G-COMPLETE REDUCIBILITY AND SATURATION

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Abstract. Let $H \subseteq G$ be connected reductive linear algebraic groups defined over an algebraically closed field of characteristic $p > 0$. In our first principal theorem we show that if a closed subgroup K of H is H-completely reducible, then it is also G-completely reducible in the sense of Serre, under some restrictions on p, generalising the known case for $G = \mathrm{GL}(V)$. Our second main theorem shows that if K is H-completely reducible, then the saturation of K in G is completely reducible in the saturation of H in G (which is again a connected reductive subgroup of G), under suitable restrictions on p, again generalising the known instance for $G = \mathrm{GL}(V)$. We also study saturation of finite subgroups of Lie type in G. We show that saturation is compatible with standard Frobenius endomorphisms, and we use this to generalise a result due to Nori from 1987 in case $G = \mathrm{GL}(V)$.

1. Introduction and main results

Let G be a connected reductive linear algebraic group over an algebraically closed field k of characteristic $p > 0$. Let H be a closed subgroup of G. Following Serre [20], we say that H is *G-completely reducible* (G-cr for short) provided that whenever H is contained in a parabolic subgroup P of G, it is contained in a Levi subgroup of P. Further, H is *G-irreducible* (G-ir for short) provided H is not contained in any proper parabolic subgroup of G at all. Clearly, if H is G-irreducible, it is trivially G-completely reducible; for an overview of this concept see [3], [19] and [20]. Note in case $G = \mathrm{GL}(V)$ a subgroup H is G-cr exactly when V is a semisimple H-module and it is G-ir precisely when V is an irreducible H-module. The same equivalence applies to $G = \mathrm{SL}(V)$.

The notion of G-complete reducibility is a powerful tool for investigating the subgroup structure of G. Any connected G-completely reducible subgroup H of G is reductive [20, Prop. 4.1]. The converse can fail in small characteristic, but it is true if p is sufficiently large. To be precise, we have the following theorem due to Serre.

Theorem 1.1 ([20, Thm. 4.4])**.** *Suppose $p \geq a(G)$ and $(H : H^\circ)$ is prime to p. Then H° is reductive if and only if H is G-completely reducible.*

Here the invariant $a(G)$ of G is defined as follows [20, §5.2]. For G simple, set $a(G) = \mathrm{rk}(G)+1$, where $\mathrm{rk}(G)$ is the rank of G. For G reductive, let $a(G) = \sup(1, a(G_1), \dots, a(G_r))$, where $G_1, \dots, G_r$ are the simple components of G. In the special case $G = \mathrm{GL}(V)$ we have $a(G) = \dim(V)$, and a subgroup H of G is G-cr if and only if V is a semisimple H-module. We recover a basic result of Jantzen [15, Prop. 3.2]: if $\rho\colon H \to \mathrm{GL}(V)$ is a representation of a connected reductive group H and $p \geq \dim(V)$ then ρ is completely reducible.

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In this paper we explore variations of Theorem 1.1 for non-connected reductive subgroups K of G when $(K : K^\circ)$ is not prime to p. One cannot expect Theorem 1.1 to carry over completely, even with extra restrictions on p: for instance, a non-trivial finite unipotent subgroup of G can never be G-cr, [20, Prop. 4.1]. Here is a version which does work.

Theorem 1.2 ([2, Thm. 1.3])**.** *Let H be a connected reductive subgroup of $\mathrm{GL}(V)$ and let K be a closed subgroup of H. Suppose $p \geq \dim(V)$. If K is H-completely reducible then K is $\mathrm{GL}(V)$-completely reducible.*

We define an invariant $d(G)$ of G as follows. For G simple let $d(G)$ be as follows:

G	A_n $(n \geq 1)$	B_n $(n \geq 3)$	C_n $(n \geq 2)$	D_n $(n \geq 4)$	E_6	E_7	E_8	F_4	G_2
$d(G)$	$n+1$	$2n+1$	$2n$	$2n$	27	56	248	26	7

For G reductive, let $d(G) = \sup(1, d(G_1), \ldots, d(G_r))$, where $G_1, \ldots, G_r$ are the simple components of G. For G simple and simply-connected and p good for G, $d(G)$ is the minimal possible dimension of a non-trivial irreducible G-module.

Our first result extends Theorem 1.2 from $\mathrm{GL}(V)$ to arbitrary connected reductive G.

Theorem 1.3. *Let $H \subseteq G$ be connected reductive groups and let K be a closed subgroup of H. Suppose $p \geq d(G)$. If K is H-completely reducible, then K is G-completely reducible.*

We note that Theorems 1.2 and 1.3 are false without the bound on p, e.g. see [3, Ex. 3.44] or Example 3.2.

Remark 1.4. Note that $d(G) \geq a(G)$. Thus if K is H-cr, so that K° is reductive, and if the index $(K : K^\circ)$ is prime to p, then Theorem 1.3 follows from Theorem 1.1 under the weaker bound $p \geq a(G)$. Thus Theorem 1.3 is only of interest when $(K : K^\circ)$ is not prime to p. For an application in such an instance, see Corollary 1.6.

Recall that a *Steinberg endomorphism* of G is a surjective morphism $\sigma : G \to G$ such that the corresponding fixed point subgroup $G_\sigma := \{g \in G \mid \sigma(g) = g\}$ of G is finite. The latter are the *finite groups of Lie type*, see Steinberg [24] for a detailed discussion. The set of all Steinberg endomorphisms of G is a subset of the set of all isogenies $G \to G$ (see [24, 7.1(a)]) that encompasses in particular all (generalised) Frobenius endomorphisms, i.e. endomorphisms of G some power of which are Frobenius endomorphisms corresponding to some $\mathbb{F}_q$-rational structure on G.

The following is an immediate consequence of [23, III 1.19(a)] and [6, Thm. 1.3].

Lemma 1.5. *Let σ be a Steinberg endomorphism of G. Then G_σ is G-irreducible.*

For a Steinberg endomorphism σ of G and a connected reductive σ-stable subgroup H of G, σ is also a Steinberg endomorphism for H with finite fixed point subgroup $H_\sigma = H \cap G_\sigma$, [24, 7.1(b)]. Thus, H_σ is H-ir, thanks to Lemma 1.5. The next result then follows from Theorem 1.3.

Corollary 1.6. *Let $H \subseteq G$ be connected reductive groups. Let $\sigma \colon G \to G$ be a Steinberg endomorphism that stabilises H. Suppose $p \geq d(G)$. Then the fixed point subgroup H_σ is G-completely reducible.*

Note that Corollary 1.6 is false without the bound on p. See Example 3.2 for an instance, when H is G-cr but H_σ is not, where $p = 3 < 8 = d(G)$.

Our next result gives a particular set of conditions on H_σ to guarantee that H_σ and H belong to the same parabolic and the same Levi subgroups of G. Note that, if $\sigma\colon H \to H$ is a Steinberg endomorphism of H, then σ stabilises a maximal torus of H, [24, Cor. 10.10]. Also, for S a torus in G, we have $C_G(S) = C_G(s)$ for some $s \in S$, see [8, III Prop. 8.18].

Proposition 1.7. *Let $H \subseteq G$ be connected reductive groups. Let $\sigma\colon G \to G$ be a Steinberg endomorphism that stabilises H and a maximal torus T of H. Suppose*

(i) *$C_G(T) = C_G(t)$, for some $t \in T_\sigma$, and*
(ii) *H_σ meets every T-root subgroup of H non-trivially.*

Then H_σ and H belong to the same parabolic and the same Levi subgroups of G. In particular, H is G-completely reducible if and only if H_σ is G-completely reducible; similarly, H is G-irreducible if and only if H_σ is G-irreducible.

In the presence of the conditions in Proposition 1.7 we can improve the bound in Corollary 1.6 considerably; the following is immediate from Theorem 1.1 and Proposition 1.7.

Corollary 1.8. *Suppose G, H and σ satisfy the hypotheses of Proposition 1.7. Suppose in addition that $p \geq a(G)$. Then H_σ is G-completely reducible.*

Note that condition (ii) in Proposition 1.7 is automatically satisfied provided σ induces a standard Frobenius endomorphism on H. In that case Example 3.1 below demonstrates that condition (i) above does hold generically. Nevertheless, Example 3.2 shows that Proposition 1.7 is false in general without condition (i) even when part (ii) is fulfilled.

