mathbf{A}}}{\rightleftarrows}} \mathcal{C}_{Einst} \\ \\ \mathcal{T}_{YM} \underset{\mathrm{Hom}_{\mathbf{A}}}{\overset{\otimes_{\mathbf{A}}}{\rightleftarrows}} \mathcal{C}_{YM} \end{gathered} \tag{27}$$

with the pair of 'opposite direction maps' $(\otimes_{\mathbf{A}}, \mathrm{Hom}_{\mathbf{A}})$ above corresponding to *the fundamental homological algebraic adjunction* that Mallios alludes to.

More technically speaking, $\otimes_{\mathbf{A}}$ is *the homological (left-adjoint) tensor product functor*[44] and $\mathrm{Hom}_{\mathbf{A}}$ *is its right-ajoint functor* [43]. When paired together, the pair:

$$\mathcal{GM}_{\mathbf{A}} := (\otimes_{\mathbf{A}}, \mathrm{Hom}_{\mathbf{A}}) \tag{28}$$

constitutes an instance of what is commonly known in category theory as a *geometric morphism* [43].[45]

[44] In the category of **A**-modules that the vector sheaves $\mathcal{E}$ of ADG belong.

[45] It is instructive here to give the definition of a general *geometric morphism* directly from [43], as it originally arose in category theory. With every continuous map f between

At the same time, the fact that the pair $(\otimes_{\mathbf{A}}, \mathrm{Hom}_{\mathbf{A}})$ indeed constitutes a categorical adjunction, derives directly from the way $\otimes_{\mathbf{A}}$ and $\mathrm{Hom}_{\mathbf{A}}$ act on the corresponding categories. Again, in the general case, we read from [43] that $-\otimes X$ is the *left-adjoint* (functor) and $\mathrm{Hom}(X,-)$ the *right-ajoint* (functor), because they act as a pair of maps as follows:

$$\mathrm{Hom}(Y \otimes X, Z) \simeq \mathrm{Hom}(Y, \mathrm{Hom}(X, Z)) \tag{29}$$

Thus, in view of the general definition of *the action of an adjunction* as in (29) above, we are now in a position to see directly that, indeed:

> *In ADG, the curvature $R(\mathcal{D})$ of a connection is the $\otimes_{\mathbf{A}}$-morph (:image) of its connection.*

which we can verify directly from the curvature's definition in terms of the action of the $\otimes_{\mathbf{A}}$ and $\mathrm{Hom}_{\mathbf{A}}$ functors in (17) and (19) earlier:

$$\begin{gathered} R:\ \mathcal{E} \xrightarrow{\mathrm{Hom}_{\mathbf{A}}} \mathbf{\Omega}^2(\mathcal{E}) \equiv \mathcal{E} \otimes_{\mathbf{A}} \mathbf{\Omega}^2 \\ R \in \mathrm{Hom}_{\mathbf{A}}(\mathcal{E}, \mathbf{\Omega}^2(\mathcal{E})) = \mathcal{H}om_{\mathbf{A}}(\mathcal{E}, \mathbf{\Omega}^2(\mathcal{E}))(X) \end{gathered} \tag{30}$$

Recalling again from footnote 26 that, for any ADG-theoretic vector sheaf $\mathcal{E}$ like the one involved in (30) above: $\mathcal{E}nd\mathcal{E} \equiv \mathcal{H}om_{\mathbf{A}}(\mathcal{E}, \mathcal{E}) \cong \mathcal{E} \otimes_{\mathbf{A}} \mathcal{E}^* = \mathcal{E}^* \otimes_{\mathbf{A}} \mathcal{E}$.

We can distill all the above discussion and express the *geometric morphism functorial equivalence* between the connection-field and curvature-field Einstein and/or Yang-Mills *functor categories* $\mathcal{T}_{Einst/YM}$ and $\mathcal{C}_{Einst/YM}$ depicted in equations (27) and (28) above, as follows:

$$\begin{gathered} \mathcal{T}_{Einst/YM} \overset{\mathcal{GM}}{\longleftrightarrow} \mathcal{C}_{Einst/YM} \\[2ex] \mathcal{T}_{Einst/YM} \underset{\mathrm{Hom}_{\mathbf{A}}}{\overset{\otimes_{\mathbf{A}}}{\rightleftarrows}} \mathcal{C}_{Einst/YM} \\[2ex] \mathcal{D}_{Einst/YM} \xrightarrow{\otimes_{\mathbf{A}}} R_{Einst/YM} \end{gathered} \tag{31}$$

with the first two lines in (31) above, reading:

two topological spaces X and Y: $f: X \to Y$, there is *a pair of adjoint functors* (f^*, f_*): $Sh(X) \underset{f^*}{\overset{f_*}{\rightleftarrows}} Sh(Y)$ (f_* is coined the *push-out* and f^* is coined the *pull-back*) between the categories of sheaves (of structureless sets) $Sh(X)$ and $Sh(Y)$ over X and Y, respectively.

The curvature (Einstein and/or Yang-Mills field categories) are the geometric morphs (:images) of the corresponding connection (field categories).

while the map in the third line of (31) above can be interpreted as stipulating that:

The (Einstein and/or Yang-Mills) curvature field is the $\otimes_{\mathbf{A}}$-image of its connection field.

which in turn vindicates what we established earlier, namely that:

The curvature is a geometrical, $\mathbf{A}$-tensorial object (:an $\otimes_{\mathbf{A}}$-tensor or an $\mathbf{A}$-invariant morphism), while its connection is not.

This gives a *raison d'être* and vindicates the epithet geometric in the *geometric morphism* denomination of the pair of maps $\mathcal{GM}_{\mathbf{A}} := (\otimes_{\mathbf{A}}, \mathrm{Hom}_{\mathbf{A}})$ above.

1.4 $\mathcal{GM}$-Dynamical Functoriality, Mallios's A-Invariance and Gauge Invariance of ADG-GT

Now that we have explicated the subtle and technical sense in which a *geometric morphism adjunction* $\mathcal{GM}$ links the ADG-theoretic Einstein/Yang-Mills connection field category $\mathcal{T}_{Einst/YM}$ with its ADG-theoretic Einstein/Yang-Mills curvature field 'counterpart-equivalent' category $\mathcal{T}_{Einst/YM}$, we are in a position to further support *Mallios's remarks on* $\mathbf{A}$*-invariance in connection with gauge invariance* in [57].

To this end, we quote *verbatim* from Section 3 of [57] the displayed paragraph before Theorem 3.1:

"...Therefore, one thus realizes that, *the fundamental adjunction*[46]*, preserves the* $\mathbf{A}$*-invariance, for any* $\mathbf{A}$*-invariant function,*[47] *pertaining to the two basic functors appearing in the aforesaid adjunction...*"[48]

[46]Our geometric morphism $\mathcal{GM}$ in equations (27) and (28) earlier.

[47]Especially, for the $\mathbf{A}$-invariant Einstein and Yang-Mills action functionals in (25) above, which, as we saw earlier, are $\mathbf{A}$-valued functionals defined on the moduli space $\mathsf{A}_{\mathbf{A}}/\mathcal{G} = \mathsf{A}_{\mathbf{A}}(\mathcal{E})/\mathcal{A}ut_{\mathbf{A}}\mathcal{E}$ of gauge equivalent $\mathbf{A}$-connections on $\mathcal{E}$.

[48]That is, the homological $\mathrm{Hom}_{\mathbf{A}}$ and $\otimes_{\mathbf{A}}$ adjoint functors constituting $\mathcal{GM}_{\mathbf{A}}$ in equations (27) and (28) earlier.

which in turn leads to the following central result (:Theorem 3.1) in [57], coined *Utiyama's Principle* therein:[49]

> "...**Theorem 3.1** *Any 'gauge invariance' of an appropriate 'Lagrangian' for (*$\mathbf{A}$*-)connections is equivalent to a similar invariance of the corresponding Lagrangian for the associated curvature with the (*$\mathbf{A}$*-)connection at issue...*"

Thus, we are now in a position to distill and re-express the deep relation between our notion of *'Dynamical' Functoriality*, with Mallios's $\mathbf{A}$*-invariance and the structure group* $\mathcal{A}ut_{\mathbf{A}}\mathcal{E}$*-gauge invariance (of the dynamical action functionals) of our ADG-theoresis on Vacuum Einstein Gravity and Free Yang-Mills Theories* in (25), as follows:

> **Fundamental Theorem of ADG-Gauge Theory.** *The 'dynamical' geometric morphism* $\mathcal{GM}_{\mathbf{A}}$ *preserves* $\mathbf{A}$*-invariance and entails the structure group sheaf* $\mathcal{G} \equiv \mathcal{A}ut_{\mathbf{A}}\mathcal{E}$*-invariance of the ADG-theoretic dynamical equations of motion for Vacuum Einstein Gravity and Free Yang-Mills Theory.*

In view of our physical interpretation of $\mathbf{A}$ earlier in this paper and throughout our work on ADG-GT [59, 60, 61, 50, 78, 79, 80, 81, 82] as *the algebra (sheaf) of our generalised coordinate measurements*,[50] an important Corollary to the Fundamental Theorem above *goes the other way around*, as follows:

> **Corollary to the Fundamental Theorem of ADG-Gauge Theory.** *The algebra (sheaf) of our generalised coordinate measurements* $\mathbf{A}$ *respects (i.e. it is 'non-perturbing' and it leaves invariant) the functorial ADG-theoretic gauge field dynamics for Vacuum Einstein Gravity and Free Yang-Mills theories, hence, in return, it entails and almost 'mandates' that the (local) Relativity Group (sheaf) of the theory is* $\mathcal{G} \equiv \mathcal{A}ut_{\mathbf{A}}\mathcal{E}$.[51] Thus, $\mathbf{A}$-invariance, via the 'dynamical' geometric morphism $\mathcal{GM}_{\mathbf{A}}$, which

[49] Again, quoted exactly as it appears in [57].

[50] That is, our localised and gauged $\mathbf{A}$-valued measurements based on an open cover $\{U\} = \mathcal{U}_i$ of local open sets U of the base topological space X employed by the theory.

[51] That the local relativity group in ADG-GT is 'naturally' $\mathcal{A}ut_{\mathbf{A}}\mathcal{E}$ (:see subsection next) has been amply expounded in [59, 60, 61, 50, 55, 56, 78, 79, 80, 81, 82].

> in turn entails the $\mathcal{G} = \mathcal{A}ut_{\mathbf{A}}\mathcal{E}$-gauge invariance of the ADG-field dynamics (expressed via the geometric morph of the connection field—the field's curvature), corresponds to a dynamical version of the *Kleinian conception of geometry*.[52]

The discussion above brings us to an important, yet subtle, technical and interpretational matter of ADG-GT.

1.4.1 Note on the Natural Transformation character of the Geometric Morphism $\mathcal{GM}_{\mathbf{A}}$: the 'Naturality' of the Functorial Dynamics of ADG-GT

Since, as we alluded to numerous times throughout this paper, the ADG-theoretic connection and curvature field categories are *functor categories* [57, 55, 56, 50], the two adjoint functors constituting the geometric morphism $\mathcal{GM}_{\mathbf{A}} := (\otimes_{\mathbf{A}}, \mathrm{Hom}_{\mathbf{A}})$ above are examples of *natural transformations* [43].[53]

This, too, was prophetically anticipated by Mallios in [57]:[54]

> "...On the other hand, the *connecting function* of a given *adjunction*, is in effect <u>*a natural transformation of functors*</u>. Consequently, the latter should still preserve '**A**-*invariance*' of the adjunction, with respect to any '**A**-*invariant function', referring to any one* of the two associated functors through the adjunction..."

The fitting physico-mathematical 'pun' here is that:

> *The* **A***-invariant and, in extenso,* $\mathcal{A}ut_{\mathbf{A}}\mathcal{E}$*-invariant functorial gauge field dynamical changes in ADG-GT are, categorically-cum-physically speaking,* <u>*Natural Transformations*</u> *of the 'dynamically equivalent' gauge connection and curvature fields involved therein, via the Natural Transformation Geometric Morphism* $\mathcal{GM}_{\mathbf{A}} := (\otimes_{\mathbf{A}}, \mathrm{Hom}_{\mathbf{A}})$ *that interlinks them.*

[52] According to Felix Klein, '*the geometry of an object is all the transformations of it that leave it invariant*' [38].

[53] In a nutshell, and quite heuristically, *a natural transformation* $\mathcal{N}$ *connects or maps one particular functor* $\mathcal{F} : A \to B$ *to another particular functor* $\mathcal{G} : A \to B$ *between two categories* A *and* B. At the same time, $\mathcal{N}$ *does not need to apply to every functor in some category of functors* [43].

[54] Excerpt from quotation earlier.

2 Intermezzo: Lateral Technical and Philosophical Offshoots and Repercussions of Functoriality

In this intermediate section, we give very brief accounts and we express them in the form of '*Aphorisms*', borrowed from previous works, of various technical, conceptual and interpretational-cum-philosophical corollaries and didactics that follow, in one way or another, from both the 'kinematical' and 'dynamical' functoriality of our ADG-theoretic perspective on Finitary Vacuum Einstein-Lorentzian Gravity and Free Yang-Mills Theories, as expounded above.

2.1 Third Gauge Auto-Gravitodynamics from Background Spacetime Manifoldlessness: Gauge Field Solipsism

The quintessential feature of ADG, especially *vis-à-vis* its novel conceptual import and potential technical applications to Quantum Gravity research [44, 45, 50, 46, 51, 48, 52, 54, 53, 58, 57, 55, 56, 59, 60, 61, 62, 63, 64, 65, 66, 67, 76, 77, 78, 79, 80, 81, 82, 83, 84, 108, 109], is arguably the following:

> **Aphorism 1: Background Spacetime Manifoldlessness.** *Mallios's Abstract Differential Geometry is a purely homological-algebraic (:sheaf and category-theoretic) way of doing and applying 'Differential Calculus', with all its technical and conceptual panoply, to many current research fronts in Theoretical and Mathematical Physics such as Quantum Gauge Theories of Matter and Quantum Gravity, but in the manifest absence of a background geometrical $\mathcal{C}^\infty$-smooth base (spacetime) manifold.*

The deep and wide spectrum of potential import of such *background differential spacetime manifoldlessness*, especially in Quantum Gauge Theory and Quantum Gravity research, has been expounded in detail over the last two decades in numerous works [46, 47, 51, 52, 55, 56, 50, 59, 60, 61, 78, 79, 80, 81, 82, 66, 67, 108, 109].

One important feature of such *a background spacetime manifold independence* is that:

Aphorism 2: Dynamical Connection Gauge Field Solipsism. ADG enables us to formulate Vacuum Einstein Gravity and Free Yang-Mills theories as *pure gauge theories of the third kind*,[55] in the sense that the sole dynamical variable in the theory is an entirely homologically-algebraically defined **A**-connection acting on (the sections of) a vector sheaf $\mathcal{E}$, without any recourse to or dependence on an external background geometrical differential (:$\mathcal{C}^{\infty}$-smooth) spacetime manifold [61, 81, 82, 50, 67]. The ADG-theoretic *Gauge Theory of the Third Kind*, which regards the **A** connection field $(\mathcal{E}, \mathcal{D})$ as the sole dynamical variable, has been coined *Half-Order Formalism*[61, 81, 82].[56] *In ADG-GT, dynamics concerns and derives solely from the stalks (of the sheaves involved), not from the base topological space X itself, which is only used for the sheaf-theoretic localisation (and continuous variation) of the 'geometrical objects' (:the algebraic connection fields) that live on that surrogate external base space.*[57] *The ADG-field dynamics is purely algebraic, smooth geometrical base spacetime manifoldless, connection field-solipsistic and autonomous* (:dynamical connection field 'self-governing' and 'self-propagating').[58]

[55]We recall directly from [61, 81, 82] that *Gauge Theory of the First Kind* is Hermann Weyl's original Global ($U(1)$) Scale Theory of the Electromagnetic Field [103]; *Gauge Theory of the Second Kind* pertains to the usual current abelian and non-abelian Yang-Mills gauge theories of matter that are localised and gauged based on an externally prescribed and fixed $\mathcal{C}^{\infty}$-smooth base spacetime continuum; while *Gauge Theory of the Third Kind* is our ADG-theoresis, which is '*field monic, solipsistic and autonomous*'. Read on.

[56]To distinguish it from the original *Second Order Formalism* of Einstein, involving the smooth spacetime metric $g_{\mu\nu}$ on a background differential spacetime manifold [70], as well as from the more recent *First Order Formalism* of Ashtekar *et al.*, which, apart from a Lorentzian spin-connection $\mathcal{A}_{\mu}$, it includes the metric-*vierbein* e^{a}_{μ} as joint gravitational dynamical variables in the theory [2, 3].

[57]To use a Wittgensteinian metaphor here: "*Once one climbs up the ladder, one throws the ladder away*" [107]. Similarly in our ADG-theoresis of gauge theory and gravity, *once we have employed the base topological space X as a surrogate scaffolding for the sheaf-theoretic localisation, gauging and continuous dynamical variation of the (Einstein and Yang-Mills) gauge connection fields on it, we formulate the gauge invariant dynamics homologically-algebraically and functorially as equations involving sheaf morphisms as we showed earlier, and we then completely forget about X, which 'atrophises' and 'dissolves' in the background* (pun intended).

[58]In this respect, it has been noted and emphasised elsewhere [61, 62, 79, 81, 82] that

We close this subsection by recalling from Brian Hatfield's prologue to Feynman's Lectures of Gravitation [19] where he discusses Feynman's prophetic intuition, *vis-à-vis* Quantum Gravity, that, in a strong sense, it was quite accidental that gravity was originally formulated in terms of a metric tensor—so that Quantum Gravity would have to involve some kind of 'quantising space-time geometry'—but rather that *gravity fundamentally reflects some kind of deep gauge invariance*. In other words, that *gravity should be regarded as a gauge theory like the other three fundamental forces of Nature*:

> "...Thus it is no surprise that Feynman would recreate general relativity from a non-geometrical viewpoint. The practical side of this approach is that one does not have to learn some 'fancy-schmanzy' (as he liked to call it) differential geometry in order to study gravitational physics. (Instead, one would just have to learn some quantum field theory.) However, when the ultimate goal is to quantize gravity, Feynman felt that the geometrical interpretation just stood in the way. From the field theoretic viewpoint, one could avoid actually defining—up front—the physical meaning of quantum geometry, fluctuating topology, space-time foam, *etc.*, and instead look for the geometrical meaning after quantization...Feynman certainly felt that the geometrical interpretation is marvellous, '*but the fact that a massless spin-*2 *field can be interpreted as a metric was simply a coincidence that might be understood as representing some kind of gauge invariance*'[a]..."
>
> [a]Our emphasis of Feynman's words as quoted by Hatfield.

the 'unitary' ADG-theoretic field-pairs $\mathcal{F} = (\mathcal{E}, \mathcal{D})$ recall and conceptually resemble *Leibniz's Monads* [40], in the sense that they are *autonomous (:self-governing), dynamically self-propagating and self-sustaining entities in no need of an external geometrical space-time continuum for their dynamical support and substenance.* Moreover, in close analogy to *Leibniz's purely algebraic (:relational) conception of the notion of derivative in Differential Calculus* [15, 7], *the ADG-theoretic connection fields are the sources of differentiation, hence the 'causes' of dynamical variability in Mallios's theory.* Read below for more detailed discussion of *the close resemblance between Mallios's homological-algebraic conception of Differential Geometry in the guise of ADG and Leibniz's relational conception of derivative and, in extenso, of Differential Calculus.*

2.1.1 The Functoriality of 3rd Quantization in ADG-GT

In ADG, the functoriality that pervades both the kinematical structures and the the purely gauge theoretic dynamical Einstein and Yang-Mills field equations as we expounded above, also extends naturally to include *geometric (pre)quantization* and *second (field) quantization*, processes which have also been developed entirely homologically-algebraically by ADG-theoretic means [46, 51, 50, 54, 57, 56, 81, 82]. The upshot of that *homological-algebraic quantisation 'procedure'* is that:

> **Aphorism 3: The ADG-Theoretic Functorial Quantum Field-Particle Duality.** The ADG-theoretic field pair $(\mathcal{E}, \mathcal{D})$ also embodies the fundamental '*Quantum Field-Particle Duality*' in the sense that from a geometric (pre)quantization and second quantisation vantage, *the (local) sections of the vector sheaves* $\mathcal{E}$ *embodied in the ADG-theoretic connection fields represent quantum states of bosonic or fermionic field-quanta*, as follows:[59]

$$\begin{array}{c} (\mathcal{E}, \mathcal{D}) \Longleftrightarrow (\text{`dynamics}', \text{`kinematics}') \\ (\mathcal{D}, \Gamma(U, \mathcal{E})) \Longleftrightarrow (\text{field, local particle states}) \\ (\mathcal{D}, \Gamma(U, \mathcal{L}_{n=1}) \Longleftrightarrow (\text{Boson field, local bosonic particle states}) \\ (\mathcal{D}, \Gamma(U, \mathcal{E}_{n>1}) \Longleftrightarrow (\text{Fermion field, local fermionic particle states}) \end{array} \tag{32}$$

Moreover, *the correspondences above have been seen to be purely functorial* [46, 51, 50, 81, 82].[60]

2.2 Demistifying and Circumventing Singularities and Field-Theoretic Unphysical Infinities.

The **A**-functorial and **A**-invariant bottom-up *aufbau* of ADG has been used to totally circumvent, 'deconstruct' and 'demistify' singularities and other associated (non-renormalisable) unphysical infinities that have hitherto seemed to mar and assail the CDG-based Einsteinian gravity and the quantum gauge

[59] The correspondences below are borrowed almost *verbatim* from [46, 51, 50, 81, 82].

[60] Additionally, the reader should note that the fermionic sheaves $\mathcal{E}_{n>1}$ in the last line above may be conventionally regarded as '*Odd-Grassmannian*' Sheaves [22, 27].

field theories of matter both of which are based on an underlying smooth geometrical spacetime manifold.

The usual physicists' consensus is that the singularities of General Relativity (GR) and the unphysical infinities of the spacetime continuum based quantum field theories of matter are indications that:

> *The Laws of Physics break down, thus Nature becomes nonsensical and unpredictable, at those sites* [30, 31, 32, 33, 25, 26, 105].

With Professor Mallios we had time and again scrutinised the statement above and *invariably it seemed fundamentally incomprehensible to him*:

> How come the Field Laws of Physics, which we normally model after differential equations, break down at 'geometrical sites' we call singularities?

It was Mallios's original and fundamental idea that:

> If we could somehow abstract, genaralise and detach the '*innate differential geometric mechanism*' of Calculus from its apparent inextricable dependence on a fixed $\mathcal{C}^{\infty}$-smooth background geometrical (spacetime) manifold, we could still do most (if not all!) of Differential Geometry without getting stuck on or breaking down at singularities or other unphysical continuous field infinities. Moreover, we could even integrate, encompass or even 'absorb' singularities into the structure sheaf $\mathbf{A}$ of ADG and still the whole differential geometric mechanism would hold in their very presence. That is, the $\mathbf{A}$-connection fields would still define and obey differential equations (laws) in the very presence of singularities, no matter how numerous, robust or pathological those singularities might be, especially when viewed from the vantage of the Classical Differential Calculus (CDG), which is based on a background smooth geometrical spacetime manifold [47, 48, 52, 53, 62, 78, 63, 64].

In other words,[61]

[61] And this was by far Tasos's favourite *motto*.

> *Don't 'blame' Nature for the singularities and the related unphysical infinities of the CDG-based field theories we have. Blame our Mathematics: pit it on CDG itself!*[62]

Thus, by showing ADG-theoretic means that *the differential geometric mechanism of Calculus is inherently or innately homological algebraic* (:sheaf and category-theoretic)[63] and not at all dependent on an external (to the fields) background geometrical spacetime continuum, Mallios abstracts and generalises the usual CDG to ADG and manifestly shows that:

> *We can actual do/use Differential Calculus in the very presence of singularities and other 'CDG/differential manifold based anomalies'*, thus, *a fortiori*, *the Laws of Physics (:the differential equations between the dynamical connection fields) do not 'break down' in any sense at their presence.* They still hold intact, and we can still calculate and 'predict' things based on them!

We close this subsection by borrowing three quotes from [62]—all three all-time favourites of Tasos—that in a sense foreshadow the development of ADG and its physical applications to gravity and gauge theories of matter and, in view of our arguments in this paper, they 'post-anticipate' how our purely algebraic (:ADG-theoretic) finitary, causal and quantal theoresis of vacuum Einstein Gravity and free Yang-Mills theory has come to 'vindicate' them:

• The first two quotes by *Albert Einstein* are taken from the very last Appendix D of *The Meaning of Relativity* [16]:

---

[62]Here, Tasos liked to use the following analogy: in much the same way as the real number line $\mathbb{R}$ comes short or 'breaks down' when we try to solve the algebraic equation $x^2 + 1 = 0$, the CDG-formulated differential equations modelling the Laws of Physics appear to break down at singularities and the latter (misleadingly) appear to be *shortcomings and blemishes* of the Physical Laws (:differential equations) themselves. However, all we had to do is to extend $\mathbb{R}$ to $\mathbb{C}$, and the 'seemingly problematic' $x^2 + 1 = 0$ is solved(!) *Mutatis mutandis* then for the extension, abstraction and generalisation of CDG to ADG: *from an ADG-theoretic perspective, the Field Laws of Physics do not break down at singularities* (see displayed statement below).

[63]More in line with Leibniz's (rather than Newton's) conception of the basic derivative operator (*viz.* connection) of Differential Calculus, Mallios preferred to call ADG *a relational theory* [15, 7], where the differential geometric mechanism derives from the algebraically modelled (dynamical) relations between the dynamical fields (connections) themselves, not from an external background geometrical spacetime continuum. Read on.

> "...One can give good reasons why reality cannot at all be represented by a continuous field. *From the quantum phenomena it appears to follow with certainty that a finite system of finite energy can be completely described by a finite set of numbers.*[a] This does not seem to be in accordance with a continuum theory, and must lead to an attempt to *find a purely algebraic theory for the description of reality*[b]...
>
> and:
>
> ...Is it conceivable that a field theory permits one to understand the atomistic and quantum structure of reality? Almost everybody will answer this question with 'no'. But I believe that at the present time nobody knows anything reliable about it. *This is so because we cannot judge in what manner and how strongly the exclusion of singularities reduces the manifold of solutions. We do not possess any method at all to derive systematically solutions that are free of singularities*[c]..."
>
> [a]Our emphasis.
> [b]Our emphasis.
> [c]Our emphasis.

• The third quote by *David Finkelstein* is taken from the introduction of his *Theory of Vacuum* [21]:

> "...The locality principle seems to catch something fundamental about nature... Having learned that the world need not be Euclidean in the large, the next tenable position is that it must at least be Euclidean in the small, a manifold.[64] The idea of infinitesimal locality presupposes that the world is a manifold.[65] *But the infinities of the manifold (the number of events per unit volume, for example) give rise to the terrible infinities of classical field theory and to the weaker but still pestilential ones of quantum field theory.*[66] The manifold postulate freezes local topological degrees of freedom which are numerous enough to account for all the degrees of freedom we actually observe.