Thanks to [20, Prop. 3.2], Levi subgroups L of G have the property that any subgroup H of L is L-cr if and only if it is G-cr. More generally, owing to [3, Thm. 3.26], assuming that p is good for G, a regular connected reductive subgroup (i.e. a subgroup normalised by a maximal torus of G) also satisfies this property. In our next result, we show that *saturated* subgroups of G also share this feature, which is a consequence of a theorem of Serre, see Theorem 4.12. For basics on the concept of saturation, see §4.

Proposition 1.9. *Let $p \geq h(G)$. Let $K \subseteq H$ be closed subgroups of G with H connected reductive and saturated in G. Then K is H-completely reducible if and only if K is G-completely reducible.*

Here the invariant $h(G)$ denotes the upper bound of the *Coxeter numbers* of the simple quotients of G, [20, (5.1)]. Recall that if G is simple, we have $h(G) + 1 = \dim(G)/\operatorname{rk}(G)$; the values of $h(G)$ for the various Dynkin types are as follows:

G	A_n $(n \geq 1)$	B_n, C_n $(n \geq 2)$	D_n $(n \geq 4)$	E_6	E_7	E_8	F_4	G_2
$h(G)$	$n+1$	$2n$	$2n-2$	12	18	30	12	6

We thus have

$$a(G) \leq h(G) \leq d(G)$$

for any reductive G.

Our second main result is a consequence of Theorem 1.3 in the context of saturation. It was derived in [2, Cor. 4.2] in the special case when $G = \mathrm{GL}(V)$. Here H^{sat} denotes the saturation of H in G, see Definition 4.4.

Theorem 1.10. *Let G, H and K be as in Theorem 1.3. Suppose $p \geq d(G)$. If K is H-completely reducible then K^{sat} is H^{sat}-completely reducible and K is G-completely reducible.*

Of particular interest in Theorem 1.10 is the case when $K = H_\sigma$ for a Frobenius endomorphism σ of G. The next result is immediate from Lemma 1.5 and Theorem 1.10.

Corollary 1.11. *Let $H \subseteq G$ be connected reductive groups. Suppose $p \geq d(G)$. Let $\sigma\colon G \to G$ be a Steinberg endomorphism that stabilises H. Then $(H_\sigma)^{\mathrm{sat}}$ is H^{sat}-completely reducible.*

We can improve the bound on p in the last corollary at the expense of imposing the conditions from Proposition 1.7, as follows.

Corollary 1.12. *Suppose G, H and σ satisfy the hypotheses of Proposition 1.7. Suppose in addition that $p \geq h(G)$. Then $(H_\sigma)^{\mathrm{sat}}$ is H^{sat}-completely reducible.*

Proof. Since $p \geq h(G) \geq a(G)$, H is G-cr, by Theorem 1.1. Thus H_σ is G-cr, by Proposition 1.7. The result now follows from Corollary 4.14 below. □

Note that in Theorem 1.10 and Corollaries 1.11 and 1.12 H^{sat} is again connected reductive. This follows from the fact that $d(G) \geq h(G) \geq a(G)$, Theorems 1.1, 4.12 and Remark 4.13.

Example 3.1 below shows that generically the conditions of Corollary 1.12 are fulfilled. Nevertheless, Example 3.2 and Corollary 4.14 show that Corollary 1.12 is false if condition (i) of Proposition 1.7 is not satisfied. In the settings of Corollaries 1.11 and 1.12, $((H_\sigma)^{\mathrm{sat}})^\circ$ is reductive.

In the context of saturation we also present a generalization of a theorem due to Nori [17, Thm. B(2)], see Theorem 6.1(ii).

Theorem 1.13. *Suppose G is simple. Let $p \geq h(G)$. Let σ be a standard Frobenius endomorphism of G and let H be a connected reductive, σ-stable, and saturated subgroup of G. Then $(H_\sigma)^{\mathrm{sat}} = H$.*

Note that [17, Thm. B(2)] is the counterpart of Theorem 1.13 for $G = \mathrm{GL}_n$ and $\sigma = \sigma_p$ the standard Frobenius endomorphism of G raising the matrix coefficients to the p^{th} power.

Theorem 1.3 and Proposition 1.7 are proved in Section 3. Results on saturation are treated in Section 4. We prove a key technical result (Proposition 4.8), which says that saturation is compatible with standard Frobenius endomorphisms. We also prove Theorem 1.10. Section 5 then explores the connection between saturation and the concept of a semisimplification of a subgroup of G from [5]. Finally, in Section 6 we study saturation of finite subgroups of Lie type of G. Here we derive Theorem 1.13 among other results.

2. Preliminaries

Throughout, we work over an algebraically closed field k of characteristic $p \geq 0$. All affine varieties are considered over k and are identified with their k-points.

A linear algebraic group H over k has identity component H°; if $H = H^\circ$, then we say that H is *connected*. We denote by $R_u(H)$ the *unipotent radical* of H; if $R_u(H)$ is trivial, then we say H is *reductive*.

Throughout, G denotes a connected reductive linear algebraic group over k. All subgroups of G that are considered are closed.

2.1. **Good and very good primes.** Suppose G is simple. Fix a Borel subgroup B of G containing a maximal torus T. Let $\Phi = \Phi(G,T)$ be the root system of G with respect to T, let $\Phi^+ = \Phi(B,T)$ be the set of positive roots of G, and let $\Sigma = \Sigma(G,T)$ be the set of simple roots of the root system Φ of G defined by B. For $\beta \in \Phi^+$ write $\beta = \sum_{\alpha\in\Sigma} c_{\alpha\beta}\alpha$ with $c_{\alpha\beta} \in \mathbb{N}_0$. A prime p is said to be *good* for G if it does not divide $c_{\alpha\beta}$ for any α and β. A prime p is said to be *very good* for G if p is a good prime for G and in case G is of type A_n, then p does not divide $n+1$. For G reductive p is *good* (*very good*) for G if p is good (very good) for every simple component of G.

2.2. **Limits and parabolic subgroups.** Let $\phi : k^* \to X$ be a morphism of algebraic varieties. We say $\lim\limits_{a\to 0}\phi(a)$ exists if there is a morphism $\hat{\phi} : k \to X$ (necessarily unique) whose restriction to k^* is ϕ; if the limit exists, then we set $\lim\limits_{a\to 0}\phi(a) = \hat{\phi}(0)$. As a direct consequence of the definition we have the following:

Remark 2.1. If $\phi : k^* \to X$ and $h : X \to Y$ are morphisms of varieties and $x := \lim\limits_{a\to 0}\phi(a)$ exists then $\lim\limits_{a\to 0}(h\circ\phi)(a)$ exists, and $\lim\limits_{a\to 0}(h\circ\phi)(a) = h(x)$.

For an algebraic group G we denote by $Y(G)$ the set of cocharacters of G. For $\lambda \in Y(G)$ we define $P_\lambda := \{g \in G \mid \lim\limits_{a\to 0}\lambda(a)g\lambda(a)^{-1} \text{ exists}\}$.

Lemma 2.2 ([3, Lem. 2.4])**.** *Given a parabolic subgroup P of G and any Levi subgroup L of P, there exists a $\lambda \in Y(G)$ such that the following hold:*

(i) $P = P_\lambda$.
(ii) $L = L_\lambda := C_G(\lambda(k^*))$.
(iii) *The map $c_\lambda : P_\lambda \to L_\lambda$ defined by*

$$c_\lambda(g) := \lim_{a\to 0}\lambda(a)g\lambda(a)^{-1}$$

is a surjective homomorphism of algebraic groups. Moreover, L_λ is the set of fixed points of c_λ and $R_u(P_\lambda)$ is the kernel of c_λ.

Conversely, given any $\lambda \in Y(G)$ the subset P_λ defined above is a parabolic subgroup of G, L_λ is a Levi subgroup of P_λ and the map c_λ as defined in (iii) has the described properties.