[64]Recall that, by definition, *a manifold is a locally Euclidean space*—that is to say, a space that is locally isomorphic to $\mathbb{R}^n$.

[65]Here, *the notion of infinitesimal locality mandates that the Laws of Physics be modelled after differential equations.*

[66]Our emphasis.

> The next bridgehead is *a dynamical topology, in which even the local topological structure is not constant but variable.*[67] The problem of enumerating all topologies of infinitely many points is so absurdly unmanageable and unphysical that *dynamical topology virtually forces us to a more atomistic conception of causality and space-time than the continuous manifold*[68]..."

2.3 ADG-Field Realism, Solipsism and Monism, and Mallios's Novel Conception of Bohr's Correspondence Principle: ADG-Field 'Unitarity' from 3rd Quantisation

In the previous section, we witnessed and we argued how the close interplay between fuctoriality and **A**-invariance of the ADG-field dynamics is tantamount to gauge invariance. That is to say, our attempts to localise and coordinatise (:'measure') the Einstein or the Yang-Mills gauge field by employing the algebra structure sheaf **A** of generalised coordinates—or '*arithmetics*', as Mallios preferred to call it—relative to, and localised over, a system of local gauges (:covers) $\mathcal{U}_i$ (of the in principle arbitrary base topological space X), does not affect the dynamical law that (the curvature of) the field—*the $\mathcal{GM}$-geometric morph of the field*—obeys.[69]

In turn, this means that that the field 'sees through' and remains undisturbed by our 'perturbing' generalised acts of measurement (:gauge localisations), hence the issue arises of what would Bohr's Correspondence Principle of the usual Quantum Theory be in our ADG-GT. Mallios had quite a fascinating, unconventional conception of, and unorthodox ideas about, that, as follows:

> Traditionally, from the very advent of Quantum Mechanics, Bohr's Correspondence Principle can be expressed as follows: observable quantum actions are represented by noncommutative 'numbers'

[67] Our emphasis.

[68] Again, our emphasis.

[69] That is to say, the action functional density and the law—the differential equation derived from it by 'variation' with respect to the local gauge connection potentials $\mathcal{A}$—remains invariant under **A**-coordinate changes and $\mathcal{A}ut_{\mathbf{A}}\mathcal{E}|_{(U\in\mathcal{U})}$ local gauge transformations.

> (:so-called *q-numbers*[70]), while our measurements thereof should correspond to (:should yield) commutative numbers (:so-called *c-numbers*—presumably, these are real numbers in $\mathbb{R}$, which are embedded in the complex numbers $\mathbb{C}$ that the usual Quantum Theory employs[71]).

Now, Mallios contended that *Bohr's c/q, commutative/noncommutative, classical/quantum dichotomy*–the *'classical/quantum divide'*, so to speak—is already embodied and structurally encoded in the ADG-conception of a field as the pair $(\mathcal{E}, \mathcal{D})$, in the following sense:

> Our generalised local measurements (:local 'arithmetics' relative to a local gauge $U \in \mathcal{U}_i$) of the ADG-fields are represented by the local sections of the *abelian algebra* structure sheaf $\mathbf{A}$ (*i.e.*, $\mathbf{A}_U = \mathbf{A}(U) = \Gamma(U, \mathbf{A})$).[72] On the other hand, the noncommutativity—the q-number Heisenberg type of indeterminacy, the quantum fuzziness and the 'quantum foam' aspect of the ADG-fields, so to speak—is already encoded in the principal group sheaf $\mathcal{A}ut_{\mathbf{A}}\mathcal{E}$ of local (gauge) automorphisms of the field, which is locally isomorphic to: $\mathcal{A}ut_{\mathbf{A}}\mathcal{E}(U) \simeq M_n^{\bullet}(\mathbf{A}(U))$, the structure group sheaf of invertible $(n \times n)$-matrices, having for entries local sections of $\mathbf{A}$ in $\mathbf{A}|_{U \in \mathcal{U}} = \mathbf{A}(U) \equiv \Gamma(U, \mathbf{A})$. Thus, $M_n^{\bullet}(\mathbf{A}(U)$ is the ADG-field theoretic version of the '*local Heisenberg group*' of the theory, which is manifestly non-abelian—elements (:local sections) of which correspond to the ADG-version of q-numbers.

Below, we are going to give briefly a 'heuristic analogy' of Mallios's seemingly unorthodox and unconventional intuition above, which has been previously noted in [61, 81, 82, 62], while in the next subsection we are going to tie

[70]Think, for instance, of Heisenberg's matrices.

[71]In that sense, *real numbers are 'real'* (pun intended), but over the years, the use of the real number continuum has been questioned and challenged in both Quantum Theory and Quantum Gravity (see for example [34] for a thoughtful exposition).

[72]The reader should note here a *key difference* between the usual conception of c-numbers (:results of measurements) in conventional Quantum Theory and the generalised c-numbers of ADG-GT. The former are sections of the constant sheaf $\mathbf{C} \equiv \mathbb{R} \subset \mathbb{C}$, while the latter are sections of the 'dynamically variable' sheaf $\mathbf{A}$, which in turn includes anyway the constant sheaves of complex numbers ($\mathbb{C}$) and real ($\mathbb{R}$) numbers as proper subsheaves. Mallios and Zafiris in [67] give a very novel operational interpretation and physical explanation of this generalisation!

our heuristics below to the *canonical, sheaf cohomological 3rd quantisation of ADG-GT first proposed in* [81].

Similar to how Bohr's Correspondence is almost tautosemous with the formal 'inverse' of the original *First Quantization Correspondence*:

$$\begin{gathered}\text{Classical Position c} - \text{number} : x \longrightarrow \hat{x} : \text{Quantum Position q} - \text{number}\\[2em] \text{Classical Momentum c} - \text{number} : p \longrightarrow \hat{p} : \text{Quantum Momentum q} - \text{number}\end{gathered} \tag{33}$$

also by imposing the following *Heisenberg Uncertainty commutation relations* between the $\hat{x}$ and $\hat{p}$ operators (:matrices), which in turn generate the usual *Heisenberg Algebra* of traditional Quantum (Matrix) Mechanics:

$$[\hat{x}, \hat{p}] = -\imath\hbar \tag{34}$$

the analogous '*position-momentum correspondence*' within the ADG-fields is, following [61, 81, 82, 62]:

$$\begin{gathered}\text{Local Particle `Position' States} \longrightarrow \text{Local Sections of } \mathcal{E}\\[1em] \text{Local Field `Momentum' Operator} \longrightarrow \mathcal{D} = d + \mathcal{A}\end{gathered} \tag{35}$$

the rationale for the heuristic semantic correspondences above being that:

> *Much in the same way that in conventional particle (Newtonian) mechanics velocity (speed) or momentum is (or measures) the (rate of) change of the position (state) of a particle,*[73] *in ADG-field theory, the* **A***-connection* $\mathcal{D}$ *in the ADG-field* $(\mathcal{E}, \mathcal{D})$ *is a generalised differential, acting as a sheaf morphism on the 'local particle states' of the ADG-field (which are in turn represented by the local sections of* $\mathcal{E}$ *within the ADG-field) as it were* <u>to change them</u>.

Then, the formal 'quantum deformation' ('Heisenberg uncertainty relation') of the usual Poisson Brackets Algebra of Classical Mechanics to the Heisenberg Algebra of Quantum Mechanics by the imposition of the following canonical commutation relations:

[73]Which momentum is, in turn, the differential (:derivative) of that position determination: $p: x \longrightarrow dx$.

$$\{x,p\} = 0 \longrightarrow [\hat{x},\hat{p}] = -\imath\hbar \tag{36}$$

can heuristically be cast within the ADG-field as follows:

$$[\mathcal{E},\mathcal{D}] = \mathcal{A}ut_{\mathbf{A}}\mathcal{E} \stackrel{\text{locally}}{\longrightarrow} [s \in \mathcal{E}(U), d+\mathcal{A}] = M_n(\mathbf{A}_U)s = M_n(\mathbf{A}(U))s \tag{37}$$

where $s \in \mathcal{E}_{\mathbf{A}}(U)$ (with $\mathcal{E}_{\mathbf{A}}(U) \equiv \Gamma(U,\mathcal{E}_{\mathbf{A}}) \simeq \mathbf{A}^n(U)$) a 'local quantum particle state' within the ADG-field $(\mathcal{E},\mathcal{D})$ and $M_n(\mathbf{A}(U))$ a local section of the structure group sheaf $\mathcal{G} = \mathcal{A}ut_{\mathbf{A}}\mathcal{E}$ of the ADG-field relative to the local open gauge U.

The 'canonical' commutator 'auto-uncertainty' relation within the ADG-field $(\mathcal{E},\mathcal{D})$ in equation (37) above can then be heuristically interpreted as follows:

> The 'canonical' commutator quantum uncertain relation between the generalised local position quantum particle states in $\mathcal{E}$ and its dua; generalised momentum field operator (:sheaf morphism) $\mathcal{D}$ generates and induces a dynamical local gauge transformation in $\mathcal{A}ut_{\mathbf{A}}\mathcal{E}|_U = M^n(\mathbf{A}(U))$, which then acts on the local quantum particle states to change them and, as it were, to 'blur' them (:'quantum fuzziness' or 'quantum foam'). In this heuristic sense, $\mathcal{E}$ is 'complementary' to $\mathcal{D}$, thus the ADG-field may be thought of as being '*self-complementary*' and '*self-quantum*'. In this sense we argued in [61, 81, 82, 62] that *the ADG-field is an already self-quantum, auto-dynamical entity.*

We can formalise the generalised ADG-theoretic quantum uncertainty/complementarity relation above as a homological $\mathbf{A}$-tensor product morphism type of map:

$$\mathcal{Q}:\ \mathcal{E}_{\mathbf{A}} \otimes_{\mathbf{A}} \mathcal{D} \longrightarrow \mathcal{A}ut_{\mathbf{A}}\mathcal{E}_{\mathbf{A}} \tag{38}$$

thus finally arrive at the definition of the *'unitary' quantal ADG-gauge field*[74] as being the following tetrad:

[74]The reader should note that, issuing from the 3rd Quantisation scheme for ADG-field theory originally presented in [81], the epithets *unitary quantum* carry standard meaning in the usual Quantum Theory, hence we use inverted single quotes around the word *unitary*, while instead of *quantum* we use *quantal* throughout our work, in order to avoid confusion of semantic reference.

$$\mathbf{U} := (\mathcal{E}, \mathcal{D}, \mathcal{A}ut_{\mathbf{A}}\mathcal{E}, \mathcal{Q}) \tag{39}$$

which encodes the following four pieces of important information:

1. The vector sheaf $\mathcal{E}$ of generalised quantum particle 'position' states;
2. The connection field $\mathcal{D}$, which effectuates dynamical 'momentum' field-like changes of the said states by acting as a sheaf morphism on $\mathcal{E}$'s local sections;
3. The structure gauge group sheaf $\mathcal{A}ut_{\mathbf{A}}\mathcal{E}$ of local gauge transformations, also effectively acting on $\mathcal{E}$ as its principal structure group sheaf[75]; and finally,
4. The quantum uncertainty operator (:morphism) $\mathcal{Q}$, which stands for an $\mathcal{A}ut_{\mathbf{A}}\mathcal{E}$-valued quantum act of perturbation on the dynamical action of the connection field on the local quantum particle states (:local sections) of $\mathcal{E}$.

The epithet '*unitary*' for $\mathbf{U}$ in (39) above indicates that it is a holistic entity, an inseparable whole, encoding state ($\mathcal{E}$), dynamical changes of state ($\mathcal{D}$), gauge symmetries and invariances of dynamical changes of state ($\mathcal{A}ut_{\mathbf{A}}\mathcal{E}$), and quantum uncertainty of 'determination' ($\mathcal{Q}$), *all-4-in-1*.

> *All there is in ADG-field theory are the Leibnizian Monad type of entities* $\mathbf{U} = (\mathcal{E}, \mathcal{D}, \mathcal{A}ut_{\mathbf{A}}\mathcal{E}, \mathcal{Q})$, *nothing else.* This is what was referred to in [61, 81, 82, 62] as '*ADG-field solipsism and monism*'.

Thus, perhaps more importantly, the adjective 'unitary' given to the tetrad $\mathbf{U} = (\mathcal{E}, \mathcal{D}, \mathcal{A}ut_{\mathbf{A}}\mathcal{E}, \mathcal{Q})$ above pertains to the fact that in ADG-field theory, *there is no reference and recourse to, no dependence whatsoever on, an external (to the ADG-fields themselves) background geometrical spacetime manifold.* It follows that the usual distinction and schism *internal/external* that is normally reserved for the *symmetries* of the usual gauge field theories

[75] And conversely, $\mathcal{E}$ can be viewed as the associated sheaf to the principal group sheaf $\mathcal{A}ut_{\mathbf{A}}\mathcal{E}$.

of matter—whether classical or quantum—loses its meaning in our ADG-theoresis of Vacuum Einstein Gravity and Free Yang-Mills theories.[76] This has also been succinctly pointed out more recently in [67].

2.3.1 Interregnum: Drawing Formal Links with 3rd Quantisation

In the discussion below, we briefly draw links between the formal 'canonical' quantisation heuristics above and the functorial sheaf cohomological 3rd Quantisation of ADG-GT scenario presented in [81].