2.3. **G-complete reducibility, products and epimorphisms.** Let $f : G_1 \to G_2$ be a homomorphism of algebraic groups. We say that f is *non-degenerate* provided $(\ker f)^\circ$ is a torus, cf. [20, Cor. 4.3]. In particular, f is non-degenerate if f is an isogeny.

Lemma 2.3 (cf. [3, Lem. 2.12])**.** *Let G_1 and G_2 be reductive groups.*

(i) *Let H be a closed subgroup of $G_1 \times G_2$. Let $\pi_i : G_1 \times G_2 \to G_i$ be the canonical projection for $i = 1, 2$. Then H is $(G_1 \times G_2)$-cr if and only if $\pi_i(H)$ is G_i-cr for $i = 1, 2$.*
(ii) *Let $f : G_1 \to G_2$ be an epimorphism. Let H_1 and H_2 be closed subgroups of G_1 and G_2, respectively.*
  (a) *If H_1 is G_1-cr, then $f(H_1)$ is G_2-cr.*
  (b) *If f is non-degenerate, then H_1 is G_1-cr if and only if $f(H_1)$ is G_2-cr, and H_2 is G_2-cr if and only if $f^{-1}(H_2)$ is G_1-cr.*

2.4. G**-complete reducibility, separability and reductive pairs.** Now we consider the interaction of subgroups of G with the Lie algebra $\operatorname{Lie} G = \mathfrak{g}$ of G. Much of this material is taken from [3].

Definition 2.4 ([3, Def. 3.27])**.** For a closed subgroup H of G, we always have $\operatorname{Lie}(C_G(H)) \subseteq \mathfrak{c}_{\mathfrak{g}}(H)$. In case of equality (that is, if the scheme-theoretic centralizer of H in G is smooth), we say that H is *separable in* G; else H is *non-separable in* G.

Of central importance is the following observation.

Example 2.5 ([3, Ex. 3.28])**.** Any closed subgroup H of $G = \mathrm{GL}(V)$ is separable in G. For separability means precisely that the centralizers of H in $\mathrm{GL}(V)$ and in $\operatorname{Lie}\mathrm{GL}(V)$ have the same dimension. For $\mathrm{GL}(V)$ this holds, because the centralizer of H in $\mathrm{GL}(V)$ is the open subset of invertible elements of the centralizer of H in $\operatorname{Lie}\mathrm{GL}(V) \cong \operatorname{End} V$.

Remark 2.6 ([3, Rem. 3.32]). The terminology in Definition 2.4 is motivated as follows. Suppose that H is topologically generated by $x_1, \dots, x_n$ in G. Then the orbit map

$$G \to G \cdot (x_1, \dots, x_n)$$

is separable if and only if

$$\mathfrak{c}_{\mathfrak{g}}(H) = \mathfrak{c}_{\mathfrak{g}}(\{x_1, \dots, x_n\}) = \operatorname{Lie} C_G((x_1, \dots, x_n)) = \operatorname{Lie} C_G(H)$$

(cf. [8, Prop. 6.7]), i.e., if and only if H is separable in G.

Definition 2.7. Following Richardson [18], we call (G, H) a *reductive pair* provided H is a reductive subgroup of G and $\operatorname{Lie} G$ decomposes as an H-module into a direct sum

$$\operatorname{Lie} G = \operatorname{Lie} H \oplus \mathfrak{m},$$

where H acts via the adjoint action Ad_G.

For a list of examples of reductive pairs we refer to P. Slodowy's article [21, I.3]. For further examples, see [3, Ex. 3.33, Rem. 3.34].

Theorem 2.8 ([3, Thm. 3.35])**.** *Suppose that (G, H) is a reductive pair. Let K be a closed subgroup of H such that K is a separable subgroup of G. If K is G-completely reducible, then it is also H-completely reducible.*

Example 2.5 shows that the separability hypothesis is automatically satisfied for the case $G = \mathrm{GL}(V)$. We obtain an immediate consequence of Theorem 2.8, which is in the spirit of a result of Serre [20, Thm. 5.4].

Corollary 2.9 ([3, Cor. 3.36])**.** *Suppose that $(\mathrm{GL}(V), H)$ is a reductive pair and K is a closed subgroup of H. If V is a semisimple K-module, then K is H-completely reducible.*

In our next example we look at the special case of the adjoint representation.

Example 2.10 ([3, Ex. 3.37])**.** Let H be a simple group of adjoint type and let $G = \mathrm{GL}(\operatorname{Lie} H)$. We have a symmetric non-degenerate Ad-invariant bilinear form on $\operatorname{Lie} G \cong \operatorname{End}(\operatorname{Lie} H)$ given by the usual trace form and its restriction to $\operatorname{Lie} H$ is just the Killing form of $\operatorname{Lie} H$. Since H is adjoint and Ad is a closed embedding, $\mathrm{ad} : \operatorname{Lie} H \to \operatorname{Lie} \mathrm{Ad}(H)$ is surjective. Thus it follows from the arguments in [3, Rem. 3.34] that if the Killing form of $\operatorname{Lie} H$ is non-degenerate, then (G, H) is a reductive pair.

Suppose first that H is a simple classical group of adjoint type and $p > 2$. The Killing form is non-degenerate for $\mathfrak{sl}(V)$, $\mathfrak{so}(V)$, or $\mathfrak{sp}(V)$ if and only if p does not divide $2 \dim V$, $\dim V - 2$, or $\dim V + 2$, respectively, cf. [11, Ex. Ch. VIII, §13.12]. In particular, for H adjoint of type A_n, B_n, C_n, or D_n, the Killing form is non-degenerate if $p > 2$ and p does not divide $n+1$, $2n-1$, $n+1$, or $n-1$, respectively.

Now suppose that H is a simple exceptional group of adjoint type. If p is good for H, then the Killing form of $\operatorname{Lie} H$ is non-degenerate; this was noted by Richardson (see [18, §5]). Thus if p satisfies the appropriate condition, then $(\mathrm{GL}(\operatorname{Lie} H), H)$ is a reductive pair and Corollary 2.9 applies.

Theorem 2.11. *For a simply connected simple algebraic group G in characteristic $p \geq 0$, consider the following conditions:*

(i) *$(\mathrm{GL}(V), G)$ is a reductive pair, where V is an untwisted irreducible G-module of least dimension;*
(ii) *$(\mathrm{GL}(V), \rho(G))$ is a reductive pair, for some irreducible representation $\rho : G \to \mathrm{GL}(V)$ with central kernel;*
(iii) *p is very good for G;*
(iv) *$(\mathrm{GL}(\mathrm{Lie}(G)), \mathrm{Ad}(G))$ is a reductive pair;*
(v) *the Killing form on $\mathrm{Lie}(G)$ is non-degenerate;*
(vi) *p is very good for G and, if G has classical type, then $p \nmid e(G)$ as follows:*

G	A_n	B_n	C_n	D_n
$e(G)$	2	$2n-1$	$n+1$	$n-1$

Then (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftarrow$ (iv) $\Leftarrow$ (v) $\Leftrightarrow$(vi).

For G of exceptional type, all these conditions are equivalent.

Proof. It is clear that (i) implies (ii). If (ii) holds, then every subgroup of $\rho(G)$ is separable, since every subgroup of $\mathrm{GL}(V)$ is separable, by Example 2.5, and this descends through reductive pairs. This means that p is pretty good for $\rho(G)$, cf. [14, Def. 2.11], which is the same as p being very good for $\rho(G)$ since G is simple. Note that p being very good is insensitive to the isogeny type of G, so (ii) implies (iii).

Next, the implication (iii) $\Rightarrow$ (i) for G of type B_n, C_n and D_n is [18, Lem. 5.1]. For $\mathrm{SL}(V)$, it is well-known that the traceless matrices $\mathfrak{sl}(V)$ and the scalar matrices $\mathfrak{c}(\mathfrak{gl}(V))$ are the only proper, nonzero $\mathrm{GL}(V)$-submodules (and $\mathrm{SL}(V)$-submodules) of $\mathfrak{gl}(V)$. Now (iii) means that p is coprime to $\dim V$, which implies that these submodules intersect trivially, so that $\mathfrak{sl}(V)$ is complemented by $\mathfrak{c}(\mathfrak{gl}(V))$. So (i) holds in type A_n.