To that end, we recall that in [81] we intuited that since the ADG-fields $\mathcal{F} = (\mathcal{E}, \mathcal{D})$ are dynamically self-supporting, autonomous monadic entities as we emphasised earlier, and moreover, since they are '*self-dual*'[77]

> *a possible quantization scenario for them should involve solely their two constitutive parts, namely, $\mathcal{E}$ and $\mathcal{D}$, without recourse to/dependence on extraneous structures (e.g., a base spacetime manifold) for its mathematical support and its self-consistent (physical) interpretation.*

Thus, in what formally looked like a canonical quantization-type of scenario,

> *in* [81] *we envisaged abstract non-trivial local commutation relations between the abstract position* (:$\mathcal{E}$) *and momentum* (:$\mathcal{D}$) *aspects of the ADG-fields.*

To that end, we recalled that

> *there are certain local (:differential) forms that uniquely characterize sheaf cohomologically the vector sheaf $\mathcal{E}$ and the connection $\mathcal{D}$ parts of the ADG-fields $\mathcal{F} = (\mathcal{E}, \mathcal{D})$*

Thus, the basic heuristic-intuitive idea in [81] was to identify the relevant forms and then posit non-trivial commutation relations between them. Moreover, for the sake of the aforementioned '*dynamical ADG-field autonomy*', we would like to require that

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[76] Traditionally, we reserve the epithet 'external' for 'the external (to the fields) spacetime symmetries', while 'internal' is normally reserved for gauge degrees of freedom and their symmetries [4, 27].

[77] In the sense that the connection momentum-like field $\mathcal{D}$ is quantum dual to the 'position quantum particle states' (represented by the local sections of) $\mathcal{E}$.

the envisaged commutation relations should not only involve just the two components (*i.e.*, $\mathcal{E}$ and $\mathcal{D}$) of the total ADG-fields $\mathcal{F} = (\mathcal{E}, \mathcal{D})$, but they should also somehow *'algebraically close' within the fields themselves*—*i.e.*, the result of their commutation relations should not take us 'outside' the total ADG-field structure (and its 'dynamical auto-transmutations'), which anyway is the only dynamical structure involved in our theory.[78]

Keeping the theoretical requirements above in mind, we recall from [44, 45, 46, 60, 50] *two important sheaf cohomological results*:

1. That, sheaf cohomologically, the vector sheaves $\mathcal{E}$ are completely characterized by a so-called *coordinate* 1*-cocycle* $\phi_{\alpha\beta} \in Z^1(\mathcal{U}, \mathcal{GL}(n, \mathbf{A}))$ associated with any system $\mathcal{U}$ of local gauges of $\mathcal{E}$. Intuitively, this can be interpreted in the following 'Kleinian symmetry-geometry' way: since any (vector) sheaf is completely determined by its (local) sections,[79] one way of knowing the latter is to know how they transform—in passing, for example, from one local gauge ($U_\alpha \in \mathcal{U}$) to another ($U_\beta \in \mathcal{U}$), with $U_\alpha \cap U_\beta \neq \emptyset$ and $\mathcal{U}$ a chosen system of local open gauges covering X.[80] To know something (*e.g.*, a 'space') is to know how it transforms, the fundamental idea underlying Klein's general conception of 'geometry' [38].

   Thus, the bottom-line here is that the characteristic cohomology classes of vector sheaves $\mathcal{E}$ are completely determined by $\phi_{\alpha\beta}$; write:

[78] This loosely reminds one of the theoretical requirement for algebraic closure of the algebra of quantum observables in canonical QG, with the important difference however that the Diff(M) group of the external (to the gravitational field) spacetime manifold must also be considered in the constraints, something that makes the desired closure of the observables' algebra quite a hard problem to overcome [95]. In [61, 81, 82] we discuss certain difficult problems that Diff(M) creates in various QG approaches, as well as how its manifest absence in ADG-gravity can help us bypass them totally. For, recall that from the ADG-perspective gravity is an external (:background) spacetime manifold unconstrained (because it is a background spacetime manifoldless) pure gauge theory (:of the 3rd kind).

[79] A basic motto (:fact) in sheaf theory is that "*a sheaf is its sections*" [6, 44]. If we know the local data (:sections), we can produce the whole sheaf space by restricting and collating them relative to an open cover $\mathcal{U}$ of the base topological space X. This is the very process of 'sheafification' (of a preasheaf) [6, 44].

[80] In particular, $\phi_{\alpha\beta}$ can be locally expressed as the $\mathbf{A}|_{U_{\alpha\beta}}$-isomorphism: $\phi_\alpha \circ \phi_\beta^{-1} \in \mathcal{A}ut_{\mathbf{A}_{\alpha\beta}}(\mathbf{A}^n|_{U_{\alpha\beta}}) = \mathrm{GL}(n, \mathbf{A}(U_{\alpha\beta})) = \mathcal{GL}(n, \mathbf{A})(U_{\alpha\beta})$, in which expression the familiar *local coordinate transition (:structure) functions* appear. Hence, also the 'natural' structure (:gauge) group sheaf $\mathcal{A}ut_{\mathbf{A}}\mathcal{E} = \mathcal{GL}(n, \mathbf{A})$ of $\mathcal{E}$ arises.

$$[\phi_{\alpha\beta}] \in H^1(X, \mathcal{GL}(n, \mathbf{A})) = \varinjlim_{\mathcal{U}} H^1(\mathcal{U}, \mathcal{GL}(n, \mathbf{A})) \tag{40}$$

where the $\mathcal{U}$s, normally assumed to be *locally finite open coverings of* X [44, 45, 59, 60, 61], constitute a *cofinal subset* of the set of all proper open covers of X.[81] *In toto*, we assume that $\phi_{\alpha\beta}$ *encodes all the (local) information we need to determine the local quantum-particle states of the field in focus* (*i.e.*, the local sections of $\mathcal{E}$).

2. On the other hand, it was observed in [81] that *locally* $\mathcal{D}$ *is uniquely determined by the so-called 'gauge potential'* $\mathcal{A}$, which is normally (*i.e.*, in CDG) defined as a Lie algebra (:vector) valued 1-form [27]. Correspondingly, in ADG $\mathcal{A}$ is seen to be an element of $M_n(\Omega(U)) = M_n(\Omega)(U) = \Omega(\mathcal{E}nd\mathcal{E})$,[82] thus it is called *the local* $\mathbf{A}$*-connection matrix* $(\mathcal{A}_{ij})$ *of* $\mathcal{D}$*, with entries local sections of* $\mathcal{E}^* = \Omega$. In turn, this means that locally $\mathcal{D}$ splits in the familiar way, as follows:

$$\mathcal{D} = d + \mathcal{A} \tag{41}$$

where ∂ is the usual 'inertial' (:flat) differential[83] and $\mathcal{A}$ the said gauge potential. In ADG-gravity, the `total field` $\mathcal{D}$ as a whole (:'globally')

[81] An assumption that has proven to be very fruitful in applying ADG to the formulation of a locally finite, causal and quantal Vacuum Einstein Gravity and Free Yang-Mills theories, as we argued in the first part of this paper and throughout our past works [59, 60, 61, 78, 79, 80, 81, 82]. *En passant*, we also note that in [81] the direct (:inductive) limit depicted in (40) above is secured by the 'cofinality' of the set of finitary (:locally finite) open coverings of X that we choose to employ [89, 83, 84, 76, 77, 59, 60, 61, 78, 79] and it was emloyed K-theoretically to link the 3rd Quantisation of ADG-fields scenario with Mallios's K-theoretic musings on topological algebra structure sheaves $\mathbf{A}$ and the 2nd Quantisation classification of the local quantum particle states (:local sections) of the vector sheaves involved in the ADG-fields into Bosons (:$\mathcal{E}$ is a line sheaf $\mathcal{L}_{n=1}$) and Fermions (:$\mathcal{E}$ is a vector sheaf of rank $n > 1$) [51].

[82] Note that, as also mentioned earlier, in ADG by definition one has: $\Omega := \mathcal{E}^* := \mathcal{H}om_{\mathbf{A}}(\mathcal{E}, \mathbf{A})$. That is, the $\mathbf{A}$-module sheaf Ω of abstract differential 1-forms is dual to the vector sheaf $\mathcal{E}$, much like in the classical theory (:CDG of $\mathcal{C}^\infty$-smooth manifolds) where differential forms (:cotangent vectors) are dual to tangent vectors [27], although as it has been empasised throughout our works, in ADG the epithet '(co)tangent' is meaningless due to the manifest absence of an operative background space(time) of any kind (and especially of a base manifold).

[83] As noted in the previous section, in ADG, the Cartan-Kähler differential d, like $\mathcal{D}$, is defined as a linear, Leibnizian $\mathbf{C}$-sheaf morphism $d: \mathbf{A} \rightarrow \Omega$, thus it is an instance of on

represents the *gravito-inertial field*, but locally it can be separated into its inertial (:d) and gravitational (:$\mathcal{A}$) parts.[84]

Thence, the envisaged sheaf cohomological canonical quantization-type of scenario for the total ADG-fields $\mathcal{F} = (\mathcal{E}, \mathcal{D})$ rests essentially on positing the following non-trivial abstract Heisenberg-type local commutation relations between (the characteristic forms that completely characterize) $\mathcal{E}$ (:abstract '*position*' particle states) and $\mathcal{D}$ (:abstract '*momentum*' field operator). Thus, heuristically we posited in [81] the following 'canonical' commutation relations:

$$[\phi, \mathcal{D}] \stackrel{loc.}{=} [\phi_{\alpha\beta}, d + \mathcal{A}_{ij}]_{U_{\alpha\beta}} = [\phi_{\alpha\beta}, d]_{U_{\alpha\beta}} + [\phi_{\alpha\beta}, \mathcal{A}_{ij}]_{U_{\alpha\beta}} \tag{42}$$

stressing also that, as highlighted in [81],

> the local commutation relations in (42) above are well defined, since they effectively *close within the noncommutative* $(n \times n)$-*matrix Klein-Heisenberg algebra* $\mathcal{E}nd\mathcal{E}(U_{\alpha\beta}) = M_n(\mathbf{A}(U_{\alpha\beta})) = M_n(\mathbf{A})(U_{\alpha\beta})$ of the field's endomorphisms—the field's 'noncommutative Kleinian geometry' we mentioned earlier representing what Mallios intuited as some kind of 'quantum field foam'—the *intrinsically noncommutative aspect of the ADG-fields*.

This '*algebraic closure*' is in accord with the theoretical requirement we imposed earlier, namely that,

> the abstract, Heisenberg-like, canonical quantum commutation relations between the two components $\mathcal{E}$ and $\mathcal{D}$ of the ADG-fields should not take us outside the fields, but should rather *close* within them.[85]

A-connection; albeit, a *flat* one (:$R(d) = d^2 = 0$), which is secured by the very definition of curvature in (17) and the niloptency of the exterior differential d.

[84] In the classical theory of gravity (:General Relativity; abbr. GR), the physical meaning of this local separation of the `total field` $\mathcal{D}$ into ∂ and $\mathcal{A}$ reflects the *local principle of equivalence*; namely, that *locally, the spacetime manifold M of GR is flat Minkowski space*, or equivalently, that *locally, GR reduces to Special Relativity*, or perhaps more importantly, that *gravity can always be 'gauged away' locally by a suitable choice of 'gauge'* (:local inertial frame). This is simply Einstein's elevator gedanken experiment [70].

[85] Here, one could envisage an abstract Heisenberg-type of algebra freely generated (locally) by ϕ (:abstract position) and $\mathcal{A}$ (:abstract momentum), modulo the (local) commutation relations (41). Plainly, it is a subalgebra of $\mathcal{E}nd\mathcal{E}(U)$, but deeper structural investigations on it must await a more complete and formal treatment.

Indeed, $\mathcal{E}nd\mathcal{E}$ *is precisely the algebra sheaf of internal/intrinsic (dynamical) self-transmutations of the (quantum particle states of the) field*—by definition, the $\mathcal{E}$-endomorphisms in $\mathcal{H}om_{\mathbf{A}}(\mathcal{E},\mathcal{E})$ (:quantum field foam).

This is another aspect of *the quantum dynamical autonomy of ADG-fields*:

> the $\mathcal{E}$ (:abstract point-particle/position) part of the ADG-field is '*complementary*', in the quantum sense of 'complementarity', to $\mathcal{D}$ (:abstract field-wave/momentum). Thus, *the total ADG-fields* $\mathcal{F} = (\mathcal{E}.\mathcal{D})$ *are 'quantum self-dual' entities* [61, 62, 78, 81].[86]

Furthermore, by choosing $\phi_{ab} = \phi_{ab}^{in}$[87] so that $\mathcal{A}$ is 'gauged away'—*i.e.*, by setting $\mathcal{A} = 0$,[88] reduces (41) to (omitting the local open gauge indices/subscripts 'α, β'):

$$[\phi^{in}, d] = \phi^{in} \circ d - d \circ \phi^{in} \tag{43}$$

Moreover, since we are sheaf cohomologically guaranteed that $d \circ \phi = 0$ globally, which is tantamount to the very *existence* of an $\mathbf{A}$-connection $\mathcal{D}$ (globally) on $\mathcal{E}$ [44, 45, 50],[89] (42) further reduces to:

[86] From our abstract and background spacetime manifoldless perspective, the de Broglie-Schrödinger wave-particle duality is almost tautosemous with the Bohr-Heisenberg momentum-position complementarity.

[87] The superscript '*in*' stands for '*inertial*', and it represents a choice (:*our* choice!) of a local change-of-gauge $\phi^{in}_{\alpha\beta} \in \mathcal{GL}(n, \mathbf{A})_{\alpha\beta} \equiv \Gamma(U_{\alpha\beta}, \mathcal{GL}(n, \mathbf{A})$ that would take us to a locally inertial frame of $\mathcal{E}$ over $U_{\alpha\beta} \subset X$.