It remains to consider the exceptional cases $(G, V) = (G_2, V_7)$, (F_4, V_{26}), (E_6, V_{27}), (E_7, V_{56}) and (E_8, V_{248}) where V_j is the appropriate minimal dimensional module in each case. Now $\mathrm{Lie}(\mathrm{GL}(V)) = V_j \otimes V_j^*$ contains $\mathrm{Lie}(G)$ as a submodule, which is irreducible since (iii) holds. The known weights of low-dimensional irreducible G-modules, given for instance in [16], now allow one to calculate the weights of $V_j \otimes V_j^*$ and determine its G-composition factors. In each case, it transpires that $\mathrm{Lie}(G)$ is in fact the unique composition factor with a particular highest weight. Since $\mathrm{Lie}(G)$ is a submodule and $V_j \otimes V_j^*$ is self-dual, $\mathrm{Lie}(G)$ also occurs as a quotient of this, and the kernel of this quotient map is then a complement to $\mathrm{Lie}(G)$, so that (i) indeed holds. We illustrate the details in case G is of type G_2 in Remark 2.12(iv).

This shows that (i), (ii) and (iii) are equivalent. Now it is clear that (iv) $\Rightarrow$ (ii). The equivalence of (v) and (vi) follows from Example 2.10. If (v) holds then the Killing form on

Lie(G) is, up to a non-zero scalar, the restriction of the trace form on Lie(GL(Lie(G))), and hence Lie(G) has an orthogonal complement, so that (iv) holds. Also, (iii) coincides with (vi) when G has exceptional type, which shows that all the conditions are equivalent in this case. $\square$

Remarks 2.12. (i). For exceptional groups in very good characteristic, one can also check, just as in the case of the minimal module, that Lie(G) is the unique G-composition factor of Lie(G) $\otimes$ Lie(G)* having a particular high weight, so that (being a submodule) it is a direct summand; this gives an alternative direct proof that (iii) $\Rightarrow$ (iv) in Theorem 2.11 for exceptional G.

(ii). For type A_n when $p = 2 \nmid n+1$, so that Lie(G) is simple and self-dual but the Killing form is degenerate, evidence suggests that $(\mathrm{GL}(\mathrm{Lie}(G)), G)$ is nevertheless a reductive pair. For instance, if $n = 2$ or 4, the module $V = \mathrm{Lie}(\mathrm{SL}_{n+1})$ appears with multiplicity 2 as a direct summand of $V \otimes V^*$; in fact

$$V \otimes V^* \cong V \oplus V \oplus W$$

for some indecomposable module W. If the first summand is the Lie algebra itself, then the kernel of the natural map $(V \otimes V^*)/V^{\perp} \to V^*$ is the sum of this and W.

(iii). For types B_n, C_n, D_n with p odd but dividing $2n-1$, $n+1$, $n-1$ respectively, considering such instances in rank up to 6 suggests that $(\mathrm{GL}(\mathrm{Lie}(G)), G)$ is never a reductive pair. In each case, it transpires that $\mathrm{Hom}(V, V \otimes V^*)$ is 1-dimensional. This gives a unique submodule isomorphic to V, which turns out to lie in a self-dual indecomposable direct summand of $V \otimes V^*$ which also has V as its head.

(iv). To illustrate the argument in the proof above, when G has type G_2 and $p \neq 2, 3$ the G-module V_7 is irreducible of highest weight λ_2, and the 49-dimensional module $V_7 \otimes V_7^*$ has high weights 0, λ_1, λ_2 and $2\lambda_2$ when $p \neq 7$; or 0, 0, λ_1, λ_2 and $2\lambda_2$ when $p = 7$. In either case, we find a unique composition factor of high weight λ_1, which is Lie(G).

3. Proofs of Theorem 1.3 and Proposition 1.7

Proof of Theorem 1.3. Let $\pi\colon G \to G/Z(G)^\circ$ be the canonical projection. Owing to Lemma 2.3(ii)(b), we can replace G with $G/Z(G)^\circ$, so without loss we can assume that G is semisimple. Let $G_1, \ldots, G_r$ be the simple factors of G. Multiplication gives an isogeny from $G_1 \times \cdots \times G_r$ to G. Again by Lemma 2.3(ii)(b), we can replace G with $G_1 \times \cdots \times G_r$, so we can assume G is the product of its simple factors. By Lemma 2.3(i) it is thus enough to prove the result when G is simple and simply connected. Of course, as well as replacing G with its (pre-)image under an isogeny, we also replace H and K with their (pre-)images under that isogeny along the way.

Since p is very good for G, it follows from Theorem 2.11 that $(\mathrm{GL}(V), G)$ is a reductive pair, where V is an untwisted irreducible G-module of least dimension. Since $p \geq d(G) = \dim V$, V is semisimple for K, thanks to Theorem 1.2, and thus K is G-cr, by Corollary 2.9. $\square$

Proof of Proposition 1.7. First assume $H_\sigma \subseteq P$ for some parabolic subgroup P of G. Then t lies in some maximal torus of P. So we can find a $\lambda \in Y(G)$ such that $P = P_\lambda$ and λ centralizes t. But then λ centralizes T, by (i), so $T \subseteq P$. Now we have a maximal torus T of H and a non-trivial part of each T-root group of H inside P, by (ii), so we can conclude that all of H belongs to P. Similarly, if $H_\sigma \subseteq L_\lambda$ for some $\lambda \in Y(G)$, we get $H \subseteq L_\lambda$. The reverse conclusions are obvious, since $H_\sigma \subseteq H$. $\square$

The following example shows that the conditions in Proposition 1.7 do hold generically.

Example 3.1. Let $\sigma_q\colon \mathrm{GL}(V) \to \mathrm{GL}(V)$ be a standard Frobenius endomorphism that stabilises the connected reductive subgroup H of $\mathrm{GL}(V)$ and a maximal torus T of H. Pick $l \in \mathbb{N}$ so that firstly all the different T-weights of V are still distinct when restricted to $T_{\sigma_q^l}$ and secondly that there is a $t \in T_{\sigma_q^l}$, such that $C_G(T) = C_G(t)$. Then for every $n \geq l$, both conditions in Proposition 1.7 are satisfied for $\sigma = \sigma_q^n$. Thus there are only finitely many powers of σ_q for which part (i) can fail. The argument here readily generalises to a Steinberg endomorphism of a connected reductive G which induces a generalised Frobenius morphism on H.

In contrast to the setting in Example 3.1, our next example demonstrates that the conclusion of Proposition 1.7 may fail, if condition (i) is not satisfied. Consequently, the conditions in Theorems 1.2 and 1.3 and in Corollary 1.6 are needed in general.

Example 3.2. Let $p = 3$, $q = 9$ and $H = \mathrm{SL}_2$. The simple H-module $V = L(1 + q + q^2)$ is isomorphic to $L(1) \otimes L(1)^{[1]} \otimes L(1)^{[2]}$, by Steinberg's tensor product theorem, where the superscripts denote q-twists. Thus, after identifying H with its image in $G = \mathrm{GL}(V)$, we have H is G-cr. Let $\sigma = \sigma_q$ be the standard Frobenius on G. Then $H_\sigma = \mathrm{SL}_2(9)$ is H-cr, by Lemma 1.5. Now as an H_σ-module, V is isomorphic to the H-module $L(1)\otimes L(1)\otimes L(1)$ which admits the non-simple indecomposable Weyl module of highest weight 3 as a constituent. As the latter is not semisimple for H_σ, V is not semisimple as an H_σ-module and so H_σ is not G-cr.

4. Saturation

Let $u \in \mathrm{GL}(V)$ be unipotent of order p. Then there is a nilpotent element $\epsilon \in \mathrm{End}(V)$ with $\epsilon^p = 0$ such that $u = 1 + \epsilon$. For $t \in \mathbb{G}_a$ we define u^t by

$$u^t := (1+\epsilon)^t = 1 + t\epsilon + \binom{t}{2}\epsilon^2 + \cdots + \binom{t}{p-1}\epsilon^{p-1}, \tag{4.1}$$

cf. [19]. Then $\{u^t \mid t \in \mathbb{G}_a\}$ is a closed connected subgroup of $\mathrm{GL}(V)$ isomorphic to $\mathbb{G}_a$.

Following [17] and [19], a subgroup H of $\mathrm{GL}(V)$ is *saturated* provided H is closed and for any unipotent element u of H of order p and any $t \in \mathbb{G}_a$ also u^t given by (4.1) belongs to H. The *saturated closure* H^{sat} of H is the smallest saturated subgroup of $\mathrm{GL}(V)$ containing H.

We now recall a notion of saturation for arbitrary connected reductive groups which generalises the one just given for $\mathrm{GL}(V)$. Suppose that $p \geq h(G)$. Then every unipotent element of G has order p, cf. [25]. Let u be a unipotent element of G. Then for $t \in \mathbb{G}_a$ there is a canonical "t^{th} power" u^t of u such that the map $t \mapsto u^t$ defines a homomorphism of the additive group $\mathbb{G}_a$ into G. We recall some results from [19] and [20, §5].

For G semisimple (and simply connected) let $\mathcal{U}$ be the subvariety of G consisting of all unipotent elements of G and let $\mathcal{N}$ be the subvariety of $\mathrm{Lie}(G)$ consisting of all nilpotent elements of $\mathrm{Lie}(G)$. Fix a maximal torus T of G, a Borel subgroup B of G containing T and let U be the unipotent radical of B. Since $p \geq h(G)$ and because the nilpotency class of $\mathrm{Lie}(U)$ is at most $h(G)$, we can view $\mathrm{Lie}(U)$ as an algebraic group with multiplication given by the Baker-Campbell-Hausdorff formula (cf. [10, Ch. II §6]).

Let $\Phi = \Phi(G,T)$ be the root system of G with respect to T. For α a root in Φ, let $x_\alpha : \mathbb{G}_a \to U_\alpha$ be a parametrization of the root subgroup U_α of G. Let $X_\alpha := \frac{d}{ds}(x_\alpha(s))|_{s=0}$

be a canonical generator of $\mathrm{Lie}(U_\alpha)$. Further, by $\mathrm{Aut}(G)$ we denote the group of algebraic automorphisms of G. We begin with the following result due to Serre; for a detailed proof, see [1, §6].

Theorem 4.2 ([19, Thm. 3]). *Let $p \geq h(G)$ (resp. $p > h(G)$ if G is not simply connected). There is a unique isomorphism of varieties $\log : \mathcal{U} \to \mathcal{N}$ such that the following hold:*

(i) $\log(\sigma u) = d\sigma(\log u)$ *for any* $\sigma \in \mathrm{Aut}(G)$ *and any* $u \in \mathcal{U}$;
(ii) *the restriction of* $\log$ *to* U *defines an isomorphism of algebraic groups* $U \to \mathrm{Lie}(U)$ *whose tangent map is the identity on* $\mathrm{Lie}(U)$;
(iii) $\log(x_\alpha(t)) = tX_\alpha$ *for any* $\alpha \in \Phi$ *and any* $t \in \mathbb{G}_a$.

Let $\exp : \mathcal{N} \to \mathcal{U}$ be the inverse morphism to log. We then define

$$u^t := \exp(t \log u), \tag{4.3}$$

for any $u \in \mathcal{U}$ and any $t \in \mathbb{G}_a$.

Definition 4.4 ([17], [19]). A subgroup H of G is *saturated (in G)* provided H is closed and for any unipotent element u of H and any $t \in \mathbb{G}_a$ also u^t belongs to H. For a subgroup H, its *saturated closure* H^{sat} is the smallest saturated subgroup of G containing H.

We give various fairly straightforward consequences of Theorem 4.2. The third is already recorded in [19] for centralizers of subgroups of G.

Corollary 4.5. *Let $p \geq h(G)$. Let $\sigma \in \mathrm{Aut}(G)$. Then the following hold:*

(i) $\sigma(u^t) = \sigma(u)^t$ *for any* $u \in \mathcal{U}$ *and* $t \in \mathbb{G}_a$;
(ii) *if H is a σ-stable subgroup of G, so is H^{sat};*
(iii) *for S a subgroup of $\mathrm{Aut}(G)$, $C_G(S)$ is saturated in G.*

Proof. (i). Since exp is the inverse to log, Theorem 4.2(i) gives $\sigma(\exp(X)) = \exp(d\sigma(X))$ for all $X \in \mathcal{N}$. Hence for any $u \in \mathcal{U}$ and $t \in \mathbb{G}_a$,

$$\sigma(u^t) = \sigma(\exp(t \log u)) = \exp(d\sigma(t \log(u))) = \exp(t d\sigma(\log u)) = \exp(t \log \sigma(u)) = \sigma(u)^t.$$

(ii). If H is σ-stable and M is any saturated subgroup of G containing H, then so is $\sigma(M)$: for, if $u \in \sigma(M)$ is unipotent, then $u = \sigma(v)$ for some $v \in M$ unipotent. Then $u^t = \sigma(v)^t = \sigma(v^t) \in \sigma(M)$ also, by (i) and the fact that M is saturated in G. Hence H^{sat}, the unique smallest saturated subgroup of G containing H, must also be σ-stable.

Part (iii) is immediate by (i). $\square$

As particular instances of Corollary 4.5(iii), we note that Levi subgroups of parabolic subgroups of G are saturated, since they arise as centralizers of tori in G, and also centralizers of graph automorphisms of G are saturated. One can also use Theorem 4.2 directly to show that parabolic subgroups of G are saturated. The following proof instead uses the language of cocharacters, which also allows us to observe that the process of "taking limits along cocharacters" commutes with saturation.

Proposition 4.6. *Let $\lambda \in Y(G)$ and let $P = P_\lambda$. Then for $u \in P$ unipotent and $v := \lim_{a \to 0} \lambda(a) u \lambda(a)^{-1}$, we have $\lim_{a \to 0} \lambda(a) u^t \lambda(a)^{-1} = v^t$. In particular, P is saturated.*

Proof. Observe that for any $t \in \mathbb{G}_a$ the map $h_t : \mathcal{U} \to \mathcal{U}$ given on points by $h_t(u) := u^t = \exp(t\log(u))$ is an isomorphism of varieties, by Theorem 4.2. Furthermore, by Corollary 4.5(i) we have for any $a \in k^*$

$$\left(\lambda(a)u\lambda(a)^{-1}\right)^t = \lambda(a)u^t\lambda(a)^{-1}.$$

The result now follows from Remark 2.1 and elementary limit calculations. $\square$

Remark 4.7. It follows from Theorem 4.2 that saturation is compatible with direct products, in the following sense. Suppose $G = G_1 \times \cdots \times G_r$ is a direct product of simple groups G_i. Then the unipotent variety $\mathcal{U}$ of G is the direct product of the unipotent varieties $\mathcal{U}_i$ of the factors, and likewise for the nilpotent variety $\mathcal{N}$ of $\mathrm{Lie}(G)$. If $p \geq h(G)$, then Theorem 4.2 says that there is a unique map $\log : \mathcal{U} \to \mathcal{N}$ satisfying properties (i), (ii) and (iii) of that theorem. Under this hypothesis, $p \geq h(G_i)$ for each i, and we have maps $\log_i : \mathcal{U}_i \to \mathcal{N}_i$ for each factor. Now the map

$$\log_1 \times \cdots \times \log_r : \mathcal{U} = \mathcal{U}_i \times \cdots \times \mathcal{U}_r \to \mathcal{N} = \mathcal{N}_1 \times \cdots \times \mathcal{N}_r$$

also has properties (ii) and (iii) of Theorem 4.2. We claim that it has property (i) as well. It is clear from the construction that (i) holds for any $\sigma \in \mathrm{Aut}(G)$ that stabilises each factor G_i. If simple factors $G_{i_1}, \ldots, G_{i_d}$ are all equal to each other then the symmetric group S_d acts on G by permuting these factors and fixing the others. It is also clear that (i) holds for this S_d-action. But $\mathrm{Aut}(G)$ is generated by automorphisms of the two types described above, so (i) holds, as claimed. Hence $\log_1 \times \cdots \times \log_r$ must be equal to log. Thus, in particular, each factor G_i is saturated when viewed as a subgroup of G, and if H is a saturated subgroup of G, then the projection $\pi_i(H)$ of H to the i^{th} factor is also saturated in G_i.