[88] As noted earlier, this is an analogue of the Equivalence Principle (EP) of GR in ADG-gravity, corresponding to the local passage to an 'inertial frame' (:one 'covarying' with the gravitational field; *e.g.*, recall Einstein's free falling elevator *gedanken* experiment) in which the curved gravito-inertial $\mathcal{D}$ reduces to its flat 'inertial' $\mathbf{A}$-connection part d [44, 45, 59, 60, 61]. As noted above, this just reflects the well known fact that *GR is locally SR*, or conversely, that when SR is localized (*ie*, 'gauged' over the base spacetime manifold) it produces GR (equivalently, the curved Lorentzian spacetime manifold of GR is locally the flat Minkowski space of SR). *In summa*, gravity (:$\mathcal{A}$) has been locally gauged away, and what we are left with is the inertial action d of the ADG-gravitational field $\mathcal{D}$. It must be also stressed here that the choice of a locally inertial frame, like all gauge choices, is an externally imposed constraint in the theory—'externally', meaning that it is *we*, the external (to the field) experimenters/theoreticians ('observers') that we impose such constraints on the field (*i.e.*, we make choices about what aspects of the field we would like to single out and, ultimately, observe/study).

[89] This essentially corresponds to the fact that the coordinate 1-cocycle $\phi_{\alpha\beta} \in Z^1(\mathcal{U}, \Omega)$ is actually a *coboundary* (:a closed form), belonging to the zero cohomology class $[d\phi_{\alpha\beta}] = 0 \in H^1(X, M_n(\Omega))$, which in turn guarantees the existence of an $\mathbf{A}$-connection on $\mathcal{E}$ as the so-called *Atiyah class* $\mathbf{A}$ of $\mathcal{E}$ vanishes (:$\mathbf{A}(\mathcal{E}) := [d\phi_{\alpha\beta}] = 0$) [44, 45, 50].

$$[\phi^{in}, d] = \phi^{in} \circ d \tag{44}$$

Now, a heuristic physical interpretation can be given to (43) if we consider its effect (:action) on a local section $s \in \mathcal{E}_{\alpha\beta} := \mathcal{E}(U_{\alpha\beta}) \equiv \mathcal{E}|_{U_{\alpha\beta}}$:

$$[\phi^{in}, d](s) = (\phi^{in} \circ d)(s) = \phi^{in}(ds) \tag{45}$$

(44) designates the inertial dynamical action of $\mathcal{D}$ (*i.e.*, the action of its locally flat, inertial part d) on (an arbitrary) s, followed by the gauge transformation of ds to an inertial frame $e_{in}^{U_{\alpha\beta}} \subset \mathcal{E}_{\alpha\beta}$ '*covarying*' with the inertio-gravitational field.

It may be interpreted as expressing what happens to a 'vacuum graviton state' s when it is first acted upon[90] by the inertial part of the `total` ADG-gravitational field $\mathcal{D}$ and then[91] to an inertial frame that in a sense '*covaries*' with the said inertial change d of s.

Heuristically, we further intuited in [81] that one can perhaps get a more adventurous (meta)physical insight into (44) by defining the *uncertainty operator* $\mathcal{U}$ as

$$\mathcal{U} := \phi^{in} \circ \partial \in \mathcal{E}nd\mathcal{E} \tag{46}$$

and by delimiting all the quantum-particle (:abstract position) states of the field (:local sections of $\mathcal{E}$) that are annihilated by it. Intuitively, these are formally the local '*classical-inertial*' states

$$\mathcal{E}_U^{cl} := \mathrm{span}_{\mathbb{C}}\{s \in \mathcal{E}(U) : \mathcal{U}(s) = 0\} =: \ker(\mathcal{U}) \tag{47}$$

for which the abstract sheaf cohomological Heisenberg uncertainty relations (42) vanish. Plainly, $\mathcal{E}^{cl}(U)$ is a $\mathbf{C}$-linear subspace of $\mathcal{E}(U)$—*the kernel of* $\mathcal{U}$.

On top of the above, intuitively it makes sense to assume that $\mathcal{U}$ is a 'projector'—a primitive idempotent (:projection operator) locally in $\mathcal{E}nd\mathcal{E}$ (*i.e.*, in $M_n(\mathbf{A}(U))$)—since the 'gedanken' operation of '*inertially covarying with a chosen local inertial frame*' must arguably be idempotent.[92] This

[90] Recall that we are considering only *vacuum* gravity, in which the non-linear gravitational field 'couples' solely to itself(!)

[91] The sequential language used here should not be interpreted in an temporal-operational sense—as it were, as 'operations carried out sequentially in time'.

[92] After all, '*inertially covarying the inertial state leaves it inertially covariant*'. Or, to use a famous Einstein '*gedanken* metaphor': '*jumping on a light-ray (in order to ride it) twice, simply leaves you riding it*'(!)

means that $\mathcal{U}^2 = \mathcal{U}$, so that $\mathcal{U}$ separates (chooses or projects out) the 'classical' ($\mathrm{eigen}_0(\mathcal{U}) \equiv \ker(\mathcal{U})$) from the quantum ($\mathrm{eigen}_1(\mathcal{U})$) local quantum gravito-inertial states.[93].

Finally, in line with [81], we would like to ask *en passant* here the following highly speculative question:

> Could the generation/emergence of (inertio-gravitational) mass be somehow accounted for by a (spontaneous) symmetry breaking-type of mechanism, whereby, the dynamical automorphism group $\mathcal{A}ut_{\mathbf{A}}\mathcal{E}$ of the ADG-gravitational field $(\mathcal{E}, \mathcal{D})$ reduces to its subroup that leaves $\ker(\mathcal{U})$ invariant? Alternatively intuited, could the emergence of inertio-gravitational mass be thought of as the result of some kind of 'quantum anomaly' of 3rd-quantized Vacuum Einstein Gravity?[94]

• **Identifying 3rd Quantisation within our 'Unitary' Quantal ADG-Gauge Field Tetrads.** The alert reader must have already noticed that the 'canonical' commutation relations (42) are the sheaf cohomological versions of our 'heuristic' canonical commutation relations (37). Also, by comparing the commutator expressions (37) and (43) and the associated definitions of the operators (:morphisms) $\mathcal{Q}$ in (38) and $\mathcal{U}$ in (46), the astute reader must have realised that $Ran(\mathcal{Q}) \subset Ran(\mathcal{U})$, as $M_n^{\bullet}(\mathbf{A}(U)) \subset \mathcal{E}nd\mathcal{E}|_U = M_n(\mathbf{A}(U))$.

Overall, and without loss of generality of mathematical structures or physical interpretation thereof, in the light of 3rd Quantisation our 'Unitary' Quantal ADG-Gauge Field Tetrads in (39) can now be identified with the tetrad:

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[93]In [81], a formal mathematical reason why we chose $\mathcal{U}$ to be a projection operator was to apply it and relate our 3rd Quantisation scenario to Mallios's K-theoretic perspective on 2nd (Field) Quantisation in [51].

[94]The epithet 'quantum' adjoined to 'anomaly' is intended to distinguish the effect intuited above from the usual anomalies. A 'quantum anomaly' is the 'converse' of an anomaly in the usual sense, in that what was a symmetry of the *quantum* theory (:an element of $\mathcal{A}ut_{\mathbf{A}}\mathcal{E}$ in our case) ceases to be a symmetry of the 'classical domain' of our theory (:$\ker(\mathcal{U})$). Let it be stressed that the emergence of gravito-inertial 'mass' in the sense intuited here has a truly relational (:algebraic) and 'global' flavour reminiscent of Mach's ideas: 'global' gravitational field symmetries in $\mathcal{A}ut_{\mathbf{A}}\mathcal{E}$ are locally reduced to inertial ones, and sheaf theory's ability to interplay between local and global comes in handy in this respect [67]. (In [67], Mallios and Zafiris do a great job in highlighting exactly how sheaf theory allows one to transit from 'local' to 'global', and vice versa.)

$$\mathbf{U} := (\mathcal{E}, \mathcal{D}, \mathcal{A}ut_{\mathbf{A}}\mathcal{E}, \mathcal{Q}) \equiv (\mathcal{E}, \mathcal{D}, \mathcal{E}nd\mathcal{E}, \mathcal{U}) \tag{48}$$

which carries the same denomination as before *'Unitary' Quantal ADG-Gauge Field* and is again symbolised by **U**.

Now, by recalling the meaning we ascribed to **U** as *an inseparable (:'unitary'), self-dual, 3rd-quantised, auto-dynamical, background spacetime manifoldless gauge field of the 3rd kind*, we can, *mutatis mutandis* address the traditional distinction and schism in the usual quantum theory between 'observer' (classical exosystem) and 'observed' (quantum endosystem) [106, 21].[95] We are not going to elaborate in detail on this subtle and important point here,[96] but it is noteworthy to mention that significant work has been done on defining by the homological algebraic (:category and topos-theoretic) means of ADG internally consistent *quantum observables* in the theory, without recourse to any external spacetime manifold, as befits the 'unitary' and quantal ADG-gauge field theory [108, 109, 67, 67].

In the light of the physical interpretation of the **U** ADG-field tetrads above, we conclude this subsection by quoting David Finkelstein from [21], making some protphetic remarks about *the future of physical laws vis-à-vis* his Quantum Relativity Theory approach to Quantum Gravity, based on 'abolishing' this external/internal field distinction and schism of the usual theory:[97]

> "*...What are we after as physicists? Once I would have said, the laws of nature; then, the law of nature. Now I wonder.*[98]
>
> A law, or to speak more comprehensively, a theory, in the ordinary sense of the word, even a quantum theory of the kind studied today

[95]This pertains to the (in)famous *Heisenberg scnitt* (:Heisenberg cut): the schism that divides and separates the classical from the quantum phenomena and it delimits the boundary across which the wave function supposedly collapses upon measurement. It is when q-numbers become c-numbers, and probability amplitudes become probability distributions. See [106] for a plethora of classic articles on the Heisenberg Schnitt and the Quantum Theory of Measurement. It's where John Wheeler noted that "*No phenomenon is a phenomenon unless it is an observed phenomenon*" [106].

[96]We are going to tackle it further in [62].

[97]This quote was one of Tasos's all-time favourites, and I recall fondly him urging me to include it in the opening talk of the first *Glafka 2004: Iconoclastic Approaches to Quantum Gravity* international theoretical physics conference that we jointly organised in Athens, Greece—setting thus '*The Spirit of the Meeting*' [80].

[98]Our emphasis.

by almost all quantum physicists, is itself not a quantum object. We are supposed to be able to know the theory completely, even if it is a theory about quanta. Its symbols and rules of inference are supposed to be essentially non-quantum. For example, ordinary quantum theory assumes that we can know the form of the equations obeyed by by quantum variables exactly, even though we cannot know all the variables exactly. This is considered consistent with the indeterminacies of quantum theory, because the theory itself is assumed to sum up conclusions from arbitrarily many experiments.

Nevertheless, since we expect that all is quantum, we cannot consistently expect such a theory to exist except as an approximation to a more quantum conception of a theory. At present we have non-quantum theories of quantum entities. Ultimately the theory too must reveal its variable nature. For example, the notion that an experiment can be repeated infinitely often is as implausible as the notion that it can be done infinitely quickly ($c = \infty$), or infinitely gently ($\hbar = 0$).

It is common to include in the Hamiltonian of (say) an electron a magnetic field that is treated as a non-quantum constant, expressing the action of electric currents in a coil that is not part of the endosystem but the exosystem. Such fields are called external fields. Upon closer inspection, it is understood, the external field resolves into a host of couplings between the original electron and those in the coil system, now part of the endosystem.

It seems likely that the entire Hamiltonian ultimately has the same status that we already give the external field. No element of it can resist resolution into further quantum variables. In pre-quantum physics the ideal of a final theory is closely connected with that of a final observer, who sees everything and does nothing. The ideal of a final theory seems absurd in a theory that has no final observer. When we renounce the ideal of a theory as a non-quantum object, what remains is a theory that is itself a quantum object. Indeed, from an experimental point of view, the usual equations that define a theory have no meaning by themselves, but only as information-storing elements of a larger system of users, as much part of the human race as our chromosomes, but responding more quickly to the environment. The fully quantum theory lies somewhere within the theorizing activity of the human race itself, or the subspecies of physicists, regarded as a quantum system. If this is indeed a quantum entity, then the goal of knowing it completely is a Cartesian fantasy, and at a certain stage in our development we will cease to be law-seekers and become law-makers.

> *It is not clear what happens to the concept of a correct theory when we abandon the notion that it is a faithful picture of nature. Presumably, just as the theory is an aspect of our collective life, its truth is an aspect of the quality of our life*[99]*...*"

3 Brief Philosophical Epilegomena: Anastasios Mallios's Original Vision and Posthumous Future Legacy

This author's earliest recollections of exchanges with Professor Anastasios Mallios in the late 1990s/early 2000s during multiple dinner evenings at a little known, quaint and cosy little tavern, fittingly called *Algebra*, situated in the northern Athens suburb of Paleo Psychiko, about *the potential import of Abstract Differential Geometry in current persistently un(re)solved technical (:mathematical), conceptual-cum-semantic and philosophical issues in Quantum Gravity and Quantum Gauge Theory research*, focused mainly on two fronts:

1. The algebraic essence and origin of 'physical space' and its 'physical geometry' and, as a 'result':

2. The non-existence of an *a priori* 'geometrical space(time)', but quite on the contrary, the emergence of 'geometrical space(time)' as an outcome of an algebraic (:relational) dynamics (:dynamical interactions) between the 'physical geometrical objects' (:the physical fields) that live on a surrogate and virtual 'space'.

Focusing on the two items above, below I will try to recall and 'reconstruct' the origins and motivations of ADG.

3.1 The Original Vision: 'Geometrical Space(-Time)' comes from 'Algebraic Dynamics'

From numerous exchanges, close collaboration and warm friendship with the creator of ADG over more than one and a half decades, this author maintains

[99] Again, our emphasis throughout.

and is willing to argue that Professor Mallios's inspired *magnum opus* (ADG) was originally motivated by two main aspects—one technical-mathematical, the other intuitive-heuristic-conceptual and physical—that synergistically feed each other, grow holistically together, and almost perfectly dovetail with each other in the *aufbau* of ADG..