We now prove the compatibility of the saturation map with standard Frobenius endomorphisms. First recall that if $\sigma_q : G \to G$ is a standard q-power Frobenius endomorphism of G, then there exists a σ_q-stable maximal torus T and Borel subgroup $B \supseteq T$, and with respect to a chosen parametrisation of the root groups as above, we have $\sigma_q(x_\alpha(s)) = x_\alpha(s^q)$ for each $\alpha \in \Phi$ and $s \in \mathbb{G}_a$, cf. [13, Thm. 1.15.4(a)].

Proposition 4.8. *Let $p \geq h(G)$. Suppose $\sigma_q : G \to G$ is a standard q-power Frobenius endomorphism of G. Then the following hold:*

(i) $\sigma_q(u^t) = \sigma_q(u)^{t^q}$ *for any* $u \in \mathcal{U}$, $t \in \mathbb{G}_a$*;*

(ii) *if H is a σ_q-stable subgroup of G, then H^{sat} is also σ_q-stable.*

Proof. (i). Fix a σ_q-stable Borel subgroup B of G as in the discussion before the statement of the proposition, with unipotent radical U. Since we have $(gug^{-1})^t = gu^tg^{-1}$ for all $u \in \mathcal{U}$, $g \in G$ thanks to Corollary 4.5(i), it is enough to show the result for $u \in U$.

There are two ways to define a Frobenius-type map on $\mathrm{Lie}(U)$. Firstly, since the X_α form a basis for $\mathrm{Lie}(U)$ as a k-space, we have the map $F_q : \mathrm{Lie}(U) \to \mathrm{Lie}(U)$ given by

$$\sum_{\alpha\in\Phi^+} c_\alpha X_\alpha \mapsto \sum_{\alpha\in\Phi^+} c_\alpha^q X_\alpha.$$

For this map, it is clear that $F_q(tX) = t^q F_q(X)$ for every $t \in \mathbb{G}_a$ and $X \in \mathrm{Lie}(U)$. Alternatively, since exp and log are mutually inverse group isomorphisms between U and $\mathrm{Lie}(U)$, there is some endomorphism $f_q : \mathrm{Lie}(U) \to \mathrm{Lie}(U)$ defined by

$$\sigma_q(\exp(X)) = \exp(f_q(X))$$

for all $X \in \mathrm{Lie}(U)$ (or, equivalently, by $\log \sigma_q(u) = f_q(\log u)$ for all $u \in U$). We claim that $f_q = F_q$.

First, note that Theorem 4.2(iii) gives equality straight away for multiples of basis elements: since $\sigma_q(x_\alpha(s)) = x_\alpha(s^q)$ for each $s \in \mathbb{G}_a$ and positive root α, we have $f_q(sX_\alpha) = s^q X_\alpha = F_q(sX_\alpha)$ for all such s and α. Now recall that if we fix some ordering of the positive roots Φ^+, then each element $u \in U$ has a unique expression as a product $u = \prod_{\alpha\in\Phi^+} x_\alpha(s_\alpha)$ with the $s_\alpha \in \mathbb{G}_a$, cf. [23, Ch. I, 1.2(b)], and hence every $X = \log(u) \in \mathrm{Lie}(U)$ has expression $X = \prod_{\alpha\in\Phi^+}(s_\alpha X_\alpha)$, where this product is calculated using the Baker-Campbell-Hausdorff formula, by Theorem 4.2(ii) and (iii). Thus, to show $f_q = F_q$ it is enough to show that F_q is a group homomorphism.

Let $X = \sum_{\alpha\in\Phi^+} s_\alpha X_\alpha$ and $Y = \sum_{\alpha\in\Phi^+} t_\alpha X_\alpha$ be two elements of $\mathrm{Lie}(U)$. In calculating XY with the Baker-Campbell-Hausdorff formula we get a number of commutators involving the $s_\alpha X_\alpha$ and $t_\beta X_\beta$ for positive roots α, β. Since the Lie bracket is bilinear, we can pull all the coefficients s_α and t_β out to the front of each commutator, and hence write XY as a linear combination of commutators in the X_α. All such commutators of degree greater than 1 can be rewritten in $\mathrm{Lie}(U)$ as a linear combination of the X_α by applying the commutator relations recursively to write any $[X_\beta, X_\gamma]$ in terms of the X_α, and then expanding out and repeating. The coefficients appearing in the commutation relations lie in the finite base field $\mathbb{F}_p$, and hence are fixed under the q-power map. Thus, for any commutator C in the root elements X_α, we may conclude that $F_q(C) = C$. This is enough to conclude that $F_q(XY) = F_q(X)F_q(Y)$ for any $X, Y \in \mathrm{Lie}(U)$, as claimed.

We can now deduce that for any $t \in \mathbb{G}_a$ and any $X \in \mathrm{Lie}(U)$, we have $f_q(tX) = t^q f_q(X)$, and hence for any $u \in U$ and any any $t \in \mathbb{G}_a$,

$$\sigma_q(u)^{t^q} = \exp(t^q \log \sigma_q(u)) = \exp(t^q f_q(\log u)) = \exp(f_q(t \log u)) = \sigma_q(\exp(t \log u)) = \sigma_q(u^t),$$

which completes the proof of (i).

(ii). This follows quickly from (i), since if H is σ_q-stable and M is any saturated subgroup of G containing H, then $\sigma_q(M)$ is another saturated subgroup of G containing H: for, if $t \in \mathbb{G}_a$ and $u \in \sigma_q(M)$ is unipotent, then we may find $s \in \mathbb{G}_a$ with $s^q = t$ (since $k = \bar{k}$ is perfect) and $v \in M$ which is unipotent such that $u = \sigma_q(v)$. Then $u^t = \sigma_q(v)^{s^q} = \sigma_q(v^s) \in \sigma_q(M)$ also. Hence H^{sat}, which is the smallest saturated subgroup containing H, must also be σ_q-stable. $\square$

Combining Corollary 4.5 and Proposition 4.8, we obtain the following.

Corollary 4.9. *Let $p \geq h(G)$. Suppose $\sigma : G \to G$ is a Steinberg endomorphism of G such that $\sigma = \tau\sigma_q$, where $\tau \in \mathrm{Aut}(G)$ and σ_q is a standard q-power Frobenius endomorphism of G. Then the following hold:*

(i) *$\sigma(u^t) = \sigma(u)^{t^q}$ for any $u \in \mathcal{U}$, $t \in \mathbb{G}_a$;*
(ii) *if H is a σ-stable subgroup of G, then H^{sat} is also σ-stable.*

Proof. (i). By Corollary 4.5(i) and Proposition 4.8(i), we have for any $u \in \mathcal{U}$, $t \in \mathbb{G}_a$

$$\sigma(u^t) = \tau(\sigma_q(u^t)) = \tau(\sigma_q(u)^{t^q}) = \tau(\sigma_q(u))^{t^q} = \sigma(u)^{t^q},$$

as desired.

Part (ii) follows from the arguments in the proofs of Corollary 4.5(ii) and Proposition 4.8(ii) along with part (i). $\square$

We note that in general, a Steinberg endomorphism of a reductive group G need not be of the form given in Corollary 4.9, e.g., see Example 6.4.

We require further observations due to Serre.

Theorem 4.10 ([19, Property 2, Thm. 4]). *Let $p \geq h(G)$. Let H be a closed connected reductive subgroup of G. Then*

(i) *$h(G) \geq h(H)$; thus saturation in both H and G makes sense;*
(ii) *for H saturated and $u \in H$ unipotent, the element u^t, with respect to H, coincides with u^t, with respect to G, i.e. saturation in H coincides with saturation in G.*

Proposition 4.11 ([20, Prop. 5.2]). *If H is saturated in G, then $(H : H^\circ)$ is prime to p.*

Theorem 4.12 ([20, Thm. 5.3]). *Let $p \geq h(G)$. For a closed subgroup H of G, the following are equivalent:*

(i) *H is G-completely reducible;*
(ii) *H^{sat} is G-completely reducible;*
(iii) *$(H^{\mathrm{sat}})^\circ$ is reductive.*

The equivalence between (i) and (ii) stems from the fact that both parabolic and Levi subgroups of G are saturated. Since $h(G) \geq a(G)$, the equivalence between (ii) and (iii) is an immediate consequence of Theorem 1.1 and Proposition 4.11.

Proof of Proposition 1.9. By Theorem 4.12, K is H-cr if and only if the connected component of its saturation in H is reductive. Thanks to Theorem 4.10, saturation in H is the same as saturation in G. Thus $(K^{\mathrm{sat}})^\circ$ is reductive which is the case if and only if K is G-cr, again by Theorem 4.12. □

Note that both implications in the equivalence in Proposition 1.9 may fail if $p < h(G)$ even when H is saturated in G, e.g. see [3, Ex. 3.45] and [7, Prop. 7.17].

For ease of reference, we recall a connectedness result for H^{sat} from [2, Cor. 4.2].

Remark 4.13. Let $p \geq h(G)$. If H is a closed connected subgroup of G, then so is H^{sat}. For, consider the subgroup M of G generated by H and the closed connected subgroups $\{u^t \mid t \in \mathbb{G}_a\} \cong \mathbb{G}_a$ of G for each unipotent element $u \in G$. Then M is connected. By definition, $M \subseteq H^{\mathrm{sat}}$. If $M \neq H^{\mathrm{sat}}$, then by repeating this process with M (possibly several times), we eventually generate all of H^{sat} by H and closed connected subgroups of G isomorphic to $\mathbb{G}_a$.

Here is a further consequence of Theorem 4.12.

Corollary 4.14. *Let $p \geq h(G)$. Let $K \subseteq H$ be closed subgroups of G with H connected reductive. Then the following are equivalent:*

(i) *K is H^{sat}-completely reducible;*
(ii) *K^{sat} is H^{sat}-completely reducible;*
(iii) *$(K^{\mathrm{sat}})^\circ$ is reductive;*
(iv) *K^{sat} is G-completely reducible;*
(v) *K is G-completely reducible.*

Proof. Owing to Remark 4.13, H^{sat} is connected. Further, since $h(G) \geq a(G)$, it follows from Theorems 1.1 and 4.12 that H^{sat} is reductive.

The equivalence of (i) through (iii) follows from Theorem 4.12 applied to $K \subseteq H^{\mathrm{sat}}$ and Theorem 4.10 and the equivalence of (iii) through (v) is just Theorem 4.12. $\square$

Proof of Theorem 1.10. Thanks to Remark 4.13, H^{sat} is connected. Since $d(G) \geq h(G) \geq a(G)$, it follows from Theorems 1.1 and 4.12 that H^{sat} is reductive.

If K is H-cr, then K is G-cr, by Theorem 1.3, and K^{sat} is H^{sat}-cr, by Corollary 4.14. $\square$

The following example illustrates that in general connected reductive subgroups are not saturated.

Example 4.15. With the explicit notion of saturation from (4.1) within $\mathrm{GL}(V)$ it is easy to check that the image H of the adjoint representation of SL_p in $G := \mathrm{GL}(\mathrm{Lie}(\mathrm{SL}_p))$ is not saturated in characteristic p, cf. [19, p18]. Evidently, H is contained in the maximal parabolic subgroup P of G which stabilises the H-submodule $\mathfrak{z}(\mathrm{Lie}(\mathrm{SL}_p))$. One checks that its saturation H^{sat} in G includes all of H but also part of the unipotent radical $R_u(P)$ of P. For instance, when $p = 2$ then the adjoint representation of $H := \mathrm{SL}_2$ with respect to a suitable basis is given by

$$\mathrm{Ad}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a^2 & b^2 & 0 \\ c^2 & d^2 & 0 \\ ac & bd & 1 \end{pmatrix}.$$

If $u = \mathrm{Ad}\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right)$ then $\log u = \begin{pmatrix} 0 & b^2 & 0 \\ 0 & 0 & 0 \\ 0 & b & 0 \end{pmatrix}$, so $u^t = \exp(t \log u) = \begin{pmatrix} 1 & tb^2 & 0 \\ 0 & 1 & 0 \\ 0 & tb & 1 \end{pmatrix}$ for any $t \in \mathbb{G}_a$. We see that if $b \neq 0, 1$ then $u^{b^2}\mathrm{Ad}\left(\begin{pmatrix} 1 & b^2 \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & b^2 + b^3 & 1 \end{pmatrix}$ is a nontrivial element of $H^{\mathrm{sat}} \cap R_u(P)$, where P is the parabolic subgroup of matrices of shape $\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ * & * & * \end{pmatrix}$.

Since the abelian unipotent radical $R_u(P)$ is an irreducible SL_p-module (of highest weight $\lambda_1 + \lambda_{p-1}$, or $2\lambda_1$ when $p = 2$), being a non-zero SL_p-submodule of $R_u(P)$, U is in fact all of $R_u(P)$. So H^{sat} is of the form $H^{\mathrm{sat}} = XR_u(P)$, where X is a subgroup of the Levi subgroup of type SL_{p^2-2} of P. In particular, H^{sat} is not reductive in this case.

We briefly revisit Example 3.2 in the context of saturation.

Example 4.16. With the hypotheses and notation from Example 3.2, a non-trivial unipotent element u from H_σ has order $p = 3$, so we can saturate u in H and in G, according to (4.1) above. Likewise for non-trivial unipotent elements in H. It turns out that H is not saturated in G. Let $u = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ be in $\mathrm{SL}_2(9) = H_\sigma$ for a fixed $b \neq 1$. Then one can check that the saturations of u in H and in G do not coincide. Thus, Theorem 4.10(ii) fails on two accounts, for H is not saturated in G and $p = 3 < h(G) = \dim V = 8$, while $p > h(H) = 2$.

5. Saturation and semisimplification

In this section we assume throughout $p \geq h(G)$.

Definition 5.1 ([5, Def. 4.1])**.** Let H be a subgroup of G. We say that a subgroup H' of G is a *semisimplification of* H *(for* G*)* if there exists a parabolic subgroup $P = P_\lambda$ of G and a Levi subgroup $L = L_\lambda$ of P such that $H \subset P_\lambda$ and $H' = c_\lambda(H)$, and H' is G-completely reducible. We say the pair (P, L) *yields* H'.

The following consequence of Proposition 4.6 shows that passing to a semisimplification of a subgroup of G and saturation are naturally compatible.

Corollary 5.2. *Let H' be a semisimplification of H yielded by (P, L). Then a semisimplification of H^{sat} is given by $(H')^{\mathrm{sat}}$ and is yielded also by (P, L). Moreover, any semisimplification of H^{sat} is G-conjugate to $(H')^{\mathrm{sat}}$.*

Proof. By Lemma 2.2 there exists a $\lambda \in Y(G)$ such that $P = P_\lambda, L = L_\lambda$ and $H' = c_\lambda(H)$. According to Proposition 4.6, we have $(H')^{\mathrm{sat}} = (c_\lambda(H))^{\mathrm{sat}} = c_\lambda(H^{\mathrm{sat}})$. Since H' is G-cr, by definition, $(H')^{\mathrm{sat}}$ is G-cr by Theorem 4.12. The final statement follows from [5, Thm. 4.5]. □

In Example 4.15, the subgroup H considered is connected, non-saturated and not G-cr. In our next example, we give a connected, non-saturated but G-cr subgroup.

Example 5.3. Consider the semisimple SL_2-module $L(1) \oplus L(p) = L(1) \oplus L(1)^{[p]}$, i.e. SL_2 acting with a Frobenius twist on the second copy of the natural module and without such on the first copy. This defines a diagonal embedding of SL_2 in $M := \mathrm{SL}_2 \times \mathrm{SL}_2 \subseteq G = \mathrm{GL}_4$. While the image H of SL_2 in G is G-cr it is not saturated in G (and neither in the saturated G-cr subgroup M of G.) The argument is similar to the one in Example 4.15. Note that M is the saturation of H in G.