3.1.1 Tracing the Mathematical 'Origins' of ADG: the Categorical Duality between Algebra and Geometry

Very early on in my crossing worldlines with Professor Mallios, he kept on bringing up one of his all-time favourite quotes by Sophie Germain [24] as being a motto at the very heart of ADG:

> "*Géometrie est une Algèbre bien figuré, mais Algèbre est une Géometrie bien écrite*"[100]

The quote above perfectly encapsulates, in a beautifully poetic way, *the fundamental mathematical (:categorical) duality between Algebra and Geometry*, which, in modern mathematical (:category-theoretic) parlance may be boiled down and reduced to two cornerstone results, both of which we have played a central role in Mallios's developing his theory:[101]

1. **Gel'fand Duality and the Gel'fand-Stone Theorem:** Generally and loosely speaking, *Gel'fand duality is a general duality between spaces and algebras of functions defined on them*. In particular, for the case of *compact topological spaces* and *abelian C^*-algebras*, Gel'fand duality roughly pertains to the result that *every commutative C^*-algebra A is equivalent to the abelian C^*-algebra of continuous functions on a suitably and 'naturally topologised' (:using the algebraic structure itself)*[102] *space called its Gel'fand Spectrum* $Spec(A)$ [23, 35, 39].[103]

[100]English translation: "*Geometry is a well figured (or designed) Algebra, while Algebra is a well written Geometry*".

[101]Tasos Mallios in numerous private communications.

[102]The set of the algebra's idecomposable, irreducible atomic elements so to speak—its (primitive or prime) ideals.

[103]Earlier in the present paper, we witnessed an instance of *'discrete' Gel'fand Duality*, when we discussed *Sorkin's finitary poset discretisations of locally compact continuous manifolds and their Gel'fand-dual incidence Rota algebras* (cf. Section 1). It must be emphasised here that the adjective '*equivalent*' in the statement of Gel'fand Duality above pertains to *the categorical equivalence (:functorial correspondence) between the category of abelian C^*-algebras and that of (compact) topological spaces.*

For Tasos, the main technical and conceptual essence of the mathematical result above is that '*Space (Geometry/Topology) can be somehow 'derived' or extracted from Algebraic Structure*'.[104] In [49], for example, Tasos goes at great lengths, by using Gel'fand Duality and the so-called *Gel'fand Transform* of an algebra, in delimiting the sort of (topological) algebra sheaves that could be used as 'good' structures sheaves **A** in ADG. As Gel'fand Duality mandates, these are algebra sheaves over the suitably topologised spectrum of the original algebra.

At the same time, at the back of Tasos's mind, back in the day of the mid-90s when he was feverishly searching for solid founding pillars to erect ADG, must have surely been his beloved *Differential Geometry* [49]: that is to say, consciously or unconsciously (and here this author only speculates in retrospect), Tasos must have asked himself:

• *How can one extract differential geometric structure (not just topological) from Algebra (:algebraic structure), thus in a sense emulate Gel'fand duality, but in a differential geometric setting?*

To that end, motivating inspiration must have come to Tasos from another celebrated mathematical result, which also elegantly depicts the categorical duality between Algebra and Geometry: *the Serre-Swan Theorem* [86, 94], to which we briefly turn next.

2. **The Serre-Swan Theorem:** Jean-Pierre Serre's version of the theorem, which to this author is more pertinent to Tasos's original *differential geometric* endeavours and quests, roughly posits that *for every commmutative unital (Noetherian) ring R, then the category of finitely generated projective R-modules (Algebra) is equivalent to the category of algebraic vector bundles $\mathcal{V}$ (i.e., locally free sheaves of structure sheaf R-modules of constant finite rank n) on the Spectrum Spec(R) of R.*

   The alert and astute reader, who is also familiar with the basic rudiments of ADG, must surely speculate that for Tasos, the result above must have come as an 'epiphany moment' in his quest for algebraic structures to model ADG, if one substitutes:

   • '*finitely generated differential A-modules*'[105] for '*finitely generated projective R-modules*' in the Serre-Swan Theorem; and also,

[104] Again, Tasos Mallios in numerous private communications.
[105] Where A here is not just a ring, but is an algebra A over a field ($\mathbb{C}$).

- '*vector sheaves $\mathcal{E}$ (i.e., locally free sheaves of structure sheaf* **A**-*differential modules of constant finite rank n)* ' instead of '*algebraic vector bundles $\mathcal{V}$ (i.e., locally free sheaves of structure sheaf R-modules of constant finite rank n)*'.

In toto, the usual $\mathcal{C}^\infty$-smooth vector bundles $\mathcal{V}$ on which the whole edifice of Classical Differential Geometry (CDG)—the so-called Classical Differential Calculus on Smooth (Differential) Manifolds—and its manifold applications to Modern Physics rest[106] is abstracted and generalised in ADG by *vector sheaves* $\mathcal{E}$ (*i.e.*, locally free **A**-modules of finite rank n) over an in principle arbitrary topological space X [44, 45, 50].[107]

As an additional bonus at this point, the ADG-theoretic extension and generalisation of the Serre-Swan Theorem was the principal move of Tasos in [46, 51, 50] towards arriving at a manifestly functorial geometric (pre)quantisation and second (:field) quantisation scheme for his ADG-theoretic field theory, with concomitant classification of the fields' elementary particle quanta into bosons and fermions, as (32) above depicts.

3.1.2 Tracing the Physical 'Origins' of ADG: Breaking Algebra-Geometry Duality in Favour of an Algebraic Physical Dynamics

The two celebrated mathematical results mentioned above—Gel'fand Duality and the Serre-Swan Theorem—were seen to express a fundamental categorical duality (:functorial equivalence) between Algebra and Geometry and, arguably, they were speculated to be centers of inspiration and motivation for Professor Mallios in developing ADG as a mathematical abstraction, expansion, generalisation and enrichment[108] of the usual CDG on smooth manifolds.

[106] We have empasised throughout our joint work with Tasos [59, 60, 61, 62] that CDG is a special case of, and can be recovered from, ADG when one assumes $\mathbf{A} \equiv \mathcal{C}^\infty(M)$—that is, when one simply assumes copies of the algebra $\mathcal{C}^\infty(M)$ of $\mathcal{C}^\infty$-smooth functions on a differential (spacetime) manifold M as occupying the stalks of the structure sheaf **A** in the theory. Equivalently, it has been recently shown that *the category $\mathcal{M}$ of $\mathcal{C}^\infty$-smooth manifolds is a full subcategory of the category $\mathcal{DT}$ of ADG-theoretic differential triads* [10].

[107] Recall from Section 1 that in our finitary case, the base space on which our finitary sheaves of differential incidence Rota algebras Ω_i are soldered is the very (primitive) spectrum $Spec(\Omega_i)$ of those algebras—a 'discrete' instance of *'Gel'fand Duality meets Serre-Swan'*.

[108] As Tasos originally preferred to call it cumulatively: *an axiomatisation* [44, 45].

In this subsection, we turn our attention to Tasos's fundamental *Physical* intuitions and motivations in applying ADG to what he used to call '*Physical Space(Time)*' and '*Physical Geometry*'.

To that end, we quote directly from [54] the following 'definition' *á-la* Mallios of what he perceived as, and, *in extenso*, what ought to qualify as being, '*Physical Geometry*':

> "*...Physical Geometry is the 'outcome' of the physical laws...*"

For which he then displayed the following 'causal nexus' for *producing Physical Geometry from Dynamical Laws*:[109]

> "...Now, by looking at the technical correspondence/association,

$$\textit{physical law} \longleftrightarrow \mathcal{A} - \textit{connection},$$

> one realizes that [the displayed expression above] might also be construed, as an *equivalent analogue of* the implication;

$$\mathcal{A} - \textit{connection}(:\ \textit{physical law}) \Longrightarrow \textit{curvature}\,(:\ `\textit{geometry}',\ \text{alias},\ `\textit{shaping}')$$

> Consequently, still to repeat the above, but state it otherwise, one concludes that:

It is actually the *physical laws*, that *make*, what we might call (physical) *'geometry'*...."

All of Mallios's prophetic musings above may be subsumed under the following distilled *Fundamental Aphorism*:

> **Fundamental Aphorism:** *Physical Space(Time)' and 'Physical Geometry' is the result or the product of Field Dynamics*, in much the same way that, as we saw earlier in the paper, *the Curvature Field is a Geometric Morphism image of the Connection Field.*

[109] Again, the quotation is borrowed *verbatim* from [49].

In this line of thought, we conclude the present paper by quoting *verbatim* below the very opening paragraph of the wonderful *Exordium* in Mallios and Zafiris's last joint research monograph [67]:[110]

> "*The major aim of the application of Analysis and Differential Geometry in Physics is the setting up of a mechanism providing a precise description of the emergence of geometric spectrums*[111] *due to dynamical interactions, which can be further used for making predictions.* In this manner, *the notion of a geometric spectrum is considered as the outcome of a physical law or more generally of a dynamical interaction of a particular form.* This raises immediately the question if there exists *an approach to physical geometric spectrums that is independent of any coordinate point manifold background, in the sense that it refers directly to the physical relations causing the appearance of these spectrums without the intervention of any ad hoc coordinate choices.* The answer provided to this question in this book is that the theory of differential vector sheaves, that is geometric vector sheaves equipped with a connectivity structure and obeying appropriate cohomological conditions, provides the sought after functorial tool for a universal and natural approach to physical geometric spectrums. The major difference of the proposed approach in comparison to the traditional ones based on classical differential calculus and differential geometry of smooth manifolds consists in the realization that a classical analytic technique is susceptible to a natural background-independent generalization if it is localizable by sheaf-theoretic means. In this case *the technique can be expressed functorially, that is by means of natural transformations of sheaf functors via the machinery of homological algebra. This is of crucial significance for setting up a mechanism describing the emergence of physical geometric spectrums where the notion of background smoothness is inapplicable...*"

[110]Some emphasised parts in the quote above are *our emphasis*.
[111]That is to say, '*geometric spaces*'.

3.2 A Wish for the Future

The whole time that I have known him, *Tasos was always with the underdog, "always supporting and taking sides with the hunted, not the hunter"*,[112] which he kept reminding me on a day-to-day basis. He also constantly urged me to *take risks* and be unconventional, unorthodox and iconoclastic in my research quests and endeavours in Quantum Gravity [80].

Thus, in paying tribute and homage to my wonderful teacher, mentor, friend and immortal companion in our joint Unending Quest, here's one of Tasos's favourite Feynman Quotes:[113]

> "...It is important that we don't all follow the same fashion. We must increase the amount of variety and the only way to do this is to implore you few guys, to take a risk with your own lives so that you will never be heard of again, and go off into the wild blue yonder to see if you can figure it out..."

May his far reaching vision, the breadth of his conceptual perception, the imagination of his mathematical and physical intuition, the depth of his philosophical enquiry, the originality and the unorthodoxy of his approach, as well as the risk and the adventurousness of his research endeavours—all coupled to the priceless legacy that Professor Anastasios Mallios leaves behind—nurture, enrich, motivate and inspire future researchers in Quantum Gravity for years to come!

Addendum I: An Anecdotal Exchange with Professor Mallios and a Conclusion Drawn from It

I would like to share with this forum a private two-part exchange that we enjoyed with Tasos way back in May 1998, actually on the day of my 30th birthday, more than two years before his first 2-volume pitch of *Abstract Differential Geometry: The Geometry of Vector Sheaves* was published by

[112] One of his own 'proverbs of wisdom': "*We are always with the hunted, not the hunter*".

[113] Taken from Richard Feynman's intro to his [19], where he talks about researchers taking the risk and '*going off into the wild blue yonder*' realm, seemingly strange and largely unexplored yet landscape, of Quantum Gravity.

Kluwer Academic Publishers [44]. That exchange on the one hand dovetails perfectly with the two main mathematical results mentioned in the last section[114] that, as I argued, must have played a pivotal role, consciously and/or unconsciously, in motivating and inspiring Tasos to develop ADG in the first place, and on the other, it casts light on Tasos's character as *a pure and young-at-heart, decent human being*.

I had just obtained my Ph.D. [75], with Professor Mallios as one of the external examiners of my thesis, and, very interested in the early developments of Topos Theory (TT) [43], I was naturally attracted by the core mathematical ideas and the working philosophy of one of TT's main early architects: *Alexandre Grothendieck* [28]. I had also just finished reading the first part (:*Fatuity and Renewal*) of Grothendieck's 'autobiographical' manuscript titled *Reaping and Sowing* [29] and we were discussing the wide range and far reaching depth of Alexandre Grothendieck's contributions to Modern Mathematics, especially to the field of Algebraic Geometry, via the introduction and application of novel Homological Algebra (:Category-theoretic) ideas, concepts and technical constructions.

Part 1 of the Exchange: Grothendieck's 'Working Philosophy' I remember I initiated the exchange by telling Tasos that I had just finished reading the first part of the *Récoltes* and made the remark that the gist of Grothendieck's working philosophy in Mathematics was to attain an epoptic—as broad, as general and as abstract—viewpoint of the entire landscape of Mathematics. Tasos agreed and added two crucial ingredients:

1. That Grothendieck, willingly or not, explicitly or not, formally or informally (:intuitively), essentially *axiomatised* Mathematics; and,

2. That Grothendieck used to encounter and stare at the (only apparently) complex and esoteric Mathematics' landscape with *the innocence and the ignorance of a child*.