We close this section by noting that in general homomorphisms are not compatible with saturation. For instance, take the inclusion of a connected reductive, non-saturated subgroup H in G. See also Examples 4.15 and 5.3.

6. Saturation of finite groups of Lie type

In this section we discuss the saturation of finite subgroups of Lie type in G. Suppose that G is simple unless specified otherwise. Then σ is a (generalized) Frobenius map, i.e. a suitable power of which is a standard Frobenius map (e.g. see [13, Thm. 2.1.11]), and the possibilities for σ are well known ([24, §11]): σ is conjugate to either σ_q, $\tau\sigma_q$, $\tau'\sigma_q$ or τ', where σ_q is a standard Frobenius morphism, τ is an automorphism of algebraic groups coming from a graph automorphism of types A_n, D_n or E_6, and τ' is a bijective endomorphism coming from a graph automorphism of type B_2 ($p = 2$), F_4 ($p = 2$) or G_2 ($p = 3$). The latter instances only occur in bad characteristic, so are not relevant here. If $\tau = 1$, then we say that G_σ is *untwisted*, else G_σ is *twisted*. Note that, since G is simple, τ and σ_q commute. Note also that $C_G(\tau)$ is again simple (e.g. see [13, Thm. 1.15.2(d)]).

Theorem 6.1. *Suppose G is simple. Let $p \geq h(G)$. Let $\sigma = \tau\sigma_q$ be a Steinberg endomorphism of G and H a connected semisimple σ-stable subgroup of G. Then H^{sat} is also σ-stable, and we have:*

(i) *if $\tau = 1$, then $(G_\sigma)^{\mathrm{sat}} = G$;*
(ii) *if $\tau = 1$ and H is saturated in G, then $(H_\sigma)^{\mathrm{sat}} = H$;*
(iii) *$(G_\sigma)^{\mathrm{sat}} = C_G(\tau)$;*

(iv) *if H is saturated in G, and both τ and σ_q stabilise H separately, then $(H_\sigma)^{\mathrm{sat}} = C_H(\tau)$;*
(v) *if H is saturated in G, then $((H_\sigma)^{\mathrm{sat}})_\sigma = H_\sigma$.*

Proof. The fact that H^{sat} is σ-stable follows from Corollary 4.9.

For the rest of the proof, there is no loss in assuming that both G and H are generated by their respective root subgroups relative to some fixed maximal σ-stable tori $T_H \subseteq T_G = T$.

(i) and (ii). If $\tau = 1$, i.e. if $\sigma = \sigma_q$ is standard, then every root subgroup of G meets G_σ non-trivially. (For, each root subgroup U_α of G is σ-stable and the σ-stable maximal torus T acts transitively on U_α. So the result follows from the Lang-Steinberg Theorem.) It thus follows from Theorem 4.2(iii) and (4.3) that $(G_\sigma)^{\mathrm{sat}}$ contains each root subgroup of G; thus (i) follows. The same argument applies for (ii) by considering the simple components of H and the fact that saturation in H coincides with saturation in G, by Theorem 4.10(ii).

(iii). Since τ and σ_q commute, we have $G_\sigma = C_G(\tau)_{\sigma_q}$. Since $C_G(\tau)$ is saturated in G, by Corollary 4.5(iii), the result follows from part (ii).

(iv). Again, since τ and σ_q commute, we have $H_\sigma = C_H(\tau)_{\sigma_q}$. Now $C_H(\tau)$ is saturated in H, by Corollary 4.5. But since H is saturated in G, saturation in H coincides with saturation in G, by Theorem 4.10(ii), so the result follows from part (ii).

(v). Thanks to Corollary 4.9, $(H_\sigma)^{\mathrm{sat}}$ is σ-stable. Thus, since $H_\sigma \subseteq (H_\sigma)^{\mathrm{sat}}$ and $(H_\sigma)^{\mathrm{sat}} \subseteq H^{\mathrm{sat}} = H$, we have $H_\sigma \subseteq ((H_\sigma)^{\mathrm{sat}})_\sigma \subseteq H_\sigma$, and equality follows. □

Remark 6.2. We note that Theorem 6.1(v) generalises [17, Thm. B(1)]: If $G = \mathrm{SL}_n(k)$, $\sigma = \sigma_q$ is a standard Frobenius endomorphism of G, and H is a σ-stable subgroup of G, then it follows directly from (4.1) that H^{sat} is again σ-stable. Thus in particular, if H is a connected, saturated semisimple σ-stable subgroup of G, then by Theorem 6.1(v) we have $((H_\sigma)^{\mathrm{sat}})_\sigma = H_\sigma$; cf. [17, Thm. B(1)].

We consider some explicit examples for Theorem 6.1.

Example 6.3. Let $G = \mathrm{SL}_n$ and let σ be the Steinberg endomorphism of G given by

$$g \mapsto \sigma_q(n_0({}^t g^{-1})),$$

where ${}^t g$ denotes the transpose of the matrix g and n_0 is a representative in G of the longest word in the Weyl group of G. Note that $\tau(g) = n_0({}^t g^{-1})$ is the graph automorphism of G. Then $\sigma^2 = \sigma_{q^2}$ is a standard Frobenius map of G given by raising coefficients to the $(q^2)^{\mathrm{th}}$ power. Note that $G_\sigma = \mathrm{SU}(q)$ is the special unitary subgroup of G. We have $\mathrm{SU}(q) = G_\sigma \subseteq G_{\sigma^2} = \mathrm{SL}(\mathbb{F}_{q^2})$, and since σ_q commutes with τ, we have (assuming $p \geq n$) $(G_\sigma)^{\mathrm{sat}} = C_G(\tau)$, by Theorem 6.1(iii), while $(G_{\sigma^2})^{\mathrm{sat}} = G$, by Theorem 6.1(i).

In the case when G is no longer simple, additional kinds of Steinberg endomorphisms are possible.

Example 6.4. Let H be a simple group defined over $\mathbb{F}_q$ and let σ_q be the corresponding standard Frobenius map of H. Let $G = H \times \cdots \times H$ (r factors) and let ΔH be the diagonal copy of H in G. Let π be the r-cycle permuting the r direct copies of H of G cyclically and let $f = (\sigma_q, id_H, \ldots, id_H) : G \to G$. Then $\sigma = \pi f$ is a Steinberg endomorphism of G where π and f do not commute. We have $G_\sigma = (\Delta H)_{\sigma_q}$, where by abuse of notation σ_q is a standard Frobenius map on ΔH. (Note that $(\Delta H)_{\sigma_q}$ is isomorphic to $H_{\sigma_q} = H(\mathbb{F}_q)$.) Now suppose $p \geq h(H) = h(G)$. Then ΔH is saturated in G, by Remark 4.7. Thus $(G_\sigma)^{\mathrm{sat}} = ((\Delta H)_{\sigma_q})^{\mathrm{sat}} = \Delta H$, by Theorem 6.1(ii).

We present an instance where Theorem 6.1(i) can be applied even though G is not simple and σ is a Steinberg endomorphism which is not a generalized Frobenius endomorphism.

Example 6.5. Let $p \geq 2$. Let σ_p, σ_{p^2} be the standard Frobenius maps of SL_2 given by raising coefficients to the p^{th} and $(p^2)^{\text{th}}$ powers, respectively. Let $G = \mathrm{SL}_2 \times \mathrm{SL}_2$. Then the map $\sigma = \sigma_p \times \sigma_{p^2} : G \to G$ is a Steinberg morphism of G that is not a (generalized) Frobenius morphism (cf. the remark following [13, Thm. 2.1.11]). We have $G_\sigma = \mathrm{SL}_2(\mathbb{F}_p) \times \mathrm{SL}_2(\mathbb{F}_{p^2})$. The image of the canonical embedding of G into $\mathrm{GL}_4(k)$ is a saturated subgroup of $\mathrm{GL}_4(k)$, and so the saturation map in G is given by the formula from (4.1). By Remark 4.7, saturating G_σ inside G amounts to saturating each factor of G_σ inside each factor of G. Now, by applying Theorem 6.1(i) to each factor of G, we get $(G_\sigma)^{\text{sat}} = G$.

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