• He backed the first point by saying that, much in the same vain as Grothendieck, he was planning to call (as he actually did!) his forthcoming work on 'Abstract Differential Geometry: The Geometry of Vector Sheaves' *an Axiomatic Approach to Differential Geometry* [44]. I asked him why did he think that the most epoptic, bird's eyeview of Mathematics could be attained by *Axiomatisation*, and he quoted me Aristotle from Nicomachean Ethics: "*He who*

[114] *Gel'fand Duality and the Serre-Swan theorem.*

can properly define, divide and distinguish is to be considered God" [1]. He said that in Mathematics one can attain the broadest and most-encompassing viewpoint only when one has properly clarified and laid down the fundamental concepts and base axioms.

• He backed his second point by quoting back to me Bertrand Russell:[115] "*Men are born ignorant, not stupid. Education makes them stupid.*". He said that Grothendieck had the gift and ability to *look at the World of Mathematics in innocent and ignorant awe and amazement*, with fresh eyes, not at all conditioned or biased by prior Knowledge or Education, but perhaps more importantly, *by not being afraid of making mistakes*.

I indeed recall reading, a couple of days earlier, from the first part of Grothendieck's *Récoltes* the following telling excerpt:

> "*...Discovery is the privilege of the child. It's the little child that I want to talk about, the child who is not afraid to be wrong, to look silly, to not be serious, to not be like everyone else. He is neither afraid that the things he looks at will have a bad taste, different from what he expects, from what they appear to be, or rather: from what he has already understood them to be. He ignores the unspoken and unwavering consensus that form part of the air we breathe - which all the grown-ups are supposed to know and they do know. God knows (I suppose the grown-ups know well) if there have been any such child, since the dawn of ages!...*
>
> *...The little child discovers the world as he breathes - the ebb and flow of his breath make him welcome the world in its delicate being, and makes him project himself into the world that also welcomes him. The adult can also discover, in those rare moments when he has forgotten his fears and his knowledge, when he looks at things or himself with eyes wide open, eager to know, new eyes - the eyes of a child...*"

Part 2 of the Exchange: Grothendieck's Work With regard to Grothendieck's main contribution to Mathematics, especially in Algebraic Geometry, Tasos maintained that *Grothendieck essentially abstracted and purely algebraicised Algebraic Geometry, and effectively substituted Hard Analysis on 'Rigid' Geometrical (:Arithmetic) Spaces by the more malleable and flexible inherently algebraic concepts and methods of Sheaf Cohomology.*

[115] This is one of Tasos's Top-3 quotes.

With the advent, blossoming and effective manifold applications of ADG to fundamental Mathematics and Physics, I can now draw confidently the following parallel in posthumous honour of Tasos:

What **Grothendieck** did for *Algebraic Geometry*, **Mallios** did for *Differential Geometry.*

Addendum II: Bohr's Poetic and Lexiplastic Imperative in Current and Future Quantum Gravity Research

Preparing 'psychologically' and 'emotionally' the reader for the heuristic, yet technical and rigorous, Glossary that follows in the Appendix next, we recall and borrow almost *verbatim* from [82],[116] nearly two decades later(!), some still significant in our opinion remarks on *the importance of using* '(onomato)poetic language*', as well as novel conceptual (:theoretical/philosophical) and new technical jargon, having manifest practical (:'calculational') implications and import, in our quest for a conceptually sound, philosophically cogent and technically creative and artful Quantum Theory of Gravity* (QG).

Descending to the quantum deep: the 'experience-to-theoretical physics-to-mathematics-to-philosophy-to-poetry' ascension. In QG research, because of the glaring absence of experimental data (in fact, of any prepared and controlled laboratory experiments!)[117] to verify—or more importantly, to falsify(!)—our theories, the theoretical/mathematical physicist finds herself in the fortuitous position of being free to roam in unconstrained, uninhibited theory making, with sole guiding tools 'aesthetic' elements such as conceptual simplicity, economy, symmetry and beauty, backed by mathematical abstraction, generality, rigor and logical consistency. This has been appreciated as early as Dirac [14], who, in trying to reason and evade singularities and unphysical infinities upon trying to quantise the electromagnetic field, implored theoretical physicists to explore and use all the *mathematical*

[116]From the very last section, titled *Poetry in Motion and in Action: the Future of Quantum Gravity Research*.

[117]Although at the same time, we are passive receptors of cosmological data from the early universe.

resources at their disposal, and temporarily divesting experiments of their theory checking and guiding role.

Ludwig Faddeev, for example, maintained fairly recently [18] that we should finally break away from the classical theory-making route followed so far by theoretical physics, which can be schematically represented by the cycle:

experiments $\longrightarrow$ predictions $\longrightarrow$ mathematical formulations $\longrightarrow$ further experiments

and instead employ all our mathematical resources to plough deeper into the foundations of 'physical reality', leaving experiments (and experimentalists!) to 'catch up' with the new mathematics (and with theoreticians!), not the other way around. In this regard, we would like to borrow from [18] some telling remarks made by Dirac from the aforementioned paper [14]:[118]

> **Part I.** "...The steady progress of physics requires for its theoretical foundation a mathematics that gets continually more advanced. This is only natural and to be expected. What, however, was not expected by the scientific workers of the last century was the particular form that the line of advancement of the mathematics would take, namely, it was expected that the mathematics would get more complicated, but would rest on a permanent basis of axioms and definitions, *while actually the modern physical developments have required a mathematics that continually shifts its foundation and gets more abstract...It seems likely that this process of increasing abstraction will continue in the future and that advance in physics is to be associated with a continual modification and generalization of the axioms at the base of mathematics rather than with logical development of any one mathematical scheme on a fixed foundation.*[119]
> **Part II.** There are at present fundamental problems in theoretical physics awaiting solution [...][120] the solution of which problems will presumably require a more drastic revision of our fundamental concepts than any that have gone before. Quite likely these changes will be so great that it will be beyond the power of human intelligence

[118]The quotation below is split into two paragraphs (I and II), on which we comment separately after it.
[119]Our emphasis.
[120]Dirac here mentions a couple of outstanding mathematical physics problems of his times. We have omitted them.

to get the necessary new ideas by direct attempt to formulate the experimental data in mathematical terms. The theoretical worker in the future will therefore have to proceed in a more indirect way. *The most powerful method of advance that can be suggested at present is to employ all the resources of pure mathematics in attempts to perfect and generalise the mathematical formalism that forms the existing basis of theoretical physics, and* after[121] *each success in this direction, to try to interpret the new mathematical features in terms of physical entities*[122]*...*"

- **Part I.** The words from this paragraph to be highlighted with ADG-GT in mind here are: '*a mathematics that gets more abstract*' and '*advance in physics is to be associated with a continual process of abstraction* [leading to a] *modification and generalization of the axioms at the base of mathematics*'. Indeed, the axiomatic ADG essentially involves an abstraction of the fundamental notions of modern differential geometry (*e.g.*, connection), resulting in an entirely algebraic (:sheaf-theoretic) modification and generalization of the latter's basic axioms [44, 45, 50]. And it is precisely this abstract and generalized character of ADG that makes us hope that its application could advance significantly (theoretical) physics, and in particular, QG research.

  For, to quote again Einstein from earlier, in the quantum deep we must look for "*a purely algebraic method for the description of reality*" [16]. [123]

- **Part II.** In this paragraph, apart from breaking from the traditional cycle 'experiment-theory-more experiment' mentioned above (*i.e.*, Dirac's anticipation that '*new ideas* [won't come] *by direct attempts to formulate the experimental data in mathematical terms*'), what should be highlighted is on the one hand Dirac's prompting us '*to generalize the mathematical formalism that forms the existing basis of theoretical physics*', and on the other, '*to try to interpret the new mathematical features in terms of physical entities*'. Again, ADG goes a long way to

[121] Dirac's own emphasis.
[122] Again, our emphasis throughout.
[123] Alas, for Einstein, the continuum spacetime and *in extenso* CDG-based field theory was simply incompatible with the finitistic-algebraic quantum theory [92], a divide that ADG has come a long way to finally bridge [59, 60, 61, 62, 78, 79, 80, 81, 82].

> fulfill Dirac's vision, since *the* (or at least the bigger part of the) mathematics that lies at the heart of current theoretical physics—namely, (the formalism of) *differential geometry* (*i.e.*, the CDG on $\mathcal{C}^\infty$-smooth manifolds)—is abstracted and generalized, while *after* this generalisation has been achieved, the physical application and interpretation (of ADG's novel concepts and features) has been carried out, especially in the theoretical physics' field of quantum gauge theories and gravity research. We believe that this is '*a powerful method of advance*' indeed.

However, *this too is not enough* in our opinion. Existing mathematical concepts, structures and techniques also come hand in hand with implicit assumptions, hidden preconceptions and prejudices associated with their historical development, *i.e.*, with past problems other than QG(!) that they were invented in order to formulate, tackle and (re)solve. Such preconceptions are very hard to forget at the primary stages of theory making, let alone to shed them altogether, especially when they have proved to be experimentally successful in the past. Again Einstein, for example, has given us a warning call regarding our almost religious abiding by old, tried-and-tested concepts [17]:

> "...Concepts which have proved useful for ordering things easily assume so great an authority over us, that we forget their terrestrial origin and accept them as unalterable facts. They then become labelled as 'conceptual necessities', 'a priori situations', etc.[124]The road of scientific progress is frequently blocked for long periods by such errors. It is therefore not just an idle game to exercise our ability to analyze familiar concepts, and to demonstrate the conditions on which their justification and usefulness depend, and the way in which these developed, little by little..."

For this, a few people have suggested to go even a bit further, past mathematics, and into the realm of *Philosophy* to look for novel QG research resources. 't Hooft, for example [96], insists that:

> "*...The problems of quantum gravity are much more than purely technical ones. They touch upon very essential philosophical issues...*"

[124]Think for instance of the apparently fundamental notion of the '*spacetime continuum*': "*time and space are modes by which we think, not conditions in which we live*" (as quoted by Manin in [68]).

For us, this will not suffice either. Philosophy too comes burdened with a host of *a priori* concepts and assumptions.[125]

Paraphrasing Finkelstein in [21], "*in the quantum deep one must travel light*". Alas, perhaps because of a deep psychological tendency towards security (and an instinctive, biological one, for survival [104]), we tend to abide by what we already know and (think) we understand (or believe to have a firm hold of backed by numerous practical applications), and we take few 'conservative risks' (pun intended) towards standing bare, ignorant (but, exactly thanks to this ignorance, uninhibited and unbiased!) before Nature. This primordial fear of the unknown must be overcome—at least it should be soothed by the Socratic stance that, *anyway, the only thing that we know for sure is that we know almost nothing*—and a way of achieving this is by engaging into imaginative, creative *poetic activity* where there is plenty of leeway for 'trial-and-error' and a lot of room for iconoclastic, unorthodox, unconventional and adventurous ideas that are unburdened by ancestral theoretical demands or traditional beaten track conventions.

Indeed, granted that QG pushes us back to theorizing about the archegonal acts of the World, what better means other than poetry (with its analogies, metaphors and allegories) do we possess for exploring, conceptually afresh and without *a priori* commitments—ultimately, to deconstruct and reconstruct anew [74]—the strange,[126] uncharted QG landscape?

Kandinsky's words echo ecophantically here [36]:

"*Poetry brings us closer to the Creator.*"

Especially regarding the unfamiliar realm of the quantum, we read from [69] (reading from [97]):

> "...In the first forty years of the twentieth century, our vision of the physical world changed radically and irretrievably. Atoms could behave like solid matter or like waves, they were made of particles with strange top-like properties, with nuclei which could disintegrate spontaneously, and, perhaps, set up chains of disintegration themselves. For many, the most interesting implication of all this new knowledge

[125] Especially the nowadays academic 'Philosophy of Science' [87, 9], which appears to be heavily (almost paracytically!) dependent on the concepts, techniques, results and current developments in science (and in particular, in theoretical physics and applied mathematics).

[126] 'Strange', of course, relative to what we already (think we) know!

> was, and still is, philosophical. We have understood that our intuitive ideas of what is possible and what is not—our common sense—are a result of the conditioning of our minds by sense-experiences. *We have had to change our ideas of what understanding consists in.*[127] As Bohr said, '*When it comes to atoms, language can only be used as in poetry. The poet, too, is not nearly so concerned with describing facts as with creating images.*'[128] The same is true of cosmological models, curved spaces and exploding universe. *Images and analogies are the keys.*[129] Not you, not I, not Einstein could interpret the universe in terms wholly related to our senses. Not that it is incomprehensible, no. *But we must learn to ignore our preconceptions concerning space, time and matter, abandon the use of everyday language and resort to metaphor. <u>We must try to think like poets</u>...*"[130]

What we have in mind here is that, in order to see and tackle the problem of QG afresh, we must foremost be able to sort of '(re)create it from scratch', forgetting for a while the voluminous body of work—the various theoretical 'evidence' that different approaches to QG provide us with—that has been gathered over the last 70+ years of research on it. The spirit of Feynman comes to mind:

"*What I cannot create, I do not understand.*" [19][131]

Of course, by 'poetry' above all we mean *creation of new conceptual terminology within a novel theoretical and technical framework'*.

In this respect, it is perhaps more important to stress that ADG is not so much a *new* theory of DG—the main '*mathematical formalism that forms the existing basis of theoretical physics*', following Dirac's expression earlier—but a theoretical framework that abstracts, generalises, revises and recasts the existing CDG on differential manifolds by isolating and capitalising on its fundamental, *essentially algebraic* (:'relational', in a Leibnizian sense) features, which are not dependent at all on a background locally Euclidean geometrical 'space(time)' (:manifold). In a way, from the novel viewpoint of ADG, we see 'old' and 'stale' problems (*e.g.*, the $\mathcal{C}^{\infty}$-smooth singularities of

[127]Midgley's emphasis.
[128]Our *emphasis*.
[129]Midgley's emphasis, and mine.
[130]*Emphasis* (and underlining) is all ours.
[131]In the '*Quantum Gravity*' prologue by Brian Hatfield.

the manifold and CDG based GR) with 'new' and 'fresh' eyes [78].

Schopenhauer's words from [85] immediately spring to mind:

> "*...Thus, the task is not so much to see what no one has yet seen, but to think what nobody yet has thought about that which everybody sees*[132]*...*"

On the 'idiosyncratic' terminology side. As we will witness in the Glossary section following next, the novel perspective on gravity that ADG enables us to entertain is inevitably accompanied by *new terminology*. We have thus not refrained from engaging into vigorous poetic, '*lexiplastic*' activity, so that our work in this paper abounds with new, 'idiosyncratic' terms for novel concepts hitherto not encountered in the standard theoretical physics' jargon and literature, such as '*gauge theory of the third kind*', '*third quantisation*', '*synvariance*' and '*autodynamics*', to name a few.

In this respect, we align ourselves with Wallace Stevens's words in [93]:

> "*...Progress in any aspect is a movement through changes in terminology...*"[133]

with the 'changes in terminology' in our case being not just superficial (:formal) 'nominal' ones introduced as it were for 'flash, effect and decor', but necessary ones coming from *a significant change in basic theoretical framework for viewing and actually doing DG in QG*: *from the usual geometrical manifold based one (CDG), to the background manifoldless and purely algebraic (:sheaf-theoretic) one of ADG.*

The bottom line is a verse: a Word for the World. According to the Biblical Genesis, '*In the beginning was the Word*', thus the ultimate task for future QG (re)search is to find the right 'words' to begin our theory making about the very beginning of the World. For, to quote Bohr (as quoted in [4]):

> "...It is wrong to think that the task of physics is to point out how nature is. *Physics concerns what we can say about nature...*"[134]

[132]All emphasis is ours.

[133]Another one of Mallios's favourite quotes.

[134]Our emphasis. What could baffle the reader here is the following apparent oxymoron: while on the one hand we seem to advocate the aforesaid principle of ADG-field realism (maintaining that the connection field $\mathcal{D}$ exists 'out there' independently of us experimenters, measurers/geometers and theoreticians), on the other we endorse Bohr's dictum

As Finkelstein notes,[135]

> "...The fully quantum theory lies somewhere within the theorizing activity of the human race itself, or the subspecies of physicists, regarded as a quantum system. If this is indeed a quantum entity, then the goal of knowing it completely is a Cartesian fantasy, and *at a certain stage in our development we will cease to be law-seekers and become law-makers*.[136]
>
> It is not clear what happens to the concept of a correct theory when we abandon the notion that it is a faithful picture of nature. Presumably, just as the theory is an aspect of our collective life, its truth is an aspect of the quality of our life..."

And what better means other than our Logos—or better, than our imaginative and creative Logos: our poetic and bardic *Mythos*—do we possess for approximating the archegonal Truth about Nature? Moreover, what a humbling thought this is: that in the end we may find out that this truth is the quintessential quality of our ellogous lives. Then, in a Nietzscheic sense [71], *we will have become what we already are: Poets true to our Nature!*

Appendix: Glossary of New Terminology and Heuristic ADG-GT Jargon

In this concluding Appendix to the paper, we outline a Glossary of the novel ADG-theoretic terminology and conceptual heuristics that abound throughout this paper, plus of some that have made recurring appearances throughout our publications in the last two and a half decades [59, 60, 61, 62, 76, 77, 78, 79, 80, 81, 82].

above. Again, there's no paradox here: what *we* can say about Nature (*ie*, in this case, about the field $\mathcal{D}$) is all encoded in the generalised arithmetics $\mathbf{A}$ that *we* choose to represent it (on $\mathcal{E}$). However, the $\mathbf{A}$-functoriality of the dynamics secures the independence of the (dynamics of the) field from our generalized measurements (and hence from our geometrical representations, *eg*, 'spacetime') in $\mathbf{A}$ (and *in extenso* $\mathcal{E}$, which is locally a power of $\mathbf{A}$).

[135] In an early draft of [21] given to this author back in 1993.

[136] For more discussion on this theme, see the section in [80], titled '*The Saviors of Physical Law*', emulating Kazantzakis' "*The Saviors of God*" [37]. Our emphasis.

The Glossary below is listed lexicographically, not in order of importance or frequency of appearance in past papers, and all the items are pre-fixed by '*ADG-*'.

1. **ADG-A-Invariance.** The functorial imperative of ADG that the dynamical laws of physics (:here, Einstein's differential equations and Yang-Mills differential equations) should be respected by (:be 'invariant' under) any of our (choices of) generalised coordinates or measurements (:arithmetics) employed in the structure sheaf $\mathbf{A}$. As we saw in this paper, $\mathbf{A}$-invariance entails local gauge invariance (see below).

2. **ADG-Autodynamics.** The idea that the ADG-fields $\mathcal{F} = (\mathcal{E}, \mathcal{D})$, whether Maxwell, Yang-Mills or Einstein, are dynamically autonomous, 'self-governing', 'unitary', 'holistic', 'self-contained' entities, with no need for externally imposed spacetime parameters or 'degrees of freedom' for their dynamical sustainance.

3. **ADG-C-Algebraized Space.** An in principle arbitrary topological space X endowed with a structure sheaf $\mathbf{A}$ of generalised arithmetics or 'coordinates' localised on it: $(X, \mathbf{A})$.

4. **ADG-Connection and Curvature Field Categories.** As we have the *Maxwell* $\mathcal{T}_{Max} = \{(\mathcal{L}, \mathcal{D}_{Max})\}$, the *Yang-Mills* $\mathcal{T}_{YM} = \{(\mathcal{E}, \mathcal{D}_{YM})\}$, and the *Einstein* category $\mathcal{T}_{Einst} = \{(\mathcal{E}, \mathcal{D}_{Einst})\}$ of ADG-fields $(\mathcal{E}, \mathcal{D})$, we also define three corresponding *ADG-curvature field functor categories*: $\mathcal{C}_{Max}$, $\mathcal{C}_{YM}$ and $\mathcal{C}_{Einst}$, whose objects are ADG-curvature fields as in (24), and whose arrows are *natural transformation* type of correspondences between their $\otimes_{\mathbf{A}}$-functorial objects.

5. **ADG-Connection/Curvature Geometric Morphism.** The pair of adjoint functors $\mathcal{GM}_{\mathbf{A}} := (\otimes_{\mathbf{A}}, \mathrm{Hom}_{\mathbf{A}})$ effectuating functorial, *natural transformation* type of correspondences between the category of ADG-fields and the corresponding category of ADG-curvature fields.

6. **ADG-Curvature Space.** This is defined as the following quintet: $(\mathbf{A}, d, \Omega^1, d, \Omega^2) \equiv (\mathbf{A}, \mathcal{D}, \Omega^1, \mathcal{D}^2, \Omega^2)$, consisting of a differential triad and a $d/\mathcal{D}$-extension of the sheaf Ω^1 of differential 1-forms to a sheaf Ω^2 of differential 2-forms so as to be able to define the curvature of a connection according to (17).

7. **ADG-Curvature Field.** A pair consisting of a structure sheaf $\mathbf{A}_X$ on a $\mathbf{C}$-algebraized space X and the curvature $R(\mathcal{D})$ of an $\mathbf{A}$-connection $\mathcal{D}$ (on a vector sheaf $\mathcal{E}$) acting as an $\mathbf{A}$-morphism: $\mathcal{R} = (\mathbf{A}, R(\mathcal{D}))$.

8. **ADG-Differential Triad.** A triplet consisting of a structure sheaf $\mathbf{A}_X$ on some $\mathbf{C}$-algebraized space X, and a flat $\mathbf{C}$-linear Leibnizian connection d acting as a $\mathbf{C}$-linear sheaf morphism that maps $\mathbf{A}_X$ to a sheaf Ω of differential $\mathbf{A}$-modules on X: $\mathfrak{T} = (\mathbf{A}_X, d, \Omega(X))$

9. **ADG-Field.** A pair consisting of a vector sheaf $\mathcal{E}$ and an $\mathbf{A}$-connection $\mathcal{D}$ on it acting as a $\mathbf{C}$-linear Leibnizian sheaf morphism: $\mathcal{F} = (\mathcal{E}, \mathcal{D})$.

10. **ADG-Field Quantal Self-Duality.** The basic intuitive-heuristic observation that for any ADG-field $\mathcal{F} = (\mathcal{E}, \mathcal{D})$, (the local sections of) $\mathcal{E}$ represent(s) some abstract kind of local quantum particle 'position' states, while the action of the connection field $\mathcal{D}$ on them represents some kind of generalised 'momentum' type of action. The two structures are said to be 'quantum complementary' aspects of the 'unitary' and 'coherent' ADG-field in the sense that they obey some abstract (sheaf cohomological) commutation Heisenberg uncertainty relations which define 3rd Quantisation in our scheme.

11. **ADG-Field Solipsism/Monadology.** The idea that the ADG-fields $\mathcal{F} = (\mathcal{E}, \mathcal{D})$ are the sole dynamical entities (:variables) in our theory—the sole *physical* entities in our ADG-GT—without any 'spacetime realm and reality' external to and separate from them. In this sense, the ADG-Field Solipsism is tantamount to *the ADG-Field Pure Realism*,[137] namely, that the auto-dynamical, self-governing and self-transforming physical laws that the ADG-fields define and obey in-themselves are invariant no matter what, independently of what, structure group sheaf $\mathbf{A}$ of generalised arithmetics—however reticular, pathological or singular—we employ to localise, coordinatise or 'measure' them. In this sense,*the ADG-fields are 'physically real' entities* [61, 62, 79, 81, 82]. This is another manifestation of Mallios's Principle of $\mathbf{A}$-Invariance.

12. **ADG-Gauge Theory of the 3rd Kind.** The idea that the ADG-field dynamics remains invariant under the 'gauge' group of dynamical

[137]We borrow the Tractarean idea of Ludwig Wittgenstein from [107], that: "*Solipsism coincides with Pure Realism*".

self-transmutations $\mathcal{A}ut_{\mathbf{A}}\mathcal{E}$ of the ADG-fields $\mathcal{F} = (\mathcal{E}, \mathcal{D})$. It follows that *all symmetries and invariances of the ADG-field dynamics are internal to* (*i.e.*, happen within) *the fields themselves*, without recourse or reference to an external (:background) spacetime manifold. There is no external (:spacetime) versus internal (:gauge) symmetries' distinction in our theory. *All transformations are pure gauge transformations, in the sense that they are changes in the generalised coordinate gauges (:arithmetics) in* $\mathbf{A}$, without reference to an external spacetime.

13. **ADG-Gel'fand Duality.** The general idea that 'differentiable space' comes from the structure sheaf $\mathbf{A}$ of our generalised arithmetics.

14. **ADG-Geometric Space.** The general idea that 'the geometry of physical space' comes from 'algebraic (:relational) dynamics' obeyed by, the dynamical relations between, the ADG-fields.

15. **ADG-Natural Transformation.** This pertains to the *Natural Transformation* character of the fundamental *Geometric Morphism* $\mathcal{GM}_{\mathbf{A}} := (\otimes_{\mathbf{A}}, \mathrm{Hom}_{\mathbf{A}})$ between the corresponding functor categories of ADG-Connection Fields and ADG-Curvature Fields (or the ADG-Curvature Spaces that the latter define). This is another expression of Mallios's Principle of $\mathbf{A}$-Invariance.

16. **ADG-Principle of A-Algebraic Relativity of Differentiability.** Since *all differentiability in ADG derives from the structure sheaf* $\mathbf{A}$ *of algebras of arithmetics or 'generalised coordinates'*, different choices of $\mathbf{A}$ entail different 'differential geometric mechanisms' (:'Calculus'), but the dynamical laws of Nature—the very differential equations that can be formulated via that differential geometric mechanism that these $\mathbf{A}$s define—remain invariant under them. This is yet another expression of Mallios's Principle of $\mathbf{A}$-Invariance.

17. **ADG-Synvariance.** The ADG-theoretic analogue of (General) Covariance in accord with ADG-Autodynamics above; namely, that in much the same way that $\mathrm{Diff}(M)$—the group of active diffeomorphisms of the 'external' base spacetime manifold of GR—represents the Principle of General Covariance (PGC) of GR, $\mathcal{A}ut_{\mathbf{A}}\mathcal{E}$—the group of $\mathbf{A}$-automorphisms of the vector sheaf $\mathcal{E}$—represents the invariance group of dynamical self-transmutations of the Einstein ADG-field $\mathcal{F}_{Einst} = (\mathcal{E}, \mathcal{D}_{Einst})$. *Mutatits mutandis* then for $\mathcal{F}_{Max}$ and $\mathcal{F}_{YM}$.

18. **ADG-'Unitary' Quantal Gauge Field.** This pertains to a tetrad of functorially and dynamically closely entwined structures $\mathbf{U} := (\mathcal{E}, \mathcal{D}, \mathcal{A}ut_{\mathbf{A}}\mathcal{E}, \mathcal{Q})$, and it subsumes under a single coherent and inseparable 'unitary whole' all the four most important functorial structural traits of ADG-GT, namely: *'local quantum particle states' represented by local sections of a vector sheaf $\mathcal{E}$, their 'dual-complementary' functorial ADG-gauge field dynamics generated by an algebraic* $\mathbf{A}$*-connection $\mathcal{D}$, the latter's local gauge invariance of the 3rd kind encoded in the principal structure sheaf $\mathcal{A}ut_{\mathbf{A}}\mathcal{E}$ of $\mathcal{E}$'s automorphisms, and the dual particle-field canonical-type of 3rd quantisation, represented by the functorial morphism $\mathcal{Q}$ between the relevant sheaf categories involved.*

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